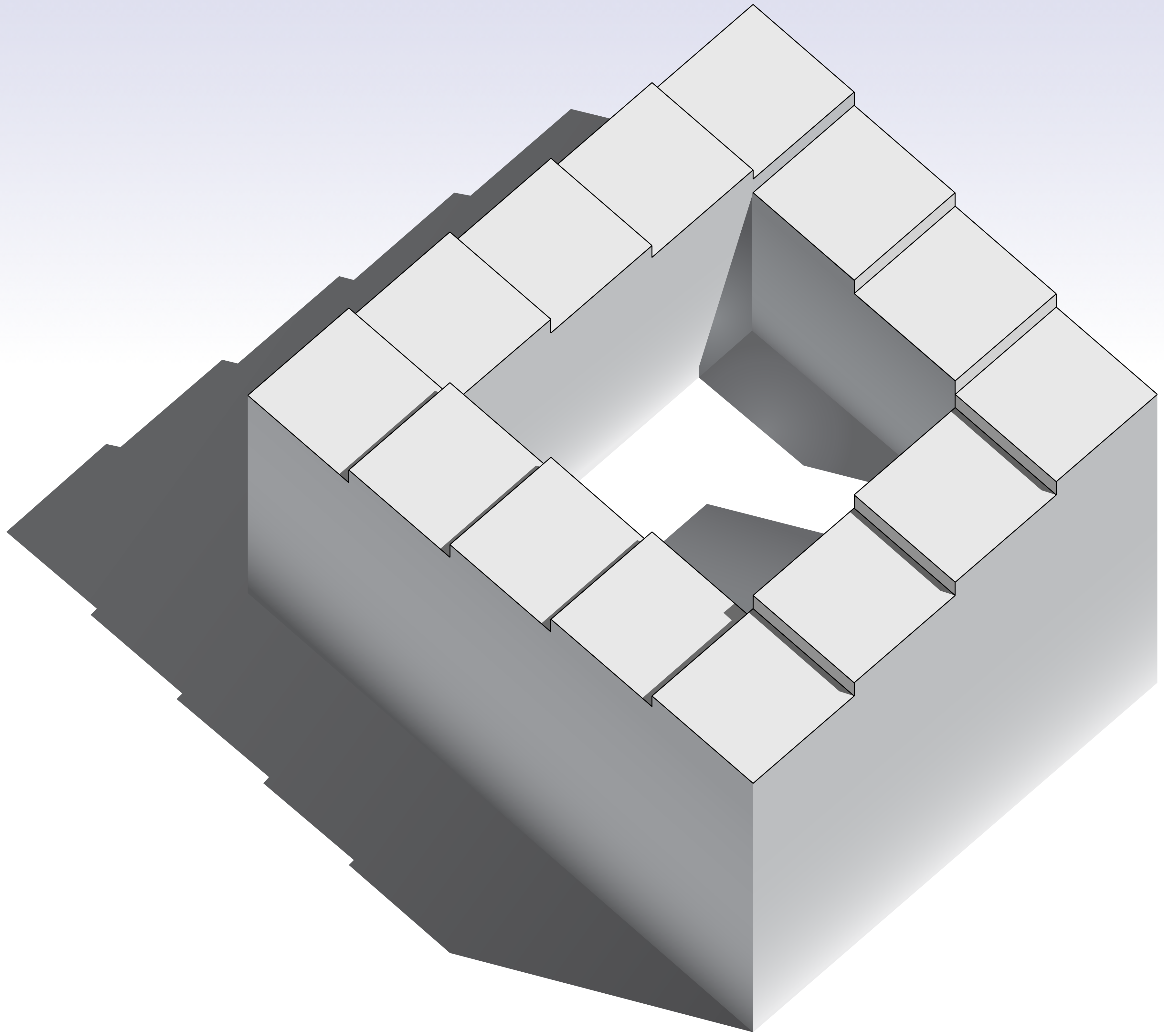
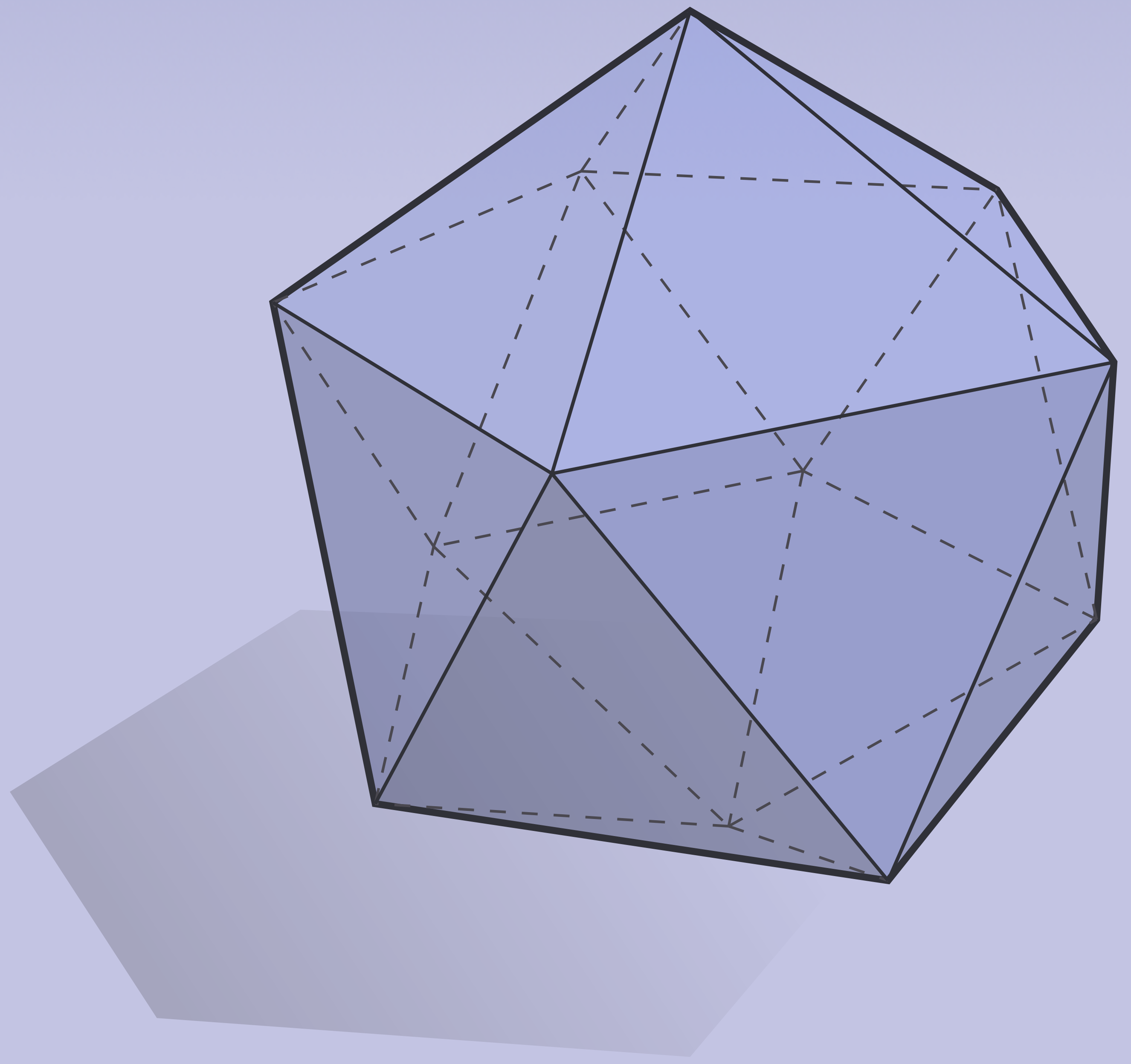


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

LECTURE 1:
OVERVIEW



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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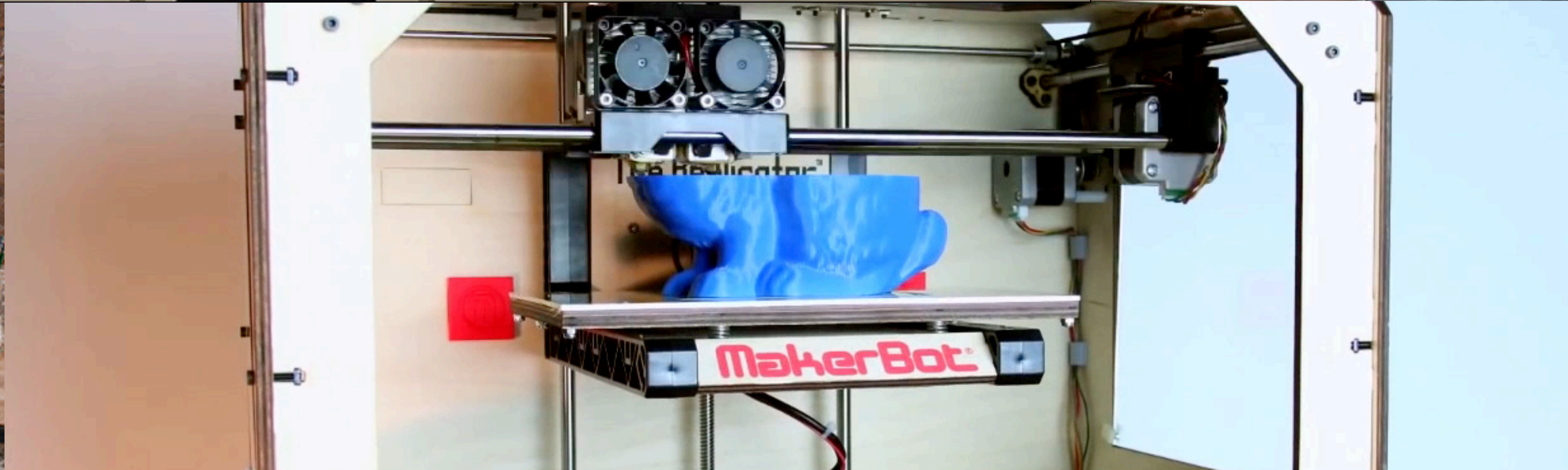
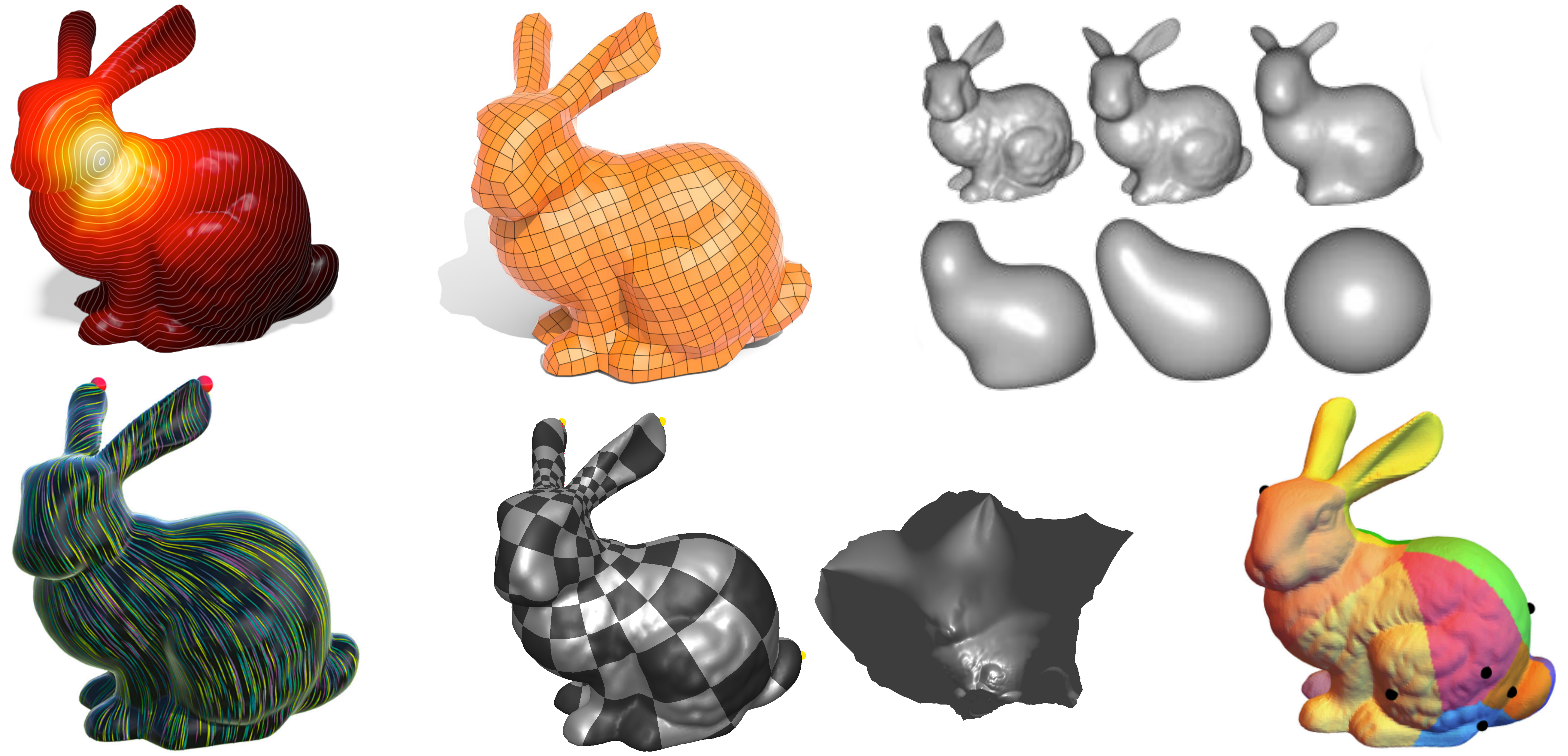
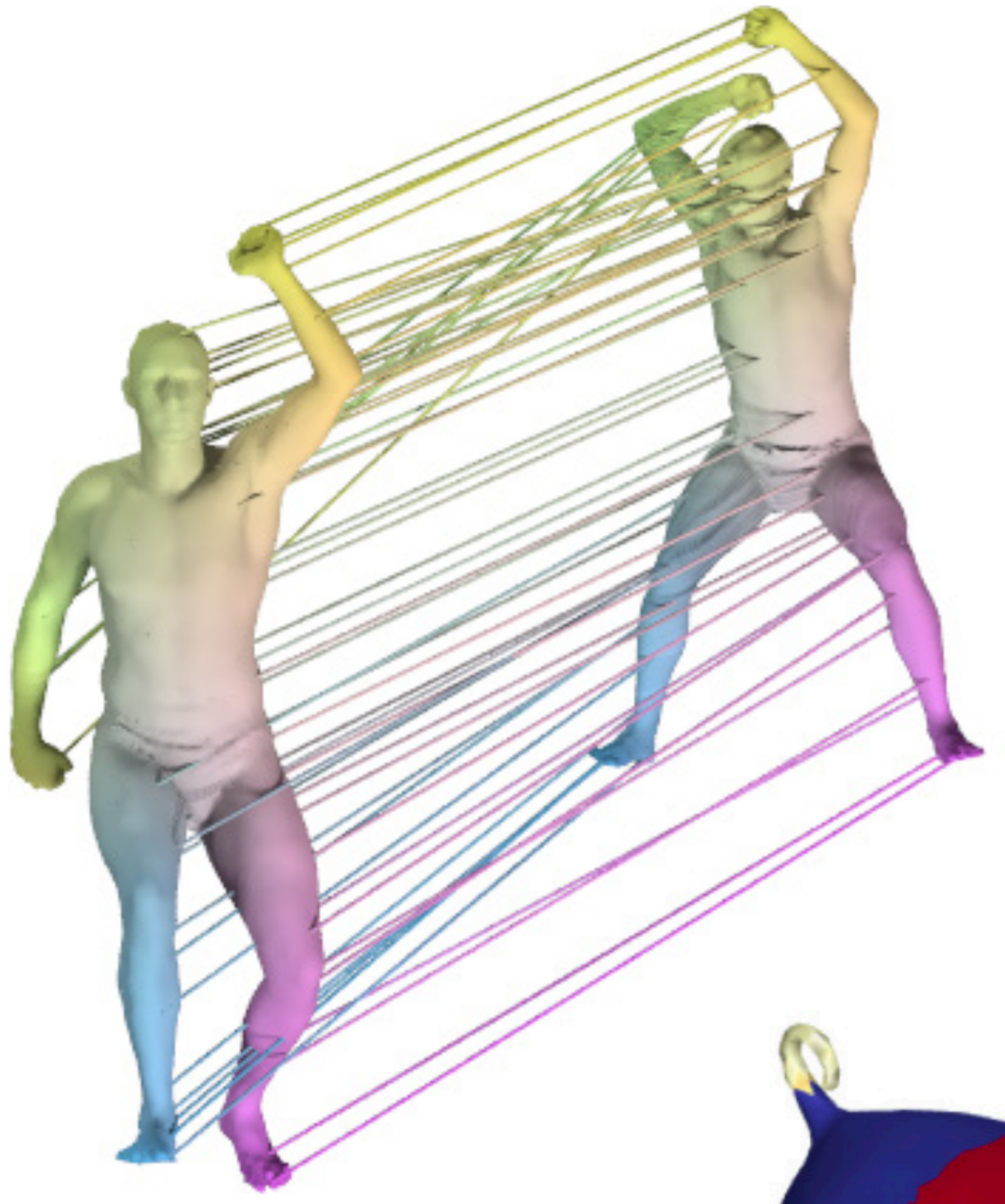
Geometry is Coming…

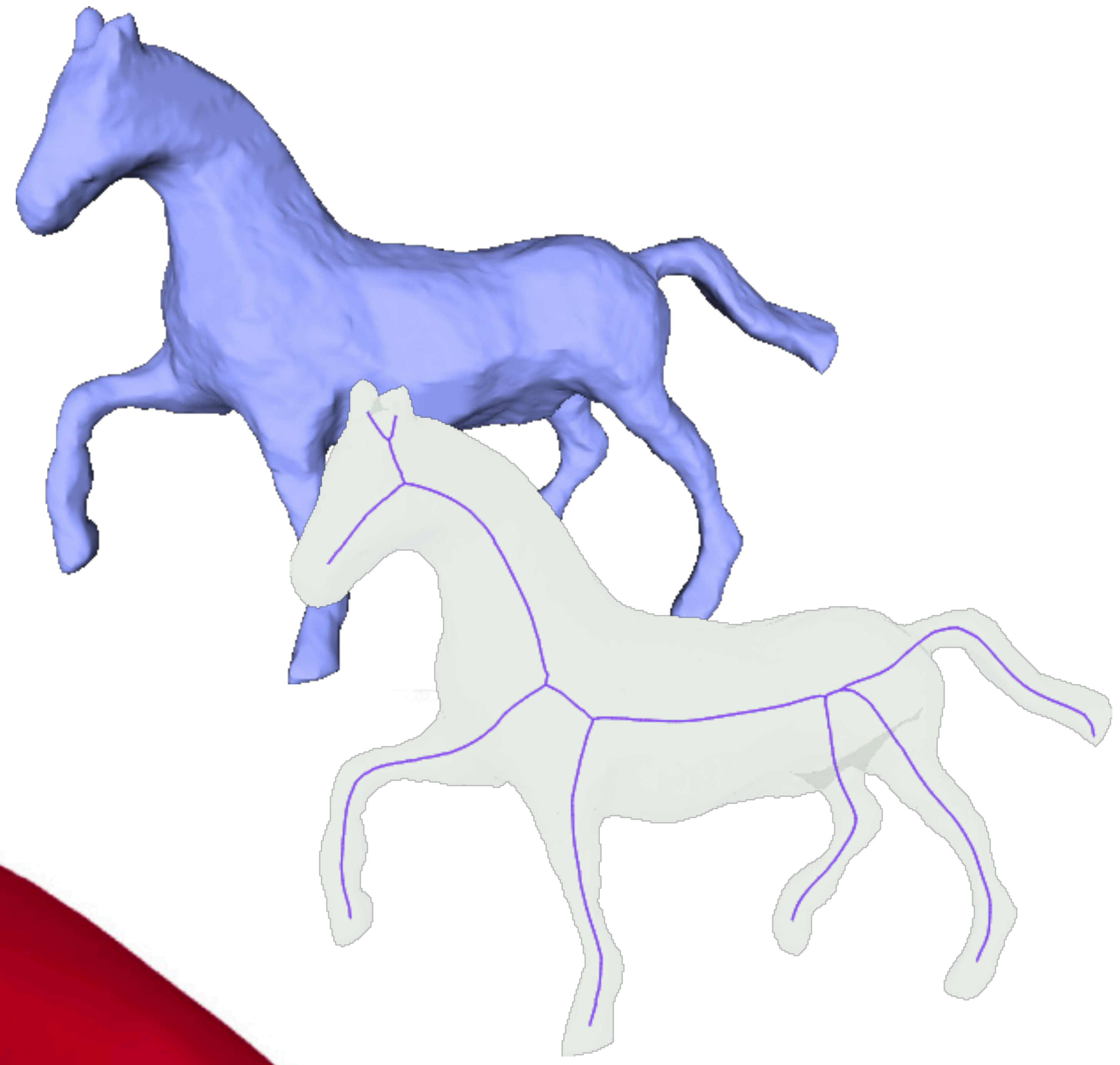


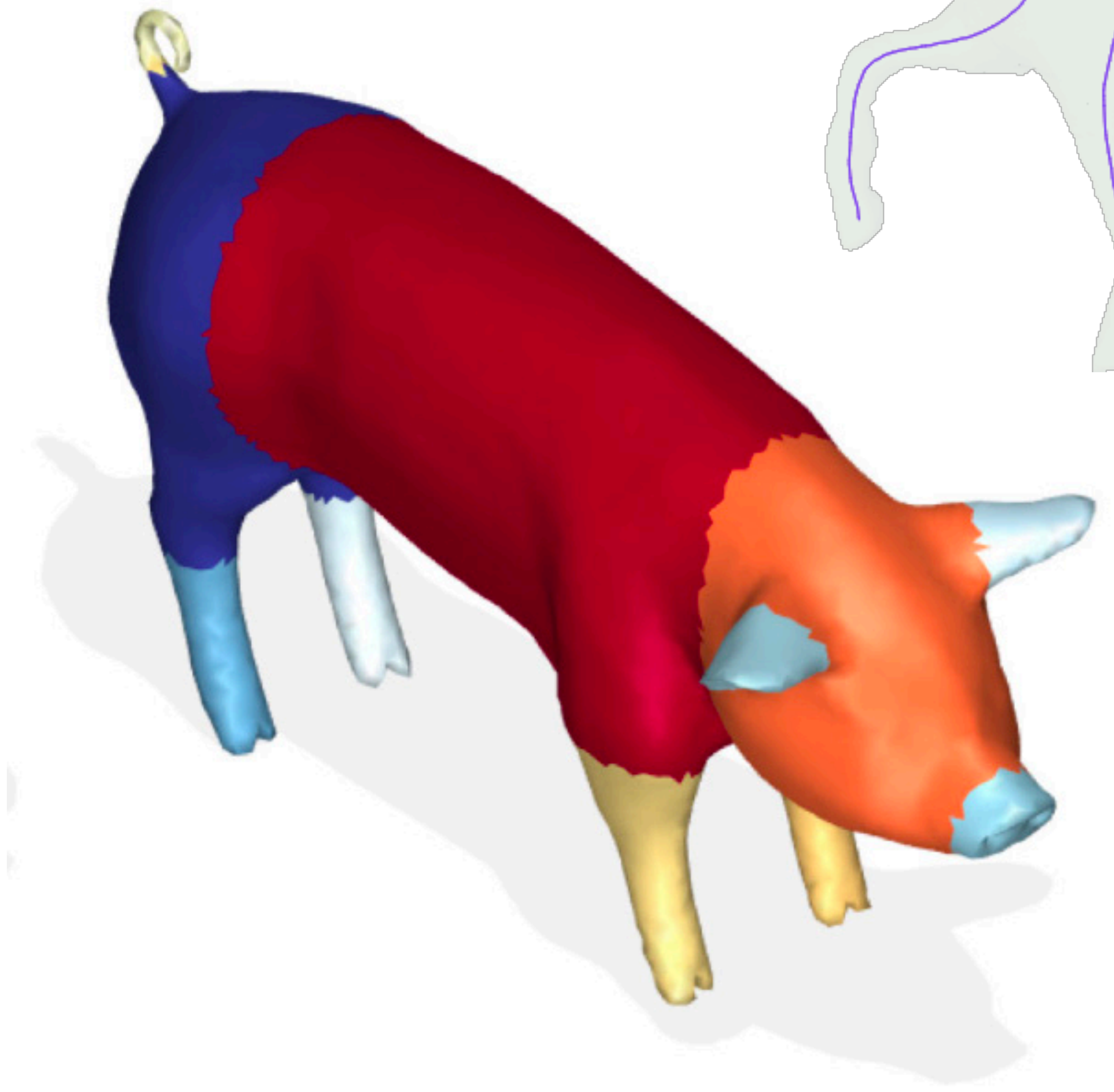
Applications of DDG: Geometry Processing

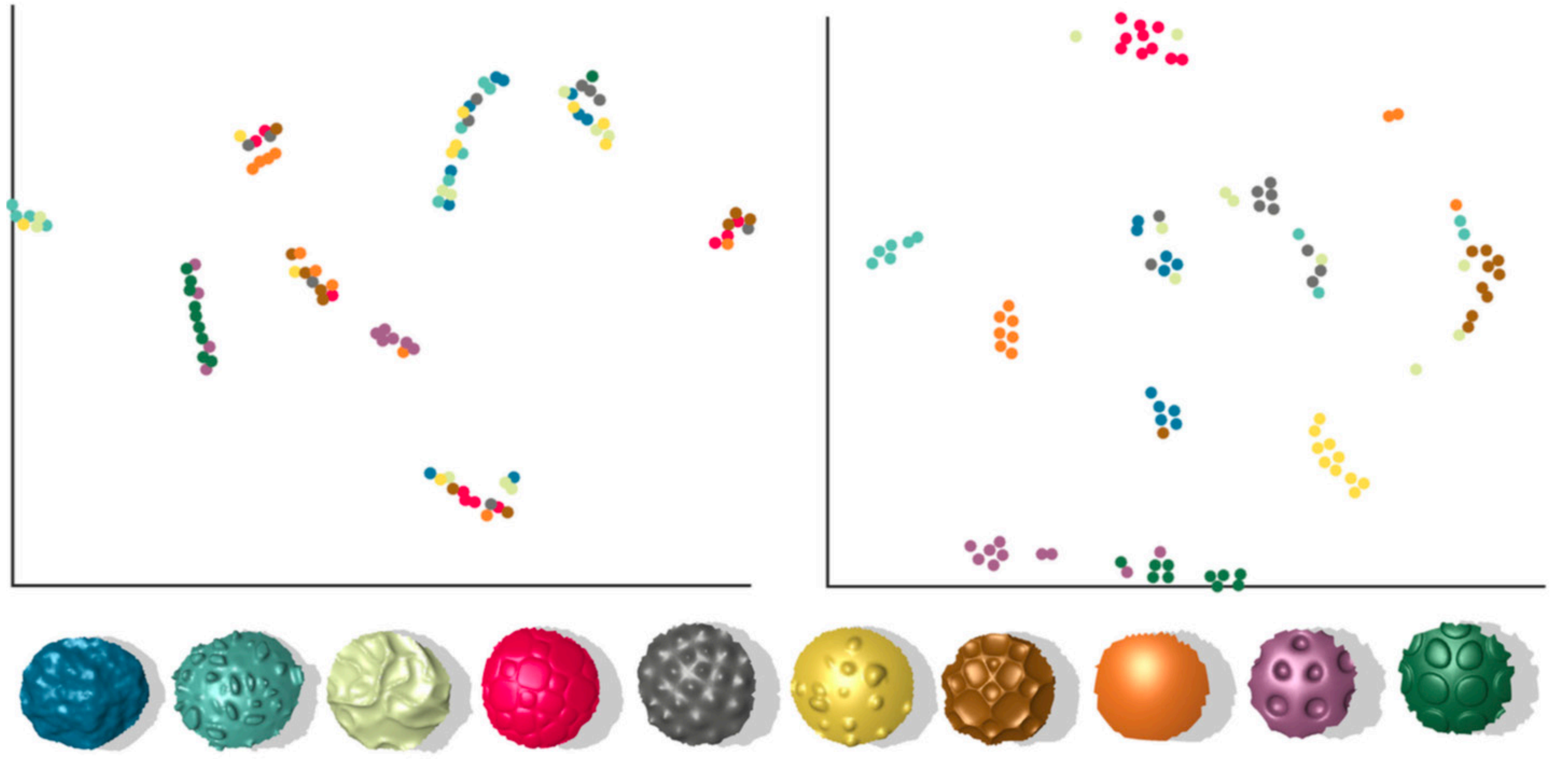


Applications of DDG: Shape Analysis

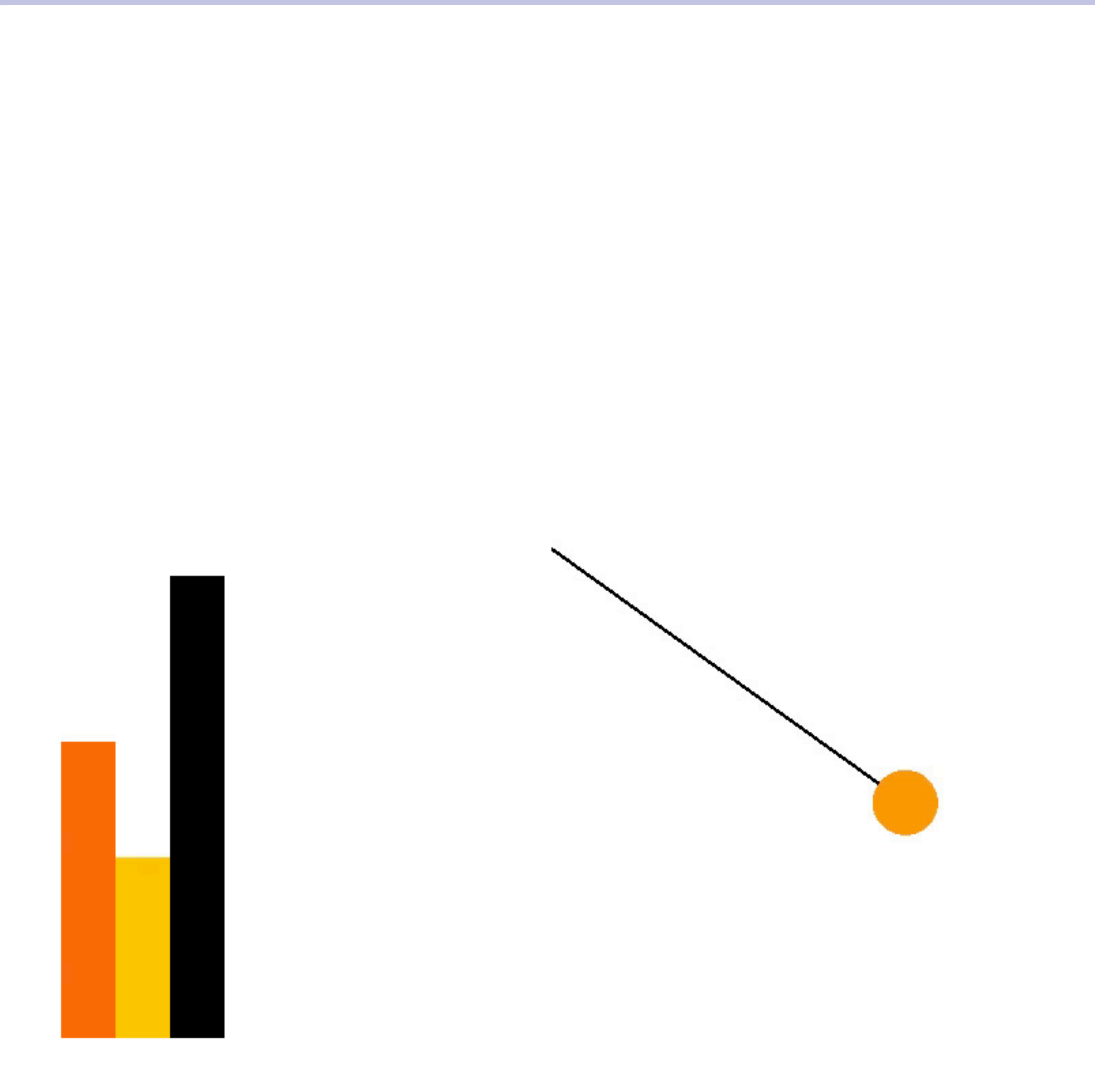


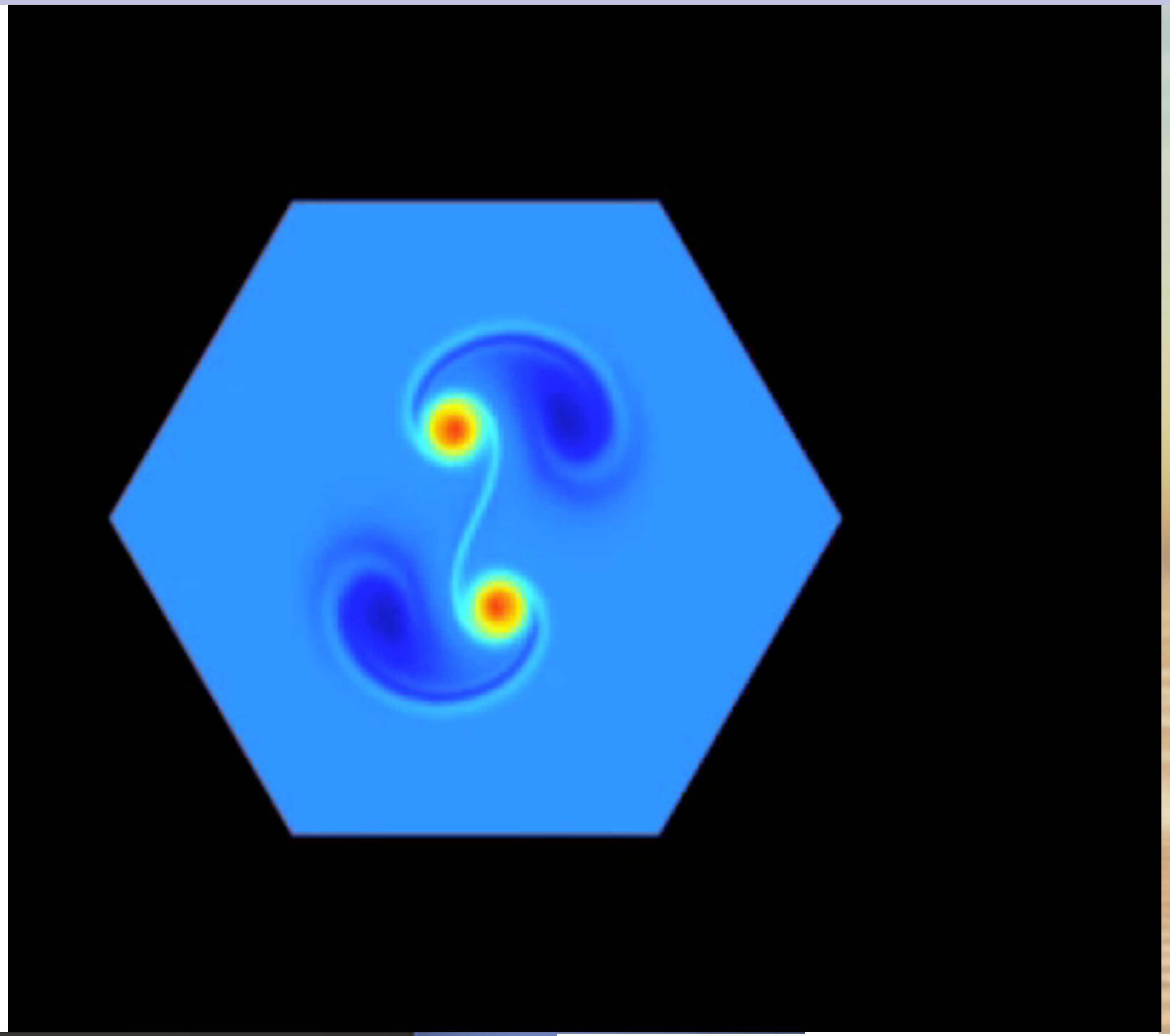


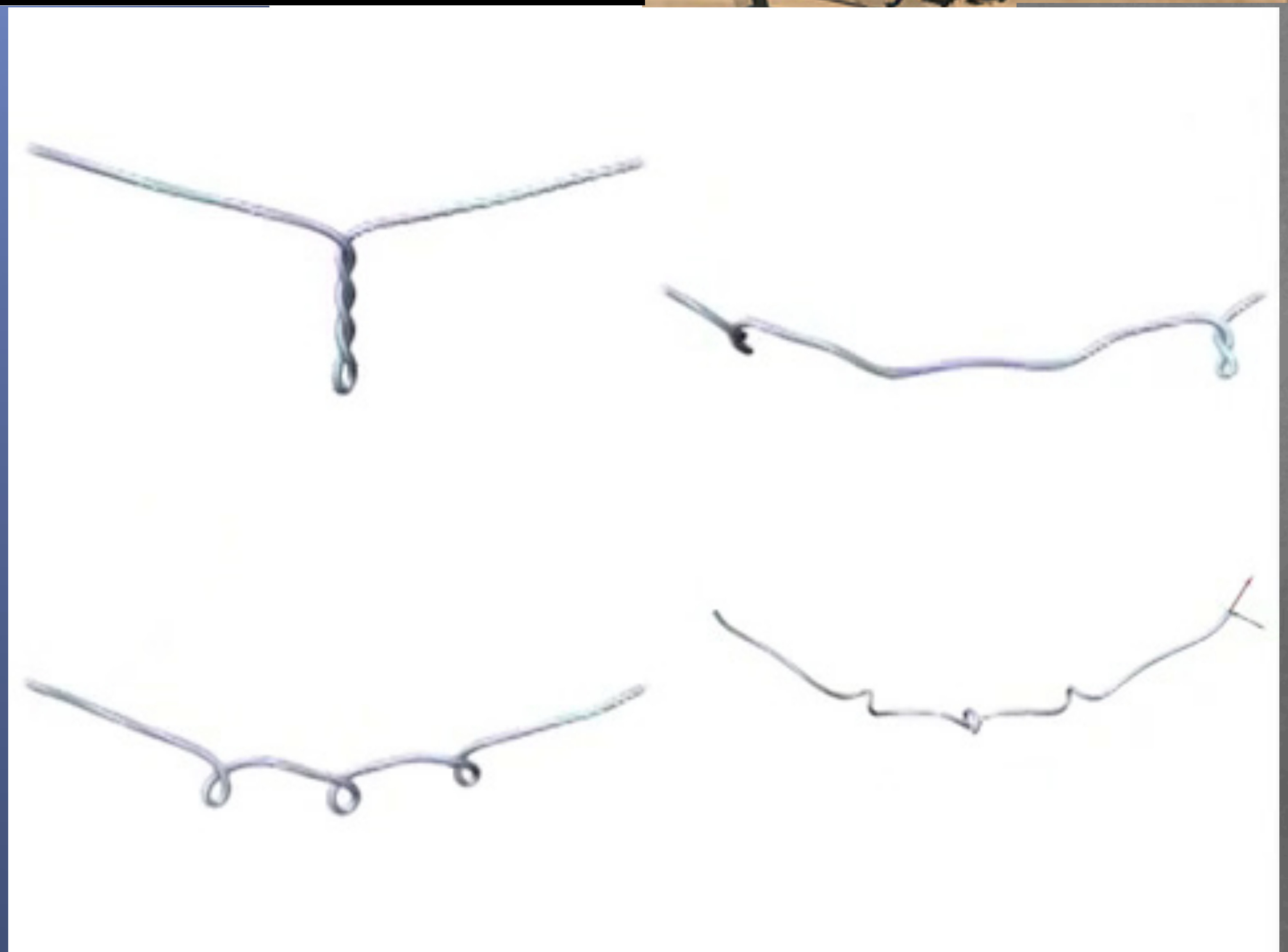


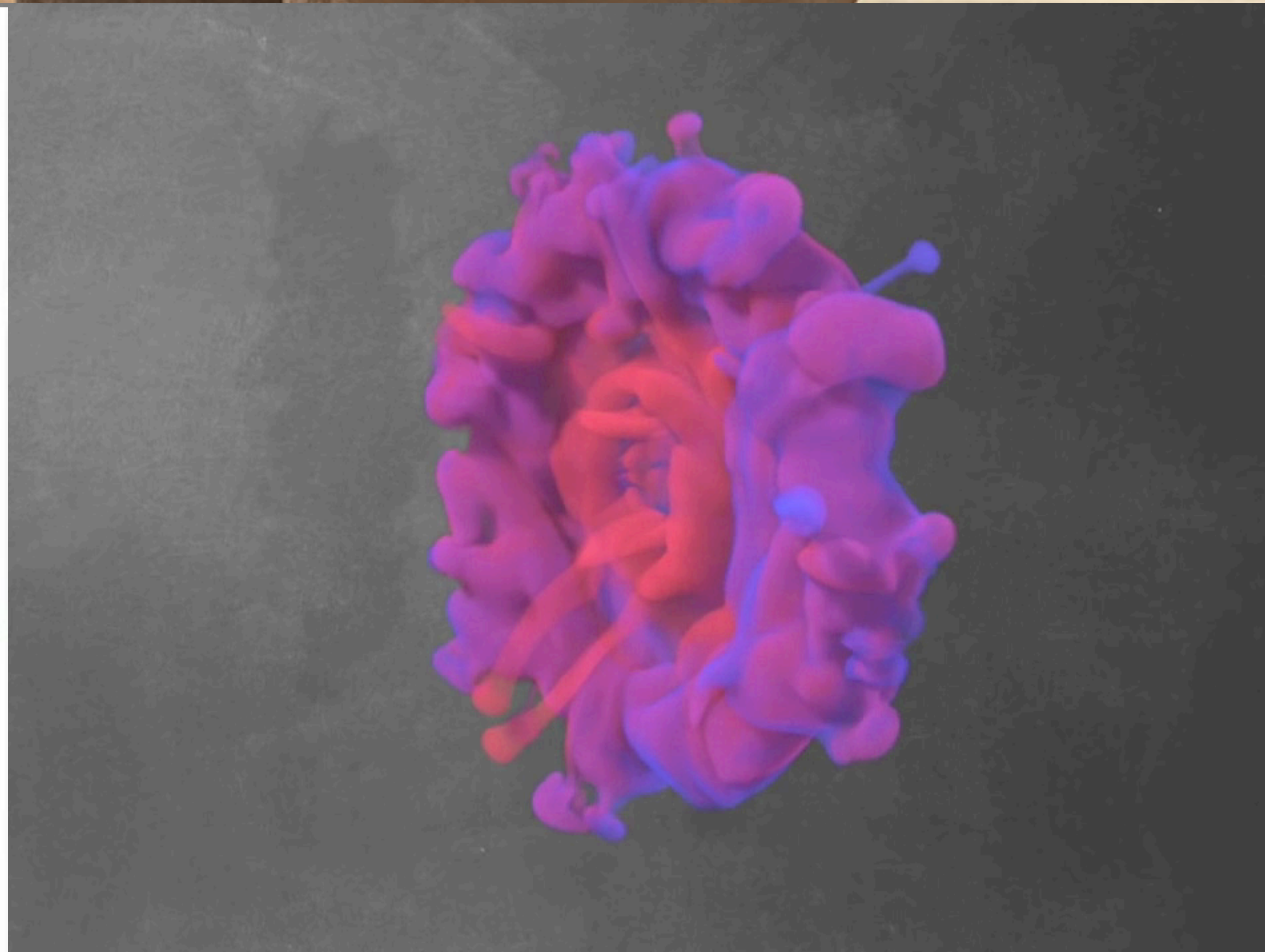


Applications of DDG: Numerical Simulation



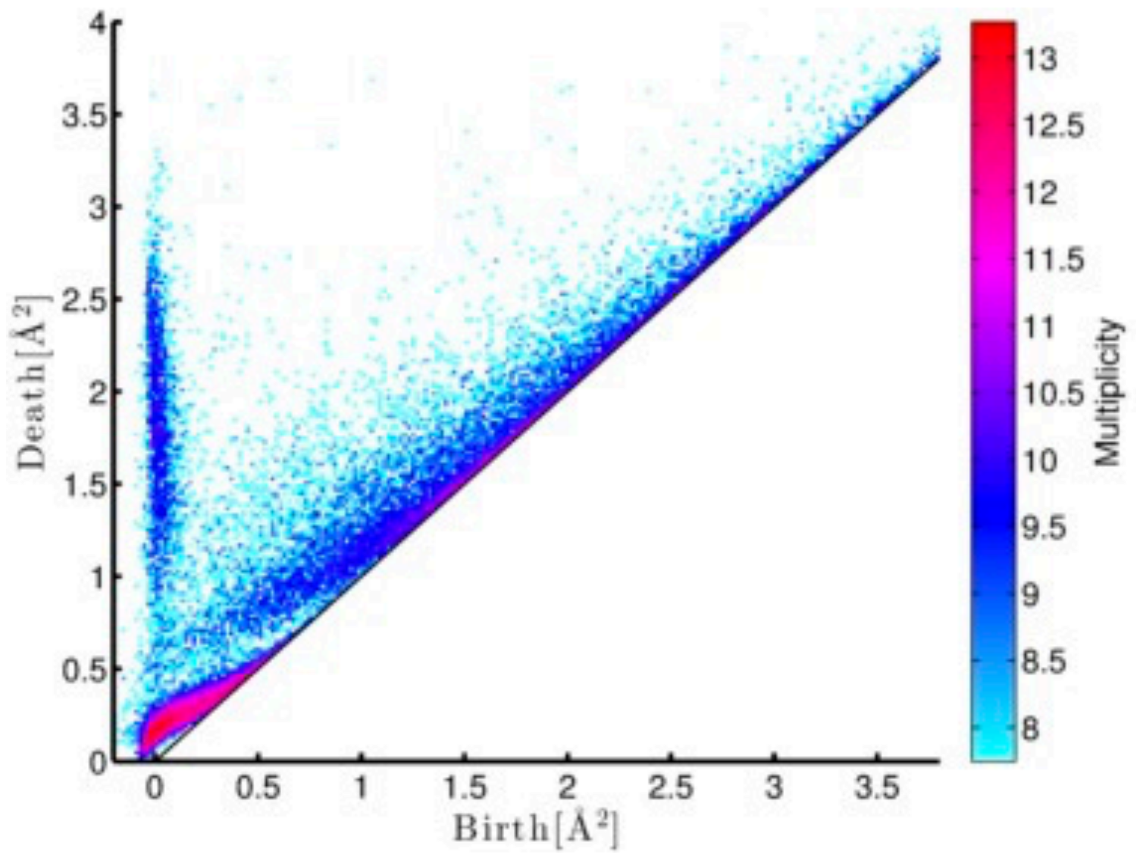


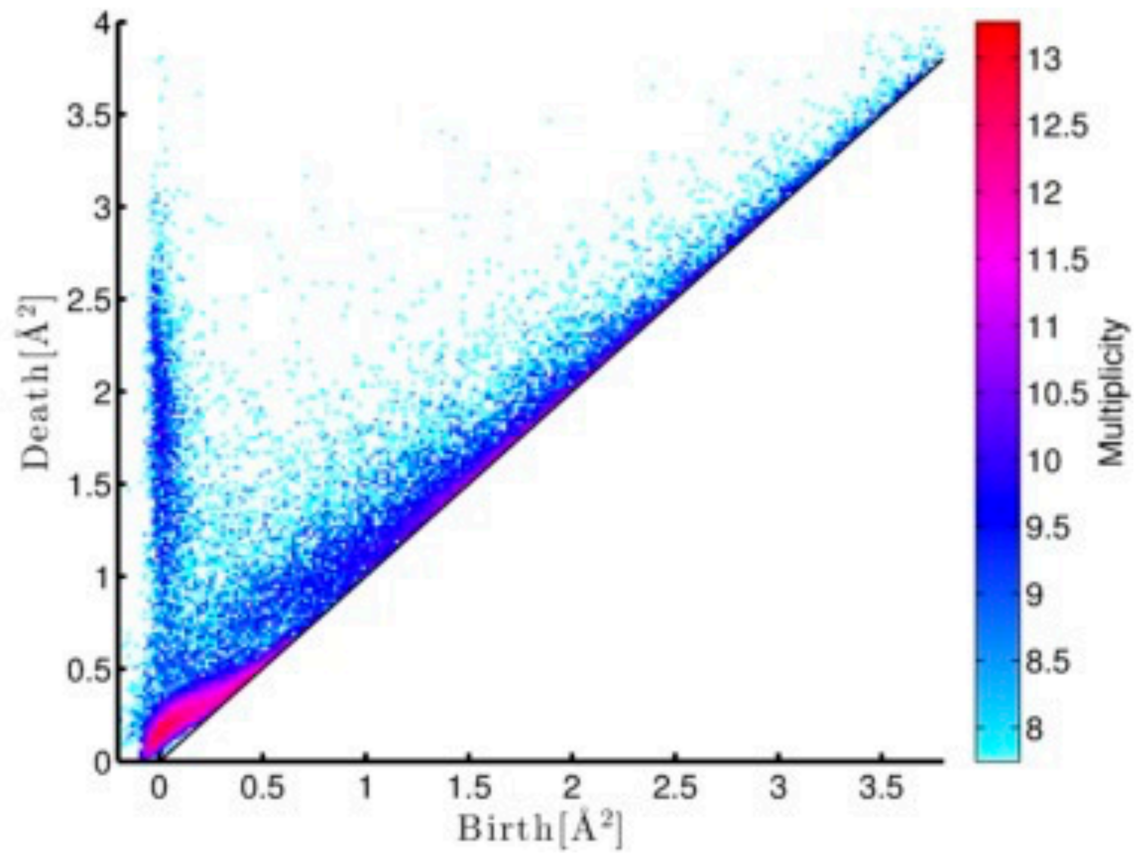


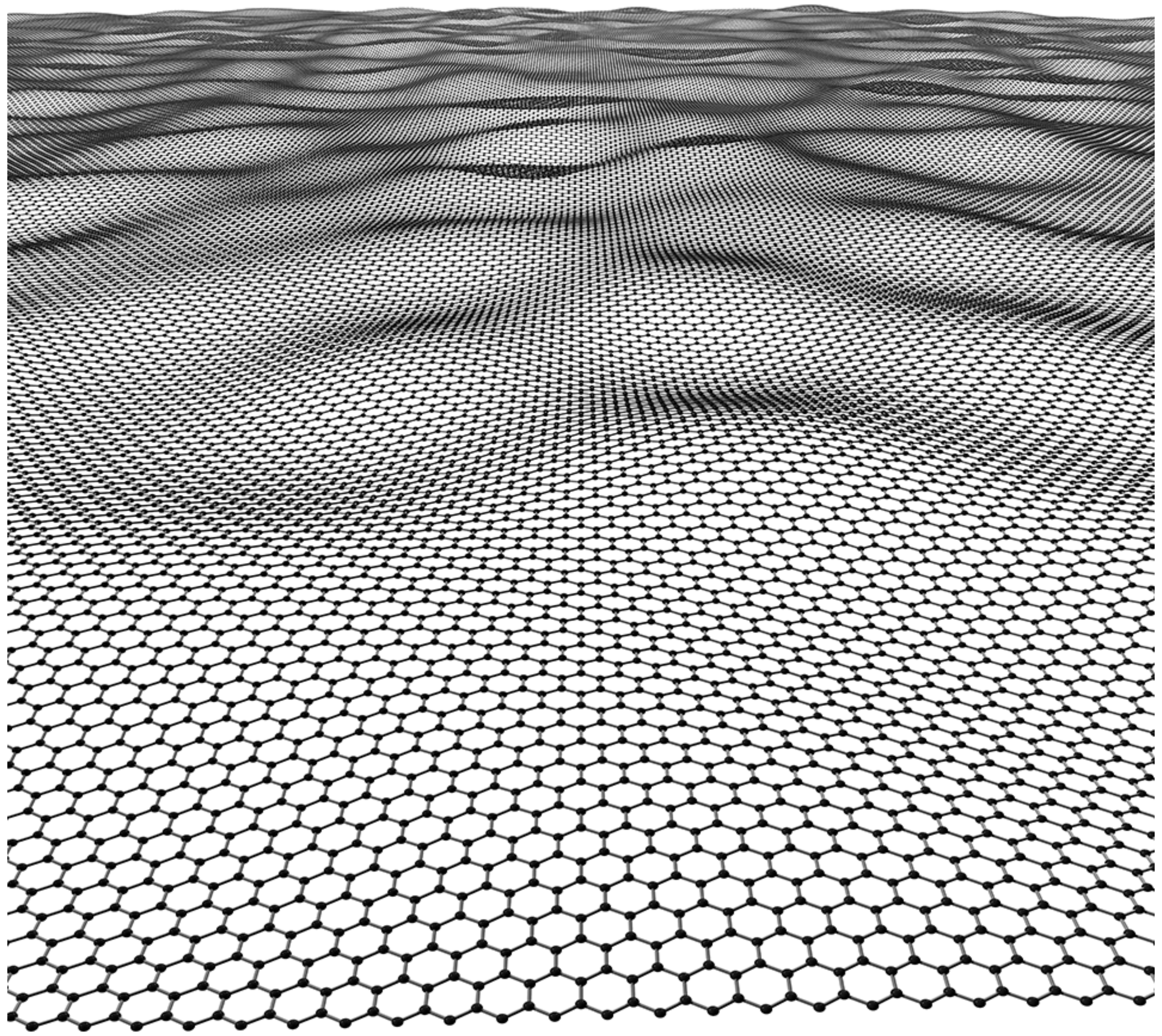


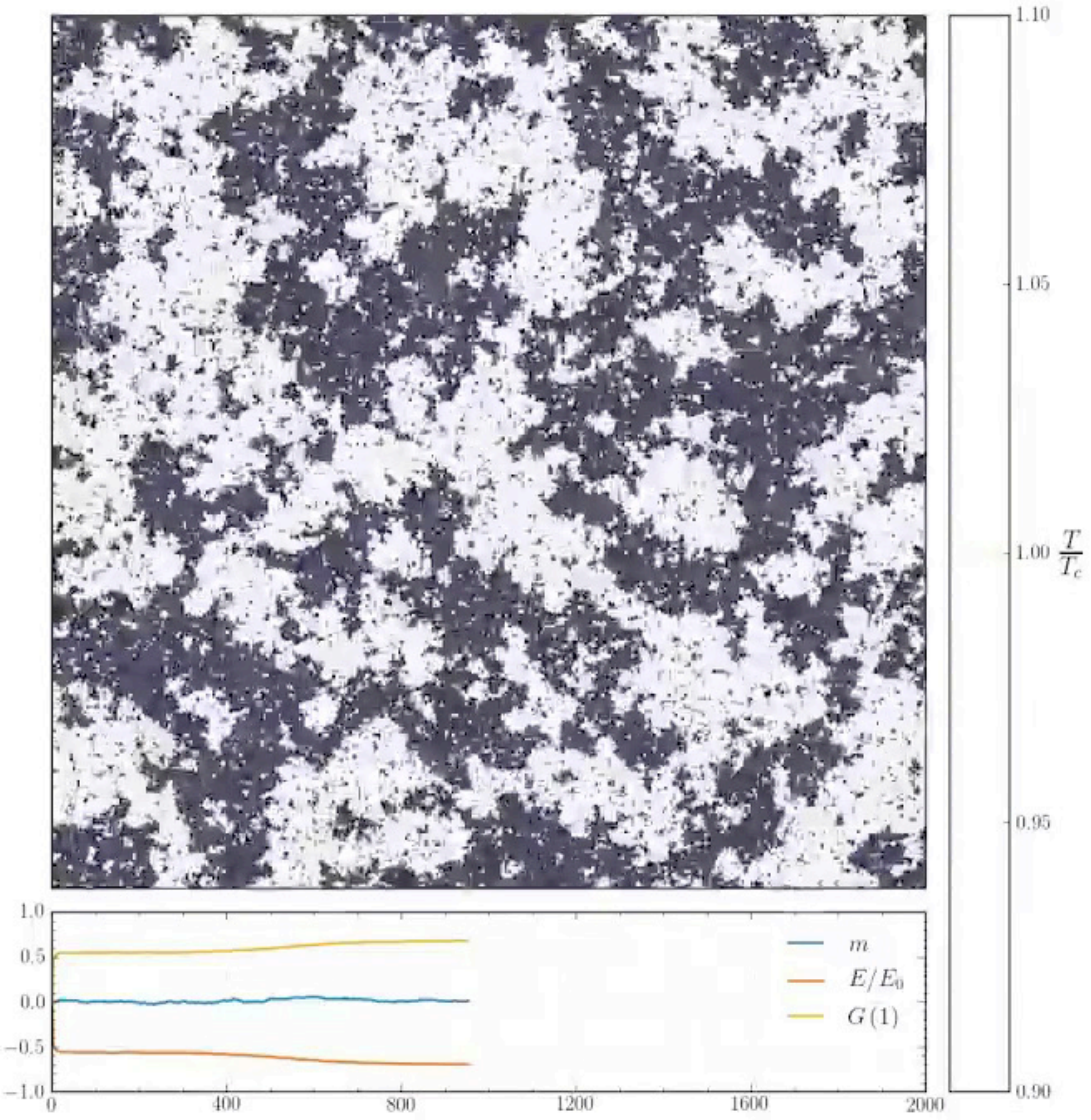
Applications of DDG: Architecture & Design

Applications of DDG: Discrete Models of Nature



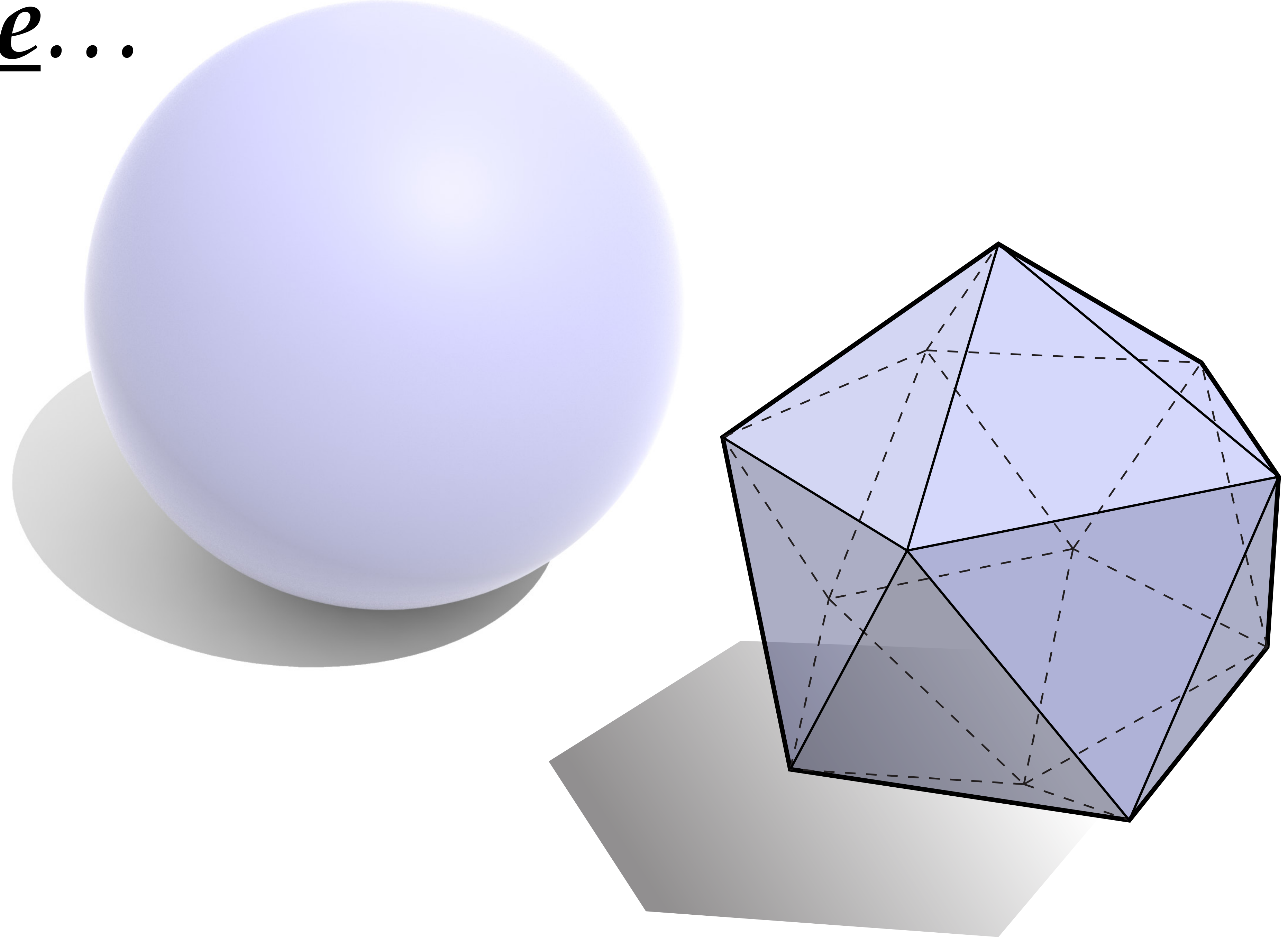






What Will We Learn in This Class?

- First and foremost: ***how to think about shape***…
  - …mathematically (differential geometry)
  - …computationally (geometry processing)
  - **Central Theme:** *link these two perspectives*



- Why? Shape is everywhere!
  - *computational biology, industrial design, computer vision, machine learning, architecture, computational mechanics, fashion, medical imaging…*
  - Flat images are old news :-)

Course Web Page

- All course information is spelled out in detail on course webpage:

`http://geometry.cs.cmu.edu/ddg`

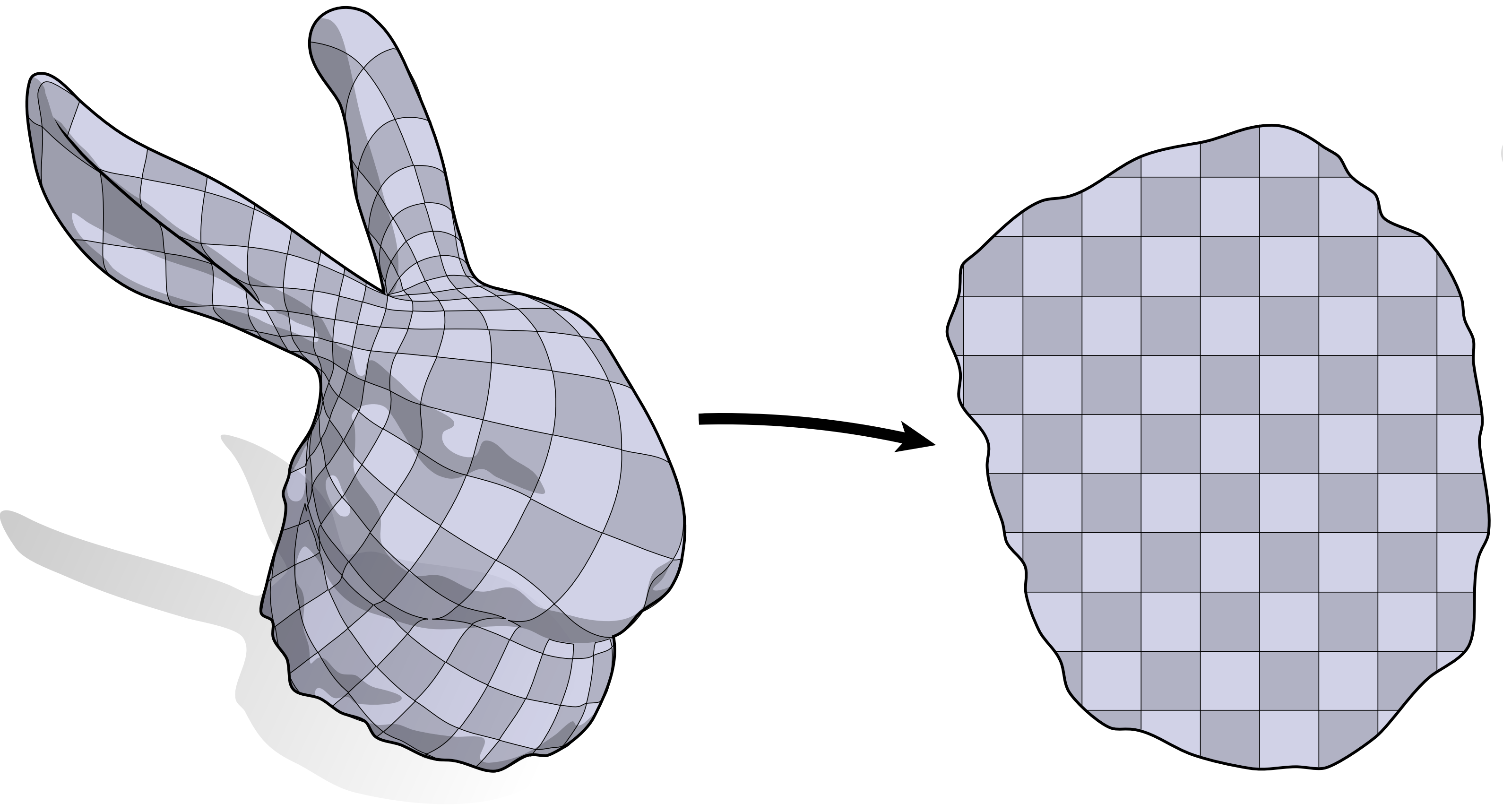
- All communication goes through this site (assignments, discussion, etc.)
- Register account from link at end of menu:

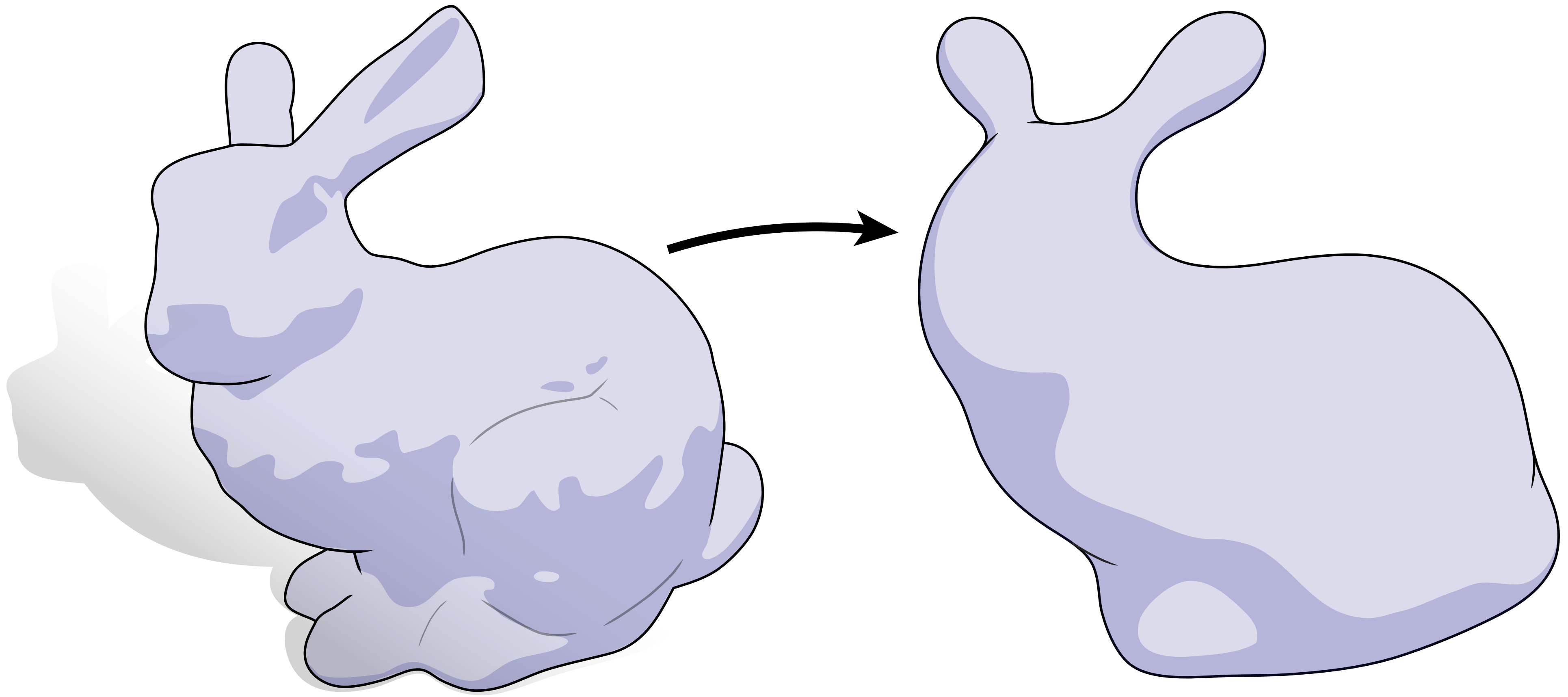
META

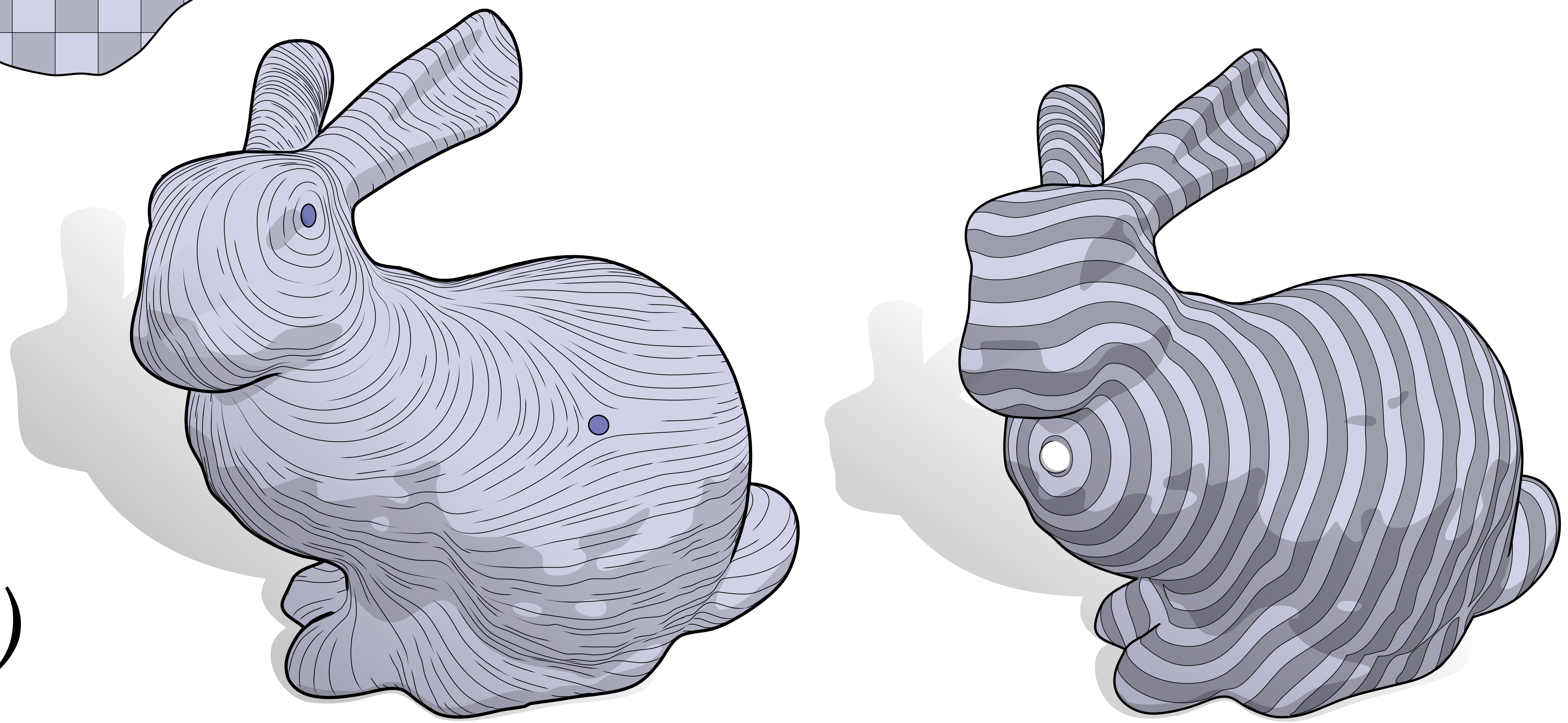
- Register
- Log in
- Entries RSS
- Comments RSS
- WordPress.org

Assignments

- **Derive** geometric algorithms from first principles (pen-and-paper)
- **Implement** geometric algorithms (coding)
  - Discrete surfaces
  - Exterior calculus
  - Curvature
  - Smoothing
  - Parameterization
  - Distance computation
  - Direction Field Design







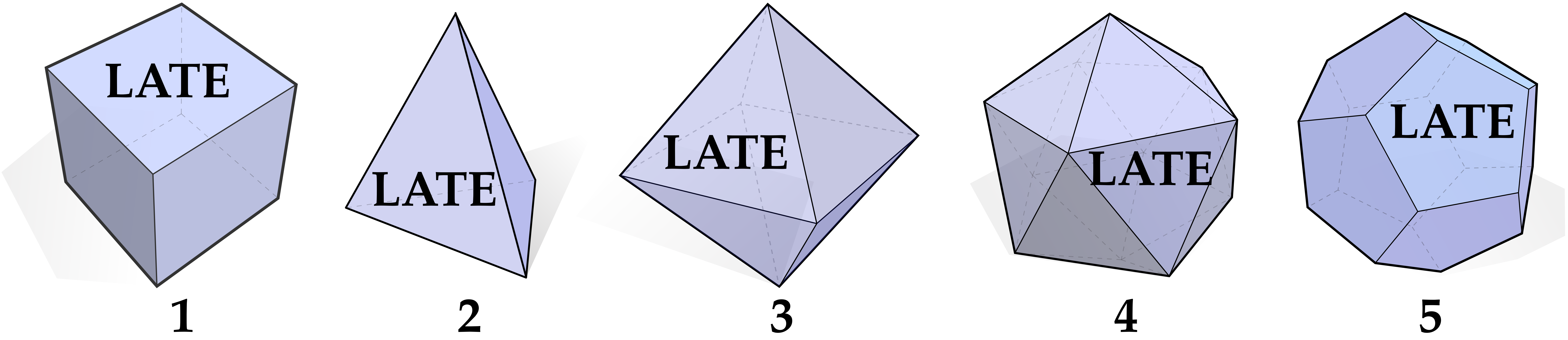
(Pick 6 of 7!)

Homework Submission

- All homework must be submitted digitally
  - All source files in a **single zip file** called `solution.zip`
  - All written exercises in a **single PDF file** called `exercises.pdf`
    - Either typeset (e.g., LaTeX) or scans/photos of written work.
    - Convert images to PDF using Preview (Mac) or imagetopdf.com
  - Email to geometry.collective@gmail.com with requested string in the subject line (e.g., `DDG19A0`)
  - Will receive written feedback via email as marked-up PDF
    - Will grade random subset of questions; you get 100% on the rest :-)

Late Policy

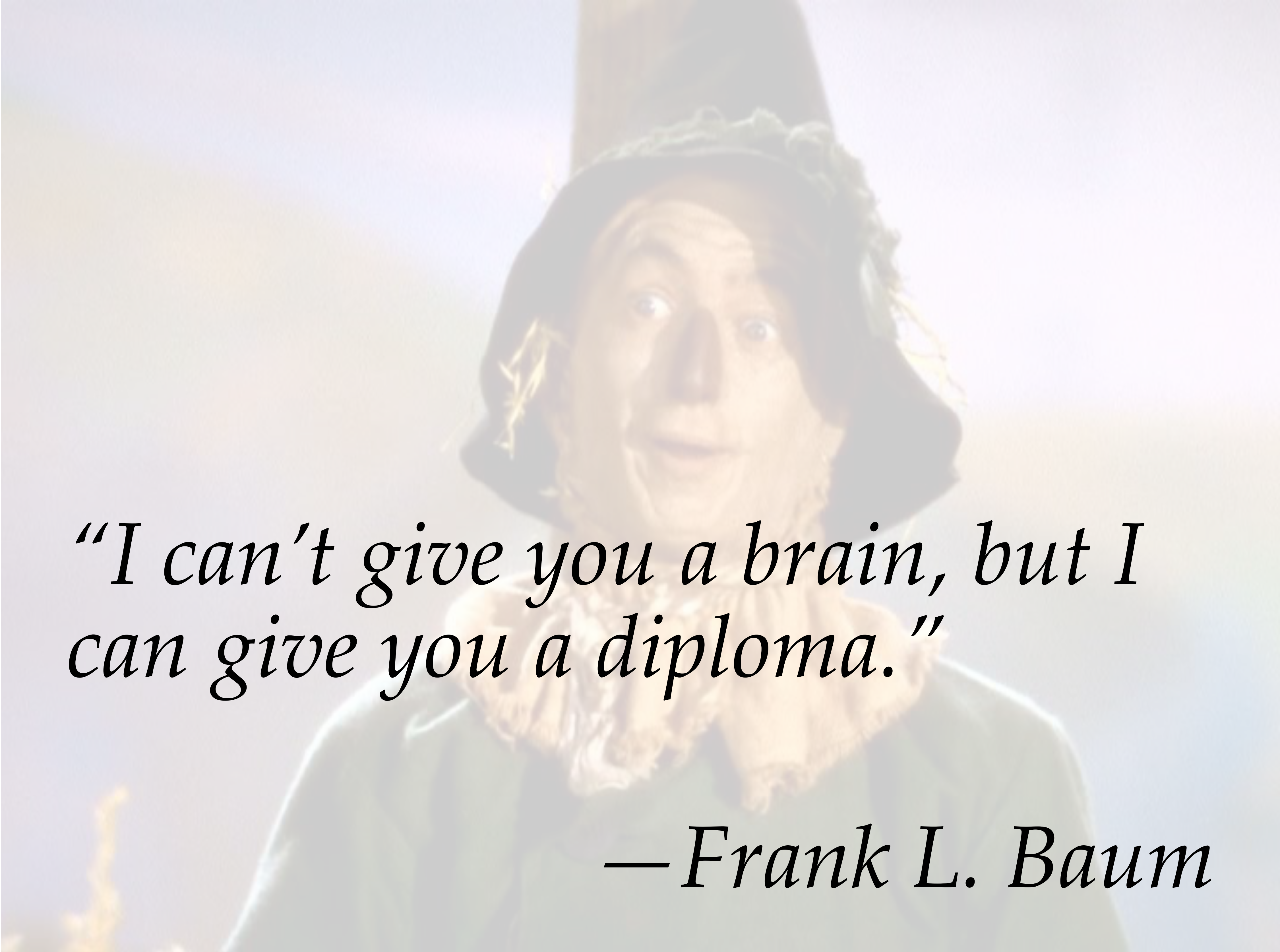
- Assignments due at 5:59:59pm on due date (Eastern time zone)
- Can use **five late days** throughout semester
  - Must indicate which late day you're using by putting one of five "Latonic" solids on your submission (draw by hand or include PDF):



- *All subsequent late work will receive a zero!*

Grade Breakdown

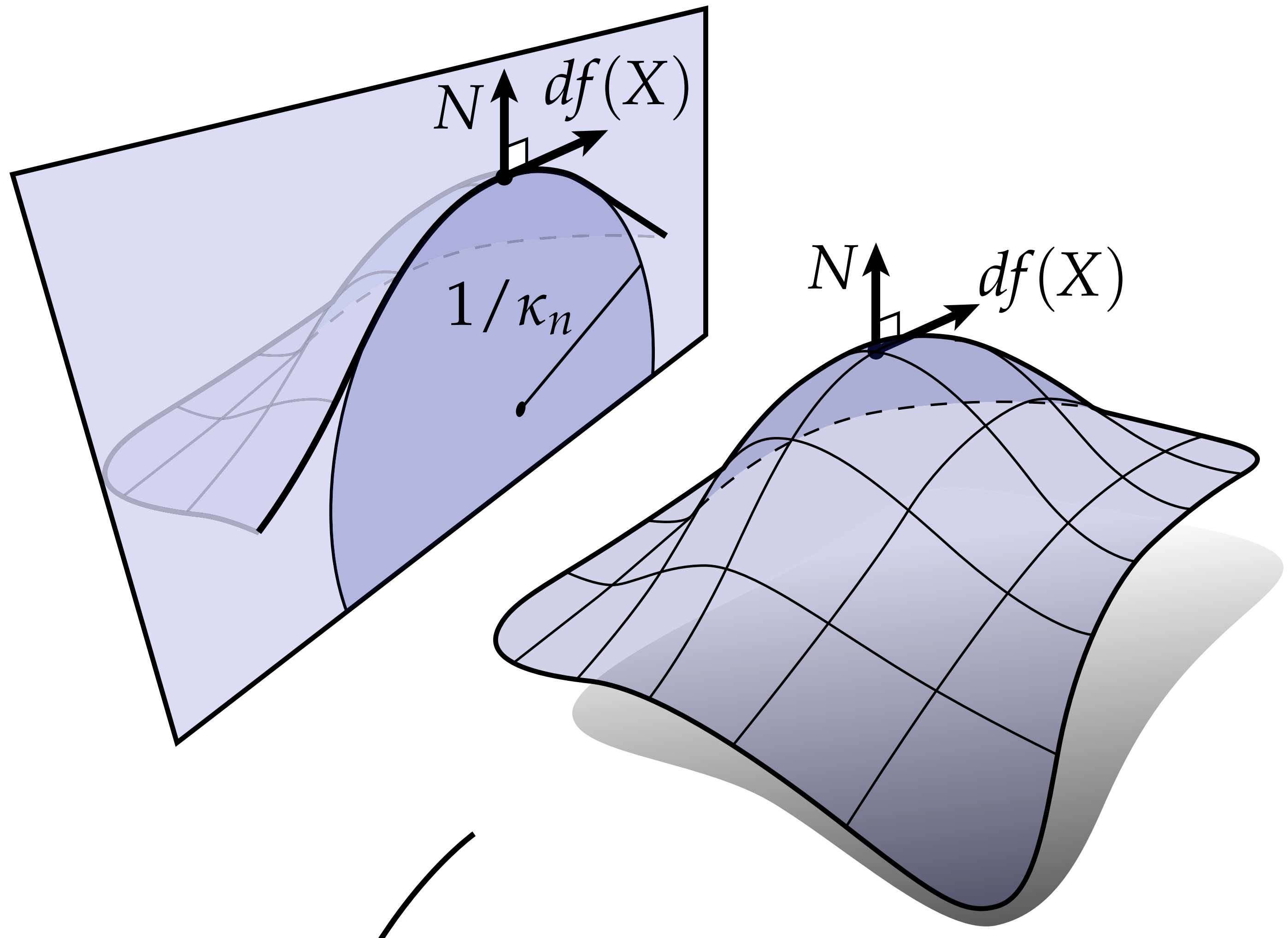
- **Assignments** – 90% (pick **6** out of 7*)
  - (15%) A0: *Combinatorial Surfaces*
  - (15%) A1: *Exterior Calculus*
  - (15%) A2: *Normals & Curvature*
  - (15%) A3: *Surface Fairing*
  - (15%) A4: *Surface Parameterization*
  - (15%) A5: *Geodesic Distance*
  - (15%) A6: *Direction Field Design*
- **Participation** – 10%
  - (5%) – in-class/web participation
  - (5%) – reading summaries/questions

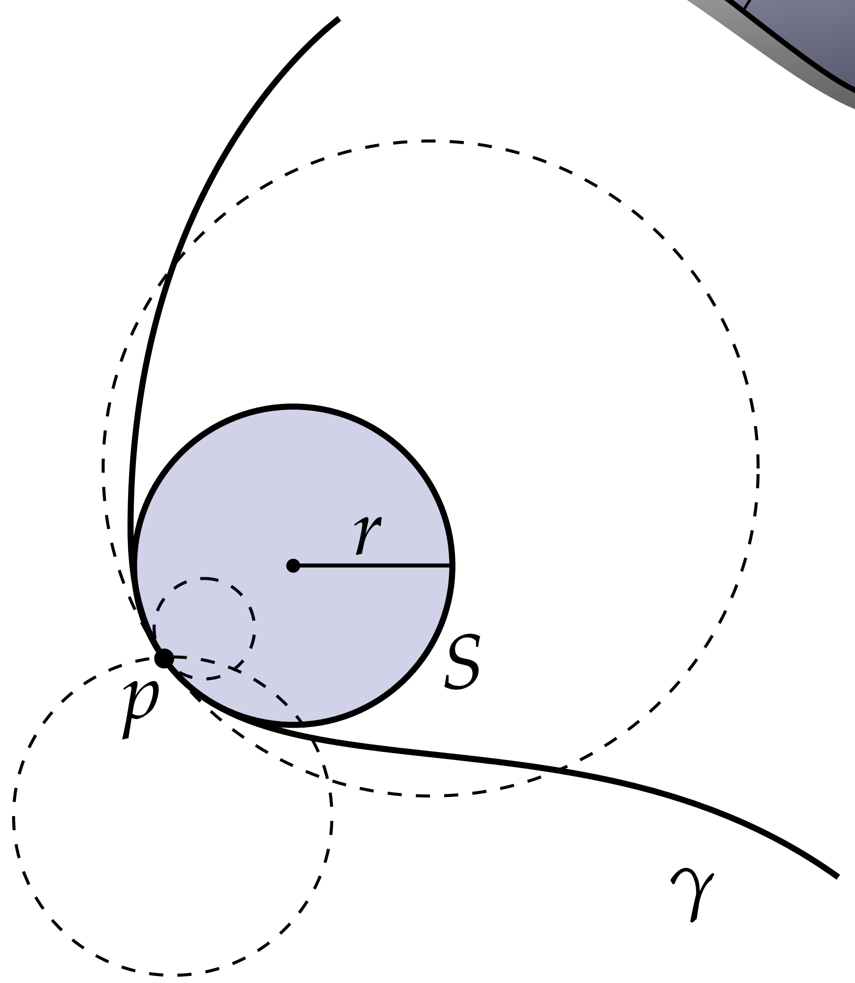


**Complete 7th assignment for up to 12% extra credit.*

What is Differential Geometry?

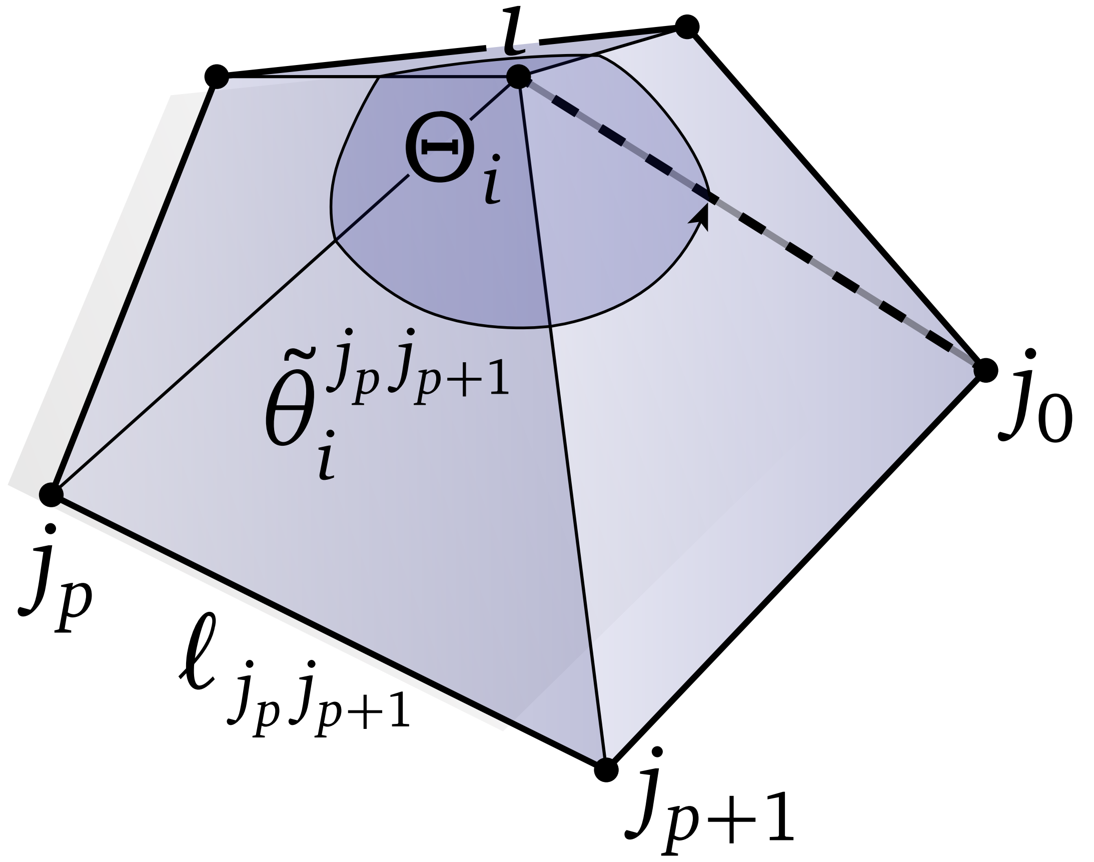
- **Language** for talking about *local properties of shape*
  - How fast are we traveling along a curve?
  - How much does the surface bend at a point?
  - etc.
- …and their connection to *global properties of shape*
  - So-called "local-global" relationships.
- Modern language of geometry, physics, statistics, …
- Profound impact on scientific & industrial development in 20th century

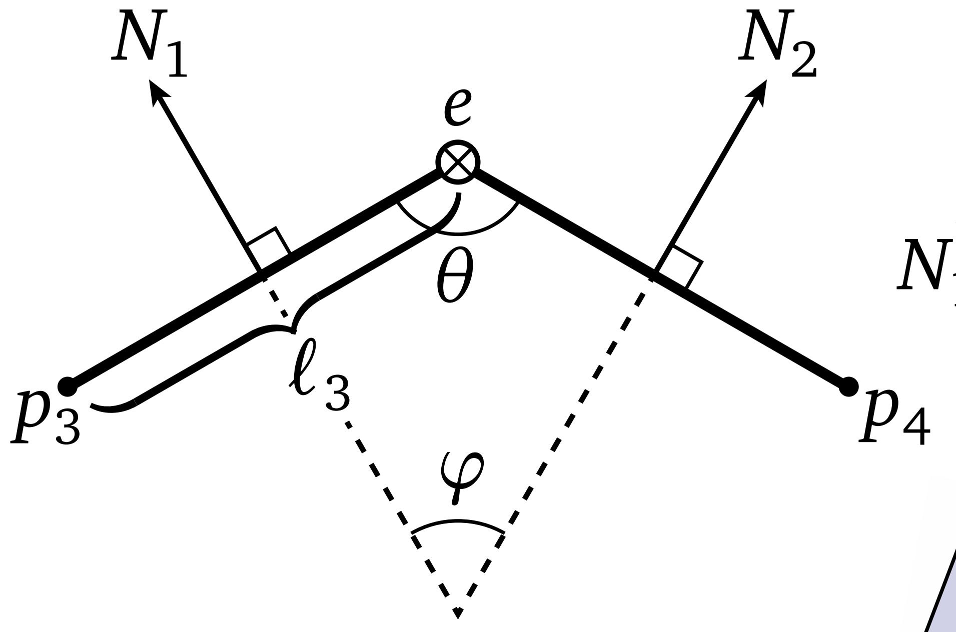


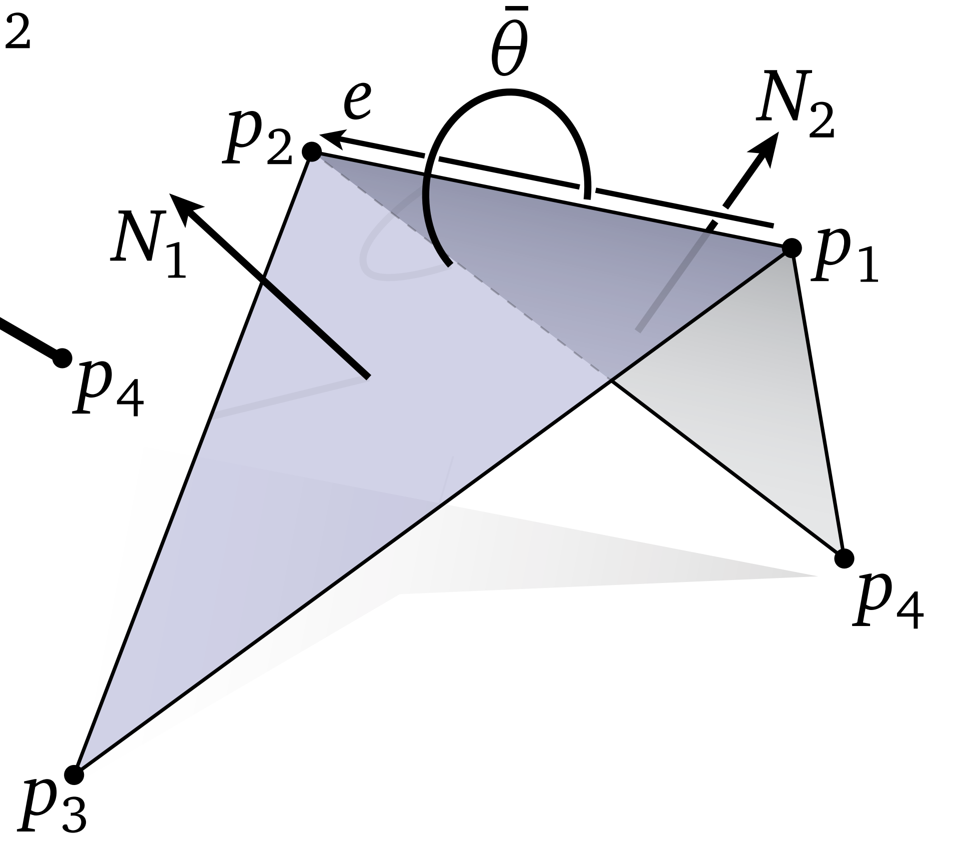


What is ***Discrete*** *Differential Geometry?*

- Also a language describing local properties of shape
  - *Infinity no longer allowed!*
  - No longer talk about derivatives, infinitesimals…
  - Everything expressed in terms of lengths, angles…
- Surprisingly little is lost!
  - Faithfully captures many fundamental ideas
- Modern language of geometric computing
- Increasing impact on science & technology in 21st century.



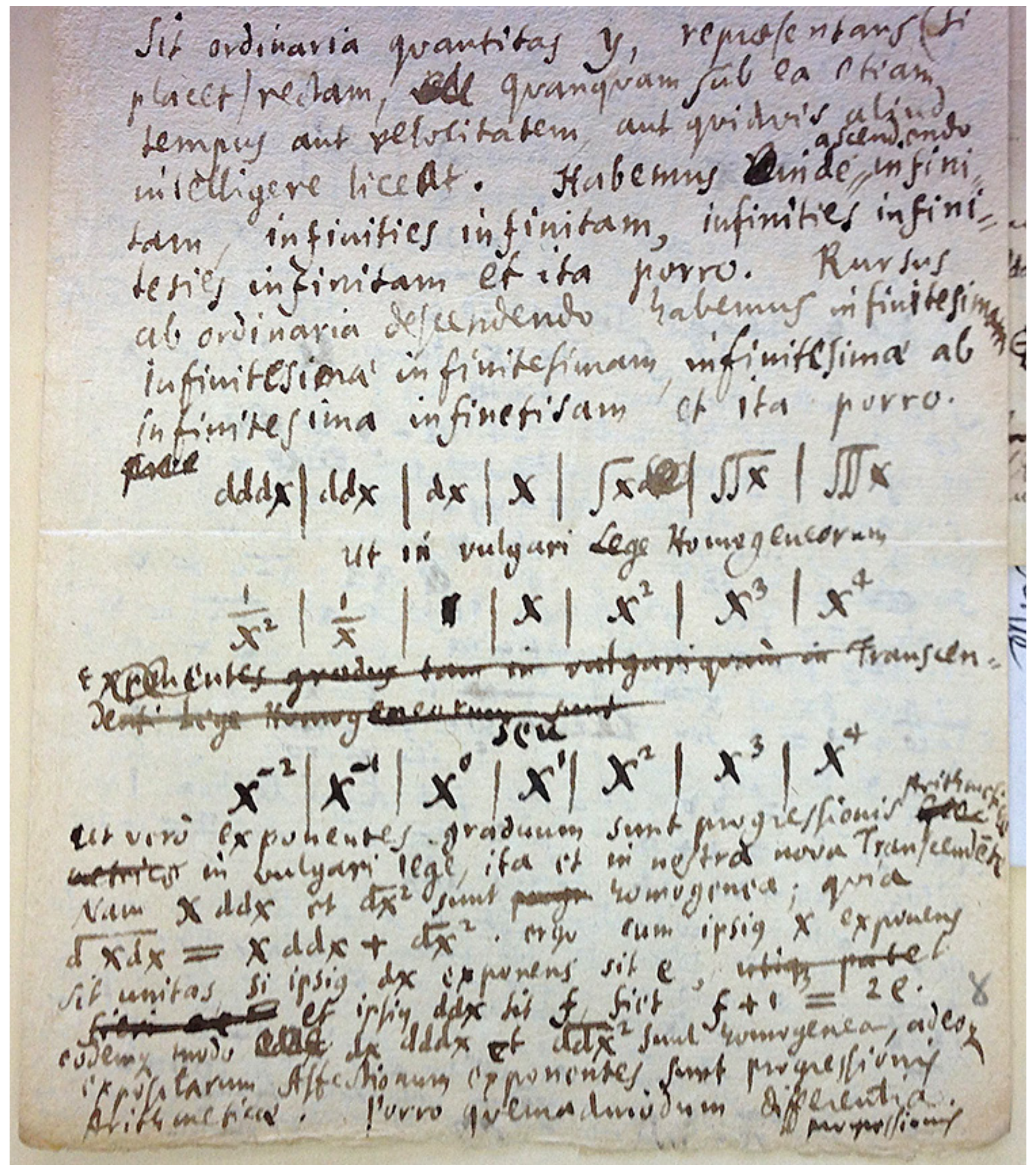


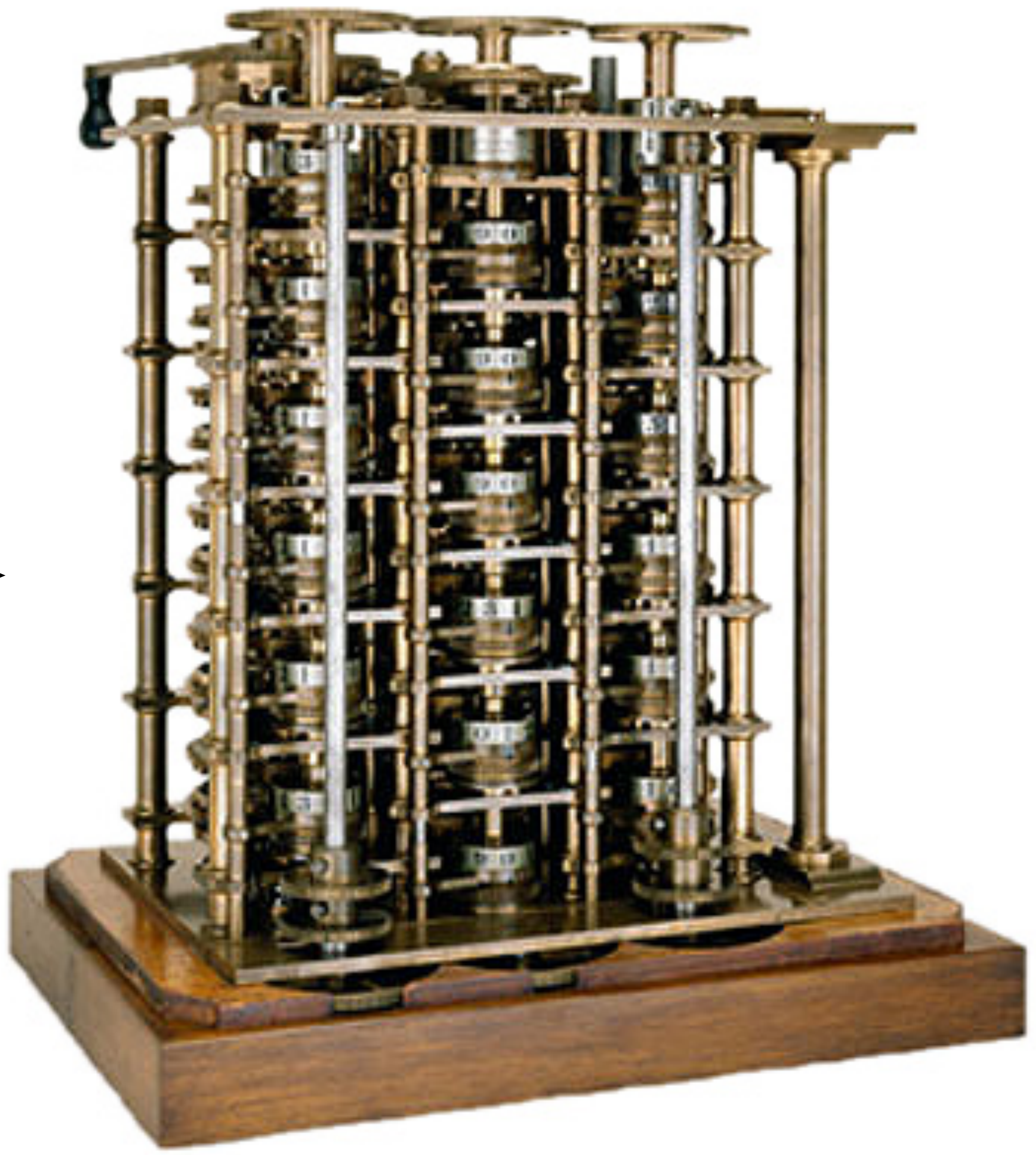


Discrete Differential Geometry—Grand Vision

Grand Vision

Translate differential geometry into **language** suitable for *computation*.





How can we get there?

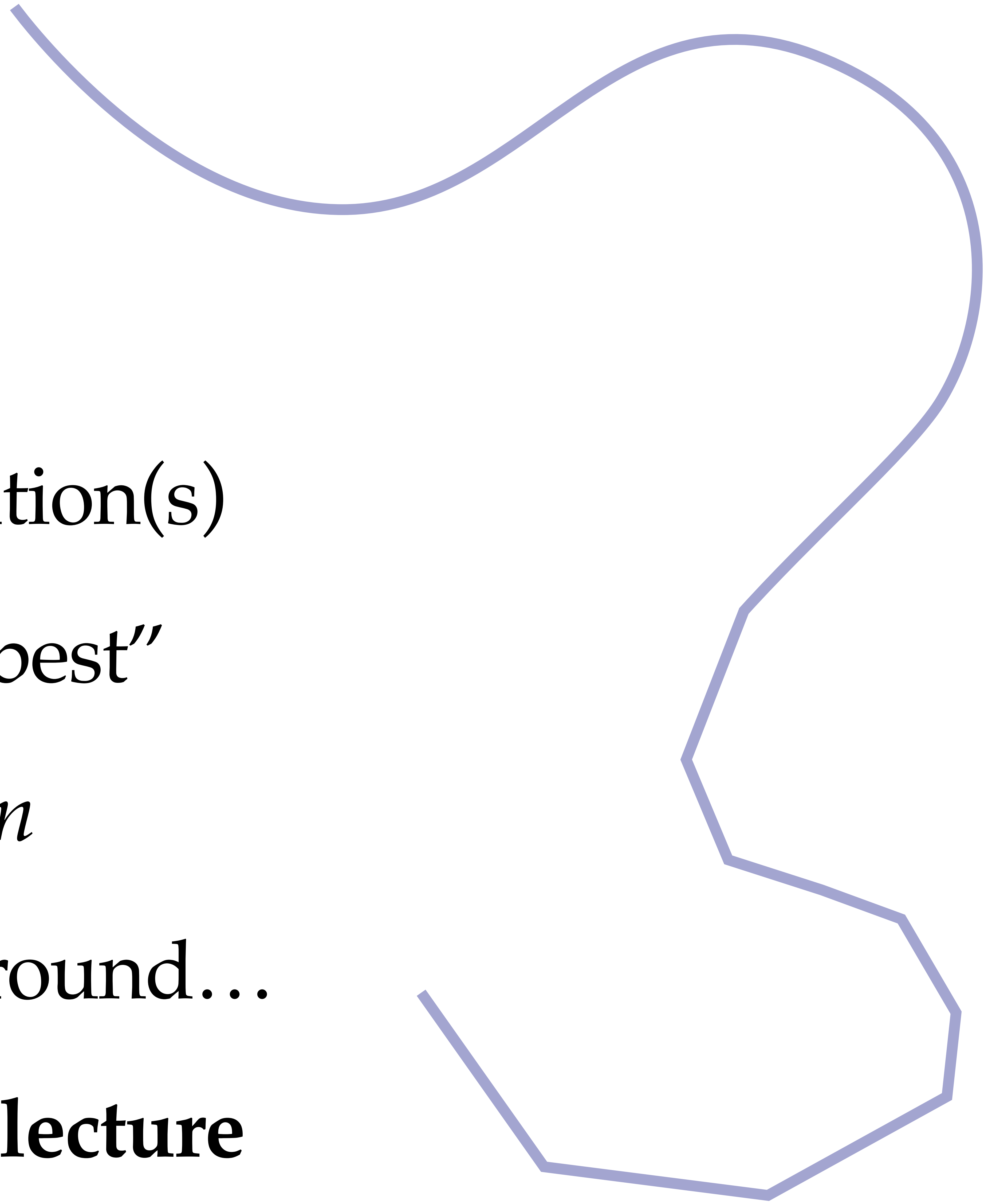
A common "game" is played in DDG to obtain discrete definitions:

1. Write down several **equivalent** definitions in the smooth setting.
2. Apply each smooth definition to an object in the discrete setting.
3. Determine which properties are captured by each resulting **inequivalent** discrete definition.

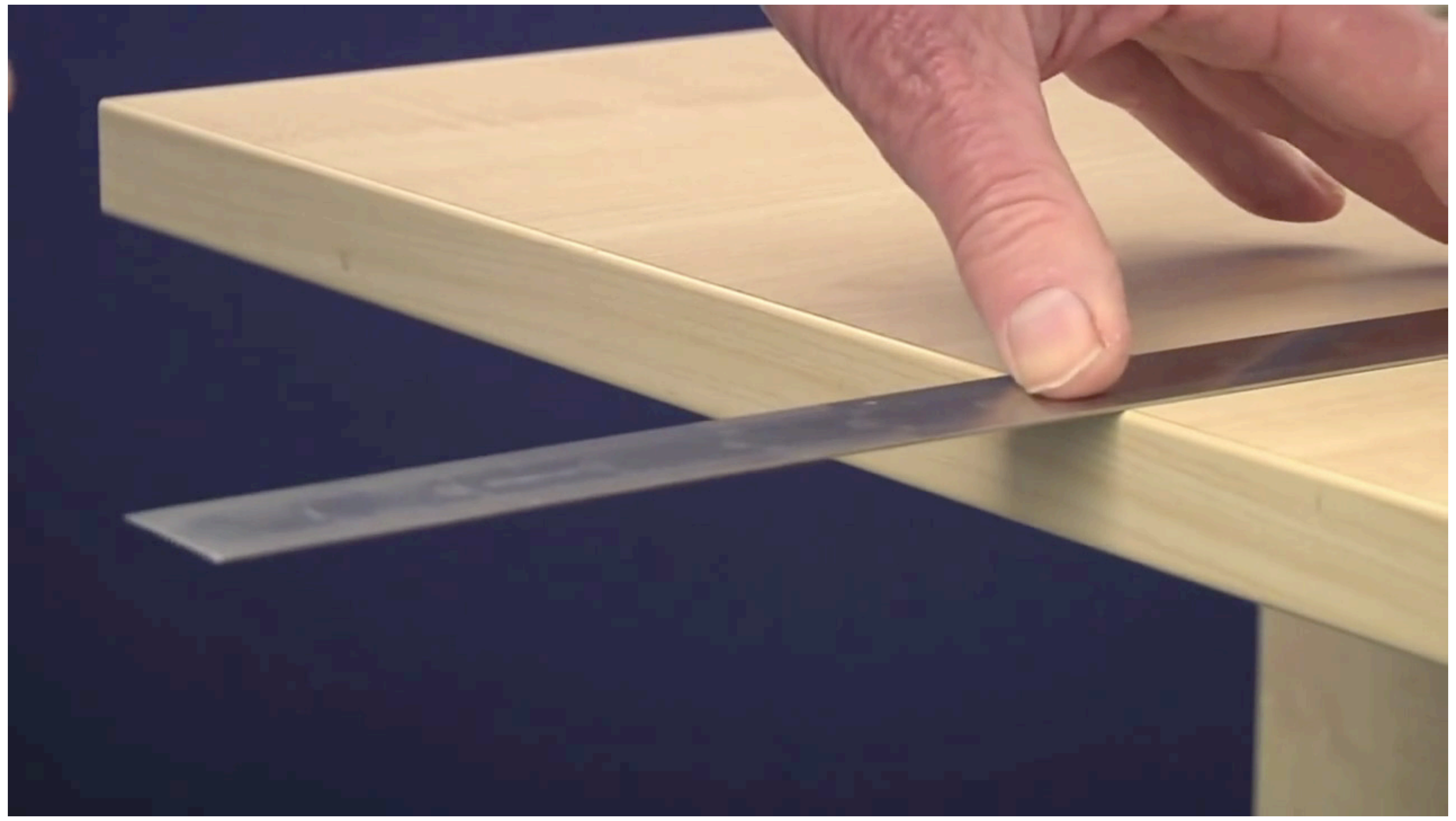
One often encounters a so-called *"no free lunch"* scenario: no single discrete definition captures *all* properties of its smooth counterpart.

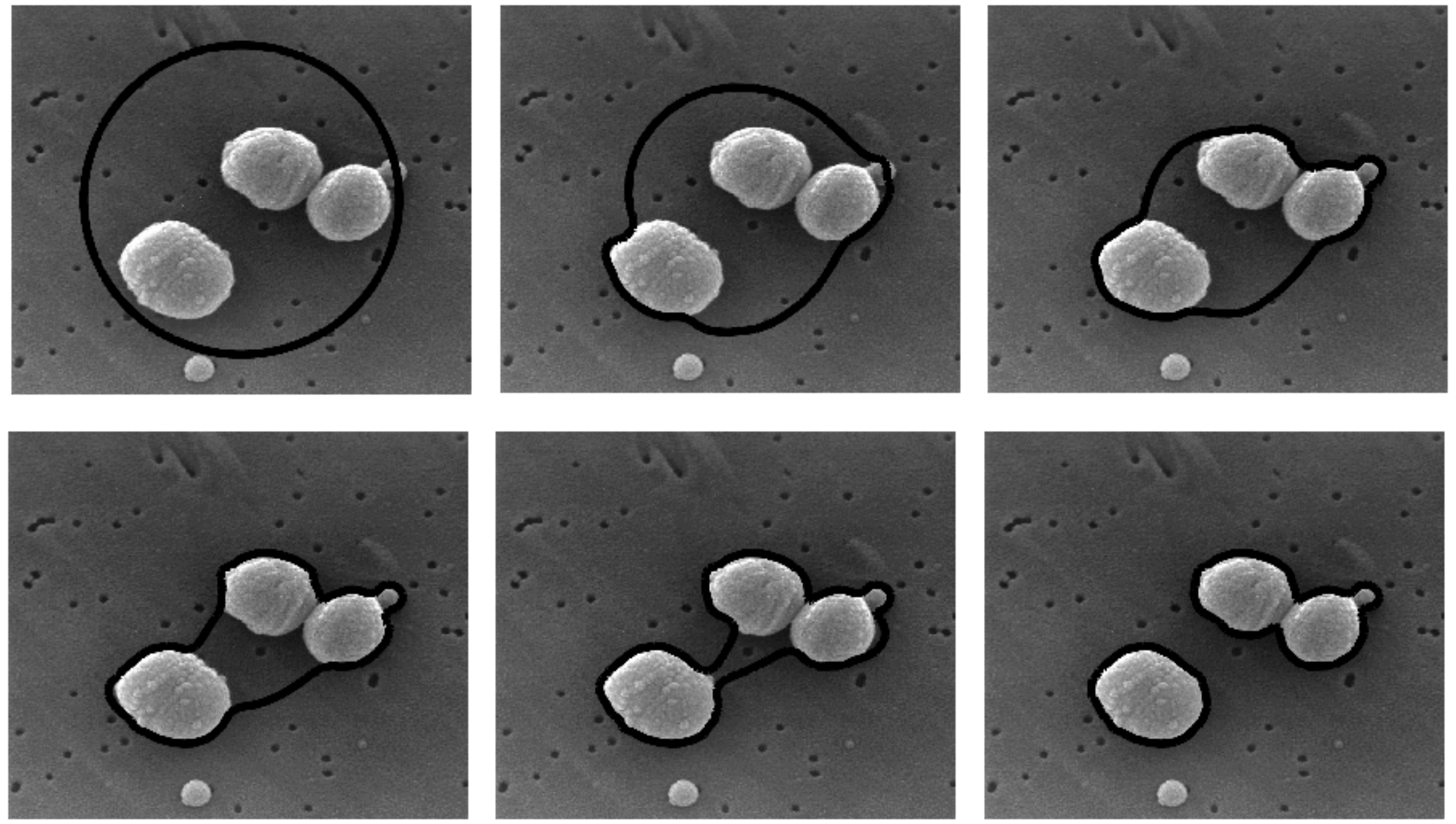
Example: Discrete Curvature of Plane Curves

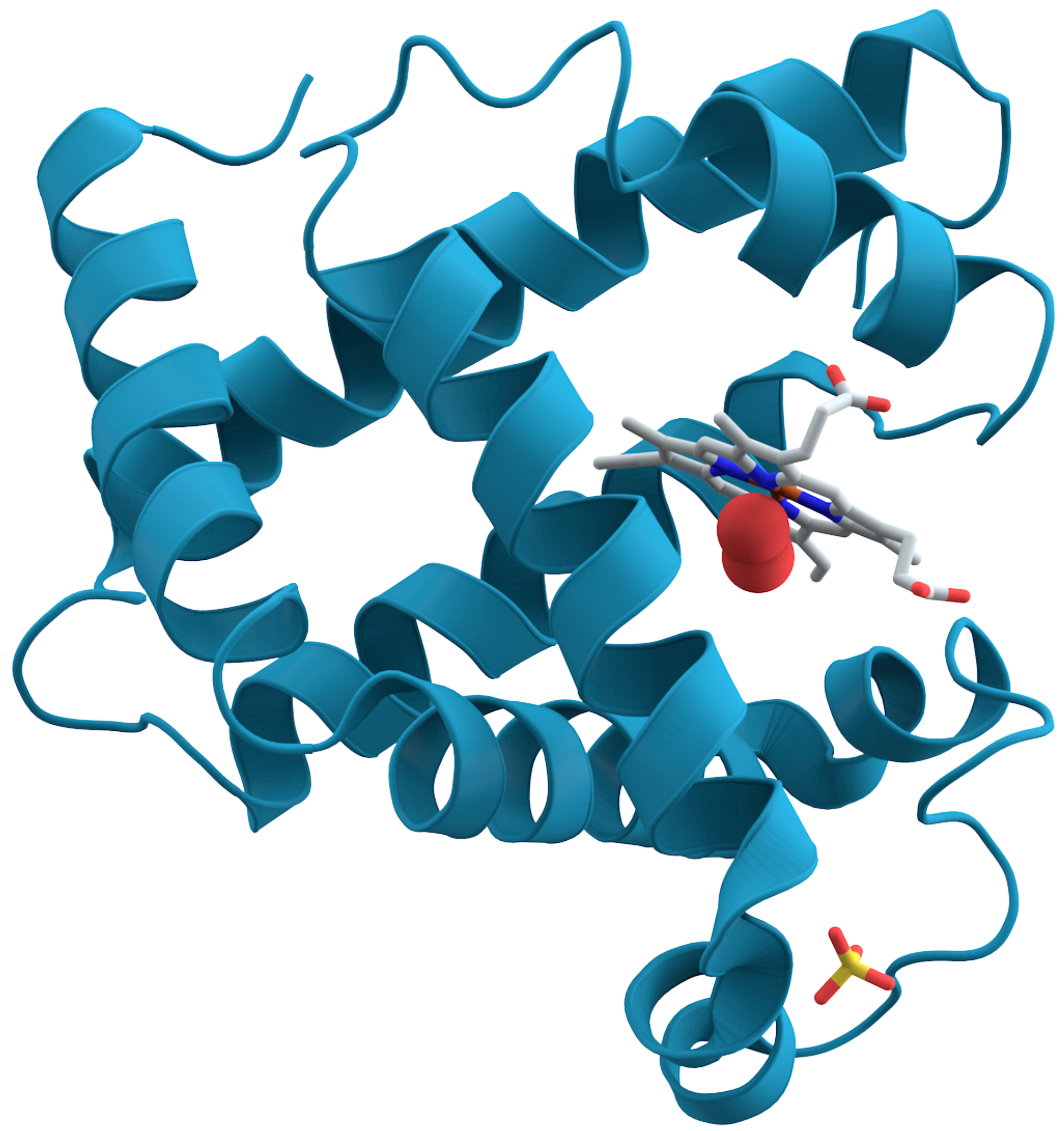
- **Toy example:** *curvature of plane curves*
  - Roughly speaking: "how much it bends"
  - First review smooth definition
  - Then play The Game to get discrete definition(s)
  - Will discover that no single definition is "best"
  - *Pick the definition best suited to the application*
- **Today** we're gonna quickly cover a lot of ground…
- Will start more slowly from the basics **next lecture**

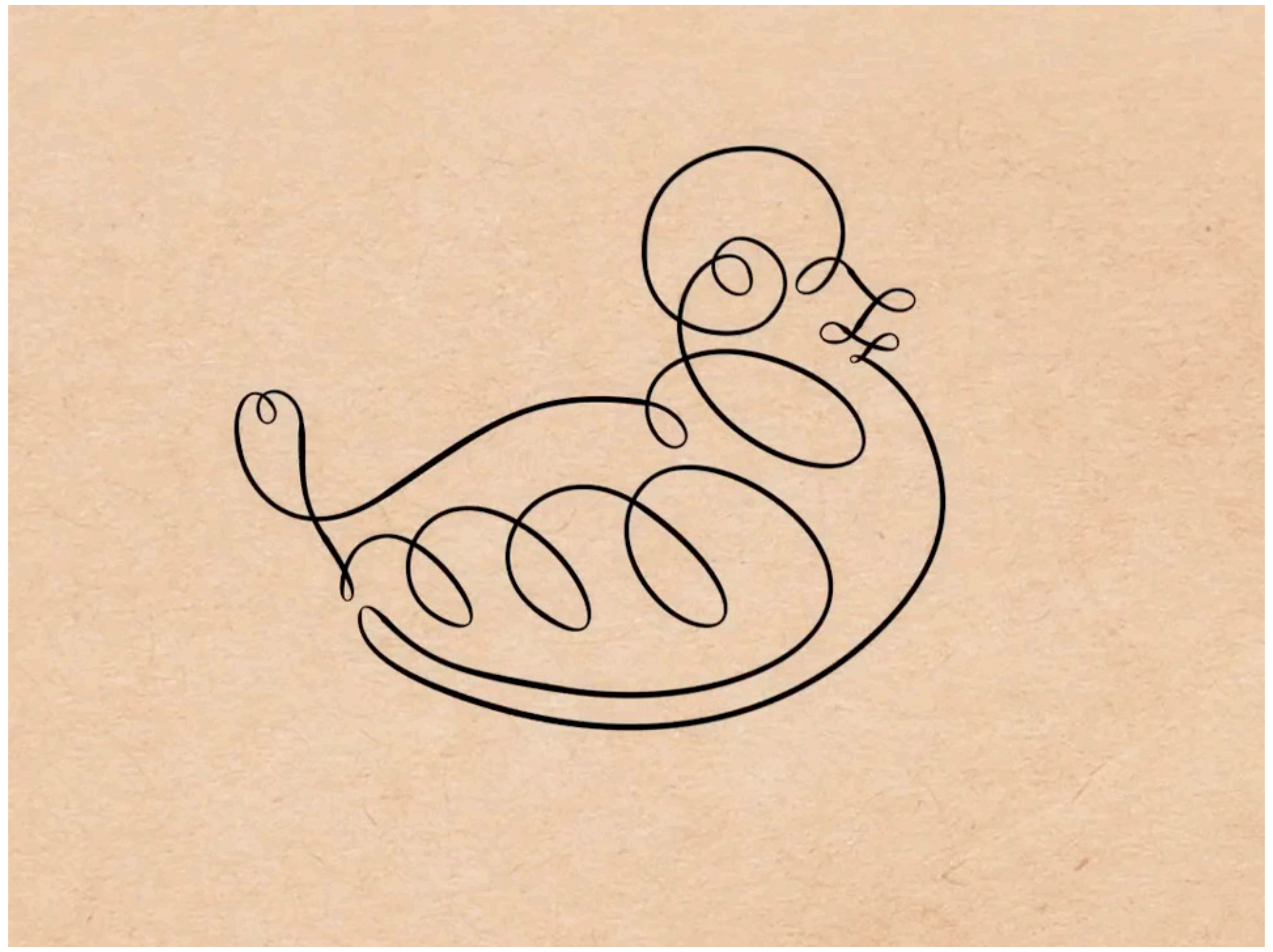


Curvature of a Curve — Motivation



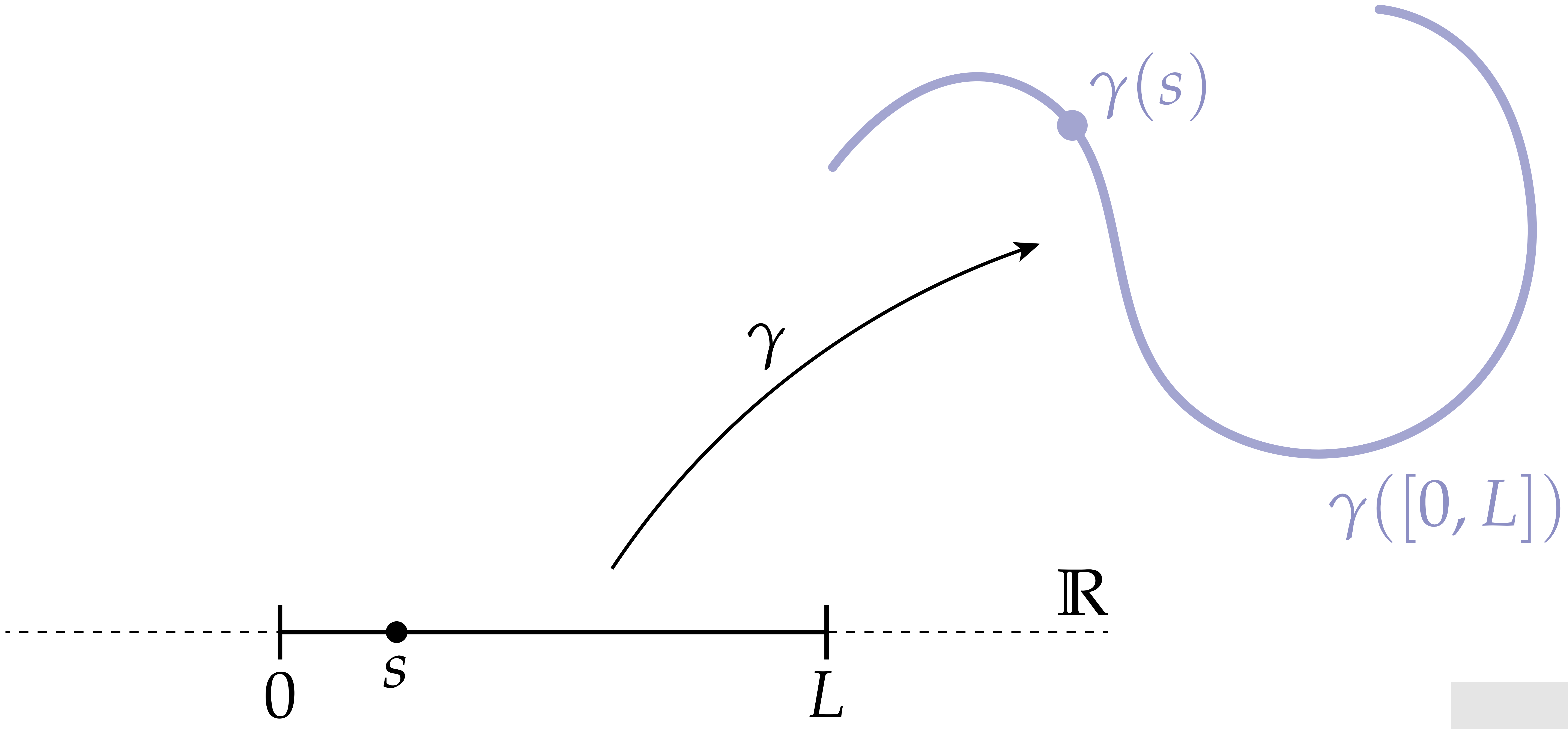






Curves in the Plane

- In the smooth setting, a **parameterized curve** is a map* taking each point in an interval $[0,L]$ of the real line to some point in the plane $\mathbb{R}^2$:



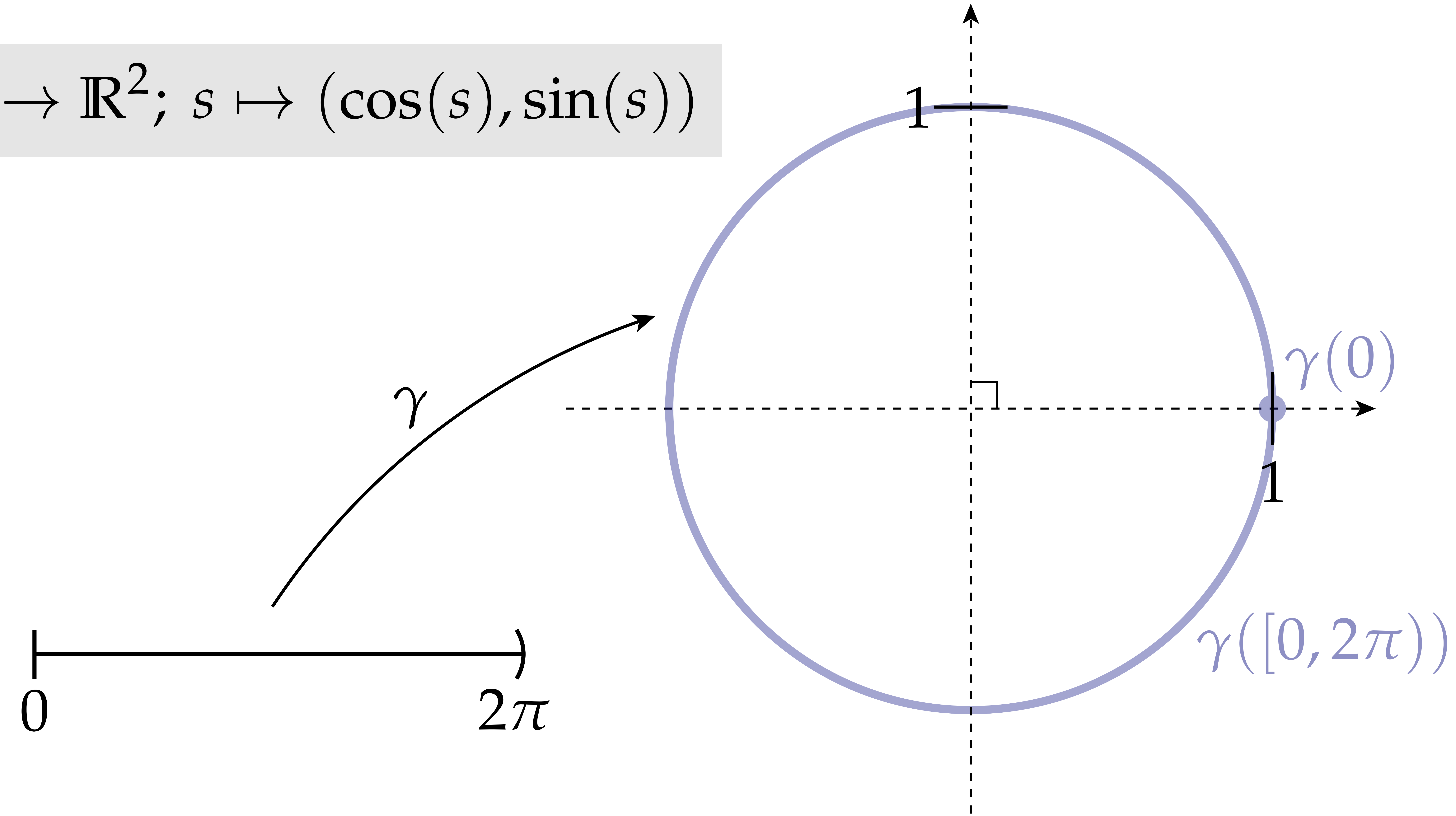
*Continuous, differentiable, smooth…

$$\gamma : [0,L] \to \mathbb{R}^2$$

Curves in the Plane—Example

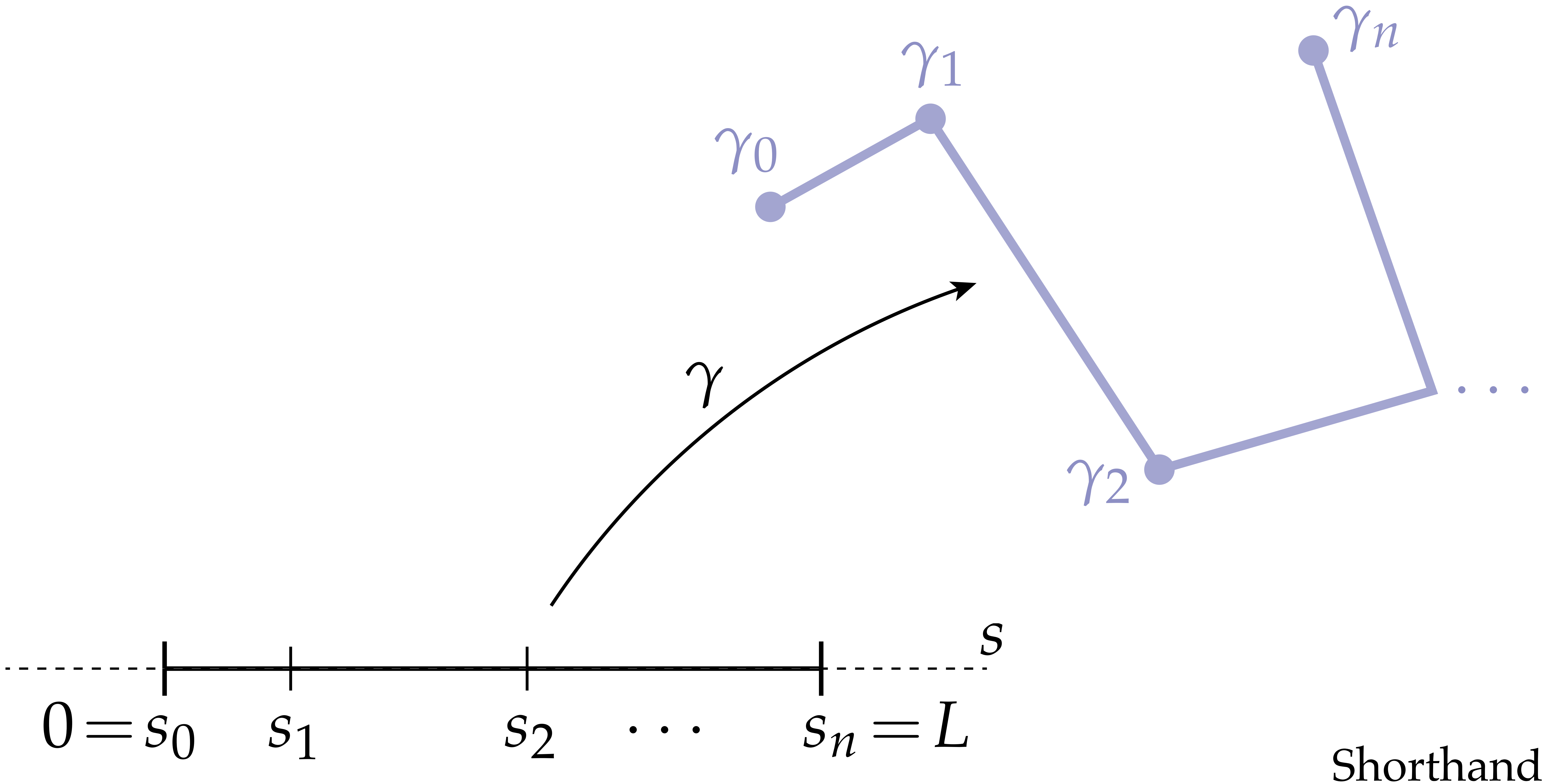
- As an example, we can express a circle as a parameterized curve γ:

$$\gamma : [0, 2\pi) \to \mathbb{R}^2;\ s \mapsto (\cos(s), \sin(s))$$



Discrete Curves in the Plane

- Special case: a **discrete curve** is a *piecewise linear* parameterized curve, *i.e.*, it is a sequence of **vertices** connected by straight line segments:

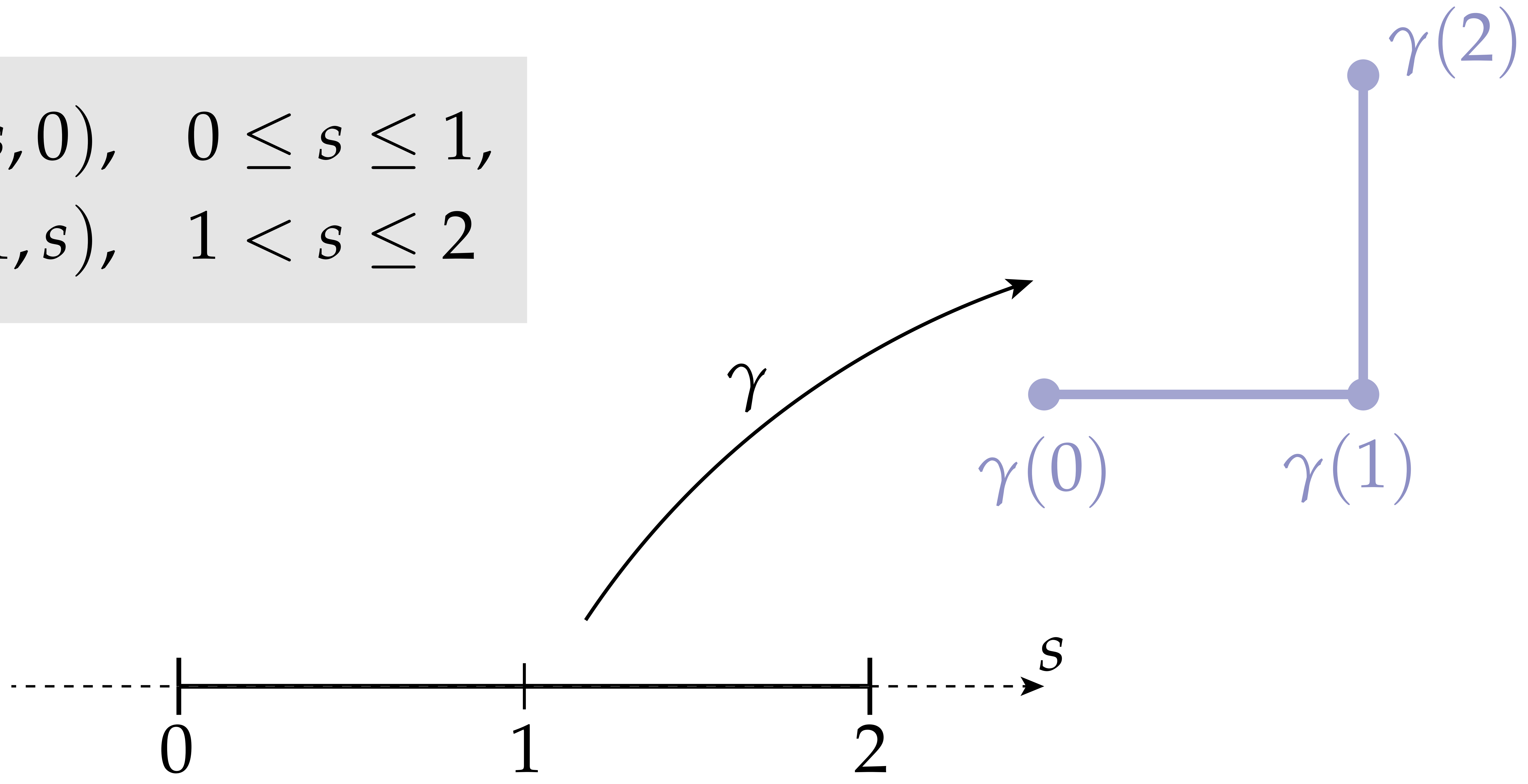


Shorthand: $\gamma_i := \gamma(s_i)$

Discrete Curves in the Plane—Example

- A simple example is a curve comprised of two segments:

$$\gamma(s) := \begin{cases} (s,0), & 0 \leq s \leq 1, \\ (1,s), & 1 < s \leq 2 \end{cases}$$

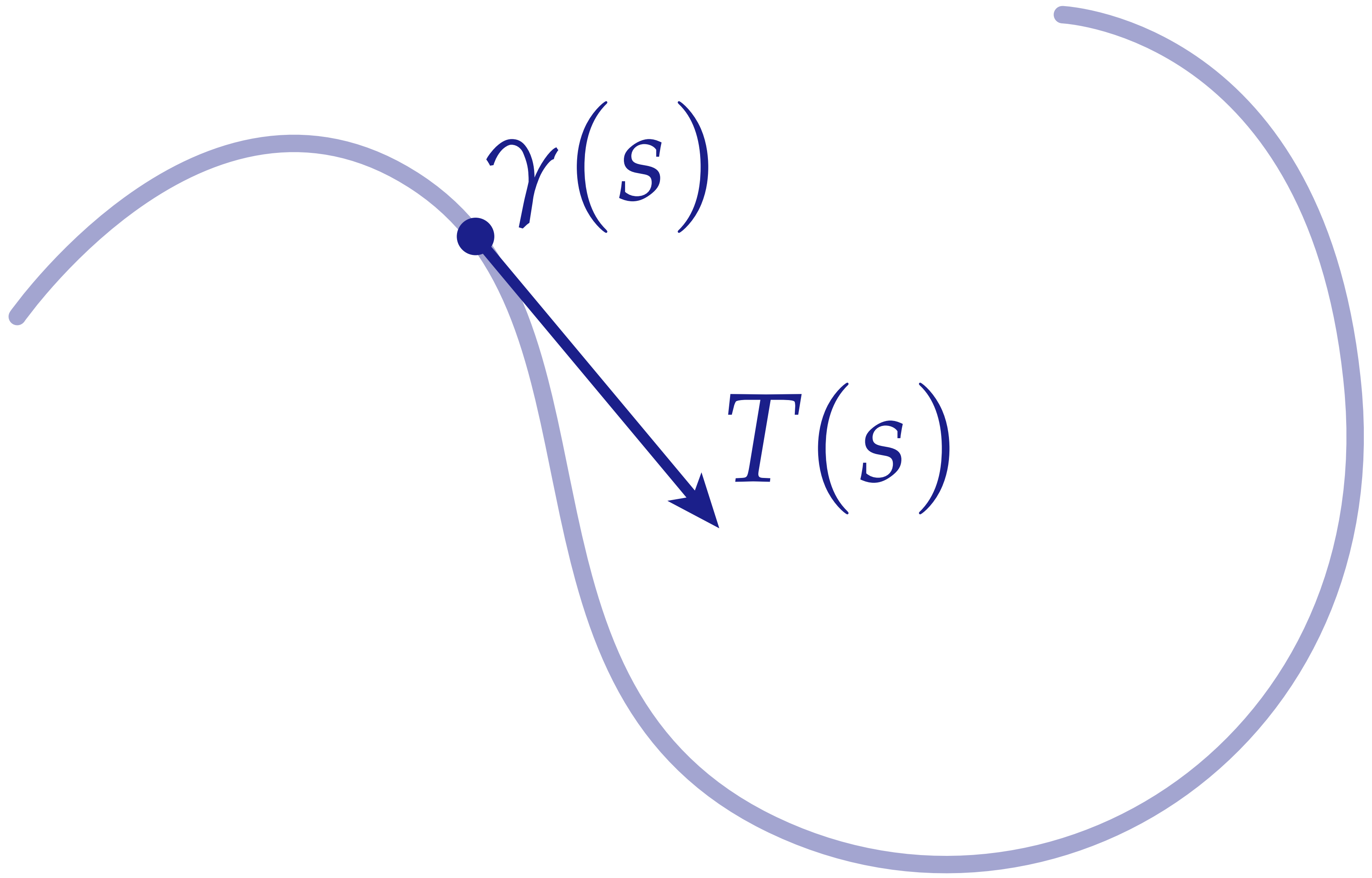


Tangent of a Curve

- Informally, a vector is *tangent* to a curve if it "just barely grazes" the curve.

- More formally, the **unit tangent** (or just **tangent**) of a parameterized curve is the map obtained by normalizing its first derivative:

$$T(s) := \tfrac{d}{ds}\gamma(s) \Big/ \left|\tfrac{d}{ds}\gamma(s)\right|$$



- If the derivative already has unit length, then we say the curve is **arc-length parameterized** and can write the tangent as just

$$T(s) := \tfrac{d}{ds}\gamma(s)$$

Tangent of a Curve—Example

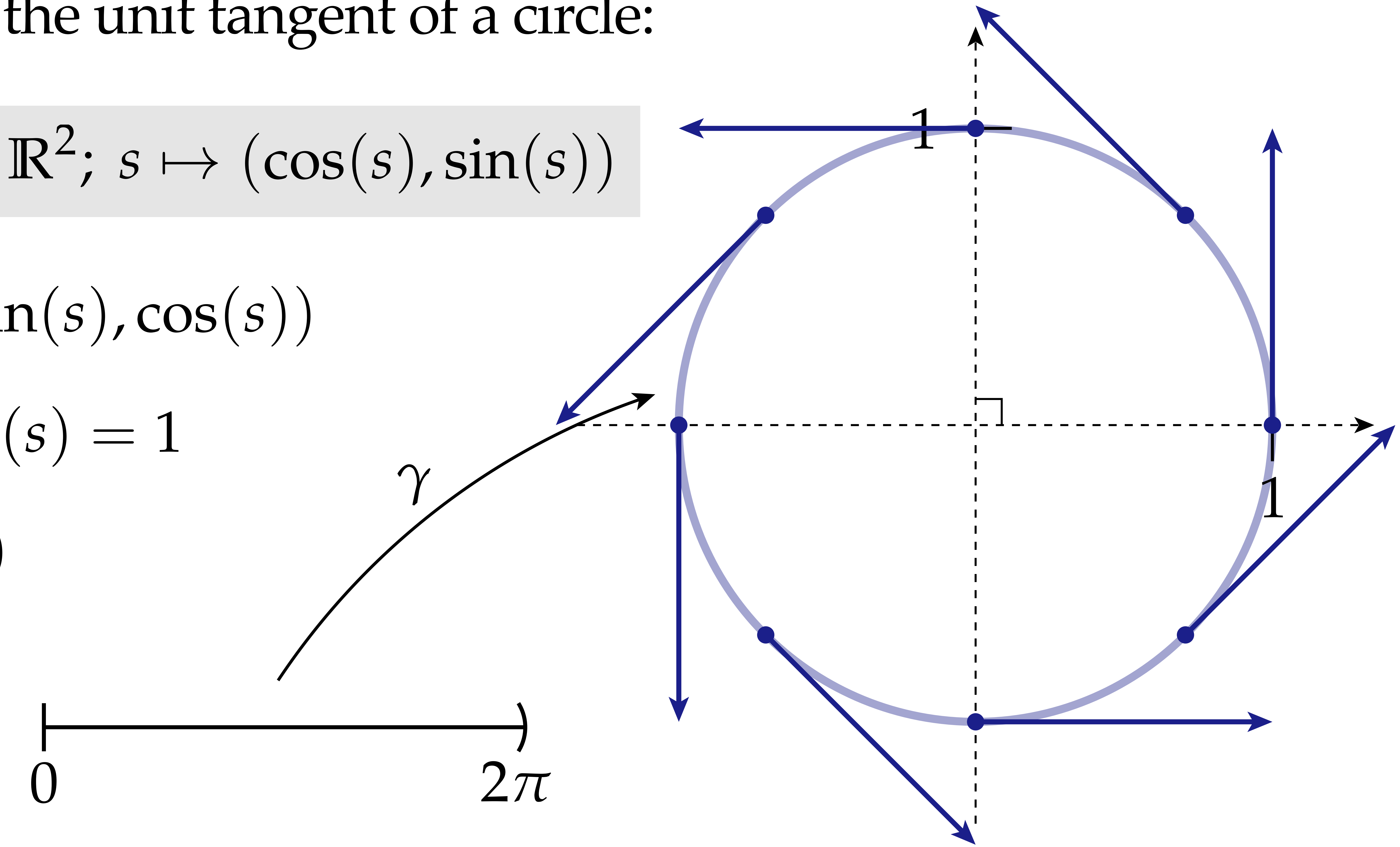
- Let's compute the unit tangent of a circle:

$$\gamma : [0, 2\pi) \to \mathbb{R}^2;\ s \mapsto (\cos(s), \sin(s))$$

$$\frac{d}{ds}\gamma(s) = (-\sin(s), \cos(s))$$

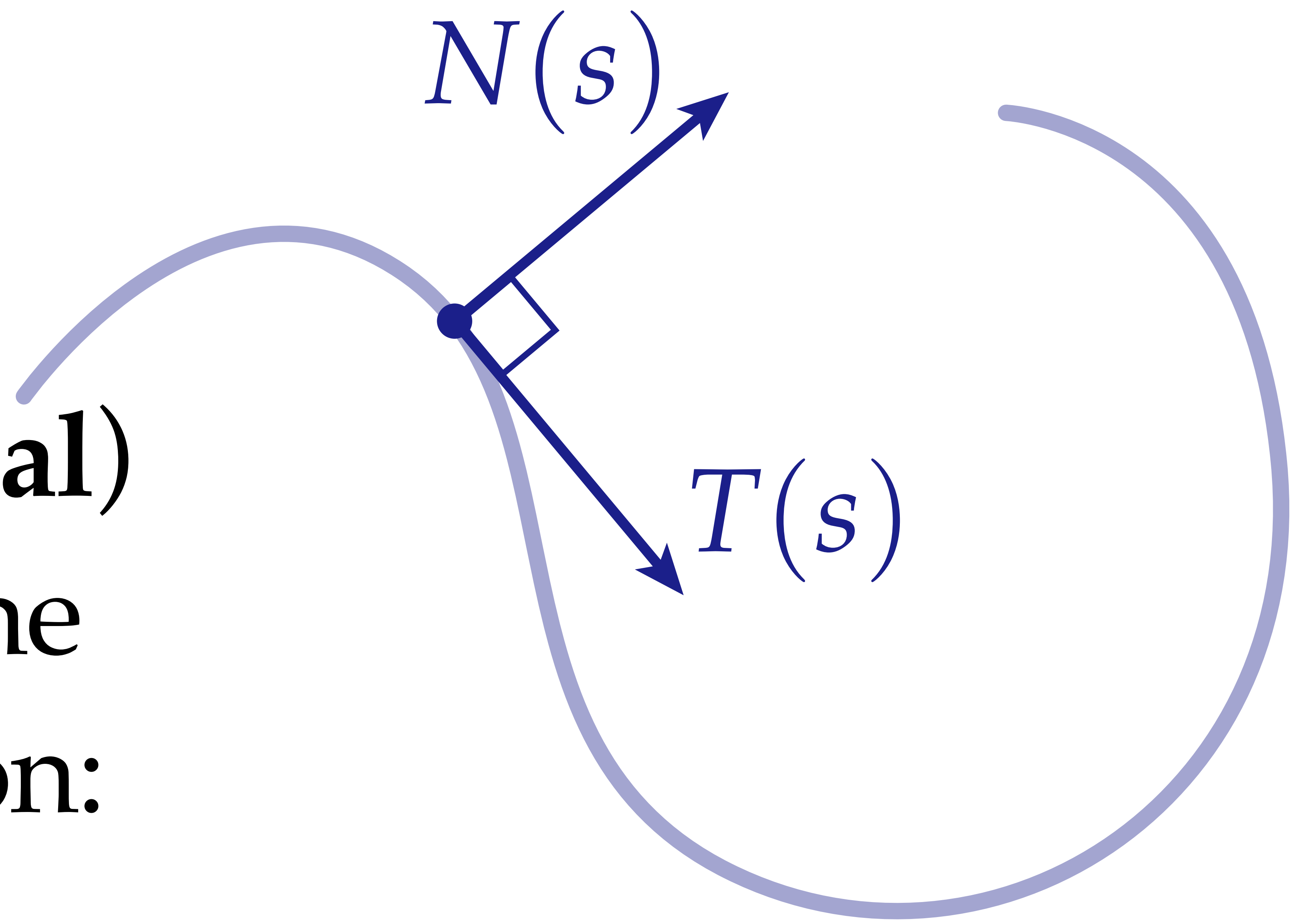
$$\cos^2(s) + \sin^2(s) = 1$$

$$\Rightarrow T = \frac{d}{ds}\gamma(s)$$



Normal of a Curve

- Informally, a vector is *normal* to a curve if it "sticks straight out" of the curve.



- More formally, the **unit normal** (or just **normal**) can be expressed as a quarter-rotation $\mathcal{J}$ of the unit tangent in the counter-clockwise direction:

$$N(s) := \mathcal{J}T(s)$$

- In coordinates (x,y), a quarter-turn can be achieved by* simply exchanging x and y, and then negating y:

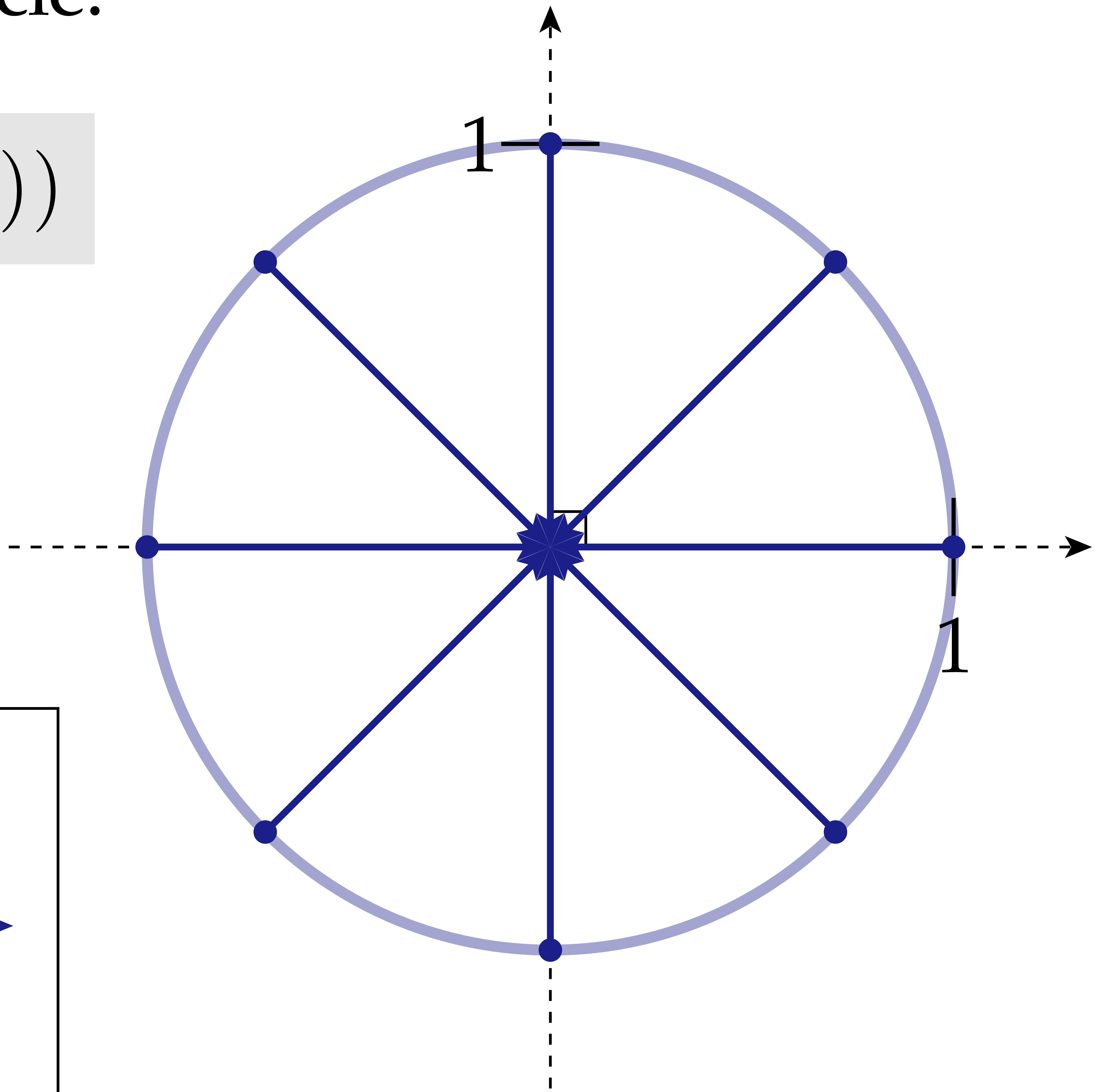
$$(x, y) \overset{\mathcal{J}}{\mapsto} (-y, x)$$

*Why does this work?

Normal of a Curve—Example

- Let's compute the unit normal of a circle:

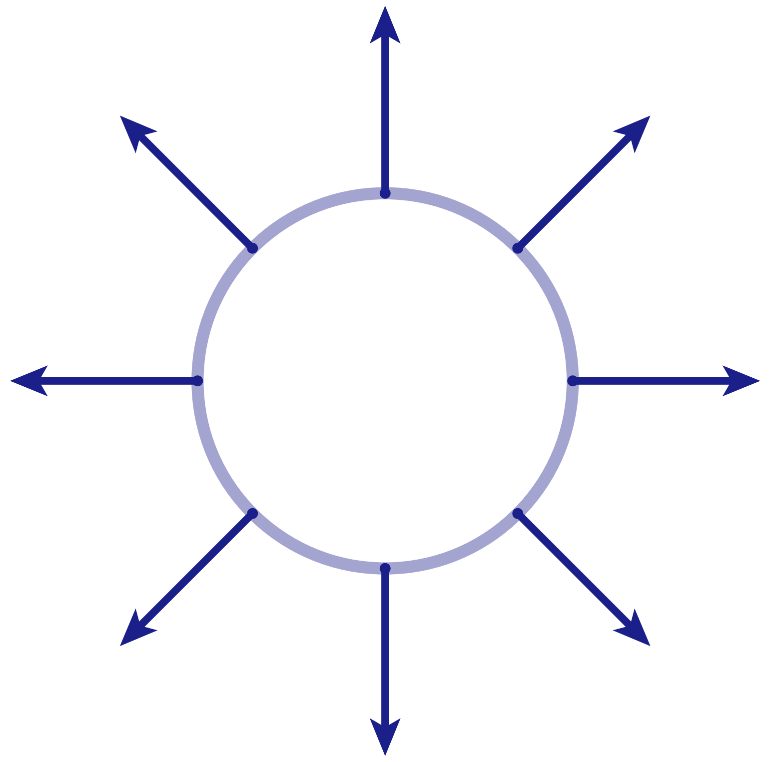
$$\gamma : [0, 2\pi) \to \mathbb{R}^2;\ s \mapsto (\cos(s), \sin(s))$$



$$T(s) = (-\sin(s), \cos(s))$$

$$N(s) = \mathcal{J}T(s) = (-\cos(s), -\sin(s))$$

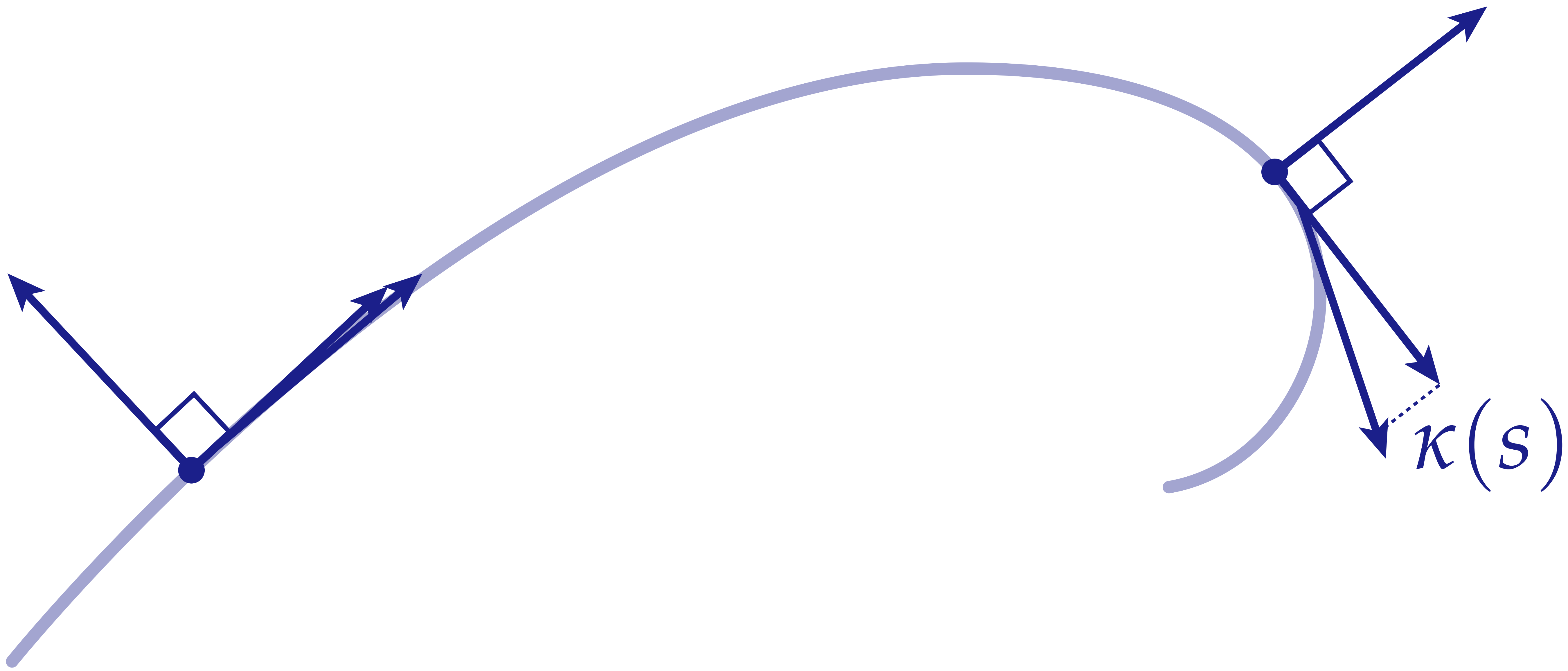
Note: could also adopt the convention $N = -\mathcal{J}T$. (Just remain consistent!)



Curvature of a Plane Curve

- Informally, curvature describes "how much a curve bends"
- More formally, the **curvature** of an arc-length parameterized plane curve can be expressed as the rate of change in the tangent*

$$\begin{aligned}\kappa(s) &:= \langle N(s), \tfrac{d}{ds}T(s)\rangle \\ &= \langle N(s), \tfrac{d^2}{ds^2}\gamma(s)\rangle\end{aligned}$$



KEY IDEA

Curvature is a *second derivative*.

*Here the angle brackets denote the usual dot product, i.e., $\langle (a,b),(x,y)\rangle := ax + by$.

Curvature: From Smooth to Discrete

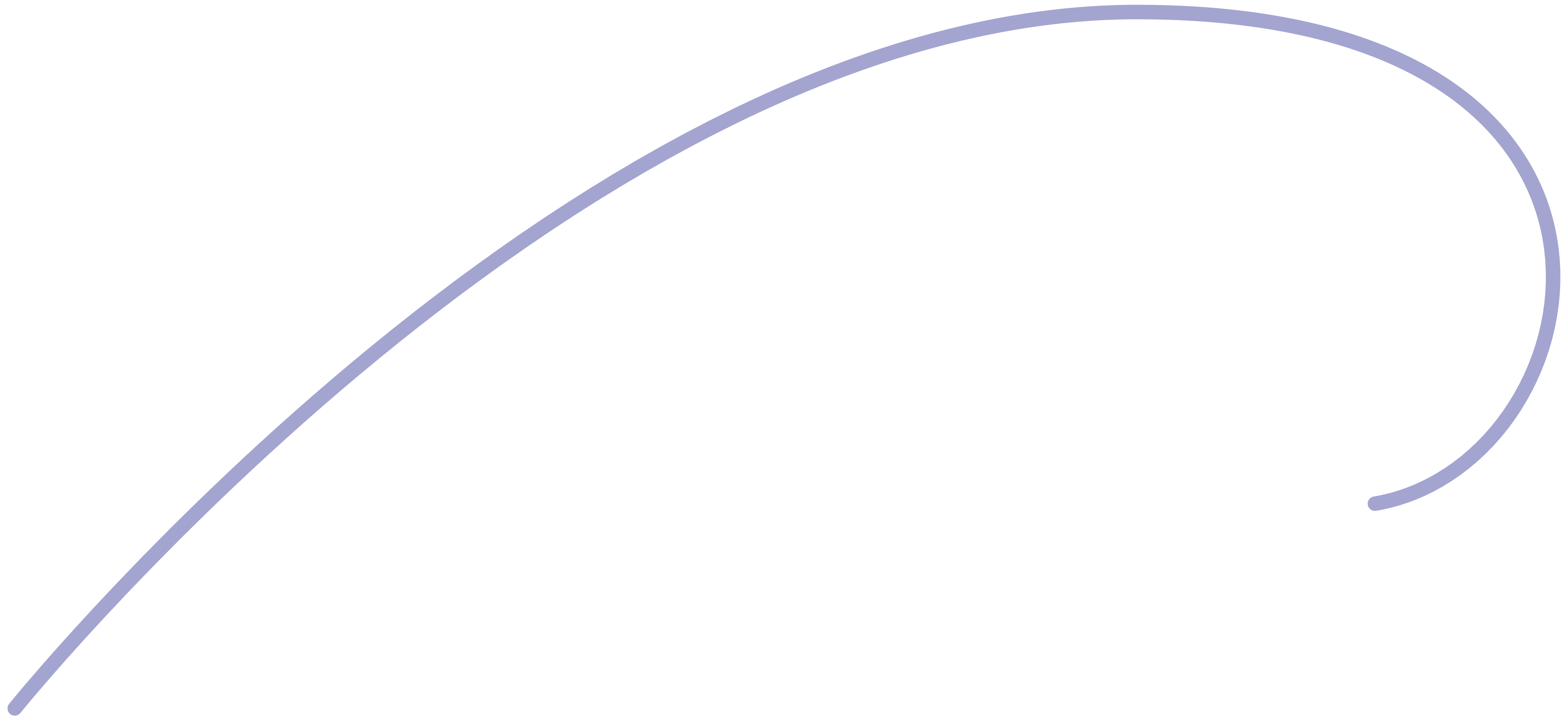
<u>Key idea</u>

Curvature is a *second derivative.*

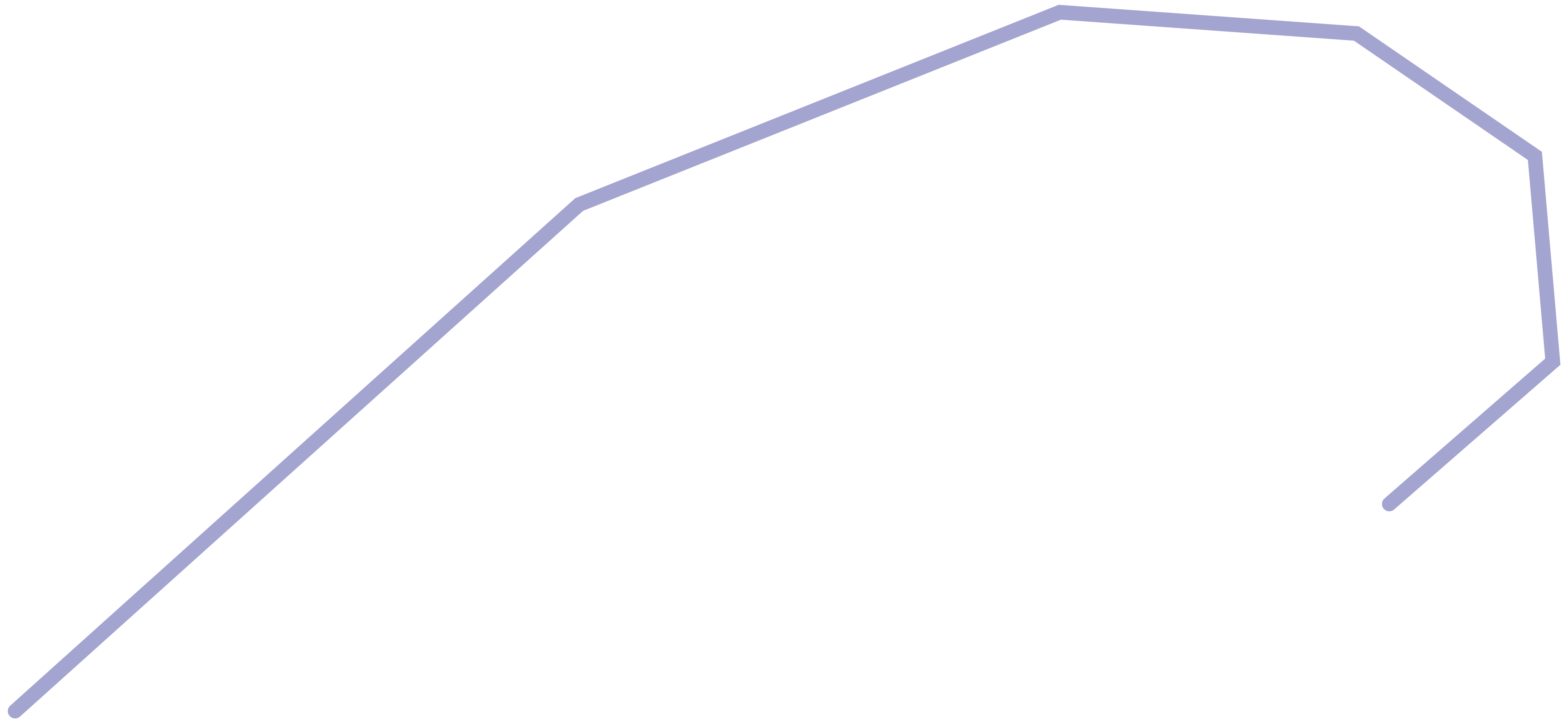
$$\kappa = \left\langle \mathcal{J}\,\frac{d}{ds}\gamma,\, \frac{d^2}{ds^2}\gamma \right\rangle$$

Can we directly apply this definition to a discrete curve?

<u>Smooth</u>



<u>Discrete</u>



No! Will get either zero or "∞". Need to think about it another way…

What is Discrete Curvature?

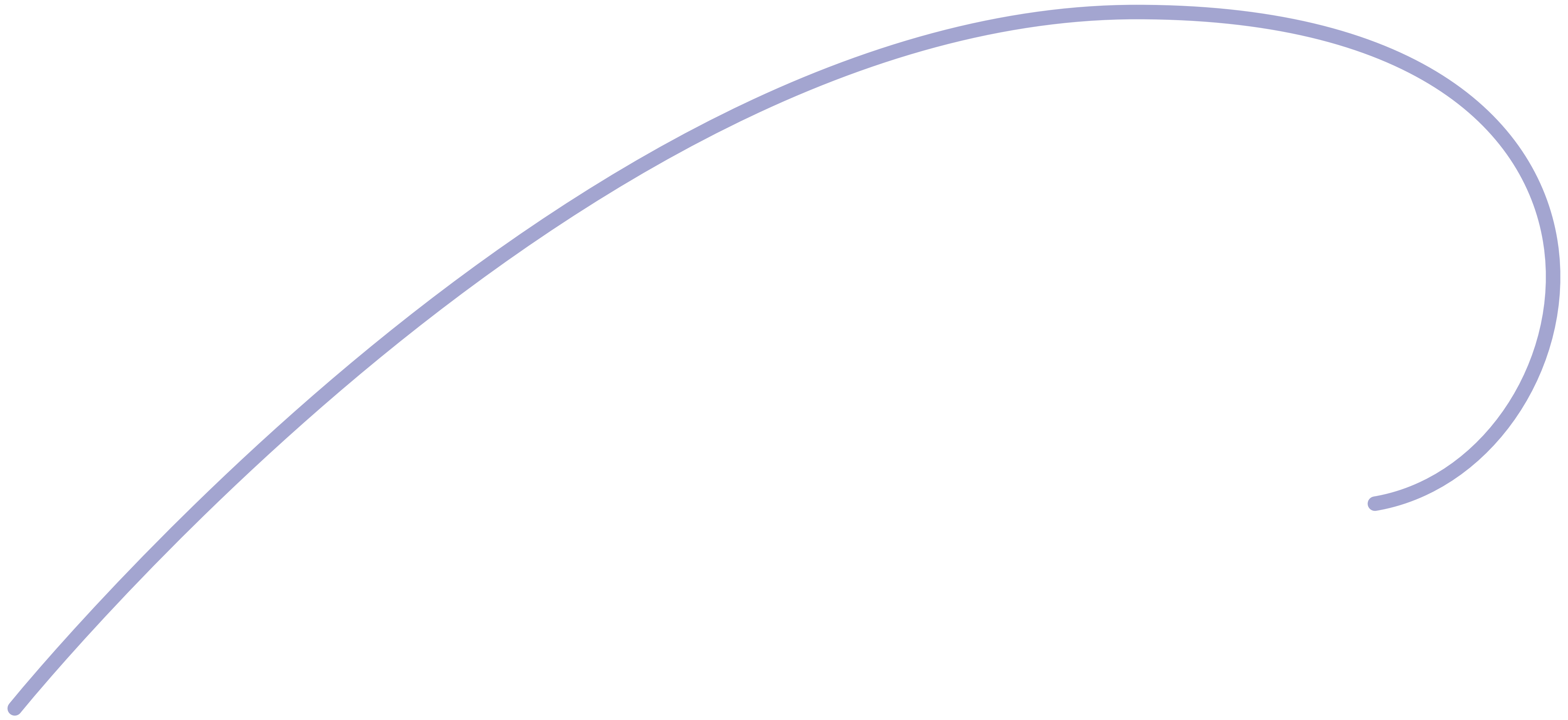
~~**KEY IDEA**~~

~~Curvature is a *second derivative*.~~

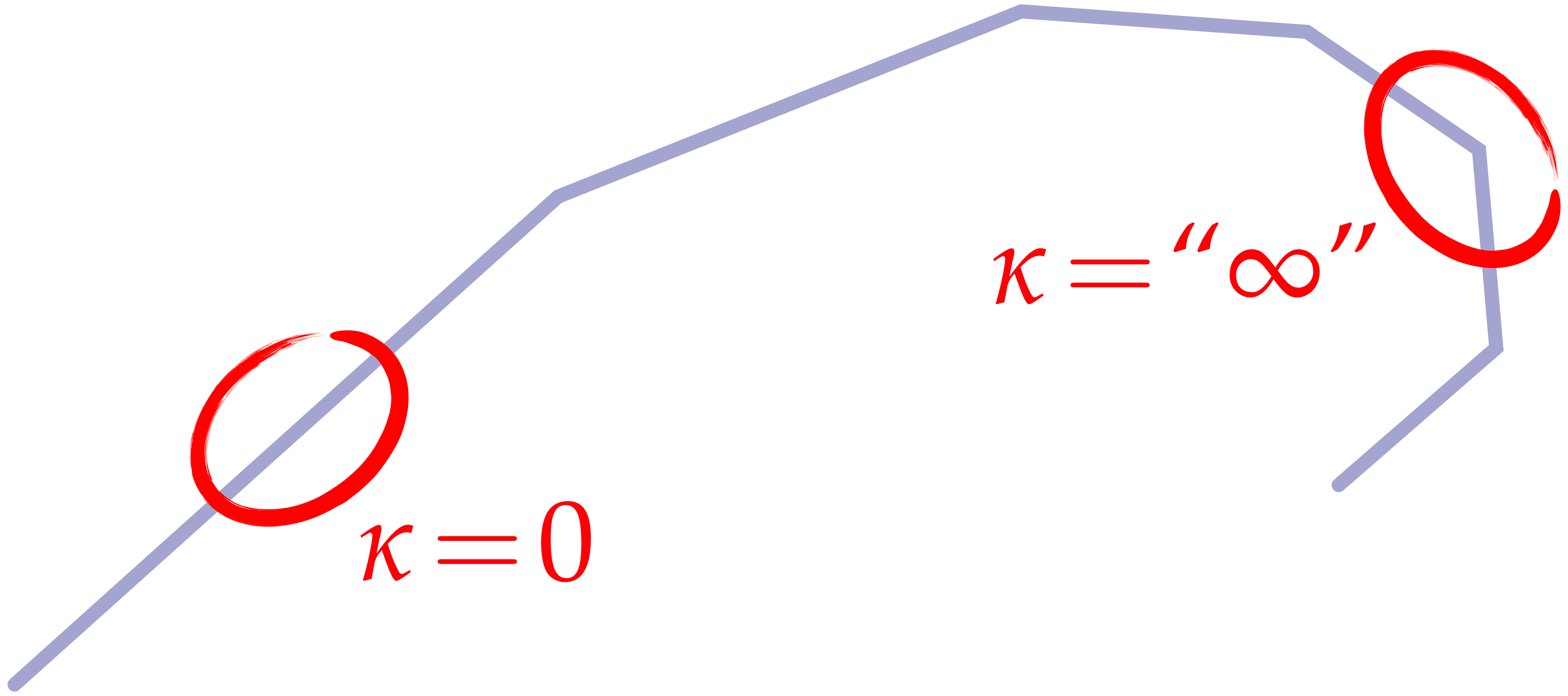
$$\kappa = \cancel{\left\langle \mathcal{J}\frac{d}{ds}\gamma, \frac{d^2}{ds^2}\gamma \right\rangle}$$

Can we directly apply this definition to a discrete curve?

SMOOTH



DISCRETE



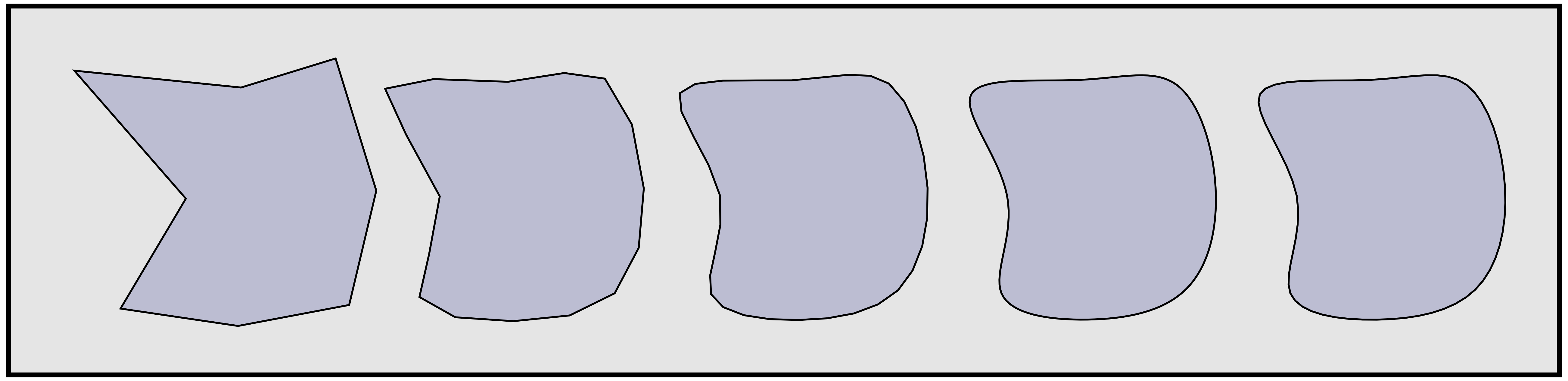
No! Will get either zero or “∞”. Need to think about it another way…

When is a Discretization "Good?"

- How will we know if we came up with a good definition?
- Many different criteria for "good":
  - *satisfies (some of the) same properties/theorems as smooth curvature*
  - *converges to smooth value as we refine our curve*
  - *efficient to compute / solve equations*
  - …

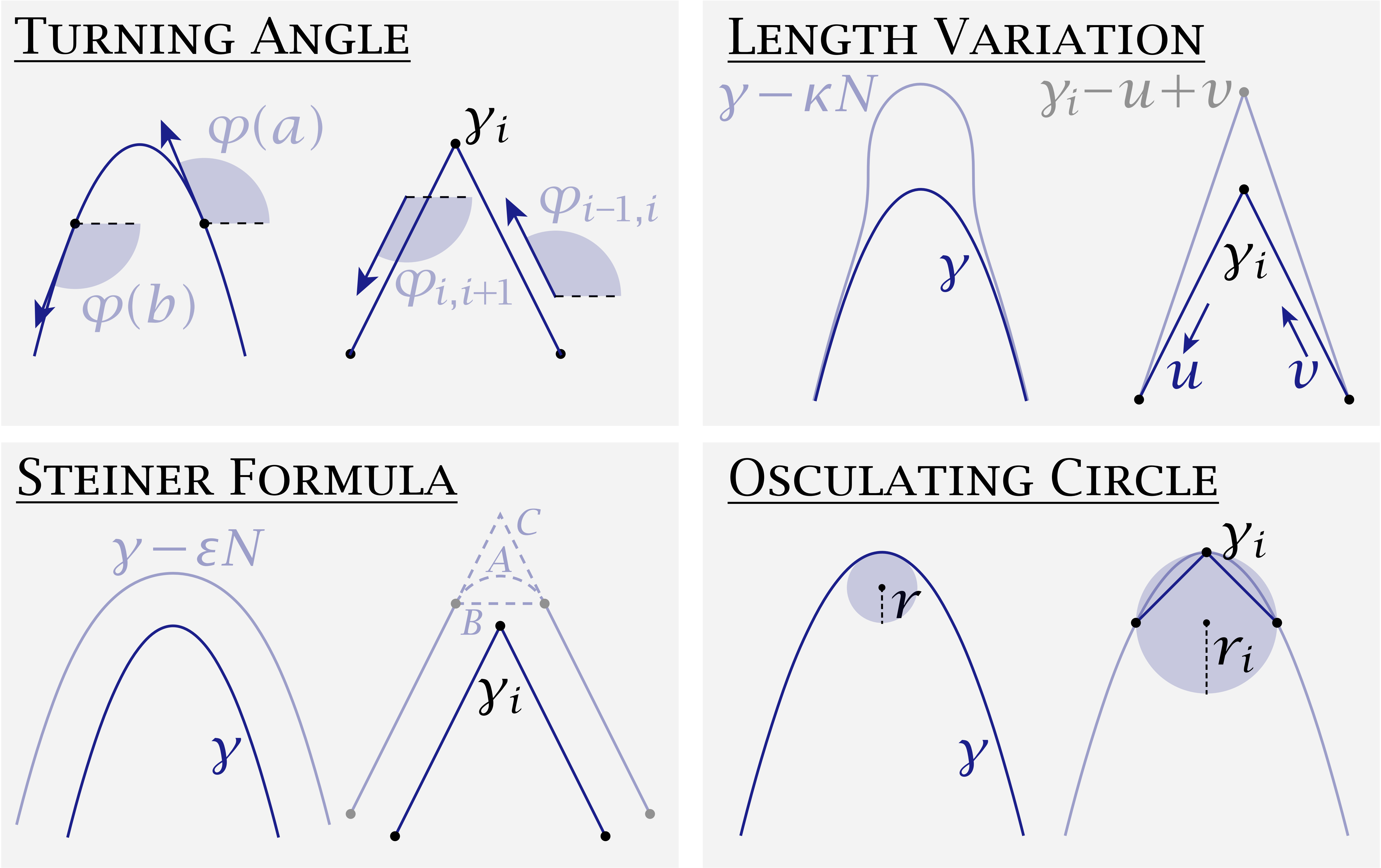
$$\oint \kappa \, ds \in 2\pi\mathbb{Z}$$

```
Complex Ta = gamma[i] - gamma[i-1];
Complex Tb = gamma[i+1] - gamma[i];
double kappa = (Tb*Ta.inv()).arg();
```



Curvature, Revisited

- In the **smooth** setting, there are several other **equivalent** definitions of curvature.
- **<u>IDEA:</u>** *perhaps some of these definitions can be applied directly to our discrete curve!*

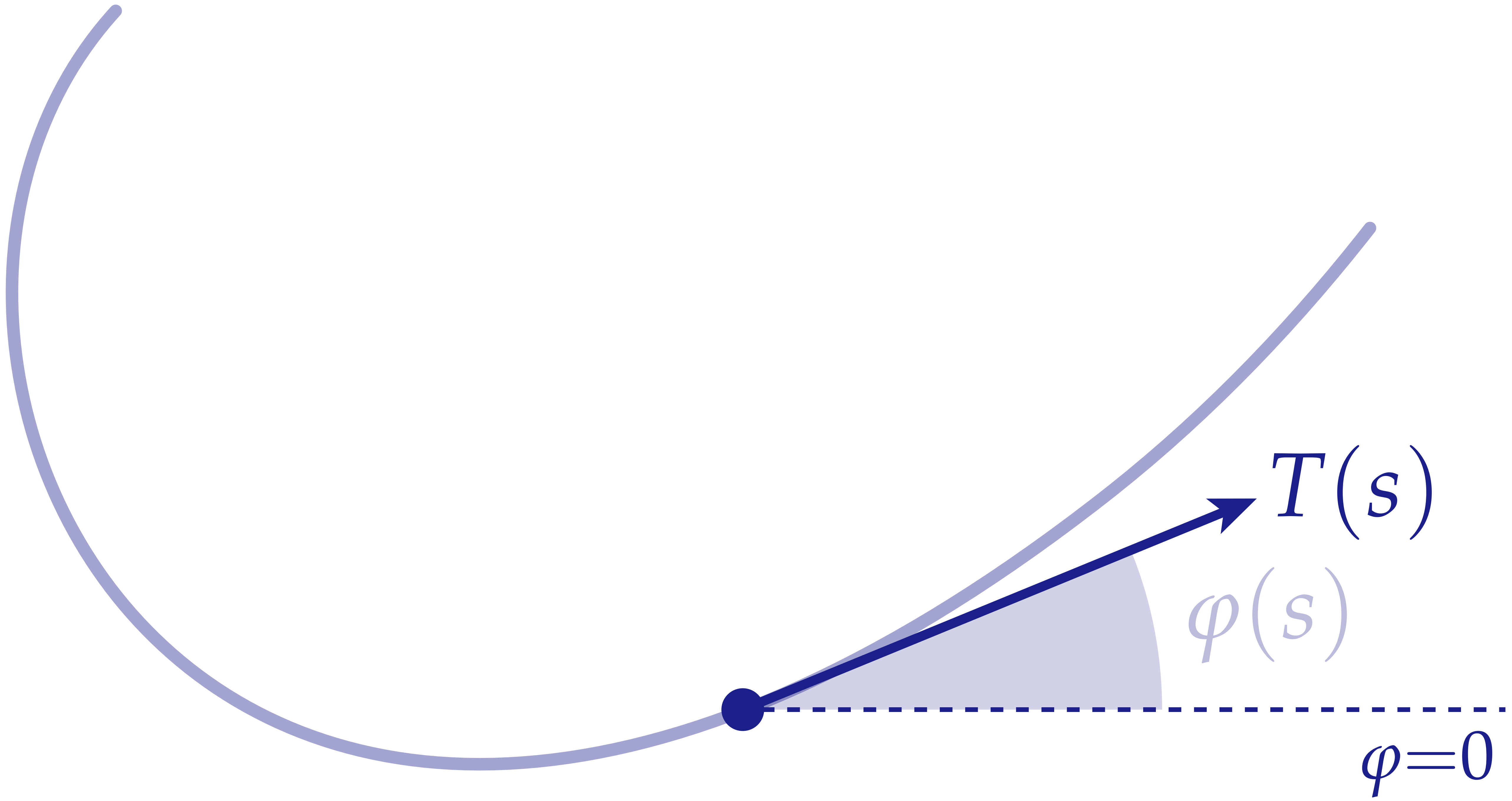


Turning Angle

- Our initial definition of curvature was the *rate of change of the tangent in the normal direction.*

$$\kappa(s) = \langle N(s), \tfrac{d}{ds}\gamma(s)\rangle$$

- Equivalently, we can measure the *rate of change of the angle the tangent makes with the horizontal*:



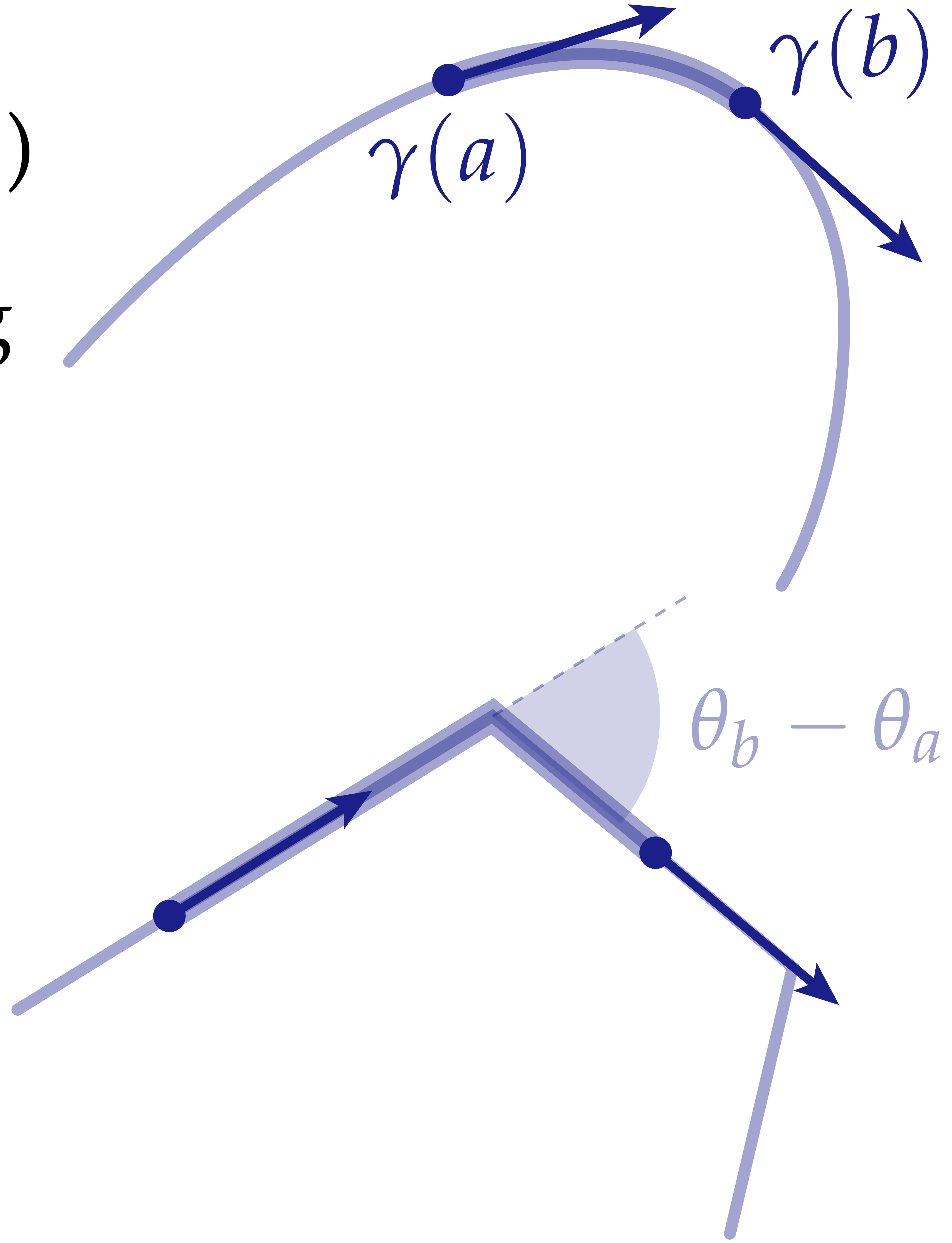
$$\boxed{\kappa(s) = \tfrac{d}{ds}\varphi(s)}$$

Integrated Curvature

- Still can't evaluate curvature at vertices of a discrete curve (*at what rate does the angle change?*)
- But let's consider the *integral* of curvature along a short segment:

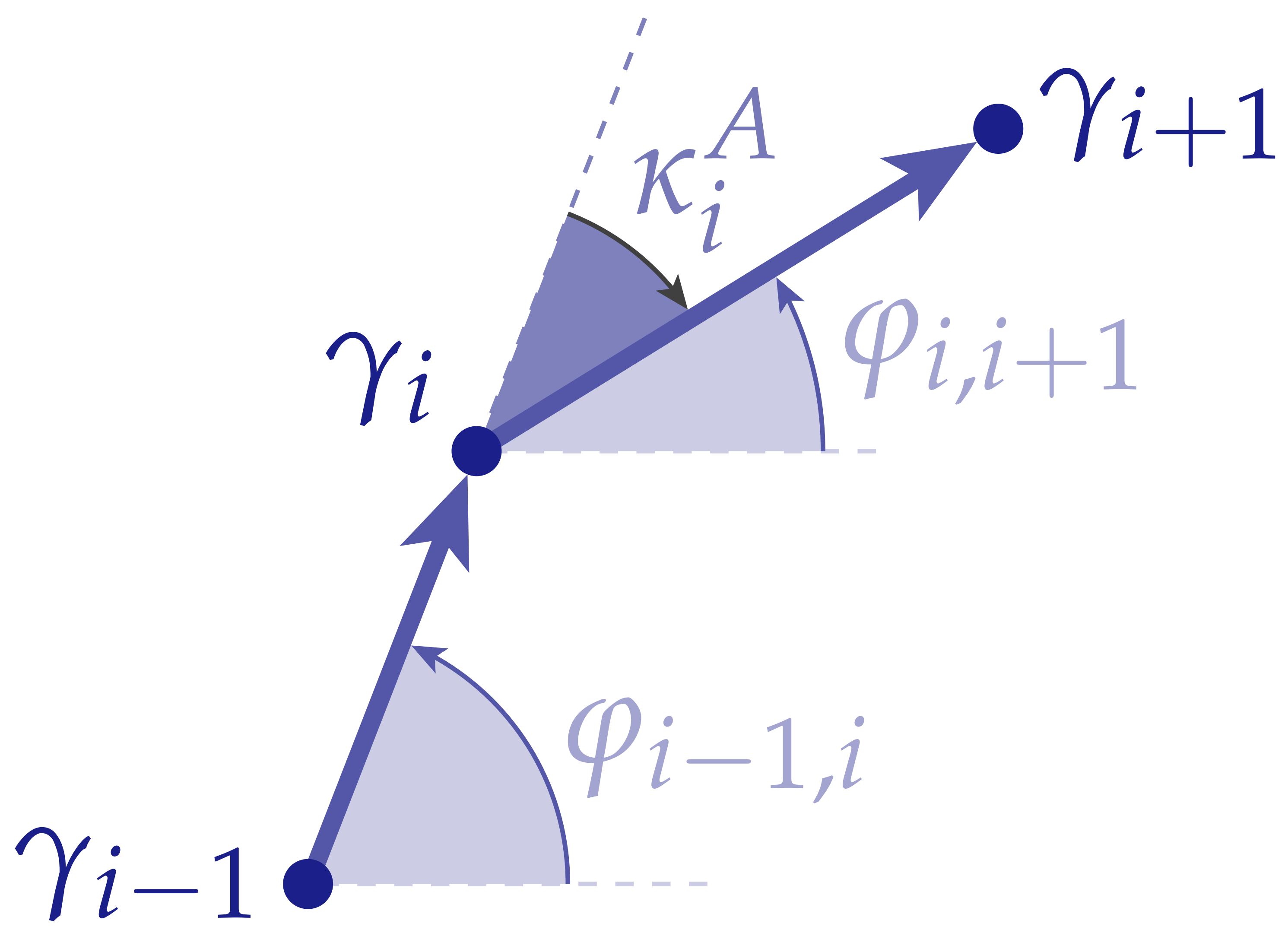
$$\int_a^b \kappa(s)\, ds = \int_a^b \left(\tfrac{d}{ds}\varphi(s)\right) ds = \varphi(b) - \varphi(a)$$

- Instead of *derivative* of angle, we now just have a *difference* of angles.
- **This definition works for our discrete curve!**



Discrete Curvature (Turning Angle)

- This formula gives us our first definition of discrete curvature, as just the *exterior angle* at the vertex of each curve:



$$\theta_i := \varphi_{i,i+1} - \varphi_{i-1,i} \quad \textbf{(exterior angle)}$$

$$\kappa_i^A := \theta_i \in (-\pi, \pi) \quad \textbf{(discrete curvature)}$$

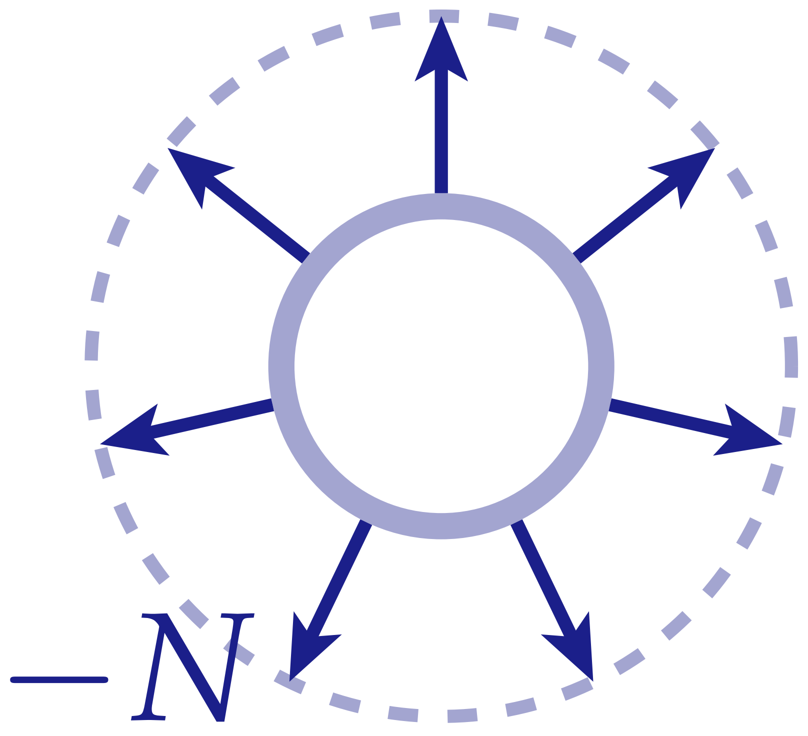
- Here we encounter another theme from discrete differential geometry: the quantities we want to work with are often more naturally expressed as <u>*integrated*</u> rather than <u>*pointwise*</u> quantities.

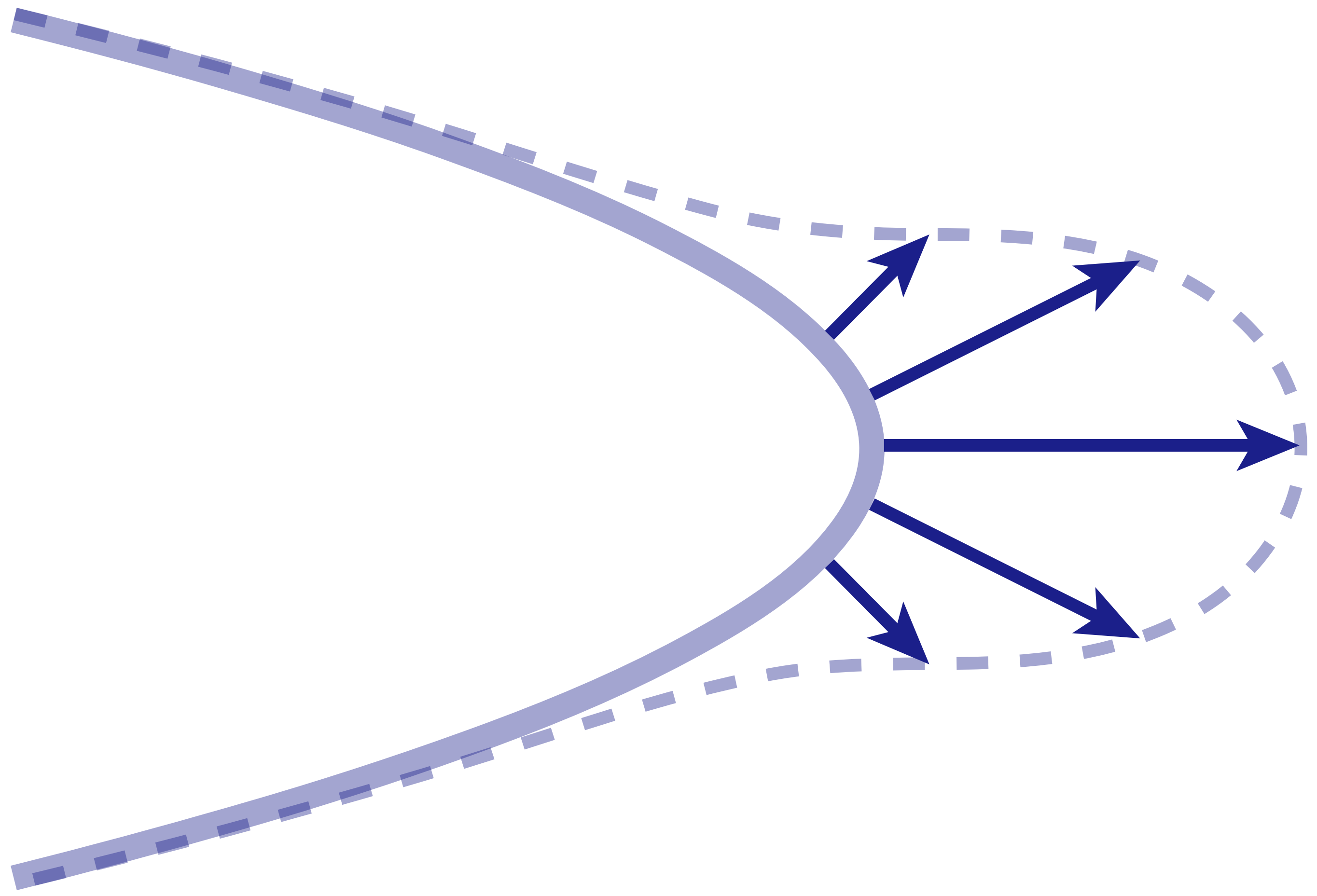
Length Variation

- Are there *other* ways to get a definition for discrete curvature?
- Well, here's a useful fact about curvature from the smooth setting:

> *The fastest way to decrease the length of a curve is to move it in the normal direction, with speed proportional to curvature.*

- **Intuition**: in flat regions, moving the curve doesn't change its length; in curved regions, the change in length (*per unit length*) is large:

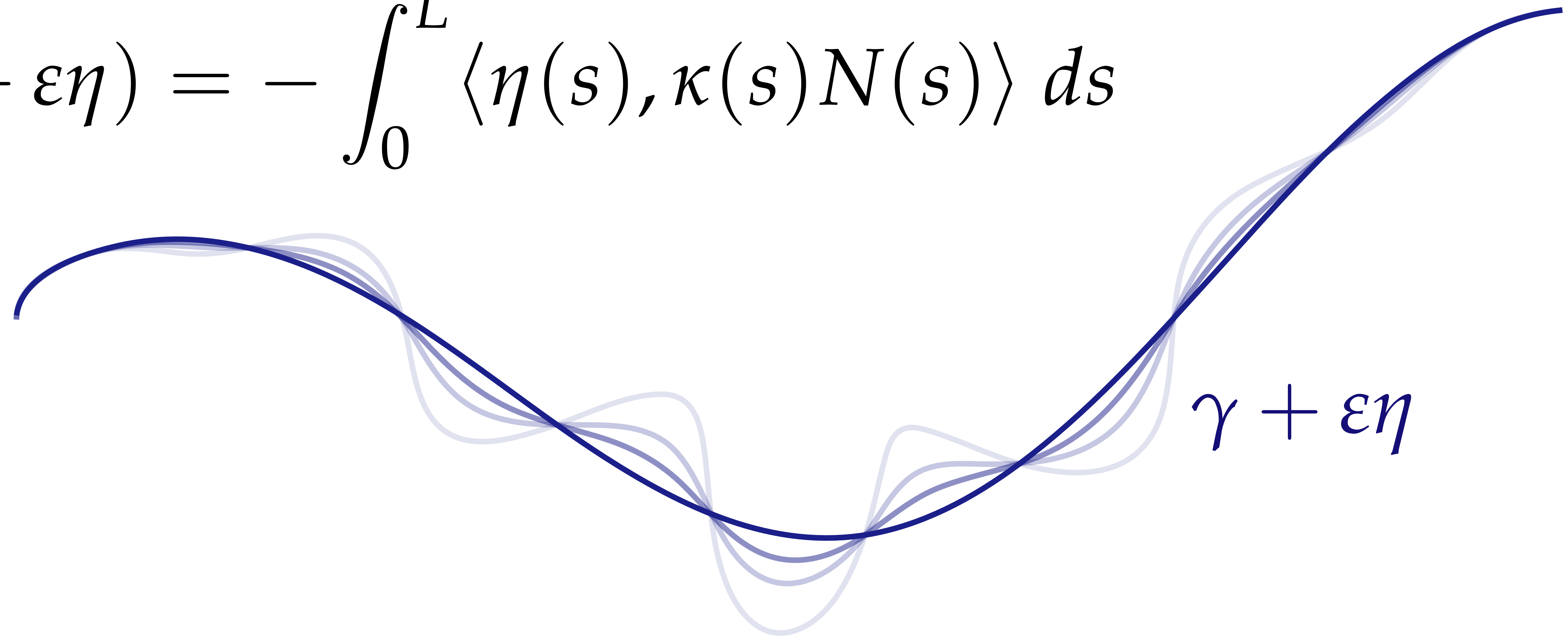




Length Variation

- More formally, consider an *arbitrary* variation of the curve. *I.e.*, suppose we have another curve* $\eta : [0, L] \to \mathbb{R}^2$. One can show that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \text{length}(\gamma + \varepsilon\eta) = -\int_0^L \langle \eta(s), \kappa(s)N(s) \rangle \, ds$$

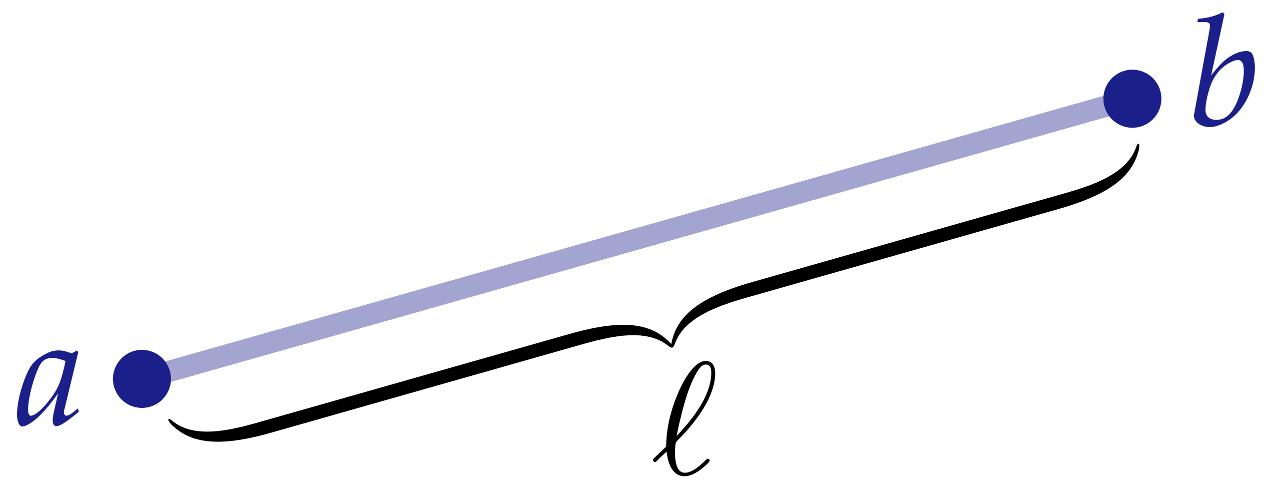


- Therefore, the motion that most quickly decreases length is $\eta = \kappa N$.

*Technical note: must go to zero at endpoints (*i.e.*, pass through the origin).

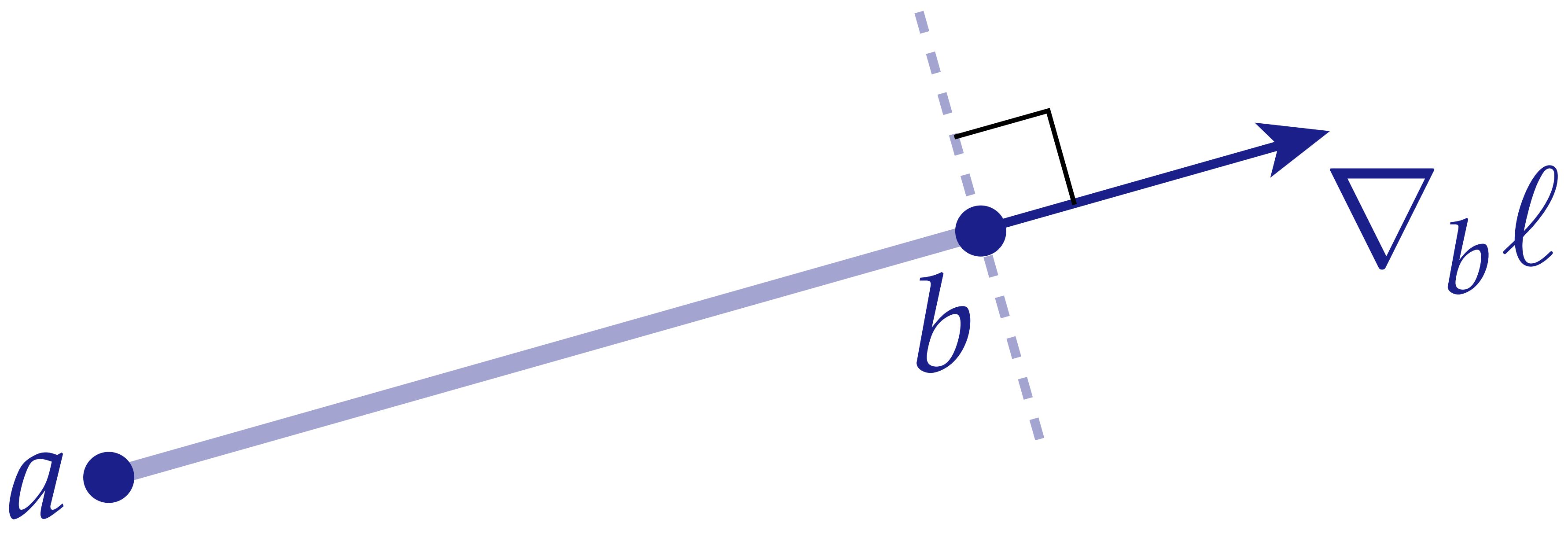
Gradient of Length for a Line Segment

- This all becomes much easier in the discrete setting: just take the gradient of length with respect to vertex positions.
- First, a warm-up exercise. Suppose we have a *single* line segment:



$$\ell := |b - a|$$

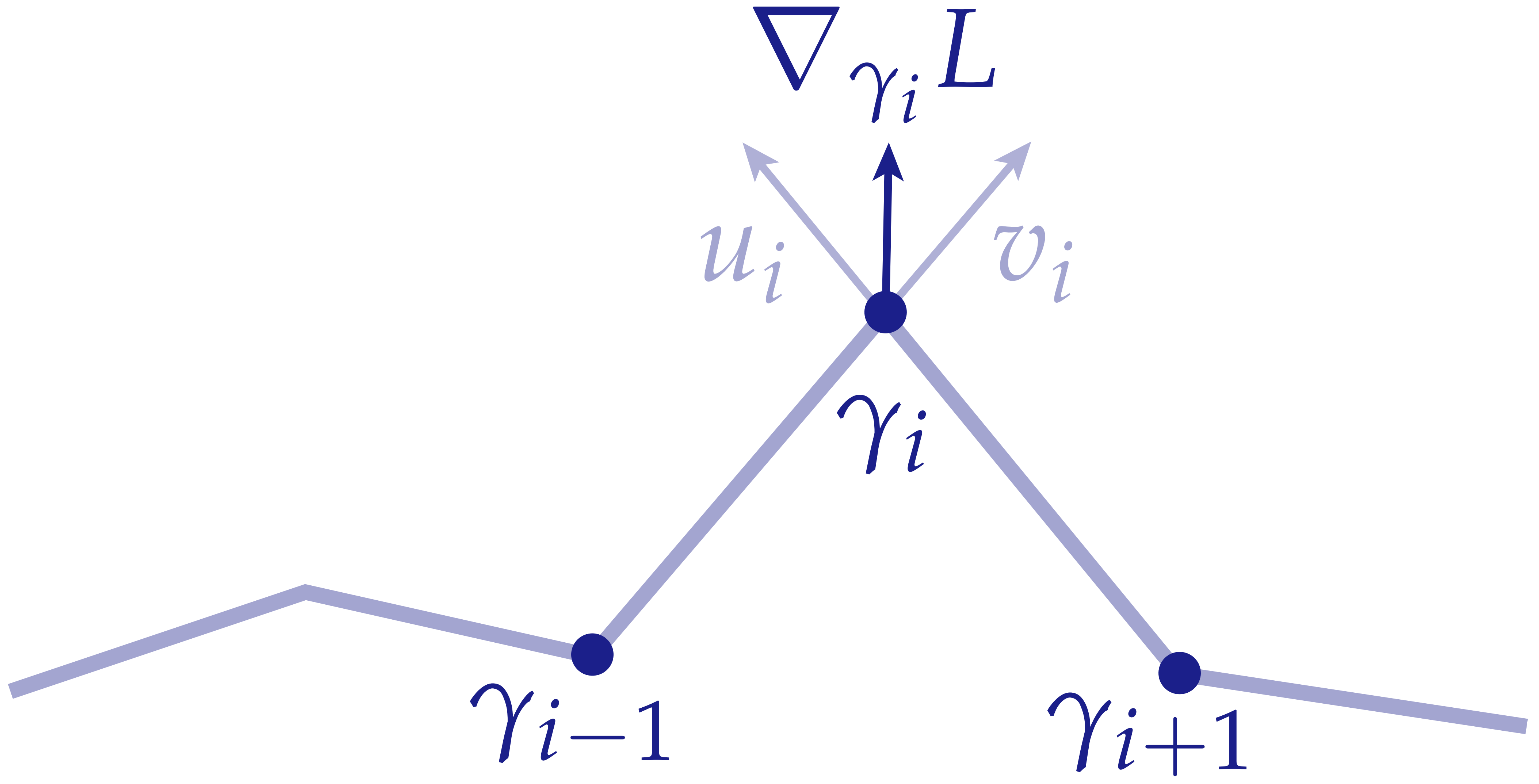
- How can we move the point b to most quickly increase its length?

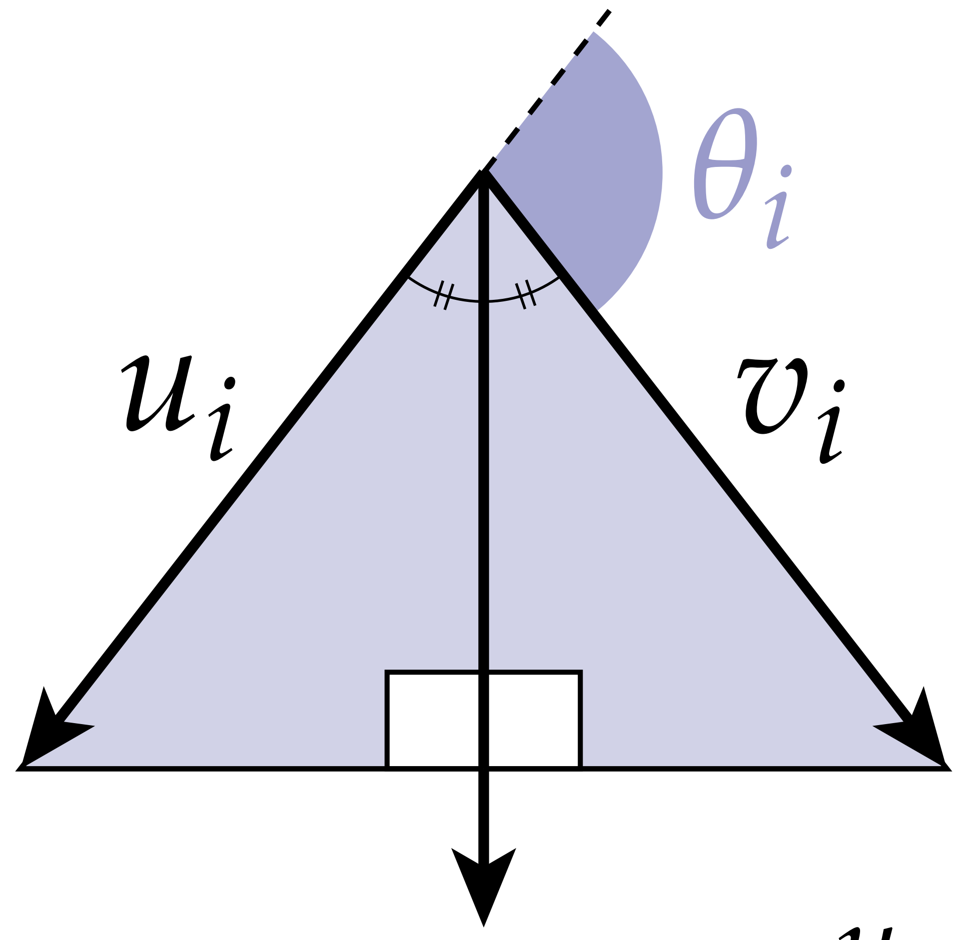


$$\nabla_b \ell = (b - a)/\ell$$

Gradient of Length for a Discrete Curve

- To find the motion that most quickly increases the *total* length L, we now just have to sum the contributions of each segment:





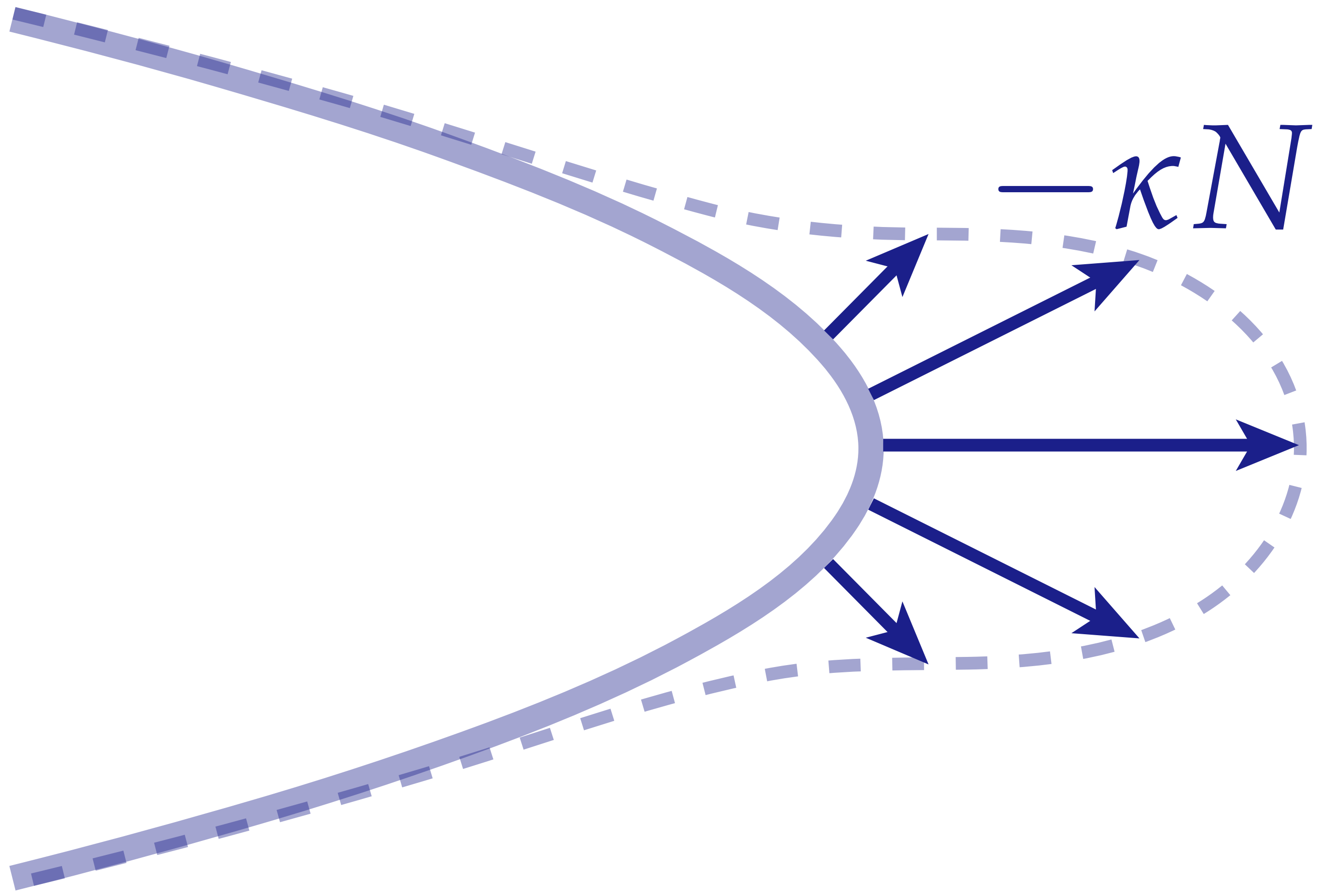
$$N_i := \frac{u_i + v_i}{|u_i + v_i|}$$

- Using some simple trigonometry, we can also express the length gradient in terms of the exterior angle θ_i and the angle bisector N_i:

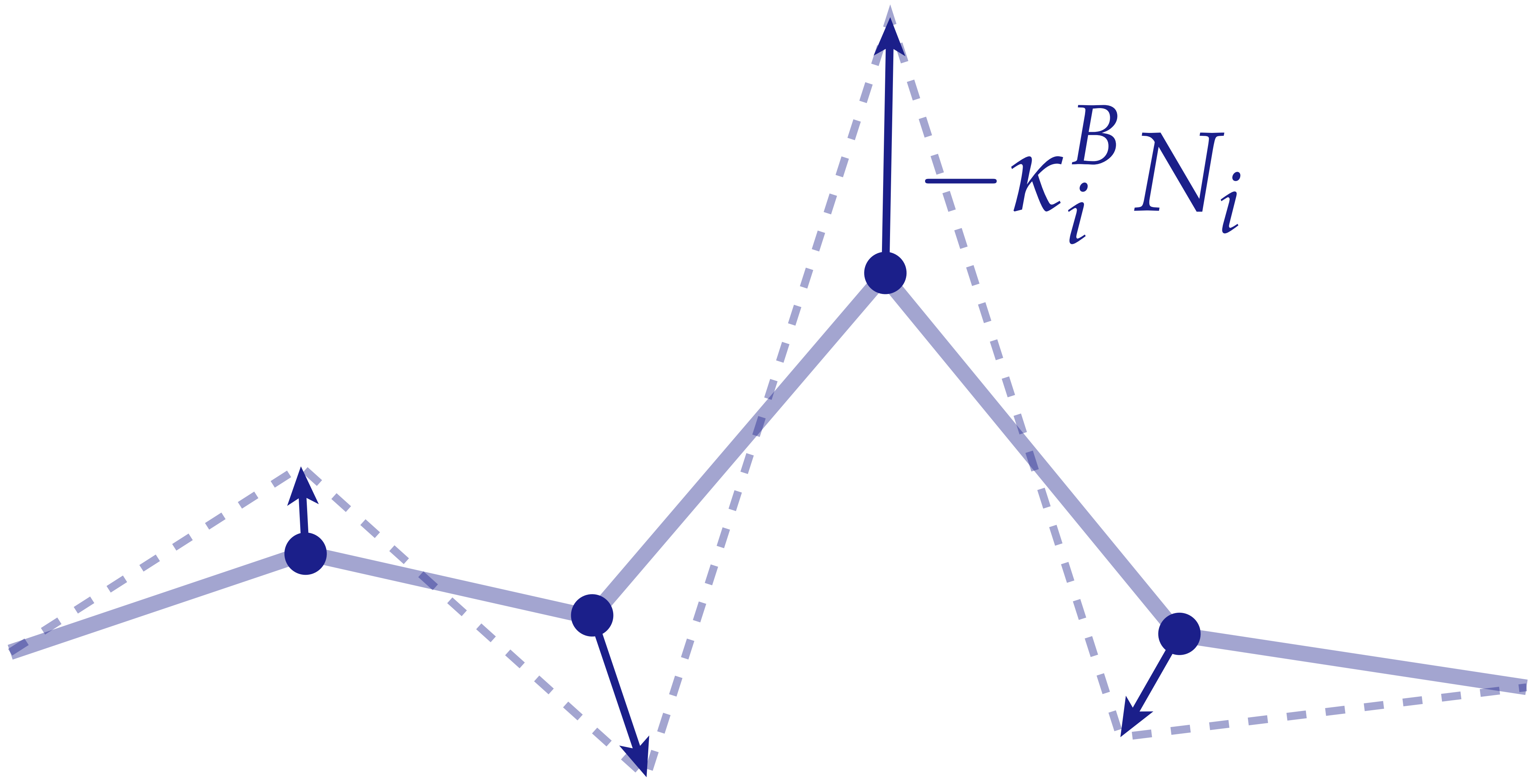
$$\nabla_{\gamma_i} L = 2\sin(\theta_i/2) N_i$$

Discrete Curvature (Length Variation)

- How does this help us define discrete curvature?
- Recall that in the smooth setting, the gradient of length is equal to the curvature times the normal.
- Hence, our expression for the *discrete* length variation provides a definition for the *discrete* curvature times the *discrete* normal.

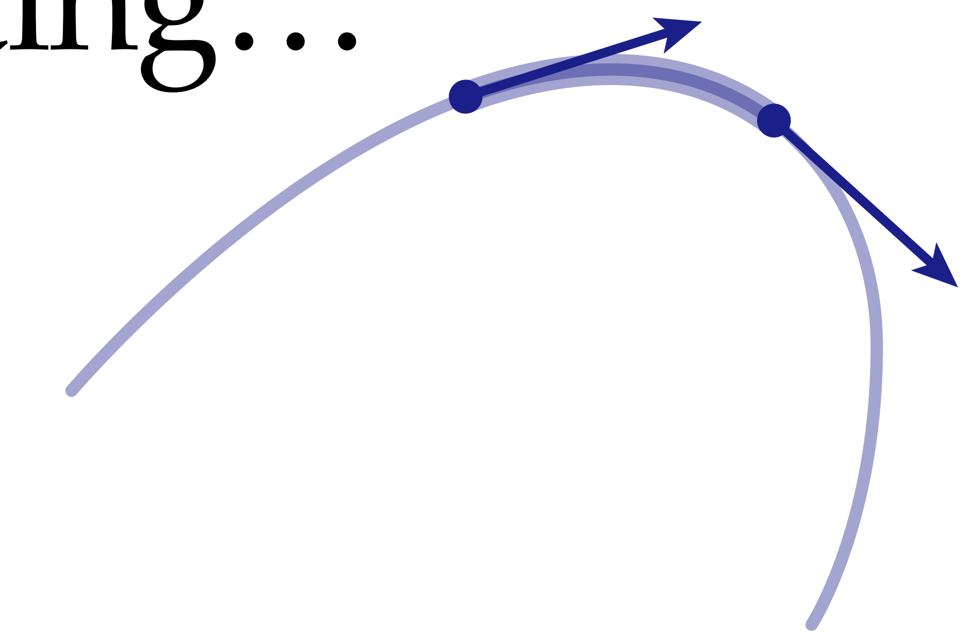


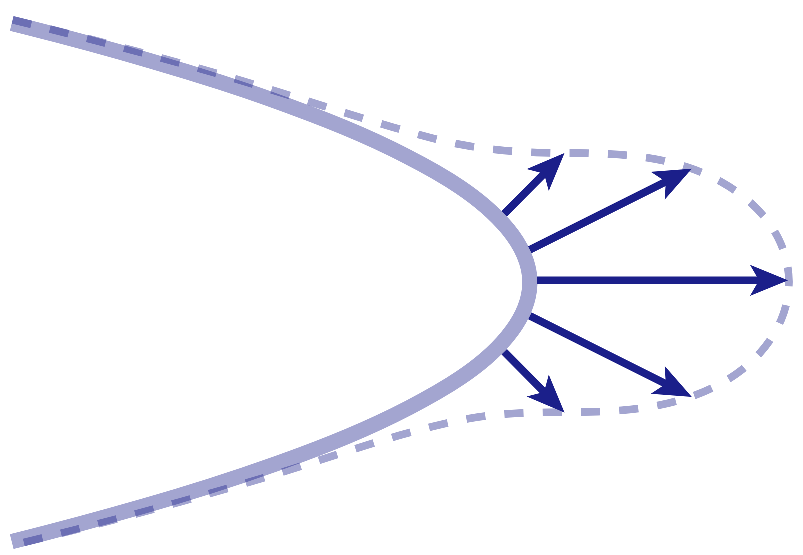
$$\kappa_i^B N_i := 2\sin(\theta_i/2) N_i$$



A Tale of Two Curvatures

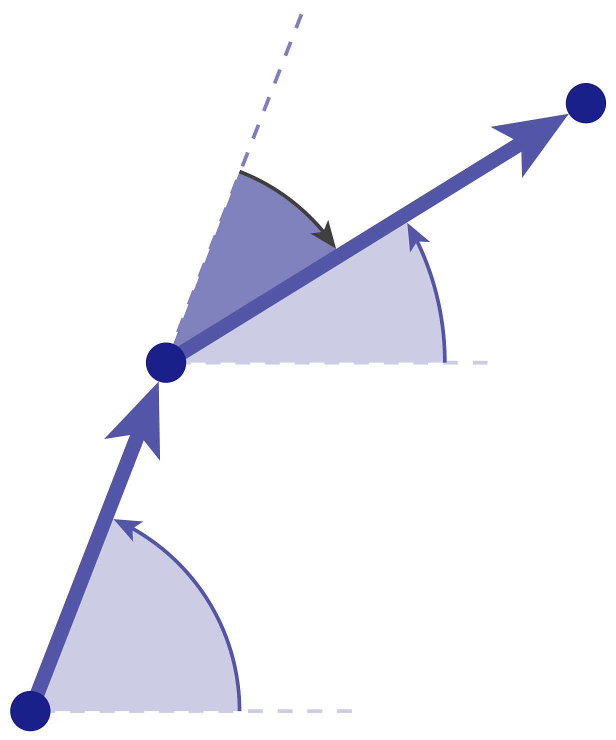
- Let's recap what we've done so far. We considered starting points that are **equivalent** in the smooth setting…

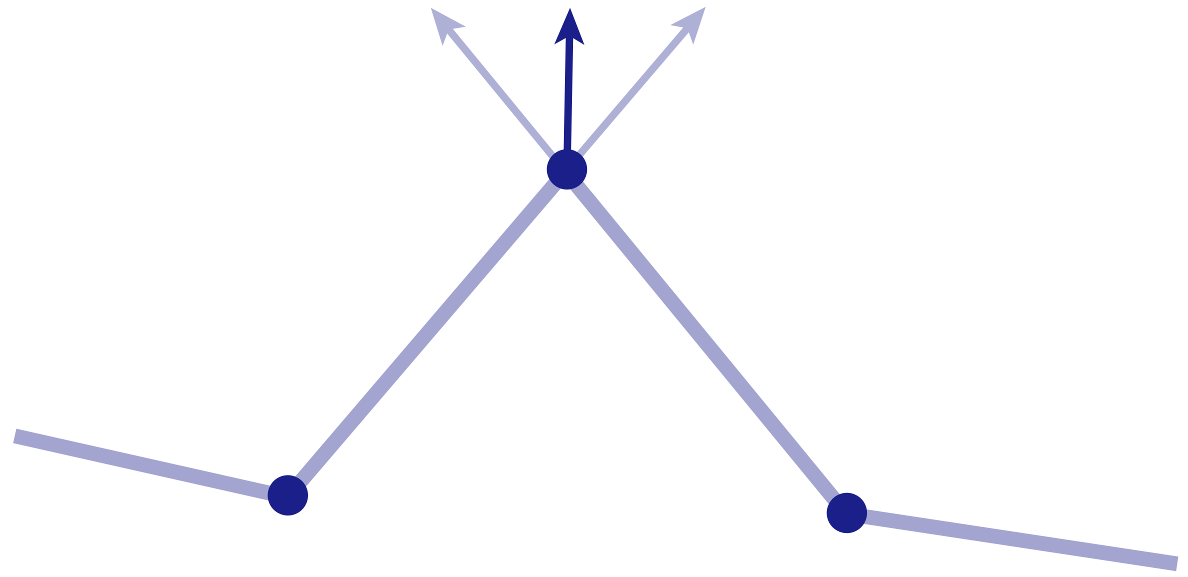




  1. turning angle
  2. length variation

- …which led to two **inequivalent** definitions of curvature in the discrete setting:





  1. $\kappa_i^A := \theta_i$
  2. $\kappa_i^B := 2\sin(\theta_i/2)$

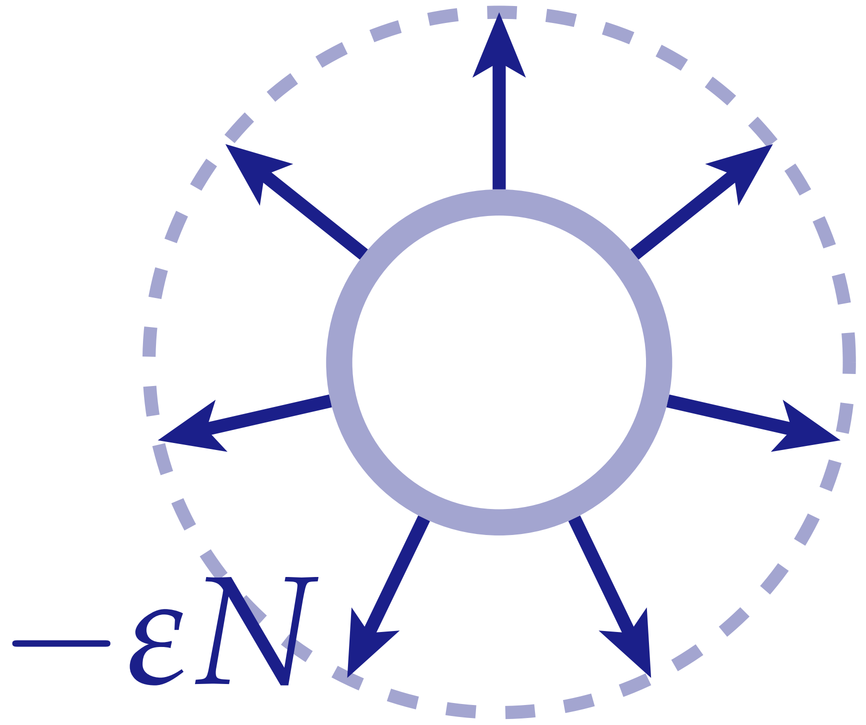
- For *small* angles, they agree. But in general, which one is "better"? And are there more possibilities? Let's keep going…

Steiner Formula

- Steiner's formula is closely related to our last approach: it says that if we move at a *constant* speed in the normal direction, then the change in length is proportional to curvature:

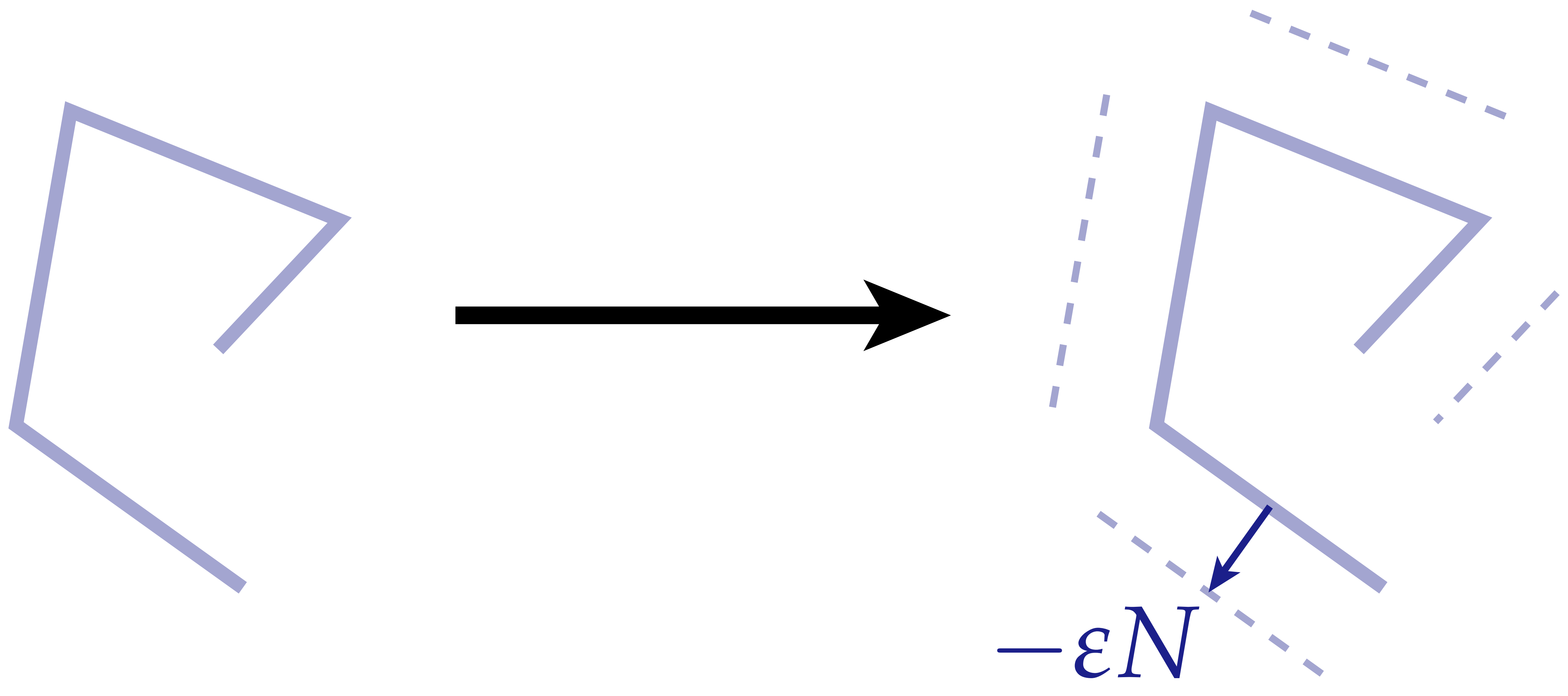
$$\text{length}(\gamma + \varepsilon N) = \text{length}(\gamma) - \varepsilon \int_0^L \kappa(s)\, ds$$

- The intuition is the same as before: for a constant-distance normal offset, length will change in curved regions but not flat regions:



Discrete Normal Offsets

- How do we apply normal offsets in the discrete case?
- The first problem is that *normals* are not defined at vertices!
- We can at very least offset individual edges along their normals:



- The question now is: how can we connect the normal-offset segments to get the final normal-offset curve?

Discrete Normal Offsets

- There are then several natural ways to connect segments:

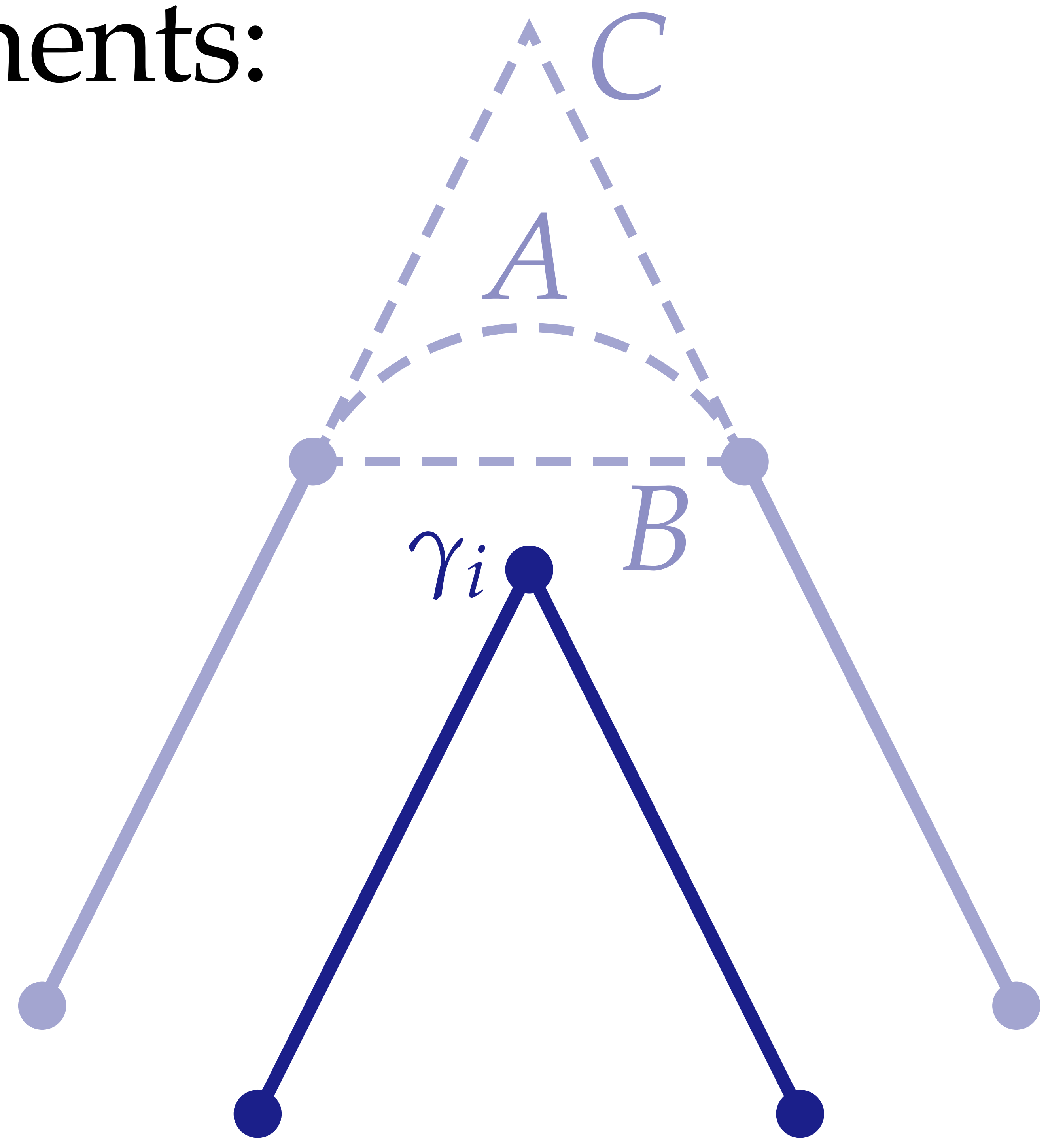
  (A) along a circular arc of radius ε

  (B) along a straight line

  (C) extend edges until they intersect

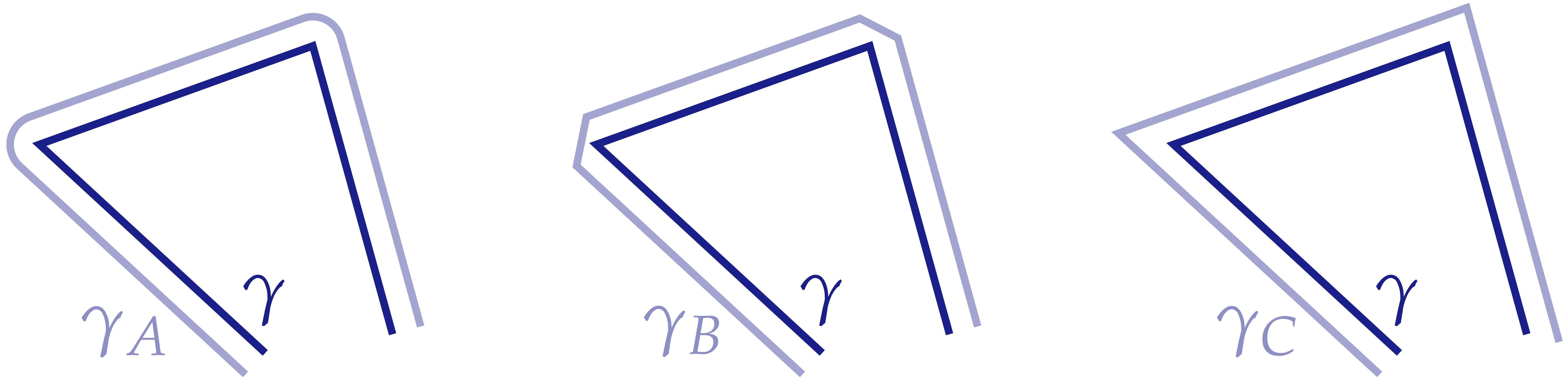
- If we now compute the total length of the connected curves, we get (after some work…):

$$\begin{aligned}
\text{length}_A &= \text{length}(\gamma) - \varepsilon \textstyle\sum_i \theta_i \\
\text{length}_B &= \text{length}(\gamma) - \varepsilon \textstyle\sum_i 2\sin(\theta_i/2) \\
\text{length}_C &= \text{length}(\gamma) - \varepsilon \textstyle\sum_i 2\tan(\theta_i/2)
\end{aligned}$$



Discrete Curvature (Steiner Formula)

- Since Steiner's formula says that the change in length is proportional to curvature, we get yet another definition for curvature by comparing the original and normal-offset lengths.

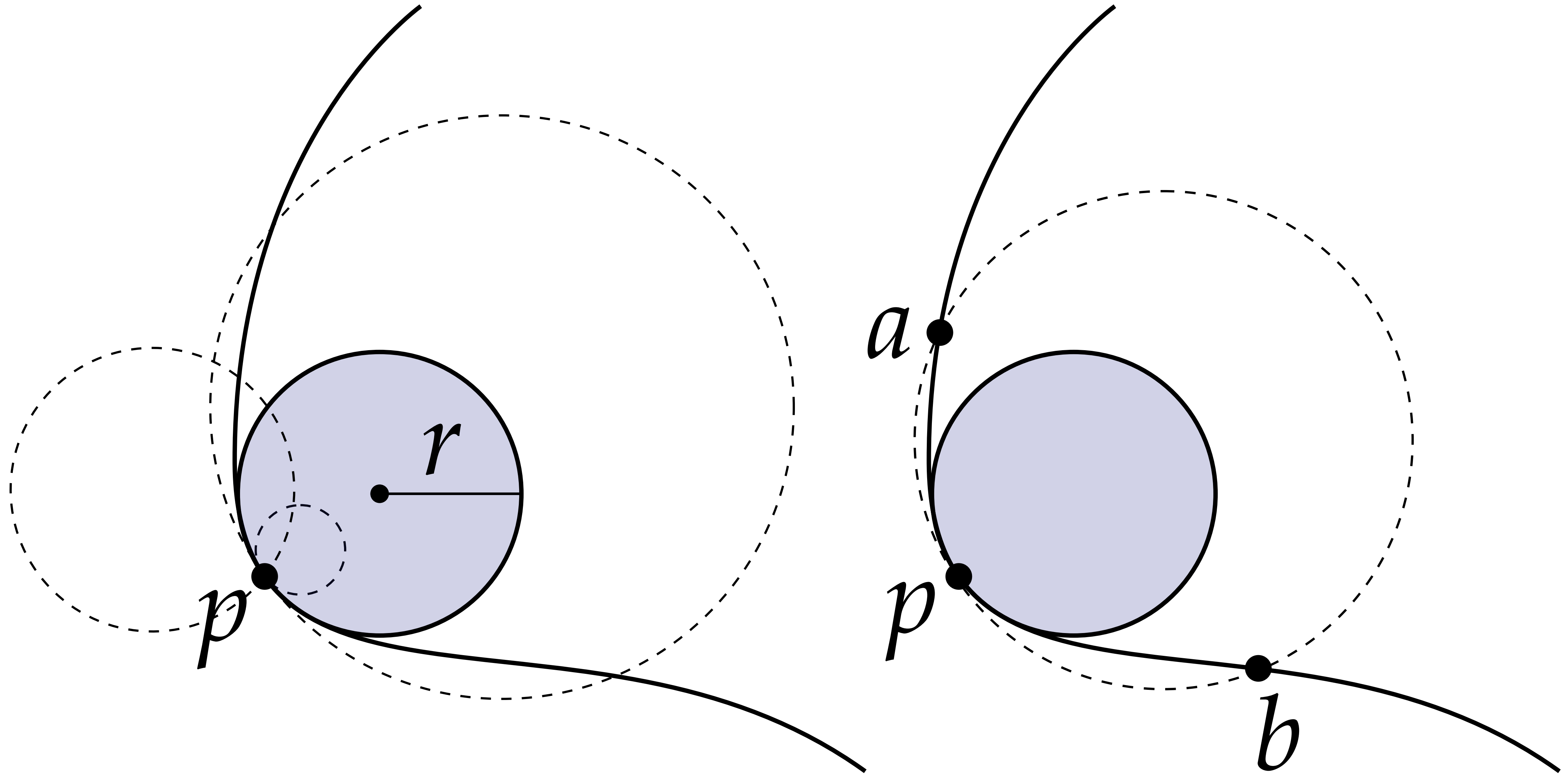


In fact, we get *three* definitions: two we've seen before and one we haven't:

$$
\begin{aligned}
\kappa_i^A &:= \theta_i \\
\kappa_i^B &:= 2\sin(\theta_i/2) \\
\kappa_i^C &:= 2\tan(\theta_i/2)
\end{aligned}
$$

Osculating Circle

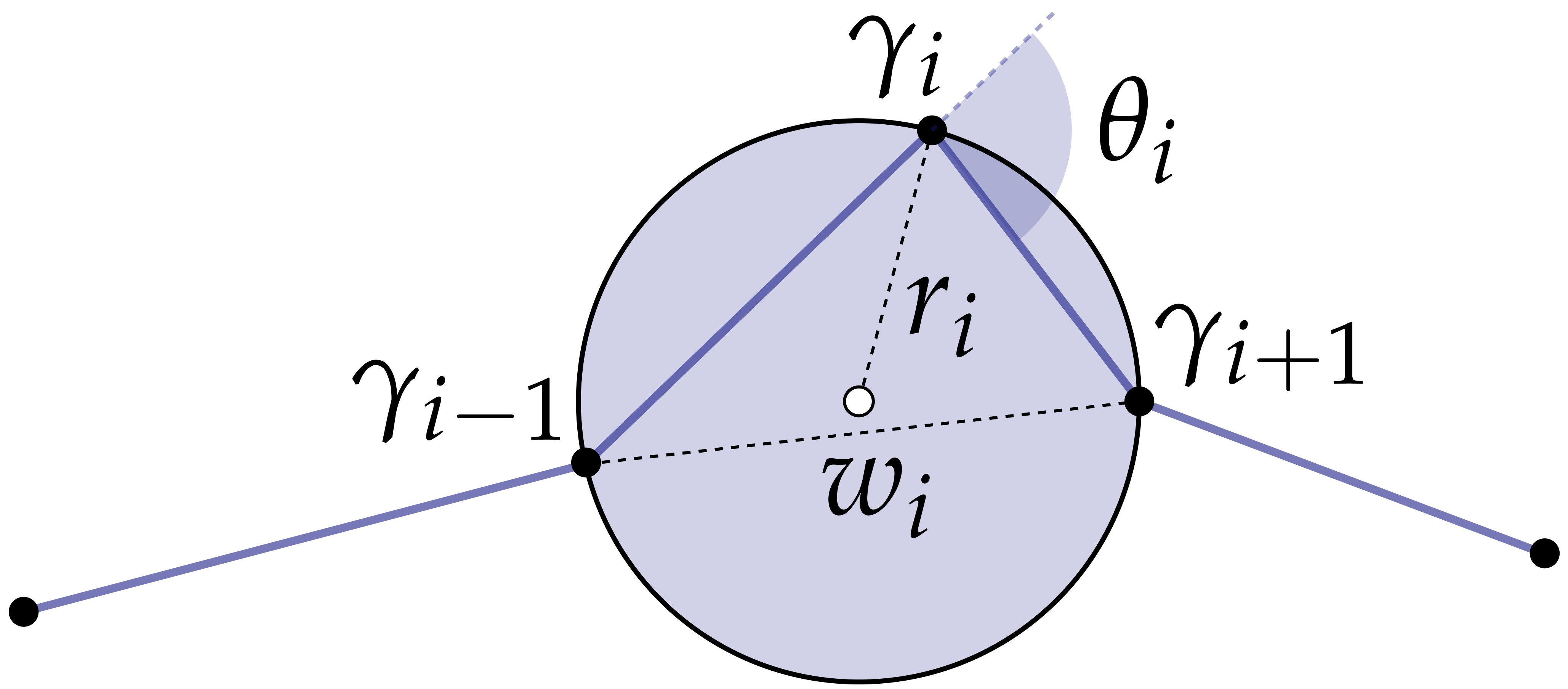
- One final idea is to consider the **osculating circle**, which is (roughly speaking) the circle that best approximates a curve at a given point p:



- More precisely, if we consider a circle passing through the point p itself and two equidistant points a and b to the "left" and "right" (resp.), the osculating circle is the limit as a and b approach p.
- The curvature is then the reciprocal of the radius: $\kappa(p) = \dfrac{1}{r(p)}$

Discrete Curvature (Osculating Circle)

- A natural idea, then, is to consider the *circumcircle* of three consecutive points on a discrete curve (*i.e.*, the circle touching all three points):



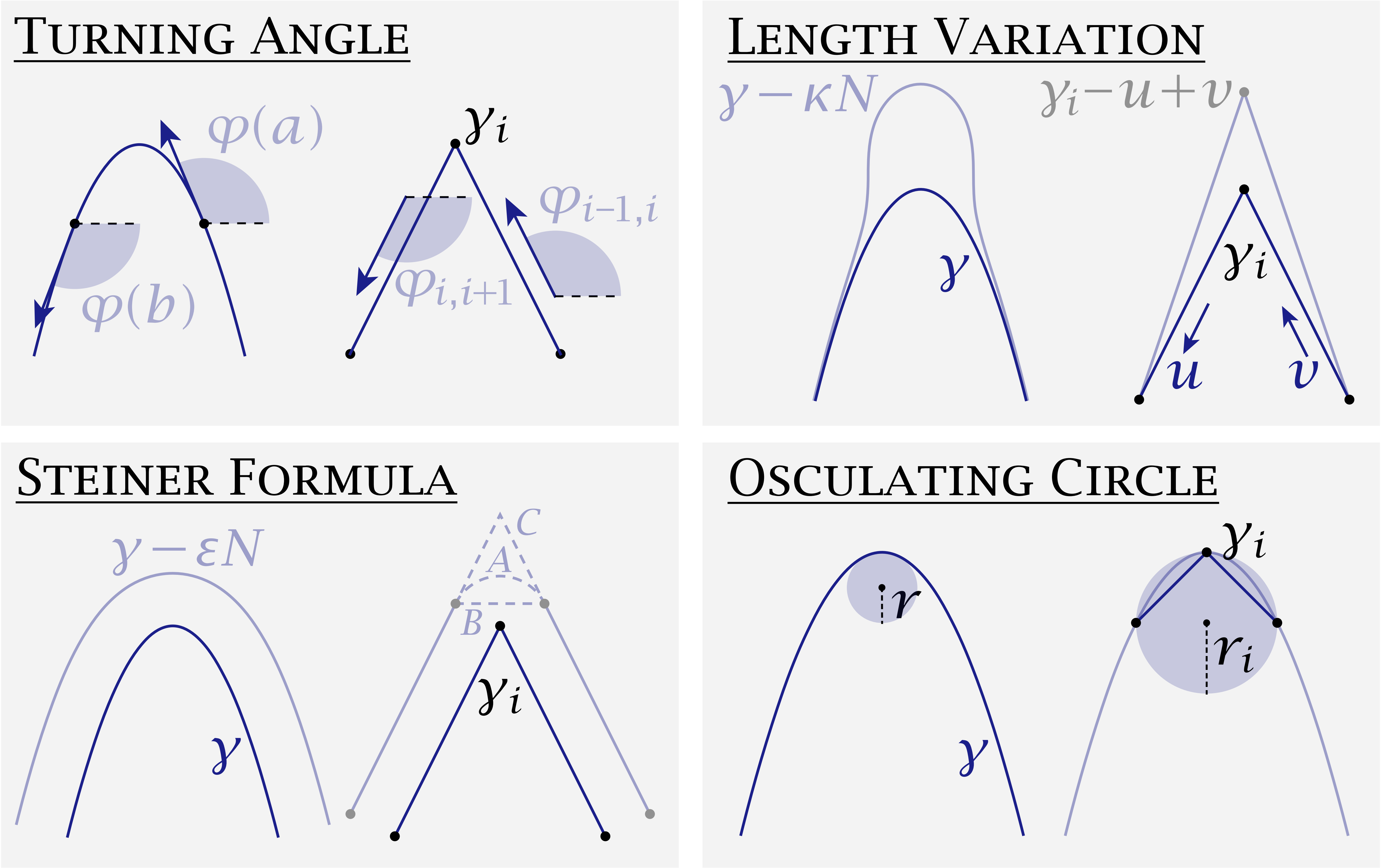
$$w_i := |\gamma_{i+1} - \gamma_{i-1}|$$

- We then get a *fourth* definition of discrete curvature:

$$\kappa_i^D := \frac{1}{r_i} = 2\sin(\theta_i)/w_i$$

A Tale of Four Curvatures

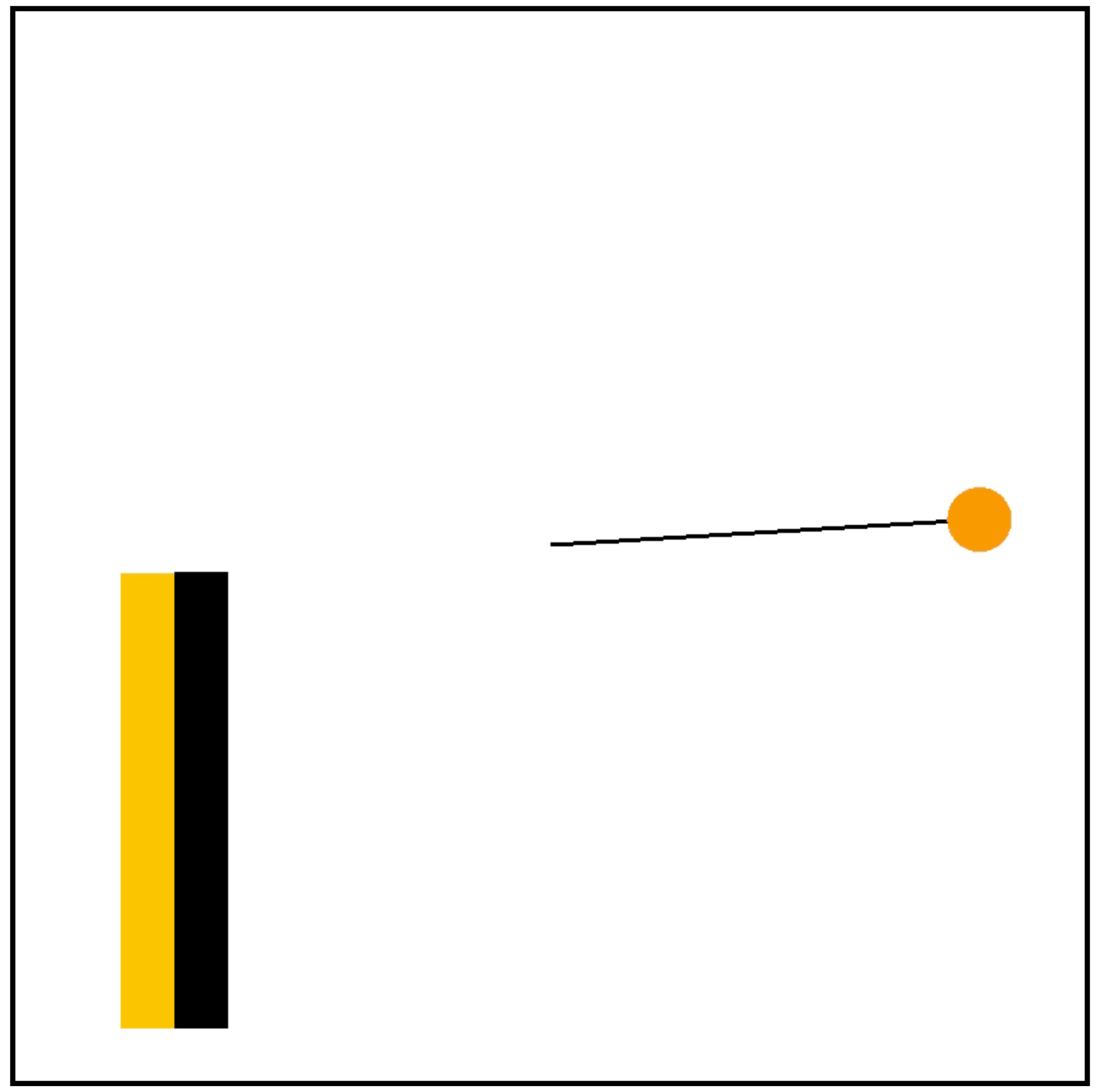
- Starting with four **equivalent** definitions of smooth curvature, we ended up with four **inequivalent** definitions for discrete curvature:



So… which one should we use?

Pick the Right Tool for the Job

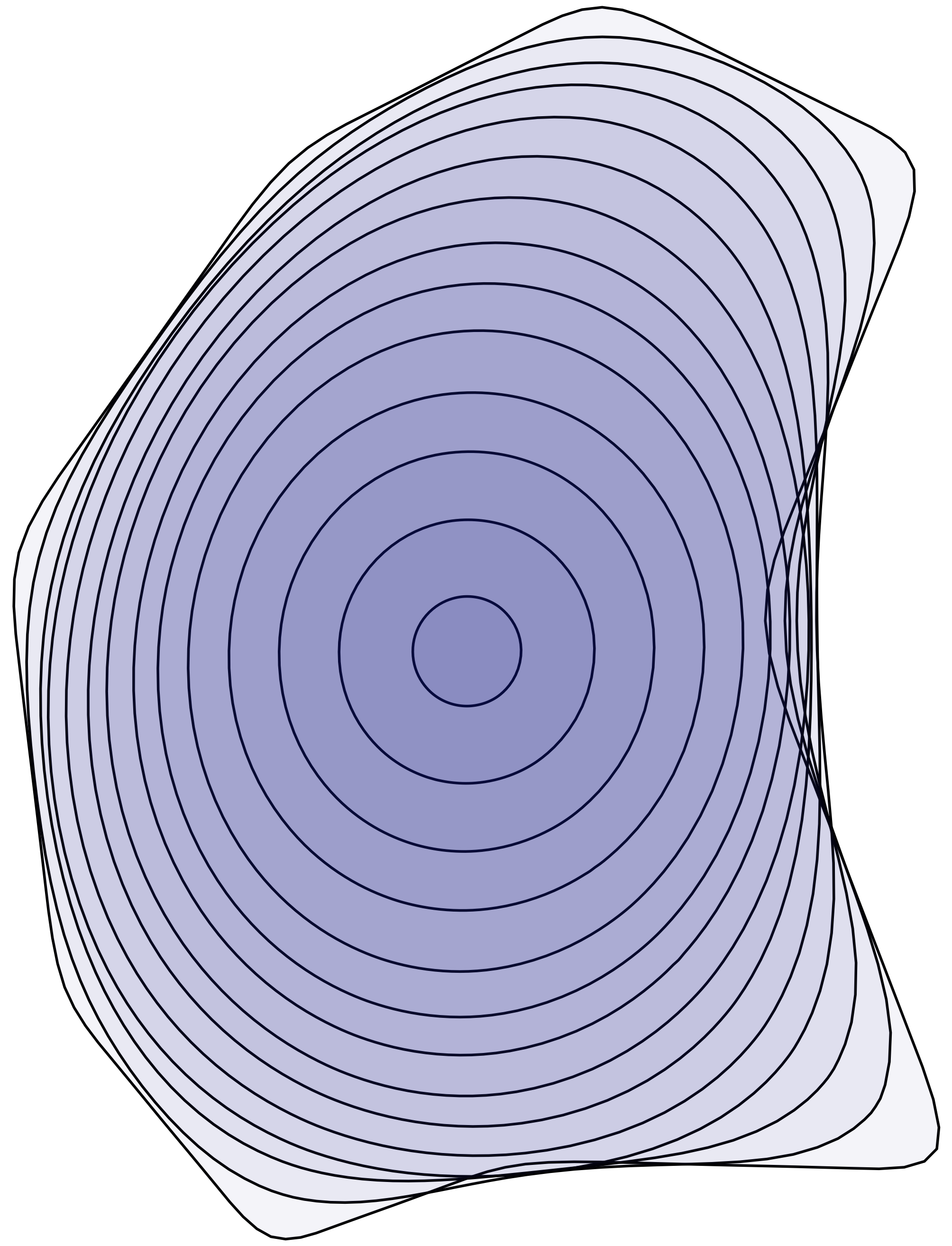
- **Answer:** pick the right tool for the job!
- For a given application, which properties are most important to us? Which quantities do we want to preserve? How much computation are we willing to do? Etc.
- *E.g.*, in a physical simulation we might care about preserving energy but not momentum—or momentum, but not energy.
- Ok… but what about curvature?



Toy Example: Curvature Flow

- A simple example is *curvature flow*, where a closed curve moves in the normal direction with speed proportional to curvature:

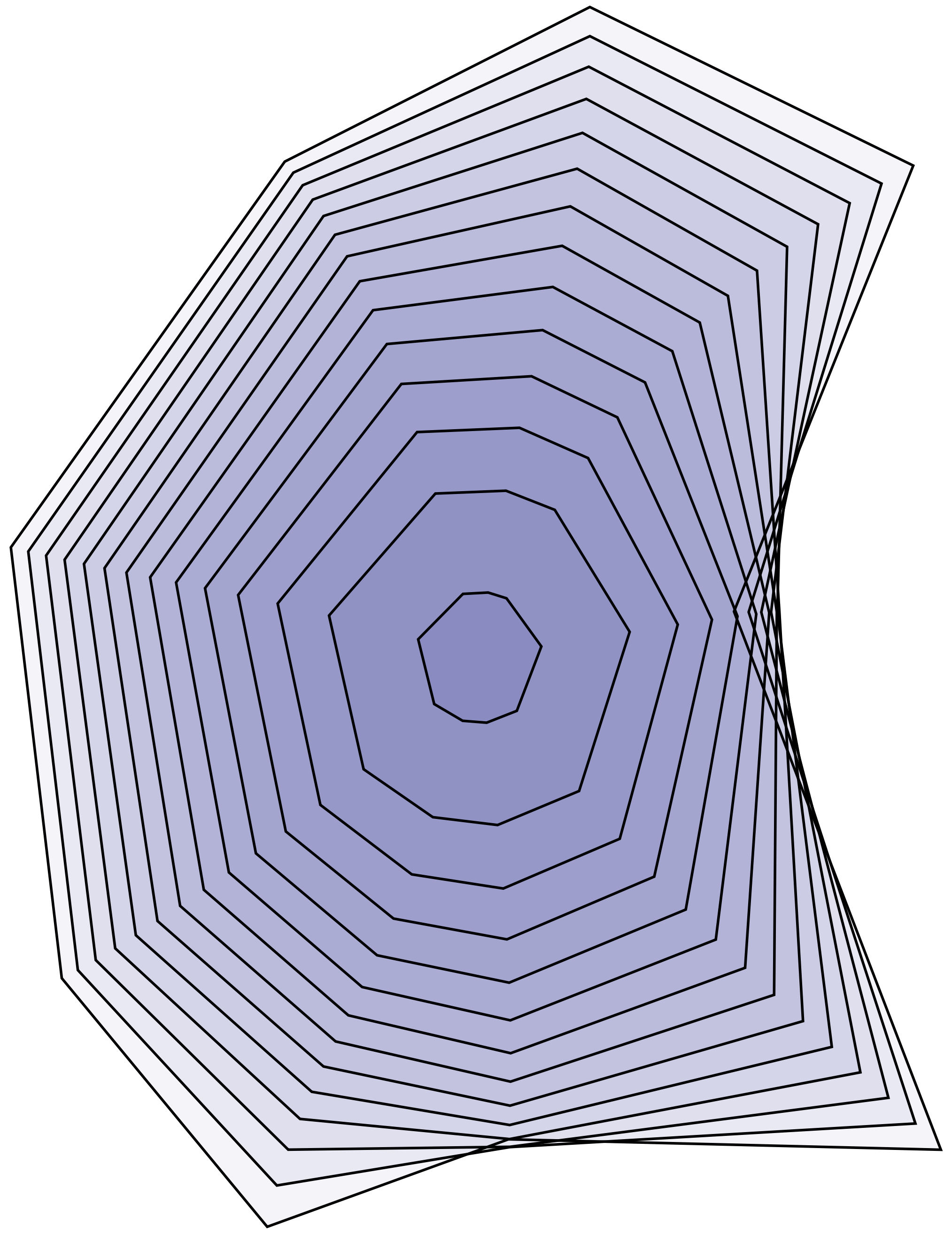
$$\frac{d}{dt}\gamma(s,t) = \kappa(s,t)N(s,t)$$

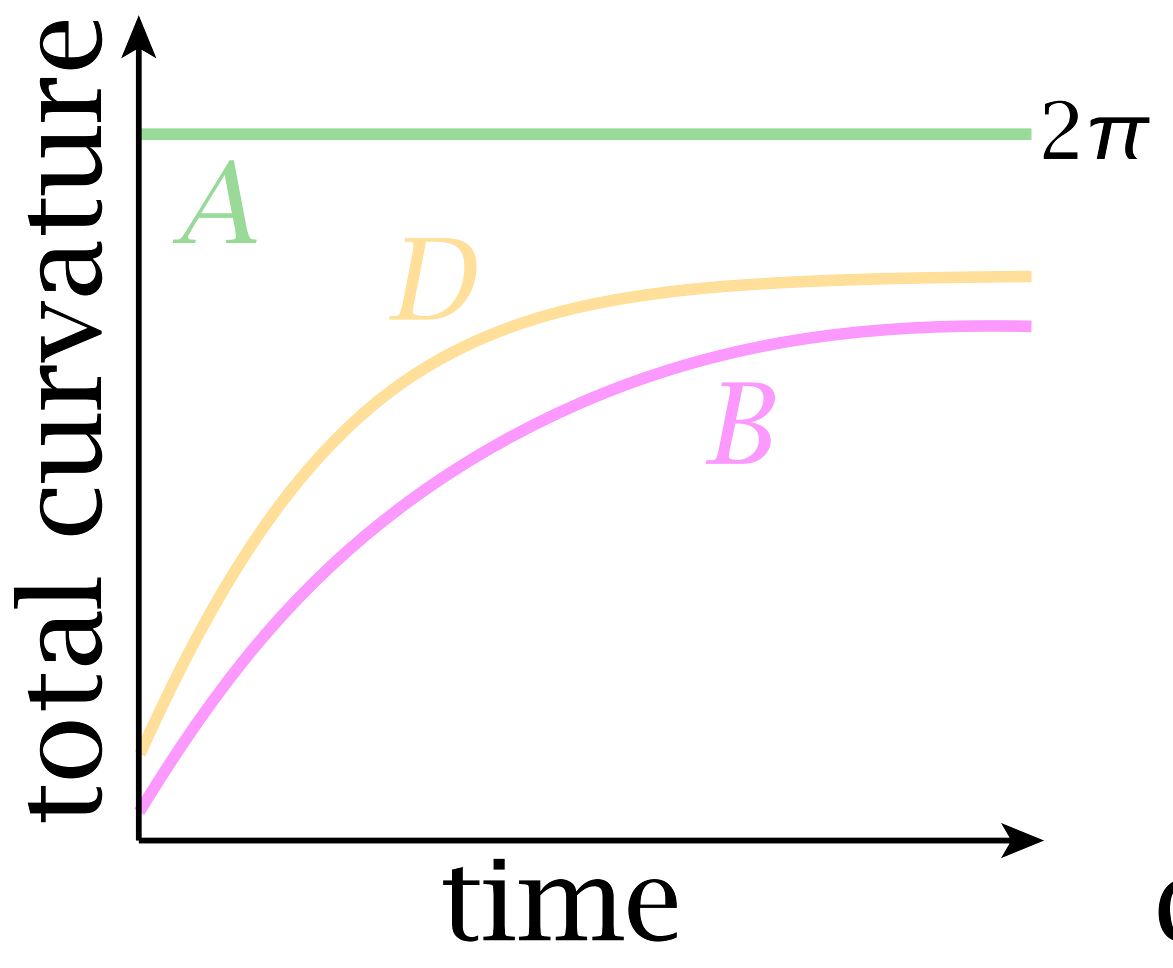


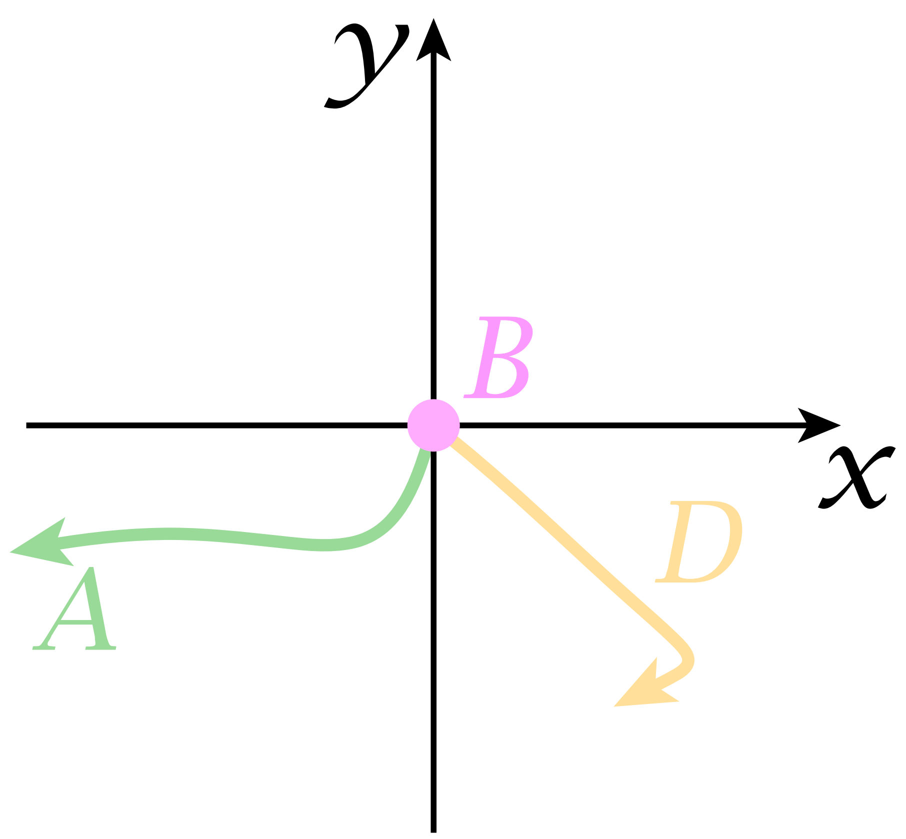
- Shows up in many places (finding silhouettes in images, annealing in metals, closed geodesics on manifolds…)
- Some key properties:
  - (**TOTAL**) Total curvature remains constant throughout the flow.
  - (**DRIFT**) The center of mass does not drift from the origin.
  - (**ROUND**) Up to rescaling, the flow is stationary for circular curves.

Discrete Curvature Flow—No Free Lunch

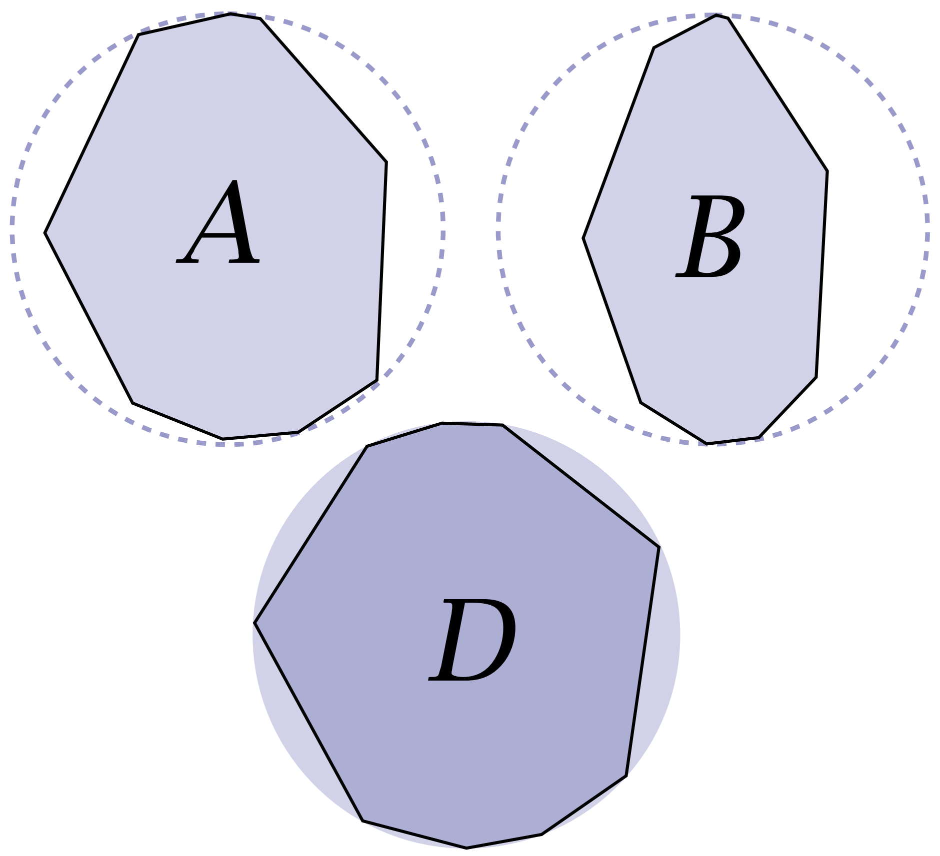
- We can approximate curvature flow by repeatedly moving each vertex a little bit in the direction of the discrete curvature normal: $$\gamma_i^{t+1} = \gamma_i^t + \tau\kappa_i N_i$$
- But **no** choice of discrete curvature simultaneously captures all three properties of the smooth flow*:







drift from center



roundness

	TOTAL	DRIFT	ROUND
κ^A	✓	×	×
κ^B	×	✓	×
κ^D	×	×	✓

*In fact, it's impossible!

No Free Lunch—Other Examples

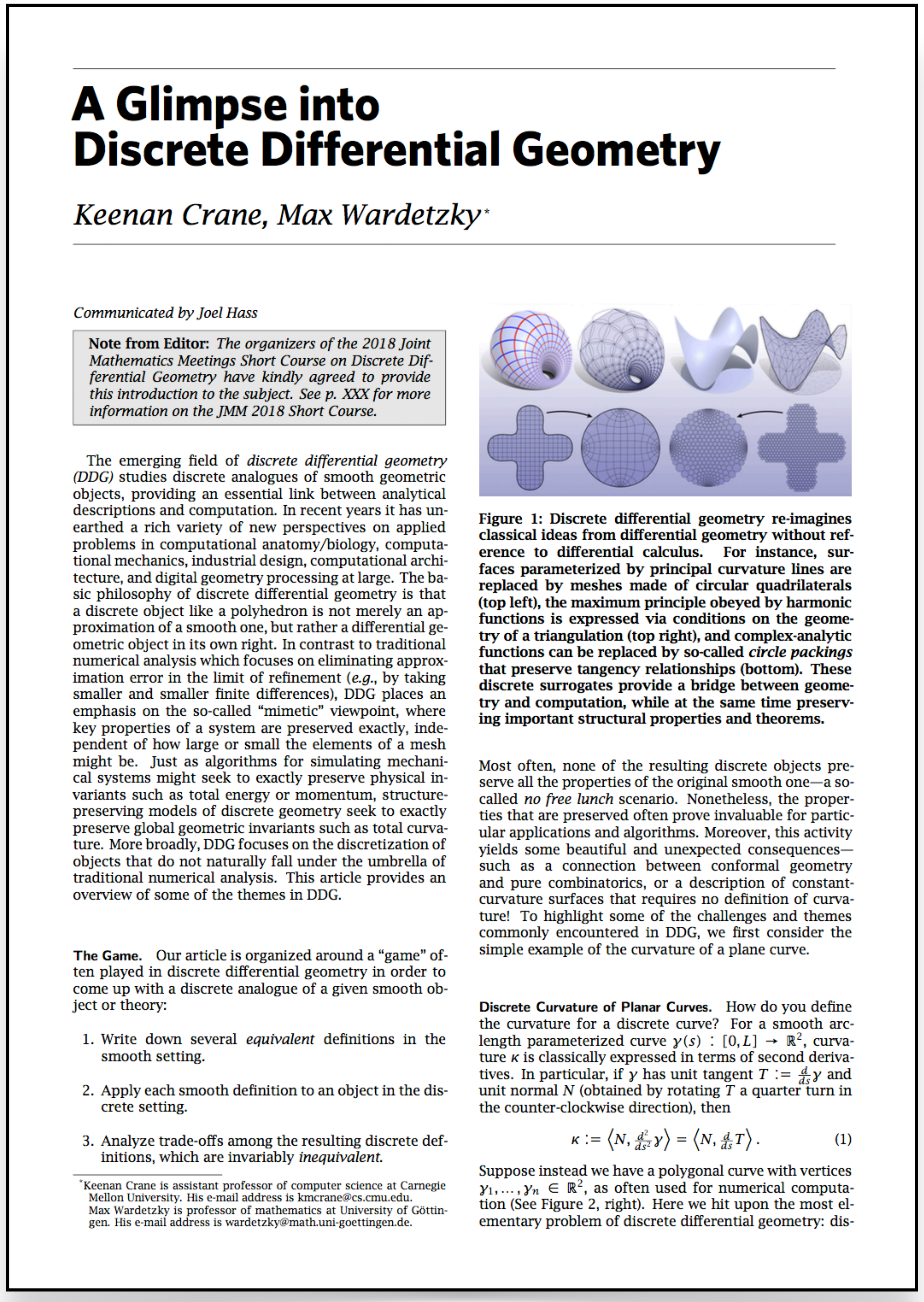
- Beyond this "toy" problem, the *no free lunch* scenario is quite common in discrete differential geometry.
- Many examples (*e.g.*, Whitney-Graustein / Kirchoff analogy for curves; conservation of energy, momentum, and symplectic form for conservative time integrators; discrete Laplace operators…)
- At a more practical level: **The Game** played in DDG often leads to new & unexpected ways of formulating geometric algorithms. (E.g., faster, simpler, clearer guarantees, …)
- Will see *much* more of this as the course continues!

First Reading Assignment

- Overview article from Notices of the AMS:

"A Glimpse into Discrete Differential Geometry"
Crane & Wardetzky (2017)
http://geometry.cs.cmu.edu/glimpse.pdf

- Written for broad mathematics audience
- Quite mathematical for non-math majors!
  - Don't sweat the details
  - Try to get the high-level story
  - *Think of it like learning a foreign language…*



A Glimpse into Discrete Differential Geometry

*Keenan Crane, Max Wardetzky**

Communicated by Joel Hass

Note from Editor: *The organizers of the 2018 Joint Mathematics Meetings Short Course on Discrete Differential Geometry have kindly agreed to provide this introduction to the subject. See p. XXX for more information on the JMM 2018 Short Course.*

The emerging field of *discrete differential geometry (DDG)* studies discrete analogues of smooth geometric objects, providing an essential link between analytical descriptions and computation. In recent years it has unearthed a rich variety of new perspectives on applied problems in computational anatomy/biology, computational mechanics, industrial design, computational architecture, and digital geometry processing at large. The basic philosophy of discrete differential geometry is that a discrete object like a polyhedron is not merely an approximation of a smooth one, but rather a differential geometric object in its own right. In contrast to traditional numerical analysis which focuses on eliminating approximation error in the limit of refinement (*e.g.*, by taking smaller and smaller finite differences), DDG places an emphasis on the so-called "mimetic" viewpoint, where key properties of a system are preserved exactly, independent of how large or small the elements of a mesh might be. Just as algorithms for simulating mechanical systems might seek to exactly preserve physical invariants such as total energy or momentum, structure-preserving models of discrete geometry seek to exactly preserve global geometric invariants such as total curvature. More broadly, DDG focuses on the discretization of objects that do not naturally fall under the umbrella of traditional numerical analysis. This article provides an overview of some of the themes in DDG.

The Game. Our article is organized around a "game" often played in discrete differential geometry in order to come up with a discrete analogue of a given smooth object or theory:

1. Write down several *equivalent* definitions in the smooth setting.
2. Apply each smooth definition to an object in the discrete setting.
3. Analyze trade-offs among the resulting discrete definitions, which are invariably *inequivalent.*

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Figure 1: Discrete differential geometry re-imagines classical ideas from differential geometry without reference to differential calculus. For instance, surfaces parameterized by principal curvature lines are replaced by meshes made of circular quadrilaterals (top left), the maximum principle obeyed by harmonic functions is expressed via conditions on the geometry of a triangulation (top right), and complex-analytic functions can be replaced by so-called *circle packings* that preserve tangency relationships (bottom). These discrete surrogates provide a bridge between geometry and computation, while at the same time preserving important structural properties and theorems.

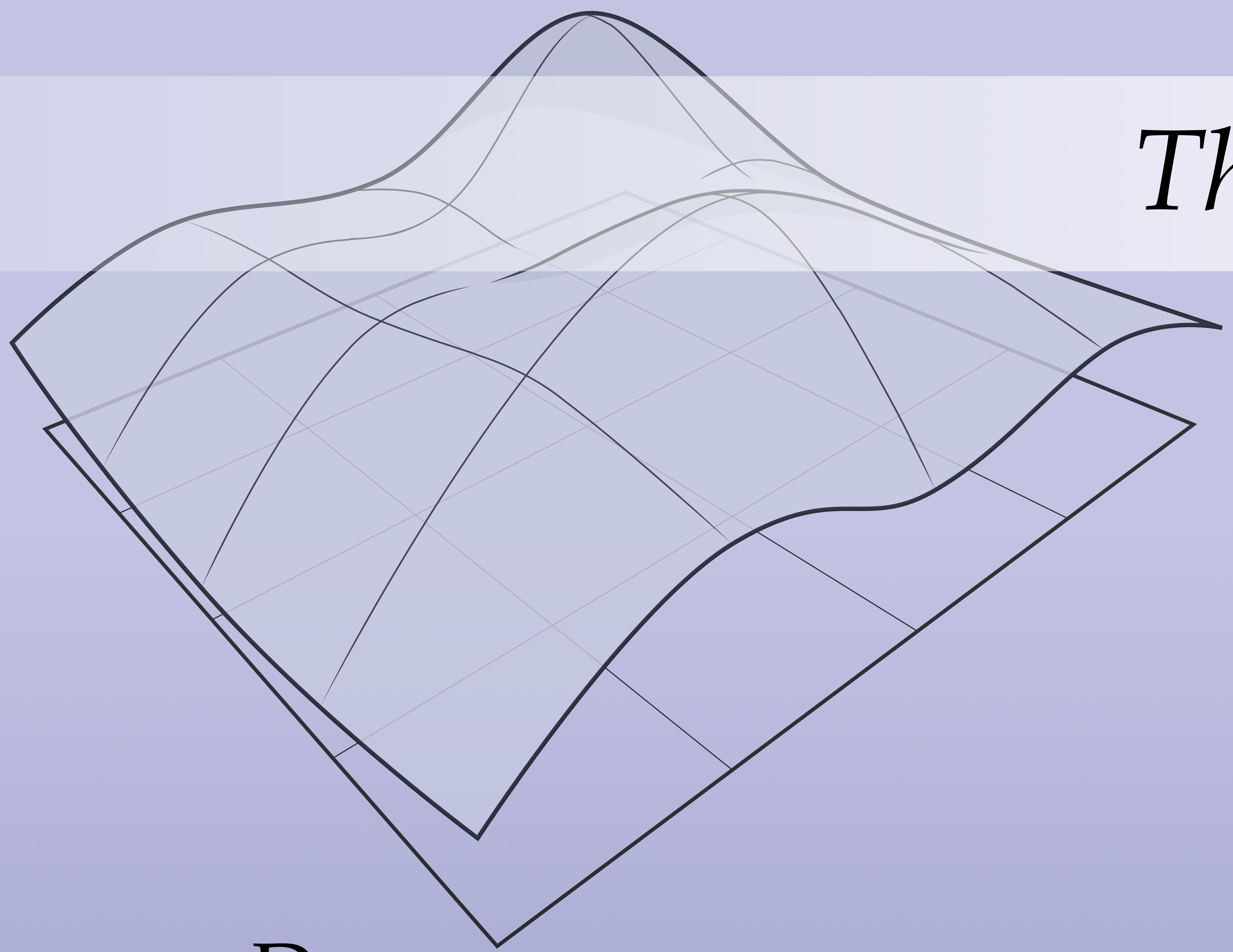
Most often, none of the resulting discrete objects preserve all the properties of the original smooth one—a so-called *no free lunch* scenario. Nonetheless, the properties that are preserved often prove invaluable for particular applications and algorithms. Moreover, this activity yields some beautiful and unexpected consequences—such as a connection between conformal geometry and pure combinatorics, or a description of constant-curvature surfaces that requires no definition of curvature! To highlight some of the challenges and themes commonly encountered in DDG, we first consider the simple example of the curvature of a plane curve.

Discrete Curvature of Planar Curves. How do you define the curvature for a discrete curve? For a smooth arc-length parameterized curve $\gamma(s) : [0,L] \to \mathbb{R}^2$, curvature κ is classically expressed in terms of second derivatives. In particular, if γ has unit tangent $T := \frac{d}{ds}\gamma$ and unit normal N (obtained by rotating T a quarter turn in the counter-clockwise direction), then

$$\kappa := \left\langle N, \tfrac{d^2}{ds^2}\gamma \right\rangle = \left\langle N, \tfrac{d}{ds}T \right\rangle . \quad (1)$$

Suppose instead we have a polygonal curve with vertices $\gamma_1, \ldots, \gamma_n \in \mathbb{R}^2$, as often used for numerical computation (See Figure 2, right). Here we hit upon the most elementary problem of discrete differential geometry: dis-

Thanks!



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

Keenan Crane • CMU 15-458/858B • Fall 2017