CHAPTER 2

DIMENSIONAL ANALYSIS

Any physical relationship must be expressible in dimensionless form. The implication of this statement is that all of the fundamental equations of physics, all approximations to these equations and, for that matter, all functional relationships between physical variables must be invariant under a dilation (or stretching) of the dimensions of the variables. This is because the variables are subject to measurement by an observer in terms of units which are selected at the *arbitrary* discretion of the observer. It is clear that a physical event cannot depend on the choice of the unit of measure used to describe the event. It cannot depend on the particular ruler used to measure space, the clock used for time, the scale used to measure mass, or any other standard of measure which might be required depending on the dimensions which appear in the problem. This *principle of covariance* is the basis for a powerful method of reduction called dimensional analysis.

2.1 INTRODUCTION

A general mathematical relationship between variables is completely devoid of symmetry. However, if the variables describe the properties of a measureable physical system, then the dimensions of the system add a symmetry property to the relationship where none existed before. In effect, assigning dimensions to the variables brings into play the principle of covariance. We can define the notion of dimension as follows.

Definition 2.1 A dimension is a measureable property of a physical system which can be varied by a dilational transformation of the units of measurement. The value of each variable of the system is proportional to a power monomial function of the fundamental dimensions.

Often dimensional analysis is carried out without any explicit consideration of the actual equations which may govern a physical phenomenon. Only the variables which affect the problem are considered. Actually, this is a little deceiving. Inevitably, the choice of variables is intimately connected to the phenomenon itself and therefore is always is connected to, and has implications for, the governing equations. In fact the most complex problems in dimensional analysis tend to be filled with ambiguity as regards the choice of variables that govern the phenomenon in question.

2.2 THE TWO-BODY PROBLEM IN A GRAVITATIONAL FIELD

First let's look at a fairly straightforward example that nicely illustrates both the power and limitations of dimensional analysis. This is the problem of determining the relationship between the mean distance from the Sun and the orbital period of the planets. The solution of this problem was published by Johannes Kepler in 1619 and has since been known as Kepler's third law. Kepler, who succeeded Tycho Brahe as the imperial mathematician of the Holy Roman Empire in 1601, was one of the truly outstanding scientists of the Age of Enlightenment. His position gave him access to Brahe's incomparable collection of astronomical data, particularly data for the movement of Mars, collected by a team of astronomers over decades of painstaking work. By 1609 he had published his first two laws (although he did not refer to them as such) that the planets follow elliptical orbits and that the movement of a planet along its orbit traces out equal areas in equal times. A decade later he published his findings that the cube of the distance from the Sun divided by the square of the period is a constant. Kepler's accomplishments at the time are all the more remarkable in that they occurred at about the same time that he had to rush to the defense of his mother who had been indicted as a witch. Only his timely defense in 1620 prevented her from being tortured and burned at the stake. Kepler remained the imperial mathematician for several more years but, through the events leading up to the thirty years war, was eventually forced to find a new patron. He fell ill and died on November 15, 1630. Kepler was the first to provide a dynamical explanation of the movements of the heavens and his results continued to have an impact long after his death. Newton relied heavily on Kepler's work in developing his theory of gravitation in the 1680's. Today we recognize that the law of equal areas applies to any pair of masses with a radially directed force between them while the first and third laws apply only to particles that obey an inverse square law; everything from the motion of satellites to the electric interactions of charged particles.

Consider the movement of one of the planets about the Sun. The orbit is elliptical with major axis, a, minor axis, b and area $A = \pi ab$. The Sun lies very close to one of the foci of the ellipse as shown in the figure below.The force between the two masses follows the Newtonian law of gravitation,

$$F = -G\left(\frac{Mm}{r^2}\right) \tag{2.1}$$

where G is the gravitational constant $G = 6.670 \times 10^{-11}$ $Newton\text{-}M^2/kg^2$ and the minus sign indicates that the force is attractive. The perturbation of the orbit by all the other planets is ignored. We wish to use dimensional analysis to rediscover Kepler's third law relating the period of the orbit to its size. Data for the solar system is included below.

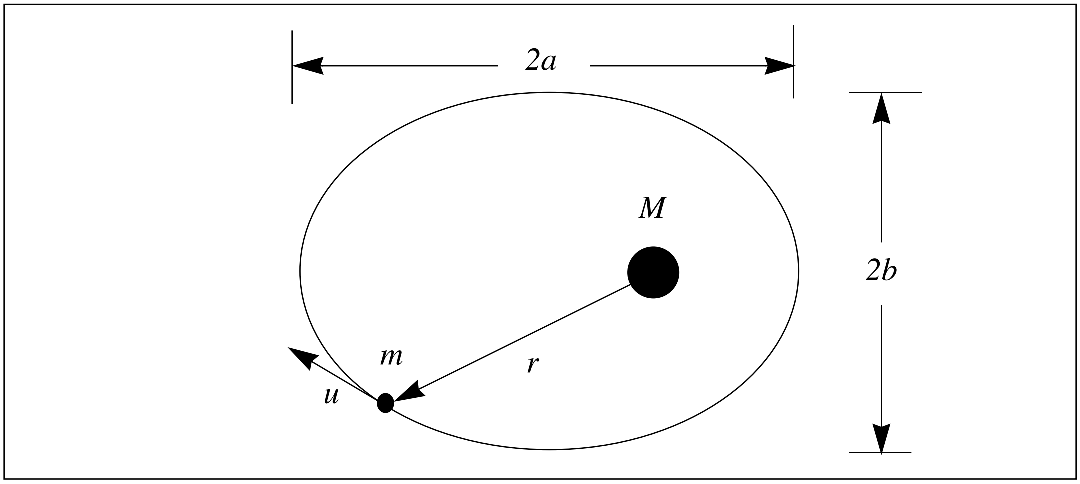


Figure 2.1 Elliptical orbit of a planet about the Sun

Heavenly Body	Number of Earth masses	Number of Earth diameters	Mean orbit radius in $km \times 10^{-6}$	Eccentricity	Orbital period in years
Sun	332,488.0	109.15	———	———	———
Mercury	0.0543	0.38	57.9	0.2056	0.241
Venus	0.8136	0.967	108.1	0.0068	0.615
Earth	1.0000	1.000	149.5	0.0167	1.000
Mars	0.1069	0.523	227.8	0.0934	1.881
Jupiter	318.35	10.97	777.8	0.0484	11.862
Saturn	95.3	9.03	1426.1	0.0557	29.458
Uranus	14.58	3.72	2869.1	0.0472	84.015
Neptune	17.26	3.38	4495.6	0.0086	164.788
Pluto	<0.1	0.45	5898.9	0.2485	247.697

Table 2.1 The planets and their orbits

The mass of the Earth is $5.975 \times 10^{24} kg$ and the mean diameter is $12742.46\ km$. The eccentricity of a planet's orbit is

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2} \tag{2.2}$$

The only parameters that can enter the problem are the lengths of the two axes, the two masses, the time of the period and the gravitational constant.

$$\hat{a} = L\ ;\ \ \hat{b} = L\ ;\ \ \hat{M} = M\ ;\ \ \hat{m} = M\ ;\ \ \hat{T} = T\ ;\ \ \hat{G} = \frac{L^3}{MT^2} \tag{2.3}$$

The "hat" over a parameter such as $\hat{a}$ in (2.3) is used to mean "dimensions of". In this problem *M-mass*, *L-length* and *T-time* are the fundamental dimensions. There are six parameters and three fundamental dimensions and so we can expect the solution to depend on three dimensionless numbers. Two of these are obviously a mass ratio and a length ratio,

$$\Pi_1 = \frac{m}{M}\ ; \qquad \Pi_2 = \frac{b}{a} \tag{2.4}$$

In view of the dimensions of G, it is clear that the third number must involve one of the masses, one of the lengths and the period. Thus, we can expect a dimensionless variable of the form.

$$\Pi_3 = \frac{GMT^2}{a^3}. \tag{2.5}$$

where we have arbitrarily chosen M and a to form Π_3 rather than m and b. According to the law of covariance one can expect all these variables to be related by a dimensionless function of the form

$$\psi = \Psi(\Pi_1, \Pi_2, \Pi_3) \tag{2.6}$$

Without loss of generality we can solve (2.6) explicitly for Π_3,

$$\frac{GMT^2}{(r_{mean})^3} = F\left(\frac{m}{M}, e\right) \tag{2.7}$$

where we have used the eccentricity in place of b/a and a mean radius defined as $r_{mean} = \sqrt{ab}$. Using the data in Table 2.1 we can plot the values of (2.7) for the various planets in the Solar System.

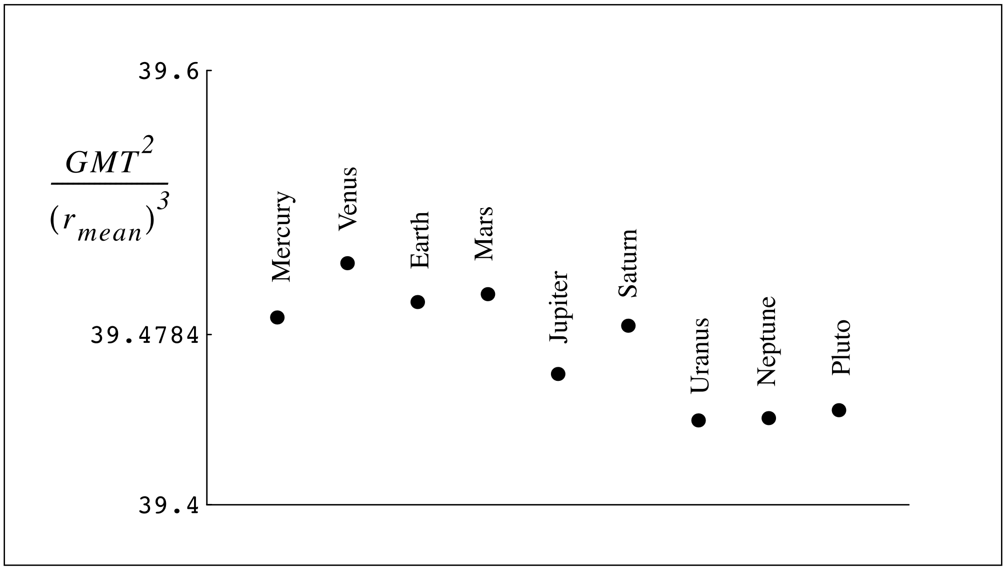


Figure 2.2 Kepler's third law for the Solar System.

Figure 2.2 provides stunning confirmation of our dimensional analysis result and indicates that the function on the right hand side of (2.7) is very nearly constant for all the planets in the Solar System. In fact, theory tells us that the right hand side of (2.7) is

$$F\left(\frac{m}{M}, e\right) = 4\pi^2 \left(\frac{1}{\left(1 + \frac{m}{M}\right)(1 - e^2)^{\frac{3}{4}}} \right) \tag{2.8}$$

For all the planets $m/M \ll 1$ and e^2 is very small for all but Mercury and Pluto. In the limits, $m/M \to 0$ and $e \to 0$ the right hand side of (2.8) approaches the finite limit $4\pi^2 = 39.4784$.

In fact we have made a lucky choice! On purely dimensional grounds, in the absence of Kepler's theory, there is absolutely no reason to select M in the definition of Π_3, m would have been just as appropriate a choice but would have produced a highly scattered plot. Dimensional analysis alone provides no information in this matter. The full theory is required.

2.3 THE DRAG OF A SPHERE

Next, we will work a problem which also illustrates the power as well as some of the pitfalls of dimensional analysis. This is the problem of viscous flow past a sphere. The previous example involved a rather simple set of basic dimensioned variables and so it could be worked out by inspection. In the present case this is not quite so easy and so we will resort to a systematic method of constructing appropriate dimensionless variables. Initially we will make the assumption that the flow is incompressible and then compressibility effects will be added. The results will then be compared with experimental data. Uniform flow of a viscous fluid past a sphere is shown in the figure below.

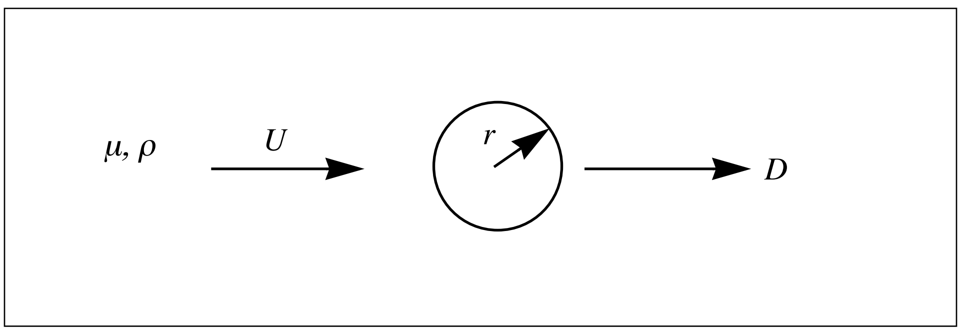


Figure 2.3 Viscous flow past a sphere

To get started let's assume that the relevant variables of the problem are the drag force, D, the fluid density, ρ, the viscosity, μ, the freestream flow velocity, U, and the radius of the sphere, r. Without loss of generality, these variables can be thought of as related to one another through a function of the form

$$\psi_0 = \Psi_0[D, \mu, \rho, U, r] \tag{2.9}$$

where ψ_0 is a pure number (ie, dimensionless) which may be zero.

Considered solely as a mathematical statement, (2.9) has no symmetry. But it is not just an abstraction! It is a physical statement in two respects. First, it states that the drag of the sphere depends only on the selected variables. This is totally at our discretion and it would be easy to argue that other quantities, for example the speed of sound in the fluid, ought to also play a role. Second, the variables in (2.9) are all measureable properties of a physical system; they have dimensions and those dimensions are measured in *arbitrarily* chosen units.

$$\hat{D} = \frac{ML}{T^2}\ ; \quad \hat{\mu} = \frac{M}{LT}\ ; \quad \hat{\rho} = \frac{M}{L^3}\ ; \quad \hat{U} = \frac{L}{T}\ ; \quad \hat{r} = L \tag{2.10}$$

Because the variables in (2.9) have dimensions, the function Ψ_0 cannot be arbitrary. If it were, the constant ψ_0 would change whenever the choice of units was changed. In effect, the drag force on the sphere would appear to depend on the choice of the units of measurement, which is impossible. To see this let's suspend the principle of covariance for a moment and imagine that the drag relationship (2.9) is

$$\begin{gathered} D = \mu + \rho + U + r \\ \textit{in terms of dimensions} \\ \frac{ML}{T^2} = \frac{M}{LT} + \frac{M}{L^3} + \frac{L}{T} + L \end{gathered} . \tag{2.11}$$

If we were to change the units of mass from kilograms to grams then μ and ρ would both be larger by a factor of 10^3 while U and r stayed the same. This would increase the term in parentheses in (2.11) but not by this factor. But D also increases by a factor of a thousand, thus the equality (2.11) cannot be maintained when the units are changed. The various terms of the drag relationship (2.11) do not vary together (they do not co-vary) as the units of mass are changed; and (2.11) cannot possibly describe the drag of a sphere!

The only way to avoid this problem is to require that the general drag relationship (2.9) satisfies the principle of covariance. Accordingly, (2.9) must be invariant under a three-parameter dilation group

$$\tilde{M} = e^m M\ ; \quad \tilde{L} = e^l L\ ; \quad \tilde{T} = e^t T \tag{2.12}$$

where the group parameters m, l and t are arbitrary real numbers. This invariance requirement severely restricts the function Ψ_0 and suggests that one can learn something important by searching for a proper invariant form of the drag relationship. We will proceed in steps. Begin by scaling the units of mass using the following one-parameter group.

$$\tilde{M} = e^{m} M \ ; \quad \tilde{L} = L \ ; \quad \tilde{T} = T \, . \tag{2.13}$$

The effect of this scaling on the variables of the problem is to transform them as follows.

$$\tilde{D} = e^{m} D \ ; \quad \tilde{\mu} = e^{m} \mu \ ; \quad \tilde{\rho} = e^{m} \rho \ ; \quad \tilde{U} = U \ ; \quad \tilde{r} = r \tag{2.14}$$

The drag relation (2.9) is required to be independent of the group parameter m and therefore must be of the form

$$\psi_0 = \Psi_1\left[\frac{D}{\rho}, \frac{\rho}{\mu}, U, r\right] \tag{2.15}$$

or something equivalent. That is, (2.15) is not unique. For example, we could have picked $D/\mu, \rho/\mu, U, r$ as the new independent variables. Either choice is invariant under (2.13). We shall return to this point in a moment. The dimensions of the variables remaining in (2.15) are

$$\frac{\hat{D}}{\hat{\rho}} = \frac{L^4}{T^2} \ ; \quad \frac{\hat{\rho}}{\hat{\mu}} = \frac{T}{L^2} \ ; \quad \hat{U} = \frac{L}{T} \ ; \quad \hat{r} = L \, . \tag{2.16}$$

Now let the units of length be scaled according to

$$\tilde{L} = e^{l} L \ ; \quad \tilde{T} = T \, . \tag{2.17}$$

The effect of this group on the variables in (2.15) is

$$\frac{\tilde{D}}{\tilde{\rho}} = e^{4l}\frac{D}{\rho}; \quad \frac{\tilde{\rho}}{\tilde{\mu}} = e^{-2l}\frac{\rho}{\mu}; \quad \tilde{U} = e^{l} U; \quad \tilde{r} = e^{l} r \, . \tag{2.18}$$

By the principle of covariance, the drag relation (2.15) must be independent of l and a functional form which accomplishes this is

$$\psi_0 = \Psi_3\left[\frac{D}{\rho U^2 r^2}, \frac{\rho U^2}{\mu}, \frac{r}{U}\right] . \tag{2.19}$$

The dimensions of the variables in (2.19) are

$$\frac{\hat{D}}{\hat{\rho}\hat{U}^2\hat{r}^2} = 1 \ ; \quad \frac{\hat{\rho}\hat{U}^2}{\hat{\mu}} = \frac{1}{T} \ ; \quad \frac{\hat{r}}{\hat{U}} = T \, . \tag{2.20}$$

Finally, scale the units of time.

$$\tilde{T} = e^{t}T\,. \tag{2.21}$$

The effect of this group on the variables in (2.19) is as follows.

$$\frac{\tilde{D}}{\tilde{\rho}\tilde{U}^2\tilde{r}^2} = \frac{D}{\rho U^2 r^2}\ ;\quad \frac{\tilde{\rho}\tilde{U}^2}{\tilde{\mu}} = e^{-t}\frac{\rho U^2}{\mu}\ ;\quad \frac{\tilde{r}}{\tilde{U}} = e^{t}\frac{r}{U}\,. \tag{2.22}$$

The drag relation (2.19) must be independent of t and this leads finally to the dimensionless form

$$\psi_0 = \Psi[C_D, Re] \tag{2.23}$$

where

$$C_D = \frac{D}{\frac{1}{2}\rho U^2(\pi r^2)}\ ;\qquad Re = \frac{\rho U(2r)}{\mu}\,. \tag{2.24}$$

The first dimensionless variable has the usual interpretation of a drag coefficient. The constants $1/2$ and π have been added to bring the definition into line with accepted usage where the drag is normalized by the free stream dynamic pressure and the frontal area of the body. The second dimensionless variable is the Reynolds number commonly defined in terms of the sphere diameter. The final result (2.23) is invariant under the three parameter group (2.12) and covariance is satisfied! The experimentally determined relationship (2.23) will be discussed in the next section. In a way (2.23) is a remarkable achievement. The drag has been found to depend on only one quantity, not four - a tremendous reduction! Furthermore we were able to reach this simple relationship without ever having to consider the equations of motion for the flow over a sphere. This does not mean we did not do any physics - there is a significant amount of physics in the identification of the relevant variables. In this respect dimensional analysis is deceptively simple. In fact it requires a deep physical understanding of the problem being addressed including the governing equations.

It is common to seek a further simplification of the problem by considering possible limiting behavior of (2.23). To illustrate this idea we will make use of the well known exact solution for the drag of a sphere in the limit of small Reynolds number,

$$C_D = \frac{24}{Re}\,. \tag{2.25}$$

If we restore the dimensioned variables in (2.25) and solve for the drag, the result is

$$\frac{D}{\mu U r} = 6\pi . \tag{2.26}$$

At very low Reynolds number the drag of a sphere is independent of the density of the surrounding fluid - a completely unexpected result and one which could not be determined without knowing the solution (2.25)! This amazing result explains a variety of phenomena. It tells us why the atmosphere of Mars, with a surface pressure which is less than one-percent of that of Earth, can support planet-wide dust storms that may take several months to settle out. The density of the atmosphere hardly matters at all, the settling speed of small dust particles is determined almost entirely by the viscosity of the Mars atmosphere which is 96% cold Carbon Dioxide at about 200K. Mariner 9 encountered such a storm when it arrived at the Red planet in 1972. At first this was thought to be a major dissapointment since the surface of the planet was totally obscured, but the optical scattering data obtained over the weeks and months as the dust settled continues to be analyzed today and will remain for a long while one of the most important sources of data on the composition of Mars (Refs [2.2] and [2.3]).

It is perfectly reasonable to try to extend this result to flow over a circular cylinder where the drag per unit length has the dimensions

$$\hat{D}_{cylinder} = \frac{M}{T^2} \tag{2.27}$$

and the drag coefficient is

$$C_{D_{cylinder}} = \frac{D_{cylinder}}{\frac{1}{2}\rho U^2 (2r)} . \tag{2.28}$$

The circle in Figure 2.3 is now interpreted as a cylinder extending to infinity. Dimensional analysis leads to a result identical to (2.23) and logic would suggest that, perhaps, in the limit of very small Reynolds number the flow over a cylinder is governed by an equation similar in form to (2.25). Let

$$C_{D_{cylinder}} = \frac{\psi}{Re} . \tag{2.29}$$

If we restore the dimensioned variables in this relation the result is

$$\frac{D_{cylinder}}{\mu U} = \psi \qquad (2.30)$$

which says that the drag of a cylinder is independent of its radius. In this case dimensional reasoning plus a little bit of experience has led us down a garden path to a nonsensical and completely incorrect result!

2.3.1 Some further physical considerations

Even when dimensional analysis succeeds in producing a physically reasonable result, that result is usually limited in very important ways. Figure 2.4 from Reference [2.1] shows measurements of circular cylinder drag versus Reynolds number taken by a variety of investigators. According to (2.23) there should be a single curve of C_D versus Re. But one can't help but be struck by the wide variation from one experiment to another depicted in Figure 2.4. Is our analysis wrong?

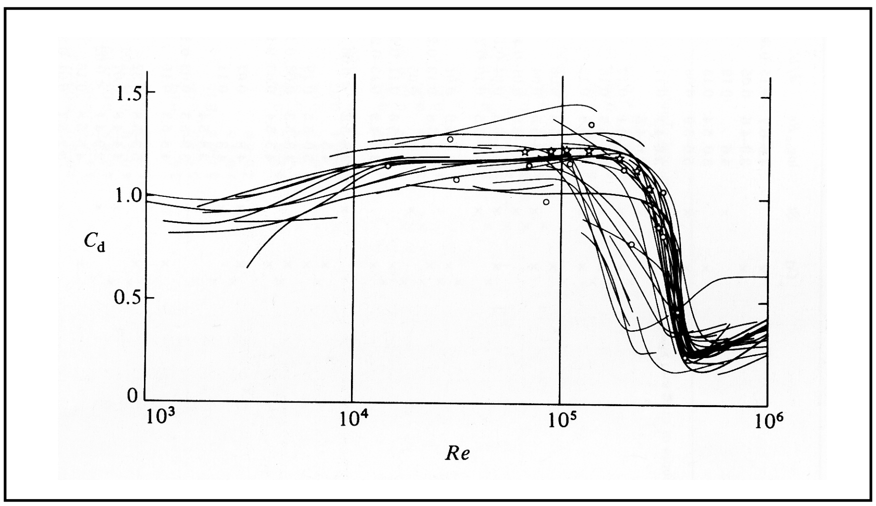


Figure 2.4 Experimental measurements of the drag of a circular cylinder from [2.1].

No, not really within the confines of the physical variables identified in (2.9). However it is pretty obvious that important variables have been ignored! The drag of a circular cylinder or a sphere is sensitive to many things. The main dependence is on the Reynolds number which is successfully identified using dimensional analysis. But in addition, the drag depends on a whole variety of velocity scales and length scales including surface roughness (measured in terms of a roughness height), the level of free stream turbulent velocity fluctuations, the length scale of turbulent eddies in the free stream, the size of the wind tunnel, the speed of sound (if U is not small enough), etc. A more complete dimensionless description of the problem would be of the form

$$\psi = \Psi\left[\frac{D}{\rho U^2 r^2}, \frac{\rho U r}{\mu}, \frac{\upsilon_1}{U}, \frac{\upsilon_2}{U}, \ldots, \frac{\lambda_1}{r}, \frac{\lambda_2}{r}, \ldots\right] \tag{2.31}$$

where $\upsilon_1, \upsilon_2, \ldots, \lambda_1, \lambda_2, \ldots$ are the neglected velocity and length scales of the problem. The point of all this is that when we formulated the original problem an implicit assumption was made that these quantities are either infinitesimally small or infinitely large and a finite limit of (2.31) exists as any one of $\upsilon_1, \upsilon_2, \ldots, \lambda_1, \lambda_2, \ldots$ goes to zero or infinity. That is, we assumed that when these variables are asymptotically small or large they have a small effect *and* that the remaining variables provide an adequate description of the physical system. The experience we gained from the Kepler problem, where the limit of (2.8) as m/M and e went to zero was $4\pi^2$ would suggest that such an assumption is justified.

But the lesson of Figure 2.4 is that not all problems are as clean as the Kepler problem. In fact fluid dynamics presents a wide variety of problems where such an assumption is a close call at best and has to be examined through experiment in each case. The drag law at low Reynolds number (2.25) is another case in point. Obviously, a finite limit at zero Reynolds number in this relationship does not exist. Only by renormalizing the drag in the form of (2.26) can a finite limit be realized. For an extensive discussion of this issue the reader is referred to the text by Barenblatt [2.4].

This example highlights a key point. A real physical system in all of its detail is devoid of perfect symmetries. We live in a universe of broken symmetries. In a sense, our mathematical physics, constructed around equations with perfect symmetry and methods which can incorporate only relatively idealized boundary conditions, simply isn't up to the task of fully describing real phenomena in all detail. Nevertheless, by incorporating as exact symmetries those approximate symmetries which play a key role in the phenomena being described, remarkably accurate models of the physical world can be developed. The identification of such symmetries is one of the main objectives of scientific inquiry.

As a final example we consider what happens to the sphere drag problem when the speed of the flow is large and the effects of compressibility are incorporated.

2.4 THE DRAG OF A SPHERE IN HIGH SPEED GAS FLOW

The figure below shows the flow that would occur when the speed of the sphere exceeds the speed of sound in the surrounding medium. In this case the pressure disturbance produced by the sphere is unable to propagate upstream to infinity. In effect the sphere continually overtakes its own sound field and the result is a shock wave standing in front of the sphere. Figure 2.5 is intended to illustrate the flow at supersonic speeds. However it is well to recognize that compressibility effects begin to come into play at a critical subsonic speed somewhat below the speed at which shock formation ahead of the body begins to occur. We have the same relevant variables that we had before, including the drag force, D, the fluid density, ρ_∞, the viscosity, μ_∞, the freestream flow velocity, U, and the radius of the sphere, r.

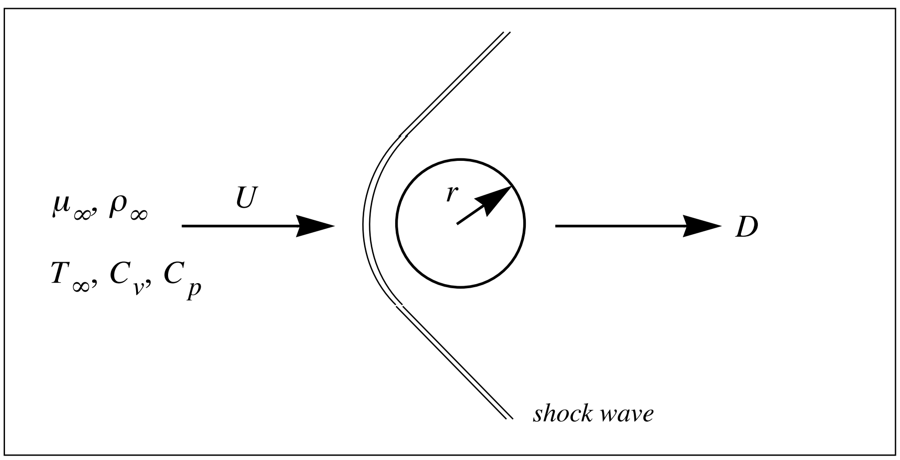


Figure 2.5 High speed flow past a sphere

At low speed, where the flow is nearly incompressible, the effect of the motion of the sphere on the internal energy of the fluid is extremely small and mainly confined to slight heating by viscous friction. At high speed, the motion of the sphere can substantially change the internal energy of the gas owing to its compressibility. The kinetic energy of the sphere ratioed to the thermal energy of the surrounding gas becomes an important measure of the degree to which the internal energy of the gas can be changed by the motion of the sphere. Moreover this ratio is correlated with the strength, shape and position of the shock and therefore the drag of the sphere. This brings into play the gas temperature and the heat capacities at constant pressure and volume indicated in Figure 2.5 as additional dimensioned variables governing the drag. Note that the temperature and density of the gas vary throughout the flow necissitating the use of subscripts to denote free stream values. The dimensions of the relevant variables are,

$$\hat{T}_{\infty} = \Theta ; \quad \hat{C}_p = \frac{M^2}{L^2\Theta} ; \quad \hat{C}_v = \frac{M^2}{L^2\Theta} . \tag{2.32}$$

Now we have one additional fundamental dimension, temperature, and three additional parameters, two of which have the same units. Note that the dimensions of the heat capacities, $speed^2/temperature$, reflect the argument just made comparing the kinetic energy of the motion to the thermal energy of the gas. When we carry through the systematic procedure used in the incompressible case, the result is two additional dimensionless variables.

$$\Pi_1 = \frac{U^2}{C_v T} \quad ; \quad \Pi_2 = \frac{C_p}{C_v} . \tag{2.33}$$

Note that $C_v T_{\infty}$ is the internal energy per unit mass of the free stream gas. Finally our drag relation is,

$$\psi = \Psi[C_D, R_e, M_{\infty}, \gamma] \tag{2.34}$$

where $\gamma = C_p/C_v$ and Π_1 is replaced by the usual form of the Mach number,

$$M_{\infty} = \frac{U}{a_{\infty}} . \tag{2.35}$$

where the speed of sound is,

$$a^2 = \gamma R T_{\infty} \tag{2.36}$$

The quantity, R is the universal gas constant divided by the molecular weight of the gas, $R = R_u/M_w$ which obeys the ideal gas law $p = \rho R T$. Without loss of generality we can write,

$$C_D = F[R_e, M_{\infty}, \gamma] . \tag{2.37}$$

Miller and Bailey [2.5] studied the available experimental data for the drag of spheres over a wide range of Reynolds and Mach numbers in air. Interestingly the most accurate high Reynolds number data for Mach numbers between 0.6 and 2.0 turned out to be the 19th century cannonball measurements by Francis Bashforth

[2.6] who was Professor of Applied Mathematics at the Royal Military Academy at Woolwich (near Greenwich) England. In 1947 the Academy was consolidated with the Royal Military Academy at Sandhurst. The Royal Artillery Barracks on Woolwich Common where many famous British military figures were trained is now the home of the Royal Artillery Museum.

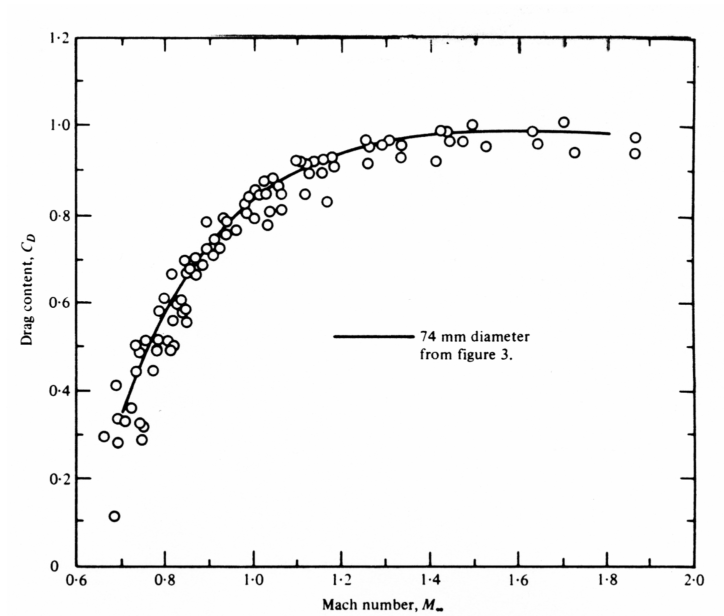


Figure 2.6 Bashforth's drag data for a 7.4 cm diameter cannonball from [2.5]

Bashforth's technique was to measure the successive times when the projectile passed through a series of ten wire screens spaced 150 feet apart and electrically connected to a chronograph consisting of a pair of pens writing on a paper-covered, rotating drum. As the projectile passed through each screen the current to the chronograph was interrupted providing a position-time history from which Bashforth could infer the velocity and deceleration of the cannonball. This information could then be used to compile an extensive set of data for the drag coefficient, Mach number and Reynolds number of spheres. The figure above (Figure 2 from Miller and Bailey) presents the data of Bashforth that shows the rapid rise in the drag coefficient of a 7.4 cm diameter sphere through the transonic Mach number regime.

Figure 2.7 shows their complete compilation of data at various Reynolds numbers and Mach numbers. The most interesting feature of the data in Figure 2.7 is the tendency for the drag coefficient to become essentially independent of Reynolds number for $M_{\infty} > 1.5$. In this regime, wave drag dominates viscous drag. In fact the data suggests that as the Mach number is increased, the drag coefficient approaches,

$$C_D \approx 1 \tag{2.38}$$

although there is a slight but systematic decrease above $M_{\infty} = 2.0$. At low Mach number the drag coefficient shows no sign of reaching an asymptotic value up to the highest Reynolds number measured.

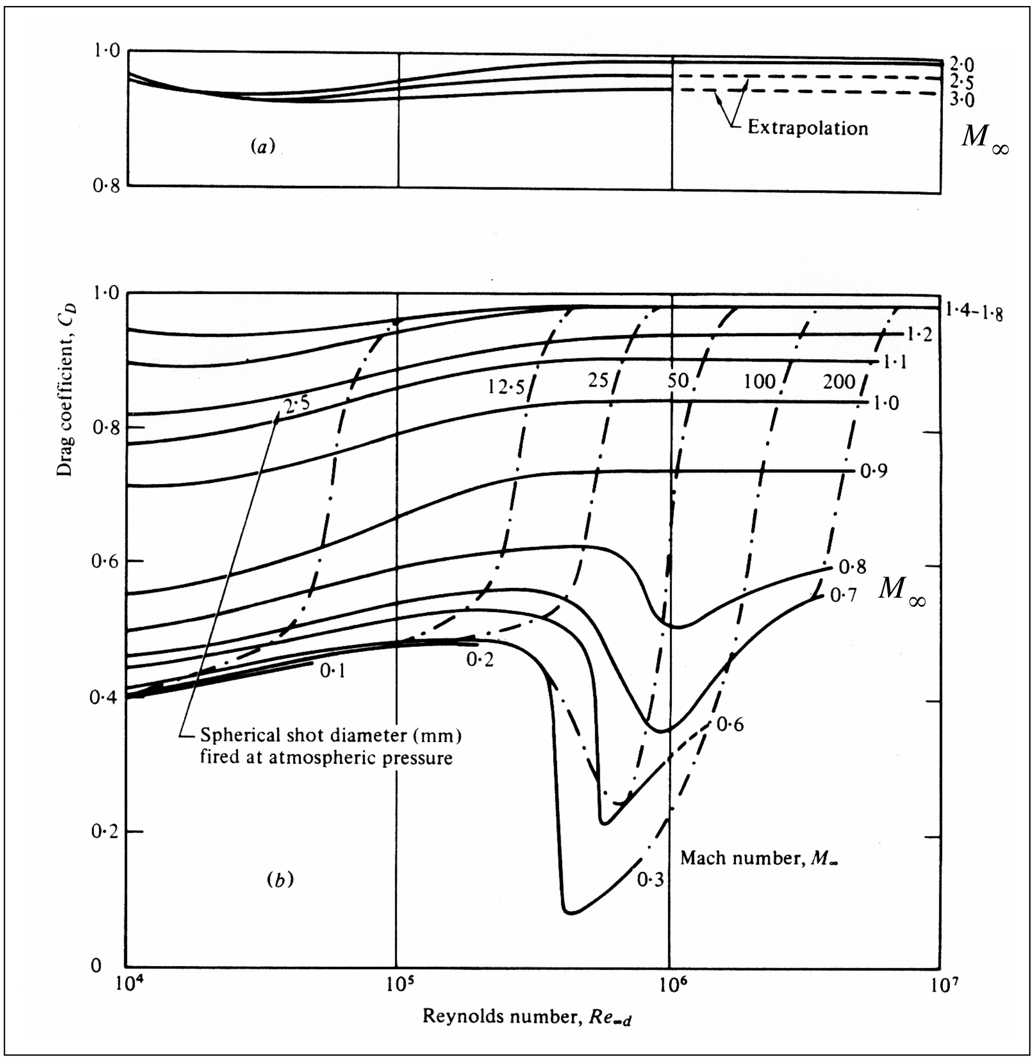


Figure 2.7 Compilation of sphere drag as a function of Mach number and Reynolds number from Miller and Bailey [2.5].

2.5 DIMENSIONLESS EQUATIONS OF MOTION

The figure below shows a wing in a compressible air stream with the various parameters of the fluid, the flow and the wing that govern the problem. In addition to the parameters shown gravity acts on the wing and the fluid. The gravitational parameter will be taken to be g_0, the gravitational acceleration at the surface of the Earth.

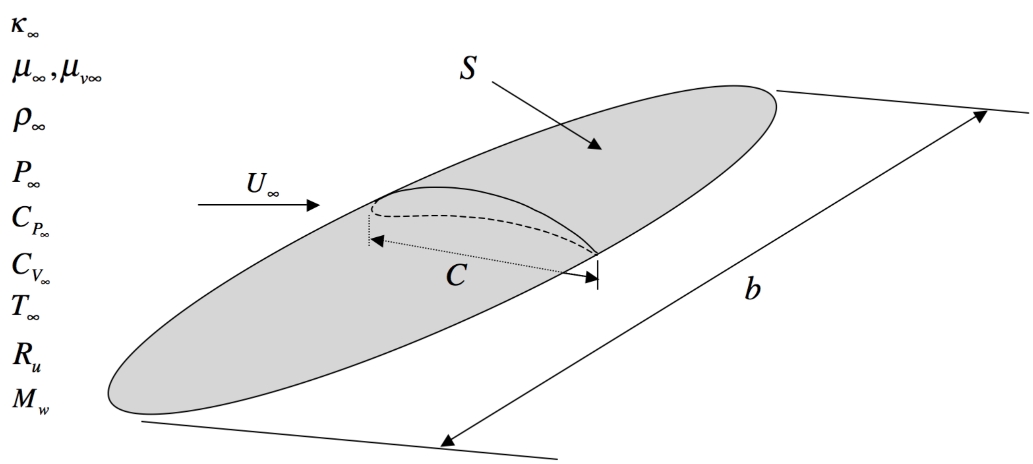


Figure 2.8 Schematic of a wing in a compressible, viscous, heat conducting flow.

The wing chord is C, the span is b and the wing planform area is S. The shear viscosity of the fluid is μ_∞ in the free stream and the bulk viscosity is $\mu_{v\infty}$The problem is completely characterized by these two viscosities, the heat conductivity of the fluid, the thermodynamic state of the fluid, the speed of the flow and the geometry of the wing. The fundamental dimensions are $M = mass$, $L = length$, $T = time$, $\theta = temperature$ and $N = moles$The units of the various parameters are

$$\hat{\kappa}_\infty = \frac{ML}{T^3\theta} \qquad \hat{\mu}_\infty, \hat{\mu}_{v\infty} = \frac{M}{LT} \qquad \hat{C}_{P_\infty} = \frac{ML^2}{T^2\theta} \qquad \hat{C}_{V_\infty} = \frac{ML^2}{T^2\theta} \tag{2.39}$$

$$\hat{P}_\infty = \frac{M}{LT^2} \qquad \hat{\rho}_\infty = \frac{M}{L^3} \qquad \hat{T}_\infty = \theta \qquad \hat{R}_u = \frac{ML^2}{T^2\theta N} \qquad \hat{M}_w = \frac{M}{N} \tag{2.40}$$

$$\hat{U}_\infty = \frac{L}{T} \qquad \hat{C} = L \qquad \hat{b} = L \qquad \hat{S} = L^2 \qquad \hat{g}_0 = \frac{L}{T^2} \tag{2.41}$$

Let the tilda on $(\tilde{\rho}, \tilde{P}, \tilde{T}, \tilde{U}_i, \tilde{e}, \tilde{h}, \tilde{\mu}, \hat{\mu}_v, \tilde{\kappa}, \tilde{x}_i, \tilde{t})$ represent dimensioned variables. The equations of motion in dimensioned variables are

Continuity

$$\frac{\partial\tilde{\rho}}{\partial\tilde{t}} + \frac{\partial(\tilde{\rho}\tilde{U}_j)}{\partial\tilde{x}_j} = 0 \tag{2.42}$$

Momentum

$$\frac{\partial\tilde{\rho}\tilde{U}_i}{\partial\tilde{t}} + \frac{\partial}{\partial\tilde{x}_j}\left(\tilde{\rho}\tilde{U}_i\tilde{U}_j + \tilde{P}\delta_{ij} - \left(\mu\left(\frac{\partial\tilde{U}_i}{\partial\tilde{x}_j} + \frac{\partial\tilde{U}_i}{\partial\tilde{x}_j}\right) - \left(\frac{2}{3}\mu - \mu_v\right)\delta_{ij}\frac{\partial\tilde{U}_k}{\partial\tilde{x}_k}\right)\right) - \tilde{\rho}\tilde{G}_i = 0 \tag{2.43}$$

Energy

$$\frac{\partial\tilde{\rho}\left(\tilde{e} + \frac{\tilde{U}_i\tilde{U}_i}{2}\right)}{\partial\tilde{t}} + \frac{\partial}{\partial\tilde{x}_j}\left(\tilde{\rho}\tilde{U}_j\left(\tilde{h} + \frac{\tilde{U}_i\tilde{U}_i}{2}\right)\right) -$$
$$\frac{\partial}{\partial\tilde{x}_j}\left(\mu\tilde{U}_i\left(\frac{\partial\tilde{U}_i}{\partial\tilde{x}_j} + \frac{\partial\tilde{U}_i}{\partial\tilde{x}_j}\right) - \left(\frac{2}{3}\mu - \mu_v\right)\delta_{ij}\frac{\partial\tilde{U}_k}{\partial\tilde{x}_k}\right) - \frac{\partial}{\partial\tilde{x}_j}\left(\tilde{\kappa}\frac{\partial\tilde{T}}{\partial\tilde{x}_j}\right) - \tilde{\rho}\tilde{G}_i\tilde{U}_i = 0 \tag{2.44}$$

In addition we have the ideal gas equation of state.

$$\tilde{P} = \tilde{\rho}\frac{R_u}{M_w}\tilde{T} \tag{2.45}$$

Now replace dimensioned variables with dimensionless variables using the scalings

$$\begin{gathered}\tilde{\kappa} = (\hat{\kappa}_\infty)\kappa \qquad \tilde{\mu} = (\mu_\infty)\mu \qquad \tilde{\mu}_v = (\mu_{v\infty})\mu_v \qquad \tilde{C}_P = (C_{P_\infty})C_P \\ \tilde{C}_V = (C_{V_\infty})C_V \qquad \tilde{P} = (\rho_\infty U_\infty^2)P \qquad \tilde{\rho} = (\rho_\infty)\rho \qquad \tilde{T} = (T_\infty)T\end{gathered} \tag{2.46}$$

$$\tilde{e} = (C_{V_\infty}T_\infty)e \qquad \tilde{h} = (C_{P_\infty}T_\infty)h \qquad \tilde{T} = (T_\infty)T \tag{2.47}$$

$$\tilde{x}_i = (C)x_i \qquad \tilde{t} = \left(\frac{C}{U_\infty}\right)t \qquad \tilde{U}_i = (U_\infty)U_i \qquad \tilde{G}_i = \frac{G_i}{g_0} \tag{2.48}$$

For example the internal energy and enthalpy become

$$\begin{aligned}\tilde{e} + \left(\frac{1}{2}\tilde{U}_i\tilde{U}_i\right) &= (C_{V_\infty}T_\infty)e + (U_\infty^2)\left(\frac{1}{2}U_iU_i\right)\\ \tilde{h} + \left(\frac{1}{2}\tilde{U}_i\tilde{U}_i\right) &= (C_{P_\infty}T_\infty)h + (U_\infty^2)\left(\frac{1}{2}U_iU_i\right)\end{aligned} \tag{2.49}$$

When the equations of motion are transformed using (2.46) to (2.48) the result is

Dimensionless continuity

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho U_j)}{\partial x_j} = 0 \tag{2.50}$$

Dimensionless momentum

$$\frac{\partial \rho U_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho U_i U_j + P\delta_{ij} - \frac{1}{R_{e_\infty}}\left(\mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \left(\frac{2}{3}\mu - \mu_v\left(\frac{\mu_{v_\infty}}{\mu_\infty}\right)\right)\delta_{ij}\frac{\partial U_k}{\partial x_k}\right)\right) - \frac{1}{F_{r_\infty}}\rho G_i = 0 \tag{2.51}$$

Dimensionless energy

$$\begin{aligned}&\frac{\partial\rho\left(e + (\gamma_\infty(\gamma_\infty - 1)M_\infty^2)\frac{U_iU_i}{2}\right)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho U_j\left(\gamma_\infty h + (\gamma_\infty(\gamma_\infty - 1)M_\infty^2)\frac{U_iU_i}{2}\right)\right) - \\ &\frac{(\gamma_\infty(\gamma_\infty - 1)M_\infty^2)}{R_{e_\infty}}\frac{\partial}{\partial x_j}\left(\mu U_i\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \left(\frac{2}{3}\mu - \mu_v\left(\frac{\mu_{v_\infty}}{\mu_\infty}\right)\right)\delta_{ij}\frac{\partial U_k}{\partial x_k}\right) - \\ &\frac{\gamma_\infty}{R_{e_\infty}P_{r_\infty}}\frac{\partial}{\partial x_j}\left(\kappa\frac{\partial T}{\partial x_j}\right) - \frac{(\gamma_\infty(\gamma_\infty - 1)M_\infty^2)}{F_{r_\infty}}\rho G_i U_i = 0\end{aligned} \tag{2.52}$$

The dimensionless form of the equation of state is

$$(\gamma_\infty M_\infty^2)P = \rho T \tag{2.53}$$

When we nondimensionalize the equations this way several important flow parameters emerge. The Reynolds number

$$R_{e_\infty} = \frac{\rho_\infty U_\infty C}{\mu_\infty} \tag{2.54}$$

The ratio of specific heats

$$\gamma_\infty = \frac{C_{P_\infty}}{C_{V_\infty}} \tag{2.55}$$

The ratio of bulk to shear viscosity in the free stream

$$\frac{\mu_{v_\infty}}{\mu_\infty} \tag{2.56}$$

The Mach number

$$M_\infty = \frac{U_\infty}{\left(\gamma_\infty \frac{R_u}{M_w} T_\infty\right)^{1/2}} \tag{2.57}$$

The Prandtl number

$$P_{r_\infty} = \frac{\mu_\infty C_{P_\infty}}{\kappa_\infty} \tag{2.58}$$

and the Froude number

$$F_{r_\infty} = \frac{U_\infty}{\sqrt{g_0 C}} \tag{2.59}$$

The flow is determined entirely by these six parameters and the wing geometry.

In dimensionless form the equations of motion become invariant under an arbitrary choice of the units of measurement. Any integrated quantity will also depend on the various properties of the flow. For example, in the absence of buoyancy effects, the drag of the wing would depend on nine parameters as well as the wing geometry.

$$D = f(\rho_\infty, T_\infty, \mu_\infty, \mu_{v\infty}, \kappa_\infty, C_{P_\infty}, C_{V_\infty}, M_w, U_\infty, Geometry) \tag{2.60}$$

But according to our dimensional study of the equations of motion the drag coefficient,

$$C_D = \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 S} \tag{2.61}$$

where S is the planform area of the wing, can depend on at most five independent quantities along with dimensionless measures of the geometry such as the wing aspect ratio $A_R = b^2/S$.

$$C_D = g(R_{e_\infty}, \gamma_\infty, M_\infty, P_{r_\infty}, (\mu_{v_\infty}/\mu_\infty), \textit{Dimensionless geometry}) \tag{2.62}$$

In practice the effects of Prandtl number and ratio of bulk to shear viscosity are small and the primary dependence is on R_{e_∞}, M_∞ and γ_∞, a tremendous simplification of the problem. All forces and moments on the wing would be normalized this way although at very low Reynolds number (a small flying insect) a more appropriate choice for normalizing forces would be that used in (2.26).

A small scale test that is intended to model a full scale flow must reproduce the values of these dimensionless quantities. Typically γ_∞ and P_{r_∞} are essentially constant and well modeled if the test fluid is air. Reproducing M_∞ is also relatively easy in a modest size test facility. However reaching realistic Reynolds numbers is much harder because of the need to match velocity and length scales. One approach is to increase the pressure or decrease the temperature of the test gas.

$$R_{e_\infty} = \frac{\rho_\infty U_\infty C}{\mu_\infty} = \frac{P_\infty}{(R_u/M_w)T_\infty}\left(\frac{U_\infty C}{\mu_\infty}\right) \tag{2.63}$$

The viscosity of a gas increases with temperature. A reasonable model is

$$\mu_\infty \sim T_\infty^{3/4} \tag{2.64}$$

Therefore

$$R_{e_\infty} \sim \frac{U_\infty C}{R_u}\left(\frac{P_\infty M_w}{T_\infty^{7/4}}\right) \tag{2.65}$$

We can reach very high Reynolds numbers on a small scale model by elevating the pressure and lowering the temperature of the test gas. This is the idea behind the National Transonic Facility, a large closed-circuit wind tunnel at the NASA Langley Research Center in Hampton Virginia. This wind tunnel uses very cold nitrogen gas at high pressures to reach flight Reynolds numbers.

2.6 BUCKINGHAM'S PI THEOREM - THE DIMENSIONAL ANALYSIS ALGORITHM

Finally, let's take a moment to formally state the systematic procedure for generating dimensionless variables. This is one way of stating the well-known Buckingham Pi Theorem (Bridgeman [2.13]).

To summarize, dimensional analysis makes use of a simple, purely algorithmic, procedure that is extremely general and can be applied to practically any physical problem. The various steps are as follows.

1) Identify the physical variables relevant to the problem $(a_1, a_2, \ldots, a_\alpha)$.

2) Determine the fundamental dimensions of each physical variable. The total number of dimensions is $(d_1, d_2, \ldots, d_\beta); \quad (\beta \le \alpha)$. Each variable is a power monomial function of its dimensions,

$$\hat{a}_i = d_1^{k_1} d_2^{k_2} \ldots d_\beta^{k_\beta}. \tag{2.66}$$

3) *Buckingham's Pi Theorem* - A relationship between physical variables $\psi = f[a_1, a_2, \ldots, a_\alpha]$ must be invariant under a β-parameter dilation group applied to the fundamental dimensions.

$$\tilde{d}_1 = e^{\delta_1} d_1 \ ; \quad \tilde{d}_2 = e^{\delta_2} d_2 \ ; \ \ldots \ ; \quad \tilde{d}_\beta = e^{\delta_\beta} d_\beta. \tag{2.67}$$

4) The algorithm for accomplishing step 3 is to apply a one-parameter dilation group to each dimension in succession. New variables are created at each step which are independent of the dimension being varied. This process is continued until all the dimensions are exhausted. In the final result, the physical problem can only depend on dimensionless variables via a function of the form, $\psi = \Psi[\Pi_1, \Pi_2, \ldots, \Pi_\gamma]$. Usually $\gamma = \alpha - \beta$. However if two or more of the phys-

ical variables have the same or commensurate units then γ can be greater than $\alpha - \beta$ by the number of variables having common units minus one. This is the case for the examples described in Sections 2.2 and 2.4.

Step 4 is a purely algorithmic process which always leads to a set of dimensionless combinations of the physical variables. The only problem is that any product of these dimensionless variables is also dimensionless, and so the reduced set is not unique and therefore not always recognizable in traditional terms. Changing the order in which various dimensions are subjected to dilation will change the form of the final variables. For example, in the case of sphere drag described above we could have wound up with

$$\phi = \Phi[C_D Re, Re] = \Phi\left[\frac{D}{\mu U r}, \frac{\rho U r}{\mu}\right] \tag{2.68}$$

as an equivalent dimensionless form of the drag equation. Note that in this renormalized form the drag law has a finite limit as the Reynolds number goes to zero.

$$\lim_{Re \to 0} \Phi\left[\frac{D}{\mu U r}, \frac{\rho U r}{\mu}\right] = 6\pi . \tag{2.69}$$

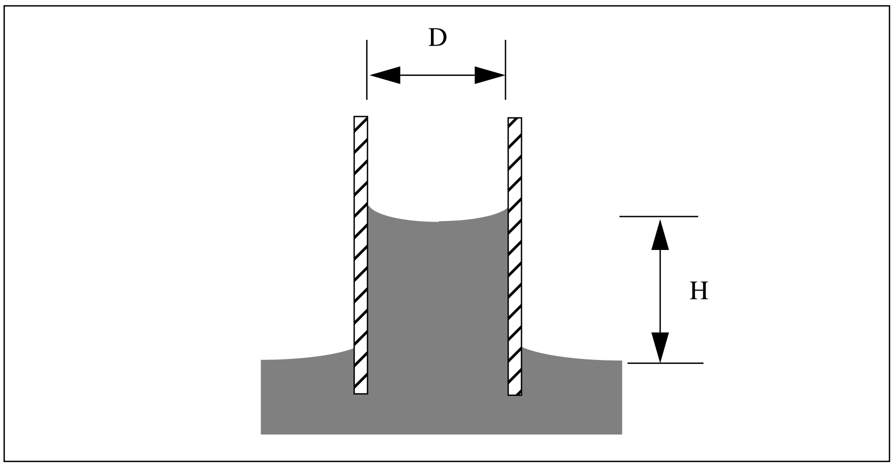
The success or failure of dimensional analysis depends entirely on Step 1; the choice of the dimensioned physical variables relevant to the problem. This comprises the art of dimensional analysis. Applied intelligently with a deep knowledge of the problem, very important and profound results can be obtained. Applied blindly, dimensional analysis can easily lead to nonsense!

2.7 CONCLUDING REMARKS

Although I have tended to emphasize the limitations of dimensional analysis this should be balanced by the recognition of the tremendous simplification achieved in converting from dimensioned to dimensionless variables. Sphere drag is a great example because, in spite of the fact that the equations governing the flow are perfectly well known and have been for over a century, we are still very far from having an adequate theory for the viscous flow past a sphere. For example, we have no idea of the asymptotic value of C_D as R_e approaches infinity at fixed Mach number or as M_∞ approaches infinity at fixed Reynolds number. Nevertheless dimensional analysis is able to reduce the number of variables in the problem from nine to five - a tremendous accomplishment! Without this all-important tool to organize the experimental data *and our thinking* rational scientific inquiry into aerodynamics and many other fields would be utterly impossible!

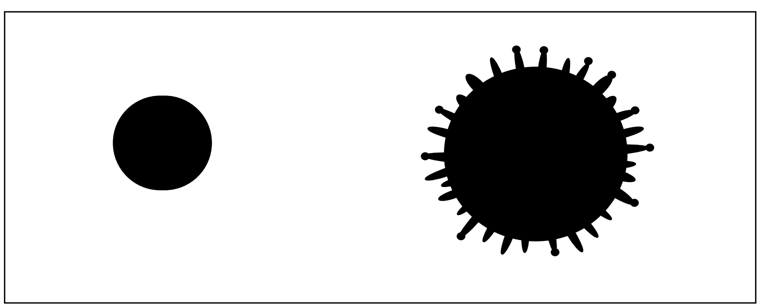
2.8 EXERCISES

EXERCISE 2.1 - Under the influence of surface tension, a liquid rises to a height H in a glass tube of diameter D. How does H depend on the parameters of the problem?



EXERCISE 2.2 - Estimate the time of oscillation of a small drop of liquid under its own surface tension.

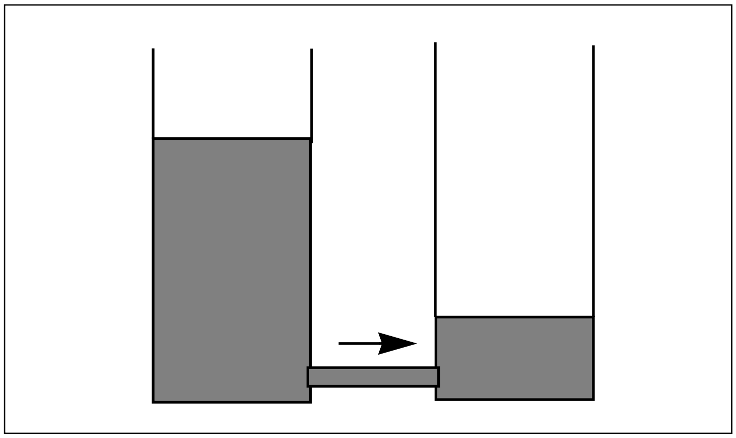
EXERCISE 2.3 - When a drop of water strikes a surface at sufficiently low speed, surface tension keeps it round, so it makes a circular spot. As the impact speed is increased, dynamic forces overcome the smoothing effect of surface tension, and the drop becomes unstable and forms a spiky shape as shown in the sketch below (Reference [2.7]).



How does the speed at which the impact becomes unstable depend on the properties of the drop? Retain only the essential properties, so that your result involves only a single unknown constant that could be determined from an experiment. Thus you may wish to assume that viscosity is negligible, the properties of the ssurrounding Air are unimportant, etc. See if your result makes sense. For example, does the critical speed depend on the surface tension in the way you would expect?

EXERCISE 2.4 - Estimate the velocity of fall of a small heavy sphere in a viscous fluid of lower density than the sphere under the influence of gravity. Compare your result with the exact solution. How long does it take the sphere to reach its terminal velocity when dropped from rest.

EXERCISE 2.5 - Liquid in an open container flows through a long horizontal pipe into a second container as shown below.



How does the time for the liquid level to reach equilibrium depend on the parameters of the problem?

EXERCISE 2.6 - Use dimensional analysis to find how rowing speed depends on the number of oarsman for racing shells. This problem is discussed by McMahon [2.8] and Barenblatt [2.4]. First identify the appropriate parameters of the problem and use the procedure developed in 2.3 to identify dimensionless Π's.

Then apply some physics using the following asssumptions.

i) The boats are geometrically similar.

ii) The boat weight, W, per oarsman is constant.

iii) Each oarsman contributes the same power, P.

iv) The only hindering force is skin friction and the friction coefficient over the wetted area is a constant.

Hint: find how the volume of the displaced water varies with the number of oarsman and the length of the boat. Equate the expenditure of energy on skin friction to the power supplied by the oarsman.

Data for men's rowing over a 2 km course from three recent Olympic summer games is presented in the following table.

Olympics	1 Oarsman	2 Oarsmen	4 Oarsmen	8 Oarsman
Atlanta	404.85	376.98	356.93	342.74
Barcelona	411.40	377.32	355.04	329.53
Seoul	409.86	381.13	363.11	____

Table 2.2 Rowing times in seconds for 1, 2, 4 and 8 man shells from four previous Olympics, the distance traveled in each case is 2000 meters.

Plot the data in logarithmic coordinates and compare with your prediction. Notice that in the context of this problem "number of oarsman" is a fundamental dimension.

EXERCISE 2.7 - Critique the assumptions in EXERCISE 2.6 particularly (i) that seems to suggest that the shells get wider as they get longer to accomodate more rowers.

(i) How does the problem work out if the width of the shell is assumed to be constant?

(ii) Suppose the drag is primarily due to the generation of waves and skin friction can be neglected, how would the speed depend on the number of oarsman? Do these results shake your confidence in the solution developed in EXERCISE 2.6?

(iii) Work the case where the race is carried out by fleas on a lake of honey.

EXERCISE 2.8 - What is the speed of the wave in a row of falling dominos on a table? Add whatever simplifying assumptions you feel are reasonable such as perfectly rigid dominos, constant coefficient of friction between the dominos and the table, etc. This problem is the subject of a pair of journal papers by Stronge [2.9] and Stronge and Shu [2.10] as well as a note in SIAM Review "problems and solutions". The problem was proposed by Daykin [2.11] and solved by McLachlan, Beaupre, Cox and Gore [2.12].

EXERCISE 2.9 - Show that if two equal size elastic spheres are pressed together, the radius of the circle of contact varies as the one-third power of the force between them. How does it vary as the radius of the spheres?

EXERCISE 2.10 - One of the well known observations in blood flow is that the viscous shear stress at the wall of an artery is approximately constant independent of the diameter of the artery. Consider a bifurcation where the flow in one large artery splits into two smaller adjoining arteries of equal size. How are the diameters of the smaller arteries related to the diameter of the large artery?

EXERCISE 2.11 - Use dimensional analysis to deduce how the weight a man can lift depends on his own weight. Assume that the strength of a muscle varies as its cross-sectional area. See if your result correlates the following data taken from the 1969 World Almanac for the 1968 Senior National AAU weightlifting championships.

Class	Body weight pounds	Lifted weight pounds
Bantam	123.5	740
Featherweight	132.25	795
Lightweight	148.75	820
Light-heavy	181.75	1025
Middle-heavy	198.25	1055
Heavyweight	?	1280

Table 2.3 Total weight lifted for different classes.

How much did the heavyweight lifter weigh?

EXERCISE 2.12 - There is a continuing interest in pushing measurements of circular cylinder drag to the highest possible Reynolds numbers. One scheme that has been proposed is to tow a submerged, high aspect-ratio cylinder behind two nuclear powered aircraft carriers pulling lines attached to each end of the cylinder. The kinematic viscosity of water is small, the cylinder diameter can be made quite large and thus high Reynolds numbers ought to be achievable. Assuming only cylinders of a given aspect ratio, say $L / r = 60$, are used, how does the required towing force vary with the Reynolds number based on cylinder diameter? What force would be required to reach a Reynolds number that exceeds the highest available data ($R_e = 10^8$, $C_d = 0.6$). The maximum towing force available is about 10^8 Newtons.

EXERCISE 2.13 - Consider the dynamics of the so-called "dead man's dive". An adult who stands rigid and topples from a three meter diving board will execute a succesful dive. How is this result changed for a child; for a man on the moon?

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