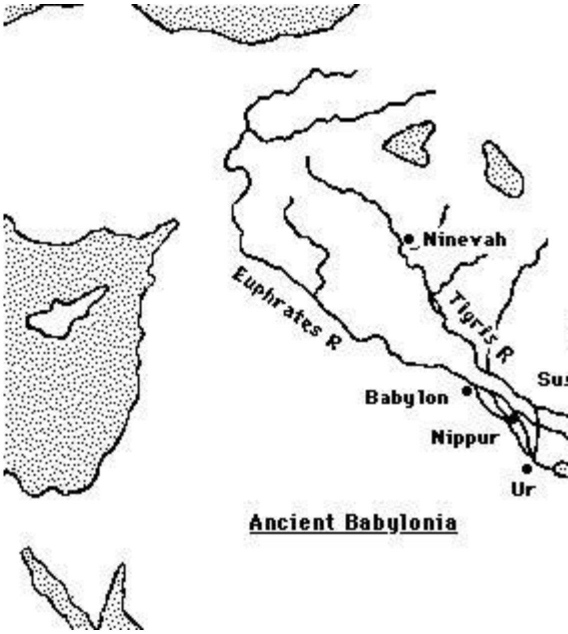
STATISTICS

DEFINITION & HISTORY OF GEOMETRY

SESSION 4

STATISTICS
DEFINITION OF GEOMETRY

Babylon (2000 BC - 500 BC)



The Babylonians replaced the older (4000 BC - 2000 BC) Sumerian civilization around 2000 BC. The Sumerians had already developed writing (cuniform on clay tablets) and arithmetic (using a base 60 number system). The Babylonians adopted both of these. But, Babylonian math went beyond arithmetic, and devloped basic ideas in number theory, algebra, and geometry. The problems they wanted to solve usually involved construction and land estimation, such as areas and volumes of rectangular objects. Some of their methods were rules that solved specialized quadratic, and even some cubic, equations. But, they didn't have algebraic notation, and there is no indication that they had logical proofs for the correctness of their rule-based methods. Nevertheless, they knew some special cases of the "Pythagorean Theorem" more than 1000 years before the Greeks (see: Pythagorean Knowledge In Ancient Babylonia and Pythagorus' theorem in Babylonian mathematics). Their durable clay tablets have preserved some of their knowledge (better than the fragile Eygptian papyri). Four specific tablets (all from the period 1900 BC - 1600 BC) good indication of Babylonian mathematical knowledge:

Yale tablet YBC 7289 - shows how to compute the diagonal of a squ

Plimpton 322 - has a table with a list of Pythagorean integer triples.

Susa tablet - shows how to find the radius of the circle through the t vertices of an isoceles triangle.

Tell Dhibayi tablet - shows how to find the sides of a rectangle with area and diagonal.

There is no direct evidence that the Greeks had access to this knowledge. But, some Babylonian math was known to the Eygptians; and probably through them, passed on to the Greeks (Thales and Pythag known to have traveled to Egypt).

Egypt (3000 BC - 500 BC)

The geometry of Egypt was mostly experimentally derived rules used by the engineers of those civilizations. They developed these rules to estimate and divide land areas, and estimate volumes of objects. Some of this was to estimate taxes for landowners. They also used these rules for construction of buildings, most notably the pyramids. They had methods (using ropes to measure lengths) to compute areas and volumes for various types of objects, various triangles, quadrilaterals, circles, and truncated pyramids. Some of their rule-based methods were correct, but others gave approximations. However, there is no evidence that the Egyptians logically deduced geometric facts and methods from basic principles. And there is no evidence that they knew a form of the "Pythagorean Theorem", though it is likely that they had some methods for constructing right angles. Nevertheless, they inspired early Greek geometers like Thales and Pythagorus. Perhaps they knew more than has been recorded, since most ancient Eygptian knowledge and documents have been lost. The only surviving documents are the Rhind and Moscow papyri.

> Ahmes (1680-1620 BC)
> wrote the ***Rhind Papyrus*** (aka the "***Ahmes Papyrus***"). In it, he claims to be the scribe and annotator of an earlier document from about 1850 BC. It contains rules for division, and has 87 problems including the solution of equations, progressi of geometric regions, volumes of granaries, etc.
>
> **Anon** (1750 BC)
> The scribe who wrote the ***Moscow Papyrus*** did not record his name. This papyrus has 25 prob solutions, some of which are geometric. One, problem 14, describes how to calculate the volum truncated pyramid (a frustrum), using a numerical method equivalent to the modern formula: $V = h(a^2 + ab + b^2)/3$, where a and b are the sides of the base and top squares, and h is the he

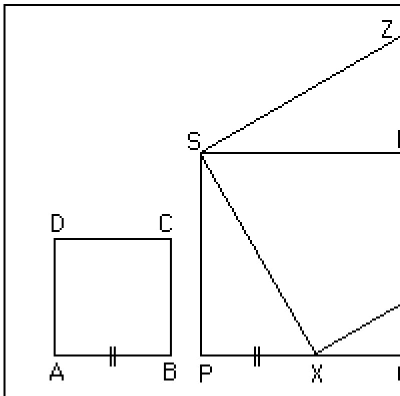
The book Mathematics in the Time of the Pharaohs gives a more detailed analysis of Egyptian mather

India (1500 BC - 200 BC)

Everything that we know about ancient Indian (Vedic) mathematics is contained in:

The Sulbasutras

These are appendices to the Vedas, and give rules for constructing sacrificial altars. To please the gods, an altar's measurements had to conform to very precise formula, and mathematical accuracy was very important. It is not historically clear whether this mathematics was developed by the Indian Vedic culture, or whether it was borrowed from the Babylonians. Like the Babylonians, results in the Sulbasutras are stated in terms of ropes; and "sutra" eventually came to mean a rope for measuring an altar. Ultimately, the Sulbasutras are simply construction manuals for some basic geometric shapes. It is noteworthy, though, that all the Sulbasutras contain a method to square the circle (one of the infamous Greek problems) as well as the converse problem of finding a circle equal in area to a given square. The main Sulbasutras, named after their authors, are:



To find a square equal to the s
given squares - given in all Su

Baudhayana (800 BC)

Baudhayana was the author of the earliest known Sulbasutra. Although he was a priest interested in c altars, and not a mathematician, his Sulbasutra contains geometric constructions for solving linear and equations, plus approximations of π (to construct circles) and $\sqrt{2}$. It also gives, often approximate, ge area-preserving transformations from one geometric shape to another. These include transforming a into a rectangle, an isosceles trapezium, an isosceles triangle, a rhombus, and a circle, and finally tran circle into a square. Further, he gives the special case of the "Pythagorean theorem" for the diagonal o square, and also a method to derive "Pythagorian triples". But he also has a construction (for a square same area as a rectangle) that implies knowing the more general "Pythagorian theorem". Some histori consider the Baudhayana as the discovery of the "Pythagorian theorem". However, the Baudhayana descriptions are all empirical methods, with no proofs, and were likely predated by the Babylonians.

Manava (750-690 BC)

contains approximate constructions of circles from rectangles, and squares from circles, which give an approximation of $\pi = 25/8 = 3.125$.

Apastamba (600-540 BC)

considers the problems of squaring the circle, and of dividing a segment into 7 equal parts. It also give accurate approximation of $\sqrt{2} = 577 / 408 = 1.414215686$, correct to 5 decimal places.

Katyayana (200-140 BC)

states the general case of the Pythagorean theorem for the diagonal of any rectangle.

The earliest recorded beginnings of geometry can be traced to early peoples, who discovered obtuse triangles in the ancient Indus Valley (see Harappan Mathematics), and ancient Babylonia (see Babylonian mathematics) from around 3000 BC. Early geometry was a collection of empirically discovered principles concerning lengths, angles, areas, and volumes, which were developed to meet some practical need in surveying, construction, astronomy, and various crafts. Among these were some surprisingly sophisticated principles, and a modern mathematician might be hard put to derive some of them without the use of calculus. For example, both the Egyptians and the Babylonians were aware of versions of the Pythagorean theorem about 1500 years before Pythagoras; the Egyptians had a correct formula for the volume of a frustum of a square pyramid;

Early geometry[edit]

The earliest recorded beginnings of geometry can be traced to early peoples, who discovered obtuse triangles in the ancient Indus Valley (see Harappan Mathematics), and ancient Babylonia (see Babylonian mathematics) from around 3000 BC. Early geometry was a collection of empirically discovered principles concerning lengths, angles, areas, and volumes, which were developed to meet some practical need in surveying, construction, astronomy, and various crafts. Among these were some surprisingly sophisticated principles, and a modern mathematician might be hard put to derive some of them without the use of calculus. For example, both the Egyptians and the Babylonians were aware of versions of the Pythagorean theorem about 1500 years before Pythagoras; the Egyptians had a correct formula for the volume of a frustum of a square pyramid;

Egyptian geometry[edit]

Main article: Egyptian mathematics

The ancient Egyptians knew that they could approximate the area of a circle as follows:[2]

Area of Circle ≈ [(Diameter) x 8/9]^2.[2]

Problem 30 of the Ahmes papyrus uses these methods to calculate the area of a circle, according to a rule that the area is equal to the square of 8/9 of the circle's diameter. This assumes that π is 4×(8/9)² (or 3.160493...), with an error of slightly over 0.63 percent. This value was slightly less accurate than the calculations of the Babylonians (25/8 = 3.125, within 0.53 percent), but was not otherwise surpassed until Archimedes' approximation of 211875/67441 = 3.14163, which had an error of just over 1 in 10,000.

Interestingly, Ahmes knew of the modern 22/7 as an approximation for pi, and used it to split a hekat, hekat x 22/x x 7/22 = hekat; however, Ahmes continued to use the traditional 256/81 value for pi for computing his hekat volume found in a cylinder.

Problem 48 involved using a square with side 9 units. This square was cut into a 3x3 grid. The diagonal of the corner squares were used to make an irregular octagon with an area of 63 units. This gave a second value for π of 3.111...

The two problems together indicate a range of values for Pi between 3.11 and 3.16.

Problem 14 in the Moscow Mathematical Papyrus gives the only ancient example finding the volume of a frustum of a pyramid, describing the correct formula:

$$V = \frac{1}{3}h(x_1^2 + x_1x_2 + x_2^2).$$

Babylonian geometry[edit]

Main article: Babylonian mathematics

The Babylonians may have known the general rules for measuring areas and volumes. They measured the circumference of a circle as three times the diameter and the area as one-twelfth the square of the circumference, which would be correct if π is estimated as 3. The volume of a cylinder was taken as the product of the base and the height, however, the volume of the frustum of a cone or a square pyramid was incorrectly taken as the product of the height and half the sum of the bases. The Pythagorean theorem was also known to the Babylonians. Also, there was a recent discovery in which a tablet used π as 3 and 1/8. The Babylonians are also known for the Babylonian mile, which was a measure of distance equal to about seven miles today. This measurement for distances eventually was converted to a time-mile used for measuring the travel of the Sun, therefore, representing time.[3]

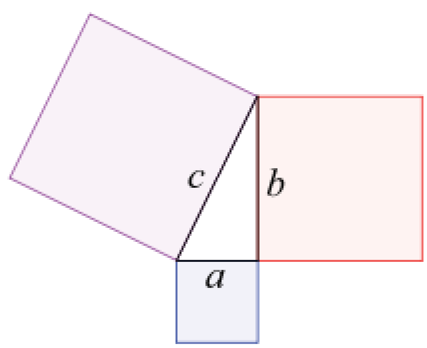
Greek geometry[edit]

See also: Greek mathematics

Classical Greek geometry[edit]

For the ancient Greek mathematicians, geometry was the crown jewel of their sciences, reaching a completeness and perfection of methodology that no other branch of their knowledge had attained. They expanded the range of geometry to many new kinds of figures, curves, surfaces, and solids; they changed its methodology from trial-and-error to logical deduction; they recognized that geometry studies "eternal forms", or abstractions, of which physical objects are only approximations; and they developed the idea of the "axiomatic method", still in use today.

Thales and Pythagoras[edit]



Pythagorean theorem: $a^2 + b^2 = c^2$

Thales (635-543 BC) of Miletus (now in southwestern Turkey), was the first to whom deduction in mathematics is attributed. There are five geometric propositions for which he wrote deductive proofs, though his proofs have not survived. Pythagoras (582-496 BC) of Ionia, and later, Italy, then colonized by Greeks, may have been a student of Thales, and traveled to Babylon and Egypt. The theorem that bears his name may not have been his discovery, but he was probably one of the first to give a deductive proof of it. He gathered a group of students around him to study mathematics, music, and philosophy, and together they discovered most of what high school students learn today in their geometry courses. In addition, they made the profound discovery of incommensurable lengths and irrational numbers.

Plato[edit]

Plato (427-347 BC), the philosopher most esteemed by the Greeks, had inscribed above the entrance to his famous school, "Let none ignorant of geometry enter here." Though he was not a mathematician himself, his views on mathematics had great influence. Mathematicians thus accepted his belief that geometry should use no tools but compass and straightedge – never measuring instruments such as a marked ruler or a protractor, because these were a workman's tools, not worthy of a scholar. This dictum led to a deep study of possible compass and straightedge constructions, and three classic construction problems: how to use these tools to trisect an angle, to construct a cube twice the volume of a given cube, and to construct a square equal in area to a given circle. The proofs of the impossibility of these constructions, finally achieved in the 19th century, led to important principles regarding the deep structure of the real number system. Aristotle (384-322 BC), Plato's greatest pupil, wrote a treatise on methods of reasoning used in deductive proofs (see Logic) which was not substantially improved upon until the 19th century.

Hellenistic geometry[edit]

Euclid[edit]

Statue of Euclid in the Oxford University Museum of Natural History.

Woman teaching geometry. Illustration at the beginning of a medieval translation of Euclid's Elements, (c. 1310)

Euclid (c. 325-265 BC), of Alexandria, probably a student of one of Plato's students, wrote a treatise in 13 books (chapters), titled *The Elements of Geometry*, in which he presented geometry in an ideal axiomatic form, which came to be known as Euclidean geometry. The treatise is not a compendium of all that the Hellenistic mathematicians knew at the time about geometry; Euclid himself wrote eight more advanced books on geometry. We know from other references that Euclid's was not the first elementary geometry textbook, but it was so much superior that the others fell into disuse and were lost. He was brought to the university at Alexandria by Ptolemy I, King of Egypt.

The Elements began with definitions of terms, fundamental geometric principles (called *axioms* or *postulates*), and general quantitative principles (called *common notions*) from which all the rest of geometry could be logically deduced. Following are his five axioms, somewhat paraphrased to make the English easier to read.

1. Any two points can be joined by a straight line.
2. Any finite straight line can be extended in a straight line.
3. A circle can be drawn with any center and any radius.
4. All right angles are equal to each other.
5. If two straight lines in a plane are crossed by another straight line (called the transversal), and the interior angles between the two lines and the transversal lying on one side of the transversal add up to less than two right angles, then on that side of the transversal, the two lines extended will intersect (also called the parallel postulate).

Archimedes[edit]

Archimedes (287-212 BC), of Syracuse, Sicily, when it was a Greek city-state, is often considered to be the greatest of the Greek mathematicians, and occasionally even named as one of the three greatest of all time (along with Isaac Newton and Carl Friedrich Gauss). Had he not been a mathematician, he would still be remembered as a great physicist, engineer, and inventor. In his mathematics, he developed methods very similar to the coordinate systems of analytic geometry, and the limiting process of integral calculus. The only element lacking for the creation of these fields was an efficient algebraic notation in which to express his concepts[citation needed].

After Archimedes[edit]

Geometry was connected to the divine for most medieval scholars. The compass in this 13th-century manuscript is a symbol of God's act of Creation.

After Archimedes, Hellenistic mathematics began to decline. There were a few minor stars yet to come, but the golden age of geometry was over. Proclus (410-485), author of *Commentary on the First Book of Euclid*, was one of the last important players in Hellenistic geometry. He was a competent geometer, but more importantly, he was a superb commentator on the works that preceded him. Much of that work did not survive to modern times, and is known to us only through his commentary. The Roman Republic and Empire that succeeded and absorbed the Greek city-states produced excellent engineers, but no mathematicians of note.

The great Library of Alexandria was later burned. There is a growing consensus among historians that the Library of Alexandria likely suffered from several destructive events, but that the destruction of Alexandria's pagan temples in the late 4th century was probably the most severe and final one. The evidence for that destruction is the most definitive and secure. Caesar's invasion may well have led to the loss of some 40,000-70,000 scrolls in a warehouse adjacent to the port (as Luciano Canfora argues, they were likely copies produced by the Library intended for export), but it is unlikely to have affected the Library or Museum, given that there is ample evidence that both existed later.

Civil wars, decreasing investments in maintenance and acquisition of new scrolls and generally declining interest in non-religious pursuits likely contributed to a reduction in the body of material available in the Library, especially in the 4th century. The Serapeum was certainly destroyed by Theophilus in 391, and the Museum and Library may have fallen victim to the same campaign.

Indian geometry[edit]

See also: Indian mathematics

Vedic period[edit]

Rigveda manuscript in Devanagari.

The *Satapatha Brahmana* (ninth century BC) contains rules for ritual geometric constructions that are similar to the *Sulba Sutras*.[4]

The *Śulba Sūtras* (literally, "Aphorisms of the Chords" in Vedic Sanskrit) (c. 700-400 BC) list rules for the construction of sacrificial fire altars.[5] Most mathematical problems considered in the *Śulba Sūtras* spring from "a single theological requirement,"[6] that of constructing fire altars which have different shapes but occupy the same area. The altars were required to be constructed of five layers of burnt brick, with the further condition that each layer consist of 200 bricks and that no two adjacent layers have congruent arrangements of bricks.[6]

According to (Hayashi 2005, p. 363), the *Śulba Sūtras* contain "the earliest extant verbal expression of the Pythagorean Theorem in the world, although it had already been known to the Old Babylonians."

The diagonal rope (*akṣṇayā-rajju*) of an oblong (rectangle) produces both which the flank (*pārśvamāni*) and the horizontal (*tiryaṇmānī*) <ropes> produce separately."[7]

Since the statement is a *sūtra*, it is necessarily compressed and what the ropes *produce* is not elaborated on, but the context clearly implies the square areas constructed on their lengths, and would have been explained so by the teacher to the student.[7]

They contain lists of Pythagorean triples,[8] which are particular cases of Diophantine equations.[9] They also contain statements (that with hindsight we know to be approximate) about squaring the circle and "circling the square."[10]

Baudhayana (c. eighth century BC) composed the *Baudhayana Sulba Sutra*, the best-known *Sulba Sutra*, which contains examples of simple Pythagorean triples, such as: $(3,4,5)$, $(5,12,13)$, $(8,15,17)$, $(7,24,25)$, and $(12,35,37)$[11] as well as a statement of the Pythagorean theorem for the sides of a square: "The rope which is stretched across the diagonal of a square produces an area double the size of the original square."[11] It also contains the general statement of the Pythagorean theorem (for the sides of a rectangle): "The rope stretched along the length of the diagonal of a rectangle makes an area which the vertical and horizontal sides make together."[11]

According to mathematician S. G. Dani, the Babylonian cuneiform tablet Plimpton 322 written c. 1850 BC[12] "contains fifteen Pythagorean triples with quite large entries, including (13500, 12709, 18541) which is a primitive triple,[13] indicating, in particular, that there was sophisticated understanding on the topic" in Mesopotamia in 1850 BC. "Since these tablets predate the Sulbasutras period by several centuries, taking into account the contextual appearance of some of the triples, it is reasonable to expect that similar understanding would have been there in India."[14] Dani goes on to say:

"As the main objective of the *Sulvasutras* was to describe the constructions of altars and the geometric principles involved in them, the subject of Pythagorean triples, even if it had been well understood may still not have featured in the *Sulvasutras*. The occurrence of the triples in the *Sulvasutras* is comparable to mathematics that one may encounter in an introductory book on architecture or another similar applied area, and would not correspond directly to the overall knowledge on the topic at that time. Since, unfortunately, no other contemporaneous sources have been found it may never be possible to settle this issue satisfactorily."[14]

In all, three *Sulba Sutras* were composed. The remaining two, the *Manava Sulba Sutra* composed by Manava (fl. 750-650 BC) and the *Apastamba Sulba Sutra*, composed by Apastamba (c. 600 BC), contained results similar to the *Baudhayana Sulba Sutra*.

Classical period[edit]

In the Bakhshali manuscript, there is a handful of geometric problems (including problems about volumes of irregular solids). The Bakhshali manuscript also "employs a decimal place value system with a dot for zero."[15] Aryabhata's *Aryabhatiya* (499) includes the computation of areas and volumes.

Brahmagupta wrote his astronomical work *Brāhma Sphuṭa Siddhānta* in 628. Chapter 12, containing 66 Sanskrit verses, was divided into two sections: "basic operations" (including cube roots, fractions, ratio and proportion, and barter) and "practical mathematics" (including mixture, mathematical series, plane figures, stacking bricks, sawing of timber, and piling of grain).[16] In the latter section, he stated his famous theorem on the diagonals of a cyclic quadrilateral:[16]

Brahmagupta's theorem: If a cyclic quadrilateral has diagonals that are perpendicular to each other, then the perpendicular line drawn from the point of intersection of the diagonals to any side of the quadrilateral always bisects the opposite side.

Chapter 12 also included a formula for the area of a cyclic quadrilateral (a generalization of Heron's formula), as well as a complete description of rational triangles (*i.e.* triangles with rational sides and rational areas).

Brahmagupta's formula: The area, *A*, of a cyclic quadrilateral with sides of lengths *a*, *b*, *c*, *d*, respectively, is given by

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where *s*, the semiperimeter, given by: $s = \frac{a+b+c+d}{2}$.

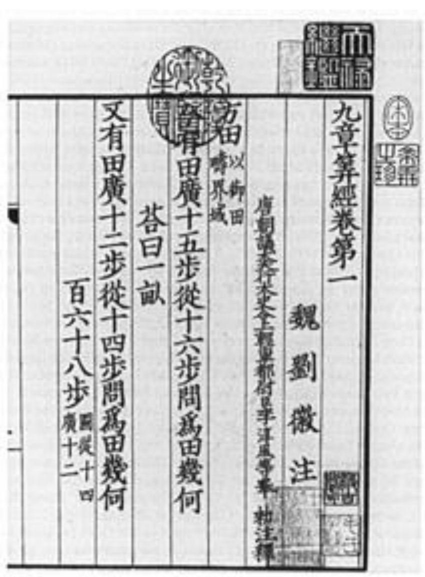
Brahmagupta's Theorem on rational triangles: A triangle with rational sides a, b, c and rational area is of the form:

$$a = \frac{u^2}{v} + v, \quad b = \frac{u^2}{w} + w, \quad c = \frac{u^2}{v} + \frac{u^2}{w} - (v + w)$$

for some rational numbers $u, v,$ and w.[17]

Chinese geometry[edit]

See also: Chinese mathematics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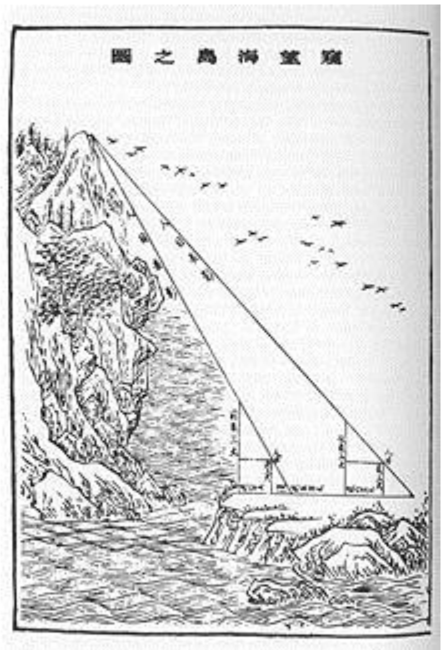
The *Nine Chapters on the Mathematical Art*, first compiled in 179 AD, with added commentary in the 3rd century by Liu Hui.



The Sea Island Mathematical Manual, Liu Hui, 3rd century.

The first definitive work (or at least oldest existent) on geometry in China was the *Mo Jing*, the Mohist canon of the early philosopher Mozi (470-390 BC). It was compiled years after his death by his later followers around the year 330 BC.[18] Although the *Mo Jing* is the oldest existent book on geometry in China, there is the possibility that even older written material exists. However, due to the infamous Burning of the Books in the political maneauver by the Qin Dynasty ruler Qin Shihuang (r. 221-210 BC), multitudes of written literature created before his time was purged. In addition, the *Mo Jing* presents geometrical concepts in mathematics that are perhaps too advanced not to have had a previous geometrical base or mathematic background to work upon.

The *Mo Jing* described various aspects of many fields associated with physical science, and provided a small wealth of information on mathematics as well. It provided an 'atomic' definition of the geometric point, stating that a line is separated into parts, and the part which has no remaining parts (i.e. cannot be divided into smaller parts) and thus forms the extreme end of a line is a point.[18] Much like Euclid's first and third definitions and Plato's 'beginning of a line', the *Mo Jing* stated that "a point may stand at the end (of a line) or at its beginning like a head-presentation in childbirth. (As to its invisibility) there is nothing similar to it."[19] Similar to the atomists of Democritus, the *Mo Jing* stated that a point is the smallest unit, and cannot be cut in half, since 'nothing' cannot be halved.[19] It stated that two lines of equal length will always finish at the same place,[19] while providing definitions for the *comparison of lengths* and for *parallels*,[20] along with principles of space and bounded space.[21] It also described the fact that planes without the quality of thickness cannot be piled up since they cannot mutually touch.[22] The book provided definitions for circumference, diameter, and radius, along with the definition of volume.[23]

The Han Dynasty (202 BC-220 AD) period of China witnessed a new flourishing of mathematics. One of the oldest Chinese mathematical texts to present geometric progressions was the *Suàn shù shū* of 186 BC, during the Western Han era. The mathematician, inventor, and astronomer Zhang Heng (78-139 AD) used geometrical formulas to solve mathematical problems. Although rough estimates for pi (π) were given in the *Zhou Li* (compiled in the 2nd century BC),[24] it was Zhang Heng who was the first to make a concerted effort at creating a more accurate formula for pi. Zhang Heng approximated pi as 730/232 (or approx 3.1466), although he used another formula of pi in finding a spherical volume, using the square root of 10 (or approx 3.162) instead. Zu Chongzhi (429-500 AD) improved the accuracy of the approximation of pi to between 3.1415926 and 3.1415927, with $^{355}/_{113}$ (密率, Milü, detailed approximation) and $^{22}/_{7}$ (约率, Yuelü, rough approximation) being the other notable approximation.[25] In comparison to later works, the formula for pi given by the French mathematician Franciscus Vieta (1540-1603) fell halfway between Zu's approximations.

The Nine Chapters on the Mathematical Art[edit]

The Nine Chapters on the Mathematical Art, the title of which first appeared by 179 AD on a bronze inscription, was edited and commented on by the 3rd century mathematician Liu Hui from the Kingdom of Cao Wei. This book included many problems where geometry was applied, such as finding surface areas for squares and circles, the volumes of solids in various three-dimensional shapes, and included the use of the Pythagorean theorem. The book provided illustrated proof for the Pythagorean theorem,[26] contained a written dialogue between of the earlier Duke of Zhou and Shang Gao on the properties of the right angle triangle and the Pythagorean theorem, while also referring to the astronomical gnomon, the circle and square, as well as measurements of heights and distances.[27] The editor Liu Hui listed pi as 3.141014 by using a 192 sided polygon, and then calculated pi as 3.14159 using a 3072 sided polygon. This was more accurate than Liu Hui's contemporary Wang Fan, a mathematician and astronomer from Eastern Wu, would render pi as 3.1555 by using $^{142}/_{45}$.[28] Liu Hui also wrote of mathematical surveying to calculate distance measurements of depth, height, width, and surface area. In terms of solid geometry, he figured out that a wedge with rectangular base and both sides sloping could be broken down into a pyramid and a tetrahedral wedge.[29] He also figured out that a wedge with trapezoid base and both sides sloping could be made to give two tetrahedral wedges separated by a pyramid.[29] Furthermore, Liu Hui described Cavalieri's principle on volume, as well as Gaussian elimination. From the *Nine Chapters*, it listed the following geometrical formulas that were known by the time of the Former Han Dynasty (202 BCE–9 CE).

Areas for the[30]

- Square
- Rectangle
- Circle
- Rhomboid
- Trapezoid
- Double trapezium

- Isosceles triangle
- Segment of a circle
- Annulus ('ring' between two concentric circles)

Volumes for the[29]

- Parallelepiped with two square surfaces
- Parallelepiped with no square surfaces
- Pyramid
- Frustum of pyramid with square base
- Frustum of pyramid with rectangular base of unequal sides
- Cube
- Prism
- Wedge with rectangular base and both sides sloping
- Wedge with trapezoid base and both sides sloping
- Tetrahedral wedge
- Frustum of a wedge of the second type (used for applications in engineering)
- Cylinder
- Cone with circular base
- Frustum of a cone
- Sphere

Continuing the geometrical legacy of ancient China, there were many later figures to come, including the famed astronomer and mathematician Shen Kuo (1031-1095 CE), Yang Hui (1238-1298) who discovered Pascal's Triangle, Xu Guangqi (1562-1633), and many others.

Islamic geometry[edit]

See also: Islamic mathematics

Page from the *Al-Jabr wa-al-Muqabilah*

Although the Islamic mathematicians are most famed for their work on algebra, number theory and number systems, they also made considerable contributions to geometry, trigonometry and mathematical astronomy, and were responsible for the development of algebraic geometry. Geometrical magnitudes were treated as "algebraic objects" by most Islamic mathematicians however.

Al-Mahani (born 820) conceived the idea of reducing geometrical problems such as duplicating the cube to problems in algebra. Al-Karaji (born 953) completely freed algebra from geometrical operations and replaced them with the arithmetical type of operations which are at the core of algebra today.

An engraving by Albrecht Dürer featuring Mashallah, from the title page of the *De scientia motus orbis* (Latin version with engraving, 1504). As in many medieval illustrations, the compass here is an icon of religion as well as science, in reference to God as the architect of creation

Thabit family and other early geometers[edit]

Thābit ibn Qurra (known as Thebit in Latin) (born 836) contributed to a number of areas in mathematics, where he played an important role in preparing the way for such important mathematical discoveries as the extension of the concept of number to (positive) real numbers, integral calculus, theorems in spherical trigonometry, analytic geometry, and non-Euclidean geometry. In astronomy Thabit was one of the first reformers of the Ptolemaic system, and in mechanics he was a founder of statics. An important geometrical aspect of Thabit's work was his book on the composition of ratios. In this book, Thabit deals with arithmetical operations applied to ratios of geometrical quantities. The Greeks had dealt with geometric quantities but had not thought of them in the same way as numbers to which the usual rules of arithmetic could be applied. By introducing arithmetical operations on quantities previously regarded as geometric and non-numerical, Thabit started a trend which led eventually to the generalisation of the number concept.

In some respects, Thabit is critical of the ideas of Plato and Aristotle, particularly regarding motion. It would seem that here his ideas are based on an acceptance of using arguments concerning motion in his geometrical arguments. Another important contribution Thabit made to geometry was his generalization of the Pythagorean theorem, which he extended from special right triangles to all triangles in general, along with a general proof.[31]

Ibrahim ibn Sinan ibn Thabit (born 908), who introduced a method of integration more general than that of Archimedes, and al-Quhi (born 940) were leading figures in a revival and continuation of Greek higher geometry in the Islamic world. These mathematicians, and in particular Ibn al-Haytham, studied optics and investigated the optical properties of mirrors made from conic sections.

Astronomy, time-keeping and geography provided other motivations for geometrical and trigonometrical research. For example Ibrahim ibn Sinan and his grandfather Thabit ibn Qurra both studied curves required in the construction of sundials. Abu'l-Wafa and Abu Nasr Mansur both applied spherical geometry to astronomy.

Geometric architecture[edit]

Recent discoveries have shown that geometrical quasicrystal patterns were first employed in the girih tiles found in medieval Islamic architecture dating back over five centuries ago. In 2007, Professor Peter Lu of Harvard University and Professor Paul Steinhardt of Princeton University published a paper in the journal *Science* suggesting that girih tilings possessed properties consistent with self-similar fractal quasicrystalline tilings such as the Penrose tilings, predating them by five centuries.[32][33]

Modern geometry[edit]

The 17th century[edit]

When Europe began to emerge from its Dark Ages, the Hellenistic and Islamic texts on geometry found in Islamic libraries were translated from Arabic into Latin. The rigorous deductive methods of geometry found in Euclid's *Elements of Geometry* were relearned, and further development of geometry in the styles of both Euclid (Euclidean geometry) and Khayyam (algebraic geometry) continued, resulting in an abundance of new theorems and concepts, many of them very profound and elegant.

DISCOURS
DE LA METHODE
Pour bien conduire sa raison, & chercher
la verité dans les sciences.
PLUS
LA DIOPTRIQVE.
LES METEORES.
ET
LA GEOMETRIE.
Qui sont des essais de cete METHODE.

A LEYDE
De l'Imprimerie de IAN MAIRE.
clɔ Iɔ c xxxvii.
Avec Privilege.

Discourse on Method by René Descartes

In the early 17th century, there were two important developments in geometry. The first and most important was the creation of analytic geometry, or geometry with coordinates and equations, by René Descartes (1596–1650) and Pierre de Fermat (1601–1665). This was a necessary precursor to the development of calculus and a precise quantitative science of physics. The second geometric development of this period was the systematic study of projective geometry by Girard Desargues (1591–1661). Projective geometry is the study of geometry without measurement, just the study of how points align with each other. There had been some early work in this area by Hellenistic geometers, notably Pappus (c. 340). The greatest flowering of the field occurred with Jean-Victor Poncelet (1788–1867).

In the late 17th century, calculus was developed independently and almost simultaneously by Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716). This was the beginning of a new field of mathematics now called analysis. Though not itself a branch of geometry, it is applicable to geometry, and it solved two families of problems that had long been almost intractable: finding tangent lines to odd curves, and finding areas enclosed by those curves. The methods of calculus reduced these problems mostly to straightforward matters of computation.

The 18th and 19th centuries[edit]

Non-Euclidean geometry[edit]

The very old problem of proving Euclid's Fifth Postulate, the "Parallel Postulate", from his first four postulates had never been forgotten. Beginning not long after Euclid, many attempted demonstrations were given, but all were later found to be faulty, through allowing into the reasoning some principle which itself had not been proved from the first four postulates. Though Omar Khayyám was also unsuccessful in proving the parallel postulate, his criticisms of Euclid's theories of parallels and his proof of properties of figures in non-Euclidean geometries contributed to the eventual development of non-Euclidean geometry. By 1700 a great deal had been discovered about what can be proved from the first four, and what the pitfalls were in attempting to prove the fifth. Saccheri, Lambert, and Legendre each did excellent work on the problem in the 18th century, but still fell short of success. In the early 19th century, Gauss, Johann Bolyai, and Lobatchewsky, each independently, took a different approach. Beginning to suspect that it was impossible to prove the Parallel Postulate, they set out to develop a self-consistent geometry in which that postulate was false. In this they were successful, thus creating the first non-Euclidean geometry. By 1854, Bernhard Riemann, a student of Gauss, had applied methods of calculus in a ground-breaking study of the intrinsic (self-contained) geometry of all smooth surfaces, and thereby found a different non-Euclidean geometry. This work of Riemann later became fundamental for Einstein's theory of relativity.

William Blake's "Newton" is a demonstration of his opposition to the 'single-vision' of scientific materialism; here, Isaac Newton is shown as 'divine geometer' (1795)

It remained to be proved mathematically that the non-Euclidean geometry was just as self-consistent as Euclidean geometry, and this was first accomplished by Beltrami in 1868. With this, non-Euclidean geometry was established on an equal mathematical footing with Euclidean geometry.

While it was now known that different geometric theories were mathematically possible, the question remained, "Which one of these theories is correct for our physical space?" The mathematical work revealed that this question must be answered by physical experimentation, not mathematical reasoning, and uncovered the reason why the experimentation must involve immense (interstellar, not earth-bound) distances. With the development of relativity theory in physics, this question became vastly more complicated.

Introduction of mathematical rigor[edit]

All the work related to the Parallel Postulate revealed that it was quite difficult for a geometer to separate his logical reasoning from his intuitive understanding of physical space, and, moreover, revealed the critical importance of doing so. Careful examination had uncovered some logical inadequacies in Euclid's reasoning, and some unstated geometric principles to which Euclid sometimes appealed. This critique paralleled the crisis occurring in calculus and analysis regarding the meaning of infinite processes such as convergence and continuity. In geometry, there was a clear need for a new set of axioms, which would be complete, and which in no way relied on pictures we draw or on our intuition of space. Such axioms were given by David Hilbert in 1894 in his dissertation *Grundlagen der Geometrie* (*Foundations of Geometry*). Some other complete sets of axioms had been given a few years earlier, but did not match Hilbert's in economy, elegance, and similarity to Euclid's axioms.

Analysis situs, or topology[edit]

In the mid-18th century, it became apparent that certain progressions of mathematical reasoning recurred when similar ideas were studied on the number line, in two dimensions, and in three dimensions. Thus the general concept of a metric space was created so that the reasoning could be done in more generality, and then applied to special cases. This method of studying calculus- and analysis-related concepts came to be known as analysis situs, and later as topology. The important topics in this field were properties of more general figures, such as connectedness and boundaries, rather than properties like straightness, and precise equality of length and angle measurements, which had been the focus of Euclidean and non-Euclidean geometry. Topology soon became a separate field of major importance, rather than a sub-field of geometry or analysis.

The 20th century[edit]

Developments in algebraic geometry included the study of curves and surfaces over finite fields as demonstrated by the works of among others André Weil, Alexander Grothendieck, and Jean-Pierre Serre as well as over the real or complex numbers. Finite geometry itself, the study of spaces with only finitely many points, found applications in coding theory and cryptography. With the advent of the computer, new disciplines such as computational geometry or digital geometry deal with geometric algorithms, discrete representations of geometric data, and so forth.

A Brief History of Greek Geometry

A Little Background:

The word geometry has its roots in the Greek work geometrein, which means "earth measuring". Before the time of recorded history, geometry originated out of practical necessity; it was the science of measuring land. Many ancient civilizations (Babylonian, Hindu, Chinese, and Egyptian) possessed geometric information. The first geometrical considerations "had their origin in simple observations stemming from human ability to recognize physical form and to compare shapes and sizes" (Historical Topics, 165). There were many circumstances in which primitive people were forced to take on geometric topics, although it may not have been recognized as such. For instance, man had to learn with situations involving distance, bounding their land, and constructing walls and homes. These types of situations were directly related to the geometric concepts of vertical, parallel, and perpendicular.

The geometry of the ancient days was actually just a collection of rule-of-thumb procedures, which were found through experimentation, observation of analogies, guessing, and sometimes even intuition. Basically, geometry in the ancient days allowed for approximate answers, which were usually sufficient for practical purposes. For example, the Babylonians took p to be equal to 3. It is said that the Babylonians were more advanced than the Egyptians in arithmetic and algebra. They even knew the Pythagorean theorem long before Pythagoras was even born. The Babylonians had an algebraic influence on Greek mathematics.

Egyptian geometry was not a science in the way the Greeks viewed geometry. It was more of a grab bag for rules for calculation without any motivation or justification. Sometimes they guessed correctly, but other times they did not. One of their greatest accomplishments was finding the correct formula for the volume of a frustum of a square pyramid. However, they thought that the formula that they had for the area of a rectangle could be applied to any quadrilateral.

Primitive people could not escape geometry in the same way that we cannot escape it today. The concept of the curve was found in flowers and the sun, a parabola was represented by tossing an object, and spider webs posed an excellent example of regular polygons. Symmetry could be seen in many living objects, including man, and the idea of volume had to be addressed when constructing a device to hold water. Historical Topics for the Mathematics Classroom calls this type of geometry “subconscious geometry”. This is the type of geometry that very young children experience as they begin to play with objects. This type of geometry involves concrete objects.

Still before the time of recorded history, man began to consider situations that were more hypothetical. They were able to take the knowledge they had learned from observation of concrete objects and come up with general algorithms and procedures to be used in particular cases. This is what Historical Topics for the Mathematics Classroom refers to as “scientific geometry” (166-7). Procedures such as trial and error, induction, and rule-of thumb were being used to discover. This was mainly the geometry of the Babylonians and the Egyptians. Although there is no evidence that they were able to deductively reason geometric facts from basic principles, it is thought that they paved the way for Greek geometry. Geometry remained this way (“scientific”) until the Greek period.

A Bit of Greek Geometry (600 BC – 400 AD):

To see a chronological outline of the work of Greek geometers, click here. (An entire mathematical chronology can be found by visiting: www-groups.dcs.st-and.ac.uk/~history/Chronology/full.html.)

The Greeks worked to transform geometry into something much different than the "scientific geometry" of the people that worked before them. "The Greeks insisted that geometric fact mush be established, … , by deductive reasoning; …" (Historical Topics, 171). They believed that geometrical truth would be found by studying rather than experimenting. They transformed the former "scientific geometry" into a more "systematic geometry".

Keep in mind that there exist virtually no first-hand sources of early Greek geometry. Hence, the following is based on manuscripts written hundreds of years after this early Greek geometry had been developed. According to these manuscripts, Thales of Miletus was the one who began early Greek geometry in the sixth century B.C. He is noted as one of the first known to indulge himself in deductive methods in geometry. His credited elementary geometrical findings resulted from logical reasoning rather than intuition and experiment. He insisted that geometric statements be established by deductive reasoning rather than by trial and error. He was familiar with the computations recorded from Egyptian and Babylonian mathematics, and he developed his logical geometry by determining which results were correct.

The next mentioned great Greek geometer is one who quite possibly studied under Thales of Miletus. This geometer is Pythagoras, who founded the Pythagorean school, which was "committed to the study of philosophy, mathematics, and natural science" (Historical Topics, 172). In the area of geometry, the members of this school developed the properties of parallel to prove that the sum of any angles of a triangle is equal to two right angles. They also worked with proportion to study similar figures. The deductive side of geometry was further developed during this time. We all think of the Pythagorean Theorem when we think of Pythagoras, however it is important to note that this theorem was used (although it may not have been proved) before his time.

As an interesting side note, Pythagoras was regarded as a religious prophet by his contemporaries. He preached the immortality of the soul and reincarnation, and he even organized a brotherhood of believers. This brotherhood had initiation rites, they were vegetarian, and they shared all property. They did, however, differ from other religious groups in one major way. They believed that elevation of the soul and union with God was achieved through the study of music and mathematics.

Hippocrates of Chios was one of these students at the Pythagorean school. It is suggested that he was the first to attempt "a logical presentation of geometry in the form of a single chain of propositions …" (Historical Topics, 172). He is credited to writing the first "Elements of Geometry" where he included geometric solutions to quadratic equations and some of the first methods of integration. He studied the problem of squaring a circle and squaring a lune. He also was the first to show that the ratio of the areas of two circles equals the ratio of the squares of the circles' radii (www.geometryalgorithms.com/history.htm).

Although Plato did not make any major mathematical discoveries himself, he did emphasize the idea of proof. He insisted on accuracy, which helped pave the way for Euclid. It is correct to say that almost every significant geometrical development can be traced back to three outstanding Greek geometers: Euclid, Archimedes, and Apollonis. Euclid collected the theorems of Pythagoras, Hippocrates, and others into a work called "The Elements". (www.geometryalgorithms.com/history.htm). Euclid is the most widely read author in the history of mankind. The teaching of geometry has been dominated by Euclid's approach to the subject. In fact, Euclid's axiomatic method is the prototype for all "pure mathematics". By "pure", it is meant that all statements can be verified through reasoning of demonstrations; no physical experiments are necessary.

Typically, the next mentioned Greek mathematician is regarded as the greatest Greek mathematician by geometryalgorithms.com. His name was Archimedes of Syracuse. He had many mathematical accomplishments as well as being the inventor of the screw, the pulley, the lever, and other mechanical devices. He perfected integration using the method of exhaustion discovered my Eudoxus, and he was able to find the areas and volumes of many objects. Inscribed on his tomb was the result he found that the volume of a sphere is two-thirds the volume of its circumscribed cylinder. (http://geometryalgorithms.com/history.htm)

Apollonius was an astronomer who had his mathematical bid to fame in his work entitled Conic Sections. It is this great Greek geometer who provided us with the terms "ellipse," "parabola," and "hyperbola." He is also accredited with showing how to construct a circle which is tangent to three objects. His approximation of π was even closer than that of Archimedes. (http://geometryalgorithms.com/history.htm)

Last, but certainly not least, Hypatia of Alexandria was the first woman to substantially contribute to mathematics. She studied under her father and assisted him in writing a new version of Euclid's Elements. She also wrote commentaries on other great Greek geometer's works. She was the "first woman in history recognized as a professional geometer and mathematician" (http://geometryalgorithms.com/history.htm).