# Lesson WEEK 9

**Definitions of the Laplace Transform, Laplace Transform Examples, and Functions)**

In this lesson, we'll introduce our last transform, the Laplace Transform. The Laplace Transform is useful in a number of different applications:

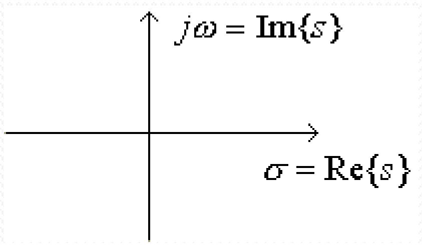
- 1. Using the Laplace Transform, differential equations can be solved algebraically.
- 2. We can use pole/zero diagrams from the Laplace Transform to determine the frequency response of a system and whether or not the system is stable.
- 3. We can transform more signals than we can with the Fourier Transform, because the Fourier Transform is a special case of the Laplace Transform.
- 4. The Laplace Transform is used for analog circuit design.
- 5. The Laplace Transform is used in Control Theory and Robotics

## Definitions of Laplace Transform

The *Bilateral* Laplace Transform of a signal $x(t)$ is defined as:

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

The complex variable $s = \sigma + j\omega$, where $\omega$ is the frequency variable of the Fourier Transform (simply set $\sigma = 0$). The Laplace Transform converges for more functions than the Fourier Transform since it could converge off of the $j\omega$ axis. Here is a plot of the $s$-plane:




The Inverse Bilateral Laplace Transform of $X(s)$ is:

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds$$

Notice that to compute the inverse Laplace Transform, it requires a contour integral. (When taking the inverse transform, the value of *c* for the contour integral must be in the region where the integral exists.) Fortunately, we will see more convenient ways (namely, Partial Fraction Expansion) to take the inverse transform so you are not required to know how to do contour integration.

If we define $x(t)$ to be 0 for $t < 0$, this gives us the unilateral Laplace transform:

$$L[x(t)] = X(s) = \int_0^{\infty} x(t)e^{-st}dt$$

As we'll see, an important difference between the bilateral and unilateral Laplace Transforms is that you need to specify the region of convergence (ROC) for the bilateral case.

We point out (without proof) several features of ROCs:

- A right-sided time function (i.e. $x(t) = 0,\ t < t_o$ where $t_o$ is a constant) has an ROC that is a right half-plane.
- A left-sided time function has an ROC that is a left half-plane.
- A 2-sided time function has an ROC that is either a strip or else the ROC does not exist, which means that the Laplace Transform does not exist.
- If the ROC contains the $j\omega$- axis, then if $x(t)$ were used as an impulse response, the system would be BIBO stable. If the boundary of the ROC is the $j\omega$-axis (i.e. $\mathrm{Re}(s) > 0$ or $\mathrm{Re}(s) < 0$), the system would be BIBO unstable.

Taking the Laplace Transform is clearly a linear operation:

$$L[ax_1(t) + bx_2(t)] = aX_1(s) + bX_2(s)$$

where $X_1(s)$ is the Laplace Transform of $x_1(t)$ and $X_2(s)$ is the Laplace Transform of $x_2(t)$.

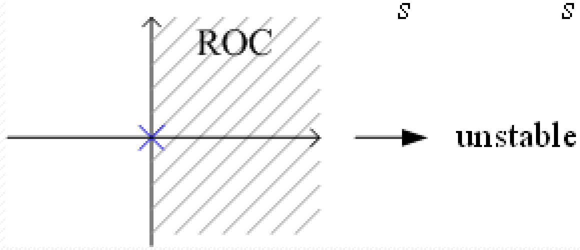
## Laplace Transform Examples

Example 1 Find $L[u(t)]$

$$L[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-st} dt$$

$$= \frac{-1}{s} e^{-st} \Big|_{0}^{\infty}$$

$$= \frac{-1}{s}(0-1) = \frac{1}{s}, \mathrm{Re}(s) > 0$$

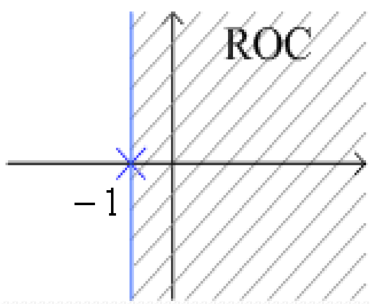



In Example 1, we needed to specify that Re($s$) > 0. If this is not the case, the integral would have not converged at the upper limit of infinity.

**Example 2** Find the Laplace Transform of $x_1(t) = e^{-t}u(t)$

$$L[x_1(t)] = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+1)t}dt$$

$$= \frac{-1}{1+s}(0-1) = \frac{1}{1+s}, \qquad \begin{array}{l} \mathrm{Re}(s+1) > 0 \\ \mathrm{Re}(s) > -1 \end{array}$$




$\longrightarrow$ stable since constant $j\omega$ axis

The general Laplace Transform for an exponential function is: $e^{-at}u(t) \leftrightarrow \frac{1}{s+a}, \mathrm{Re}(s+a) > 0$

## Laplace Transforms of Functions

Example 3 Find the Laplace Transform of $\delta(t - t_0)$ **where** $t_0 \geq 0.$

$\delta(t - t_0)$ where $t_0 \geq 0.$

$$x(t) = \delta(t - t_0) \quad t_0 \geq 0$$

$$L[x(t)] = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-st} dt$$

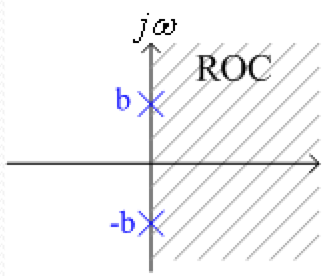
$$= e^{-st_0} \qquad \forall s \Rightarrow \text{stable}$$

**Example 4** Find the Laplace Transform of $\sin(bt)u(t)$

$$x(t) = \sin(bt) = \frac{1}{2j}\left[e^{jbt} - e^{-jbt}\right]$$

$$X(s) = \frac{1}{2j}\left[\int_0^\infty e^{jbt}e^{-st}dt + \int_0^\infty e^{-jbt}e^{-st}dt\right]$$

$$= \frac{1}{2j}\left[\int_0^\infty e^{(jb-s)t}dt + \int_0^\infty e^{-(jb+s)t}dt\right]$$




$$= \frac{1}{2j}\left[\frac{1}{jb-s}e^{(jb-s)t}\Big|_0^\infty + \frac{-1}{jb+s}e^{-(jb+s)t}\Big|_0^\infty\right]$$

$$= \frac{1}{2j}\left[\frac{1}{jb-s}(0-1) - \frac{1}{jb+s}(0-1)\right]$$

$\mathrm{Re}(jb-s) < 0$

$\mathrm{Re}(jb+s) > 0$

BIBO unstable $\mathrm{Re}(s) > 0$

$$= \frac{1}{2j}\left[\frac{1}{jb+s} - \frac{1}{jb-s}\right] \quad = \frac{b}{s^2+b^2}$$

## Laplace Transform Properties

As we saw from the Fourier Transform, there are a number of properties that can simplify taking Laplace Transforms. I'll cover a few properties here and you can read about the rest in the textbook.

### Real Time Shifting

$$x(t)u(t) \leftrightarrow X(s)$$
$$x(t-t_0)u(t-t_0) \leftrightarrow e^{-t_0 s} X(s)$$

Derive this:
Plugging in the time-shifted version of the function into the Laplace Transform definition, we get:

$$\int_{t=-\infty}^{\infty} x(t-t_0)u(t-t_0)e^{-st}\,dt$$
$$= \int_{t=t_0}^{\infty} x(t-t_0)e^{-st}\,dt$$

Letting $\tau = t - t_o$, we get:

$$= \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau+t_0)}\,d\tau$$
$$= e^{-st_0}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}\,d\tau$$
$$= e^{-st_0}X(s)$$

**Example 1** Find the Laplace Transform of $x(t) = \sin[b(t - 2)]u(t - 2)$

$$x(s) = e^{-s2}\frac{b}{b^2 + s^2}, \quad \mathbf{Re}(s) > 0$$

## Differentiation

$$x(t) \leftrightarrow X(s)$$

$$x'(t) \leftrightarrow sX(s) - x(0^+)$$

for Unilateral Laplace Transform only

Recall the equation for the voltage of an inductor:

$$V_L(t) = L\frac{di_L(t)}{dt}$$

If we take the Laplace Transform of both sides of this equation, we get:

$$V_L(s) = sL\,I_L(s)$$

which is consistent with the fact that an inductor has impedance *sL*.

## Proof of the Differentiation Property:

1) First write $x(t)$ using the Inverse Laplace Transform formula:

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)\, e^{st}\, ds$$

2) Then take the derivative of both sides of the equation with respect to $t$ (this brings down a factor of $s$ in the second term due to the exponential):

$$\frac{d}{dt} x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} s X(s)\, e^{st}\, ds$$

3) This shows that $x'(t)$ is the Inverse Laplace Transform of $s\,X(s)$:

$$\frac{d}{dt} x(t) \leftrightarrow sX(s)$$

The Differentiation Property is useful for solving differential equations.

## Integration

$$x(t) \leftrightarrow X(s)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{s} X(s)$$

Recall the equation for the voltage of a capacitor turned on at time 0:

$$V_c(t) = \frac{1}{C}\int_{0}^{t} i_c(\tau)\,d\tau$$

If we take the Laplace Transform of both sides of this equation, we get:

$$V_c(s) = \frac{1}{(sC)} I_c(s)$$

which is consistent with the fact that a capacitor has impedance $\frac{1}{sC}$ .

# ADDITIONAL PROPERTIES

## Multiplication by *t*

$$x(t) \leftrightarrow X(s)$$
$$t\,x(t) \leftrightarrow -\frac{dX(s)}{ds}$$

Derive this:

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt$$

Take the derivative of both sides of this equation with respect to *s*:

$$\frac{d}{ds}X(s) = \int_{-\infty}^{\infty} x(t)(-t\,e^{-st})dt \; = \int_{-\infty}^{\infty} (-t\,x(t))\,e^{-st}\,dt$$

This is the expression for the Laplace Transform of $-t\,x(t)$. Therefore,

$$t\,x(t) \leftrightarrow -\frac{dX(s)}{ds}$$

## Initial Value

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

(Given without proof)

## Final Value

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

(Given without proof)

# Independent-Variable Transformation (for Unilateral Laplace Transform)

$$x(t) \leftrightarrow X(s)$$
$$x(at-b) \leftrightarrow \frac{1}{a} e^{\frac{-sb}{a}} X\left(\frac{s}{a}\right)$$

Derive this:
Plugging in the definition, we find the Laplace Transform of $x(at\ \text{-}b)$:

$$\int_{-\infty}^{\infty} x(at-b)\, e^{-st} dt$$

Let $u = at - b$ and $du = adt$, we get:

$$= \int_{-\infty}^{\infty} x(u)\, e^{\frac{-s(u+b)}{a}} \frac{du}{a}$$
$$= \frac{1}{a} e^{\frac{-sb}{a}} \int_{-\infty}^{\infty} x(u)\, e^{\frac{-su}{a}} du$$
$$= \frac{1}{a} e^{\frac{-sb}{a}} X\left(\frac{s}{a}\right)$$