A New Axiomatic Geometry: Cylindrical (or Periodic) Geometry

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ABSTRACT

Historically, new geometries have developed by making changes to current axiom systems and then developing a model that illustrates the new geometry. The most classic example of this is the development of hyperbolic geometry, which came from negating Euclid's fifth postulate. This project inverts the process: we start with a geometric object and attempt to find a set of axioms that characterizes the geometry modeled by the object. This project investigates the infinite cylinder and develops an axiom system that captures the geometry on the surface of an infinite cylinder. This paper sets up a logical system which describes this geometry and which is used to prove theorems about the geometry.

1. INTRODUCTION

Historically, new geometries have developed by making changes to current axiom systems and then developing a model that illustrates the new geometry. The most classic example of this is the development of hyperbolic geometry. Before hyperbolic geometry developed, the mathematical community was attempting to prove that Euclid's fifth postulate was actually derivable from the other four postulates; Saccheri made the most rigorous effort at this in 1733 [5].

The founders of non-euclidean geometry took the view that the fifth postulate was actually an independent postulate and attempted to form a geometric system with Euclid's first four postulates and a negation of the fifth postulate, G.S. Klugel was the first person to publish that he believed that the fifth postulate was independent of the other four. He published this in 1763 [6]. In 1766, Lambert started considering what geometries would result by replacing the fifth postulate with one of its alternatives. He discovered another form of non-euclidean geometry called elliptic geometry, but he did not discover hyperbolic geometry [5]. Carl Friedrich Gauss and his associates worked on discovering hyperbolic geometry at the beginning of the 19^{th} century. It is believed from his correspondence that Gauss had the main ideas of hyperbolic geometry sometime near 1813 but he never published them [6].

The discovery of hyperbolic geometry was first published independently by both Janos Bolyai (1832) and Nikolai Ivanovich Lobachevsky (1829) [5]. It was later proven by Bolyai and Lobachevsky that hyperbolic geometry was logically as consistent as real analysis [6]. This means that hyperbolic geometry is exactly as consistent as Euclidean geometry.

The new geometry was named “hyperbolic” by Felix Klein in 1871 [6]. It now is modeled most often by the hyperbolic plane, an infinite surface with constant negative curvature. Studies in non-euclidean geometry, especially hyperbolic geometry, have become very interesting recently because the field of astronomy has realized that Einstein’s space-time continuum is not necessarily Euclidean; the theory of relativity is inconsistent with Euclidean geometry. [4]

This project inverts the process historically used to discover new geometries; we start with a geometric object and attempt to find a set of axioms that characterizes the geometry modeled by the object. In this case, we chose the infinite cylinder to serve as our model and interpreted "straight" lines to be geodesics (lines which would feel straight to a creature living on the surface) on the cylinder. We describe the thirty-seven axioms that compose cylindrical geometry as well as the necessary definitions. We model our axioms off of Hilbert's Axioms of Euclidean geometry [3]. We also discuss and prove several theorems within cylindrical geometry.

Thus, in the following sections of this paper, we will give the thirty-three axioms for cylindrical geometry divided into six sections based on content; required definitions will be interspersed as necessary. After some sections of axioms, we will give some theorems in cylindrical geometry that follow from those axioms and demonstrate their power. The conclusion of this paper gives information about the overall geometry and poses open problems in cylindrical geometry as well as ideas for other geometries that could be formed.

2. UNDEFINED TERMS AND MODELS

All logical systems must begin with primitive undefined terms. Primitive terms prevent a logical infinite regression caused by words within definitions being defined in terms of other new words. The primitive terms in cylindrical geometry are: "Point", "Line", "Between", "Lies on", "Separate", "Measure", and "Union".

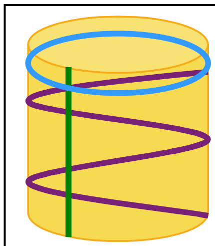


Figure 1

While these terms cannot be rigorously defined within the geometric system that we are developing, they do correspond to certain aspects of the infinite cylinder we are using as our model. Given an infinite cylinder in Euclidean 3-space, a "point" refers to a Euclidean point on the surface of the cylinder. A "line" refers to a geodesic on the cylinder. Intuitively, the geodesics on the cylinder can be found by drawing straight lines on a piece of transparency paper and then rolling that piece of paper into a cylinder. Anyone who tries this will find that a horizontal line becomes a horizontal circle, a vertical line stays a vertical line, and that a diagonal line becomes a helix (See Figure 1). These three kinds of lines are the straight lines on an infinite cylinder. Because of shared properties between them, the author clusters vertical lines and helixes into one group, which is called open lines, and calls the group containing the horizontal circles closed lines.

The following undefined words have a specific syntax. "Between" is used in contexts such as, "Point *B* is between points *A* and *C*." The properties of betweenness are given as axioms. "Lies on" is used to say, "Point *A* lies on line *m*." "Separate" is used in sentences such as, "Points *A* and *B* separate points *C* and *D*." This is similar to how this term is used in elliptic geometry. The "measure" of a line segment is its length and the "measure" of an angle is the degrees it encompasses.

There is an alternative way of visualizing this geometry; the axioms and theorems proved in this paper will hold equally well on both models. The second model we call the periodic model. In this model, the Euclidean plane is broken into vertical "stripes" and the content of each stripe in terms of points and lines is identical. On Figure 2, this separation is represented by the pink dotted lines. Thus, a dot made in the middle panel of Figure 2 would also appear simultaneously in the same place on the left and right panels and every other panel on the Euclidean plane. Thus, this model is called the *periodic model* because the pattern repeats with a fixed period. Under

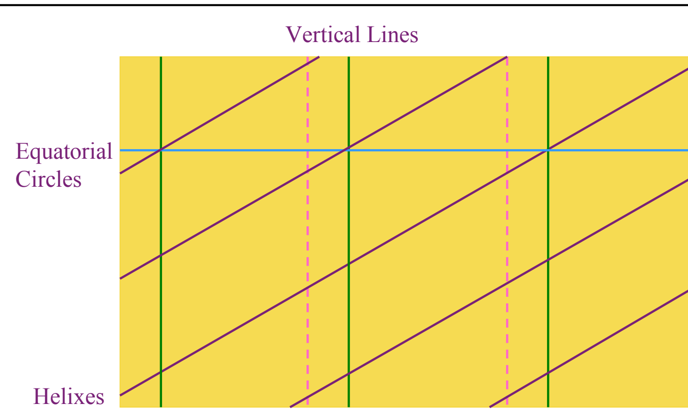


Figure 2

the periodic model, lines are the same as straight lines on the standard Euclidean plane.

It is easy to see that the cylindrical and periodic models represent the same geometry. An isomorphism can be created from the periodic model to the cylindrical model by considering only one panel from the periodic model and then gluing opposite horizontal edges of the panel together to form the cylinder. Then, horizontal lines become equatorial circles, diagonal lines become helices, and vertical lines remain vertical lines (See Figure 2).

3. AXIOMS OF ORDER

The first collection of axioms are the *Axioms of Order*. These axioms concern where points lie along a line in relation to one another. These concepts are needed to define the different types of lines that are found on the infinite cylinder. The first four axioms formalize our intuitions about what it means for a point to be between two others on a line. The next six axioms give the properties of the separation relation. The last axiom concerns triangles on the infinite cylinder. The axioms will be enumerated with a letter and then a number. The letter will represent which section the axiom is from, and the number will tell where in the section the axiom is located.

If point B were between points A and C, then we would also want point B to be between points C and A. Axiom A-1 formalizes this notion.

A-1: If point B is between points A and C (we will denote this by A-B-C), then A, B, and C are distinct points on the same line and C-B-A.

For axiom A-2 to read nicely, we will now have to make our first definition:

Definition 1: A line ***contains*** a point A if A lies on that that line. If a line l contains a point A, we write $A \in l$. If l does not contain A, we write $A \notin l$.

Axiom A-2 allows us to establish the existence of points other than our given points on a line. It intuitively allows us to assume that point B on a line has a point to its left and to its right.

A-2: Given any two distinct points B and D, there exist distinct points A, C, and E lying on each line that contains B and D such that A-B-D, B-C-D, and B-D-E.

Axiom A-3 formalizes an intuitive idea about how betweenness carries from one line to another:

A-3 If m and n are two lines containing the same points A, B, and C, and A-B-C on line m, then A-B-C on line n.

As an example, imagine that A, B, and C all lie on a vertical line on our vertical cylinder such that B is between A and C and A, B, and C are equally spaced. Now imagine a spiral (helix) that spirals around once from A to B and then one more time from B to C. B is still between A and C on that helix. Note that without the hypothesis that A, B, and C all lie on both lines, there is no reason to believe that if $\overline{\boldsymbol{AB}_m}$ and $\overline{\boldsymbol{AB}_n}$ with a point C lying on the line segment $\overline{\boldsymbol{AB}_m}$ that the point C lies on the line segment $\overline{\boldsymbol{AB}_n}$; C may not lie on line n.

Now that there is a notion of betweenness we can formally distinguish among, and thus define, open and closed lines as they were used in Section 2:

Definition 2: A line is ***closed*** if for all distinct points A, B, and C on the line, A-B-C, B-C-A and C-A-B.

Definition 3: A line is ***open*** if, for all points A, B and C on the line, there is some betweenness relationship between A, B and C, without loss of generality, say A-B-C, and A-B-C precludes all other orderings of A, B, and C except C-B-A.

The next axiom ensures that, given three collinear points, there is some betweenness relationship concerning them. It also ensures that all the betweenness relationships are satisfied (a closed line) or that one element is exclusively between the

A-4 A line is either open or closed.

other two (open line):

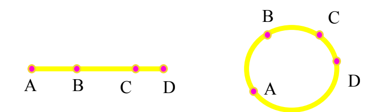
Note that a line cannot be both open and closed because the definitions of open lines and closed lines are mutually exclusive.

At this time it seems appropriate to make some other definitions about lines that will be useful throughout this project:

Definition 4: Two lines ***intersect*** if they contain a common point.

Definition 5: Two or more points are ***collinear*** if they all lie on the same line.

The next six axioms in the *Axioms of Order* concern the undefined term "separate". The idea for this term was taken from the standard axiomatization of elliptic geometry. This order term can be used for any line in this geometry, but it is especially useful for closed lines because betweenness does not give any point-location information on closed lines. It is necessary to have four points to talk about order of points on a closed line. For the intuitive idea of separation, see Figure 3.



Separation: *A* and *C* separate *B* and *D*

Figure 3

The next few axioms will seem very similar to the previous axioms. Axiom A-5 is analogous to Axiom A-1:

> **A-5** If points A and B separate points C and D (we denote this $(A,B|C,D)$), then points A, B, C, and D are collinear and distinct.

Axiom A-6 is the separation version of the second part of Axiom A-1:

> **A-6** If $(A,B|C,D)$, then $(C,D|A,B)$ and $(B,A|C,D)$.

Axiom A-7 guarantees that separation divides point positions into more than one equivalence class. The definition of open lines provides this role for betweenness. This is not a property of betweenness and closed lines:

> **A-7** If $(A,B|C,D)$, then A and C do not separate B and D.

Axiom A-8 is similar to axiom A-5:

> **A-8** If points A, B, C, and D are collinear and distinct, then $(A,B|C,D)$, $(A,C|B,D)$, or $(A,D|B,C)$.

Axiom A-9 gives the possible relationships between four points with a specified separation relationship and a fifth point:

A-9 For five distinct collinear points $A,B,C,D,$ and $E,$ if $(A,B|D,E)$, then $(A,B|C,D)$ or $(A,B|C,E)$.

Axiom A-10 gives the association connecting betweenness and separation. This axiom is very important so that a problem phrased in one order relation can be solved using properties of the other order relation. Any four points must have both a betweenness ordering and a separation equivalance class. Intuitively, we know this relationship but it needs to be formalized in the geometry so that it can be used in proving theorems. First, we need to define what a betweenness ordering on four points is:

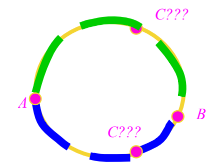
Definition 6: A ***betweenness ordering on four points, A-B-C-D*** means *A-B-C, B-C-D, A-B-D*, and *A-C-D.*

A-10 If $(A,C|B,D)$, then *A-B-C-D, A-D-C-B, C-D-A-B,* or *C-B-A-D.*

For Axiom A-11, we must first define line segments. The intuitive definition of line segments given the machinery we have already set up would be to call a line segment from A to B the points A, $B,$ and all points between A and B.

Definition 7: An ***open line segment***, AB, is the points A, B and all points C on a specified open line that contains A and B such that *A-C-B* is true. We call A and B ***endpoints*** of AB.

While this definition works for open lines, it fails to uniquely determine a line segment on a closed line (See Figure 4). Thus, it actually takes three points to uniquely determine a line segment on a closed line and to do this we use the separation relation.



Is AB the blue segment or the green segment?

Figure 4

Definition 8: For points A, B, and D on a closed line, a ***closed line segment***, $\overline{AB}_{\not{D}}$**,** is the points A, B and all points C on the specified closed line such that $(A,B|C,D)$ is true. We call A and B ***endpoints*** of $\overline{AB}_{\not{D}}$ and we call D the ***exclusion point*** for $\overline{AB}_{\not{D}}$.

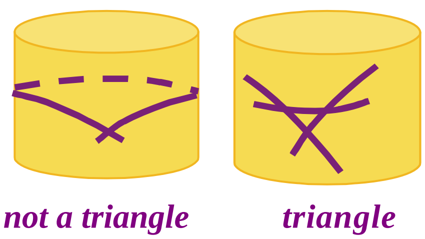
Note that if A and B do not separate D and some point E on the same closed line, then $\overline{AB}_{\not{D}}$ and $\overline{AB}_{\not{E}}$ contain the same points. Closed line segments generate equivalence classes based on their exclusion points; this will be formalized in theorem 1. Since every segment in the same equivalence class contains exactly the same points, we represent the equivalence class by any element in it.

Now we define a line segment in general.

Definition 9: In general, a ***line segment***, $\overline{AB}$, refers to an open line segment with endpoints A and B or a specific closed line segment with endpoints A and B.

Next, we want to state the cylindrical axiom that is equivalent to the Euclidean axiom that guarantees that a triangle has a definite inside and outside. The problem is that the question of "What is a triangle?" is not as easy to answer in cylindrical geometry. If a triangle is defined as the union of any three line segments with pairwise common endpoints (our intuitive notion of a triangle), this leads to many strange formations that would not obey the premise that a triangle must have an inside and an outside (See Figure 5); this premise is similar to the Jordan Curve Theorem of differential geometry (which would not be true in general in cylindrical geometry). Ensuring that a triangle has an inside and an outside enables us to state axiom A-11, that a line that passes through one side of the triangle must pass through another as well. This guarantees that the lines that make up the edges of any triangle, and thus any line, must be solid and not dashed or broken.

Figure 5



Thus, we need a more complicated concept for a triangle. To solve this problem, we need to ensure that the triangle does not completely encircle the cylinder or overlap itself and that it contains an inside and outside. To do this, we will determine that there must exist an open line that does not hit any segment whose union defines the triangle. Definition 10 formalizes this idea:

> **Definition 10:** A ***triangle,*** denoted ΔABC is the union of line segments $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ such that these segments do not intersect except at points *A, B,* or *C* and such that there exists an open line *n* that does not intersect $\overline{AB}$, $\overline{AC}$, or $\overline{BC}$. We call $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ the ***sides of the triangle.***

A-11 Let ΔABC be a triangle. Then if line m contains a point on the segment $\overline{AB}$, then m will also contain a point on the segment $\overline{AC}$ or $\overline{BC}$.

Axiom A-11 states that a triangle has an inside and an outside. This is because if line n crosses one side of the triangle, it must cross one of the other two as well.

These axioms appear to give every desirable property of the order relations. Axioms A-1 through A-4 are modeled after Hilbert's axioms of betweenness for Euclidean geometry [3]. Axioms A-5 through A-9 are modeled after the separation axioms for elliptic geometry [1].

4. THEOREMS CONCERNING THE AXIOMS OF ORDER

The following theorems show that some other desirable and intuitive notions of ordering follow from these axioms. The proofs also give a good example of the use of the interaction between separation and betweenness. This interaction is new in cylindrical geometry because neither the Euclidean, nor hyperbolic, nor elliptic geometries have both open and closed lines.

Theorem 1: Given a closed line, *m*, and two points, *A* and *B*, on *m*, then the line segments $\overline{AB}$ on *m* can be partitioned into two sets based on their exclusion points and these partitions form two equivalence classes of line segments with every line segment in the same equivalence class containing the same points.

Proof: Let A, B, and D be distinct points on the closed line m. Then $\overline{AB}_D$ is a specific closed line segment on m. Let C be a point on $\overline{AB}_D$. Let E be a point such that $(A,B|C,E)$. Note that by axiom A-9, E does not lie on $\overline{AB}_D$ since A and B do not separate D and E. Let F be any point on the line segment $\overline{AB}_D$. Then, $(A,B|D,F)$. By axiom A-9, either $(A,B|E,F)$ or $(A,B|D,E)$. Since we know that A and B do not separate D and E, this implies that $(A,B|E,F)$. Therefore, any point on $\overline{AB}_D$ is also on $\overline{AB}_E$ by definition. Similarly, if G is any point on $\overline{AB}_E$, G will also be on $\overline{AB}_D$.

Consider $\overline{AB}_C$. Note that D and E are on $\overline{AB}_C$. Since E could be any point on $\overline{AB}_C$, every point on $\overline{AB}_C$, if used as an exclusion point for endpoints A and B creates a line segment that contains the same points as $\overline{AB}_D$. Note that by applying the reasoning in the paragraph above, any point not on $\overline{AB}_C$ which lies on line m can be used as an exclusion point with the endpoint A and B to create a line segment that contains the same points as $\overline{AB}_C$. Therefore, it is possible to use the exclusion points of segments to classify the closed line segments on m into two partitions with every member of the partition containing exactly the same points.

All that is left is to show that this partition creates an equivalence class. Let $\overline{AB}_D$ ◙ $\overline{AB}_E$ indicate that $\overline{AB}_E$ and $\overline{AB}_D$ are members of the same element of the partition. Then $\overline{AB}_D$ ◙ $\overline{AB}_D$ so ◙ is reflexive. If $\overline{AB}_D$ ◙ $\overline{AB}_E$, then $\overline{AB}_E$ ◙ $\overline{AB}_D$ by the above paragraphs so ◙ is symmetric. If $\overline{AB}_D$ ◙ $\overline{AB}_E$ and $\overline{AB}_E$ ◙ $\overline{AB}_G$, then A and B do not separate D and E and A and B do not separate E and G. Then, by the contrapositive of axiom A-9, A and B do not separate D and G. Therefore, $\overline{AB}_D$ ◙ $\overline{AB}_G$. Thus, ◙ is transitive so ◙ is an equivalence relation. î

Theorem 2: Given *A-B-C* and *A-C-D* on an open line *m*, *A-B-C-D* on *m*.

Proof: *A-B-C* and *A-C-D* on an open line *m*. By Axiom A-1, *A*, *B, C,* and *D* are all distinct points on the line *m* with the exception that *B* may equal *D*. For the sake of contradiction, assume that *B* equals *D*. Then, by substitution of *D* into *A-B-C,* we get *A-D-C*, which is a contradiction of *A-C-D* because the line *m* is open. Thus, *A*, *B, C,* and *D* are all distinct points. Thus, (*A,B*|*C,D*), (*A,C*|*B,D*), or (*A,D*|*B,C*) by axiom A-8.

Case 1: If (*A,B*|*C,D*), then *A-C-B-D, A-D-B-C, B-D-A-C,* or *B-C-A-D* by A-10. *A-C-B-D* implies *A-C-B* by definition 6, which is a contradiction of *A-B-C* by the definition of open. *A-D-B-C* implies *A-D-C* by definition 6, which is a contradiction of *A-C-D* because the line *m* is open. *B-D-A-C* implies *B-A-C* by definition 6, which is a contradiction of *A-B-C* because the line *m* is open. *B-C-A-D* implies *B-C-A* by definition 6, which is a contradiction of *A-B-C* by the definition of open. Thus, *A* and *B* do not separate *C* and *D.*

Case 2: If (*A,D*|*B,C*), then *A-B-D-C, A-C-D-B, D-C-A-B,* or *D-B-A-C.* *A-B-D-C* implies *A-D-C* by definition 6, which is a contradiction of *A-C-D* because the line *m* is open. *A-C-D-B* implies *A-C-B* by definition 6, which is a contradiction of *A-B-C* by the definition of open. *D-C-A-B* implies *C-A-B* by definition 6, which is a contradiction of *A-B-C* because the line *m* is open. *D-B-A-C* implies *B-A-C* by definition 6, which is a contradiction of *A-B-C* by the definition of open. Thus, *A* and *D* do not separate *B* and *C.*

Case 3: If (*A,C*|*B,D*), then *A-B-C-D, A-D-C-B, C-D-A-B,* or *C-B-A-D.* *A-D-C-B* implies *A-C-B* by definition 6, which is a contradiction of *A-B-C* because the line *m* is open. *C-D-A-B* implies *C-A-B* by definition 6, which is a contradiction of *A-B-C* by the definition of open. *C-B-A-D* implies *C-A-D* by definition 6, which is a contradiction of *A-C-D* because the line *m* is open. Thus, *A-B-C* and *A-C-D* imply *A-B-C-D* because *A-B-C-D* is the only option that does not lead to a contradiction. أ

Theorem 3: Let *A,B,C,* and *D* be points such that *A-B-D* and *A-C-D* on an open line *m.* Then either *A-B-C-D, A-C-B-D,* or *B*=*C.* Moreover, exactly one of the relations holds.

Proof: If $B = C$, then we are done so assume that $B \neq C$. Thus, *A*, *B, C,* and *D* are all distinct points on the line *m.* Thus, (*A,B*|*C,D*), (*A,C*|*B,D*), or (*A,D*|*B,C*) by axiom A-8.

Case 1: If (*A,B*|*C,D*), then *A-C-B-D, A-D-B-C, B-D-A-C,* or *B-C-A-D*. *A-D-B-C* implies *A-D-C* by definition 6, which is a contradiction of *A-C-D* by the definition of open. *B-D-A-C* implies *D-A-C* by definition 6, which is a contradiction of *A-C-D* because the line *m* is open. *B-C-A-D* implies *C-A-D* by definition 6, which is a contradiction of *A-C-D* because the line *m* is open. *A-C-B-D* implies *A-B-D* and *A-C-D* by definition 6, so *A-C-B-D* is possible.

Case 2: If (*A,D*|*B,C*), then *A-B-D-C, A-C-D-B, D-C-A-B,* or *D-B-A-C*. *A-B-D-C* implies *A-D-C* by definition 6, which is a contradiction of *A-C-D* by the definition of open. *A-C-D-B* implies *A-D-B* by definition 6, which is a contradiction of *A-B-D* because the line *m* is open. *D-C-A-B* implies *D-A-B* by definition 6, which is a contradiction of *A-B-D* because the line *m* is open. *D-B-A-C* implies *D-B-A* by definition 6, which is a contradiction of *A-B-D* by the definition of open.

Case 3: If (*A,C*|*B,D*), then *A-B-C-D, A-D-C-B, C-D-A-B,* or *C-B-A-D*. *A-D-C-B* implies *A-D-C* by definition 6, which is a contradiction of *A-C-D* by the definition of open. *C-D-A-B* implies *C-D-A* by definition 6, which is a contradiction of *A-C-D* because the line *m* is open. *C-B-A-D* implies *C-A-D* by definition 6, which is a contradiction of *A-C-D* because the line *m* is open. *A-B-C-D* implies *A-B-D* and *A-C-D* by definition 6, so *A-B-C-D* is possible. Thus, we have eliminated all possibilities but *A-B-C-D* or *A-C-B-D* in the case where $B \neq C$.

If $B=C$, *A-B-C-D* and *A-C-B-D* are not possible because the betweenness relation implies that all points are distinct. Likewise, if one of the betweenness relations holds, $B \neq C$. *A-B-C-D* implies *A-B-C* and *A-C-B-D* implies *A-C-B*. Since *A-B-C* and *A-C-B* are mutually exclusive, so are *A-B-C-D* and *A-C-B-D*. أ

We would now like to generalize the notion of betweenness by defining a betweenness relation on *n* points.

Definition 11: A ***n*-betweenness relationship**, A_1- A_2-A_3-…- A_n, among *n* points means that A_i- A_j-A_k **whenever** $i < j < k$.

We would like *n*-betweenness to give us a linear ordering of these *n* points on a given line; that is, we would like to be able to talk about the order the points appear on the line. Thus, we prove the following lemma and theorem.

Lemma 4: Given 3 points on a line, there is a betweenness relationship among them.

Proof: There are three distinct points *A*, *B*, and *C* on line *m*. By A-2, there exists points *D,E,* and *F* such that *D-A-E-B-F.* Since all five of these points, *A, B,D,E,* and *F*, are distinct, *C* can equal at most one of these points. Thus, there are at least five distinct points on line *m* including points A, B, and C. Let X be a point distinct from A, B, and C. Then there exists a separation relationship between points A, B, C, and X by A-8. But then there exists a 4-betweeness relationship between A, B, C, and X by A-10. But then, there is a betweenness relationship among A, B, and C. It is unique by the definition of an open line. î

Theorem 5: Given *n* points, $A_1 \ldots A_n$, on an open line, there exists a betweenness relationship among them which is unique up to reversal.

Proof: Assume that there is a unique (up to reversal) (n–1)-betweenness relationship on any (n-1) distinct points $A_1 \ldots A_{n-1}$. Consider the n distinct points $A_1 \ldots A_{n-1}$ and Q. Given any two distinct points in A_i, $A_j \in \{A_1 \ldots A_{n-1}\}$, there is a distinct betweenness relationship on A_i, A_j, and Q. Without loss of generality, assume that A_1-A_2- … - A_{n-1} (if not, re-label). If A_i-A_j-Q for all A_i and A_j in $\{A_1 \ldots A_{n-1}\}$ with $i < j$, then A_1-A_2- … - A_{n-1}-Q by definition. If A_i-A_j-Q for all A_i and A_j in $\{A_1 \ldots A_{n-1}\}$ with $i > j$, then Q-A_1-A_2- … - A_{n-1} by definition. Assume for the sake of contradiction, that A_i-A_j-Q for all A_i and A_j in $\{A_1 \ldots A_{n-1}\}$ and there exists w,x,y,z such that A_w-A_x-Q, $w < x$ and A_z-A_y-Q, $y < z$. Then there is a separation relationship between A_w, A_x, Q, and A_y by axiom A-8. Thus, $(A_w,A_x|Q,A_y)$, $(A_w,Q|A_x,A_y)$, or $(A_w,A_y|A_x,Q)$.

Case 1: $(A_w,A_x|Q,A_y)$. Therefore, by axiom A-10, A_w-Q-A_x-A_y, A_x-A_y-A_w-Q, A_w-A_y-Q-A_x, or A_x-Q-A_w-A_y. But A_w-A_x-Q, so this implies that none of these choices are possible.

Case 2: $(A_w,Q|A_x,A_y)$. Therefore, by axiom A-10, A_w-A_x-Q-A_y, A_w-A_y-Q-A_x, Q-A_y-A_w-A_x, or A_x-Q-A_w-A_y. But A_w-A_x-Q, so this implies A_w-A_x-Q-A_y. But A_z-A_y-Q so A_w-A_x-A_z-

A_y-Q. Thus, A_w-A_x-A_z-A_y. But A_w-A_x-A_y-A_z. by the induction hypothesis which is a contradiction since all these points are being considered on an open line.

Case 3: (A_w,A_y|A_x,Q). Therefore, by axiom A-10, A_w-A_x-A_y-Q, A_w-A_x-A_y-Q, or A_y-Q-A_w-A_x, A_y-A_x-A_w-Q. But A_w-A_x-Q, so this implies A_w-A_x-A_y-Q or A_w-A_y-A_x-Q. Assume A_w-A_x-A_y-Q. But A_z-A_y-Q so A_w-A_x-A_z-A_y-Q, A_w-A_z-A_x -A_y-Q, or A_z-A_w-A_x-A_y-Q. Each of these possibilities contradicts A_w-A_x-A_y-A_z,.which is true by the induction hypothesis. Now consider A_w-A_y-A_x-Q. But A_z-A_y-Q. Then, A_w-A_z-A_y-A_x-Q or A_z-A_w- A_y-A_x-Q. Both of these contradict A_w-A_x-A_y-A_z,.which is true by the induction hypothesis.

Therefore, we are done or there exists two points A_i, $A_j \in \{A_1 \ldots A_{n-1}\}$ such that A_i -Q-A_j.. We want to show that there exist two points A_k, A_{k+1} such that A_k-Q-A_{k+1}. If we could do that, it would be straightforward to establish the n-betweenness relation on Q and $A_1 \ldots A_{n-1}$.

Let A_i-Q-A_j. with $i<j$. If $j=i+1$, we are done. Assume by induction that an n-betweenness relationship exists if $j=i+p$. Let $j=i+p+1$. Then, if Q is between A_i and $A_{i+1} \ldots A_{i+p}$, we are done. Thus, A_{i+p}-Q-A_j. But $A_{i+p} = A_{j-1}$ so A_{j-1} -Q-A_j.

Since, given two endpoints, we cannot assume that there is only one line segment between them, we need to make the following notational definition so that we know along which line our line segment lies.

Definition 12: Let $\overline{AB}_m$ represent the line segment with endpoints A and B such that the points contained in $\overline{AB}_m$ that are not A or B lie on line m between points A and B.

We would now like to argue that $\overline{AB}_m$ is unique if m is an open line. If m is a closed line, we would like to show that $\overline{AB}_m$ has exactly two possibilities and that if $\overline{AB}_m$ has a given exclusion point, then $\overline{AB}_m$ is unique.

Theorem 6: $\overline{AB}_m$ is unique if *m* is an open line.

Proof: $\overline{AB}_m$ represents the open line segment with endpoints A and B such that the points contained in $\overline{AB}_m$ that are not A or B lie on line m between points A and B by definition. Yet, an open line segment, AB, is the points A, B and all points C on a specified open line that contains A and B such that A-C-B is true. We have already specified the line, and the points C such that A-C-B in line m is a well-defined set. Since $\overline{AB}_m$ is the union of all such points, $\overline{AB}_m$ is unique. ∎

Theorem 7: If *m* is a closed line, $\overline{AB}_m$ has exactly two possibilities and if $\overline{AB}_m$ has a given exclusion point, *D*, then $\overline{AB}_{m\not{D}}$ is unique.

Proof: $\overline{AB}_m$ represents the open line segment with endpoints A and B such that the points contained in $\overline{AB}_m$ that are not A or B lie on line m between points A and B by definition. Every closed line segment has an exclusion point. Let us call the exclusion point of our line segment D. Then $\overline{AB}_{m\not{D}}$**,** is the points A, B and all points C on the specified closed line such that $(A,B|C,D)$ is true by definition. We have already specified the line, and the points C such that $(A,B|C,D)$ in line m is a well-defined set. Since $\overline{AB}_{m\not{D}}$ is the union of all such points, $\overline{AB}_{m\not{D}}$ is unique.

Now, we show that if an exclusion point is not given, there are only too possible line segments. Let D be a point on the closed line m. Let C be a point on the segment $\overline{AB}_m$. Let F be the exclusion point for $\overline{AB}_m$. Then $(A,B|C,F)$ by definition. But then by A-9, $(A,B|C,D)$ or $(A,B|D,F)$. If $(A,B|C,D)$, then D is also an exclusion point for $\overline{AB}_{m\cancel{F}}$ and $\overline{AB}_{m\cancel{D}}$ is one possibility for the line segment $\overline{AB}_m$. If $(A,B|D,F)$, then D lies on $\overline{AB}_{m\cancel{F}}$**.** But if D lies on line F, then if E is any point such that $(A,B|E,D)$, then E lies on $\overline{AB}_{m\cancel{D}}$ and does not lie on $\overline{AB}_{m\cancel{F}}$**.** Thus, $\overline{AB}_{m\cancel{D}}$ is the other option for segment $\overline{AB}_m$. ∎

Theorem 8: Given *A-B-C* along an open line *m*, then $\overline{AC}_m = \overline{AB}_m \cup \overline{BC}_m$ where $\overline{AC}_m$ is the shortest line along *m* that has *A* and *C* as its endpoints and contains *B*, and *B* is the only point common to the segments $\overline{AB}_m$ and $\overline{BC}_m$.

Proof: Consider $\overline{AC}_m$, $\overline{AB}_m$, and $\overline{BC}_m$ as sets of the points they contain. Then, $\overline{AC}_m = \overline{AB}_m \cup \overline{BC}_m$ means that (a) any point in $\overline{AC}_m$ is in either $\overline{AB}_m$ or $\overline{BC}_m$ and that (b) any point in $\overline{AB}_m$ or $\overline{BC}_m$ is in $\overline{AC}_m$.

(a) By definition of an open line segment, $\overline{AC}_m$ is the points A, C and all points D on m such that *A-D-C* is true. Since B is on $\overline{AC}_m$, then *A-B-C* on m. Let E be some other point on $\overline{AC}_m$. By definition, *A-E-C*. By Theorem 2, either *A-B-E-C, A-E-B-C,* or $B=E$. If *A-B-E-C,* then *B-E-C* so E is in $\overline{BC}_m$. If *A-E-B-C,* then *A-E-B* so E is in $\overline{AB}_m$. If $B=E$, E is in both $\overline{BC}_m$ and $\overline{AB}_m$. Therefore, $\overline{AC}_m$ is a subset of $\overline{AB}_m \cup \overline{BC}_m$.

(b) Let F be a point on $\overline{AB}_m$. Then *A-F-B* by the definition of an open line segment. Since *A-F-B* and *A-B-C,* then *A-F-B-C* by Theorem 1. Thus, *A-F-C* by definition 7. Thus, F is on $\overline{AC}_m$ by the definition of an open line segment. Let G be a point on $\overline{BC}_m$. Then B-G-C by the definition of an open line segment. Since A-B-C and B-G-C, then C-G-B and C-B-A by Axiom A-1. Thus C-G-B-A by Theorem 1. Thus, C-G-A by definition 6. Therefore, A-G-C by Axiom A-1. Thus, G is on $\overline{AC}_m$ by the definition of an open line segment. Therefore, $\overline{AB}_m \cup \overline{BC}_m$ is a subset of $\overline{AC}_m$. Thus, $\overline{AC}_m = \overline{AB}_m \cup \overline{BC}_m$. ∎

Theorem 9: Given (A,C |B,D) along a closed line m, then $\overline{AC}_{\not D} = \overline{AB}_{\not D} \cup \overline{BC}_{\not D}$ and B is the only point common to the segments $\overline{AB}_{\not D}$ and $\overline{BC}_{\not D}$.

Proof: Consider $\overline{AC}_{\not D}$, $\overline{AB}_{\not D}$, and $\overline{BC}_{\not D}$ as sets of the points they contain. Then, $\overline{AC}_{\not D} = \overline{AB}_{\not D} \cup \overline{BC}_{\not D}$ means that (a) any point in $\overline{AC}_{\not D}$ is in either $\overline{AB}_{\not D}$ or $\overline{BC}_{\not D}$ and that (b) any point in $\overline{AB}_{\not D}$ or $\overline{BC}_{\not D}$ is in $\overline{AC}_{\not D}$.

(a) By definition, $\overline{AC}_{\not D}$ is the points A, C and all points X on the specified closed line such that ($A,C|X,D$) is true. Let E be some point on $\overline{AC}_{\not D}$. If $B = E$ then E is contained in $\overline{AB}_{\not D}$ and $\overline{BC}_{\not D}$ as an endpoint. Assume $E \neq B$. Then A, B, C, D, and E are all distinct points on the same closed line. Since E is on $\overline{AC}_{\not D}$, ($A,C|E,D$) by the definition of a closed line segment. Since ($A,C|E,D$), ($E,D|A,C$) by axiom A-6. By axiom A-9, since ($E,D|A,C$), either ($E,D|B,A$) or ($E,D|B,C$). If ($E,D|B,A$), then ($B,A|E,D$) by axiom A-6. Thus, also by axiom A-6, ($A,B|E,D$) which means that E is on $\overline{AB}_{\not D}$ by the definition of a closed line segment. If ($E,D|B,C$), then (B,C |E,D) by axiom A-6.

Thus, E is on $\overline{BC}_{\cancel{D}}$ by the definition of a closed line segment. Thus, $\overline{AC}_{\cancel{D}}$ is a subset of $\overline{AB}_{\cancel{D}} \cup \overline{BC}_{\cancel{D}}$.

(b) Let F be a point on $\overline{AB}_{\cancel{D}}$. If F equals A or B then F is on $\overline{AC}_{\cancel{D}}$ by the hypothesis, so assume that $F \neq A$ and $F \neq B$. Then, $(A,B|F,D)$ by the definition of a closed line segment. $(A,B|F,D) = (F,D|A,B)$ by axiom A-6. Since $A, B, C, D,$ and F are all distinct points on a closed line segment, $(F,D|A,B)$ implies $(F,D|C,B)$ or $(F,D|C,A)$ by axiom A-9. $(A,C|B,D)$ implies $(B,D|A,C)$ by axiom A-6. $(B,D|A,C)$ implies $(B,D|F,A)$ or $(B,D|F,C)$. $(B,D|F,A) = (F,A|B,D) = (A,F|B,D)$ by axiom A-6 which contradicts $(A,B|F,D)$ by axiom A-7. Thus, $(B,D|F,C)$. But $(B,D|F,C) = (D,B|F,C) = (F,C|D,B)$ which contradicts $(F,D|C,B)$ by axiom A-7. Thus, $(F,D|C,A)$. $(F,D|C,A) = (C,A|F,D) = (A,C|F,D)$ by axiom A-6. Thus, F is on $\overline{AC}_{\cancel{D}}$ by the definition of a closed line segment. Thus, $\overline{AB}_{\cancel{D}}$ is a subset of $\overline{AC}_{\cancel{D}}$.

Let G be a point on $\overline{BC}_{\cancel{D}}$. If G equals B or C then G is on $\overline{AC}_{\cancel{D}}$ by the hypothesis, so assume that $G \neq B$ and $G \neq C$. Then, $(B,C|G,D)$ by the definition of a closed line segment. $(B,C|G,D) = (G,D|B,C)$ by axiom A-6. Since $A, B, C, D,$ and F are all distinct points on a closed line segment, $(G,D|B,C)$ implies $(G,D|A,B)$ or $(G,D|A,C)$ by axiom A-9. $(A,C|B,D)$ implies $(B,D|A,C)$ by axiom A-6. $(B,D|A,C)$ implies $(B,D|G,A)$ or $(B,D|G,C)$. $(B,D|G,C) = (G,C|B,D) = (C,G|B,D)$ by axiom A-6 which contradicts $(C,B|G,D)$ by axiom A-7 and $(C,B|G,D) = (B,C|G,D)$ by axiom A-6. Thus, $(B,D|G,A)$. But $(B,D|G,A) = (D,B|G,A) = (G,A|D,B)$ which contradicts $(G,D|A,B)$ by axiom A-7. Thus, $(G,D|G,C)$. $(G,D|A,C) = (A,C|G,D)$ by axiom A-6. Thus, G is on $\overline{AC}_{\cancel{D}}$ by the definition of a closed line segment. Thus, $\overline{BC}_{\cancel{D}}$ is a subset of $\overline{AC}_{\cancel{D}}$. Therefore,

$\overline{AB}_{\text{Đ}} \cup \overline{BC}_{\text{Đ}}$ is a subset of $\overline{AC}_{\text{Đ}}$. But $\overline{AC}_{\text{Đ}}$ is also a subset of $\overline{AB}_{\text{Đ}} \cup \overline{BC}_{\text{Đ}}$ so $\overline{AC}_{\text{Đ}}$ = $\overline{AB}_{\text{Đ}} \cup \overline{BC}_{\text{Đ}}$. أ

5. AXIOMS OF CONNECTION

The axioms of connection are the axioms concerning the interaction between points and lines. While these axioms come first in Hilbert's axioms of Euclidean geometry, it was necessary for us to first develop our *Axioms of Order* because the relations between points and open lines are not always the same as the relationship between points and closed lines.

It is clear in the first two axioms of this section that cylindrical geometry and the proofs and the foundation of cylindrical geometry will be significantly different from those in Euclidean geometry. While the first axiom of Euclidean geometry given by both Euclid and Hilbert reads, "Between two points there is one and only one straight line," the related axioms (yes, multiple axioms) in cylindrical geometry read as follows:

B-1	Through any point *A* there is always one and only one closed line. We denote it Φ_A.

B-2	Through any two distinct points A and B, exactly one of the following occurs: *(i)* *A* and *B* lie on a countably infinite number of open lines. *(ii)* *A* and *B* lie on the same closed line and there is no open line that contains both *A* and *B*.

In terms of the cylindrical model axiom B-1 states that for any given point on the cylinder, there is exactly one horizontal circle on which that point lies. Axiom B-2 is less intuitive. On the infinite cylinder, picture two points that are not at the same height (i.e. not on the same closed line) and lie on the same vertical line. These two points are also connected by the helix that twists up and right and rotates 360°, the helix that twists up and left 360°, the helix that twists up and right 720°, the helix that twists up and left 720°, etc. Continuing this pattern, we can see that this would identify a countable number of lines connecting our two points. Yet, these are the only possible lines because only helices with regular spacing are allowed as straight lines in our model. It is then easy to see that this concept generalizes to two points that are not on the same closed circle or on the same vertical line.

The following definition allows us to talk about the segment(s) with the shortest length between points *A* and *B*. This concept is particularly useful since line segments are not unique. Many proofs from Euclidean geometry that use Axiom 1 can only be

> **Definition 13:** The line segment(s) from *A* to *B* with minimal measure are called
>
> $\overline{AB}_0$.

phrased in terms of the shortest segment in cylindrical geometry.

Now we are in a position to state axiom B-3.

B-3 Through any two points A and B there exist 1 (or 2) line segments m (and n) such that A and B are the endpoints of m (and n) and the length of m (and n) is less than the length of any other line segment with points A and B as its endpoints. If there are two such lines m and n, then $\overline{AB}_m$ has the same measure as $\overline{AB}_n$. We call $\overline{AB}_{m(n)}$ the ***shortest segment*** from A to B and denote it $\overline{AB}_0$.

On the model of the cylinder, two points have two shortest segments between them if they are diametrically opposed. That is, if the straight line that passes through them in three-space intersects the axis of the infinite cylinder. See Figure 6.

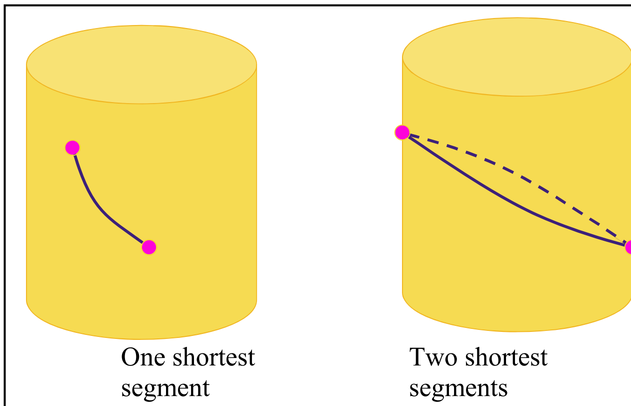


Figure 6

Now we can define the shortest line(s) between two points.

Definition 14: The ***shortest line(s)*** from A to B is the line(s) that contains all the points of the shortest segment.

Now we need to establish that our geometry is not the empty set. In particular, we need to make sure that our geometry contains at least enough points and lines to create a full geometry such as the one on the model. These axioms are necessary for the completeness of the axiom system.

B-4 On every line there exists at least two distinct points.

B-5 Given any line, there exists a point that is not on that line.

B-6 There exists a point.

6. THEOREMS BASED ON THE AXIOMS OF CONNECTION

The following theorems flesh out the existence theorems and establish basic facts of cylindrical geometry that we would certainly like to be true.

Theorem 10: For every point there is at least one open line that does not contain it.

Proof: Let A be a point (B-6). Let *a* be the closed line through *A* (B-1). Let *B* be another point on *a* (B-4). Let *C* be a point not on *a* (B-5). There exists a line through *B* and *C* (B-2). Call it *x*. *x* is either open or closed. *x* is not closed because *x* is clearly distinct from *a* and if *x* were closed, point *B* would lie on two distinct closed lines which violates axiom B-1. Therefore, *x* is open. If *x* contained point *A*, then points *A* and *B* would both be contained in both an open line and a closed line which violates axiom B-2.
ٱ

Theorem 12: For every point there exists at least one open line and one closed line that contain it.

Proof: Let A be a point (B-6). Let a be the closed line through A (B-1). Let B be another point on a (B-4). Let C be a point not on a (B-5). There exists a line through A and C (B-2). Call it m. $m \neq a$ because C is on m and C is not on a. m is an open line because there is only one closed line through the point by axiom B-1 and every line is either open or closed by axiom A-4. ∎

Theorem 13: There are an infinite number of points on an open line.

Proof: Let m be an open line. Suppose, for the sake of contradiction, that m contains only n points. By axiom B-4, $n \geq 2$. If n=2, let the points be A and B. Then, by axiom A-2 there exist distinct points $C, D,$ and E such that C-A-D-B-E so there are more than 2 distinct points which is a contradiction. Assume $n > 2$. Let the points be named $A_1 \ldots A_n$ such that A_1- A_2-A_3-…- A_n by theorem 5. By axiom A-4, there exists a point Q such that Q- A_1- A_2. By the openness of the line m, Q cannot be any of n points given which is a contradiction. Therefore, there are an infinite number of points on an open line. ∎

Corollary 14: There are an infinite number of closed lines.

Proof: There exists an open line, m, with an infinite number of points by theorem 13. Each of those points lies on exactly one closed line. For the sake of contradiction, assume that there are two points, A and B, on m that both lie on the same closed line, c.

That would contradict axiom B-2. Therefore, all points on the open line lie on different closed lines. Thus, there are an infinite number of closed lines. î

Corollary 15: All points on the open line lie on distinct closed lines.

Theorem 16: There are an infinite number of open lines.

Proof: Let A be a point (B-6). Let *a* be the closed line through *A* (B-1). Let *B* be another point on *a* (B-1). Let *C* be a point not on *a* (B-5). Through *A* and *C* there exists a countably infinite number of open lines by axiom B-2. î

Before our next existence theorem, we need a preliminary theorem telling that all closed lines are parallel. Thus, we define parallel.

> **Definition 15:** Two lines *m* and *n* are ***parallel*** if there exists no point, *A,* that is on both line *m* and line *n*.

Theorem 17: All closed lines are parallel.

Proof: Let *c* and *d* be two distinct closed lines. Assume *c* and *d* intersect at a point *A*. Then there are two closed lines through *A*. This contradicts axiom B-1. Thus, all closed lines are parallel. î

Theorem 18: There exist three distinct lines that are not concurrent.

Proof: Let A be a point (B-6). There exists an open line, *o*, and a closed line, *c*, through *A* by theorem 12. There exists a second point *C* on *o* by axiom B-4. *C* is not on

c by corollary 15. There exists a closed line, c_2, through C by axiom B-1. c_2 does not intersect c by theorem 17. î

Theorem 19: Let A and B be distinct points and let m be an open line. Then $\overline{AB}_m = \overline{BA}_m$.

Proof: Suppose that $\overline{AB}_m$ is an open line segment. Then, by definition, $\overline{AB}_m$ is the points A, B and all points C on the open line m such that A-C-B is true. Let D be any of the points C so A-D-B. By axiom A-1, B-D-A on the same line m. Thus, D is on segment $\overline{BA}_m$ by definition of open line segment. But D could be any of the points C, and the endpoints of $\overline{AB}_m$ and $\overline{BA}_m$ are the same, so $\overline{AB}_m \subseteq \overline{BA}_m$. By a similar argument, $\overline{BA}_m \subseteq \overline{AB}_m$. Thus, $\overline{AB}_m = \overline{BA}_m$ if $\overline{AB}_m$ is an open line segment. î

Theorem 20: Let A and B be distinct points and let m be a closed line. Then $\overline{AB}_{m\cancel{D}} = \overline{BA}_{m\cancel{D}}$.

Proof: Suppose that $\overline{AB}_{m\cancel{D}}$ is a closed line segment. (Definition of line segment). Then, by definition, there exists a point D collinear to points A and B on a specific closed line m such that $\overline{AB}_{m\cancel{D}}$ is the points A, B and all points C on the specified closed line such that $(A,B|C,D)$ is true. Let E be any of the points C so $(A,B|E,D)$. By axiom A-6, $(B,A|E,D)$ on the same line. Thus, E is on segment $\overline{BA}_{m\cancel{D}}$ by definition of closed line segment. But E could be any of the points C, and the endpoints of $\overline{AB}_{m\cancel{D}}$ and $\overline{BA}_{m\cancel{D}}$ are the same, so $\overline{AB}_{m\cancel{D}} \subseteq \overline{BA}_{m\cancel{D}}$. By a similar argument, $\overline{BA}_{m\cancel{D}} \subseteq \overline{AB}_{m\cancel{D}}$. Thus, $\overline{AB}_{m\cancel{D}} = \overline{BA}_{m\cancel{D}}$ if $\overline{AB}_{m\cancel{D}}$ is a closed line segment. î

Theorem 21: If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.

Proof: Let $\overline{AB}_m \cong \overline{CD}_n$. $\overline{CD} \cong \overline{CD}$ by axiom B-2. Then $\overline{CD} \cong \overline{AB}$ by axiom B-2. أ

7. METRIC AXIOMS FOR ANGLES AND SEGMENTS

The metric axioms of a geometry establish the structure necessary to make measurements in a geometry. These metric axioms will establish a measuring system for the length of segments and for the width of an angle. The first metric axiom establishes the concept of the measure of a line segment as a positive real number.

C-1 The measure of a segment $\overline{AB}$ is denoted **m**(AB)**.** Given a segment $\overline{AB}$, **m**(AB) assigns one and only one real number to $\overline{AB}$.

C-2 states a property of the measure of a segment that we would certainly want to be true. Before we can state axiom C-2, we need to specify what we mean by a "certain direction" on a line. Thus, we make the following definitions. To do this, we start with defining a ray.

Definition 16: A ***ray***, $\overrightarrow{AB}$, is the points *A, B*, and all points *C* on a line such that *A-C-B* or *A-B-C* is true.

We proceed from here to make a series of definitions which culminates with the definition of a direction on a line. Because a ray on a closed line would be all of the closed line, it is necessary to be able to restrict a ray to a local area. Thus, we define a local ray.

Definition 17: A ***local ray*** (or a ***ray locally***), $\overrightarrow{AB}$, along line *m,* is the union of the line segment $\overline{AB}$ and all points *D* on the segment $\overline{EC}_m$, where *E* and *C* are both ε away from *B* along *m* and $E \neq C$ for any real number ε.

Definition 18: Let the ***shortest ray*** from *A* to *B* be the ray $\overrightarrow{AB}$ along the line that contains $\overline{AB}_0$**.** Denote this $\overrightarrow{AB}_0$**.**

Definition 19: To specify a line, *m*, in a certain ***direction*** from point *A* means to choose between local rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ where *B*-*A*-*C* on *m*.

We are now ready for axiom C-2.

C-2 Let *m* be an open line. Let *A* be a point on *m*. For every real number α, there exists one and only one point *B* in a given direction on *m* so that **m**(AB) = α.

C-2 is necessary to ensure the continuity of the measures of segments. That is, axiom E-2 ensures that if we fix one point, *A*, and vary a second point, *B*, then as the point *B* slides back and forth, the measure of $\overline{AB}$ varies in a continuous way.

Since the length of a closed line is finite, it is necessary to establish a fixed value for the length of a closed line.

C-3 Let m be a closed line and let points A and B lie on m. Then, $0 < \mathbf{m}(AB_0) \leq \mathbf{180}$.

Be careful to note that 180 is the measure of *half* of the length of a closed line. This is because the measure of the shortest segment between A and B is between 0 and 180. Thus, the total length of a closed line is 360. Therefore, the length of a closed line segment $\overline{AB}$ is the same as the degree measure of the angle $\angle AOB$ in Euclidean 3-space where O is the point on the axis of the infinite cylinder which lies in the same plane as the closed circle.

Axiom C-4 is the equivalent of axiom C-2 for closed lines.

C-4 Let m be any closed line. Let A be a point on m. For every real number α between 0 and 180, there exists one and only one point B in a given direction on m so that $\mathbf{m}(AB_0) = \alpha$.

C-5 If A-B-C on line m, then $\mathbf{m}(AC_m) = \mathbf{m}(AB_m) + \mathbf{m}(BC_m)$.

C-5 provides the additivity of the measure of segments.

We now define inequality relations on segments.

Definition 20: $\overline{AB} < \overline{CD}_m$ ($\overline{AB}$ ***less than*** $\overline{CD}_m$) means that there exists a point E on segment $\overline{CD}_m$ such that $\mathbf{m}(AB)=\mathbf{m}(CE_m)$.

Definition 21: $\overline{CD}_m > \overline{AB}$ ($\overline{CD}_m$ ***greater than*** $\overline{AB}$) means $\overline{AB} < \overline{CD}_m$.

Definition 22: $\overline{AB} \leq \overline{CD}_m$ ($\overline{AB}$ ***less than or equal to*** $\overline{CD}_m$) means $\overline{AB} < \overline{CD}_m$ or **m**(AB)=**m**(CD$_m$).

Definition 23: $\overline{CD}_m \geq \overline{AB}$ ($\overline{CD}_m$ ***greater than or equal to*** $\overline{AB}$) means $\overline{AB} < \overline{CD}_m$ or **m**(AB)=**m**(CD$_m$).

We now give similar axioms for how measure acts on angles. First we have to define what we mean by an angle.

Definition 24: Let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ be two rays such that A is the only point they have in common. Then $\overrightarrow{AB} \cup \overrightarrow{AC}$ is an ***angle.*** We denote this angle $\angle BAC$ or $\angle CAB$. The rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called the ***sides of the angle*** and the point A is called the ***vertex of the angle.***

Definition 25: The ***interior of the angle*** is the set of points that lie on a ray $\overrightarrow{AZ_0}$ such that X-Z-Y on $\overline{XY_0}$ where X is ε away from A along $\overrightarrow{AB}$ and Y is δ away from A along $\overrightarrow{AC}$ (See Figure 7). ε and δ must be chosen small enough that the interior determined by ε and δ is equal to the interior determined by ε_1 and δ_1 where $\varepsilon_1 < \varepsilon$ and $\delta_1 < \delta$.

The ***exterior of the angle*** is the set of points that are not in the interior of the angle or the sides of the angle.

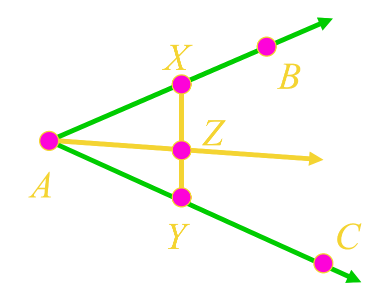


Figure 7

This definition of the interior of an angle is different from the standard definition in Euclidean geometry. The standard Euclidean definition states, "If X is a point on ray $\overrightarrow{AB}$ and Y is a point on ray $\overrightarrow{AC}$, and Z is a point on the line segment $\overline{XY}$, then Z is in the interior of angle $\angle BAC$." This definition is not adequate in cylindrical geometry for many reasons. A line segment $\overline{XY}$ in cylindrical geometry is not unique. The obvious solution to this problem would be only to consider the shortest segment between X and Y. This is not adequate, though, because at some point the interior of the angle $\angle BAC$ will become wider than the longest possible shortest line segment (see Figure 8). Thus, some sections that we would like to consider to be the interior of the angle would no longer be in the interior of the angle. For example, in Figure 8, we would like the entire cylinder above point D to be in the interior of the angle, but it would not be under the shortest-segment definition of the interior of an angle.

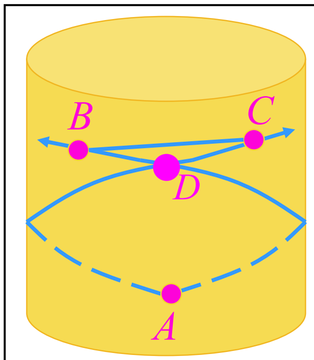


Figure 8

The definition of interior of an angle that is used in this paper ensures that the points X and Y are chosen close enough to the vertex of angle $\angle BAC$ that the surface that angle $\angle XAY$ will be considered on will be locally flat. This local definition is then extended to the entire cylinder by considering the rays $\overrightarrow{AZ_0}$. Visually, as Z moves along segment $\overline{XY_0}$, these rays will scan the intended interior of our angle like a laser beam.

We can now state our first axiom concerning the measure of angles.

C-6 The measure of an angle $\angle ABC$ is denoted $\prec ACB$. Given an angle $\angle ABC$, $\prec$ assigns one and only one real number to $\angle ABC$.

Axiom C-7 sets the scale for angle measurements. It causes the measure of an angle to be given in degrees.

C-7 For every angle $\angle ABC$, $\prec$ is between 0 and 180.

Axiom C-8 is similar to axiom C-2 in regard to angles instead of segments. Therefore we need a similar concept to the direction on a line that is applicable for angles. Therefore, we need to define a given side of a line or ray. That is, our first axiom concerning congruent angles will be concerned with the halves of the surface that are created by a line. To do this, it is necessary to be able to find a point X specific given line such that X is the closest possible point to a given point A. To do this, it is of course necessary to define perpendicular lines on the infinite cylinder. Thus, we make the following definitions.

Definition 26: Two rays are ***opposite rays*** if they lie on the same line and locally have only their vertex in common for some choice of ε.

Note that choosing a direction on a line means choosing between two local, opposite rays.

Definition 27: Two angles are ***supplementary angles*** if they have a common vertex, have exactly one common side, and their non-common sides are opposite rays.

Definition 28: An angle is called a ***right angle*** if it is the member of a pair of supplementary angles that have the same measure.

Therefore, if we are given an angle $\angle ABC$ that we know is a right angle, than we know that if we consider the line along the ray $\overrightarrow{BA}$ and any point D on that line such that D-B-A, then angle $\angle DBC$ has the same measure as $\angle ABC$.

Definition 29: Two rays are ***perpendicular rays*** if their union is a right angle.

Definition 30: Two lines, m and n, are ***perpendicular lines*** if they contain a pair of perpendicular rays. We denote this $n \perp m$.

Definition 31: A ***nearest point*** on a line m to a point A (not necessarily on line m) is a point, X, on line m such that line a through A perpendicular to m contains X and, for any point Y on both m and a, $\overline{AX} \leq \overline{AY}$.

Thus, given a line m and a point A, the closest point to A that lies on line m is the "nearest point" of m and A.

Definition 32: The ***partner of an open line***, m, is the set of all points, A, such that A has exactly two nearest points with regard to line m.

Thus, the partner of an open line will be the line that has the same slope and is evenly spaced from the first open line. (See Figure 9: The green line and blue line are partner lines). Now, we are ready to define whether two points are on the same side of an open line. Notice that an open line and its partner divide the surface of the cylinder into

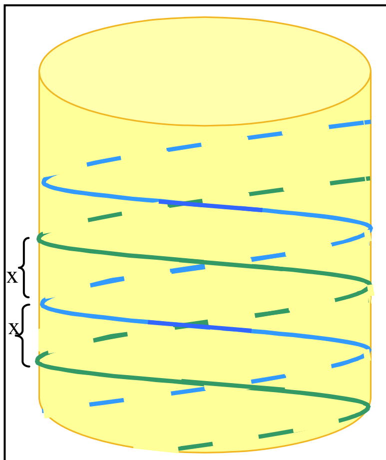


Figure 9

two sets. In Figure 9, these two sets would be the strip of cylinder which had the blue line on top and the green line on the bottom and the strip of cylinder with the green line on the top and the blue line on the bottom. Intuitively, two points are on the same side of an open line if they are in the same set of the partition created by an open line and its partner. Technically, the definition looks a little different.

Definition 33: Let m be an open line and let A and B be any points that do not lie on m. If A=B or if the $\overline{AB}_0$ contains no point lying on m or the partner of m, we say that A and B lie ***on the same side of*** m. If $A \neq B$ and if $\overline{AB}_0$ contains a point on m or the partner of m, we say that A and B are ***on opposite sides of*** m.

It is easy to see that this definition, while easier to word in a technically precise way, is equivalent to the partition definition given in the paragraph above.

We now consider what it means for two points to be on the same side of a closed line. Intuitively, it is easy to see that a closed line divides the cylinder into two subsets, the top and the bottom. We would like two points to be on the same side of a closed line if they are both in the same subset. Thus, we make the following definition.

Definition 34: Let m be a closed line and let A and B be any points that do not lie on m. If $A=B$ or if the $\overline{AB}_0$ contains no points lying on m, we say that A and B lie ***on the same side of m***. If $A \neq B$ and if $\overline{AB}_0$ contains a point on m, we say that A and B are ***on opposite sides of m.***

Once again, it is easy to see that this technical definition gives us exactly what we want.

We are now ready to state axiom C-8.

C-8 (Angle Construction Axiom) Let $\overrightarrow{AB}_m$ be a ray. For every real number α between 0 and 180, there is exactly one ray $\overrightarrow{AC}_0$ where C is on a given side of m such that $\prec CAB = \alpha$.

Axiom C-9 ensures the additivity of the measure of angles.

C-9 (Angle Addition Postulate) If D is in the interior of $\angle ABC$ and we consider ray $\overrightarrow{BD}_0$, then $\prec ABC = \prec ABD + \prec DBC$.

Axiom C-10 establishes that the measure of the angle formed by three points on the same line is 180.

C-10 $\angle ABC$ and $\angle ABD$ are supplementary angles if and only if $\prec ABC + \prec ABD = 180$.

We now define inequality relations on angles.

Definition 35: $\angle ABC < \angle DEF$ ($\angle ABC$ ***less than*** $\angle DEF$) means that there exists a ray $\overrightarrow{EG}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ so that $\prec ABC = \prec DEG$.

Definition 36: $\angle ABC > \angle DEF$ ($\angle ABC$ ***greater than*** $\angle DEF$) means $\angle DEF < \angle ABC$.

Definition 37: $\angle ABC \leq \angle DEF$ ($\angle ABC$ ***less than or equal to*** $\angle DEF$) means $\angle ABC < \angle DEF$ or $\prec ABC = \prec DEF$.

Definition 38: $\angle ABC \geq \angle DEF$ ($\angle ABC$ ***greater than or equal to*** $\angle DEF$) means $\angle ABC > \angle DEF$ or $\prec ABC = \prec DEF$.

8. AXIOM OF CONGRUENCE

We define congruence for segments and angles.

Definition 39: Two segments $\overline{AB}$ and $\overline{A'B'}$ are ***congruent*** ($\overline{AB} \cong \overline{A'B'}$) if **m**(AB)=**m**(A′B′).

Definition 40: Two angles $\angle ABC$ and $\angle DEF$ are ***congruent*** if $\prec ABC = \prec DEF$.

Axiom D-1 is the Side-Angle-Side (SAS) property of congruent triangles. To prepare for this, we define congruent triangles.

Definition 33: Two triangles are ***congruent triangles*** if there exists a one-to-one correspondence between their vertices so that the corresponding sides and corresponding angles are congruent.

D-1 If, in the two triangles ΔABC and $\Delta A'B'C'$, $\overline{AB}$ is congruent to $\overline{A'B'}$, $\overline{AC}$ is congruent to $\overline{A'C'}$, and $\angle BAC$ is congruent to $\angle B'A'C'$, then $\angle A'B'C'$ is congruent to the angle, $\angle ABC$ and $\angle ACB$ is congruent to $\angle A'C'B'$, and $\overline{BC}$ is congruent to $\overline{B'C'}$.

9. THEOREMS OF CONGRUENCE

The first theorem determines that the congruence relation on line segments is an equivalence relation.

Theorem 22: If a segment $\overline{A'B'}$ and a segment $\overline{A''B''}$ are congruent to the same segment $\overline{AB}$ then $\overline{A'B'}$ is congruent to the segment $\overline{A''B''}$. Every segment is congruent to itself. Segment $\overline{AB}$ is congruent to segment $\overline{BA}$.

Proof: If $\overline{A'B'}$ is congruent to $\overline{AB}$, then **m**(AB)=**m**(A′B′). Similarly, since $\overline{A''B''}$ is congruent to $\overline{AB}$, then **m**(AB)=**m**(A″B″). Thus, **m**(A′B′)=**m**(A″B″). Therefore, by definition, $\overline{A'B'} \cong \overline{A''B''}$. For all segments, $\overline{AB}$, **m**(AB)= **m**(AB). Thus, $\overline{AB} \cong \overline{AB}$. Also, by theorems 19 and 20, $\overline{AB} = \overline{BA}$ setwise. Thus, $\overline{AB}$ and $\overline{BA}$ are two different names for the same set of points. Therefore, by axiom C-1, **m**(AB)=**m**(BA). Thus, $\overline{AB} \cong \overline{BA}$. Thus, $\cong$ is an equivalence relation. ∎

The next theorem solidifies the additive property of line segments.

Theorem 23: On the line *m*, let $\overline{AB}$ and $\overline{BC}$ be two line segments which, except for *B*, have no point in common. Furthermore, on the same or another line m', let $\overline{A'B'}$ and $\overline{B'C'}$ be two segments which except for B' have no point in common. In that case, if $\overline{AB}$ is congruent to $\overline{A'B'}$ and $\overline{BC}$ is congruent to $\overline{B'C'}$, then $\overline{AC}$ is congruent to $\overline{A'C'}$.

Proof: By the definition of congruence, **m**(AB)= **m**(A′B′) and **m**(BC)=**m**(B′C′). By Theorem 2, *A-B-C* on line *m* and *A′-B′-C′* on line *m′*. Therefore, by axiom C-5, **m**(AC)=**m**(AB)+**m**(BC)= **m**(A′B′)+**m**(B′C′)= **m**(A′C′). Thus, by definition, $\overline{AC} \cong \overline{A'C'}$.

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The next theorem determines that the congruence relation on angles is an equivalence relation.

Theorem 24: If an angle $\angle ABC$ and an angle $\angle DEF$ are congruent to the same segment $\angle GHI$ then $\angle ABC$ is congruent to $\angle DEF$. Every angle is congruent to itself. Angle $\angle ABC$ is congruent to angle $\angle CBA$.

Proof: If $\angle ABC$ is congruent to $\angle GHI$, then $\prec ABC = \prec GHI$. Similarly, since $\angle DEF$ is congruent to $\angle GHI$, then $\prec DEF = \prec GHI$. Thus, $\prec ABC = \prec DEF$. Therefore, by definition, $\angle ABC \cong \angle DEF$. For all angles, $\angle ABC$, $\prec ABC = \prec ABC$. Thus, $\angle ABC \cong \angle ABC$. By definition, $\angle ABC = \overrightarrow{AB} \cup \overrightarrow{AC} = \overrightarrow{AC} \cup \overrightarrow{AB} = \angle CBA$**.** Therefore, $\sphericalangle ABC = \sphericalangle CBA$. Thus, $\cong$ is an equivalence relation. Î

Theorem 25: All right angles have a measure of 90 and thus are congruent.

Proof: Let $\angle ABC$ be a right angle. Therefore, $\angle ABC$ is a member of a pair of supplementary angles whose measures are equal. Without loss of generality, let $\angle ABD$ be the other angle in that pair. Since $\angle ABC$ and $\angle ABD$ are supplementary, $\sphericalangle ABC + \sphericalangle ABD = 180$. Since $\angle ABC$ and $\angle ABD$ have the same measure, $\sphericalangle ABC = \sphericalangle ABD$. Therefore, $2 \times \sphericalangle ABC = 180$ so $\sphericalangle ABC = 90$. Î

10. AXIOM OF PARALLELS

The axiom of parallels for a geometry is dependent on the Gaussian curvature of the surface of the model the geometry describes. An infinite cylinder has the same Gaussian curvature as the Euclidean plane. Recall that a second model for cylindrical geometry is a periodic Euclidean plane. Therefore, the axiom of parallels for cylindrical geometry is the same as the parallel axiom for Euclidean geometry.

E-1	Given a line *m* and a point *A* that is not on line *m,* then there is one and only one line that can be drawn through point *A* that is parallel to line *m*.

11. THEOREMS CONCERNING THE AXIOM OF PARALLELS

The axiom of parallels enables us to prove the transitivity of parallelism.

Theorem 26: If line *a* is parallel to line *b* and line *b* is parallel to line *c*, then line *a* is parallel to line *c* or line *a* is line *c*.

Proof: Assume line a is distinct from line c; if not, we are done. Assume line a is not parallel to line c. Then line a and line c intersect in a point. Call it Q. Consider the line b and the point Q. There are two lines, namely a and c, through point Q that do not intersect with line B. This contradicts the parallel axiom. Thus, a and c are parallel. ∎

Theorem 27: Given an open line and a closed line, they intersect exactly once.

Proof: Let o be an open line and let c be a closed line. Assume o and c intersect more than once. Say they intersect at points A and B. Then A and B both lie on the same closed line and A and B both lie on the same open line. This violates axiom B-2. Thus, o and c intersect at most at one point. Assume that o and c do not intersect. Therefore, they are parallel. There exists a point, A, on o by axiom B-4. There exists a closed line, a, through A by axiom B-1. a is parallel to c by theorem 17. But c is parallel to o so a is parallel to o by theorem 25. But this is a contradiction so o and c intersect exactly once. ∎

12. AXIOMS OF CONTINUITY

The first axiom of continuity is Dedekind's Axiom.

F-1 Axiom of Continuity for Open Lines (Dedekind's Axiom): Suppose that the set of points of a line m is a disjoint union of two non-empty subsets S and T such that no point of either subset is between two points of the other. Let $s \in S$ and $t \in T$ be points. Then there is a unique point O on m such that S is equal to a ray $\overrightarrow{Os}$ - O, and T is equal to $\overrightarrow{Ot}$ - O.

Note that the hypothesis of Dedekind's Axiom will never hold for points on a closed line; every point is between every other. Therefore, it is necessary to have another axiom to ensure the continuity of closed lines.

F-2 Axiom of Continuity for Closed Lines: Suppose that the set of points of a closed line m is a disjoint union of two non-empty subsets S and T such that no two points of one subset are separated by two points of the other. Then there are unique points O and P on m such that for all $s \in S$ and for all $t \in T$, $(O,P|s,t)$.

The third axiom of continuity is unique to Cylindrical Geometry.

F-3 Given a line m, there exists a number α_m such that, given points A and B on m, if the length of the segment $\overline{AB}_m$ is less than α_m, then m is the shortest line from A to B.

Axiom F-2 ensures that the infinite cylinder which is the model for cylindrical geometry does not have infinite diameter. This is because, given the diameter of the infinite cylinder, it is pretty straightforward to calculate, in Euclidean 3-space, the value of α . If we consider the periodic model of cylindrical geometry, an infinite diameter in the cylindrical model would translate to a period of infinite width in the periodic model. Thus, axiom F-2 ensures that the Euclidean plane cannot be a model for cylindrical geometry.

The final axiom of continuity will ensure that the only open lines allowed in cylindrical geometry fit with our conception of open lines by ensuring that all open lines have a constant slope. First, though, we need to define slope and prove that it is well defined.

Definition 42: Lines perpendicular to closed lines are called ***vertical lines***.

Definition 43: If *m* is a non-vertical open line, and *A* and *B* are two distinct points on *m* with m($\overline{AB}_m$) < $\boldsymbol{\alpha}_m$, then the ***rise*** from *A* to *B* can be found by drawing the closed line, *c,* through *A* and the vertical line, *v,* through *B*. The ***rise*** from *A* to *B* is measure along *v* from *B* to *C*, where *C* is the intersection point of *v* and *c*.

The existence of a unique point *C* is guaranteed by theorem 26. Thus, the only thing that it is necessary to show to show that rise is well defined is to make sure that there is exactly one vertical line through *B*. That result will be a corollary of the next theorem.

Theorem 28: Given a point *A*, and a line *m* through *A*, there is exactly one line perpendicular to *m* that contains *A*.

Proof: Let *A* be a point. Let *m* be the line through *A* (B-1). Let *B* be a second point on *m* (B-4). Consider the ray $\overrightarrow{AB}_m$ By axiom C-8, it is possible to find exactly one ray on each side of m such that the measure of the angle formed by $\overrightarrow{AB_m}$ and that ray is 90. Let $\overrightarrow{AD}$ and $\overrightarrow{AE}$ be these two rays. Then $\prec DAB + \prec BAE = 90 + 90 = 180$. Therefore, $\prec DAB$ and $\prec BAE$ are supplementary angles. Therefore, by definition, rays $\overrightarrow{AD}$ and $\overrightarrow{AE}$ lie on the same line. Say line n. Since $\overrightarrow{AD}$ and $\overrightarrow{AE}$ are also congruent, they are right angles and so lines n and m are perpendicular by definition. Therefore, there exists a line through A perpendicular to m. Assume there exists two such lines. Call them n and p. Then p would contain two rays, say $\overrightarrow{AF}$ and $\overrightarrow{AG}$ with F on the same side of m as D and G on the same side of m as E, that are perpendicular to $\overrightarrow{AB_m}$. Therefore, $\angle FAB$ and $\angle GAB$ are right angles. Therefore, $\prec FAB = \prec GAB = 90$ by theorem 25. But by axiom C-8, $\overrightarrow{AD}$ in the only ray on its side of m such that the angle formed by the ray and $\overrightarrow{AB_m}$ has the measure of 90. Thus, $\overrightarrow{AD} = \overrightarrow{AF}$. Similarly, $\overrightarrow{AE} = \overrightarrow{AG}$. Thus, $n = p$. Thus, there is only one line perpendicular to m through A. ︎

Definition 44: If m is a non-vertical open line, and A and B are two distinct points on m with $m(\overline{AB_m}) < \alpha_m$, then the ***run*** from A to B can be found by drawing the closed line, c, through A and the vertical line, v, through B. The ***run*** from A to B is $m(\overline{AC_{c0}})$, the measure of the shortest line from A to C along c, where C is the intersection point of v and c.

Definition 45: Let m be a line. If m is closed, define m to have a ***slope*** of zero everywhere. If m is vertical, define m to have an infinite ***slope*** everywhere. If m is a non-vertical open line, and A and B are two distinct points on m with $\mathrm{m}(\overline{AB_m}) < \alpha_m$, then the slope from A to B along m is equal to the rise from A to B divided by the run from A to B

We can now state the final axiom of cylindrical geometry.

F-4 If m is a line, then there will exists a constant, slp(m), such that for any two distinct points A and B on m (If m is a non-vertical open line, then for A and B with $\mathrm{m}(\overline{AB_m}) < \alpha_m$), the slope from A to B will equal slp(m).

13. AN ADDITIONAL TOPIC IN CYLINDRICAL GEOMETRY: SENSE

There is one more topic in cylindrical geometry that should be covered in this paper since this paper sets up the machinery for working in cylindrical geometry. Throughout the development of this paper, the author often wanted to consider whether two open lines were twisting in the same or opposite directions or whether the ray from A to B pointed in the same direction as the ray from A to C. In short, the author was looking for a way to formally define clockwise and counterclockwise within cylindrical geometry. The problem with defining the problem this way is that it is impossible to define the concepts of clockwise and counterclockwise on the cylinder because these classifications are dependent on the viewpoint of the viewer. Thus, the best we can do is determine if two lines twist in the same or opposite directions.

This issue is addressed in cylindrical geometry by comparing the relative direction of a sequence of three points on a given line to another sequence of three points on a line using the notion of "sense" as described by Coxeter in reference to projective geometry [2]. This notion requires a lot of machinery to be built in the system of cylindrical geometry before it can be made rigorous.

Definition 46: The set of all points on the same side of a vertical line m is called a ***halfplane with respect to m.*** We denote this $\mathbb{H}_m$.

Thus, it is possible to visualize a halfplane by visualizing a vertical plane cutting the infinite cylinder and considering the section of the cylinder visible on one side of the plane. (See Figure 10).

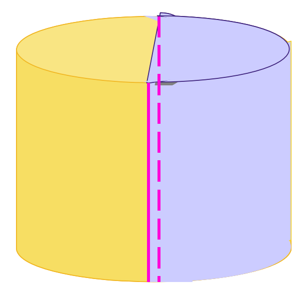

Figure 10

Definition 47: Let A and B be two points on the same halfplane. Consider the local ray $\overrightarrow{AB_0}$. Let C be a point that is on $\overrightarrow{AB_0}$ but not $\overline{AB_0}$. C is called a ***direction point*** of the ray $\overrightarrow{AB_0}$**.**

Therefore, C is a point on the small section of the local ray that is not on the line segment. C is called a direction point because since C is on the section of the ray which is not the line segment, the position of C distinguishes between $\overrightarrow{AB}$ and $\overrightarrow{BA}$**.** Therefore, C distinguishes the direction of the ray.

Definition 48: Let the ***direction number**, n,* of a point C with respect to a point A along a ray $\overrightarrow{AB}$ be the length of $\overline{AC_0}$ along the local ray $\overrightarrow{AB}$. We will denote this C_n.

Definition 49: Let $\{A,B,C\}$ and $\{D,E,F\}$ be two sets of distinct points along a closed line m. Let the direction number of each of these points be determined with respect to A along ray $\overrightarrow{AB_0}$ ($A_a, B_b, C_c, D_d, E_e, F_f$). If A and B do not lie in the same halfplane for any vertical line, re-label *A to B, B to C,* and *C to A* so that they do. Then, $\{A,B,C\}$ and $\{D,E,F\}$ have ***the same sense*** if $d<e<f$, $e<f<d$, or $f<d<e$. We denote this $s\{A,B,C\} = s\{D,E,F\}$. Otherwise, $\{A,B,C\}$ and $\{D,E,F\}$ have ***opposite sense.*** We denote this $s\{A,B,C\} \neq s\{D,E,F\}$. (See Figure 11)

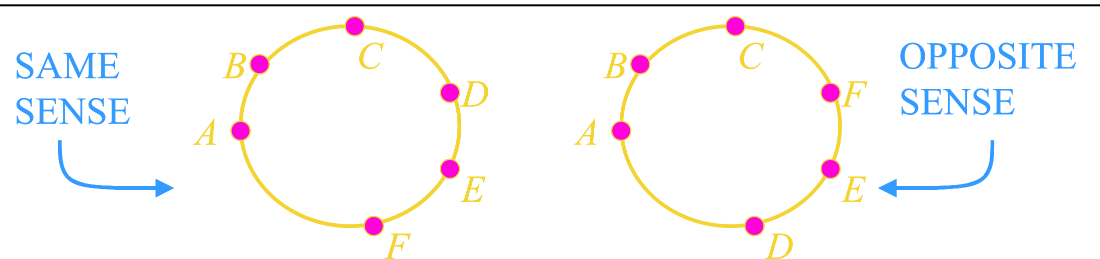

Figure 10

In this way, we can determine if two rays are pointing in the same rotational direction on a closed line.

To be able to apply this notion in a more general fashion, we use the concept of a nearest point. If $\{A,B,C\}$ and $\{D,E,F\}$ are two sets of distinct points along the same line or two separate lines, then let $\{A',B',C'\}$ and $\{D',E',F'\}$ be the nearest points of $\{A,B,C\}$

and $\{D,E,F\}$ on the closed line through point A. Then, $s\{A,B,C\} = s\{D,E,F\}$ if $s\{A',B',C'\}= s\{D',E',F'\}$ and $s\{A,B,C\} \neq s\{D,E,F\}$ if $s\{A',B',C'\}\neq s\{D',E',F'\}$. Since both nearest points and halfplanes depend on vertical lines, it is quite easy to see that this generalization is the result that we want.

14. CONCLUSION

Cylindrical geometry has two types of lines that are distinctly different. Open lines extend forever and closed lines have finite length, wrapping back on themselves. This difference causes the axiomatization of cylindrical geometry to be distinct from elliptic or Euclidean geometry since Euclidean geometry contains lines that are all infinitely extendable and elliptic geometry contains all finite lines. As expected, some axioms in cylindrical geometry resemble axioms from Euclidean geometry and some axioms from cylindrical geometry resemble axioms from elliptic geometry.

The two axioms in the *Axioms of Order* that do not resemble axioms from Euclidean or elliptic geometry are Axioms A-10 and A-11. Axiom A-10 clarifies the relationship between the notions of betweenness and separation. Since Euclidean geometry relies on betweenness and elliptic geometry relies on separation, the need for both these terms is new in cylindrical geometry. Axiom A-11 guarantees that triangles have both an inside and an outside. There is an axiom that guarantees this in Hilbert's axiomatization of Euclidean geometry [3], but it is much less complicated than the axiom for cylindrical geometry. This is because, on the infinite cylinder, not all unions of three line segments with common endpoints have an inside and an outside.

The impact of having both open and closed lines in the same geometry becomes even clearer as we look at the remaining groups of axioms. One rather remarkable result, which forms the basis for the first two axioms of the *Axioms of Connection*, is the observation that unlike Euclidean geometry, where two points uniquely determine a line, one point uniquely determines a closed line in cylindrical geometry and two points not on the same closed line do not uniquely determine a line; in fact, two such points determine a countably infinite number of lines. They determine the shortest line (or lines) from one point to the other, but there is also a helix with each orientation (think clockwise versus counterclockwise) that rotates around once, twice, or *n* times before connecting the first point to the second.

Since lines are not unique, the *Axioms of Connection* also establish the concept of the shortest line between two points, which is unique except for when the two points are diametrically opposed; then there are two shortest lines. This concept helps make up for the lack of unique lines in cylindrical geometry. The *Axioms of Connection* section also contains enough existence axioms to ensure that cylindrical geometry is a flushed out geometry.

The *Metric Axioms of Segments and Angles* ensures that there is enough structure within cylindrical geometry to take measurements of the lengths of segments and the degrees of angles. Congruence is defined based on the undefined term measure. Theorems prove that congruence is an equivalence relation. As in Hilbert's axioms of Euclidean Geometry, Side-Angle-Side is an axiom.

Cylindrical geometry has the same Gaussian curvature and the same parallel axiom as Euclidean geometry; recall the periodic model. Cylindrical geometry has four axioms of continuity. There is the standard Dedekind axiom to ensure that each open line has the structure of the real numbers as well as a modified version of Dedekind's axiom which insures the structure of closed lines. The third axiom ensures that the cylinder which models cylindrical geometry does not have an infinite diameter. The fourth axiom ensures that every line has a constant slope. Lastly, the direction that a line twists can be determined relative to another line by the notion of sense, a relationship on six points.

It is easy to see that the axioms of cylindrical geometry are as consistent as Euclidean geometry because of the natural map taking cylindrical geometry into three-dimensional Euclidean space. The completeness of these axioms has not been proven. It is reasonable to believe that these axioms are complete or nearly complete because of their similarity to Hilbert's axioms of Euclidean geometry, but rigorously proving the completeness of this axiom system is left as an open question by the author. Other interesting surfaces to axiomatize include the cone, the cube, the donut-shaped torus, and the topological torus (flat square with the opposite edges glued together).

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