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Default Costs and Self-fulfilling Fiscal Limits in a Small Open Economy*

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Abstract

We revisit the link between the risk of sovereign default and default costs. Contrary to prior literature, we show that higher costs of default may result in higher default probabilities, lower bond prices, and fiscal limits that are not pinned down by economic fundamentals. Government debt sustainability depends on private investment behavior, which is affected by expectations about defaults and capital returns. We argue that this feedback loop gives rise to multiple equilibria. In 'bad' equilibria, investors expect default and low capital returns; their low capital investment tightens the governments' fiscal constraints and reduces the probability of debt repayment, validating investor pessimism. In 'good' equilibria, optimistic investors choose higher capital investment; this results in higher future fiscal surpluses, raises the probability of debt repayment and validates investor optimism.

Keywords: sovereign default, default costs, fiscal limit, multiple equilibria

JEL Codes: E62, H30, H60

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1 Introduction

It has long been understood that sovereign defaults must be associated with some sort of losses that governments perceive—otherwise governments would not be able to borrow. Defaults on government debt may lead to political sanctions, damage the policymakers' chances of reelection, and restrict governments' ability to borrow further from the international capital markets.[1] But aside from these consequences, sovereign defaults also appear to have an immediate negative effect on economic activity within the country that results in losses of output (see e.g. Sturzenegger 2004, De Paoli *et al.* 2006). Contemporary models of sovereign default incorporate such economic costs when specifying the government's strategic decision to default on foreign debt.[2] In these setups, higher economic costs of default inevitably lead to lower probabilities of default, as benevolent governments choose to repay their debts more often. This reasoning suggests that high economic punishment from sovereign default perceived *ex ante* should allow for cheaper sovereign borrowing and enhance financial stability within the country.

In this paper we revisit the link between the costs of default and the default risk. Contrary to the preceding literature, we show that higher costs of default may in fact be associated with higher default probabilities and lower bond prices. These results arise because the costs that follow a default on government debt are perceived not only by the government, but also by private agents who invest in domestic capital. When defaults are possible, high potential default costs reduce expected capital returns and induce a capital outflow, depressing the economy and making debt repayment less feasible.

We obtain these results in the context of general equilibrium in a small open economy with capital. The government borrows on the international capital market and uses tax revenues to service its debt. The government repays the debt whenever repayment is feasible, and defaults otherwise. The governments' fiscal surplus is procyclical, in any given period it depends on the capital stock of the economy.[3] A default on government debt triggers a productivity loss (default cost) that shrinks the returns on domestic capital.

Crucially, investment in domestic capital is carried out by autonomous private agents that do not coordinate with each other. This setup gives rise to a feedback loop between the expectations of private agents and the future fiscal surpluses, which generates multiple equilibria. In a 'bad' equilibrium agents perceive default probability to be high. Anticipating capital returns dampened by the costs of default, they curb investment into domestic capital. This, in turn, reduces future output and depresses future fiscal surpluses. Tighter fiscal constraints in the future validate the expectation of a likely default, completing the vicious circle. By contrast, in a 'good' equilibrium agents perceive the chances of debt repayment to be high; they therefore also expect relatively high returns on capital. These expectations boost capital investment

[1] See e.g. Borensztein & Panizza (2009).

[2] See e.g. Aguiar & Gopinath (2006), Arellano (2008), Asonuma & Trebesch (2016), Hatchondo *et al.* (2016) and others. In Mendoza & Yue (2012) these losses arise endogenously, as a result of firms loosing access to foreign credit and being forced to alter the structure of working capital.

[3] There is evidence of fiscal policy procyclicality in developing countries (e.g. Frankel *et al.* 2013), and, more recently, in advanced economies (see Poghosyan & Tosun 2019).

today and result in relatively high output and fiscal surplus in the future; this makes the probability of the government hitting the fiscal constraint relatively low, which then validates investor optimism. We show that the duality described above effectively means that the fiscal limit that the government faces is not pinned down by the fundamentals of the economy. We also show that this particular type of multiplicity only arises when default costs are positive.

Our work is closely related to the literature on defaults in the context of multiple equilibria. Calvo (1988) points out that a given amount of governments' financing needs can be met with either a combination of high bond price and low quantity of new debt (in the 'good' equilibrium), or a low bond price together with high volume of new debt obligations (in the 'bad' equilibrium). When the government cannot commit to a particular bond issuance, this duality gives rise to multiple equilibria and creates a potential for a slow moving debt crisis examined by Lorenzoni & Werning (2019). Cole & Kehoe (2000) emphasize that when the government cannot commit to debt repayment, pessimistic investors anticipating default may refuse to buy newly issued bonds, thereby preventing the government from rolling over the debt. This framework has been further extended and quantitatively assessed by Aguiar *et al.* (2017), Conesa & Kehoe (2017) and Bocola & Dovis (2019). Aguiar & Amador (2018) emphasize the presence of a dynamic multiplicity caused by self-fulfilling beliefs about future sovereign borrowing and bond prices that arise under limited commitment with respect to fiscal policy.

In this paper we develop a distinct mechanism for equilibrium multiplicity and point out that multiple equilibria may arises even when the government can fully commit to specific fiscal policy rule as well as choose a particular bond issuance. In our framework, equilibrium multiplicity is driven by the lack of coordination between individual capital investors which prevails even when the government commits to debt repayment whenever it is feasible. Our framework can be extended to allow for other types of multiplicity. We therefore view the mechanism presented here as complementary to other results developed in the literature. Our argument is particularly relevant for economies that are close to their fiscal limits and are expected to face sizable losses in the event of default.

In this paper we assert that higher costs of default may lead to lower equilibrium bond prices. This is at odds with the literature that models default as a strategic choice made by the government (e.g. Eaton & Gersovitz 1981, Aguiar & Gopinath 2006, Arellano 2008, Yue 2010). Unlike this literature, we focus on the government's capacity (rather than willingness) to repay the debt. A similar approach is taken by Uribe (2006), Bi (2012), Ghosh *et al.* (2013), Sokolova (2015), Bi *et al.* (2018), Reis (2017), Battistini *et al.* (2019), and others; fiscal constraints are also emphasized by works within the scope of the fiscal theory of the price level (see Leeper 1991). Similar to the literature featuring fiscal constraints, our model yields an equilibrium fiscal limit, i.e. the maximum debt level that can be backed by future fiscal surpluses—however, we show that this limit may not be pinned down by the economic fundamentals because of the feedback loop between the capital investment decisions driven by investors' expectations about the future and the future fiscal surpluses.

Recent empirical evidence shows the importance of fiscal limits for both developed and developing countries—the limits that may prevent governments from running surpluses and keeping debts sustainable even when costs of default are substantial and incentives to repay are strong. Trabandt & Uhlig (2011) argue that over the preceding 20 years developed economies have drawn closer to the peaks of their Laffer curves, suggesting limited scope for raising fiscal surpluses via increases in tax rates. Cochrane (2011) asserts that even if an economy is supposed to operate well below the Laffer curve peak, a small rise in the tax rate may cause a prominent slowdown of economic growth thereby reducing future taxable income. But even when the Laffer curve limitations play no significant role, it is still difficult to implement austere fiscal policy in democratic environments without a prominent delay (see Alesina & Drazen 1991). Thus, even when governments want to repay their debts, they may not be able to do so. Our environment allows us to examine debt sustainability when governments are constrained by such limitations.

The results obtained in this paper stem from the assumption that defaults reduce productivity of capital investments. Recent empirical literature appears to support this notion: Hebert & Schreger (2017) show that increases in perceived sovereign default probability in Argentina have prominent negative effect on the value of domestic firms, with a particularly strong effect for exporters and foreign-owned companies; Chari *et al.* (2017) point out a similar negative relationship between government default risk and firm equity returns for Puerto Rico. There is a number of channels through which these adverse effects of default could arise. On the one hand, defaults appear to harm firms' activity on international markets by depressing bilateral trade (Rose 2005, Martinez & Sandleris 2011), reducing foreign credit to the private sector (Arteta & Hale 2008, Zymek 2012) and FDI flows (Fuentes & Saravia 2010). On the other hand, defaults have strong adverse effects on financial intermediation within the country.[4] Gennaioli *et al.* (2018) study a sample of 20,000 banks in 191 countries for the period 1998-2012 and conclude that, on average, banks hold about 9% of their assets in government bonds, and that during sovereign default the contraction of lending is especially severe for banks with high domestic bond holdings. Popov & Horen (2015) show that in 2010 during the European debt crisis, banks holding substantial amounts of government bonds perceived to be risky reduced syndicated lending by 21.3% relative to banks with low levels of exposure; such credit crunches have been shown to have a particularly severe effect on small businesses, prompting them to curtail investments and employment (Iyer *et al.* 2014, Dwenger *et al.* 2020). Andrade & Chhaochharia (2018) find that the stocks of firms vulnerable to disruptions of financial intermediation are priced as if the market anticipates a 12% reduction in these firms' value following a sovereign default.

We conclude that high perceived costs of sovereign default make governments that operate close to their respective fiscal limits vulnerable to self-fulfilling crises of investor confidence. In such an environment, a decrease in perceived default costs can enhance debt sustainability. The perceived default costs related to a reduction in firms' international activity could perhaps be influenced by the stance of the IMF in its role as an arbiter in debt renegotiations, as

[4]This credit channel is explored in some recent literature modeling sovereign default (e.g. Gennaioli *et al.* 2014, Bocola 2016, Arellano *et al.* 2017).

well as by the conduct of the government officials.[5] Perceived costs related to the anticipation of the domestic credit crunch could potentially be mitigated if the central bank commits to accommodative policies in the event of a fiscal crisis. The famous 'whatever it takes' statement by Mario Draghi together with the Outright Monetary Transactions program for conditional purchases of risky sovereign bonds can be viewed as an example of such an intervention. Reis (2017) argues that sovereign default costs can be reduced through Quantitative Easing (QE) that could alter the composition of banks' balance sheets away from risky government bond holdings, which in turn would likely result in a less severe credit crunch in the event of default. In our model, committing to such measures *ex ante* can reduce the perceived default costs and loosen the fiscal constraints the government is likely to face in the future.

The paper is organized as follows. In Section 2 we describe the model setup. In Section 3 we demonstrate our core result in a two-period perfect foresight equilibrium context. In Section 4 we study its dynamic implications; in subsection 4.1 we derive fiscal limits for an infinite-horizon version of the model and show that there are multiple limits consistent with sequential equilibrium; in subsection 4.2 we construct multiple fiscal limits for a recursive equilibrium structure featuring sunspots. In Section 5 we examine equilibrium multiplicity under uncertainty over fundamentals of the economy. Section 6 concludes.

2 The model

We consider a model of a small open economy featuring competitive domestic firms, the government that follows a fiscal rule, domestic households and foreign households that are risk-neutral. Households invest in capital and trade in assets with foreign creditors. Firms employ capital to produce goods; capital productivity that firms are facing depends on whether or not the country is currently in default, as default triggers a productivity loss. The government trades in bonds with risk-neutral foreign investors, generates revenues by taxing capital and labor income and makes transfer payments to the households.

Whether or not the government repays its debt depends on the current fiscal surpluses as well as on the revenue from the auction of bonds that the government can generate. The bond pricing schedule faced by the government depends on what economic agents expect the fiscal surpluses to be in the future. Fixed losses to productivity triggered by default give rise to equilibrium multiplicity: when 'optimistic' private agents perceive the chances of full repayment to be high, they invest heavily in capital which results in higher future revenues from tax collection that validate the original 'optimistic' expectations. Alternatively, investor pessimism leads to capital flight, lower future GDP, lower revenues from tax collection and a lower debt repayment rate that validate investors' pessimism.

[5] There is evidence suggesting that output losses during default are more severe for governments that behave coercively toward foreign creditors, see Trebesch & Zabel (2017)

2.1 The Government

Each period the government taxes capital income $r_t^K k_t$ and labor income $w_t l_t$, collecting tax revenue $\tau^K r_t^K k_t + \tau^w w_t l_t$. The government also pays households tr_t in transfers. We assume the tax rates τ^K and τ^w to be constant and positive, and the transfers tr_t to be exogenously determined. In Section 3 and Section 4 the transfers follow a deterministic path; Section 5 introduces uncertainty over future transfer payments, e.g. arising prior to an election.[6]

In any given period, the fiscal surplus equals

$$S_t \equiv \tau^K r_t^K k_t + \tau^w w_t l_t - tr_t. \tag{1}$$

The government trades in one-period bonds with foreign risk-neutral creditors to cover its operational deficit:

$$q_t B_{t+1} = B_t - S_t, \tag{2}$$

where B_t is debt issued in period $t-1$ and q_t is the bond price for debt issued in period t. [7] When issuing new bonds the government faces a bond pricing schedule $Q^t(B)$; the government selects B_{t+1} and a corresponding $q_t = Q^t(B_{t+1})$ to satisfy (2). If there is more than one debt level at which (2) can be satisfied, the government selects the smallest bond issuance. We therefore assume that the government can commit to issuing a specific level of debt.[8]

The repayment of debt B_t is feasible if there exists a B_{t+1} for which the budget constraint is satisfied. We assume that the government repays the debt whenever possible, that is, when $B_t \leq \max_{B_{t+1}}\{Q^t(B_{t+1})B_{t+1}\} + S_t$. Conversely, if the maximum amount of revenues that could be raised from auctioning bonds is not enough to cover the gap between the pre-existing debt and the current fiscal surplus, the government is forced to default. In the event of default the government permanently looses access to the international financial market and transfers a share γ of its discounted sum of future fiscal surpluses to the foreign creditors as partial payment. The debt recovery rate in period t is therefore

$$\chi_t = \begin{cases} 1, & if \;\; B_t \leq \max_{B_{t+1}}\{Q^t(B_{t+1})B_{t+1}\} + S_t; \\ \gamma \cdot \frac{\sum_{i=0}^{\infty} S_{t+i}/(1+r)^i}{B_t}, & if \;\; B_t > \max_{B_{t+1}}\{Q^t(B_{t+1})B_{t+1}\} + S_t. \end{cases} \tag{3}$$

where $0 \leq \gamma \leq 1$. The risk-neutral foreign creditors purchase government debt at the bond price

[6] We use this simple structure for fiscal policy for ease of exposition; our results can be generalized to the case in which tax rates and transfer payments depend on the level of debt—as long as there exists an upper bound on tax rates and a lower bound on transfer payments. While this more robust fiscal policy would affect quantitative implications of the model, the qualitative implications will remain unchanged, as it is the behavior of the government finances at the boundary that would determine the fiscal limits.

[7] In this paper we abstract from questions related to the effects of debt maturity structure and long-term debt. These questions are explored in, e.g. Hatchondo & Martinez (2009), Hatchondo *et al.* (2016) and Aguiar & Amador (2018).

[8] Relaxing this assumption in our model would yield more multiplicity in equilibrium outcomes. See Lorenzoni & Werning (2019) for a detailed discussion of this assumption in the context of multiple equilibria.

that matches expected returns on a government bond with returns on risk-free investment:

$$q_t = \frac{E_t[\chi_{t+1}]}{1+r}. \tag{4}$$

where r is the risk-free world interest rate.

2.2 Households

We model two groups of households: domestic and foreign, and assume that capital investment is carried out by the risk-neutral foreign households. Foreigners are assumed to have unlimited access to the international capital market. By contrast, domestic households do not have access to financial instruments that could hedge them from the domestic income risk.[9] We discuss the significance of these assumptions in Online Appendix B.

Domestic households. A representative domestic household inelastically supplies a unit of labor ($l_t = 1$) to domestic firms and receives labor income. The household also receives transfers tr_t from the government and pays the labor income tax. Its budget constraint is

$$c_t \leq (1-\tau^w)w_t + tr_t. \tag{5}$$

The expected lifetime utility of the domestic household is given by $E_t[\sum_{i=0}^{\infty} \beta^i u(c_{t+i})]$, where β is the subjective discount factor of the domestic households.

Foreign Households. A representative foreign household is risk-neutral; each period it receives an endowment $\tilde{y}_t$ and trades with other risk-neutral foreign investors in risk-free assets, $\tilde{s}_t$, at the risk-free world interest rate r. Foreign households invest in domestic capital k_t and receive interest income $r_t^K k_t$ on which they pay taxes to the domestic government. They also trade in the domestic governments' bonds. The households' flow budget constraint is

$$\tilde{c}_t + k_{t+1} + \tilde{s}_{t+1} + q_t B_{t+1} \leq \left[1-\delta + r_t^K(1-\tau^K)\right] k_t + (1+r)\tilde{s}_t + \tilde{y}_t + \chi_t B_t \tag{6}$$

where δ is the depreciation rate, and τ^K is the tax rate on capital.[10] Foreign households choose $\tilde{c}_t$, k_{t+1}, B_{t+1} and $\tilde{s}_{t+1}$ to maximize their expected welfare, $E_t[\sum_{i=0}^{\infty} \tilde{\beta}^i \tilde{c}_{t+i}]$, where $\tilde{\beta}$ is the subjective discount factor of the foreign households. The first-order conditions yield:

$$1 = \tilde{\beta}[1-\delta + E_t[r_{t+1}^K](1-\tau^K)], \tag{7}$$

$$1 = \tilde{\beta}[1+r], \tag{8}$$

$$q_t = \frac{E_t[\chi_{t+1}]}{1+r}. \tag{9}$$

[9]The assumption that it is only the foreigners who invest in domestic capital is not strictly necessary for the results obtained in this paper, and neither is the assumption about domestic households not having access to the international capital markets, see details in Online Appendix B. The argument developed in this paper is most appropriate for countries where either the foreign creditors have access to the domestic capital markets, or the domestic capital investors can avoid being taxed on their interest earnings received from abroad.

[10]Here, we assume that capital depreciation is not tax-deductible, but this assumption is immaterial for the arguments developed in this paper, which would also hold under the alternative assumption.

Equation (8) places a restriction on model parameters, implying that the subjective discount factor of foreign households must equal $\frac{1}{1+r}$. Moving forward we will assume this to be the case.

Combining (7) and (8) we find that the expected return on domestic capital corrected for tax and depreciation, $E_t[r^K_{t+1}](1-\tau^K)-\delta$, must match the foreign risk-free rate r:

$$E_t[r^K_{t+1}](1-\tau^K)-\delta=r. \tag{10}$$

2.3 Firms

The firms in the economy are infinitesimally small and competitive; each representative firm is endowed with a production technology:

$$Y_t = A_t k_t^{\alpha} l_t^{1-\alpha}, \tag{11}$$

where A_t is the overall level of productivity. Absent default, A_t is at the underlying productivity level a_t. Default is associated with a permanent productivity loss of $\psi\cdot 100$ percent.[11] The overall productivity is thus given by

$$A_t = \begin{cases} a_t, & \textit{if} \ \ \chi_i = 1, \forall i \leq t; \\ a_t(1-\psi), & \textit{else}. \end{cases} \tag{12}$$

In equilibrium firms' labor demand equals the inelastic labor supply $l_t = 1$, and firms' profit maximization yields:

$$r^K_t = \alpha A_t k_t^{\alpha-1}, \tag{13}$$

$$w_t = (1-\alpha)A_t k_t^{\alpha}. \tag{14}$$

2.4 Fiscal surplus and expectations

The fiscal surplus crucially depends on the amount of capital investment in the economy: in period t capital stock k_t determines households' wage w_t through (14) and the interest on capital r^K_t through (13). Substituting profit maximization conditions into the definition of the fiscal surplus given by (1) we can rewrite:

$$S_t = \tau A_t[k_t]^{\alpha} - tr_t \tag{15}$$

where $\tau \equiv [\alpha\tau^K + (1-\alpha)\tau^w]$. Higher capital stock is associated with higher output, and since tax revenues comprise a share τ of the GDP, higher accumulated capital in period t also translates into higher fiscal surplus in t. Importantly, capital stock in period t is pre-determined by

[11] Our results crucially depend on there being a one-period loss to productivity; the simplifying assumption we make her of the productivity loss being permanent has quantitative implications but does not change any of the qualitative results. Default cost in the form of a productivity loss has previously been used in Cole & Kehoe (2000), Da-Rocha *et al.* (2013), Conesa & Kehoe (2017) and others. Alonso-Ortiz *et al.* (2017) estimate the productivity loss to be around $3.70-5.88\%$.

capital investment choices made by households in period $t-1$. This capital investment decision primarily depends on expectations about future productivity formed in the preceding period:

$$k_t = \left[\frac{\alpha E_{t-1}[A_t](1-\tau^K)}{r+\delta}\right]^{\frac{1}{1-\alpha}}. \tag{16}$$

Because productivity A_t may be depressed by a default penalty, the expected productivity $E_{t-1}[A_t]$ crucially depends on whether or not households expect default on sovereign debt.

This is the channel through which self-fulfilling prophecies arise. Households that perceive the probability of default to be low will simultaneously expect higher productivity in the future and invest more in capital today through (16). Higher capital would lead to higher expected fiscal surplus through (15) and higher probability of full repayment—validating the original optimistic expectation. Such optimism would also translate into higher bond prices today due to higher expected recovery rates. By contrast, when investors expect default to occur with high probability, they anticipate productivity to be lower due to the default penalty ψ and thereby invest less in domestic capital; this leads to lower future revenues from tax collection and higher probability of default which validates investor pessimism and feeds into current bond prices.

3 Multiple equilibria with perfect foresight: a two-period example

In this section we examine a deterministic version of the model discussed above, which features an absorbing state: suppose there is a period $T < \infty$ such that in all periods following T productivity and transfers are assumed to equal the levels achieved in period T. If the government does not default in period T, then the debt can be rolled over indefinitely at the risk-free bond price, i.e. there is no threat of 'bad' self-fulfilling equilibria after period T. In this section we consider the dynamics between periods $T-1$ and T and show that the model gives rise to multiple equilibria with perfect foresight.

In period $T-1$ the government makes a decision about issuing new debt B_T given a bond pricing schedule $Q^{T\text{-}1}(B)$. In equilibrium, bond prices along the schedule $Q^{T\text{-}1}(B)$ must be consistent with rational expectations about whether the government will repay its debt. In this subsection we show that an equilibrium bond pricing schedule satisfying these requirements exists. We also assert that whenever the cost of default is positive, there exist more than one such schedule—in other words, there are multiple equilibria.

The debt acquired in period $T-1$ will be repaid in T whenever repayment is feasible. Full repayment is feasible if the government is able to roll over the debt indefinitely at the risk-free interest rate given the constant fiscal surplus S_T:

$$B_T \leq \frac{1+r}{r} S_T. \tag{17}$$

The fiscal surplus S_T on the right-hand side depends on current productivity, as well as on capital investment decisions made in period $T-1$ that are guided by period $T-1$ expectations about period T productivity. We introduce the following notation:

$$S_T = \tau A_T[k_T(A_T^e)]^\alpha - tr_T \equiv S(A_T, k_T(A_T^e), tr_T), \tag{18}$$

where $k_T(A_T^e)$ is determined through (16) denoting $A_T^e \equiv E_{T-1}[A_T]$ for brevity. We now give definition to the equilibrium bond pricing schedule $Q^{T\text{-}1}(B)$.

Definition 1A. Equilibrium bond pricing schedule $Q^{T\text{-}1}(B)$: perfect foresight. *The equilibrium bond pricing schedule is a function $Q^{T\text{-}1}(B)$ such that for each $B_T \in B$ the bond price $q_{T-1} \in Q^{T\text{-}1}(B_T)$ satisfies*

$$q_{T-1} = \frac{\chi_T}{1+r} \tag{19}$$

where

$$\chi_T = \begin{cases} 1, & if \quad B_T \le S(a_T, k_T(A_T^e), tr_T) \cdot \frac{1+r}{r}; \\ \frac{\gamma \cdot S(a_T[1-\psi], k_T(A_T^e), tr_T) \cdot (1+r)/r}{B_T}, & if \quad B_T > S(a_T, k_T(A_T^e), tr_T) \cdot \frac{1+r}{r}, \end{cases} \tag{20}$$

and

$$k_T(A_T^e) = \left[\frac{\alpha A_T^e(1-\tau^K)}{r+\delta}\right]^{\frac{1}{1-\alpha}} \tag{21}$$

with

$$A_T = \begin{cases} a_T, & if \quad B_T \le S(a_T, k_T(A_T^e), tr_T) \cdot \frac{1+r}{r}; \\ a_T(1-\psi), & if \quad B_T > S(a_T, k_T(A_T^e), tr_T) \cdot \frac{1+r}{r}. \end{cases} \tag{22}$$

and

$$A_T^e = A_T. \tag{23}$$

Under perfect foresight rational foreign creditors correctly predict the recovery rate when pricing bonds through (19). Condition (20) states that the recovery rate equals 1 (i.e. the government fully repays its debt) if the debt can be backed by the stream of future fiscal surpluses; this stream is affected by the capital $k_T(A_T^e)$. The capital investment decision depends on expectations A_T^e through (21); (23) requires these expectations to match the actual realized productivity. Whether or not this productivity is affected by the default penalty ψ is contingent on whether or not there is default (as stated in 22).

Finding $Q^{T\text{-}1}(B)$ means assigning to each $B_T \in B$ a price q_{T-1} that satisfies *Definition 1A*. This task is simplified by the fact that A_T^e, the expected productivity, can only take one of two values: a_T or $a_T[1-\psi]$. If investors expect $A_T^e = a_T$, they must also anticipate full debt repayment in T and high productivity unaffected by default penalties; when they expect $A_T^e = a_T[1-\psi]$, they would also predict default in T accompanied by a productivity penalty. Armed with this insight, we proceed as follows.

We start by picking one of two values, a_T or $a_T[1-\psi]$, as an initial guess for A_T^e; we then use this guess together with (21) to characterize the capital investment decision. Given $k_T(A_T^e)$, we check whether our guess about A_T^e complies with rational expectations by verifying that (22) selects the matching productivity value. It is straightforward to show that at least one of the two guesses will always be correct: if under $A_T^e = a_T$ (22) selects $A_T = a_T(1-\psi)$, then it would also select $A_T = a_T(1-\psi)$ under $A_T^e = a_T(1-\psi)$ because the function $S(a_T, k_T(A_T^e), tr_T)$ is increasing in A_T^e; similarly, if under $A_T^e = a_T(1-\psi)$ (22) chooses $A_T = a_T$, then $A_T = a_T$ would also be chosen under $A_T^e = a_T$. After finding a valid A_T^e, we can determine the corresponding recovery rate through (20) and the bond price q_{T-1} through (19). Note that the reasoning above suggests that we can find a valid q_{T-1} for *any* B_T, which means that a bond pricing function $Q^{T\text{-}1}(B)$ satisfying *Definition 1A* can be constructed for any set of model parameters.

Lemma 1. *The bond pricing schedule* $Q^{T\text{-}1}(B)$ *that satisfies Definition 1A exists.*

We now follow the steps outlined above to locate *all* q_{T-1} that satisfy *Definition 1A* for each B_T. Intuitively, for low levels of debt agents should rationally expect full repayment and price the bonds at a high risk-free price; for high levels of debt they should expect default, and bonds should trade at a lower price. In what follows we derive the threshold values of debt that delineate these intervals and show that the intervals intersect.

Suppose agents expect full debt repayment and $A_T^e = a_T$. They would then choose capital $k_T(A_T^e) = k_T(a_T)$ through (21). The expectation of full repayment and $A_T^e = a_T$ is rational if (20) and (22) select $\chi_T = 1$ and $A_T = a_T$, which will be the case if the debt does not exceed the discounted sum of future fiscal surpluses under the capital choice $k_T(a_T)$:

$$B_T \leq S(a_T, k_T(a_T), tr_T)\frac{1+r}{r} \equiv \bar{B}_T. \tag{24}$$

When debt exceeds $\bar{B}_T$, the expectation of debt repayment and $A_T^e = a_T$ cannot be validated; when $B_T \leq \bar{B}_T$, the equilibrium with full debt repayment can be constructed, and in this equilibrium the expectation $A_T^e = a_T$ is rational. We therefore arrive at a conclusion that there is an upper bound on debt, $\bar{B}_T$, beyond which we cannot construct an equilibrium bond price consistent with an anticipated full repayment. At the same time, for any $B_T \leq \bar{B}_T$ this can be done: the bond price $q_{T-1}^f = \frac{1}{1+r}$ that anticipates full repayment together with the expected productivity $A_T^e = a_T$ satisfy *Definition 1A*.

Suppose agents expect default and $A_T^e = a_T[1-\psi]$. They then choose capital $k_T(A_T^e) = k_T(a_T[1-\psi])$ through (21). These expectations are validated if (20) and (22) choose $\chi_T < 1$ and $A_T = a_T(1-\psi)$, which will happen if debt is too high relative to the corresponding fiscal surplus:

$$B_T > S(a_T, k_T(a_T[1-\psi]), tr_T)\frac{1+r}{r} \equiv \underline{B}_T. \tag{25}$$

There is therefore a threshold $\underline{B}_T$ such that whenever debt exceeds this threshold, expectations of default with productivity $A_T^e = a_T[1-\psi]$ can be an equilibrium outcome. The corresponding bond price that satisfies *Definition 1A* equals $q_{T-1}^d = \frac{\gamma \cdot S(a_T[1-\psi], k_T(a_T[1-\psi]))/r}{B_T}$.

The results discussed above crucially depend on how the *ex ante* expectation of a productivity loss in conjuncture with default affects the capital investment decision. When investors expect debt repayment, they also anticipate returns on capital investments to be unhindered by the default penalty; they accordingly make substantial capital investments which translate into higher future capital stock, output and fiscal surpluses. When $B_T \leq \bar{B}_T$, these relatively high fiscal surpluses are enough to ensure full debt repayment; this validates investors' expectations of full debt repayment. At the same time, when investors expect future fiscal surpluses to be insufficient for full debt repayment, they anticipate capital returns diminished by default penalties and underinvest in capital; this leads to lower output and fiscal surpluses. When $B_T > \underline{B}_T$, the diminished surpluses make debt repayment infeasible, which validates the original pessimistic expectation.

Crucially, whenever the productivity loss triggered by default is positive (i.e. $\psi > 0$) we have $\underline{B}_T < \bar{B}_T$.[12] This means that the debt intervals for which an equilibrium with default and an equilibrium with repayment can be constructed intersect. Armed with this insight and taking into account the above discussion about debt thresholds, we arrive at the following results.

First, for each $B_T \leq \underline{B}_T$ it is also true that $B_T < \bar{B}_T$, and therefore an equilibrium with anticipated full repayment and bond price q^f_{T-1} can be constructed. At the same time, an expectation of default cannot be validated as an equilibrium outcome, because even if in period $T-1$ investors expect default penalties ($A^e_T = a_T[1-\psi]$) and underinvest in capital choosing $k_T(a_T[1-\psi])$, debt repayment is still feasible in T (as condition 25 is violated), which means that the pessimistic expectation of default cannot be rational, and that the price q^d_{T-1} would not arise in equilibrium.

Second, for each $B_T > \bar{B}_T$ it is also true that $B_T > \underline{B}_T$, hence we can construct an equilibrium with anticipated default and a corresponding bond price q^d_{T-1}. At the same time, an equilibrium with anticipated repayment cannot be constructed: even if *ex ante* investors optimistically expect high productivity ($A^e_T = a_T$) and invest $k_T(a_T)$, debt repayment is still infeasible *ex post*, as debt exceeds fiscal surpluses, violating condition 24. Therefore, for this region the price q^f_{T-1} is not consistent with equilibrium.

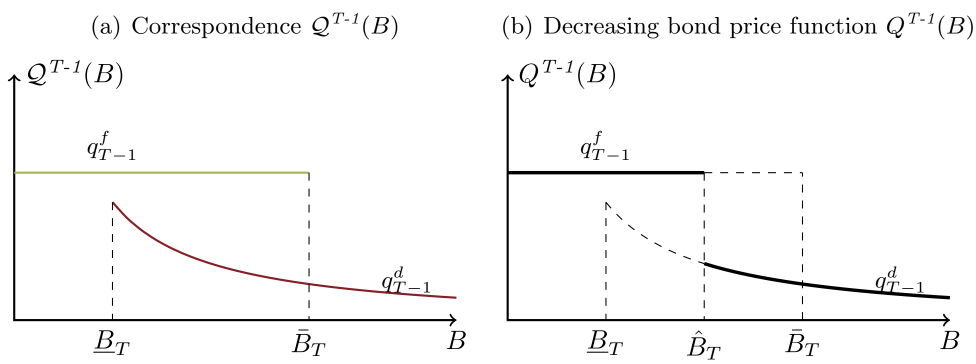
Finally, for each $B_T \in (\underline{B}_T, \bar{B}_T]$ we can construct *both* an equilibrium with anticipated default and an equilibrium with anticipated repayment. On the one hand, $B_T > \underline{B}_T$ means that if in period $T-1$ investors expect default with productivity penalties ($A^e_T = a_T[1-\psi]$) and invest $k_T(a_T[1-\psi])$ in capital, then in period T, due to low fiscal surplus, debt repayment is infeasible (as condition 25 holds) and the government defaults, which validates investor pessimism. On the other hand, $B_T \leq \bar{B}_T$ implies that if in period $T-1$ investors expect repayment and high productivity ($A^e_T = a_T$) choosing $k_T(a_T)$, then high surpluses in period T make repayment feasible, validating investor optimism. Therefore, for each $B_T \in (\underline{B}_T, \bar{B}_T]$ there exist two distinct bond prices that satisfy *Definition 1A*: q^f_{T-1}, a bond price consistent with rational

[12] To see this, note that, first, capital investment described in (21) is strictly increasing in expected productivity A^e_T, which means $k_T(a_T[1-\psi]) < k_T(a_T)$; second, the fiscal surplus is strictly increasing in the capital stock, implying $S(a_T, k_T(a_T[1-\psi]), tr_T) < S(a_T, k_T(a_T), tr_T)$ and therefore $\underline{B}_T < \bar{B}_T$.

expectations of full debt repayment and $A^e_T = a_T$, and q^d_{T-1}, a bond price that corresponds to a rationally anticipated default with $A^e_T = a_T[1-\psi]$.

Denote by $\mathcal{Q}^{T\text{-}1}(B)$ the correspondence that collects *all* bond prices q_{T-1} that satisfy *Definition 1A* for each $B_T \in B$. Figure 1(a) depicts this correspondence; the correspondence assigns both q^f_{T-1} and q^d_{T-1} to each $B_T \in (\underline{B}_T, \bar{B}_T]$, as both bond prices can be supported as equilibrium under perfect foresight. The equilibrium bond pricing *function* $Q^{T\text{-}1}(B)$ can then be constructed by choosing a selection from $\mathcal{Q}^{T\text{-}1}(B)$ that assigns unique values to each B_T (e.g. see Figure 1b). Because $\mathcal{Q}^{T\text{-}1}(B)$ has two distinct bond prices corresponding to each $B_T \in (\underline{B}_T, \bar{B}_T]$, we can construct an infinite number of distinct selections $Q^{T\text{-}1}(B)$ from $\mathcal{Q}^{T\text{-}1}(B)$, i.e. there are in fact infinitely many bond pricing schedules that can be supported as equilibrium with perfect foresight.

Figure 1: Constructing bond pricing schedule



Proposition 1. *For $\psi > 0$ there exist infinitely many bond pricing schedules $Q^{T\text{-}1}(B)$ that satisfy Definition 1A. The bond prices along these schedules 1) equal q^f_{T-1} for all $B_T \leq \underline{B}_T$ 2) assume one of two values for each $B_T \in (\underline{B}_T, \bar{B}_T]$, either q^f_{T-1} or $q^d_{T-1} < q^f_{T-1}$ 3) equal q^d_{T-1} for each $B_T > \bar{B}_T$, where*

$$
\begin{aligned}
q^f_{T-1} &= 1/(1+r), \\
q^d_{T-1} &= \frac{\gamma \cdot S(a_T[1-\psi], k_T(a_T[1-\psi]), tr_T)/r}{B_T}.
\end{aligned}
$$

Proof. see Appendix A.

To see the intuition behind this result, suppose the economy is very close to its fiscal limit: investors choose capital $k_T(a_T)$ expecting debt repayment, but the resulting fiscal surplus is just enough to cover the debt, i.e. $B_T = \bar{B}_T$. At this boundary any decrease in fiscal surpluses would make debt no longer sustainable—therefore, if capital investment drops below $k_T(a_T)$, default is guaranteed. This opens a door for a self-fulfilling debt crisis: if investors start expecting default and reduce their capital investment to $k_T(a_T[1-\psi])$, then the debt $B_T = \bar{B}_T$ will not be repaid, and the expectations of default will be validated. Because of the wedge introduced by the default cost to productivity this logic can be extended to lower debt levels that are close to $\bar{B}_T$; however, for debt levels at or below $\underline{B}_T$ this is no longer true: even if capital investment falls to the 'pessimistic' level $k_T(a_T[1-\psi])$ reducing the fiscal surplus, the debt can still be repaid, which means that such investor pessimism cannot be a valid equilibrium outcome. Therefore, the multiplicity brought forth by the presence of default costs is most acute when the economy is close to its fiscal capacity. In other words, we assert that the fiscal limit cannot be pinned down by the fundamentals of the economy, as it crucially depends on investor sentiment.

The debt interval for which multiple equilibria are possible increases with the value of default cost ψ. Intuitively, a high productivity loss in the event of default means that there is a large gap between capital investments chosen when investors anticipate default and when they expect repayment. This implies that even relatively low debt levels can become unsustainable as expectations of default cause a dramatic decline in fiscal surpluses. *Corollary 1A* below establishes this result.

Corollary 1A. *The lower bound of the multiplicity interval, $\underline{B}_T$, decreases with ψ, i.e. higher default costs expand the region of debt in which default can be an equilibrium outcome.*
Proof. see Appendix A.

It is plain to see that when there is no loss of productivity associated with default, this multiplicity does not arise. At $\psi = 0$, capital investment decisions with- and without default is exactly the same; there is therefore no ambiguity in what the fiscal limit might be: investor sentiment pertaining to default has no bearing on the future fiscal surplus.

We have established that there are many possible bond pricing schedules that satisfy *Definition 1A*; however, not all such schedules are equally intuitive. For example, consider a function $Q^{T\text{-}1}(B)$ that attains q^f_{T-1} on $B_T \leq \underline{B}_T$, returns q^d_{T-1} for $B_T > \bar{B}_T$, but switches back and forth between q^d_{T-1} and q^f_{T-1} on $(\underline{B}_T, \bar{B}_T]$. This schedule technically satisfies *Definition 1A*—however, it is counterintuitive, as there will be values $B^*_T > B^{**}_T$ such that $Q^{T\text{-}1}(B^*_T) > Q^{T\text{-}1}(B^{**}_T)$, i.e. the function $Q^{T\text{-}1}(B)$ will sometimes return higher prices for higher debt levels, implying that the more debt the government sells, the higher bond prices it faces.

We will now show that the multiplicity caused by the productivity loss associated with default arises even if we exclude such counterintuitive cases and restrict $Q^{T\text{-}1}(B)$ to be decreasing in B. To construct such decreasing bond pricing schedule we can select some $\hat{B}_T \in [\underline{B}_T, \bar{B}_T]$ and set

$$Q^{T\text{-}1}(B_T) = \begin{cases} q^f_{T-1}, & if \;\; B_T \leq \hat{B}_T; \\ q^d_{T-1}, & if \;\; B_T > \hat{B}_T. \end{cases} \tag{26}$$

Proposition 1 established that $q^d_{T-1} < q^f_{T-1}$ for all $\hat{B}_T \in (\underline{B}_T, \bar{B}_T]$; furthermore, $q^d_{T-1} = \gamma \frac{S(a_T[1-\psi], k_T(a_T[1-\psi]), tr_T)/r}{B_T}$ decreases in B_T. This means that a function described in (26) is decreasing regardless of the specific choice of the threshold $\hat{B}_T$. Note that we can construct infinitely many such decreasing functions choosing different $\hat{B}_T \in [\underline{B}_T, \bar{B}_T]$ each time. Figure 1(b) shows an example of one such decreasing bond pricing schedule.

Note that the function family described by (26) does not capture *all* possible selections from $\mathcal{Q}^{T\text{-}1}(B)$ that are decreasing functions: we could construct a decreasing function with a different behavior around the jump point. The function family in (26) sets the bond price at the jump point to $Q^{T\text{-}1}(\hat{B}_T) = q^f_{T-1}$. We could alternatively assign $Q^{T\text{-}1}(\hat{B}_T) = q^d_{T-1}$ as the value at the jump point. However, such definition has a serious drawback: the corresponding revenue function would have no maximum on B. *Proposition 2* below makes this distinction, establishing that (26) describes all decreasing bond pricing schedules that satisfy *Definition 1A* and at the same time permit there to be a maximum revenue from the auction of bonds. In what follows we focus on the function family in (26) when constructing decreasing equilibrium bond pricing schedules.

Proposition 2. *For $\psi > 0$ there exist infinitely many decreasing bond pricing functions $Q^{T\text{-}1}(B)$ that satisfy Definition 1A such that $\max_{B_T}\{Q^{T\text{-}1}(B_T) \cdot B_T\}$ exists; all such functions belong to the function family described by (26).*
Proof. see Appendix A.

We will now consider how equilibrium multiplicity affects the government's ability to roll over its debt in period $T-1$, focusing on the family of decreasing bond pricing schedules described in *Proposition 2*. The government enters period $T-1$ carrying a pre-existing debt B_{T-1}. The capital stock in period $T-1$, $k(A^e_{T-1})$, is predetermined by period $T-2$ investment decisions based on period $T-2$ expectations, A^e_{T-1}. In period $T-1$ the government attempts to repay its debt; it chooses B_T to satisfy

$$B_{T-1} - S(a_{T-1}, k(A^e_{T-1}), tr_{T-1}) = Q^{T\text{-}1}(B_T) \cdot B_T. \tag{27}$$

As we discuss in subsection 2.1, we assume that if there is more than one value of debt that solves this equation, the government chooses the lowest B_T that solves (27).

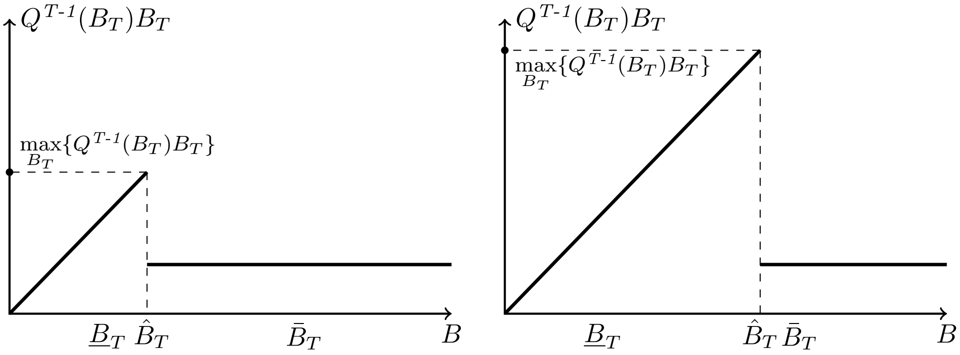
Repayment of debt B_{T-1} is feasible if the maximum of $Q^{T\text{-}1}(B_T) \cdot B_T$ lies above the value of the pre-existing debt less the current fiscal surplus shown on the left-hand side of (27). Using the decreasing bond pricing schedule described in (26) with some threshold $\hat{B}_T \in [\underline{B}_T, \bar{B}_T]$ we can derive the revenue from the auction of bonds for different bond issuances:

$$Q^{T\text{-}1}(B_T)B_T = \begin{cases} q^f_{T-1}B_T = \frac{1}{1+r}B_T, & if\ \ B_T \leq \hat{B}_T; \\ q^d_{T-1}B_T = \gamma S(a_T[1-\psi], k_T(a_T[1-\psi]), tr_T)/r, & if\ \ B_T > \hat{B}_T. \end{cases} \tag{28}$$

Figure 2 shows this function given different choices of the threshold $\hat{B}_T$. Through *Proposition 1* we know that $q^f_{T-1} > q^d_{T-1}$ for $\hat{B}_T \in [\underline{B}_T, \bar{B}_T]$, which means that at the threshold $\hat{B}_T$ there is a fall in revenue; to the left of the threshold, the revenue is increasing in B_T; to the right of the threshold, the revenue is constant. It follows that the maximum revenue from the auction of bonds that the government can extract lies at the threshold $\hat{B}_T$, and equals $\frac{1}{1+r}\hat{B}_T$. Since the threshold $\hat{B}_T$ is arbitrary (in the sense that it is not pinned down by the fundamentals of the economy), this implies indeterminacy over the maximum revenue from the auction of bonds.

Figure 2: Revenue from the auction of bonds and multiplicity

(a) 'Pessimistic' expectations (low $\hat{B}_T$) (b) 'Optimistic' expectations (high $\hat{B}_T$)



Corollary 2A. *Given $\psi > 0$ and a decreasing bond pricing schedule, the maximum revenue from the auction of government bonds can take any value in $[\frac{1}{1+r}\underline{B}_T, \frac{1}{1+r}\bar{B}_T]$, i.e. there is indeterminacy over the maximum revenue from the bonds auction.*
Proof. see Appendix A

We conclude that in the presence of positive fixed costs of default economic fundamentals do not pin down the maximum revenue that the government can extract from auctioning bonds. We can interpret this result as suggesting that the equilibrium outcome is driven by investor sentiment. Figure 2 illustrates this point. If 'pessimistic' investors perceive the threshold $\hat{B}_T$ to be low (Figure 2a), the maximum revenue from selling bonds is lower compared to that associated with the 'optimistic' investor expectations and a corresponding high $\hat{B}_T$ (Figure 2b).

The width of the interval $[\frac{1}{1+r}\underline{B}_T, \frac{1}{1+r}\bar{B}_T]$ increases with the value of the perceived productivity loss from default, ψ. Higher costs of default decrease the lower bound $\underline{B}_T$, widening the multiplicity interval (see *Corollary 1A*). Intuitively, the bound $\underline{B}_T$ reflects discounted revenues from tax collection given low capital investment under expectations of default; the higher the default penalties, the lower the capital investment and the tax revenue under such expectations. More specifically:

$$\frac{1}{1+r}\bar{B}_T - \frac{1}{1+r}\underline{B}_T = \tau a_T \left[\frac{\alpha a_T(1-\tau^K)}{r+\delta}\right]^{\frac{\alpha}{1-\alpha}} \left[1-(1-\psi)^{\frac{\alpha}{1-\alpha}}\right]\frac{1}{r} \tag{29}$$

When $\psi = 0$ there is no multiplicity and $\bar{B}_T = \underline{B}_T$; as ψ rises, so does the gap between $\frac{1}{1+r}\bar{B}_T$ and $\frac{1}{1+r}\underline{B}_T$.

To make sense of these results numerically, suppose that transfers tr_T are set to be a fixed share of GDP: $tr_T = \tau^{tr} \cdot y_T$. With this assumption, $\frac{1}{1+r}\underline{B}_T$ is lower than $\frac{1}{1+r}\bar{B}_T$ by $[1-(1-\psi)^{\frac{\alpha}{1-\alpha}}] \times 100$ percent. Assuming $\alpha = 1/3$ and a 10% anticipated reduction in productivity in the event of default, this gap would be at 5%; with a 5% productivity loss the gap would be about 3%.

Note that none of the qualitative results presented so far depend on the specific assumptions about the recovery rate and the fraction γ of the fiscal surplus being transferred to foreign creditors. On Figure 2 the value of γ affects the hight of the horizontal line, but not the value at the peak. The above results would hold under $\gamma = 1$, i.e. if the government transfers the entirety of the fiscal surplus to foreign creditors in the event of default. They would also hold under $\gamma = 0$, a full default.[13]

Interestingly, it is straightforward to show that the bond pricing schedule described in (26) is decreasing in ψ in the regions where it is differentiable. That is, aside from introducing equilibrium multiplicity, the default cost in this model has a negative effect on the equilibrium bond price—once a specific bond pricing schedule is selected. This result appears to go against conventional intuition: in models where the government is not constrained by fiscal limits, but optimally chooses whether to repay the debt or default, higher default costs dampen incentives to default, raising the probability of repayment and the associated risky bond price. In (26), while the risk-free bond price q^f_{T-1} is not affected by the default cost, the bond price that anticipates default, q^d_{T-1}, is strictly decreasing in ψ, provided that $\gamma > 0$. This is due to our assumption that in the event of default, the debt recovery rate depends positively on the government's fiscal surplus. A high cost of default decreases this amount through two channels: first, it has a direct negative effect on GDP when the country is in default and thereby on the fiscal surplus; second, in the period preceding default, high ψ discourages capital investment, resulting in lower capital stock when default occurs and, again, lower fiscal surplus.

This latter result depends on the value of γ: if nothing is transferred to the creditors in the event of default (i.e. $\gamma = 0$), then the bond price anticipating default would not decrease as ψ goes up because the repayment rate would be unaffected by the default cost. In Section 5 we show that once we introduce uncertainty over fundamentals, there emerges another channel that can make the bond price decreasing in ψ. Suppose that in some states of the world the government is expected to default, while in other states it repays the debt. As default costs rise, expected productivity *ex post* decreases, which depresses capital investment *ex ante*. This results in a reduction of the expected future surplus, followed by an increase in the number of states of the world in which the government is forced to default on its debt. As a result, the bond price would decrease even under $\gamma = 0$, as the number of states in which default is expected would go up.

4 Dynamic Implications of Equilibrium Multiplicity

We now turn to study the dynamic implications of the equilibrium multiplicity in this model. In subsection 4.1 we use a sequential equilibrium framework with perfect foresight to derive fiscal

[13]In this paper we do not examine how the default cost ψ may be affected by the renegotiation process over γ: our results hold as long as *some* fixed loss resulting from default exists. For papers examining the link between default costs and the debt restructuring process, see e.g. Asonuma *et al.* 2019.

limits for an arbitrary $t < T$. We show that the fiscal limit in period t, i.e. the maximum amount of debt that can be backed by fiscal surpluses, is not pinned down by the fundamentals of the economy—due to the feedback loop between investment decisions and future fiscal surpluses arising under a positive default cost ψ. We construct an example featuring two equilibrium fiscal limits: a fiscal limit that corresponds to the 'optimistic' expectations about future bond prices and a fiscal limit associated with investor pessimism. We show that the gap between the two limits widens as default costs go up.

In subsection 4.1 we thus show that there are multiple possible paths that could be supported as equilibria under perfect foresight. Importantly, alongside any of these paths the future equilibrium allocations are correctly anticipated by the economic agents, i.e. there is no uncertainty over which allocation will be realized. In subsection 4.2 we introduce sunspots that shift the economy between 'optimistic' and 'pessimistic' states. We consider a recursive equilibrium structure in which the agents perceive a possibility of a self-fulfilling debt crisis and assign probability values to different equilibria being selected. We construct an example in which the equilibrium fiscal limits depend negatively on the value of the default cost in both the 'optimistic' and the 'pessimistic' states of the world.

4.1 Perfect foresight equilibria and fiscal limits in infinite horizon

We now turn to study dynamic implications of equilibrium multiplicity in a sequential equilibrium framework with perfect foresight. We start by pinning down the conditions under which debt repayment is feasible in period $T-1$. We then work backwards and study fiscal limits arising arbitrary periods prior to period T. Finally, we set T to infinity and examine multiplicity in a setting where the specific assumptions made about the absorbing state T cease to matter.

We have previously established that in period $T-1$ there is an infinite number of bond pricing schedules consistent with the perfect foresight equilibrium, i.e. schedules that satisfy *Definition 1A*. We will now extend this argument to other periods; to this end we define an equilibrium bond pricing schedule for an arbitrary $t \leq T-2$ in *Definition 1B* in Appendix A. Solving the model by backward induction requires producing a sequence of bond pricing schedules consistent with perfect foresight equilibrium. In *Proposition 3* below we present a formal characterization of such a sequence and show that for an arbitrary $t \leq T-2$ the feasibility of debt repayment in period t may be driven by self-fulfilling expectations: specifically, there is a non-empty interval of debt levels for which 1) there is a sequence of equilibrium bond pricing schedules consistent with default on current debt and at the same time 2) there is a sequence of equilibrium bond pricing schedules that allows for debt repayment.

Because we are constructing a perfect foresight equilibrium in which the future outcomes are always correctly anticipated, when the government issues debt B_{t+1} in any given period $t \leq T-1$, it faces one of two bond prices: the price $q_t^f = \frac{1}{1+r}$ consistent with anticipated repayment, or the price $q_t^d = \gamma S(a_{t+1}[1-\psi], k(a_{t+1}[1-\psi]), tr_{t+1})/(rB_{t+1})$ consistent with anticipated default. In Section 3 we established that for $t = T-1$ there is an interval of debt levels, $(\underline{B}_T, \bar{B}_T]$, on which both q_{T-1}^f and q_{T-1}^d can be supported as equilibrium prices for debt levels B_T. Suppose we find that in some period $t < T-1$ there is also a set of new debt levels B_{t+1} for which two equilibrium bond prices can be constructed. We will now denote such a set by $\hat{\mathcal{B}}_{t+1}$.

In Section 3 we show that in period $T-1$ the set of debt levels for which both q^f_{T-1} and q^d_{T-1} are consistent with equilibrium, $\hat{\mathcal{B}}_T$, is non-empty (see *Propostion 1*). Because of the sequential nature of the equilibrium constructed here, it is reasonable to expect that for $t < T-1$ the sets $\hat{\mathcal{B}}_{t+1}$ would be non-empty as well. This means that at any given $t < T-1$, there might be multiple bond pricing schedules that can be supported as equilibrium. Therefore, to construct a specific equilibrium path we would need to come up with a selection mechanism that assigns unique prices to each $B_{t+1} \in \hat{\mathcal{B}}_{t+1}$ whenever $\hat{\mathcal{B}}_{t+1}$ is non-empty.

Before proceeding with a formal argument, we construct an intuitive example in which we compare equilibrium paths under two alternative mechanisms for selecting bond pricing schedules—we term them the 'optimistic' rule and the 'pessimistic' rule. These selection mechanisms will operate as follows. Suppose we find that in period t there is a set $\hat{\mathcal{B}}_{t+1}$ of new debt levels such that for each $B_{t+1} \in \hat{\mathcal{B}}_{t+1}$ both q^f_t and q^d_t are consistent with equilibrium. We will call the 'optimistic' selection, $Q^{t,o}(B)$, the bond pricing schedule such that $Q^{t,o}(B_{t+1}) = q^f_t$ for all $B_{t+1} \in \hat{\mathcal{B}}_{t+1}$. In other words, under the 'optimistic' rule, whenever multiple bond prices are possible, the highest bond price gets selected. By contrast, we will denote with $Q^{t,p}(B)$ the 'pessimistic' bond pricing schedule, a function for which $Q^{t,p}(B_{t+1}) = q^d_t$ for all $B_{t+1} \in \hat{\mathcal{B}}_{t+1}$.

We first use backward induction to solve for the equilibrium path under the 'pessimistic' rule. In period $T-1$ the set $\hat{\mathcal{B}}_T$ for which two bond prices can realize as equilibrium outcomes is given by $(\underline{B}_T, \bar{B}_T]$ where $\underline{B}_T$ is defined in (25) and $\bar{B}_T$ is given by (24). The 'pessimistic' bond pricing schedule $Q^{T\text{-}1,p}(B)$ assigns q^d_{T-1} to each $B_T \in (\underline{B}_T, \bar{B}_T]$; for $B_T \notin (\underline{B}_T, \bar{B}_T]$ the rule assigns the corresponding unique values. This is equivalent to setting the bond pricing schedule according to (26) under $\hat{B}_T = \underline{B}_T$. Per *Proposition 2* this schedule satisfies *Definition 1A.*

Under the 'pessimistic' schedule $Q^{T\text{-}1,p}(B)$, the maximum revenue that can be extracted from the auction of bonds in period $T-1$ equals $\frac{1}{1+r}\underline{B}_T$, or $S(a_T, k_T(a_T[1-\psi]), tr_T)/r$. In period $T-1$, the pre-existing debt B_{T-1} is repaid if it does not exceed the sum of the current fiscal surplus and the maximum revenue from the auction of new bonds, that is, if:

$$B_{T-1} \leq S(a_{T-1}, k(A^e_{T-1}), tr_{T-1}) + S(a_T, k_T(a_T[1-\psi]), tr_T)/r. \tag{30}$$

where $S(a_{T-1}, k(A^e_{T-1}), tr_{T-1})$ is the fiscal surplus in period $T-1$ that is conditional on the concurrent capital stock $k(A^e_{T-1})$.

We now move to derive the 'pessimistic' bond pricing schedule for period $T-2$, $Q^{T\text{-}2,p}(B)$, corresponding to debt B_{T-1}. Condition (30) suggests that the feasibility of debt repayment depends on expectations about productivity in period $T-1$ formed in period $T-2$, A^e_{T-1}. As before, we have two options for A^e_{T-1}: either agents expect repayment, in which case $A^e_{T-1} = a_{T-1}$, or agents expect default and $A^e_{T-1} = a_{T-1}[1-\psi]$. The expectation of repayment can be validated if debt does not exceed the present value of future fiscal surpluses conditional on expected repayment and capital $k(a_{T-1})$:

$$B_{T-1} \leq S(a_{T-1}, k(a_{T-1}), tr_{T-1}) + S(a_T, k_T(a_T[1-\psi]), tr_T)/r \equiv \bar{B}^p_{T-1}. \tag{31}$$

The expectation of default can be rationalized if debt is higher than the present value of future fiscal surpluses that arises when agents expect default and invest $k(a_{T-1}[1-\psi])$ in capital:

$$B_{T-1} > S(a_{T-1}, k(a_{T-1}[1-\psi]), tr_{T-1}) + S(a_T, k_T(a_T[1-\psi]), tr_T)/r \equiv \underline{B}^p_{T-1}. \tag{32}$$

Note that $\underline{B}^p_{T-1} < \bar{B}^p_{T-1}$ for any $\psi > 0$. Similarly to what we observed for period $T-1$, in period $T-2$ for all $B_{T-1} \in (\underline{B}^p_{T-1}, \bar{B}^p_{T-1}] \equiv \mathcal{B}_{T-1}$ there are two bond prices consistent with rational expectations: q^f_{T-2} and q^d_{T-2}. We can now construct the 'pessimistic' bond pricing schedule $Q^{T\text{-}2,p}(B)$ assigning q^d_{T-2} to each $B_{T-1} \in (\underline{B}^p_{T-1}, \bar{B}^p_{T-1}]$, q^f_{T-2} to each $B_{T-1} \leq \underline{B}^p_{T-1}$ and q^d_{T-2} to each $B_{T-1} > \bar{B}^p_{T-1}$.

Repeating this backward induction process we arrive at the 'pessimistic' bond pricing schedule consistent with perfect foresight equilibrium for an arbitrary $t \leq T-2$:

$$Q^{t,p}(B_{t+1}) = \begin{cases} q^f_t = \frac{1}{1+r}, & if \;\; B_{t+1} \leq \underline{B}^p_{t+1}; \\ q^d_t = \gamma S(a_{t+1}[1-\psi], k(a_{t+1}[1-\psi]), tr_{t+1})/(rB_{t+1}), & if \;\; B_{t+1} > \underline{B}^p_{t+1}, \end{cases} \tag{33}$$

where

$$\underline{B}^p_{t+1} = \sum_{i=1}^{T-t-1} \frac{S(a_{t+i}, k(a_{t+i}[1-\psi]), tr_{t+i})}{(1+r)^{i-1}} + \frac{1}{r}\frac{S(a_T, k(a_T[1-\psi]), tr_T)}{(1+r)^{T-t-2}}.$$

The maximum revenue that the government can extract from the auction of bonds in period t is then given by $\frac{1}{1+r}\underline{B}^p_{t+1}$. We deduce that under the 'pessimistic' bond pricing schedule and given the accumulated capital $k(A^e_t)$, repayment in any period $t \leq T-2$ is feasible if the preexisting debt does not exceed the following fiscal limit:

$$\begin{aligned} B_t \;\; &\leq \;\; S(a_t, k(A^e_t), tr_t) + \sum_{i=1}^{T-t-1} \frac{S(a_{t+i}, k(a_{t+i}[1-\psi]), tr_{t+i})}{(1+r)^i} + \\ &+ \frac{1}{r}\frac{S(a_T, k(a_T[1-\psi]), tr_T)}{(1+r)^{T-t-1}} \equiv \underline{B}^T_t. \end{aligned} \tag{34}$$

The value of the fiscal limit $\underline{B}^T_t$ on the right-hand side depends negatively on default costs. This is because under the 'pessimistic' bond pricing rule agents expect debt repayment to be infeasible whenever there is room for self-fulfilling expectations of default. When default costs are high, high expected productivity losses associated with default make self-fulfilling defaults possible under lower levels of debt. The government becomes unable to extract high revenues from auctioning bonds in the future, which results in a lower threshold for debt sustainability today.

We now dispense with the terminal period T and set $T \to \infty$. With this modification the specific assumptions we made about the absorbing state T cease to matter for determining the debt limit. Repayment of accumulated debt in period t is now feasible if:

$$B_t \leq S(a_t, k(A_t^e), tr_t) + \sum_{i=1}^{\infty} \frac{S(a_{t+i}, k(a_{t+i}[1-\psi]), tr_{t+i})}{(1+r)^i} \equiv \underline{B}_t^{\infty}. \tag{35}$$

As in the case of finite horizon, this fiscal limit depends negatively on the value of default costs. Again, this is due to the 'pessimistic' bond pricing schedule always selecting the worst case scenario.

We now contrast these results with the fiscal limit that would obtain under the 'optimistic' selection rule that assigns q_t^f to all $B_{t+1} \in \hat{\mathcal{B}}_{t+1}$, i.e. the bond pricing schedule that anticipates full debt repayment whenever this expectation can be validated. Repeating the previous steps we derive $Q^{t,o}(B)$, the 'optimistic' bond pricing schedule for an arbitrary $t \leq T-2$:

$$Q^{t,o}(B_{t+1}) = \begin{cases} q_t^f = \frac{1}{1+r}, & if \;\; B_{t+1} \leq \bar{B}_{t+1}^o; \\ q_t^d = \gamma S(a_{t+1}[1-\psi], k(a_{t+1}[1-\psi]), tr_{t+1})/(rB_{t+1}), & if \;\; B_{t+1} > \bar{B}_{t+1}^o, \end{cases} \tag{36}$$

where

$$\bar{B}_{t+1}^o \quad = \quad \sum_{i=1}^{T-t-1} \frac{S(a_{t+i}, k(a_{t+i}), tr_{t+i})}{(1+r)^{i-1}} + \frac{1}{r}\frac{S(a_T, k(a_T), tr_T)}{(1+r)^{T-t-2}}.$$

Clearly, $\bar{B}_{t+1}^o > \underline{B}_{t+1}^p$, and therefore bond prices along the 'optimistic' schedule are higher or equal to the corresponding 'pessimistic' prices, i.e. $Q^{t,o}(B_{t+1}) \geq Q^{t,p}(B_{t+1})$.

As in the case of the bond pricing schedule $Q^{T\text{-}1}(B)$ discussed in Section 3, the schedules $Q^{t,p}(B)$ and $Q^{t,o}(B)$ are decreasing in the level of default costs (where differentiable). The sections of the schedules to the right of the respective jump points with bond prices q_t^d are strictly decreasing in ψ if $\gamma > 0$. Once again, this result arises due to the assumption that in the event of default the government transfers a share γ of the fiscal surplus to the foreign creditors. Higher default costs reduce this surplus (by imposing a direct loss *ex post* and depressing capital investment *ex ante*), thereby lowering the debt recovery rate and the corresponding bond price.

We can also deduce that under the 'optimistic' bond pricing schedule repayment in period t is feasible if:

$$B_t \leq S(a_t, k(A_t^e), tr_t) + \sum_{i=1}^{T-t-1} \frac{S(a_{t+i}, k(a_{t+i}), tr_{t+i})}{(1+r)^i} + \frac{1}{r}\frac{S(a_T, k(a_T), tr_T)}{(1+r)^{T-t-1}} \equiv \bar{B}_t^T. \tag{37}$$

Under $T \to \infty$ this condition becomes:

$$B_t \leq S(a_t, k(A_t^e), tr_t) + \sum_{i=1}^{\infty} \frac{S(a_{t+i}, k(a_{t+i}), tr_{t+i})}{(1+r)^i} \equiv \bar{B}_t^{\infty}. \tag{38}$$

The fiscal limits $\bar{B}_t^T$ and $\bar{B}_t^{\infty}$ are higher compared to the corresponding limits $\underline{B}_t^T$ and $\underline{B}_t^{\infty}$

obtained under the 'pessimistic' selection mechanism. Furthermore, the limits $\bar{B}_t^T$ and $\bar{B}_t^\infty$ are unaffected by the value of default costs: this is because the 'optimistic' rule selects bond prices consistent with anticipated repayment whenever possible.

We have shown that there is a stark difference between the fiscal limit that corresponds to the bond pricing schedules reflecting 'optimistic' self-fulfilling expectations (i.e. the 'optimistic' fiscal limit $\bar{B}_t^\infty$) and the fiscal limit that arises when expectations are always 'pessimistic' (i.e. the 'pessimistic' fiscal limit $\underline{B}_t^\infty$). In either case the debt sustainability criterion states that debt must not exceed the sum of discounted future fiscal surpluses. Under the 'optimistic' rule these surpluses are calculated using high levels of capital investment, $k(a_t)$. Under the 'pessimistic' rule the surpluses are computed using lower capital investment levels that arise when productivity is expected to be impaired by default penalties, $k(a_t[1-\psi])$. This latter result emerges because under the 'pessimistic' rule investor pessimism is presumed to always be self-fulfilling: whenever expectations of default can be validated as equilibrium outcome, they get reflected in the bond pricing schedules—in other words, whenever there is room for a self-fulfilling debt crisis, default is expected. When debt exceeds $\underline{B}_t^\infty$, capital outflow can trigger a budget shortage; therefore, debt repayment hinges on whether or not future capital investment is expected to be depressed by the anticipated default penalties. This is when pessimistic expectations can become self-fulfilling.

Proposition 3 below makes a more general argument about debt repayment in period t in conjuncture with self-fulfilling expectations about future bond pricing schedules. We show that for any B_t in $(\underline{B}_t^T, \bar{B}_t^T]$ (or, under $T \to \infty$, in $(\underline{B}_t^\infty, \bar{B}_t^\infty]$) we can construct some sequence of decreasing equilibrium bond pricing schedules such that the government is forced to default in t; we can also construct some sequence of decreasing equilibrium bond pricing schedules such that the government is able to repay the debt in t.

Proposition 3 *Consider the economy entering period $t \leq T-2$ with pre-existing debt B_t and capital $k(A_t^e)$ where $\psi > 0$. There exists an infinite number of perfect foresight equilibria in which each period the government faces a decreasing bond pricing schedule that satisfies Definition 1B; furthermore, for each debt level B_t in $(\underline{B}_t^T, \bar{B}_t^T]$ there is at least one such equilibrium where B_t is consistent with default in t, and at least one equilibrium in which it is consistent with full debt repayment in t. As $T \to \infty$ this interval converges to $(\underline{B}_t^\infty, \bar{B}_t^\infty]$.*
Proof. see Appendix A.

The result that we can construct both equilibrium with repayment and equilibrium with default for any B_t between the 'pessimistic' and the 'optimistic' fiscal limits derived above is intuitive: we derived these limits assuming most extreme selections from the space of plausible equilibrium bond pricing schedules, featuring the 'always optimistic' or 'always pessimistic' choices; however, there is an infinite number of choices between these two alternatives which result in fiscal limits that fall between the two extremes.

The results uncovered in this section crucially depend on the value of default costs. The equilibrium multiplicity discussed in *Proposition 3* disappears for zero costs of default: for $\psi = 0$ we have $\underline{B}_t^T = \bar{B}_t^T$ and $\underline{B}_t^\infty = \bar{B}_t^\infty$. It is therefore the presence of the anticipated losses to productivity in the event of default that allows for multiple equilibria with distinct fiscal limits. Furthermore, the range of fiscal limits that can be validated as equilibria expands with the value of default costs: as ψ increases, the lower bounds of the multiplicity intervals ($\underline{B}_t^T$ and $\underline{B}_t^\infty$) go down, and self-fulfilling debt crises fueled by investor pessimism become possible for lower levels of debt.

4.2 Sunspot equilibria in infinite horizon

In subsection 4.1 we show that when defaults are associated with productivity losses the model generates infinitely many perfect foresight sequential equilibria with distinct fiscal limits. Each of these equilibria gives rise to a unique sequence of bond pricing schedules. We constructed an example that compares fiscal limits arising under two specific sequences of equilibrium bond pricing schedules: the 'pessimistic' equilibrium (with bond prices that anticipate default whenever this expectation can be self-fulfilling) and the 'optimistic' equilibrium (with bond prices that anticipate repayment whenever it can be rationalized). Crucially, whichever sequence of bond pricing schedules is selected, agents predict future prices perfectly, i.e. even though multiple equilibrium paths are possible, there is not uncertainty along a specific path. Therefore, the fact that multiple equilibria co-exist does not factor into the decisions economic agents make.

In this section we consider a version of the model in which agents perceive the possibility of self-fulfilling debt crises, and assign probability values to different equilibria being selected. We consider a sunspot Markov equilibrium that mirrors the example discussed in subsection 4.1, in which the bond pricing schedule depends on the realization of extrinsic uncertainty that switches the economy between two states, state 'o' and state 'p'. Each period the bond pricing schedule that the government faces when issuing new bonds B', $Q(B')$, may take one of two forms, $Q^o(B')$ or $Q^p(B')$, such that for $Q^o(B') \geq Q^p(B')$ for all B'. The exogenous probability of the state 'p' realizing is $0 < \pi^p < 1$; the probability of state 'o' is $(1 - \pi^p)$. We will show that, in contrast to the results obtained in subsection 4.1, here, the default cost affects fiscal limits that arise in both the 'optimistic' and the 'pessimistic' equilibria—because the agents internalize the possibility of sunspots that switch the economy from state 'o' to state 'p'.

We assume that unpenalized productivity a_T and transfers tr_T are time-invariant and denote them by a and tr; for brevity in this section we will write $S(A', k(E[A']))$ instead of $S(A', k(E[A']), tr)$. *Definition 2* below summarizes what we consider to be equilibrium bond pricing schedules in this setup.

Definition 2. Equilibrium bond pricing schedules $Q^i(B)$: sunspots. *The equilibrium is characterized by a pair of bond pricing schedules $Q^p(B'), Q^o(B')$ such that for each B' the bond price $q^i \in Q^i(B')$ for $i = \{p, o\}$ satisfies*

$$q^i = \frac{E[\chi'^j]}{1 + r} \tag{39}$$

where

$$\chi'^j = \begin{cases} 1, & if \quad B' \leq S(a, k(E[A'])) + \max_{B''}\{Q^j(B'')B''\}; \\ \frac{\gamma \cdot S(a[1-\psi], k(E[A'])) \cdot (1+r)/r}{B'}, & if \quad B' > S(a, k(E[A'])) + \max_{B''}\{Q^j(B'')B''\}, \end{cases} \tag{40}$$

and

$$k(E[A']) = \left[\frac{\alpha E[A'](1-\tau^K)}{r+\delta}\right]^{\frac{1}{1-\alpha}} \tag{41}$$

with

$$A'^j = \begin{cases} a, & if \quad B' \leq S(a, k(E[A'])) + \max_{B''}\{Q^j(B'')B''\}; \\ a(1-\psi), & if \quad B' > S(a, k(E[A'])) + \max_{B''}\{Q^j(B'')B''\} \end{cases} \tag{42}$$

where $j = p$ *with probability* π^p *and* $j = o$ *with probability* $1-\pi^p$*, and* $E[A'] = \pi^p A'^p + (1-\pi^p)A'^o$*. Furthermore, for any* B'*,* $Q^o(B') \geq Q^p(B')$*.*

The bond pricing schedules that satisfy *Definition 2* must be such that the maxima of $Q^j(B')B'$ exist. Denote these maxima by:

$$X^i \equiv \max_{B'}\{Q^i(B')B'\} \tag{43}$$

where $i = \{p, o\}$. The fact that $Q^o(B') \geq Q^p(B')$ implies that $X^o \geq X^p$.

In what follows we will construct an example featuring bond pricing schedules $Q^p(B')$ and $Q^o(B')$ that comply with *Definition 2.* One technical difficulty is that we cannot find X^i without determining $Q^i(B')$ first. We thus proceed in three steps. First, we characterize all combinations of debt B', bond prices and expectations $E[A']$ that satisfy (39)-(42) given *some* arbitrary $\check{X}^o$ and $\check{X}^p$ (to be determined later). Second, using these results and conditioning on $\check{X}^o$ and $\check{X}^p$ we construct an example featuring decreasing functions $Q^o(B'|\check{X}^o, \check{X}^p)$ and $Q^p(B'|\check{X}^o, \check{X}^p)$ such that $Q^o(B'|\check{X}^o, \check{X}^p) \geq Q^p(B'|\check{X}^o, \check{X}^p)$ for all B'. Third, we find X^o and X^p by substituting $Q^o(B'|\check{X}^o, \check{X}^p)$ and $Q^p(B'|\check{X}^o, \check{X}^p)$ into (43) and solving the resulting system of equations; we then use X^o and X^p to construct $Q^o(B')$ and $Q^p(B')$ and verify that the functions satisfy *Definition 2.*

We start by looking for combinations of debt, bond prices and expectations about productivity $E[A']$ that can be supported as equilibrium in *Definition 2*, given *some* choice of $\check{X}^o$ and $\check{X}^p$ such that $\check{X}^o \geq \check{X}^p$. With the sunspot switching the economy between two states there are only three possible values that $E[A']$ could take.

Suppose agents expect full repayment in both states. If next period agents anticipate full repayment in both states 'o' and 'p', then $E[A'] = a$. They would then choose $k(a)$ (capital choice under certain full repayment) as the next period's capital. This expectation is rational for levels of borrowing B' that, given this capital choice, can be repaid fully next period even in the pessimistic state:

$$B' \leq S(a, k(a)) + \check{X}^p \equiv \bar{B}(\check{X}^p). \quad (44)$$

If (44) holds, then under $E[A'] = a$ repayment of B' is feasible in both state 'p' and state 'o', as by construction $\check{X}^p \leq \check{X}^o$. The threshold $\bar{B}(\check{X}^p)$ denotes the maximum value of debt consistent with anticipated repayment in both states. For any debt level within $\bar{B}(\check{X}^p)$ a risk-free bond price $q^f = \frac{1}{1+r}$ can be supported as an equilibrium outcome.

Suppose agents expect default in both states. This results in expectations $E[A'] = a[1-\psi]$, and a capital choice $k(a[1-\psi])$. These expectations can be rationalized for debt levels that cannot be repaid even in state 'o':

$$B' > S(a, k(a[1-\psi])) + \check{X}^o \equiv \underline{B}(\check{X}^o). \quad (45)$$

For debt that exceeds the threshold $\underline{B}(\check{X}^o)$ we can construct an equilibrium bond price consistent with default in both states: $q^d = \gamma \frac{S(a[1-\psi], k(a[1-\psi]))}{rB'}$.

Suppose agents expect default in state 'p' and full repayment in state 'o'. The expected productivity then reads $E[A'] = a[1-\pi^p\psi]$ (and the capital choice is $k(a[1-\pi^p\psi])$). This can be supported as equilibrium for debt levels in the following interval:

$$B' > S(a, k(a[1-\pi^p\psi])) + \check{X}^p \equiv \tilde{B}^l(\check{X}^p) \quad (46)$$

$$B' \leq S(a, k(a[1-\pi^p\psi])) + \check{X}^o \equiv \tilde{B}^h(\check{X}^o). \quad (47)$$

Note that the reverse (i.e. default in 'o' and full repayment in 'p') is not possible, as by construction $\check{X}^o \geq \check{X}^p$. When debt lies in $(\tilde{B}^l(\check{X}^p), \tilde{B}^h(\check{X}^o)]$, a bond price consistent with expectations of default in 'p' and repayment in 'o' can be supported as an equilibrium outcome: $q^\pi = \frac{1}{1+r}\left[(1-\pi^p) + \pi^p\gamma\frac{S(a[1-\psi], k(a[1-\psi]))(1+r)}{rB'}\right]$.

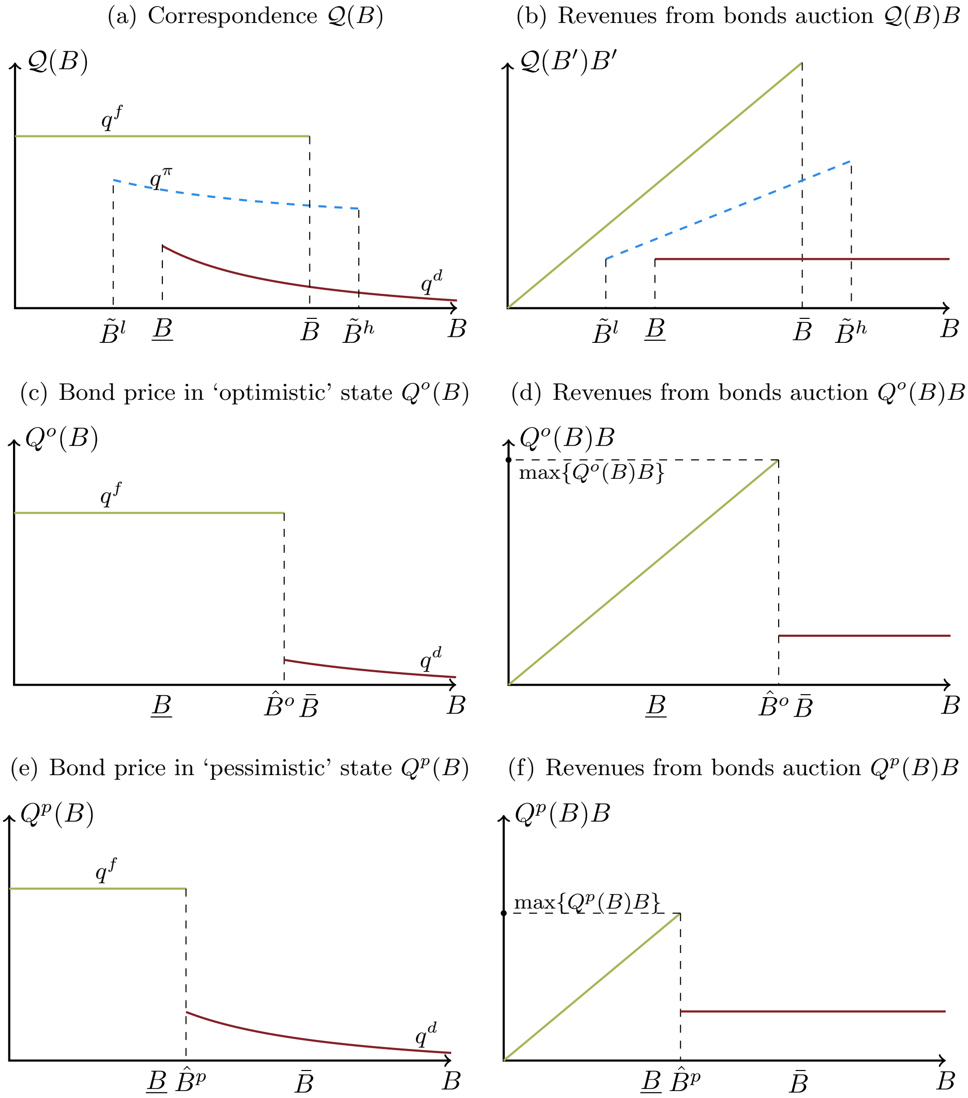
Unlike in the perfect foresight case, here it is not necessarily true that $\underline{B}(\check{X}^o) < \bar{B}(\check{X}^p)$; however, we will show that there exists an infinite number of function pairs $Q^o(B'), Q^p(B')$ featuring decreasing bond pricing schedules that satisfy *Definition 2* for which this is true—see the example below.

Example: Equilibrium with certain default or certain repayment

Suppose $\underline{B}(\check{X}^o) < \bar{B}(\check{X}^p)$. We will rely on this assumption to construct $Q^o(B'|\check{X}^o, \check{X}^p)$ and $Q^p(B'|\check{X}^o, \check{X}^p)$. We will then use (43) to find X^o, X^p and construct $Q^o(B'), Q^p(B')$; we then verify that, consistent with the assumption made above, $\underline{B}(X^o) < \bar{B}(X^p)$ holds.

Figure 3(a) presents a correspondence $\mathcal{Q}(B|\check{X}^o, \check{X}^p)$ that collects all bond prices that can be supported as equilibria given some $\check{X}^o$ and $\check{X}^p$, and assuming $\underline{B}(\check{X}^o) < \bar{B}(\check{X}^p)$—in accordance with the reasoning presented above; Figure 3(b) shows corresponding revenues from the auction of bonds. The green lines show the constant bond price corresponding to anticipated full repayment in both states, q^f, and the associated revenue from the auction of bonds; the red

Figure 3: Revenue from bonds auction and multiplicity



lines correspond to expectations of default in both states with a price q^d; the blue dashed line represents expectations of default in state 'p' and repayment in state 'o'; given arbitrary $\check{X}^o$ and $\check{X}^p$ we do not know the exact positioning of thresholds $\tilde{B}^l(\check{X}^p)$ and $\tilde{B}^h(\check{X}^o)$ (and therefore the blue lines are dashed), but it is immaterial for the argument presented below.

We proceed by constructing two decreasing bond pricing schedules as selections from the correspondence $\mathcal{Q}(B|\check{X}^o,\check{X}^p)$. Each schedule combines q^f and q^d such that $Q^o(B'|\check{X}^o,\check{X}^p) \geq Q^p(B'|\check{X}^o,\check{X}^p)$, and features a downward jump at some $\hat{B}^j \in [\underline{B}(\check{X}^o),\bar{B}(\check{X}^p)]$:

$$Q^j(B'|\check{X}^o,\check{X}^p) = \begin{cases} q^f = \frac{1}{1+r}, & if \;\; B' \leq \hat{B}^j; \\ q^d = \gamma S(a[1-\psi],k(a[1-\psi]))/(rB'), & if \;\; B' > \hat{B}^j. \end{cases} \tag{48}$$

with $\hat{B}^o > \hat{B}^p$ (we do not consider $\hat{B}^o = \hat{B}^p$ to ensure that the schedules $Q^o(B'|\check{X}^o, \check{X}^p)$ and $Q^p(B'|\check{X}^o, \check{X}^p)$ are distinct). Figure 3(c) and Figure 3(e) depict examples of such bond pricing schedules $Q^o(B|\check{X}^o, \check{X}^p)$ and $Q^p(B|\check{X}^o, \check{X}^p)$.

We are now in position to determine X^o and X^p. Set $\hat{B}^j = m^j \underline{B}(X^o) + (1 - m^j)\bar{B}(X^p)$ with $m^j \in [0, 1]$; require that $m^p > m^o$. Crucially, the maxima of the revenues under schedules $Q^o(B|\check{X}^o, \check{X}^p)$ and $Q^p(B|\check{X}^o, \check{X}^p)$ above will lie at the respective jump points, as shown on Figure 3(d) and Figure 3(f). Combining this insight with the definition (43) we get:

$$X^o = q^f \cdot \hat{B}^o = \frac{1}{1+r}[m^o \underline{B}(X^o) + (1 - m^o)\bar{B}(X^p)] \tag{49}$$

$$X^p = q^f \cdot \hat{B}^p = \frac{1}{1+r}[m^p \underline{B}(X^o) + (1 - m^p)\bar{B}(X^p)] \tag{50}$$

Substituting these results into (44) and (45), we obtain two linear equations from which we can solve for $\bar{B}, \underline{B}$:

$$\bar{B}(X^p) = S(a, k(a)) + \frac{1}{1+r}[m^p \underline{B}(X^o) + (1 - m^p)\bar{B}(X^p)] \tag{51}$$

$$\underline{B}(X^o) = S(a, k(a[1-\psi])) + \frac{1}{1+r}[m^o \underline{B}(X^o) + (1 - m^o)\bar{B}(X^p)] \tag{52}$$

This system yields:

$$\underline{B} = \frac{(1+r)(r+m^p)}{r(1+r+m^p-m^o)} S(a, k(a[1-\psi])) + \frac{(1+r)(1-m^o)}{r(1+r+m^p-m^o)} S(a, k(a))$$

$$\bar{B} = \frac{(1+r)m^p}{r(1+r+m^p-m^o)} S(a, k(a[1-\psi])) + \frac{(1+r)}{(r+m^p)}[1 + \frac{m^p(1-m^o)}{r(1+r+m^p-m^o)}] S(a, k(a))$$

We can substitute this result into (49) and (50) to find X^o and X^p; we can also construct $Q^o(B')$ and $Q^p(B')$ substituting $\hat{B}^j = m^j \underline{B}(X^o) + (1 - m^j)\bar{B}(X^p)$ into (48). Finally, we can verify that our original guess that $\bar{B} > \underline{B}$ is correct: with some algebra we can show that $\bar{B} > \underline{B}$ whenever $S(a, k(a)) > S(a, k(a[1-\psi]))$, which is always the case under $\psi > 0$.

By construction, the functions $Q^o(B')$ and $Q^p(B')$ in our example satisfy (39)-(42) given our solutions for X^o and X^p. Furthermore, X^o and X^p represent the maxima of $Q^o(B')B'$ and $Q^p(B')B'$. Because $m^p > m^o$, it follows that $\hat{B}^o > \hat{B}^p$ and (48) implies that $Q^o(B')B' \geq Q^p(B')B'$ for all B'. We therefore conclude that $Q^o(B')$ and $Q^p(B')$ in our example satisfy *Definition 2*. We have shown that the equilibrium bond pricing schedules described in *Definition 2* exist and that we can construct equilibria in which the economy switches between two states, each associated with a distinct decreasing bond pricing schedule.

The argument above works for any choices of $\hat{B}^o > \hat{B}^p$ in $[\underline{B}, \bar{B}]$. But within this class of equilibria, if $\hat{B}^o$ and $\hat{B}^p$ lie relatively close to each other, then the two states do not differ much in the associated economic outcomes and the maximum revenues from the auction of bonds. We will now consider an extreme case featuring the most distinct bond pricing schedules $Q^o(B)$ and $Q^p(B)$, in which $\hat{B}^o$ and $\hat{B}^p$ lie on the opposite ends of the multiplicity interval, with $\hat{B}^o = \bar{B}$ (and $m^o = 0$) and $\hat{B}^p = \underline{B}$ (and $m^p = 1$). This equilibrium mirrors the example presented in subsection 4.1, in which in the 'pessimistic' equilibrium default was anticipated whenever possible, while in the 'optimistic' equilibrium repayment was always expected if it could be rationalized.

With $m^o = 0$ and $m^p = 1$, in state 'o' given capital k, debt B can be repaid if:

$$B \leq S(a,k) + \frac{1}{r(2+r)} S(a, k(a[1-\psi])) + \frac{1+r}{r(2+r)} S(a, k(a)) \tag{53}$$

while in state 'p' debt B can be repaid if:

$$B \leq S(a,k) + \frac{1+r}{r(2+r)} S(a, k(a[1-\psi])) + \frac{1}{r(2+r)} S(a, k(a)) \tag{54}$$

Unlike in the example considered in subsection 4.1, here the default cost ψ affects the fiscal limits that arise in both states—not only when investors are pessimistic, but also in the 'optimistic' state. This is because, unlike in the sequential perfect foresight equilibrium discussed in subsection 4.1, here in state 'o' agents perceive the possibility of the sunspot occurring in the future, and the potential for a switch between 'o' and 'p'.

To see the intuition behind this result, consider X^o defined in (49). Suppose that the economy is currently in state 'o'. In state 'o', the maximum of the revenue function, X^o, corresponds to the biggest debt level that can be repaid fully in both states in the future (which trades at the risk-free price q^f)—as illustrated on Figure 3(d). If state 'p' realizes in the future, investors will anticipate default and productivity penalties whenever such expectation can be rationalized; higher default costs push down future X^p along with the maximum level of debt that can be repaid in the future state 'p'. As a result, in the current state 'o' the maximum debt that can be repaid in both future states goes down as default cost increases—this has a negative impact on the maximum revenue X^o that can be extracted in state 'o' and, as a result, on the maximum debt that can be repaid in state 'o' defined in (53).

To sum up, the fiscal limit, i.e. the maximum level of debt that can be repaid, depends on the maximum revenue that can be extracted from auctioning bonds, which, in turn, is conditional on default costs. Note that under $\psi = 0$, the fiscal limits in states 'o' and 'p' become identical; furthermore, they would match the 'optimistic' fiscal limit in the sequential equilibrium model specified in (38). For $\psi > 0$ fiscal limits in both states are negatively affected by the value of the default cost.[14]

[14] Note that the probability of the sunspot occurring, π^p, does not affect the fiscal limits—because they depend on the maximum revenues from auctioning bonds that are achieved at risk-free borrowing levels. But the example presented here is not the only equilibrium satisfying *Definition* 2 that we can construct. We found that for high γ and low π^p we can also construct an equilibrium in which the maximum revenue from the auction of bonds is achieved at price q^π which anticipates repayment in state 'o' and default in state 'p'. In this equilibrium, the feasibility of debt repayment is affected by both γ and π^p.

5 Multiplicity and Uncertainty over Fundamentals

The results presented so far describe bond prices and multiplicity under the assumption that there is no uncertainty over the fundamentals of the economy: Section 3 and subsection 4.1 consider equilibria with perfect foresight, while subsection 4.2 introduces a sunspot that switches the economy between states characterized by distinct expectations about the future. In both these setups we constructed bond prices that were decreasing in the level of debt, as well as in the default cost (where differentiable). We also pointed out intervals on which the bond prices were strictly decreasing in ψ (i.e. where $q_t = q_t^d$).

This latter result emerged because of our assumption about debt recovery rates: we assume that after a default the government transfers a fraction γ of its fiscal surpluses to the foreign creditors. The default cost has a two-fold effects on this post-default surplus: first, it creates a direct loss to output when default occurs which results in a lower recovery rate; second, the anticipation of low returns following a default depresses capital investment and through it, the capital stock and the fiscal surplus that is transferred to the foreign creditors when default occurs. Importantly, these two channels disappear if $\gamma = 0$, i.e. assuming full default on debt.

In this section we introduce uncertainty over the fundamentals of the economy and show that, provided that uncertainty is continuously distributed, there emerges an additional channel through which the default cost can negatively affect the bond price that is not tied to the value of γ. Intuitively, when repayment is uncertain, higher default costs *ex post* reduce expected returns on capital and depress capital investment *ex ante*. This results in lower capital stock and fiscal surplus *ex post*. In consequence, *ex ante* default is expected to occur in more states of the world, resulting in a lower associated bond price.

In what follows we present an extension of the sequential version of the model with the absorbing state T, in which we introduce uncertainty over the transfer payments in period T. Our interpretation of this setup is similar to that discussed in Lorenzoni & Werning (2019): for example, there could be a political event taking place in period T (e.g. an election) that could prompt changes to fiscal policy. We assume that the transfers are distributed continuously with pdf $f(tr_T)$ and support $[tr^{min}, tr^{max}]$. While the results presented below are specific to this setup, the mechanism discussed above would also arise under uncertainty over other fundamentals that affect the fiscal surplus (e.g. uncertainty over productivity or tax rates). Here, we prefer to use uncertainty over transfer payments because it does not impact the capital accumulation decision through any other channel aside from expectations of default and default penalties.

As in the deterministic case, the realized return $r_T^K = \alpha A_T k_T^{\alpha-1}$ depends on whether default happens in period T and, in consequence, on whether there is a default penalty. Capital investment in period $T-1$ is determined by expectations of period $T-1$ about how different realization of transfers relate to default decisions and default penalties in period T. The realized fiscal surplus equals:

$$S_T \equiv S(A_T, k_T(E_{T-1}[A_T]), tr_T) = \tau A_T [k_T(E_{T-1}[A_T])]^\alpha - tr_T. \tag{55}$$

The surplus depends on the current realization of transfer payments as well as the capital choice that, in turn, is conditional on past expectations about productivity. We assume that $S(a_T(1-\psi), k_T(a(1-\psi)), tr^{max}) > 0$ which ensures that the surplus is always positive—for ease of exposition.

Definition 3. Equilibrium bond pricing schedule $Q^{T\text{-}1}(B)$: uncertainty. *A bond pricing schedule is a function $Q^{T\text{-}1}(B)$ such that for each $B_T \in B$ the bond price $q_{T-1} \in Q^{T\text{-}1}(B_T)$ satisfies*

$$q_{T-1} = \frac{E_T[\chi_T]}{1+r} \tag{56}$$

where

$$\chi_T = \begin{cases} 1, \ if & B_T \leq S(a_T, k_T(E_{T-1}[A_T]), tr_T) \cdot \frac{1+r}{r}; \\ \gamma \frac{S(a_T[1-\psi], k_T(E_{T-1}[A_T]), tr_T)\cdot(1+r)/r}{B_T}, \ if & B_T > S(a_T, k_T(E_{T-1}[A_T]), tr_T) \cdot \frac{1+r}{r}. \end{cases} \tag{57}$$

and

$$k_T = \left[\frac{\alpha E_{T-1}[A_T](1-\tau^K)}{r+\delta}\right]^{\frac{1}{1-\alpha}} \tag{58}$$

with

$$A_T = \begin{cases} a_T, \ if & B_T \leq S(a_T, k_T(E_{T-1}[A_T]), tr_T) \cdot \frac{1+r}{r}; \\ a_T(1-\psi), \ if & B_T > S(a_T, k_T(E_{T-1}[A_T]), tr_T) \cdot \frac{1+r}{r}. \end{cases} \tag{59}$$

The bond pricing schedule faced by the government depends on expectations about the realization of transfers, tr_T. First, these expectations directly affect the future fiscal surplus, and through it, the expected recovery rate. This mechanism is similar to other models of default. Second, the expected recovery rate affects expected returns on capital which then determine present capital choice. This choice, in turn, has an effect on future fiscal surpluses and the expected recovery rate. This feedback loop results in a fixed-point problem that may have several solutions. As before, we will show that positive default costs may cause equilibrium multiplicity.

Note that expressions (57) and (59) give recovery rates and default penalties conditional on the realization of the shock to transfers, tr_T. Using the definition of the fiscal surplus $S(a_T, k_T(E_{T-1}[A_T]), tr_T)$ we can rewrite them as follows:

$$\chi_T = \begin{cases} 1, \ if & tr_T \leq \hat{tr}_T; \\ \gamma \frac{S(a_T[1-\psi], k_T(E_{T-1}[A_T]), tr_T)\cdot(1+r)/r}{B_T}, \ if & tr_T > \hat{tr}_T; \end{cases} \tag{60}$$

and

$$A_T = \begin{cases} a_T, \ if & tr_T \leq \hat{tr}_T; \\ a_T(1-\psi), \ if & tr_T > \hat{tr}_T; \end{cases} \tag{61}$$

where

$$\hat{tr}_T \equiv \tau a_T(k_T(E_{T-1}[A_T]))^{\alpha} - B_T\frac{r}{1+r}. \tag{62}$$

Therefore, default and productivity penalties occur whenever the realization of transfers surpasses an endogenous threshold $\hat{tr}_T$. We can use this insight to express the bond pricing equation (56) and the capital investment decision (58) in terms of expectations of the realization of tr_T relative to the threshold $\hat{tr}_T$:

$$q_{T-1} = F(\hat{tr}_T)\frac{1}{1+r} + \frac{\gamma}{r}\cdot\frac{\int_{\hat{tr}_T}^{\infty}[\tau a_T(1-\psi)(k_T(E_{T-1}[A_T])^{\alpha} - tr_T]f(tr_T)dtr_T}{B_T} \tag{63}$$

and

$$k_T(E_{T-1}[A_T]) = \left[\frac{\alpha a_T\left[1-(1-F(\hat{tr}_T))\psi\right](1-\tau^K)}{r+\delta}\right]^{\frac{1}{1-\alpha}} \tag{64}$$

where $F(.)$ is the cdf of tr_T. Bond prices depend on the probability of transfers staying within $\hat{tr}_T$, $F(\hat{tr}_T)$, and on the expected surplus that will be transferred to foreign creditors if they do surpass $\hat{tr}_T$ (the right term in 63). This latter value is affected by the expected transfers as well as the capital choice. When $\hat{tr}_t$ is such that the probability of default is positive (i.e. when $F(\hat{tr}_T) < 1$), capital investment decreases with the default cost ψ; it also decreases with the probability of default, $(1 - F(\hat{tr}_T))$. This is intuitive: capital returns drop in times of default by the magnitude which is predicated on the value of the default cost ψ. Consequently, the realized fiscal surplus also depends on $\hat{tr}_T$:

$$S(A_T, k_T(E_{T-1}[A_T]), tr_T) = \tau A_T\left[\frac{\alpha a_T\left[1-(1-F(\hat{tr}_T))\psi\right](1-\tau^K)}{r+\delta}\right]^{\frac{\alpha}{1-\alpha}} - tr_T. \tag{65}$$

To construct bond prices that satisfy *Definition 3* we will use reasoning similar to that employed in Section 3 when constructing bond prices under perfect foresight: we will make guesses about expected productivity $E_{T-1}[A_T]$, and then find values of B_T for which these guesses can be correct in equilibrium.

There are three possible values that expected productivity $E_{T-1}[A_T]$ could take. First, agents could be anticipating repayment in every state, in which case they would expect $E_{T-1}[A_T] = a_T$; below we derive necessary and sufficient conditions under which an equilibrium featuring these expectations can be constructed. Second, agents could be expecting default in every state with $E_{T-1}[A_T] = a_T[1-\psi]$—again, we derive necessary and sufficient conditions for this to be an equilibrium. Third, agents may be anticipating default in some states and repayment in other states, in which case $E_{T-1}[A_T] = a_T\left[1-(1-F(\hat{tr}_T))\psi\right]$. Unlike with the previous two sets of expectations, here we will only define sufficient (but not necessary) conditions for such an equilibrium to exist.

Suppose agents expect full repayment in all states and $E_{T-1}[A_T] = a_T$. These expectations are validated if (60) and (61) select $\chi_T = 1$ and $A_T = a_T$ in every state. This is the case if $\hat{tr}_T \geq tr^{max}$ or, equivalently, if under the capital choice $k_T(a_T)$ associated with anticipated repayment in all states, in each of those states the debt B_T does not exceed the discounted sum of future fiscal surpluses, $S(a_T, k_T(a_T), tr_T) \cdot (1+r)/r$. Since $S(.)$ is decreasing in tr_T, this condition holds for each state if and only if it holds in the state with the highest transfers possible, i.e. under $B_T \leq S(a_T, k_T(a_T), tr^{max}) \cdot (1+r)/r$, or:

$$B_T \leq \left(\tau a_T \left[\frac{\alpha a_T (1-\tau^K)}{r+\delta} \right]^{\frac{\alpha}{1-\alpha}} - tr^{max} \right) \frac{1+r}{r} \equiv \bar{B}_T \tag{66}$$

As in the previous sections, we conclude that there is an upper bound on debt B_T beyond which one cannot construct an equilibrium bond price consistent with full repayment in all states. At the same time, for debt levels below this threshold such a bond price can be constructed and would equal $q^f_{T-1} = \frac{1}{1+r}$. As in the previous sections, $B_T \leq \bar{B}_T$ is a necessary and sufficient condition for the existence of an equilibrium bond price that anticipates full repayment in all states: if $B_T \leq \bar{B}_T$ is violated, expectations of full repayment cannot be validated as equilibrium, as given the corresponding capital choice in some of the states the debt will not be repaid.

Suppose agents expect default in all states and $E_{T-1}[A_T] = a_T[1-\psi]$. Again, these expectations are validated if (60) and (61) select $\chi_T < 1$ and $A_T = a_T[1-\psi]$ in every state, which is the case if $\hat{tr}_T < tr^{min}$ or, equivalently, if under the capital choice $k_T(a_T[1-\psi])$ consistent with expected default in all states, in each of those states the debt B_T exceeds the discounted sum of future fiscal surpluses, $S(a_T, k_T(a_T[1-\psi]), tr_T) \cdot (1+r)/r$. Again, because $S(.)$ is decreasing in tr_T, this condition holds if and only if under capital $k_T(a_T[1-\psi])$ debt cannot be repaid even in the state with the lowest realization of transfers, i.e. under $B_T > S(a_T, k_T(a_T[1-\psi]), tr^{min}) \cdot (1+r)/r$, or:

$$B_T > \left(\tau a_T \left[\frac{\alpha a_T (1-\psi)(1-\tau^K)}{r+\delta} \right]^{\frac{\alpha}{1-\alpha}} - tr^{min} \right) \frac{1+r}{r} \equiv \underline{B}_T \tag{67}$$

There is therefore a lower bound on debt B_T such that for all debt levels exceeding it we can construct an equilibrium bond price consistent with default in all states, while for $B_T \leq \underline{B}_T$ this cannot be done. Again, $B_T > \underline{B}_T$ is a necessary and sufficient condition for the bond price corresponding to default in all states to satisfy *Definition 3.*

Suppose agents expect default in some states and full repayment in other states, and $E_{T-1}[A_T] = a_T\left[1-(1-F(\hat{tr}_T))\psi\right]$. These expectations are validated if there exists some $\hat{tr} \in [tr^{min}, tr^{max})$ such that (60) and (61) select $\chi_T = 1$ and $A_T = a_T$ for all $tr_T \leq \hat{tr}_T$ and $\chi_T < 1$ and $A_T = a_T[1-\psi]$ for all $tr_T > \hat{tr}_T$. The capital choice is then determined by the expected productivity $E_{T-1}[A_T] = a_T\left[1-(1-F(\hat{tr}_T))\psi\right]$. For $tr_T \leq \hat{tr}_T$ the discounted realized surplus $S(a_T, k_T(a_T\left[1-(1-F(\hat{tr}_T))\psi\right]), tr_T) \cdot (1+r)/r$ must not exceed B_T, while for $tr_T > \hat{tr}_T$ it must be greater than B_T. Because the surplus is continuously decreasing over tr_T and since tr_T is continuously distributed, at $tr_T = \hat{tr}_T$ the debt must exactly match the value of discounted future fiscal surpluses:

$$B_T = \left[\tau a_T \left[\frac{\alpha a_T \left[1 - (1 - F(\hat{tr}_T))\psi\right](1 - \tau^K)}{r + \delta} \right]^{\frac{\alpha}{1-\alpha}} - \hat{tr}_T \right] \frac{1+r}{r} \equiv \Omega(\hat{tr}_T). \tag{68}$$

To find $\hat{tr}_T$ that can support expectations of default under $tr_T > \hat{tr}_T$ and repayment under $tr_T \leq \hat{tr}_T$, we need to find $\hat{tr}_T \in [tr^{min}, tr^{max})$ that solves problem (68). The function $\Omega(\hat{tr}_T)$ captures the discounted sum of fiscal surpluses at the threshold $tr_T = \hat{tr}_T$. It is not clear whether the function $\Omega(\hat{tr}_T)$ is decreasing or increasing: on the one hand, higher $\hat{tr}_T$ means higher expected productivity and, as a result, higher level of capital investment and fiscal surplus; on the other hand, higher $\hat{tr}_T$ also implies larger transfer payments and lower fiscal surplus. We cannot derive necessary conditions for the existence of $\hat{tr}_T \in [tr^{min}, tr^{max})$ that solves problem (68) without making additional assumptions about the structure of uncertainty. Nevertheless, we will derive sufficient conditions that guarantee the problem (68) to have a solution on $\hat{tr}_T \in [tr^{min}, tr^{max})$.

Note that $\Omega(tr^{min}) = \underline{B}_T$ and $\Omega(tr^{max}) = \bar{B}_T$. Note also that the function $\Omega(\hat{tr}_T)$ is continuous on $[tr^{min}, tr^{max})$. Consider the following two possibilities: $\underline{B}_T > \bar{B}_T$ and $\underline{B}_T < \bar{B}_T$. Suppose $\underline{B}_T > \bar{B}_T$; it follows that $\Omega(tr^{min}) > \Omega(tr^{max})$ and that on $[tr^{min}, tr^{max})$, because of continuity, the function $\Omega(\hat{tr}_T)$ assumes each value in $(\Omega(tr^{max}), \Omega(tr^{min})]$ at least once. It follows that for any $B_T \in (\bar{B}_T, \underline{B}_T]$ problem (68) has at least one solution on $\hat{tr}_T \in [tr^{min}, tr^{max})$. Consequently, whenever $\underline{B}_T > \bar{B}_T$, we can find $\hat{tr}_T$ for any $B_T \in (\bar{B}_T, \underline{B}_T]$ such that there is default under $tr_T > \hat{tr}_T$ and repayment otherwise.

Suppose now that $\underline{B}_T < \bar{B}_T$; this implies that $\Omega(tr^{min}) < \Omega(tr^{max})$ and that, again, because of continuity, on $[tr^{min}, tr^{max})$ the function $\Omega(\hat{tr}_T)$ assumes each value in $[\Omega(tr^{min}), \Omega(tr^{max}))$ at least once. It follows that for any $B_T \in [\underline{B}_T, \bar{B}_T)$ problem (68) has at least one solution on $\hat{tr}_T \in [tr^{min}, tr^{max})$. Therefore, for any $B_T \in [\underline{B}_T, \bar{B}_T)$ we can find $\hat{tr}_T$ such that there is default under $tr_T > \hat{tr}_T$ and repayment otherwise.

The above considerations yield an important result. For each B_T we can construct at least one $\hat{tr}_T$ that satisfies (60)-(62) and (64); for some B_T there is more than one such $\hat{tr}_T$ (e.g. for $B_T \in (\underline{B}_T, \bar{B}_T]$ when $\underline{B}_T < \bar{B}_T$). We can therefore construct a correspondence $\Phi^{T\text{-}1}(B)$ that links each $B_T \in B$ with a threshold $\hat{tr}_T$ that satisfies (60)-(62) and (64). We can then construct a bond pricing schedule that satisfies *Definition 3* through one of two alternative routes. First, we can construct a correspondence $\mathcal{Q}^{T\text{-}1}(B)$ by applying (63) to each pair $(B_T, \hat{tr}_T) : \hat{tr}_T = \Phi^{T\text{-}1}(B_T)$; we can then construct a function $Q^{T\text{-}1}(B)$ by specifying a selector from the correspondence $\mathcal{Q}^{T\text{-}1}(B)$. Alternatively, we can specify a selector function $\phi^{T\text{-}1}(B)$ from the correspondence $\Phi^{T\text{-}1}(B)$ that assigns each B_T a unique $\hat{tr}_T$; we can then obtain $Q^{T\text{-}1}(B)$ by applying (63) to each pair $(B_T, \hat{tr}_T) : \hat{tr}_T = \phi^{T\text{-}1}(B_T)$. Because (63) describes a function, these two routes result in the same set of bond pricing schedules $Q^{T\text{-}1}(B)$ that satisfy *Definition 3*. We have therefore established that a bond pricing schedule that satisfies *Definition 3* exists (see proof of *Proposition 4* below for additional details).

Importantly, there may be more than one bond pricing schedule that satisfies *Definition 3.* Specifically, when $\underline{B}_T < \bar{B}_T$, correspondences $\Phi^{T\text{-}1}(B)$ and $\mathcal{Q}^{T\text{-}1}(B)$ assign more then one value to each $B_T \in (\underline{B}_T, \bar{B}_T]$ as there are several $\hat{tr}_T$ that satisfy (60)-(62) and (64) and therefore several bond prices satisfying *Definition 3.* In this case, multiple distinct selections from $\Phi^{T\text{-}1}(B)$ and $\mathcal{Q}^{T\text{-}1}(B)$ are possible, and more than one distinct bond pricing schedule satisfying *Definition 3* can be constructed.

Proposition 4 below summarizes these results and establishes that a sufficient condition for multiplicity, $\underline{B}_T < \bar{B}_T$, requires that the default cost ψ is 'high enough'. *Proposition 4* also asserts that under this condition multiple *decreasing* schedules exist. Denote

$$\tilde{\psi} \equiv 1 - \left[1 - \frac{tr^{max} - tr^{min}}{\tau a_T \left[\frac{\alpha a_T (1-\tau^k)}{r+\delta} \right]^{\frac{\alpha}{1-\alpha}}} \right]^{\frac{1-\alpha}{\alpha}}. \tag{69}$$

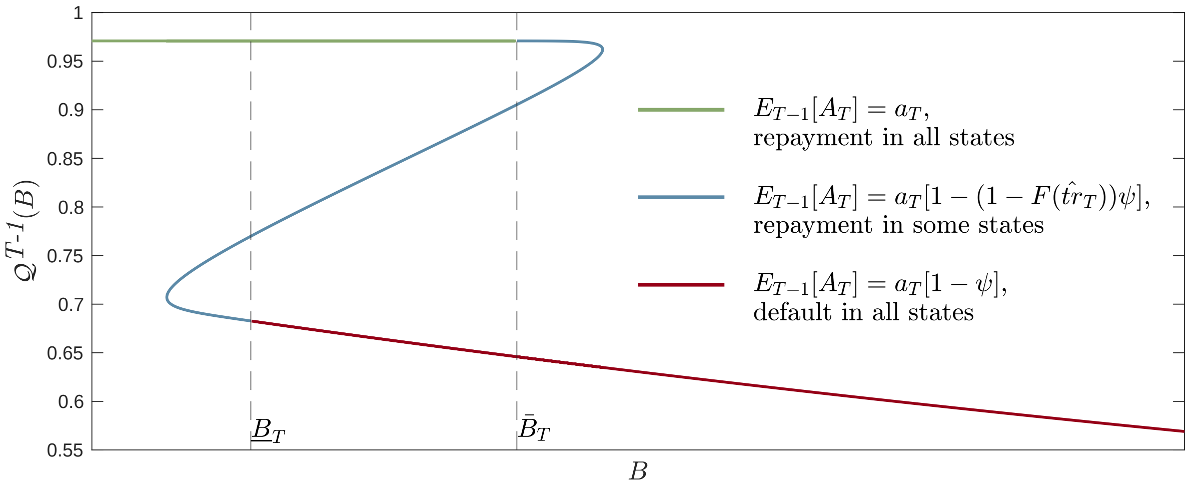
Proposition 4. *Equilibrium bond pricing schedule $Q^{T\text{-}1}(B)$ that satisfies Definition 3 exists. Furthermore, if $\psi > \tilde{\psi}$, then $\underline{B}_T < \bar{B}_T$ and there exist multiple decreasing bond pricing schedules $Q^{T\text{-}1}(B)$ that satisfy Definition 3 with each schedule taking distinct values on the interval $(\underline{B}_T, \bar{B}_T]$. When the cost ψ goes up, the interval $(\underline{B}_T, \bar{B}_T]$ expands as the threshold $\underline{B}_T$ decreases.*

Proof. see Appendix A.

Note that the model with uncertain transfers nests the deterministic model presented in Section 3: under $tr^{min} = tr^{max}$ we would have $\tilde{\psi} = 0$, and the multiplicity region $(\underline{B}_T, \bar{B}_T]$ would exist for any $\psi > 0$—*Proposition 4* would then state the results from *Proposition 1*, namely, that it is always possible to construct an equilibrium bond price consistent with default alongside the price consistent with full repayment in the region $(\underline{B}_T, \bar{B}_T]$. In addition to these results, *Proposition 4* states that a similar multiplicity can also arise under uncertainty (i.e. $tr^{min} < tr^{max}$), provided the costs of default are high enough relative to the range of possible realizations of transfer payments. This is intuitive: with a large range of possible tr_T it is likely that for some particularly small levels of transfers repayment is feasible even if investors are pessimistic and expect default in all states, and in consequence invest little in capital; this then would invalidate expectation of default in all states. As in the deterministic case discussed in Section 3, the higher the cost ψ, the lower $\underline{B}_T$ and the wider the region $(\underline{B}_T, \bar{B}_T]$ in which multiplicity is guaranteed.

Figure 4 plots the correspondence $\mathcal{Q}^{T\text{-}1}(B)$ from a Matlab simulation in which transfers tr_T are distributed according to a truncated normal distribution, for $\gamma = 1$ and $\psi = 0.15$. The flat part of the correspondence marked in green represents bond prices consistent with expectations of repayment in all states and expected productivity $E_{T-1}[A_T] = a_T$; these expectations can be validated for $B_T \leq \bar{B}_T$. The monotonically decreasing segment on the right marked in red corresponds to expectations of default in every state and $E_{T-1}[A_T] = a_T[1-\psi]$; such expectations can be validated for $B_T > \underline{B}_T$. Because in this example $\psi > \tilde{\psi}$ (and, in consequence,

Figure 4: Bond price correspondence $\mathcal{Q}^{T\text{-}1}(B)$: a simulation.



$\bar{B}_T > \underline{B}_T$), on $(\underline{B}_T, \bar{B}_T]$ both expectations of default in all states and repayment in all states can be validated—these two segments of $\mathcal{Q}^{T\text{-}1}(B)$ also appear on Figure 1(a) describing $\mathcal{Q}^{T\text{-}1}(B)$ for the deterministic case.

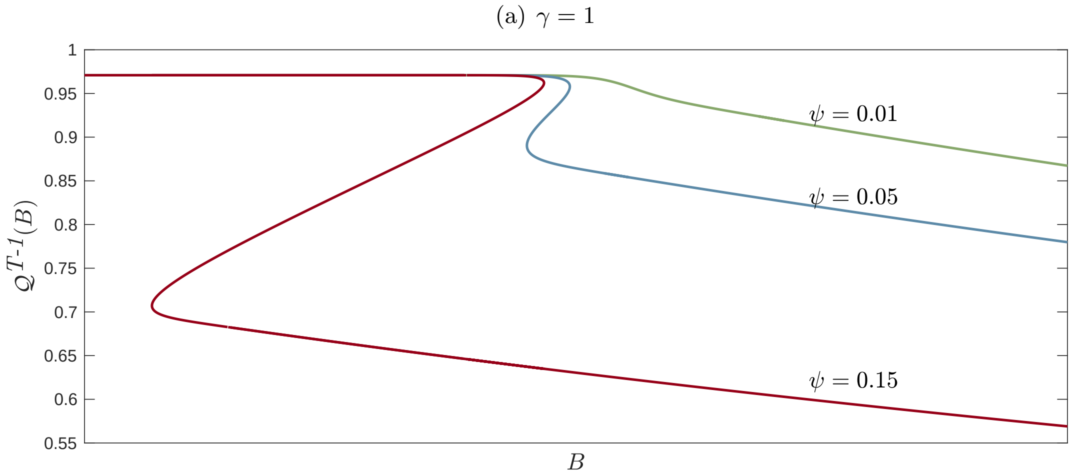
The middle segment of $\mathcal{Q}^{T\text{-}1}(B)$ marked in blue on Figure 4 represents bond prices consistent with repayment in some states and default in other states. Unlike the correspondence $\mathcal{Q}(B)$ constructed in subsection 4.2 (see Figure 3a), this correspondence is 'smooth' (i.e. there are no discrete jumps)—this is due to the distribution of transfers being continuous. The graph shows intuition behind the sufficient conditions for existence of equilibrium bond price corresponding to uncertain repayment: because of continuity, the blue line must connect the green and red sectors of $\mathcal{Q}^{T\text{-}1}(B)$; whenever $\bar{B}_T > \underline{B}_T$, we can be sure that there will be at least one such bond price corresponding to each B_T between $\underline{B}_T$ and $\bar{B}_T$ (and we can make a similar argument for when $\bar{B}_T < \underline{B}_T$). At the same time, the graph demonstrates that this sufficient condition is not necessary: the bond prices marked in blue can also be constructed for some $B_T < \underline{B}_T$ and for some $B_T > \bar{B}_T$.

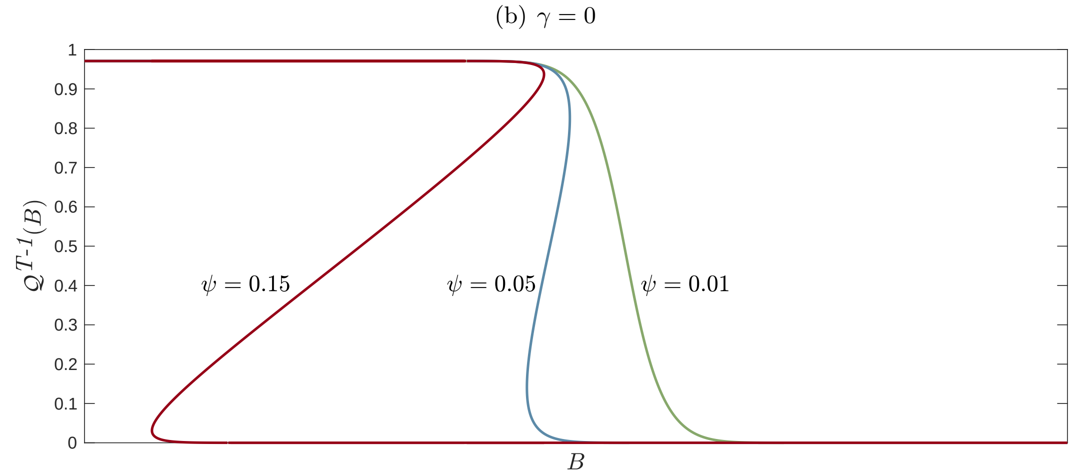
In line with *Proposition 4* we observe that a bond price consistent with *Definition 3* can be constructed for any B_T—therefore, we can construct $Q^{T\text{-}1}(B)$ by specifying a selection from $\mathcal{Q}^{T\text{-}1}(B)$ that assigns a unique price to each B_T. Furthermore, we can construct a decreasing selector function. For example, we can specify a function similar to that discussed in Section 3, that assigns risk-free bond prices for $B_T \leq \hat{B}_T$ and prices anticipating default in all states for $B_T > \hat{B}_T$, with $\hat{B}_T$ set between $\underline{B}_T$ and $\bar{B}_T$.

Figure 5(a) and Figure 5(b) show the correspondence $\mathcal{Q}^{T\text{-}1}(B)$ for different values of default costs ψ. In line with *Proposition 4*, while multiplicity arises for values of default costs that are relatively high, it disappears as ψ tends to zero: the correspondence $\mathcal{Q}^{T\text{-}1}(B)$ is uniquely-valued under $\psi = 0.01$. Furthermore, the range of values of debt for which multiplicity arises shrinks as the default cost is reduced.

The bond price correspondences plotted on Figure 5(a) and (b) appear to be moving upward

Figure 5: Bond price correspondences and default costs





as ψ decreases, and higher equilibrium bond prices become available for some debt levels. This, however, does not necessarily mean that the bond pricing schedules constructed as selectors from these correspondences would be decreasing in ψ: because of multiplicity, schedules $Q^{T\text{-}1}(B)$ constructed under high ψ would feature discrete jumps—their behavior around the jump points cannot be generalized without specifying how each jump point is selected. At the same time, we can examine the behavior of $Q^{T\text{-}1}(B)$ for debt regions in which the function is 'smooth'.

As we mention above, for a given ψ the function $Q^{T\text{-}1}(B)$ can be constructed by choosing a selector $\phi^{T\text{-}1}(B)$ from the correspondence $\Phi^{T\text{-}1}(B)$ that links each B_T with all appropriate thresholds $\hat{tr}_T$, and applying the function (63) to each pair $(B_T, \hat{tr}_T) : \hat{tr}_T = \phi^{T\text{-}1}(B_T)$. To reflect the fact that the threshold $\hat{tr}_T$ returned by $\phi^{T\text{-}1}(B)$ may be affected by the default cost ψ, we will now specify the threshold function in terms of both the level of debt and the default cost: $\phi^{T\text{-}1}(B, \psi)$. The bond pricing schedule constructed using $\phi^{T\text{-}1}(B, \psi)$ may be affected by the default cost as well—we now denote it with $Q^{T\text{-}1}(B, \psi)$. *Proposition 5* below considers a neighborhood of some point (B_T^0, ψ^0) in which the function $\phi^{T\text{-}1}(B, \psi)$ is differentiable in both arguments. It asserts that if $\phi^{T\text{-}1}(B,\psi)$ is decreasing in B_T (i.e. if higher debt results in default being triggered by lower realizations of transfers), then in that neighborhood the bond pricing schedule is decreasing in both B_T and ψ.

Proposition 5. *Suppose there exists a neighborhood of* (B_T^0,ψ^0) *in which the function* $\phi(B,\psi)$ *is differentiable in both* B_T *and* ψ *and is decreasing in* B_T*. Then in this neighborhood the bond pricing schedule* $Q^{T\text{-}1}(B,\psi)$ *decreases in* B_T *and* ψ*, i.e. higher default costs are associated with lower bond price.*
Proof. see Appendix A.

The assumption that $\phi^{T\text{-}1}(B,\psi)$ is decreasing in B_T is intuitive: it means that as debt rises, default happens in more states of the world. *Proposition 5* shows that whenever this is the case for some interval, then the bond pricing schedule on that interval is decreasing in both B_T and the value of the default cost. The latter result arises for two reasons. First, as in Section 3, higher costs of default mean that in the event of default the fiscal surplus would be lower: *ex post*, higher default penalty triggers a larger direct loss of output; *ex ante*, anticipated high penalties depress capital investment. This reduces pledgeable funds and the expected repayment rate whenever full repayment is uncertain. Second, as default costs increase, for each B_T associated with uncertain repayment, default can now be triggered in more states of the world—again, due to an *ex ante* decrease in capital investment. This mechanism is not present in the deterministic version of the model.

To make this contrast more evident, *Corollary 5A* shows that under $\gamma = 0$, i.e. when in the event of default the repayment rate is zero, we can assert that in neighborhoods where the bond pricing schedule is decreasing in B_T, it is also decreasing in ψ. Furthermore, if the bond pricing schedule is strictly decreasing in B_T, it is strictly decreasing in ψ as well.

Corollary 5A. *Suppose* $\gamma = 0$ *and that there exists a neighborhood of* (B_T^0,ψ^0) *in which the function* $\phi(B,\psi)$ *is differentiable over both* B_T *and* ψ*, and the bond pricing schedule is decreasing in* B_T*. Then in this neighborhood the bond pricing schedule* $Q^{T\text{-}1}(B,\psi)$ *is decreasing over* ψ*, i.e. higher default costs are associated with lower bond price. In addition, if* $Q^{T\text{-}1}(B,\psi)$ *is strictly decreasing over* B_T *in this neighborhood, it is also strictly decreasing in* ψ
Proof. see Appendix A.

Figure 5(b) depicts simulated bond pricing schedules for different default costs under $\gamma = 0$. Similarly to Figure 5(a), we see that the segments of the schedule associated with uncertainty over debt repayments move up as default costs decrease, suggesting that lower default costs lead to higher bond prices. Unlike on Figure 5(a), there is only one mechanism at work here: an increase in ψ causes a reduction in the expected capital return for B_T associated with uncertain repayment—this reduces capital accumulation *ex ante* and fiscal surpluses *ex post*, resulting in default in more states *ex post* and lower bond prices *ex ante*. We have shown that uncertainty over fundamentals of the economy brings forth an additional mechanism through which an increase in the default cost may adversely affect bond prices on newly issued debt.

6 Concluding Remarks

Defaults on government debt impair economic activity in debtor countries. Previous literature has asserted that these perceived default costs may deter debtor countries from defaulting on their debts when repayment is feasible, as governments would optimally choose to avoid default costs by repaying their creditors. In this paper—contrary to this intuition—we show that the presence of default costs may adversely affect the *feasibility* of debt repayment. This result arises because the losses that follow defaults are perceived not only by the governments, but also by the private capital investors; when the latter anticipate that default costs may dampen future capital returns, they cut their investment into domestic capital. This, in turn, lowers future output and revenues from tax collection, tightening the fiscal limits that the governments are facing, raising the probability of default for high levels of borrowing and reducing the associated bond prices.

We identify three channels through which an increase in the perceived default cost may result in lower bond prices. Two of these channels arise under the assumption that the amount of debt recovered after default depends positively on the concurrent level of economic activity within the debtor country (e.g. because it affects its fiscal surplus). First, high default costs have a direct negative effect on the post-default output. Second, the anticipation of capital returns hindered by default penalties discourages capital investment; this lowers future capital stock and output. Both these effects reduce perceived recovery rates following default (assuming they depend positively on output) and bond prices. The third channel emerges under uncertainty over the fundamentals of the economy. Consider high levels of debt that the government can repay in some future states of the world, but is forced to default on in the other states. If the perceived losses to productivity associated with 'default' states increase, this would depress current capital investment and cause reductions in future output and tax revenues. As a result, repayment would become infeasible in more states of the world, which would, again, reduce the expected recovery rate and the associated bond price.

We also demonstrate that the very presence of default costs may result in equilibrium multiplicity: with positive default costs the fiscal limits that the governments face may not be pinned down by the fundamentals of the economy. Intuitively, future fiscal surpluses depend positively on output, which, in turn, depends on future capital stock. But capital accumulation decisions are driven by expectations of how capital returns might be affected by default penalties, which can be self-fulfilling. On the one hand, when investors are 'optimistic', they invest more in capital; this raises future fiscal surpluses and reduces the probability of the government hitting the fiscal limit in the future, which validates investor optimism. On the other hand, investor pessimism translates into lower capital investment, lower future fiscal surpluses, tighter fiscal limits and default in more states—validating pessimistic expectations. The higher the default cost, the larger the gap between the 'pessimistic' and the 'optimistic' fiscal limits.

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Appendix A Proposition proofs

Proposition 1. *For $\psi > 0$ there exist infinitely many bond pricing schedules $Q^{T\text{-}1}(B)$ that satisfy Definition 1. The bond prices along these schedules 1) equal q^f_{T-1} for all $B_T \leq \underline{B}_T$ 2) assume one of two values for each $B_T \in (\underline{B}_T, \bar{B}_T]$, either q^f_{T-1} or $q^d_{T-1} < q^f_{T-1}$ 3) equal q^d_{T-1} for each $B_T > \bar{B}_T$, where*

$$\begin{aligned} q^f_{T-1} &= 1/(1+r), \\ q^d_{T-1} &= \frac{\gamma \cdot S(a_T[1-\psi], k_T(a_T[1-\psi]), tr_T)/r}{B_T}. \end{aligned}$$

Proof. Note that under $\psi > 0$ for $\underline{B}_T$ in (25) and $\bar{B}_T$ in (24) we have $\underline{B}_T < \bar{B}_T$, and therefore the interval $(\underline{B}_T, \bar{B}_T]$ is nonempty. Notice also that the definitions of the bond price and the recovery rate together imply that the bond price can only assume one of two values, q^d_{T-1} or q^f_{T-1}. As there are no other candidate bond prices, to characterize all possible bond pricing schedules it is sufficient to establish for each B_T whether q^d_{T-1} and q^f_{T-1} can be supported as equilibrium.

First, we show that for $B_T > \underline{B}_T$ the bond price q^d_{T-1} satisfies *Definition 1* with $A^e_T = a_T(1-\psi)$; furthermore, $q^d_{T-1} < q^f_{T-1}$. To check this, suppose that agents expect a default and anticipate productivity to be lowered by the default penalty, i.e. $A^e_T = a_T(1-\psi)$. Under these expectations the capital investment decision will be $k_T(a_T[1-\psi])$ in accordance with (21). Given this capital stock, the expectation of default is validated if (20) selects $\chi_T < 1$ and (22) selects $A_T = a_T(1-\psi)$. Note that $B_T > \underline{B}_T$ implies $B_T > S(a_T, k_T(a_T[1-\psi]), tr_T)\frac{1+r}{r}$, which given the expectation $A^e_T = a_T(1-\psi)$ we can rewrite as $B_T > S(a_T, k_T(A^e_T), tr_T)\frac{1+r}{r}$. We can now observe that for $B_T > \underline{B}_T$ and $A^e_T = a_T(1-\psi)$ (22) selects $A_T = a_T(1-\psi)$ (therefore satisfying 23) and (20) selects $\chi_T = \frac{\gamma \cdot S(a_T[1-\psi], k_T(a_T[1-\psi]), tr_T)(1+r)/r}{B_T}$. Finally, (19) implies the bond price of $q_{T-1} = \frac{\gamma \cdot S(a_T[1-\psi], k_T(a_T[1-\psi]), tr_T)/r}{B_T} \equiv q^d_{T-1}$. We have therefore established that for $B_T > \underline{B}_T$, the price q^d_{T-1} is consistent with (19)-(23) under an expectation of $A^e_T = a_T(1-\psi)$. Finally, note that on the interval $B_T > \underline{B}_T$ the price q^d_{T-1} is smaller than $q^f_{T-1} = \frac{1}{1+r}$. This follows from $\gamma \leq 1$ and the fact that $B_T > S(a_T, k_T(a_T[1-\psi]), tr_T)\frac{1+r}{r}$ on this interval.

Second, we show that for $B_T \leq \bar{B}_T$ the bond price q^f_{T-1} satisfies *Definition 1.* To see this, suppose there is an expectation of full repayment with $A^e_T = a_T$. Capital investment given this expectation will be $k_T(a_T)$ in line with (21). Note that $B_T \leq \bar{B}_T$ implies $B_T \leq S(a_T, k_T(a_T), tr_T)\frac{1+r}{r}$, or $B_T \leq S(a_T, k_T(A^e_T), tr_T)\frac{1+r}{r}$; this means that (22) selects $A_T = a_T$ (therefore satisfying 23) and (20) selects $\chi_T = 1$. Finally, (19) implies the bond price of $q_{T-1} = \frac{1}{1+r} \equiv q^f_{T-1}$. We have therefore shown that for $B_T \leq \bar{B}_T$, the price q^f_{T-1} is consistent with (19)-(23) under an expectation of $A^e_T = a_T$.

Third, we show that for $B_T \leq \underline{B}_T$ the bond price q^d_{T-1} does not satisfy *Definition 1.* From (20), the bond price q^d_{T-1} can only be equilibrium outcome if there are expectations of default; suppose it is the case, and $A^e_T = a_T(1-\psi)$, and therefore $k_T(A^e_T)$=$\mathrm{k}_T(a_T[1-\psi])$ and $S(a_T, k_T(A^e_T), tr_T) = S(a_T, k_T(a_T[1-\psi]), tr_T)$. Notice that $B_T \leq \underline{B}_T$ implies $B_T \leq S(a_T, k_T(a_T[1-\psi]), tr_T)\frac{1+r}{r}$. Given this, (22) implies $A_T = a_T$ and (20) implies full repayment. We reach a contradiction.

Fourth, we show that for $B_T > \bar{B}_T$ the bond price q^f_{T-1} does not satisfy *Definition 1.* From (20), the bond price q^f_{T-1} can only be supported as equilibrium under expectations of full repayment; suppose there are such expectationsd, and $A^e_T = a_T$, and therefore $k_T(A^e_T)$=$\mathrm{k}_T(a_T)$ and $S(a_T, k_T(A^e_T), tr_T) = S(a_T, k_T(a_T), tr_T)$. Notice that $B_T > \bar{B}_T$ implies $B_T > \bar{S}(a_T, k_T(a_T), tr_T)\frac{1+r}{r}$. Given this, (22) implies $A_T = a_T(1-\psi)$ and (20) implies default, a contradiction.

Combining these findings we can now state the following. For $B_T \leq \underline{B}_T$ the only price that satisfies *Definition 1* is q^f_{T-1}, i.e. the correspondence $\mathcal{Q}^{T\text{-}1}(B)$ is single-valued; for $B_T > \bar{B}_T$ the only price that satisfies *Definition 1* is q^d_{T-1}, and the correspondence $\mathcal{Q}^{T\text{-}1}(B)$ is single-valued in this region as well; for $B_T \in (\underline{B}_T, \bar{B}_T]$ both q^f_{T-1} and $q^d_{T-1} < q^f_{T-1}$ satisfy *Definition 1*, and the correspondence $\mathcal{Q}^{T\text{-}1}(B)$

connects each B_T with two admissible bond prices. As this is true for the whole interval $B_T \in (\underline{B}_T, \bar{B}_T]$, there is an infinite number of distinct selections $Q^{T\text{-}1}(B)$ from the correspondence $\mathcal{Q}^{T\text{-}1}(B)$ that all satisfy *Definition 1* and differ from one another on the interval $B_T \in (\underline{B}_T, \bar{B}_T]$. □

Corollary 1A. *The lower bound of the multiplicity interval, $\underline{B}_T$, decreases with ψ, i.e. higher default costs expand the region of debt in which default can be an equilibrium outcome.*

Proof. To show that $\underline{B}_T$ decreases with ψ, recall:

$$\underline{B}_T \equiv S(a_T, k_T(a_T[1-\psi]))\frac{1+r}{r} = \left[\tau a_T \left[\frac{\alpha a_T[1-\psi](1-\tau^K)}{r+\delta}\right]^{\frac{\alpha}{1-\alpha}} - tr_T\right]\frac{1+r}{r}. \tag{70}$$

The expression on the right-hand side decreases in ψ. The upper bound $\bar{B}_T$ is not affected by the value of default costs. Therefore, the length of the interval $(\underline{B}_T, \bar{B}_T]$ increases with ψ. □

Proposition 2. *For $\psi > 0$ there exist infinitely many decreasing bond pricing functions $Q^{T\text{-}1}(B)$ that satisfy Definition 1A such that $\max_{B_T}\{B_T \cdot Q^{T\text{-}1}(B_T)\}$ exists; all such functions belong to the function space described by (26).*

Proof. The proof consists of two parts. First, we show that For $\psi > 0$ there exist infinitely many decreasing bond pricing functions $Q^{T\text{-}1}(B)$ that satisfy *Definition 1A*, and describe a function space $\mathfrak{Q}$ that contains all such functions (and includes functions defined by 26). Second, we show that within $\mathfrak{Q}$, only functions defined by (26) permit existence of $\max_{B_T}\{B_T Q^{T\text{-}1}(B_T)\}$.

Denote by $\mathfrak{Q}$ the function space that includes two function families: first, a family described by (26) with $\hat{B}_T \in [\underline{B}_T, \bar{B}_T]$; second, a family described by

$$Q^{T\text{-}1}(B_T) = \begin{cases} q^f_{T-1}, & if \;\; B_T < \hat{B}_T; \\ q^d_{T-1}, & if \;\; B_T \geq \hat{B}_T. \end{cases} \tag{71}$$

with $\hat{B}_T \in (\underline{B}_T, \bar{B}_T]$.

In *Proposition 1* we established that $q^d_{T-1}(B_T) = \gamma \frac{S(a_T[1-\psi], k_T(a_T[1-\psi]), tr_T)/r}{B_T}$ satisfies *Definition 1A* for $B_T > \underline{B}_T$, that $q^f_{T-1} = \frac{1}{1+r}$ satisfies *Definition 1A* for any $B_T \leq \bar{B}_T$, and that $\underline{B}_T < \bar{B}_T$ for any $\psi > 0$. It follows that $Q^{T\text{-}1}(B)$ constructed in accordance with (26) satisfies *Definition 1A* for an arbitrary $\hat{B}_T \in [\underline{B}_T, \bar{B}_T]$ and that $Q^{T\text{-}1}(B)$ constructed in accordance with (71) satisfies *Definition 1A* for any $\hat{B}_T \in (\underline{B}_T, \bar{B}_T]$. Because on $B_T \in (\underline{B}_T, \bar{B}_T]$ we have $q^d_{T-1}(B_T) < q^f_{T-1}$, and because $q^d_{T-1}(B_T)$ is decreasing in $B_T > \underline{B}_T$, the functions described in (26) and (71) are decreasing. We conclude that all functions in $\mathfrak{Q}$ satisfy *Definition 1A* and are decreasing on B.

We will now show that there are no decreasing schedules $Q^{T\text{-}1*}(B)$ satisfying *Definition 1A* that cannot be described using (26) or (71). In other words, we show that $\mathfrak{Q}$ describes all decreasing selections from $\mathcal{Q}^{T\text{-}1}(B)$. Let us attempt to construct a decreasing selection $Q^{T\text{-}1*}(B)$ from $\mathcal{Q}^{T\text{-}1}(B)$ that does not belong to $\mathfrak{Q}$. On $B_T \leq \underline{B}_T$ and $B_T > \bar{B}_T$ the correspondence $\mathcal{Q}^{T\text{-}1}(B)$ is single-valued, and therefore there is only one possible selection from $\mathcal{Q}^{T\text{-}1}(B)$; it follows that on $B_T \leq \underline{B}_T$ and $B_T > \bar{B}_T$ the graph of $Q^{T\text{-}1*}(B)$ must be the same as the graphs of all functions in $\mathfrak{Q}$.

On $(\underline{B}_T, \bar{B}_T]$ the correspondence $\mathcal{Q}^{T\text{-}1}(B)$ returns two distinct values for each B_T, q^f_{T-1} and $q^d_{T-1}(B_T)$, with $q^f_{T-1} > q^d_{T-1}(B_T)$—the function $Q^{T\text{-}1*}(B)$ must assign one of the two values to each B_T. There are three possible ways in which a decreasing selection can be constructed on this interval. First, we can set $Q^{T\text{-}1*}(B) = q^f_{T-1}$—this possibility is incorporated in (26). Second, we can set $Q^{T\text{-}1*}(B) = q^d_{T-1}(B_T)$; this is accounted for by both (26) and (**??**). Third, we can permit the function $Q^{T\text{-}1*}(B)$ to jump *once* from q^f_{T-1} to $q^d_{T-1}(B_T)$. Denote by $\hat{B}_T$ the jump point; we can define the jump such that $\lim_{B_T\to\hat{B}_T^-} = Q^{T\text{-}1*}(B)$ and $\lim_{B_T\to\hat{B}_T^+} \neq Q^{T\text{-}1*}(B)$, this is accounted for in (26). Alternatively, the jump can be set such that $\lim_{B_T\to\hat{B}_T^+} = Q^{T\text{-}1*}(B)$ and $\lim_{B_T\to\hat{B}_T^-} \neq Q^{T\text{-}1*}(B)$; this possibility is incorporated in (**??**).

We have shown that the graph of $Q^{T\text{-}1*}(B)$ matches the graphs of all functions in $\mathfrak{Q}$ on $B_T \leq \underline{B}_T$ and $B_T > \bar{B}_T$; we have also shown that the graph of $Q^{T\text{-}1*}(B)$ matches the graph of at least one function in $\mathfrak{Q}$ on $B_T \in (\underline{B}_T, \bar{B}_T]$. It follows that $Q^{T\text{-}1*}(B) \in \mathfrak{Q}$. We reach a contradiction and conclude that $\mathfrak{Q}$ must describe all decreasing functions that satisfy *Definition 1A*.

Note that for both (26) and (71) the supremum of $\{B_T \cdot Q^{T\text{-}1}(B_T)\}$ lies at the jump point and equals $\lim_{B_T\to\hat{B}_T^-} B_T \cdot Q^{T\text{-}1}(B_T) = \hat{B}_T q^f_T \equiv \hat{B}_T \frac{1}{1+r}$, the value of debt at the jump point multiplied by the risk-free price. Under (26), $Q^{T\text{-}1}(\hat{B}_T) = q^f_T$ and therefore $\hat{B}_T \cdot Q^{T\text{-}1}(\hat{B}_T) = \hat{B}_T q^f_T$, which is the maximum of $\{B_T \cdot Q^{T\text{-}1}(B_T)\}$. In other words, the revenue attains a maximum at $\hat{B}_T$. Under (71), $Q^{T\text{-}1}(\hat{B}_T) = q^d_T(\hat{B}_T) < q^f_T$ and therefore $\hat{B}_T \cdot Q^{T\text{-}1}(\hat{B}_T) < \hat{B}_T q^f_T$. But we can always find a $B^*_T < \hat{B}_T$ such that $B^*_T \cdot Q^{T\text{-}1}(B^*_T) = B^*_T \cdot q^f_T > \hat{B}_T \cdot Q^{T\text{-}1}(\hat{B}_T)$. However, for any such B^*_T there always exists a $B^{**}_T > B^*_T$ such that $B^{**}_T \cdot Q^{T\text{-}1}(B^{**}_T) > B^*_T \cdot Q^{T\text{-}1}(B^*_T)$—in other words, under (71) the revenue function has no maximum value, as for any candidate B^*_T we can find a $B^{**}_T > B^*_T$ that delivers a higher revenue. We conclude that within $\mathfrak{Q}$ only the function family in (26) corresponds to a revenue function $B_T \cdot Q^{T\text{-}1}(B_T)$ that has a maximum value.

□

Corollary 2A. *Given $\psi > 0$ and a decreasing bond pricing schedule, the maximum revenue from the auction of government bonds can take any value in $[\frac{1}{1+r}\underline{B}_T, \frac{1}{1+r}\bar{B}_T]$, i.e. there is indeterminacy over the maximum revenue from the bonds auction.*

Proof. Proposition 2 establishes that all decreasing bond pricing schedules $Q^{T\text{-}1}(B_T)$ consistent with *Definition 1A* such that the corresponding revenue function $Q^{T\text{-}1}(B_T)B_T$ has a maximum can be described by (26). Given (26), the revenue from the auction of bonds can therefore be written as

$$Q^{T\text{-}1}(B_T)B_T = \begin{cases} q^f_{T-1}B_T = \frac{1}{1+r}B_T, & if\;\; B_T \leq \hat{B}_T; \\ q^d_{T-1}B_T = \gamma S(a_T[1-\psi], k_T(a_T[1-\psi]), tr_T)/r, & if\;\; B_T > \hat{B}_T, \end{cases} \tag{72}$$

with $\hat{B}_T \in [\underline{B}_T, \bar{B}_T]$. Note that $q^f_{T-1}B_T$ is strictly increasing in B_T while $q^d_{T-1}B_T$ is constant. Furthermore, at $\hat{B}_T$ there is a downward jump because by *Proposition 1*, $q^f_{T-1} > q^d_{T-1}$. It follows that for the bond pricing schedule described in (26) the maximum revenue is at $\frac{1}{1+r}\hat{B}_T$. Depending on $\hat{B}_T$, the maximum revenue can assume any value in $[\frac{1}{1+r}\underline{B}_T, \frac{1}{1+r}\bar{B}_T]$. □

Definition 1B. Equilibrium bond pricing schedule $Q^{t\text{-}1}(B)$: perfect foresight. *For $t \leq T-1$ the equilibrium bond pricing schedule is a function $Q^{t\text{-}1}(B)$ such that for each $B_t \in B$ the bond price $q_{t-1} \in Q^{t\text{-}1}(B_t)$ satisfies*

$$q_{t-1} = \frac{\chi_t}{1+r} \tag{73}$$

where

$$\chi_t = \begin{cases} 1, & if\;\; B_t \leq S(a_t, k_T(A^e_t), tr_t) + \max_{B_{t+1}}\{Q^t(B_{t+1})B_{t+1}\}; \\ \frac{\gamma\cdot S(a_t[1-\psi], k(A^e_t), tr_t)\cdot(1+r)/r}{B_t}, & if\;\; B_t > S(a_t, k_T(A^e_t), tr_t) + \max_{B_{t+1}}\{Q^t(B_{t+1})B_{t+1}\}, \end{cases} \tag{74}$$

and

$$k(A_t^e) = \left[\frac{\alpha A_t^e(1-\tau^K)}{r+\delta}\right]^{\frac{1}{1-\alpha}} \tag{75}$$

with

$$A_t = \begin{cases} a_t, & if \quad B_t \leq S(a_t, k_T(A_t^e), tr_t) + \max\limits_{B_{t+1}}\{Q^t(B_{t+1})B_{t+1}\}; \\ a_t(1-\psi), & if \quad B_t > S(a_t, k_T(A_t^e), tr_t) + \max\limits_{B_{t+1}}\{Q^t(B_{t+1})B_{t+1}\}. \end{cases} \tag{76}$$

and

$$A_t^e = A_t \tag{77}$$

where for $t < T-1$ the function $Q^t(B)$ must satisfy Definition 1B, and for $t = T-1$ the function $Q^t(B)$ must satisfy Definition 1A.

Proposition 3 *Consider the economy starting period $t \leq T-2$ with pre-existing debt B_t and capital $k(A_t^e)$ where $\psi > 0$. There exists an infinite number of perfect foresight equilibria in which each period the government faces a decreasing bond pricing schedule that satisfies Definition 1B; furthermore, for each debt level B_t in $(\underline{B}_t^T, \bar{B}_t^T]$ we can construct at least one such equilibrium where B_t is consistent with default in t, and at least one equilibrium in which it is consistent with full debt repayment in t. As $T \to \infty$ this interval converges to $(\underline{B}_t^\infty, \bar{B}_t^\infty]$.*

Proof. We need to show that we can construct a sequence of bond pricing schedules such that each schedule satisfies *Definition 1B*. We will solve this problem by backward induction: starting in $T-1$ we construct some $Q^{T\text{-}1*}(B)$ that satisfies *Definition 1A*; we then construct a $Q^{T\text{-}2*}(B)$ that satisfies *Definition 1B* given $Q^{T\text{-}1*}(B)$; we then construct a $Q^{T\text{-}3*}(B)$ that satisfies *Definition 1B* given $Q^{T\text{-}2*}(B)$, etc. This exercise is simplified by the fact that *Definition 1B* is very similar to the *Definition 1B*, except that the conditions that select χ_t and A_t, (74) and (76), include not only the current surplus as in *Definition 1A*, but also the maximum revenue from the auction of bonds in the next period, $\max\limits_{B_{t+1}}\{Q^t(B_{t+1})B_{t+1}\}$. Therefore, when solving the model with backward induction we will make sure to select bond pricing schedules that are such that $\max\limits_{B_{t+1}}\{Q^t(B_{t+1})B_{t+1}\}$ exists.

We start by looking for $Q^{T\text{-}1}(B)$ that satisfies *Definition 1A*. Per the statement of *Proposition 3* we must choose a decreasing $Q^{T\text{-}1}(B)$. Furthermore, to be able to solve the problem with backward induction we need to make sure that the $Q^{T\text{-}1}(B)$ we construct is such that the corresponding revenue from the auction of bonds has a maximum on B_T. By *Proposition 2*, all functions that comply with these criteria (i.e. satisfy *Definition 1A*, are decreasing and are such that the revenue function has a maximum) belong to the function family described by (26) with thresholds $\hat{B}_T$ in $[\underline{B}_T, \bar{B}_T]$.

Denote by $Q^{T\text{-}1*}(B)$ a specific selection from this family of functions with a corresponding $\hat{B}_T^* \in [\underline{B}_T, \bar{B}_T]$. We now show that given this bond pricing schedule in period $T-1$ we can construct a decreasing bond pricing schedule arising in period $T-2$ consistent with *Definition 1B*; with some abuse of notation we denote this schedule with $Q^{T\text{-}2}(B|\hat{B}_T^*)$ to indicate that it is conditional on a specific selection $Q^{T\text{-}1*}(B)$ with a distinct corresponding $\hat{B}_T^* \in [\underline{B}_T, \bar{B}_T]$.

Given a choice of $\hat{B}_T^*$, the maximum revenue from the auction of bonds the government can extract in period $T-1$ equals $\hat{B}_T^*/(1+r)$. Note that now that we have a specific $\max\limits_{B_{t+1}}\{Q^t(B_{t+1})B_{t+1}\} = \hat{B}_T^*/(1+r)$, the *Definition 1B* is almost the same as *Definition 1A*, with the addition of a constant term in the conditions selecting χ_t and A_t. We can therefore use the proofs in *Propositions 1* and *2* to establish existence of $Q^{T\text{-}2}(B|\hat{B}_T^*)$ and its properties. We briefly demonstrate the intuition below.

The repayment in $T-1$ is possible if:

$$B_{T-1} \leq S(a_{T-1}, k(A^e_{T-1}), tr_{T-1}) + \hat{B}^*_T/(1+r). \tag{78}$$

There are two possible expectations about productivity, A^e_{T-1}, that can be supported as equilibria in $T-2$: $A^e_{T-1} = a_{T-1}$ and $A^e_{T-1} = a_{T-1}(1-\psi)$. Accordingly, there are two possible values that $k(A^e_{T-1})$ in (78) can take. The expectation $A^e_{T-1} = a_{T-1}$ can be validated as equilibrium if:

$$B_{T-1} \leq S(a_{T-1}, k(a_{T-1}), tr_{T-1}) + \hat{B}^*_T/(1+r) \equiv \bar{B}_{T-1}(\hat{B}^*_T), \tag{79}$$

while the expectation $A^e_{T-1} = a_{T-1}(1-\psi)$ can be validated as equilibrium if

$$B_{T-1} > S(a_{T-1}, k(a_{T-1}[1-\psi]), tr_{T-1}) + \hat{B}^*_T/(1+r) \equiv \underline{B}_{T-1}(\hat{B}^*_T). \tag{80}$$

Note that (79) and (80) mirror (24) and (25), and that for $\psi > 0$ we have $\bar{B}_{T-1}(\hat{B}^*_T) > \underline{B}_{T-1}(\hat{B}^*_T)$.

Retracing the steps detailed in the proof of *Proposition 1* we can ascertain that given $\hat{B}^*_T/(1+r)$ there exist infinitely many bond pricing schedules $Q^{T\text{-}2}(B|\hat{B}^*_T)$ that satisfy *Definition 1B*; the bond prices along these schedules 1) equal q^f_{T-2} for all $B_{T-1} \leq \underline{B}_{T-1}(\hat{B}^*_T)$; 2) assume one of two values for each $B_{T-1} \in (\underline{B}_{T-1}(\hat{B}^*_T), \bar{B}_{T-1}(\hat{B}^*_T)]$, either q^f_{T-2} or q^d_{T-2}; 3) equal q^d_{T-2} for each $B_{T-1} > \bar{B}_{T-1}(\hat{B}^*_T)$, where $q^f_{T-2} = 1/(1+r)$ and $q^d_{T-2} = \frac{\gamma \cdot S(a_{T-1}[1-\psi], k_T(a_{T-1}[1-\psi]), tr_{T-1})/r}{B_{T-1}}$.

We can now use the arguments made in *Proposition 2* to construct a decreasing bond pricing schedule $Q^{T\text{-}2}(B|\hat{B}^*_T)$ that satisfies *Definition 1B*, such that the corresponding revenue function has a maximum on B_{T-1}. Following the steps of *Proposition 2* we verify that there are infinitely many such schedules; a subset of them can be described as a function family similar to (26):

$$Q^{T\text{-}2}(B_{T-1}|\hat{B}^*_T) = \begin{cases} q^f_{T-2}, & if \;\; B_{T-1} \leq \hat{B}_{T-1}; \\ q^d_{T-2}, & if \;\; B_{T-1} > \hat{B}_{T-1}. \end{cases} \tag{81}$$

where $\hat{B}_{T-1}$ is any value in $[\underline{B}_{T-1}(\hat{B}^*_T), \bar{B}_{T-1}(\hat{B}^*_T)]$.

We verified that for any $\hat{B}^*_T \in [\underline{B}_T, \bar{B}_T]$ we can use (81) to construct a decreasing function $Q^{T\text{-}2}(B|\hat{B}^*_T)$ that is consistent with *Definition 1B*. We will now use this result to construct a family of decreasing functions $Q^{T\text{-}2}(B)$, such that for each selection $Q^{T\text{-}2*}(B)$ from this family there is at least one $Q^{T\text{-}1*}(B)$ consistent with *Definition 1A* such that $Q^{T\text{-}2*}(B)$ satisfies *Definition 1B* given $Q^{T\text{-}1*}(B)$.

Within the family of functions described by (81) the thresholds $\hat{B}_{T-1}$ can vary between $[\underline{B}_{T-1}(\hat{B}^*_T), \bar{B}_{T-1}(\hat{B}^*_T)]$ where $\hat{B}^*_T$ can take any value in $[\underline{B}_T, \bar{B}_T]$. Hence, the threshold $\hat{B}_{T-1}$ in (81) can take any value in $[\underline{B}_{T-1}(\underline{B}_T), \bar{B}_{T-1}(\bar{B}_T)]$. In other words, for each $\hat{B}^*_{T-1} \in [\underline{B}_{T-1}(\underline{B}_T), \bar{B}_{T-1}(\bar{B}_T)]$ we can use (81) to construct an equilibrium bond pricing schedule $Q^{T\text{-}2*}(B)$, as there will always be some counterpart $Q^{T\text{-}1*}(B)$ that satisfies *Definition 1A* such that given $Q^{T\text{-}1*}(B)$, $Q^{T\text{-}2*}(B)$ satisfies *Definition 1B*. The resulting family of decreasing equilibrium bond pricing schedules can be described by setting

$$Q^{T\text{-}2}(B_{T-1}) = \begin{cases} q^f_{T-2}, & if \;\; B_{T-1} \leq \hat{B}_{T-1}; \\ q^d_{T-2}, & if \;\; B_{T-1} > \hat{B}_{T-1}. \end{cases} \tag{82}$$

and requiring that $\hat{B}_{T-1} \in [\underline{B}_{T-1}, \bar{B}_{T-1}]$ where

$$\begin{aligned} \underline{B}_{T-1} &\equiv \underline{B}_{T-1}(\underline{B}_T) = S(a_{T-1}, k(a_{T-1}[1-\psi]), tr_{T-1}) + S(a_T, k(a_T[1-\psi]), tr_T)/r, \\ \bar{B}_{T-1} &\equiv \bar{B}_{T-1}(\bar{B}_T) = S(a_{T-1}, k(a_{T-1}), tr_{T-1}) + S(a_T, k(a_T), tr_T)/r. \end{aligned}$$

Repeating the steps outlined above we confirm that a set of decreasing equilibrium bond pricing schedules arising in $T-3$ can be constructed by setting

$$Q^{T\text{-}3}(B_{T-2}) = \begin{cases} q^f_{T-3}, & if \;\; B_{T-2} \leq \hat{B}_{T-2}; \\ q^d_{T-3}, & if \;\; B_{T-2} > \hat{B}_{T-2}. \end{cases} \tag{83}$$

and demanding that $\hat{B}_{T-2} \in [\underline{B}_{T-2}, \bar{B}_{T-2}]$ where

$$\begin{aligned}
\underline{B}_{T-2} &= S(a_{T-2}, k(a_{T-2}), tr_{T-2}) + \frac{S(a_{T-1}, k(a_{T-1}), tr_{T-1})}{1+r} + \frac{S(a_T, k(a_T), tr_T)/r}{1+r}, \\
\bar{B}_{T-2} &= S(a_{T-2}, k(a_{T-2}[1-\psi]), tr_{T-2}) + \frac{S(a_{T-1}, k(a_{T-1}[1-\psi]), tr_{T-1})}{1+r} + \\
&\quad + \frac{S(a_T, k(a_T[1-\psi]), tr_T)/r}{1+r}.
\end{aligned}$$

For each function $Q^{T\text{-}3^*}(B)$ in (83) we can construct a decreasing schedule $Q^{T\text{-}1^*}(B)$ consistent with *Definition 1A* and a decreasing schedule $Q^{T\text{-}2^*}(B)$ consistent with *Definition 1B* given $Q^{T\text{-}1^*}(B)$, such that given $Q^{T\text{-}2^*}(B)$ the function $Q^{T\text{-}3^*}(B)$ satisfies *Definition 1B*.

Repeating these steps and iterating backwards we deduce that for bonds issued in period t we can construct a family of decreasing equilibrium bond pricing schedules by setting

$$Q^{t}(B_{t+1}) = \begin{cases} q^f_{t}, & if \;\; B_{t+1} \leq \hat{B}_{t+1}; \\ q^d_{t}, & if \;\; B_{t+1} > \hat{B}_{t+1}. \end{cases} \tag{84}$$

and demanding that $\hat{B}_{t+1} \in [\underline{B}_{t+1}, \bar{B}_{t+1}]$ with

$$\underline{B}_{t+1} = \sum_{i=1}^{T-t-1} \frac{S(a_{t+i}, k(a_{t+i}[1-\psi]), tr_{t+i})}{(1+r)^{i-1}} + \frac{S(a_T, k(a_T[1-\psi]), tr_T)/r}{(1+r)^{T-t-2}}, \tag{85}$$

$$\bar{B}_{t+1} = \sum_{i=1}^{T-t-1} \frac{S(a_{t+i}, k(a_{t+i}), tr_{t+i})}{(1+r)^{i-1}} + \frac{S(a_T, k(a_T), tr_T)/r}{(1+r)^{T-t-2}}. \tag{86}$$

All functions in the family described by (84) satisfy *Definition 1B*, in that for any such function $Q^{t^*}(B)$ we can construct a sequence of functions $Q^{t+1^*}(B), Q^{t+2^*}(B), ..., Q^{T\text{-}1^*}(B)$ such that $Q^{T\text{-}1^*}(B)$ satisfies *Definition 1A*, $Q^{T\text{-}2^*}(B)$ satisfies *Definition 1B* given $Q^{T\text{-}1^*}(B)$,..., $Q^{t+1^*}(B)$ satisfies *Definition 1B* given $Q^{t+2^*}(B)$, and $Q^{t^*}(B)$ satisfies *Definition 1B* given $Q^{t+1^*}(B)$.

We have now shown that any bond pricing schedule described by (84) can be supported as equilibrium. For each such function the maximum revenue from the auction of bonds that the government can extract in period t equals $\hat{B}_{t+1}/(1+r)$. It follows that the equilibrium maximum revenue from the auction of bonds can take any value in $[\underline{B}_{t+1}/(1+r), \bar{B}_{t+1}/(1+r)]$. Therefore, in period t default on debt B_t can be supported as equilibrium if:

$$B_t > S(a_t, k(A^e_t), tr_t) + \sum_{i=1}^{T-t-1} \frac{S(a_{t+i}, k(a_{t+i}[1-\psi]), tr_{t+i})}{(1+r)^{i}} + \frac{S(a_T, k(a_T[1-\psi]), tr_T)/r}{(1+r)^{T-t-1}} \equiv \underline{B}^T_t$$

while repayment can be supported as equilibrium if:

$$B_t \leq S(a_t, k(A_t^e), tr_t) + \sum_{i=1}^{T-t-1} \frac{S(a_{t+i}, k(a_{t+i}), tr_{t+i})}{(1+r)^i} + \frac{S(a_T, k(a_T), tr_T)/r}{(1+r)^{T-t-1}} \equiv B_t^T.$$

Since $S(a_T, k(a_T), tr_T)$ is bounded, taking a limit $T \to \infty$ yields

$$\begin{aligned}
\lim_{T\to\infty} \underline{B}_t^T &= S(a_t, k(A_t^e), tr_t) + \sum_{i=1}^{\infty} \frac{S(a_{t+i}, k(a_{t+i}[1-\psi]), tr_{t+i})}{(1+r)^i} \equiv \underline{B}_t^{\infty}, \\
\lim_{T\to\infty} \bar{B}_t^T &= S(a_t, k(A_t^e), tr_t) + \sum_{i=1}^{\infty} \frac{S(a_{t+i}, k(a_{t+i}), tr_{t+i})}{(1+r)^i} \equiv \bar{B}_t^{\infty}.
\end{aligned}$$

□

Proposition 4. *Equilibrium bond pricing schedule $Q^{T\text{-}1}(B)$ that satisfies Definition 3 exists. Furthermore, if $\psi > \tilde{\psi}$, then there exist multiple decreasing bond pricing schedules $Q^{T\text{-}1}(B)$ that satisfy Definition 3 and take distinct values on the interval $(\underline{B}_T, \bar{B}_T]$. When the cost ψ goes up, the interval $(\underline{B}_T, \bar{B}_T]$ expands as the threshold $\underline{B}_T$ decreases.*

Proof. Here we will show formally that for any value of $B_T \in B$ a bond price satisfying *Definition 3* can be constructed. We will split B into three regions and show that at least one bond price satisfying *Definition 3* can be constructed in each region.

First, consider $B_T \leq \bar{B}_T$. For this region we can construct a bond price consistent with anticipated repayment in every state. If repayment is anticipated in every state, then according to (58) capital must equal

$$k_T = \left[\frac{\alpha a_T(1-\tau^K)}{r+\delta}\right]^{\frac{1}{1-\alpha}} \tag{87}$$

Under this capital choice, fiscal surplus defined in (92) is

$$S(A_T, k_T(E_{T-1}[A_T]), tr_T) = \tau A_T \left[\frac{\alpha a_T(1-\tau^K)}{r+\delta}\right]^{\frac{\alpha}{1-\alpha}} - tr_T. \tag{88}$$

Conditions (57) and (59) select $\chi_T = 1$ and $A_T = a_T$ for states in which $B_T \leq S(a_T, k_T(E_{T-1}[A_T]), tr_T) \cdot \frac{1+r}{r}$, or

$$B_T \leq \left[\tau a_T \left[\frac{\alpha a_T(1-\tau^K)}{r+\delta}\right]^{\frac{\alpha}{1-\alpha}} - tr_T\right] \cdot \frac{1+r}{r} \tag{89}$$

The right-hand side of the above inequality is decreasing in tr_T; to ensure that $\chi_T = 1$ and $A_T = a_T$ are selected in all states it is therefore sufficient to check that it is selected in the state with the highest possible transfers, $tr_T = tr^{max}$ which will be true when:

$$B_T \leq \left[\tau a_T \left[\frac{\alpha a_T(1-\tau^K)}{r+\delta}\right]^{\frac{\alpha}{1-\alpha}} - tr^{max}\right] \cdot \frac{1+r}{r} \tag{90}$$

which holds for any $B_T \leq \bar{B}_T$. We have therefore verified that for $B_T \leq \bar{B}_T$ expectations of repayment in all states are consistent with (58), (57) and (59). We can now use (56) to construct the corresponding bond price $q = 1/(1+r)$ that is consistent with all requirements of *Definition 3*.

Second, consider $B_T > \underline{B}_T$. For this region we can construct a bond price consistent with anticipated default in every state. Given expectations of default penalties in every state, capital in (58) must equal

$$k_T = \left[\frac{\alpha a_T(1-\psi)(1-\tau^K)}{r+\delta}\right]^{\frac{1}{1-\alpha}} \tag{91}$$

Given this capital choice, fiscal surplus defined in (92) is

$$S(A_T, k_T(E_{T-1}[A_T]), tr_T) = \tau A_T \left[\frac{\alpha a_T(1-\psi)(1-\tau^K)}{r+\delta}\right]^{\frac{\alpha}{1-\alpha}} - tr_T. \tag{92}$$

Conditions (57) and (59) select $\chi_T < 1$ and $A_T = a_T(1-\psi)$ for states in which $B_T > S(a_T, k_T(E_{T-1}[A_T]), tr_T) \cdot \frac{1+r}{r}$, or in this case

$$B_T > \left[\tau a_T \left[\frac{\alpha a_T(1-\psi)(1-\tau^K)}{r+\delta}\right]^{\frac{\alpha}{1-\alpha}} - tr_T\right] \cdot \frac{1+r}{r} \tag{93}$$

Again, the right-hand side of the above inequality is decreasing in tr_T; to ensure that $\chi_T < 1$ and $A_T = a_T(1-\psi)$ are selected in all states it is sufficient to check that it is selected in the state with the lowest possible transfers, $tr_T = tr^{min}$ which will be true when:

$$B_T > \left[\tau a_T \left[\frac{\alpha a_T(1-\psi)(1-\tau^K)}{r+\delta}\right]^{\frac{\alpha}{1-\alpha}} - tr^{min}\right] \cdot \frac{1+r}{r} \tag{94}$$

which holds for any $B_T < \underline{B}_T$. We verified that for $B_T \leq \underline{B}_T$ expectations of default in all states are consistent with (58), (57) and (59). We can now use (56) to construct the corresponding bond price that is consistent with all requirements of *Definition 3*.

Third, suppose $\underline{B}_T > \bar{B}_T$ and consider $B_T \in (\bar{B}_T, \underline{B}_T]$, or that $\underline{B}_T < \bar{B}_T$ and consider $B_T \in [\underline{B}_T, \bar{B}_T)$. For this region we can construct a bond price consistent with anticipated default in some states and repayment in other states. Suppose that there is a $\hat{tr}_T \in [tr^{min}, tr^{max})$ such that repayment is expected for states with $tr_T \leq \hat{tr}_T$, while default is expected in states where $tr_T > \hat{tr}_T$. We will show that 1) If such $\hat{tr}_T$ exists, we can construct a bond price satisfying *Definition 3* and that 2) such $\hat{tr}_T$ exists.

Given expectations discussed above, capital choice in (58) equals capital in (64), while the fiscal surplus is given by (65). For the expectations above to be validated, it must be true that (57) and (59) select $\chi_T = 1$ and $A_T = a_T$ for $tr_T \leq \hat{tr}_T$ and $\chi_T < 1$ and $A_T = a_T(1-\psi)$ for $tr_T > \hat{tr}_T$. In other words, for $tr_T \leq \hat{tr}_T$ it must be that

$$B_T \leq \left[\tau a_T \left[\frac{\alpha a_T\left[1-(1-F(\hat{tr}_T))\psi\right](1-\tau^K)}{r+\delta}\right]^{\frac{\alpha}{1-\alpha}} - tr_T\right] \frac{1+r}{r} \tag{95}$$

and for $tr_T < \hat{tr}_T$ we must have

$$B_T > \left[\tau a_T \left[\frac{\alpha a_T\left[1-(1-F(\hat{tr}_T))\psi\right](1-\tau^K)}{r+\delta}\right]^{\frac{\alpha}{1-\alpha}} - tr_T\right] \frac{1+r}{r} \tag{96}$$

Note that the right-hand side of the two expressions above is decreasing in tr_T. This means that if there is a $\hat{tr}_T$ such that

$$B_T = \left[\tau a_T \left[\frac{\alpha a_T\left[1-(1-F(\hat{tr}_T))\psi\right](1-\tau^K)}{r+\delta}\right]^{\frac{\alpha}{1-\alpha}} - \hat{tr}_T\right] \frac{1+r}{r} \equiv \Omega(\hat{tr}_T) \tag{97}$$

which lies in $[tr^{min}, tr^{max})$, then it is indeed true that (57) and (59) select $\chi_T = 1$ and $A_T = a_T$ for $tr_T \leq \hat{tr}_T$ and $\chi_T < 1$ and $A_T = a_T(1-\psi)$ for $tr_T > \hat{tr}_T$.

It remains to show that a solution to problem (97), $B_T = \Omega(\hat{tr}_T)$, can be found in $[tr^{min}, tr^{max})$. The function $\Omega(\hat{tr}_T)$ is continuous on $[tr^{min}, tr^{max})$ which means that on $[tr^{min}, tr^{max})$ it takes each value in $[\Omega(tr^{min}), \Omega(tr^{max}))$ at least once.

If $\underline{B}_T > \bar{B}_T$ then $\Omega(tr^{min}) > \Omega(tr^{max})$; in that case we would consider $B_T \in (\bar{B}_T, \underline{B}_T]$ and therefore know that $B_T \in (\Omega(tr^{max}), \Omega(tr^{min})]$. It follows that there is at least on point of intersection between B_T and $\Omega(\hat{tr}_T)$ in $[tr^{min}, tr^{max})$.

If $\underline{B}_T < \bar{B}_T$ then $\Omega(tr^{min}) < \Omega(tr^{max})$; in that case we would consider $B_T \in [\underline{B}_T, \bar{B}_T)$,which would mean that $B_T \in [\Omega(tr^{min}), \Omega(tr^{max}))$. Again, it follows that there is at least on point of intersection between B_T and $\Omega(\hat{tr}_T)$ in $[tr^{min}, tr^{max})$.

We have shown that if $\underline{B}_T < \bar{B}_T$, then for any $B_T \in [\underline{B}_T, \bar{B}_T)$ we can find a $\hat{tr}_T$ such that given the capital choice in (58), (57) and (59) select $\chi_T = 1$ and $A_T = a_T$ for $tr_T \leq \hat{tr}_T$ and $\chi_T < 1$ and $A_T = a_T(1-\psi)$ for $tr_T > \hat{tr}_T$. We have shown that the same is true for $\underline{B}_T > \bar{B}_T$ and $(\bar{B}_T, \underline{B}_T]$. We can then use (56) to construct the corresponding bond price that is consistent with all requirements of *Definition 3*.

The steps above ascertain that 1) if $\underline{B}_T \leq \bar{B}_T$, then we can construct at least one bond price consistent with *Definition 3* for each B_T and 2) the same is true if $\underline{B}_T > \bar{B}_T$. There, the bond pricing schedule described in *Definition 3* exists.

We have also shown that if $\underline{B}_T < \bar{B}_T$, then for each B_T on the interval $[\underline{B}_T, \bar{B}_T]$ there exist multiple distinct bond prices satisfying *Definition 3*. This means that we can construct multiple bonds pricing schedules that satisfy *Definition 3*. The condition $\underline{B}_T < \bar{B}_T$ holds if

$$\psi > 1 - \left[1 - \frac{tr^{max} - tr^{min}}{\tau a_T \left(\frac{\alpha a_T(1-\tau^k)}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}}\right]^{\frac{1-\alpha}{\alpha}} \equiv \tilde{\psi}. \tag{98}$$

Note that because we assumed that $S(a_T(1-\psi)k_T(a(1-\psi)), tr^{max}) > 0$, the term inside the brackets is always positive, and the right hand side is smaller than unity, i.e. there is always some $\psi < 1$ such that $\psi > \tilde{\psi}$

Finally, we show that under $\psi > \tilde{\psi}$ we can construct multiple decreasing bond pricing schedules with distinct values in $(\underline{B}_T, \bar{B}_T]$. We will show this by constructing an example. We have established that for $B_T > \underline{B}_T$ we can construct a bond pricing schedule that corresponds to anticipated default in all states:

$$Q^{T\text{-}1,p}(B) = \frac{\gamma}{r} \frac{\int_{-\infty}^{\infty} [\tau a_T(1-\psi)(k_T^p)^\alpha - tr_T] f(tr_T) dtr_T}{B_T} \tag{99}$$

with

$$k_T^p = \left[\frac{\alpha a_T(1-\psi)(1-\tau^K)}{r+\delta}\right]^{\frac{1}{1-\alpha}} \tag{100}$$

For $B_T \leq \bar{B}_T$ we can also construct a bond pricing schedule that corresponds to anticipated full repayment in all states:

$$Q^{T\text{-}1,o}(B) = \frac{1}{1+r}. \tag{101}$$

$Q^{T\text{-}1,p}(B)$ satisfies *Definition 3* on the interval $B_T > \underline{B}_T$; $Q^{T\text{-}1,o}(B)$ satisfies *Definition 3* on the interval $B_T \leq \bar{B}_T$. On the interval $B_T \in (\underline{B}_T, \bar{B}_T]$ both satisfy *Definition 3*. On this interval $B_T > [\tau a_T(k_T^p)^\alpha - tr_T]\frac{1+r}{r}$ for all tr_T because (57) selects default given k_T^p in each state. It follows that $[\tau a_T(1-\psi)(k_T^p)^\alpha -$

$tr_T]$ is smaller than $\mathrm{B}_T \frac{r}{1+r}$ for each tr_T, and that the same is true for the expected surplus appearing in the numerator of (63). With this we can deduce that $Q^{T\text{-}1,p}(B) < \frac{\gamma}{1+r} \leq \frac{1}{1+r} = Q^{T\text{-}1,o}(B)$ on the interval $(\underline{B}_T, \bar{B}_T]$.

We can now choose an arbitrary $\hat{B}_T \in (\underline{B}_T, \bar{B}_T]$ and construct a bond pricing schedule that anticipates full repayment in all states for $B_T \leq \hat{B}_T$ and full default in all states for $B_T > \hat{B}_T$:

$$Q^{T\text{-}1}(B) = \begin{cases} Q^{T\text{-}1,o}(B) & if\ B_T \leq \hat{B}_T \\ Q^{T\text{-}1,p}(B) & if\ B_T > \hat{B}_T \end{cases}$$

Because the choice of $\hat{B}_T \in (\underline{B}_T, \bar{B}_T]$ is arbitrary, there is an infinite number of such possible bond pricing schedules. Furthermore, each such schedule is decreasing over B_T, as both $Q^{T\text{-}1,o}(B)$ and $Q^{T\text{-}1,p}(B)$ are decreasing and because there is a downward at $\hat{B}_T$. □

Proposition 5. *Suppose there exists a neighborhood of (B_T^0, ψ^0) in which the function $\phi(B, \psi)$ is differentiable over both B_T and ψ and is decreasing over B_T. Then in this neighborhood the bond pricing schedule $Q^{T\text{-}1}(B, \psi)$ is decreasing over B_T and ψ, i.e. higher default costs are associated with lower bond price.*

Proof. Denote the neighborhood with $N(B_T^0, \psi^0)$. Consider some $(B_T^{0*}, \psi^{0*}) \in N(B_T^0, \psi^0)$. There are three possibilities with respect to values returned by $\phi(B_T^{0*}, \psi^{0*})$ and the corresponding bond price.

First, if $\phi(B_T^{0*}, \psi^{0*}) < tr^{min}$, then the corresponding bond price anticipates default in all states, and (63) becomes:

$$q_{T-1} = \frac{\gamma}{r} \cdot \frac{\tau a_T(1-\psi)(k_T(a_T[1-\psi]))^\alpha - E_{T-1}[tr_T]}{B_T}. \tag{102}$$

An increase in ψ results in a decrease in the bond price: first, it directly decreases the surplus that can be recovered in default by lowering the productivity; second, the lowered anticipated productivity negatively affects the capital stock, further reducing the pledgeable surplus. The function is also decreasing in B_T. Therefore, if $\phi(B_T^{0*}, \psi^{0*}) < tr^{min}$ then the bond pricing schedule is decreasing at (B_T^{0*}, ψ^{0*}) over both B_T and ψ.

Second, if $\phi(B_T^{0*}, \psi^{0*}) \geq tr^{min}$, then the corresponding bond price anticipates full repayment in all states, and (63) becomes:

$$q_{T-1} = \frac{1}{1+r}. \tag{103}$$

This function is also decreasing in B_T and ψ.

Finally, suppose $\phi(B_T^{0*}, \psi^{0*}) \in [tr^{min}, tr^{max})$. This corresponds to an equilibrium in which there is default in some states and repayment in other states. As discussed in the text, in this equilibrium (68) must yield a solution with $\hat{tr}_T \in [tr^{min}, tr^{max})$. Therefore, it must be that (68) holds under (B_T^{0*}, ψ^{0*}) and $\hat{tr}^{0*} = \psi(B_T^{0*}, \psi^{0*}) \in [tr^{min}, tr^{max})$. Using (68) and applying the implicit function theorem we can calculate the derivatives of $\phi(B_T, \psi)$ over B_T and ψ:

$$\frac{\partial \phi}{\partial B_T} = -\frac{r}{1+r} \cdot \theta \tag{104}$$

$$\frac{\partial \phi}{\partial \psi} = -\frac{\alpha}{1-\alpha} \frac{\tau a_T k(\hat{tr}^{0*})^\alpha (1-F(\hat{tr}^{0*}))}{1-(1-F(\hat{tr}^{0*}))\psi} \cdot \theta \tag{105}$$

where $\theta = 1/\left[1 - \frac{\alpha}{1-\alpha} \frac{\tau a_T k(\hat{tr}^{0*})^\alpha \psi f(\hat{tr}^{0*})}{1-(1-F(\hat{tr}^{0*}))\psi}\right]$. If $\theta \geq 0$, then both $\frac{\partial \phi}{\partial B_T} \leq 0$ and $\frac{\partial \phi}{\partial \psi} \leq 0$; if $\theta < 0$, then both $\frac{\partial \phi}{\partial B_T} > 0$ and $\frac{\partial \phi}{\partial \psi} > 0$. It follows that for the $\frac{\partial \phi}{\partial B_T} \leq 0$ to hold it must be that $\theta \geq 0$. If $\theta \geq 0$, we also have $\frac{\partial \phi}{\partial \psi} \leq 0$. In other words, if the threshold $\hat{tr}$ is decreasing with the level of debt (i.e. higher debt leads to lower transfer levels causing default), then it must be that the threshold $\hat{tr}$ is also decreasing in ψ (i.e. higher default costs lead to lower transfer levels causing default).

The derivatives of the bond price defined in (63) over B_T and ψ are:

$$\frac{\partial q}{\partial B_T} = \frac{\partial \phi}{\partial B_T} \cdot \Omega - \frac{\gamma}{r} \frac{\int_{\hat{tr}^{0*}}^{\infty} [\tau a_T (1-\psi) k(\hat{tr}^{0*})^{\alpha} - tr] f(tr) d(tr)}{B_T^2} \tag{106}$$

$$\frac{\partial q}{\partial \psi} = \frac{\partial \phi}{\partial \psi} \cdot \Omega - \frac{\gamma}{r} \frac{\tau a_T k(\hat{tr}^{0*})^{\alpha} (1 - F(\hat{tr}^{0*}))}{B_T} \tag{107}$$

where $\Omega = f(\hat{tr}^{0*}) \left[\frac{1-\gamma}{1+r} + \frac{\gamma}{r} \frac{\tau a_T k(\hat{tr}^{0*})^{\alpha}}{B_T} \left(\frac{\alpha}{1-\alpha} \frac{(1-\psi)(1-F(\hat{tr}^{0*}))}{1-(1-F(\hat{tr}^{0*}))\psi} + 1 \right) \right]$. It is straightforward to see that $\Omega \geq 0$. From $\frac{\partial \phi}{\partial B_T} \leq 0$ and $\frac{\partial \phi}{\partial \psi} \leq 0$ it follows that $\frac{\partial q}{\partial B_T} \leq 0$ and $\frac{\partial q}{\partial \psi} \leq 0$, i.e. the bond pricing schedule is decreasing in both B_T and ψ. We have therefore shown that regardless of the particular location of the neighborhood $N(B_T^0, \psi^0)$, if the function $\phi(B, \psi)$ is differentiable and decreasing in B_T in that neighborhood, then the associated bond pricing schedule is decreasing in both B_T and ψ. □

Corollary 5A. *Suppose $\gamma = 0$ and that there exists a neighborhood of (B_T^0, ψ^0) in which the function $\phi(B, \psi)$ is differentiable over both B_T and ψ, and the bond pricing schedule is decreasing in B_T. Then in this neighborhood the bond pricing schedule $Q^{T\text{-}1}(B, \psi)$ is decreasing over ψ, i.e. higher default costs are associated with lower bond price. In addition, if $Q^{T\text{-}1}(B, \psi)$ is strictly decreasing over B_T in this neighborhood, it is also strictly decreasing in ψ.*

Proof. Following the proof of *Proposition 5* we can again consider three possible locations for $(B_T^{0*}, \psi^{0*}) \in N(B_T^0, \psi^0)$. For $\phi(B_T^{0*}, \psi^{0*}) < tr^{min}$ and $\phi(B_T^{0*}, \psi^{0*}) \geq tr^{max}$ the bond pricing schedule is decreasing in ψ (see poof of *Proposition 5*). For $\phi(B_T^{0*}, \psi^{0*}) \in [tr^{min}, tr^{max})$ computing the derivatives of the bond pricing schedule we get:

$$\frac{\partial q}{\partial B_T} = \frac{\partial \phi}{\partial B_T} \cdot \Omega \tag{108}$$

$$\frac{\partial q}{\partial \psi} = \frac{\partial \phi}{\partial \psi} \cdot \Omega \tag{109}$$

where $\Omega = f(\hat{tr}^{0*}) \frac{1}{1+r} > 0$ for $\phi(B_T^{0*}, \psi^{0*}) \in [tr^{min}, tr^{max})$. Clearly, $\frac{\partial q}{\partial B_T} \leq 0$ is only possible if $\frac{\partial \phi}{\partial B_T} \leq 0$. The derivatives $\frac{\partial \phi}{\partial B_T}$ and $\frac{\partial \phi}{\partial \psi}$ can be derived as in the proof of *Proposition 5*:

$$\frac{\partial \phi}{\partial B_T} = -\frac{r}{1+r} \cdot \theta \tag{110}$$

$$\frac{\partial \phi}{\partial \psi} = -\frac{\alpha}{1-\alpha} \frac{\tau a_T k(\hat{tr}^{0*})^{\alpha} (1 - F(\hat{tr}^{0*}))}{1 - (1 - F(\hat{tr}^{0*}))\psi} \cdot \theta \tag{111}$$

where $\theta = 1/\left[1 - \frac{\alpha}{1-\alpha} \frac{\tau a_T k(\hat{tr}^{0*})^{\alpha} \psi f(\hat{tr}^{0*})}{1-(1-F(\hat{tr}^{0*}))\psi} \right]$. If $\frac{\partial \phi}{\partial B_T} \leq 0$, then $\theta \geq 0$; it follows that $\frac{\partial \phi}{\partial \psi} \leq 0$. Substituting this result into the derivative $\frac{\partial q}{\partial \psi}$ we assert that $\frac{\partial q}{\partial \psi} \leq 0$. We have therefore shown that regardless of the particular location of the neighborhood $N(B_T^0, \psi^0)$, if the function $\phi(B, \psi)$ is differentiable and the bond price is decreasing in B_T in that neighborhood, then the associated bond pricing schedule is decreasing in ψ.

It remains to show that if the bond pricing schedule is strictly decreasing over B_T in $N(B_T^0, \psi^0)$, then it is also strictly decreasing in ψ. Consider the three options discussed above. For $\phi(B_T^{0*}, \psi^{0*}) < tr^{min}$ and $\phi(B_T^{0*}, \psi^{0*}) \geq tr^{max}$ the bond pricing schedule is not strictly decreasing in B_T. Consider now $\phi(B_T^{0*}, \psi^{0*}) \in [tr^{min}, tr^{max})$. In (108), if $\frac{\partial q}{\partial B_T} < 0$ then $\frac{\partial \phi}{\partial B_T} < 0$. From (110) it then follows that $\theta > 0$. Then, from (111) it follows that $\frac{\partial \phi}{\partial \psi} < 0$, in which case (109) implies that $\frac{\partial q}{\partial \psi} < 0$. □

Online Appendix B A model with foreign capital investment

Here we show that the assumption that it is only the foreigners who invest in domestic capital is not strictly necessary for the results obtained in this paper, and neither is the assumption about domestic households not having access to the international capital markets. We re-state the model assuming that domestic capital investment is carried out by domestic households, and that the domestic households can borrow and lend freely on international markets.

Suppose that, unlike in the baseline model, the domestic households can trade in risk-free assets on the international capital market. To make the contrast with the baseline more stark, we further assume that foreigners cannot access the domestic capital market: only domestic households can invest in domestic capital. We additionally assume that the domestic households are risk-neutral, and that their interest incomes received from abroad are not taxed by the domestic government.

Domestic Households
A representative domestic household trades with risk-neutral foreign investors in risk-free assets, s_t, at the the risk-free world interest rate. The households invest in domestic capital and receive interest income; they also supply a unit of labor and receive wages; they pay taxes to the government and receive transfers. The households' flow budget constraint is

$$c_t + k_{t+1} + s_{t+1} \leq \left[1 - \delta + r_t^K(1 - \tau^K)\right] k_t + (1 + r)s_t + (1 - \tau^w)w_t + tr_t. \tag{112}$$

The households choose c_t, k_{t+1} and s_{t+1} to maximize their expected welfare, $E_t[\sum_{i=0}^{\infty} \beta^i c_{t+i}]$. The first-order conditions characterizing the solution of the households' problem are:

$$1 = \beta[1 - \delta + E_t[r_{t+1}^K](1 - \tau^K)], \tag{113}$$

$$1 = \beta[1 + r]. \tag{114}$$

These conditions exactly mirror (7) and (8). Combining them we obtain an expression identical to (10) that pins down the expected capital return:

$$E_t[r_{t+1}^K](1 - \tau^K) - \delta = r. \tag{115}$$

Assumptions about the household sector affect the results derived in the text insofar as they alter the capital accumulation decision. Here, it is clear that capital investment ends up being exactly the same as the one we arrived at when solving the baseline model. Therefore, the results obtained for the baseline specification would also apply here.

We have therefore shown that this alternative setup yields results that are identical to those in our baseline model if 1) domestic households are risk-neutral and 2) the government does not tax interest income from abroad.

If the first condition fails, i.e. if capital investment decisions are made by risk-averse domestic households, this would alter the optimality condition (10). The first order condition obtained under risk-aversion would take into account the correlation between returns to capital and fluctuations in the households' utility. We expect this would amplify the effect that expectations have on capital investment decisions that is at the core of the argument made in this paper.

Violation of the second condition may have more prominent consequences for our argument. In this paper we argue that anticipated productivity losses in the event of default may trigger a capital outflow causing an output drop. If only domestic households can invest in capital and the government can tax whatever interest income they receive from abroad, the capital outflow could potentially be prevented, which could undercut the mechanism discussed here. Therefore, the argument developed in this paper is most appropriate for countries where either the foreign creditors have access to the domestic capital markets, or the domestic capital investors can avoid being taxed on their interest earnings received from abroad.

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