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Deflections and rotations in rectangular beams with straight haunches under uniformly distributed load considering the shear deformations

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Abstract. This paper presents a model of the elastic curve for rectangular beams with straight haunches under uniformly distributed load and moments in the ends considering the bending and shear deformations (Timoshenko Theory) to obtain the deflections and rotations on the beam, which is the main part of this research. The traditional model of the elastic curve for rectangular beams under uniformly distributed load considers only the bending deformations (Euler-Bernoulli Theory). Also, a comparison is made between the proposed and traditional model of simply supported beams with respect to the rotations in two supports and the maximum deflection of the beam. Also, another comparison is made for beams fixed at both ends with respect to the moments and reactions in the support A, and the maximum deflection of the beam. Results show that the proposed model is greater for simply supported beams in the maximum deflection and the traditional model is greater for beams fixed at both ends in the maximum deflection. Then, the proposed model is more appropriate and safe with respect the traditional model for structural analysis, because the shear forces and bending moments are present in any type of structure and the bending and shear deformations appear.

Keywords: elastic curve; rectangular beams; straight haunches; uniformly distributed load; moments in the ends; bending and shear deformations; Timoshenko Theory; Euler-Bernoulli Theory; deflections and rotations on the beam

1. Introduction

Deformation of reinforced concrete beams and steel beams is an important measure of their serviceability performance, already is specifically required in the current performance-based design codes. Generally, deformations of beams consist in the bending and shear deformations.

In structural engineering, there are two design criteria for the beams: the strength and the serviceability.

The problem of the elastic curve to obtain the deflections and rotations anywhere for the non-prismatic beams subjected to any type of load and to different boundary conditions has been investigated by many researchers.

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The relevant publications of non-prismatic cantilever beams are those of Lee (2002), Dado and Al-Sadder (2005), Borboni and De Santis (2006), Banerjee *et al*. (2008), Solano-Carrillo (2009), Chen (2010), Yau (2010), Brojan *et al*. (2012).

Yuksel (2009) studied the behaviour of symmetrically haunched non-prismatic members subjected to temperature changes. Yuksel (2012) investigated the non-prismatic beams having symmetrical parabolic haunches with constant haunch length ratio of 0.5. Ponnada and Vipparthy (2013) used the improved method of estimating deflection in prestressed steel I-beams. Luévanos-Rojas (2014) presented a mathematical model of the elastic curve for simply supported beams subjected to a uniformly distributed load taking into account the shear deformations for a prismatic section. Ponnada and Thonangi (2015) showed the deflections in non-prismatic simply supported prestressed concrete beams. Bouchafa *et al*. (2015) analyzed the thermal stresses and deflections of functionally graded sandwich plates using a new refined hyperbolic shear deformation theory. Razavi *et al*. (2015) presented the load-deflection analysis prediction of CFRP strengthened RC slab using RNN. Akbas (2015) analyzed the large deflection of edge cracked simple supported beams. Luévanos-Rojas *et al*. (2016a) investigated a mathematical model of the elastic curve for simply supported beams subjected to a concentrated load taking into account the shear deformations for a prismatic section. Li and Chen (2016) showed the deflection of battened beams with shear

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and discrete effects. Unsal *et al*. (2017) presented a study on load-deflection behavior of two-span continuous concrete beams reinforced with GFRP and steel bars. Ju *et al*. (2017) studied the flexural behaviour and a modified prediction of deflection of concrete beam reinforced with ribbed GFRP bars.

There are some papers that show methods of structural analysis for statically indeterminate beams and frames for prismatic sections taking into account the bending and shear deformations (Luévanos-Rojas 2012, 2013a, b).

Also, there are papers that present the fixed-end moments of a beam subjected to a uniformly or triangularly distributed load taking into account bending and shear deformations for prismatic sections (Luévanos-Rojas 2013c, d).

The papers that describe the modeling for beams cross section "I" or "rectangular" subjected to a uniformly distributed load or concentrated load with straight haunches taking into account the bending and shear deformations show the fixed-end moments, carry-over factors and stiffness factors (Luévanos-Rojas 2015, Luévanos-Rojas *et al*. 2016b, c, Luévanos-Soto and Luévanos-Rojas 2017).

The literature reviews of researches developed and/or compared by software are shown below:

Majumder and Kumar (2013) analyzed the maximum deflection of a simply supported beam under different types of loading. The loads are: a) Concentrated load at the mid span; b) uniformly distributed load; c) Triangularly distributed load. The theoretical analysis was realized by the Euler-Bernoulli Theory, and compared with the ANSYS 14.0 software. On comparing the numerical results to those obtained from the commercial software ANSYS 14.0, excellent accuracy of the present method has been demonstrated. Moreover while using ANSYS it has also been noted that in case of deflection the Element 2 i.e., TET 8 Node element gives a closer value in all types of loading than the Element 1 i.e. BRICK 8 Node element. This inference is exactly opposite in case of stress analysis. There ELEMENT 1 gives a better result than ELEMENT 2. Hence it can be concluded that when the deflection of a solid structure is to be ascertained the user can use, in case other preferences are absent, 8 Node TET Element. But when stress analysis is essential 8 Node BRICK element should be preferred. Note: If results were most accurate between the Euler-Bernoulli Theory and the ANSYS 14.0 software as shown in the conclusions, then the shear deformations are not considered in the ANSYS 14.0 software.

Debnath and Debnath (2014) studied the maximum deflection for different uniform rectangular cross section beams, and beams types are: a) Simply supported beam with a uniformly distributed load; b) Simply supported beam with a concentrated load at center; c) Cantilever beam with a uniformly distributed load; d) Cantilever beam with a concentrated load at the end. The theoretical calculation was made according to the Euler-Bernoulli Theory, and the computational analysis was realized by the ANSYS 14.0 software. The data considered for all beams are: L = 100 m, b = 10 m, h = 10 m, ν = 0.3, E = 2×10^7 N/m^2, F = 500 N. The studied solid elements were 188, 189, 185, and 285. The most accurate result was measured by solid 189 element followed by solid 188 element and other solid elements. Note: If results were accurate the Euler-Bernoulli Theory with the ANSYS14.0 software, then the shear deformations are neglected in the ANSYS 14.0 software.

Sihua *et al*. (2015) investigated the nonlinear analysis of a reinforced concrete beam was conducted based on the finite element analysis software ABAQUS. This simply supported beam is 1500 mm long; with a section of 180 mm × 100 mm. the Concrete strength is C25. Longitudinal reinforcement and stirrups adopted HPB235 reinforced. In this simply supported beam analysis, the plasticity model of concrete damage in ABAQUS has been introduced thoroughly. Finally, the results of the experimentation and the ABAQUS analysis were compared in a diagram, and the load reached the capacity of 24 kN, the value of mid-span deflection is 10.521 mm of ABAQUS and 12.795 mm of test. If the shear deformations in the ABAQUS software would have been considered, the results of the ABAQUS software would be closer to the experimental test, because these deformations tend to increase to the total deformations for simply supported beams.

This paper presents a model of the elastic curve for rectangular beams with straight haunches under uniformly distributed load and moments in the ends considering the bending and shear deformations (Timoshenko Theory) to obtain the deflections and rotations on the beam, which is the main part of this research. The proposed model is shown in three parts for the beam of 0 ≤ a, a ≤ L – c and L – c ≤ L. The traditional model of the elastic curve for rectangular beams under uniformly distributed load considers only the bending deformations (Euler-Bernoulli Theory). Also, a comparison is made between the proposed and traditional model of simply supported beams with respect to the rotations in two supports and the maximum deflection of the beam, another comparison is made for beams fixed at both ends with respect to the moments and reactions in the support A, and the maximum deflection of the beam to observe the differences of the two models.

2. Methodology

Fig. 1 shows the difference between the Timoshenko theory and Euler-Bernoulli theory. The first theory includes the effect of bending and shear stresses on the deformation ($dy/dx = dy_s/dx + dy_b/dx$), and the second theory includes the effect of bending stresses on the deformation ($dy/dx = dy_b/dx$) (Timoshenko 1947, Timoshenko and Gere 1972).

Timoshenko theory considers the bending and shear deformations, i.e., also is valid for the short and long members. The equation of the elastic curve is presented as follows (Timoshenko 1947, Timoshenko and Gere 1972)

$$\frac{d^2y}{dx^2} = \frac{d^2y_s}{dx^2} + \frac{d^2y_b}{dx^2} \tag{1}$$

$$\frac{d^2y}{dx^2} = -\frac{w}{GA_{sx}} - \frac{M_z}{EI_z} \tag{2}$$

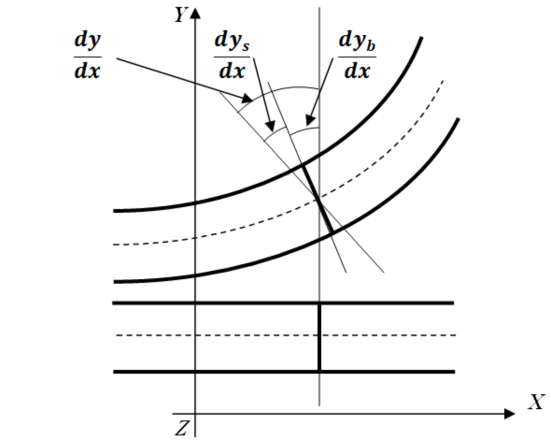


Fig. 1 Deformation of a structural member

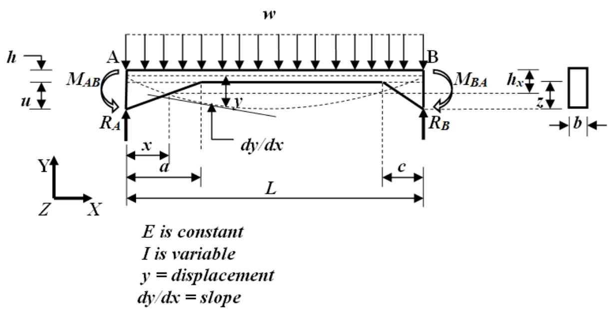


Fig. 2 Beam subjected to moments at its ends and uniformly distributed load

where: G is shear modulus, y is total displacement, y_s is shear displacement due to the shear, y_b is bending displacement due to the moment, A_{sx} is shear area, E is the modulus of elasticity, M_z is bending moment around of the axis "Z", I_z is moment of inertia around of the axis "Z".

The rotations in anywhere of the beam by integration of Eq. (2) are obtained

$$\frac{dy}{dx} = -\int \frac{w}{GA_{sx}}dx - \int \frac{M_z}{EI_z}dx \tag{3}$$

Fig. 2 shows the beam "AB" subjected to moments at its ends and uniformly distributed load, and also its rectangular cross-section taking into account that the width "b" is constant and height "h_x" is variable of straight shape in three different parts (Luévanos-Rojas 2015, Luévanos-Rojas *et al*. 2016a, b, Luévanos-Soto and Luévanos-Rojas 2017).

2.1 For the interval of the beam of $0 \le x \le a$

Substituting the properties of the Table 1 in Eq. (8), and the rotations anywhere are obtained.

The values of R_A and R_B are obtained from the following equations

$$R_A = \frac{M_{AB} - M_{BA}}{L} + \frac{wL}{2} \tag{4}$$

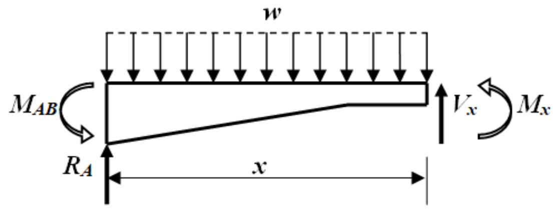


Fig. 3 Diagram of free body of the beam

$$R_B = \frac{wL}{2} - \frac{M_{AB} - M_{BA}}{L} \tag{5}$$

Fig. 3 shows the diagram of free body to find shear force and moment at anywhere of the beam on axis "x".

The shear forces and moments at anywhere of the beam are

$$V_x = R_A - wx \tag{6}$$

$$M_x = R_A x - \frac{wx^2}{2} - M_{AB} \tag{7}$$

where: L = beam length, $V_x = V_y$, and $M_x = M_z$.

Substituting Eq. (7) in Eq. (3) is obtained

$$\frac{dy}{dx} = -\int \frac{w}{GA_{sx}}dx - \frac{1}{E}\int \frac{\left(R_A x - \frac{wx^2}{2} - M_{AB}\right)}{I_z}dx \tag{8}$$

Table 1 shows the equations of the heights "h_x", shear areas "A_{sx}" to a distance "x", and the moment of inertia "I_z" around of the axis "z" to a distance "x" for each interval (Luévanos-Rojas *et al*. 2016a).

$$\frac{dy}{dx} = -\frac{6}{5Gb}\int \frac{aw}{a(h+u)-ux}dx + \frac{12}{Eb}\int \frac{a^3\left(\frac{wx^2}{2} - R_A x + M_{AB}\right)}{[a(h+u)-ux]^3}dx \tag{9}$$

Eq. (9) is simplified, and this is presented in Eq. (10). Subsequently the Eq. (10) is integrated to obtain the deflections of the beam

$$\frac{dy}{dx} = -\frac{6}{5Gb}\left[-\frac{wa}{u}ln[a(h+u)-ux]\right] + \frac{12}{Eb}\left[-\frac{wa^3}{2u^3}ln[a(h+u)-ux] + \frac{a^3x[wa(h+u)-R_A u]}{u^2[a(h+u)-ux]^2} - \frac{a^3[3wa^2(h+u)^2 - 2R_A au(h+u) - 2M_{AB}u^2]}{4u^3[a(h+u)-ux]^2}\right] + C_1 \tag{10}$$

$$y = -\frac{6}{5Gb}\left[\frac{wa[a(h+u)-ux]}{u^2}ln[a(h+u)-ux] + \frac{wax}{u}\right] + \frac{12}{Eb}\left[-\frac{a^3[wux - 3wa(h+u) + 2R_A u]}{2u^4}ln[a(h+u)-ux] + \frac{a^3[wa^2(h+u)^2 - 2R_A au(h+u) + 2M_{AB}u^2]}{4u^4[a(h+u)-ux]} + \frac{wa^3x}{2u^3}\right] + C_1 x + C_2 \tag{11}$$

Table 1 Properties of the rectangular section

Concept	Equations		
Interval	$0 \le x \le a$	$a \le x \le L - c$	$L - c \le x \le L$
h_x	$\frac{a(h+u)-ux}{a}$	h	$\frac{c(h+z)-z(L-x)}{c}$
A_{sx}	$\frac{5b[a(h+u)-ux]}{6a}$	$\frac{5bh}{6}$	$\frac{5b[c(h+z)-z(L-x)]}{6c}$
I_z	$\frac{b[a(h+u)-ux]^3}{12a^3}$	$\frac{bh^3}{12}$	$\frac{b[c(h+z)-z(L-x)]^3}{12c^3}$

2.2 For the interval of the beam of a ≤ x ≤ L – c

Substituting, the properties of the Table 1 in Eq. (8), and the rotations anywhere are obtained

$$\frac{dy}{dx} = -\frac{6}{5Gb}\int \frac{w}{h}dx + \frac{12}{Eb}\int\left[\frac{wx^2}{2h^3} - \frac{R_A x}{h^3} + \frac{M_{AB}}{h^3}\right]dx \quad (12)$$

Eq. (12) is simplified, and this is presented in Eq. (13). Subsequently the Eq. (13) is integrated to obtain the deflections of the beam

$$\frac{dy}{dx} = -\frac{6}{5Gb}\left[\frac{wx}{h}\right] + \frac{12}{Eb}\left[\frac{wx^3}{6h^3} - \frac{R_A x^2}{2h^3} + \frac{M_{AB}x}{h^3}\right] + C_3 \quad (13)$$

$$y = -\frac{6}{5Gb}\left[\frac{wx^2}{2h}\right] + \frac{12}{Eb}\left[\frac{wx^4}{24h^3} - \frac{R_A x^3}{6h^3} + \frac{M_{AB}x^2}{2h^3}\right] + C_3 x + C_4 \quad (14)$$

2.3 For the interval of the beam of L – c ≤ x ≤ L

Substituting, the properties of the Table 1 in Eq. (8), and the rotations anywhere are obtained

$$\frac{dy}{dx} = -\frac{6}{5Gb}\int \frac{cw}{c(h+z)-z(L-x)}dx + \frac{12}{Eb}\int \frac{c^3\left(\frac{wx^2}{2} - R_A x + M_{AB}\right)}{[c(h+z)-z(L-x)]^3}dx \quad (15)$$

Eq. (15) is simplified, and this is presented in Eq. (16). Subsequently the Eq. (16) is integrated to obtain the deflections of the beam

$$\frac{dy}{dx} = -\frac{6}{5Gb}\left[\frac{wc}{z}ln[c(h+z)-z(L-x)]\right] + \frac{12}{Eb}\left[\frac{wc^3}{2z^3}ln[c(h+z)-z(L-x)] + \frac{c^3x[wc(h+z)+(R_A-wL)z]}{z^2[c(h+z)-z(L-x)]^2} + \frac{c^3\{3w[c(h+z)-Lz]^2 + 2R_A z[c(h+z)-Lz] - 2M_{AB}z^2\}}{4z^3[c(h+z)-z(L-x)]^2}\right] + C_5 \quad (16)$$

$$y = -\frac{6}{5Gb}\left[\frac{wc[c(h+z)-z(L-x)]}{z^2}ln[c(h+z)-z(L-x)] - \frac{wcx}{z}\right] + \frac{12}{Eb}\left[\frac{c^3[wzx + 3wc(h+z) + (2R_A - 3wL)z]}{2z^4}ln[c(h+z) - z(L-x)] + \frac{c^3\{w[c(h+z)-Lz]^2 + 2R_A z[c(h+z)-Lz] + 2M_{AB}z^2\}}{4z^4[c(h+z)-z(L-x)]} - \frac{wc^3x}{2z^3}\right] + C_5 x + C_6 \quad (17)$$

The six known conditions that the beam must meet are:

1) Substituting, the condition *x* = 0 and *y* = 0 in Eq. (11).

2) Substituting, the condition *x* = *a* in Eqs. (10)-(13), and the values of *dy/dx* are equal, because the rotations at this point must be same.

3) Substituting, the condition *x* = *a* in Eqs. (11)-(14), and the values of *y* are equal, because the deflections at this point must be same.

4) Substituting, the condition *x* = *L* – *c* in Eqs. (13)-(16), and the values of *dy/dx* are equal, because the rotations at this point must be same.

5) Substituting, the condition *x* = *L* – *c* in Eqs. (14)-(17), and the values of *y* are equal, because the deflections at this point must be same.

6) Substituting, the condition *x* = *L* and *y* = 0 in Eq. (17).

Now, substituting the six known conditions into corresponding equations are generated six equations to obtain the integration constants.

The integration constants are shown in Eqs. (18)-(23), which are presented in the appendix.

Substituting, the integration constants to obtain the equations of the rotations and the deflections are presented in Eqs. (24)-(29), which are shown in the appendix.

3. Verification of the proposed model

A way to verify the proposed model is as follows:

1. Substituting the value of "$M_{AB} = 0$, $R_A = wL/2$, a = c, u = z and dy/dx = 0" for simply supported rectangular cross-section beams in Eq. (25) is obtained "x = L/2", i.e., when the rotation is zero, the maximum deflection is produced (symmetrical beam).

2. Substituting the value of "x = a" in Eqs. (24) and (25), the values obtained of the two equations are equals, i.e., the continuity is guaranteed in this point for the rotations.

3. Substituting the value of "x = L − c" in Eqs. (25) and (26), the values obtained of the two equations are equals, i.e., the continuity is guaranteed in this point for the rotations.

4. Substituting the value of "$M_{AB} = 0$, $R_A = wL/2$, a = 0, c = 0 and x = L/2" for simply supported rectangular beams of constant cross-section in Eq. (28) is obtained the maximum deflection "$y_{max} = wL^2(24Eh^2 + 25GL^2)/160bh^3EG = 5wL^4(1 + 48EI/5GA_sL^2)/384EI$" (Timoshenko and Gere 1972), for constant cross section (bending and shear deformations are considered).

Table 2 Simply supported beam for h = 0.1 L

c	z/h	Factors for the rotations in the supports						Factors for the maximum displacements				
		α_A			α_B			β		γ		
		PM	TM	PM/TM	PM	TM	PM/TM	PM	TM	PM	TM	PM/TM
a = 0.2L; u/h = 0.5												
0.2L	0.5	477.61	464.30	1.0287	−477.61	−464.30	1.0287	0.5000	0.5000	156.01	152.49	1.0231
	1.0	475.46	462.21	1.0287	−461.51	−448.82	1.0283	0.4986	0.4986	154.94	151.44	1.0231
	1.5	474.08	460.86	1.0287	−452.00	−439.74	1.0279	0.4977	0.4977	154.25	150.77	1.0231
	2.0	473.11	459.92	1.0287	−445.72	−433.79	1.0275	0.4971	0.4971	153.77	150.31	1.0230
0.4L	0.5	455.41	442.33	1.0296	−408.84	−396.38	1.0314	0.4855	0.4853	145.07	141.67	1.0240
	1.0	441.09	428.22	1.0301	−361.22	−349.91	1.0323	0.4761	0.4759	138.19	134.89	1.0245
	1.5	431.98	419.27	1.0303	−333.97	−323.44	1.0326	0.4701	0.4699	133.88	130.65	1.0247
	2.0	425.65	413.07	1.0305	−316.44	−306.49	1.0325	0.4660	0.4658	130.92	127.75	1.0248
a = 0.2L; u/h = 1.0												
0.2L	0.5	461.51	448.82	1.0283	−475.46	−462.21	1.0287	0.5014	0.5014	154.94	151.44	1.0231
	1.0	459.36	446.73	1.0283	−459.36	−446.73	1.0283	0.5000	0.5000	153.86	150.40	1.0230
	1.5	457.98	445.38	1.0283	−449.85	−437.65	1.0279	0.4991	0.4991	153.17	149.72	1.0230
	2.0	457.00	444.44	1.0283	−443.57	−431.69	1.0275	0.4985	0.4985	152.69	149.25	1.0230
0.4L	0.5	439.31	426.85	1.0292	−406.69	−394.29	1.0314	0.4869	0.4867	143.97	140.59	1.0240
	1.0	424.99	412.75	1.0297	−359.07	−347.81	1.0324	0.4775	0.4773	137.06	133.79	1.0244
	1.5	415.88	403.80	1.0299	−331.82	−321.35	1.0326	0.4715	0.4713	132.74	129.54	1.0247
	2.0	409.55	397.60	1.0301	−314.29	−304.40	1.0325	0.4674	0.4672	129.77	126.63	1.0248

where: θ_A (Rotations in the support A) = $\alpha_A w/bE$; θ_B (Rotations in the support B) = $\alpha_B w/bE$; x (Location of the maximum displacement) = βL; y_{max} (Maximum displacement) = $\gamma wL/bE$.

5. Substituting the value of "x = 0" in Eq. (27), the deflection is zero.

6. Substituting the value of "x = L" in Eq. (29), the deflection is zero.

7. Substituting the value of "x = a" in Eqs. (27) and (28), the values obtained of the two equations are equals, i.e., the continuity is guaranteed in this point for the deflections.

8. Substituting the value of "x = L − c" in Eqs. (28) and (29), the values obtained of the two equations are equals, i.e., the continuity is guaranteed in this point for the deflections.

9. If shear deformations are neglected, the above conditions also are verified and the maximum deflection for the symmetry condition is: "$y_{max} = 5wL^4/32bh^3E = 5wL^4/384EI$", for constant cross section (bending deformations are considered).

10. If shear deformations are neglected, and the value of "$M_{AB} = wL^2/12$, $R_A = wL/2$ and a = c = 0" for rectangular beams with fixed supports of constant cross-section are substituted in Eq. (24) is obtained "dy/dx = 0", i.e., the rotation in support A is zero.

11. If shear deformations are neglected, and the value of "$M_{AB} = wL^2/12$, $R_A = wL/2$ and a = c = 0" for rectangular beams with fixed supports of constant cross-section are substituted in Eq. (26) is obtained "dy/dx = 0", i.e., the rotation in support B is zero.

Therefore the proposed model in this paper of the elastic curve for rectangular cross-section beams with straight haunches under uniformly distributed load and moments at its ends considering the bending and shear deformations (Timoshenko theory) is valid.

4. Results

Tables 2 and 3 show the comparison of the two models to obtain the factors for the rotations in the supports and the maximum deflections for a simply supported beam, the proposed model (PM) is the mathematical model presented in this paper taking into account the bending and shear deformations, and the traditional model (TM) considering only the bending deformations. Table 2 presents to h = 0.1L. Table 3 shows to h = 0.2L. These comparisons were realized for G = 5E/12 for concrete, a = 0.2L; u/h = 0.5, u/h = 1.0; c = 0.2L, 0.4L; z = 0.5h, h, 1.5h, 2h.

Just as shown in Tables 2 and 3, the factors in the rotations for the support "A" and "B" are influenced by the height "h". As the height of the haunches is increased in support "B" is seen a decrease (absolute value) in these factors for the same support and in the support "A" occurs a decrease, this is for the two models. Also, the factors in the maximum deflections are influenced by the height "h". As the height of the haunches is increased in support "B" is seen a decrease in these factors, this is for the two models. According to the results, the proposed model is greater in all cases for the rotations in the supports and the maximum deflections, and for the two cases in h = 0.2L, where the biggest difference is of 13.68% for the rotation in support "B", and for the maximum deflections is of 9.92%.

Tables 4 and 5 present the comparison of the two models to find the factors for the moments and reactions in the support A and the maximum deflections for a beam fixed at both ends. Table 4 presents to h = 0.1L. Table 5 shows to h = 0.2L. These comparisons were realized for G = 5E/12 for concrete, a = 0.2L; u/h = 0.5, u/h = 1.0; c = 0.2L, 0.4L; z = 0.5h, h, 1.5h, 2h.

Table 3 Simply supported beam for h = 0.2 L

c	z/h	Factors for the rotations in the supports						Factors for the maximum displacements				
		α_A			α_B			β		γ		
		PM	TM	PM/TM	PM	TM	PM/TM	PM	TM	PM	TM	PM/TM
a = 0.2L; u/h = 0.5												
0.2L	0.5	64.69	58.04	1.1146	−64.69	−58.04	1.1146	0.5000	0.5000	20.82	19.06	1.0923
	1.0	64.40	57.78	1.1146	−62.45	−56.10	1.1132	0.4986	0.4986	20.68	18.93	1.0924
	1.5	64.22	57.61	1.1147	−61.09	−54.97	1.1113	0.4976	0.4977	20.58	18.85	1.0918
	2.0	64.08	57.49	1.1146	−60.19	−54.22	1.1101	0.4970	0.4971	20.52	18.79	1.0921
0.4L	0.5	61.83	55.29	1.1183	−55.77	−49.55	1.1255	0.4858	0.4853	19.41	17.71	1.0960
	1.0	59.96	53.53	1.1201	−49.39	−43.74	1.1292	0.4765	0.4759	18.51	16.86	1.0979
	1.5	58.76	52.41	1.1212	−45.96	−40.43	1.1368	0.4706	0.4699	17.94	16.33	1.0986
	2.0	57.92	51.63	1.1218	−43.28	−38.31	1.1297	0.4664	0.4658	17.55	15.97	1.0989
a = 0.2L; u/h = 1.0												
0.2L	0.5	62.45	56.10	1.1132	−64.40	−57.78	1.1146	0.5014	0.5014	20.68	18.93	1.0924
	1.0	62.16	55.84	1.1132	−62.16	−55.84	1.1132	0.5000	0.5000	20.53	18.80	1.0920
	1.5	61.97	55.67	1.1132	−60.81	−54.71	1.1115	0.4991	0.4991	20.44	18.72	1.0919
	2.0	61.84	55.55	1.1132	−59.90	−53.96	1.1101	0.4984	0.4985	20.37	18.66	1.0916
0.4L	0.5	59.58	53.36	1.1166	−55.49	−49.29	1.1258	0.4872	0.4867	19.26	17.57	1.0962
	1.0	57.71	51.59	1.1186	−49.11	−43.48	1.1295	0.4780	0.4773	18.36	16.72	1.0981
	1.5	56.51	50.47	1.1197	−45.40	−40.17	1.1302	0.4720	0.4713	17.79	16.19	1.0988
	2.0	55.68	49.70	1.1203	−42.99	−38.05	1.1298	0.4679	0.4672	17.40	15.83	1.0992

Table 4 Beam fixed at both ends for h = 0.1 L

c	z/h	Factors for the moments and reactions in the support A						Factors for the maximum displacements				
		ψ_{AB}			ξ_{AB}			β		γ		
		PM	TM	PM/TM	PM	TM	PM/TM	PM	TM	PM	TM	PM/TM
a = 0.2L; u/h = 0.5												
0.2L	0.5	0.0968	0.0941	1.0287	0.5000	0.5000	1.0000	0.5000	0.5000	18.56	18.87	0.9836
	1.0	0.0898	0.0872	1.0298	0.4802	0.4805	0.9994	0.4805	0.4809	15.57	15.92	0.9780
	1.5	0.0855	0.0830	1.0301	0.4679	0.4685	0.9987	0.4672	0.4682	13.85	14.25	0.9758
	2.0	0.0827	0.0802	1.0312	0.4600	0.4605	0.9989	0.4584	0.4593	12.89	13.21	0.9758
0.4L	0.5	0.0932	0.0908	1.0264	0.4892	0.4903	0.9978	0.4924	0.4924	17.06	17.46	0.9771
	1.0	0.0819	0.0799	1.0250	0.4551	0.4575	0.9948	0.4696	0.4661	12.88	13.36	0.9641
	1.5	0.0736	0.0719	1.0236	0.4292	0.4326	0.9921	0.4423	0.4447	10.23	10.77	0.9499
	2.0	0.0676	0.0660	1.0242	0.4097	0.4136	0.9906	0.4230	0.4266	8.36	8.96	0.9330
a = 0.2L; u/h = 1.0												
0.2L	0.5	0.1096	0.1067	1.0272	0.5198	0.5195	1.0006	0.5195	0.5191	15.57	15.92	0.9780
	1.0	0.1021	0.0993	1.0282	0.5000	0.5000	1.0000	0.5000	0.5000	12.97	13.36	0.9708
	1.5	0.0975	0.0947	1.0296	0.4877	0.4878	0.9998	0.4867	0.4870	11.46	11.91	0.9622
	2.0	0.0945	0.0917	1.0305	0.4796	0.4798	0.9996	0.4775	0.4781	10.51	11.02	0.9537
0.4L	0.5	0.1060	0.1034	1.0251	0.5101	0.5108	0.9986	0.5119	0.5112	14.42	14.80	0.9743
	1.0	0.0940	0.0919	1.0229	0.4766	0.4788	0.9954	0.4854	0.4857	10.87	11.34	0.9586
	1.5	0.0852	0.0834	1.0216	0.4508	0.4540	0.9930	0.4631	0.4647	8.46	9.00	0.9400
	2.0	0.0786	0.0769	1.0221	0.4311	0.4349	0.9913	0.4440	0.4469	6.88	7.54	0.9125

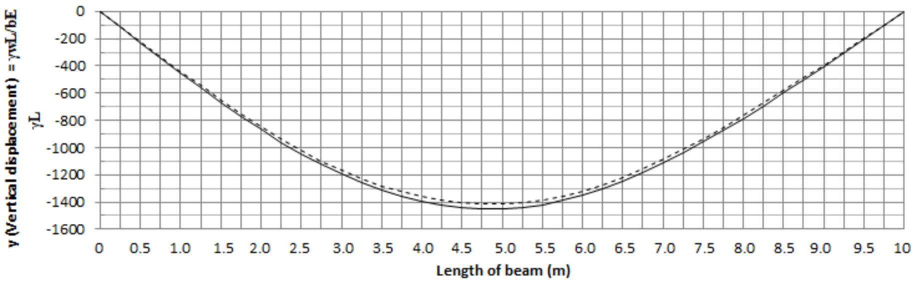
where: M_{AB} (Fixed-end moments in the support A) = $\psi_{AB}wL^2$; R_A (Reactions in the supports A) = $\xi_{AB}wL$.

Tables 4 and 5 are obtained by substituting x = 0; G = 5E/12; a = 0.2L; u/h = 0.5; c = 0.2L or c = 0.4L; z/h = 0.5h, h, 1.5h and 2h; h = 0.1L (Table 4) and h = 0.2L (Table 5) in Eqs. (24)-(26) and these are made equals to zero. Subsequently, these two equations are solved to obtain the fixed-end moment in the support "A" "$M_{AB} = \psi_{AB}wL^2$" and the reaction in the support "A" "$R_A = \xi_{AB}wL$". The fixed-end moment in the support "B" "$M_{BA} = \psi_{BA}wL^2$" and the reactions in the support "B" "$R_B = \xi_{BA}wL$" are obtained by static balance.

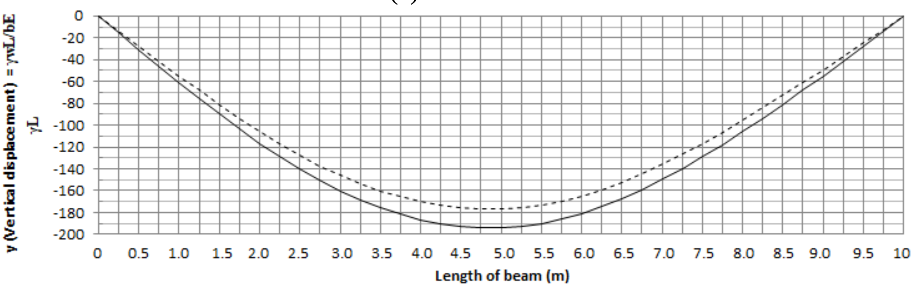
Just as presented in Tables 4 and 5, the factors for the moments and reactions in the support "A" are influenced by the height "h". As the height of the haunches is increased in support "B" is seen as a decrease in these factors for the support "A" in both factors, this is for the two models. Also, the factors in the maximum deflections are influenced by the height "h". As the height of the haunches is increased in support "B" is seen a decrease in these factors, this is for

Table 5 Beam fixed at both ends for h = 0.2 L

c	z/h	Factors for the moments and reactions in the support A						Factors for the maximum displacements				
		ψ_{AB}			ξ_{AB}			β		γ		
		PM	TM	PM/TM	PM	TM	PM/TM	PM	TM	PM	TM	PM/TM
a = 0.2L; u/h = 0.5												
0.2L	0.5	0.1049	0.0941	1.1148	0.5000	0.5000	1.0000	0.5000	0.5000	2.20	2.36	0.9322
	1.0	0.0975	0.0872	1.1181	0.4790	0.4805	0.9969	0.4786	0.4809	1.81	1.99	0.9095
	1.5	0.0931	0.0830	1.1217	0.4664	0.4685	0.9955	0.4646	0.4682	1.59	1.78	0.8933
	2.0	0.0903	0.0802	1.1259	0.4583	0.4605	0.9952	0.4550	0.4593	1.46	1.65	0.8848
0.4L	0.5	0.1004	0.0908	1.1057	0.4861	0.4903	0.9914	0.4927	0.4924	2.00	2.18	0.9174
	1.0	0.0879	0.0799	1.1001	0.4477	0.4575	0.9786	0.4616	0.4661	1.42	1.67	0.8503
	1.5	0.0788	0.0719	1.0960	0.4190	0.4326	0.9686	0.4339	0.4447	1.06	1.35	0.7852
	2.0	0.0723	0.0660	1.0955	0.3978	0.4136	0.9618	0.4099	0.4266	0.83	1.12	0.7411
a = 0.2L; u/h = 1.0												
0.2L	0.5	0.1185	0.1067	1.1106	0.5210	0.5195	1.0029	0.5213	0.5191	1.81	1.99	0.9095
	1.0	0.1105	0.0993	1.1128	0.5000	0.5000	1.0000	0.5000	0.5000	1.47	1.67	0.8802
	1.5	0.1057	0.0947	1.1162	0.4873	0.4878	0.9990	0.4857	0.4870	1.29	1.49	0.8658
	2.0	0.1027	0.0917	1.1200	0.4791	0.4798	0.9985	0.4759	0.4781	1.17	1.38	0.8478
0.4L	0.5	0.1139	0.1034	1.1015	0.5079	0.5108	0.9943	0.5144	0.5112	1.64	1.85	0.8865
	1.0	0.1005	0.0919	1.0936	0.4700	0.4788	0.9816	0.4844	0.4857	1.15	1.42	0.8099
	1.5	0.0907	0.0834	1.0875	0.4413	0.4540	0.9720	0.4574	0.4647	0.85	1.13	0.7522
	2.0	0.0836	0.0769	1.0871	0.4197	0.4349	0.9650	0.4329	0.4469	0.63	0.94	0.6702



(a) h = 0.1 L



(b) h = 0.2 L

——— PM - - - - - TM

Fig. 4 Deflections in a simply supported beam

the two models. According to the results, the proposed model is greater for the moments in the support "A" and for the reactions, and for the maximum deflections are lower, and for the two cases in h = 0.2L, where the biggest difference is of 12.59% for the moments, for the reactions in the support "A" is of 0.9618 times the proposed model compared to the traditional model, and for the maximum deflections is of 0.6702 times the proposed model with to the traditional model.

Fig. 4 shows the elastic curve for a simply supported rectangular beam with straight haunches under uniformly distributed load for values of G = 5E/12; a = 0.2L; u/h = 0.5; c = 0.4L; z/h = 0.5; h = 0.1L and h = 0.2L of the two models.

Fig. 5 shows the elastic curve for a rectangular beam fixed at both ends with straight haunches under uniformly distributed load for values of G = 5E/12; a = 0.2L; u/h = 0.5; c = 0.4L; z/h = 0.5; h = 0.1L and h = 0.2L of the two

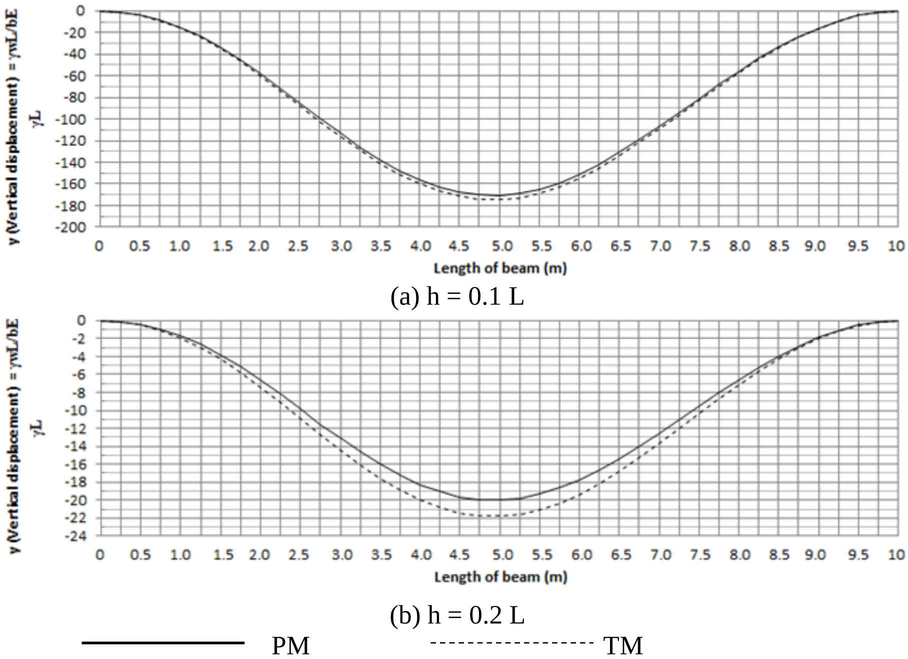


Fig. 5 Deflections in a beam fixed at both ends

models.

The values of the elastic curve (deflections) for the simply supported beam is greater for the proposed model with respect to the traditional model for h = 0.1 L and h = 0.2 L, and the values greater are for h = 0.2 L (see Fig. 4). For the beam fixed at both ends is lower for the proposed model with respect to the traditional model for h = 0.1L and h = 0.2 L, and the values greater are for h = 0.2 L (see Fig. 5).

5. Conclusions

The model proposed to obtain the rotations and the deflections anywhere of the rectangular cross-section beam with straight haunches under a uniformly distributed load and moments in the ends considering the bending and shear deformations (Timoshenko theory) has been developed for the general case.

The mathematical technique presented in this research is adequate to find the rotations and the deflections anywhere of the beam subjected to a uniformly distributed load and any type of moments applied to its ends, because the mathematical formulas of the elastic curve are shown.

The main conclusions are:

1. The traditional model is not influenced by the relationship of “h/L” for the beams fixed at both ends for the moments and reactions in the support “A”, and the location of the maximum displacements, and also for the simply supported beams in the location of the maximum displacements.

2. The greater difference is presented in “h/L = 0.20” than in “h/L = 0.10”, i.e., to a greater relationship of “h/L” appear the greater difference for the simply supported beams in the rotations for the supports “A” and “B”, and the maximum displacements, and for the beams fixed at both ends for the moments and reactions in the support “A”, and the maximum displacements.

3. The proposed model for the simply supported beams is greater in all cases for the rotations in the supports and for the maximum deflections.

4. The proposed model for the beams fixed at both ends is greater for the moments in the support “A” and for the reactions, and for the maximum deflections are lower.

The maximum deflections by the proposed model (bending and shear deformations are considered) are greater for the simply supported beams, and for the beams fixed at both ends are lower with respect to the traditional model (bending deformations are considered). Then maximum deflections acting on the beams of the proposed model in this paper must be compared against the maximum deflections permitted by building codes, because in some conditions could be that does not meet the standards set by building codes or in other conditions could be conservative.

Then, the proposed model is more appropriate and safe with respect the traditional model for structural analysis, because the shear forces and bending moments are present in any type of structure and the bending and shear deformations appear.

The suggestions for future research may be: 1) when the applied load is different to a uniformly distributed load; 2) when the cross-section of the beam is different to a rectangular; 3) when the beam is constructed with parabolic haunches.

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