Decidable Problems

Represent problem using language

Problem: Is w accepted by DFA B?

A_{DFA} = {<B,w> | *B is a DFA that accepts w*}

< > denotes encoding as string

Language decidable ⟷ Problem decidable

Th. A_{DFA} is a decidable language.

Proof idea: Decider M will simulate D on input w.

If D would accept w, then M accepts; D rejects, M rejects.

q0q1q2q3 # 01# <table># q0#q2q3##0111
0111 # q0
▲

Decidable Problems

A_{NFA} = {<B,w> | B is an NFA, B accepts w}

Th. A_{NFA} is decidable.

Proof idea: Define N to use TM M (last th) as subroutine.

1. *Convert NFA B to DFA D.*
2. *Run TM M on input <D,w>*
3. *If M accepts, then N accepts; M rejects, N rejects.*

q0q1q2q3 # 01#<nd.table>#....
{q0} {q0 q1} .....#<det. table>#

Decidable Problems

E_{DFA} = {<B> | B is a DFA, L(B) is nonempty}

Th. E_{DFA} is decidable.

Lemma: For DFA B, L(B) is nonempty iff B accepts a string of length at most |Q|.

$\leftarrow$ Trivial.

$\rightarrow$ Suppose B accepts some word. Let w be a shortest word accepted by B.
If |w| is $\leq$ |Q|, then we are done. So suppose |w| > |Q|. Consider the sequence of states of B on w.

Decidable Problems

w = w1 w2 w3 .... wm, m > |Q|

$q0 \xrightarrow{w1} q1 \xrightarrow{w2} q2$ ..... $\xrightarrow{wm} qm$

Since there are more that |Q| states in this sequence, there must be a repetition of states, by the pigeonhole principle. But then we can take a snippet z out of w, and get a shorter string that is accepted by B.
Contradiction. Therefore w is of length $\leq$ |Q|.

Back to Theorem:

Decider D will, on input <B>, simulate B on inputs of length 0 to |Q|. If B accepts any string of that length, D accepts. If no such string is found, D rejects. By the Lemma, D accepts E_{DFA}.

Decidable Problems

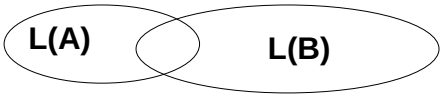
EQ_{DFA} = {<A,B> | A,B are DFA, and L(A) = L(B) }

Th: EQ_{DFA} is decidable.

Proof: We construct a DFA C with

$L(C) = (L(A) \cap \overline{L(B)}) \cup (\overline{L(A)} \cap L(B))$

(symmetric difference)



C can be constructed by TM M using algorithms we developed for closure under $\cup$, $\cap$, $\overline{\ }$

Since L(C) is empty iff. L(A) = L(B), we can use previous theorem for deciding L(C) nonempty.

Decidable Problems for CFG's

A_{CFG} = {<G,w> | G is a CFG, G generates w}

Th. A_{CFG} is decidable.

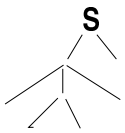
Proof idea: TM could start with start var. S and try all derivations. If w is not in L(G), there will be none!
So trying all derivations to see if get w would not yield decider.
To define a decider, we first put G in Chomsky normal form.
Then we can bound the number of steps in a derivation TM needs to try to $2|w|-1$. QU: WHY?
The decider D will then list all derivations of length 0 to $2|w|-1$. If any derivation generates w, D accepts; ow, reject.

Decidable Problems on CFG's

E_{CFG} = {<G> | G is a CFG, L(G) is nonempty}

Th. E_{CFG} is decidable.

Proof idea: Can't try all strings (won't be decidable). Consider any parse tree for a string w.



If parse tree has path of length > |V|, then there is a repetition of variables along that path, and we can produce a shorter parse tree. We can repeat this process so that there is a parse tree with no path of length > |V|. Therefore, if G generates any string, it generates a string whose parse tree has no path > |V|.

Decidable Problems on CFG's

Th. E_{CFG} is decidable. ***(continued)***

A decider D for this problem will generate a collection C of parse trees on its tape. C will initially contain **S** *. D repeatedly adds to C any tree that can be obtained from one already in C by applying a single rule, such that*

1. *the new tree is not in C*
2. *the new tree does not have any path of len. > |V|*

Since there are only a finite number of trees of fixed length, the TM D will eventually complete C. L(G) is nonempty iff at least one tree in C has only terminals as leaves.

Decidable Problems for CFG's

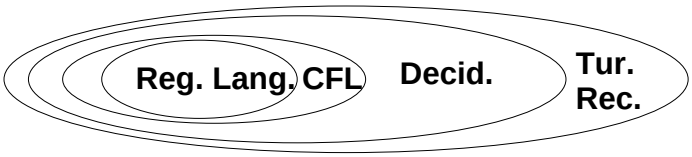
EQ_{CFG} = {<G,H> | G,H are CFG, and L(G) = L(H) }

EQ_{CFG} decidable??

Can't use same idea we used for DFA's (symmetric diff.) because CFL's are NOT closed under complement.

Turns out EQ_{CFG} is NOT decidable. We can't prove it yet.

Th. Every CFL is decidable. (HW 3)



Undecidable Problems

Showed these problems for FA and CFG's decidable:
Acceptance, Emptiness, Equivalence

Qu: What about for TM?

A_{TM} = {<M,w> | M is a TM that accepts w}

Will show that A_{TM} is not decidable (ie, undecidable)

What's wrong with:

Define TM U (Universal TM) on input <M,w>

1. *Simulate M on input w*
2. *If M accepts, U accepts; if M rejects, U rejects.*

U can act like any TM on any input

Is U a decider????

Universal TM U is not a decider

U has no way to determine if M halts on input w.

Lemma: A_{TM} is Turing-recognizable.

Proof: U is the recognizer.

Halting Problem: A_{HALT} = {<B,w> | B halts on input w}

A_{HALT} is decidable $\longleftrightarrow$ A_{TM} is decidable

We show A_{TM} is undecidable (then A_{HALT})

Doesn't mean that we CAN'T determine for a particular <B,w> whether B halts (accepts) on input w.

Does mean that there is no algorithm that works for all input <B,w>. Gives fundamental limitation of Algorithms/TM.

Diagonalization for TM

Suppose there was a super-diagonalizer TM D

D on input <M>:

1. **Simulate M on input <M>**
2. **If M accepts, D rejects; if M rejects, D accepts. If M never halts, D accepts.**

Note D has power to decide if M halts or not on <M>.

QU: Can D be in the list (ie = Mi for some i)?

	M1	***M2***	***M3***	***.....***
<M1>				
<M2>				
<M3>				
:				