Various Techniques for Nonlinear Energy-Related Optimizations

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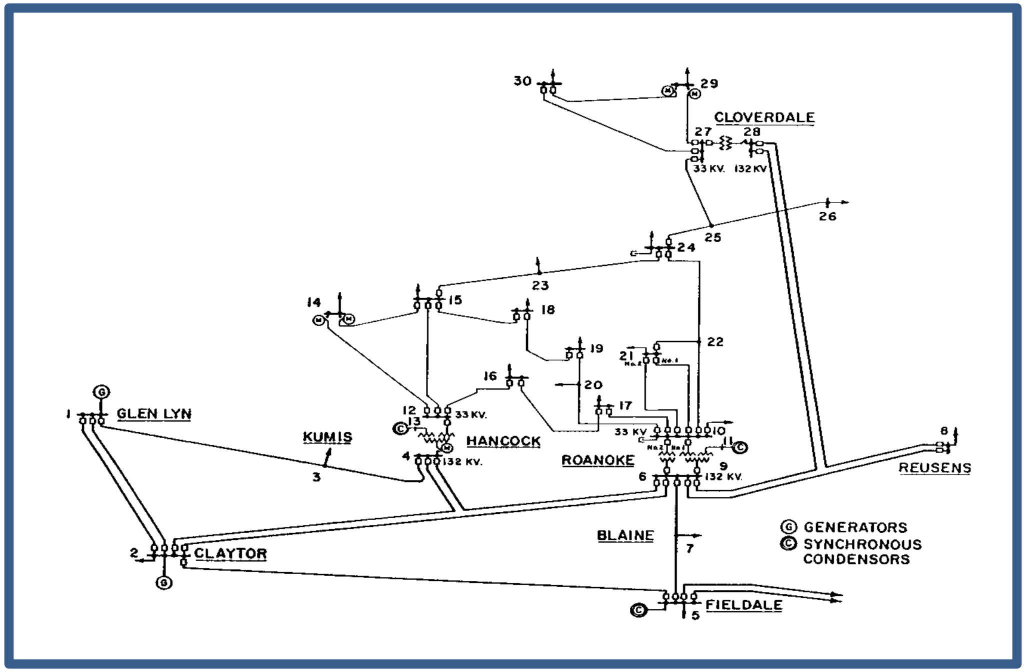
- J. Lavaei and S. Low, "Zero Duality Gap in Optimal Power Flow Problem," IEEE Transactions on Power Systems, 2012.
- J. Lavaei, D. Tse and B. Zhang, "Geometry of Power Flows in Tree Networks," in IEEE Power & Energy Society General Meeting, 2012.
- S. Sojoudi and J. Lavaei, "Physics of Power Networks Makes Hard Optimization Problems Easy To Solve," in IEEE Power & Energy Society General Meeting, 2012.
- M. Kraning, E. Chu, J. Lavaei and S. Boyd, "Message Passing for Dynamic Network Energy Management," Submitted for publication, 2012.
- S. Sojoudi and J. Lavaei, "Semidefinite Relaxation for Nonlinear Optimization over Graphs with Application to Optimal Power Flow Problem," Working draft, 2012.
- S. Sojoudi and J. Lavaei, "Convexification of Generalized Network Flow Problem with Application to Optimal Power Flow," Working draft, 2012.

Power Networks *(CDC 10, Allerton 10, ACC 11, TPS 11, ACC 12, PGM 12)*

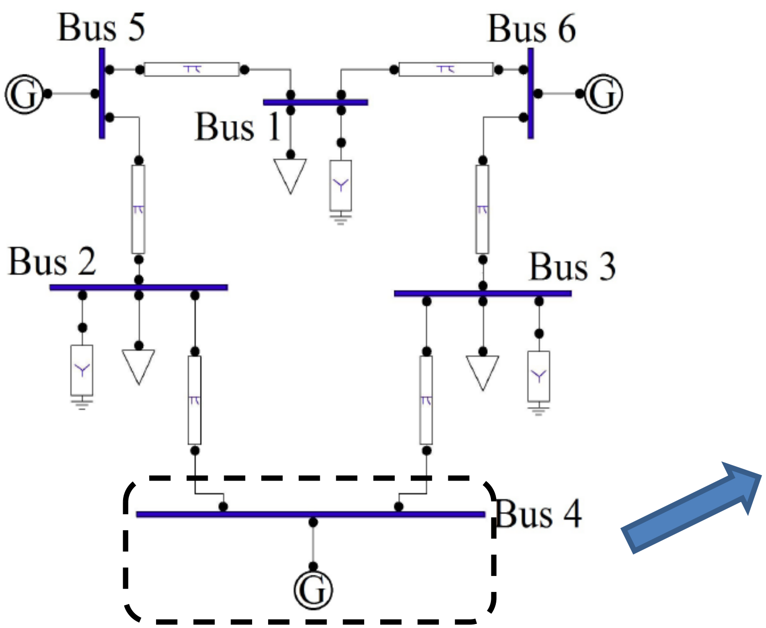
- **Optimizations:**
  - Resource allocation
  - State estimation
  - Scheduling

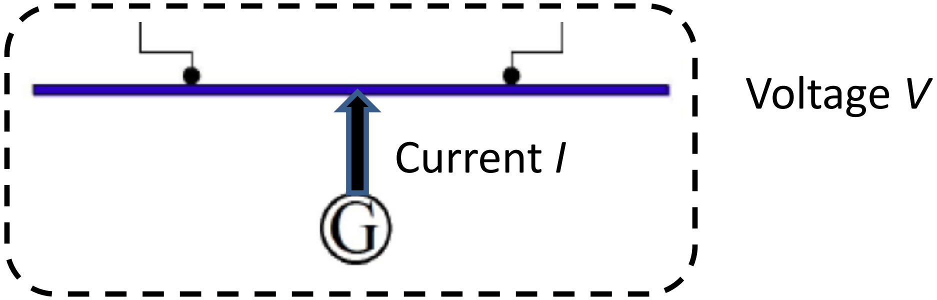


- **Issue:** Nonlinearities

- **Transition from traditional grid to smart grid:**
  - More variables (10X)
  - Time constraints (100X)

Resource Allocation: Optimal Power Flow (OPF)





Complex power = $VI^{*}=P + Q\,\mathrm{i}$

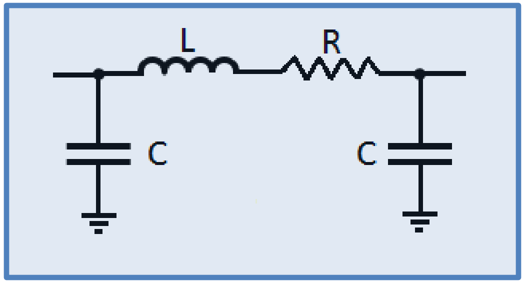
OPF: Given constant-power loads, find optimal *P*'s subject to:

- Demand constraints
- Constraints on *V*'s, *P*'s, and *Q*'s.

Summary of Results

Project 1: How to solve a given OPF in polynomial time? (joint work with Steven Low)

- ❑ A sufficient condition to globally solve OPF:
  - Numerous randomly generated systems
  - IEEE systems with 14, 30, 57, 118, 300 buses
  - European grid



- ❑ **Various theories:** It holds widely in practice

Project 2: Find network topologies over which optimization is easy? (joint work with Somayeh Sojoudi, David Tse and Baosen Zhang)

- ❑ Distribution networks are fine.
- ❑ Every transmission network can be turned into a good one.

Summary of Results

Project 3: How to design a parallel algorithm for solving OPF? (joint work with Stephen Boyd, Eric Chu and Matt Kranning)

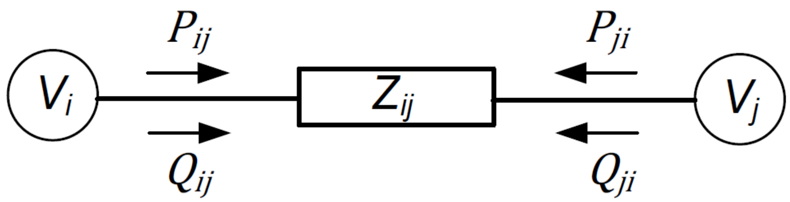
- A practical (infinitely) parallelizable algorithm
- It solves 10,000-bus OPF in 0.85 seconds on a single core machine.

Project 4: How to do optimization for mesh networks? (joint work with Ramtin Madani)

Project 5: How to relate the polynomial-time solvability of an optimization to its structural properties? (joint work with Somayeh Sojoudi)

Project 6: How to solve generalized network flow (CS problem)? (joint work with Somayeh Sojoudi)

Convexification



- **Flow:** $$P_{ij} + Q_{ij}\sqrt{-1} = V_i(V_i - V_j)^* \frac{1}{Z_{ij}^*}$$
- **Injection:** $$P_i = \sum_{j \in \mathcal{N}(i)} P_{ij}$$

- **Polar:** $$V_i \Longrightarrow (|V_i|, \theta_i)$$
- **Rectangular:** $$V_i \Longrightarrow (\mathrm{Re}\{V_i\}, \mathrm{Im}\{V_i\})$$

Convexification in Polar Coordinates

$P_{ij} = \|V_i\|^2 G_{ij} - \|Y_{ij}\|\|V_i\|\|V_j\| \cos(\theta_{ij} + \measuredangle Z_{ij})$	$Q_{ij} = \|V_i\|^2(-B_{ij}) - \|Y_{ij}\|\|V_i\|\|V_j\| \sin(\theta_{ij} + \measuredangle Z_{ij})$
$P_{ji} = \|V_j\|^2 G_{ij} - \|Y_{ij}\|\|V_i\|\|V_j\| \cos(-\theta_{ij} + \measuredangle Z_{ij})$	$Q_{ji} = \|V_j\|^2(-B_{ij}) - \|Y_{ij}\|\|V_i\|\|V_j\| \sin(-\theta_{ij} + \measuredangle Z_{ij})$

Theorem

Having fixed $|V_1|, \ldots, |V_n|$, the functions P_{ij}, Q_{ij}, P_i and Q_i's are all convex in $\theta_1, \ldots, \theta_n$ if

$$0 \leq \pm\theta_{ij} + \measuredangle Z_{ij} \leq 90^\circ$$

Similar to the condition derived in Ross Baldick's book

$\frac{X_{ij}}{R_{ij}}$	3	5	7	9
$\max \|\theta_{ij}\|$	18.43°	11.30°	8.13°	6.34°

- ❑ Imposed implicitly (thermal, stability, etc.)
- ❑ Imposed explicitly in the algorithm

Convexification in Polar Coordinates

$P_{ij} = \|V_i\|^2 G_{ij} - \|Y_{ij}\|\|V_i\|\|V_j\| \cos(\theta_{ij} + \measuredangle Z_{ij})$	$Q_{ji} = \|V_i\|^2(-B_{ij}) - \|Y_{ij}\|\|V_i\|\|V_j\| \sin(\theta_{ij} + \measuredangle Z_{ij})$
$P_{ji} = \|V_j\|^2 G_{ij} - \|Y_{ij}\|\|V_i\|\|V_j\| \cos(-\theta_{ij} + \measuredangle Z_{ij})$	$Q_{ji} = \|V_j\|^2(-B_{ij}) - \|Y_{ij}\|\|V_i\|\|V_j\| \sin(-\theta_{ij} + \measuredangle Z_{ij})$

Theorem

Having fixed θ_{ij}'s satisfying

$$0 \leq \pm\theta_{ij} + \measuredangle Z_{ij} \leq 90^\circ,$$

the functions P_{ij}, Q_{ij}, P_i and Q_i's are all convex in $\sqrt{|V_1|}, \ldots, \sqrt{|V_n|}$.

❑ **Idea:**

$$|V_i|^2 \Longrightarrow X_i$$
$$-|V_i||V_j| \Longrightarrow -\sqrt{X_i}\sqrt{X_j}$$

❑ **Algorithm:**

- Fix magnitudes and optimize phases
- Fix phases and optimize magnitudes

Convexification in Polar Coordinates

$P_{ij} = \|V_i\|^2 G_{ij} - \|Y_{ij}\|\|V_i\|\|V_j\| \cos(\theta_{ij} + \measuredangle Z_{ij})$	$Q_{ji} = \|V_i\|^2(-B_{ij}) - \|Y_{ij}\|\|V_i\|\|V_j\| \sin(\theta_{ij} + \measuredangle Z_{ij})$
$P_{ji} = \|V_j\|^2 G_{ij} - \|Y_{ij}\|\|V_i\|\|V_j\| \cos(-\theta_{ij} + \measuredangle Z_{ij})$	$Q_{ji} = \|V_j\|^2(-B_{ij}) - \|Y_{ij}\|\|V_i\|\|V_j\| \sin(-\theta_{ij} + \measuredangle Z_{ij})$

- Can we jointly optimize phases and magnitudes?

Change of variables:

$$|V_i| \Longrightarrow X_i^{\frac{1}{m}}$$

Assumption (implicit or explicit):

$$45^\circ < \pm\theta_{ij} + \measuredangle Z_{ij} < 90^\circ$$

- **Observation 1:** Bounding $|V_i|$ is the same as bounding X_i.
- **Observation 2:** $-|V_i||V_j| \sin(\pm\theta_{ij} + \measuredangle Z_{ij})$ is convex for a large enough m.
- **Observation 3:** $-|V_i||V_j| \cos(\pm\theta_{ij} + \measuredangle Z_{ij})$ is convex for a large enough m.
- **Observation 4:** $|V_i|^2$ is convex for $m \le 2$.

Convexification in Polar Coordinates

Strategy 1: Choose *m=2*.

$$P_{ij} = |V_i|^2 G_{ij} - |Y_{ij}||V_i||V_j| \cos(\theta_{ij} + \measuredangle Z_{ij}) \quad \Rightarrow \quad P_{ij} \approx |V_i|^2 G_{ij} - |Y_{ij}| \cos(\theta_{ij} + \measuredangle Z_{ij})$$

Strategy 2: Choose *m* large enough

- P_{ij}, Q_{ij}, P_i and Q_i become convex after the following approximation:

Replace $|V_i|^2$ with its nominal value.

Example 1

$$\min_{x_1,x_2} x_1^4 + ax_2^2 + bx_1^2x_2 + cx_1x_2$$

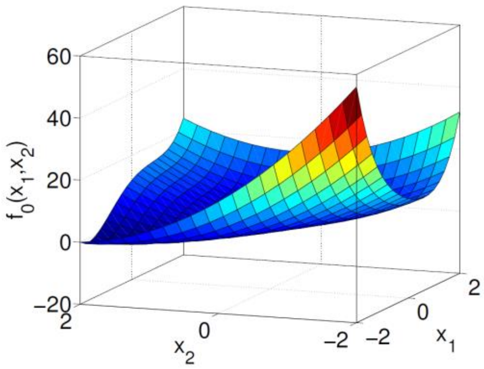
Trick: $x_1^4 = (x_1^2)^2 = x_3^2$

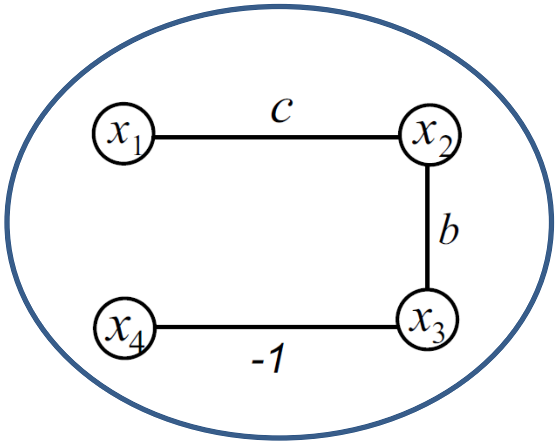
$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{R}^4} \quad & x_3^2 + ax_2^2 + bx_3x_2 + cx_1x_2 \\ \text{s.t.} \quad & x_1^2 - x_3x_4 = 0 \\ & x_4^2 - 1 = 0 \end{aligned}$$

SDP relaxation: $\mathbf{x}\mathbf{x}^* \longrightarrow W$

$x_ix_j \longrightarrow W_{ij}$

$$\begin{aligned} \min_{W\in\mathcal{S}^4} \quad & W_{33} + aW_{22} + bW_{32} + cW_{12} \\ \text{s.t.} \quad & W_{11} - W_{34} \leq 0 \\ & W_{44} - 1 = 0 \\ & W \succeq 0 \end{aligned}$$

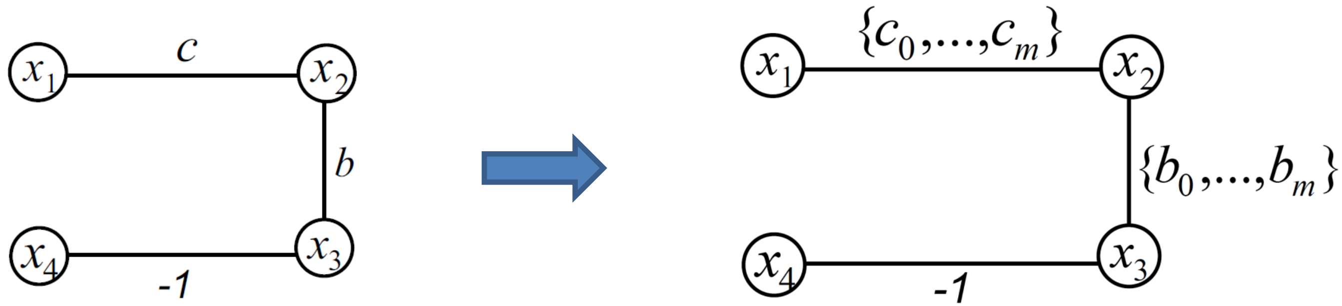




❖ Guaranteed rank-1 solution!

Example 1

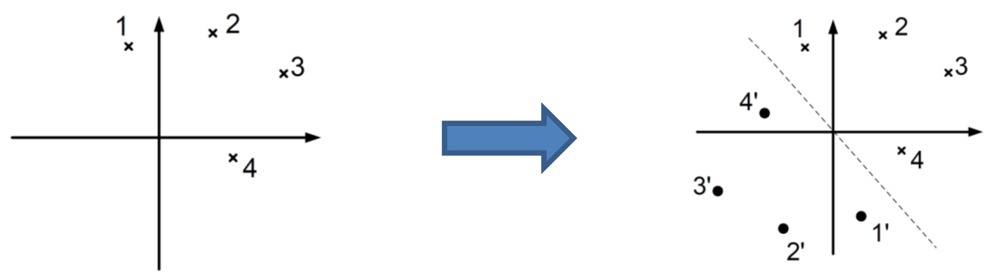
Opt:
$$\min_{x_1,x_2} \quad x_1^4 + a_0 x_2^2 + b_0 x_1^2 x_2 + c_0 x_1 x_2$$
$$\text{s.t.} \quad x_1^4 + a_j x_2^2 + b_j x_1^2 x_2 + c_j x_1 x_2 \leq \alpha_j \quad j = 1, \ldots, m$$



❖ **Sufficient condition for exactness:** Sign definite sets.

❖ **What if the condition is not satisfied?** Rank-2 **W** (but hidden)

Complex case:



Formal Definition: Optimization over Graph

Optimization of interest:

(real or complex)

$$\min_{\mathbf{x}\in\mathcal{D}^n} \quad f_0(\mathbf{y},\mathbf{z})$$
$$\text{s.t.} \quad f_j(\mathbf{y},\mathbf{z}) \leq 0, \qquad j = 1, 2, \ldots, m$$

Define:

$$\mathbf{y} = \left\{ |x_i|^2 \mid \forall i \in \mathcal{G} \right\}$$
$$\mathbf{z} = \left\{ \mathrm{Re}\{c_{ij}^1 x_i x_j^*\}, \ldots, \mathrm{Re}\{c_{ij}^k x_i x_j^*\} \;\middle|\; \forall (i,j) \in \mathcal{G} \right\}$$

❖ SDP relaxation for **y** and **z** (replace $\mathbf{x}\mathbf{x}^*$ with W) .

❖ f ($\mathbf{y}$, $\mathbf{z}$) is increasing in **z** (no convexity assumption).

❖ **Generalized weighted graph:** weight set $\{c_{ij}^1, \ldots, c_{ij}^k\}$ for edge (i,j).

Highly Structured Optimization

Theorem (Real Case)

Exact relaxation if

$$\sigma_{ij} \neq 0, \qquad (i,j) \in \mathcal{G}$$

$$\prod_{(i,j)\in\mathcal{O}_r} \sigma_{ij} = (-1)^{|\mathcal{O}_r|}, \qquad r \in \{1, \ldots, p\}$$

Edge

Cycle

Theorem (Complex Case)

Exact relaxation for acyclic graphs with sign-definite weight sets.

Theorem (Imaginary Case)

Exact relaxation for weakly cyclic graphs with homogeneous weight sets.

Convexification in Rectangular Coordinates

$$\min_{\mathbf{V},\mathbf{P}_G,\mathbf{Q}_G} \sum_{k\in\mathcal{G}} f_k(P_{G_k}) \tag{1a}$$

Subject to

$$P_k^{\min} \leq P_{G_k} \leq P_k^{\max} \tag{1b}$$

$$Q_k^{\min} \leq Q_{G_k} \leq Q_k^{\max} \tag{1c}$$

$$V_k^{\min} \leq |V_k| \leq V_k^{\max} \tag{1d}$$

$$\text{Re}\left\{V_l(V_l - V_m)^* y_{lm}^*\right\} \leq P_{lm}^{\max} \tag{1e}$$

$$\text{trace}\{\mathbf{V}\mathbf{V}^*\mathbf{Y}^* e_k e_k^*\} = P_{G_k} - P_{D_k} + (Q_{G_k} - Q_{D_k})\text{i} \tag{1f}$$

Cost (1a)

Operation (1b)–(1d)

Flow (1e)

Balance (1f)

❖ Express the last constraint as an inequality.

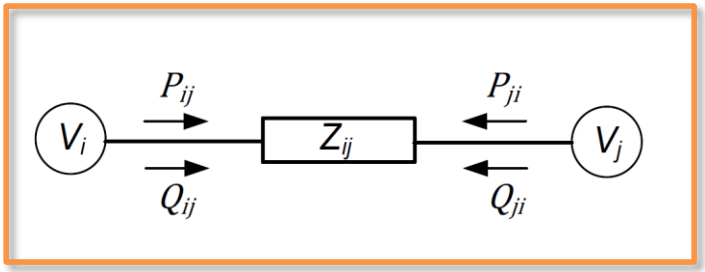
Trick: Replace $\mathbf{V}\mathbf{V}^*$ with a matrix $\mathbf{W} \succeq 0$ subject to $\text{rank}\{\mathbf{W}\} = 1$.

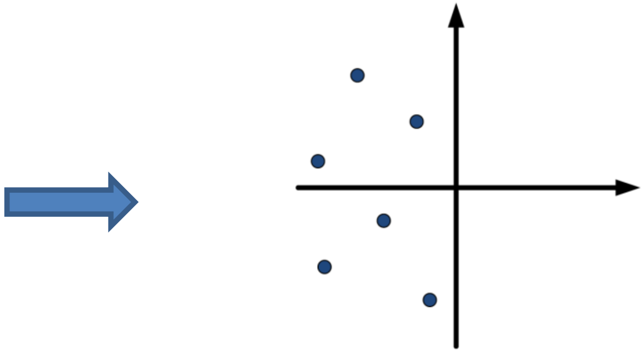
Convexification in Rectangular Coordinates

$$\min_{\mathbf{V}} \quad h_0(\mathbf{P}, \mathbf{Q}, |\mathbf{V}|)$$
$$\text{s.t.} \quad h_j(\mathbf{P}, \mathbf{Q}, |\mathbf{V}|) \leq 0, \quad j = 1, \ldots, m$$

Theorem

Exact relaxation for DC/AC distribution and DC transmission networks.



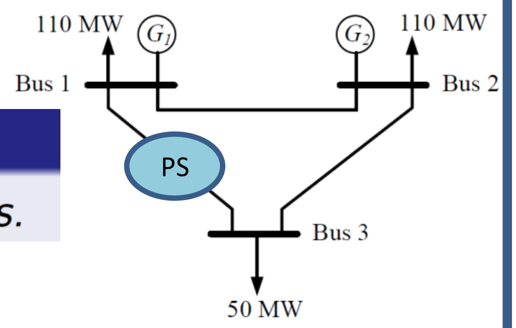


- ❑ Partial results for AC lossless transmission networks.

Phase Shifters

Theorem

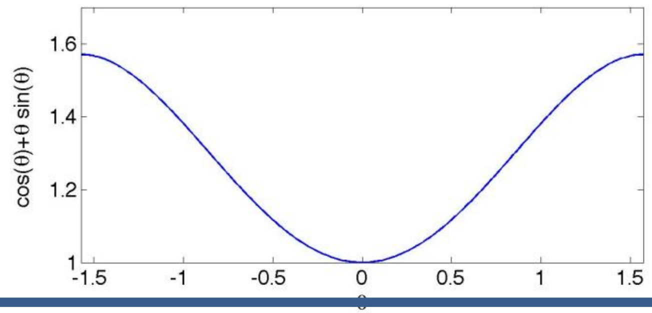
Exact relaxation for AC networks with virtual phase shifters.



- **Practical approach:** Add phase shifters and then penalize their effects.

$$\sum_{i \in \mathcal{G}} f_i(P_i) \quad \Rightarrow \quad \sum_{i \in \mathcal{G}} f_i(P_i) + \lambda(|\phi_1| + \cdots + |\phi_k|)$$

- **Stephen Boyd's function for PF:**



Integrated OPF + Dynamics

- Synchronous machine with interval voltage $|E|e^{j\delta}$ and terminal voltage $|V|e^{j\theta}$.

- **Swing equation:**

$$\frac{d\delta(t)}{dt} = \omega(t) \qquad \boxed{\frac{d(|E|\sin(\delta(t)))}{dt} = |E|\cos(\delta(t))\omega(t)}$$

$$M\frac{d\omega}{dt} = -D\omega(t) + P_M(t) - \frac{|E||V(t)|\sin(\delta(t) - \theta(t))}{\alpha}$$

- **Define:** $\mathbf{x}(t) = \left[\; 1 \quad \omega(t) \quad \mathrm{Re}\{E\} \quad \mathrm{Im}\{E\} \quad \mathrm{Re}\{V(t)\} \quad \mathrm{Im}\{V(t)\} \;\right]^H$

- **Linear system:**

$$\frac{dW_{14}(t)}{dt} = W_{32}(t)$$

$$\frac{dW_{12}(t)}{dt} = -\frac{D}{M}W_{12}(t) - \frac{1}{M\alpha}(W_{45}(t) - W_{36}(t)) + \frac{1}{M}P_M(t)$$

Sparse Solution to OPF

- Unit commitment:

  1- $\alpha_i P_i^{\min} \leq P_i \leq \alpha_i P_i^{\max}$

  2- $\alpha_i \in \{0,1\}$

- Unit commitment:

  1- $\alpha_i P_i^{\min} \leq P_i \leq \alpha_i P_i^{\max}$

  2- $\alpha_i(\alpha_i - 1) = 0$

- Sparse solution to OPF:

  1- $0 \leq P_i \leq P_i^{\max}$

  2- Sparse vector $[P_1 \ P_2 \ \cdots \ P_n]$

- Minimize:

$$\sum_{i=1}^{n} f_i(P_i) + \sum_{i=1}^{n} w_i P_i$$

IEEE system	14 bus	30 bus	118 bus
No. of "on" generators	4-1	6-3	54-9

Lossy Networks

- **Relationship between polar and rectangular?**

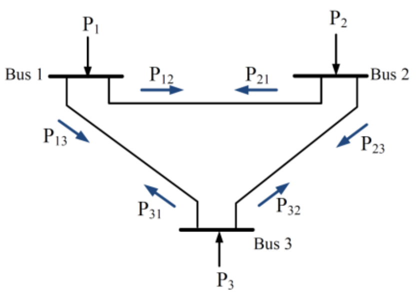
- **Assumption (implicit or explicit):** $45^\circ < \pm\theta_{ij} + \measuredangle Z_{ij} < 90^\circ$

- **Conjecture:** This assumptions leads to convexification in rectangular coordinates.

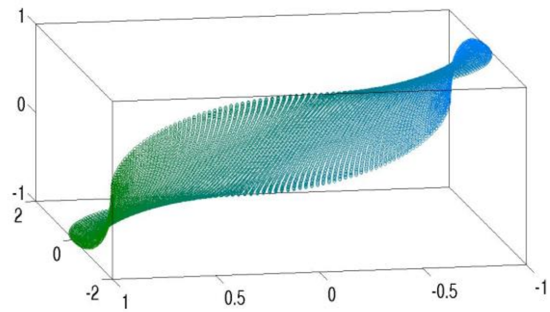
- **Partial Result:** Proof for optimization of reactive powers.

Lossless Networks

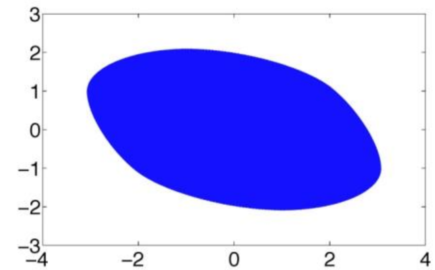
- Consider a lossless AC transmission network.



Lossless 3 bus



(P_{12}, P_{23}, P_{31})

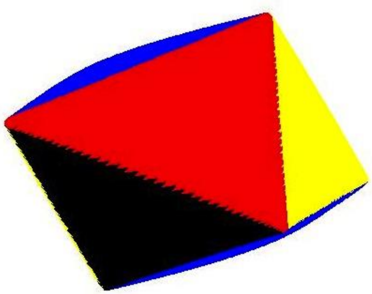


(P_1, P_2)

(P_1, P_2, P_3) for a

4-bus cyclic

Network:



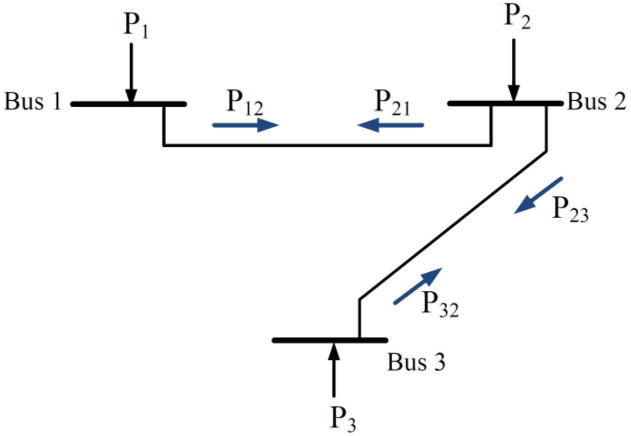
Theorem: The injection region is never convex for $n \geq 5$ if

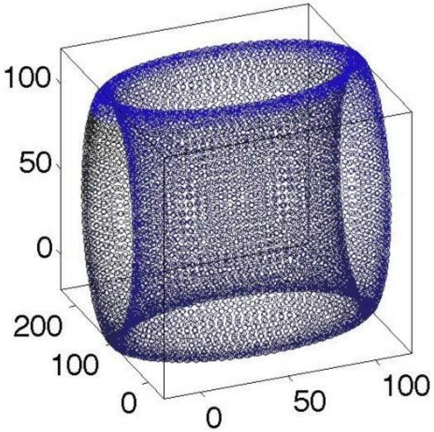
$$|\theta_{ij}| \leq \theta_{ij}^{\max} < 90^\circ, \quad (i,j) \in \mathcal{E}$$

- Current approach: Use polynomial Lagrange multiplier (SOS) to study the problem

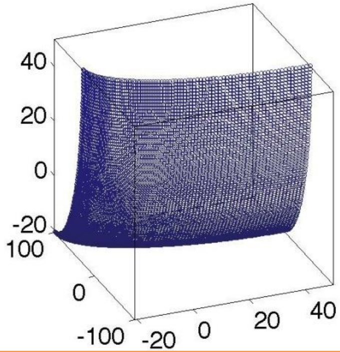
OPF With Equality Constraints

- **Injection region under fixed voltage magnitudes:**



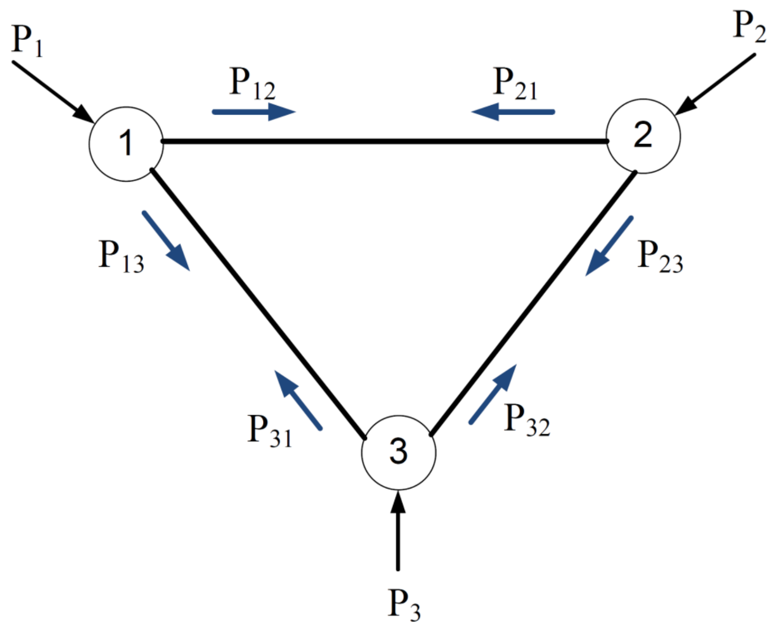


- **When can we allow equality constraints?** Need to study Pareto front



$$\theta_{ij}^{\max} \leq \tan^{-1}\left(\frac{x_{ij}}{r_{ij}}\right)$$

Generalized Network Flow (GNF)



injections

$$p_i = \sum_{j \in \mathcal{N}(i)} p_{ij}$$

flows

$$p_{ji} = f_{ij}(p_{ij}),$$
$$p_{ij} \in [p_{ij}^{\min}, p_{ij}^{\max}]$$

limits

$$p_i^{\min} \leq p_i \leq p_i^{\max}$$

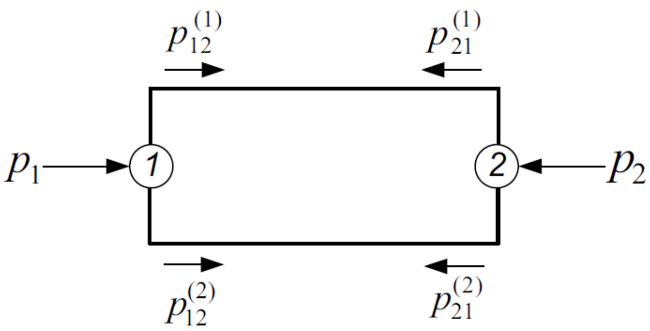
❖ **Goal:**

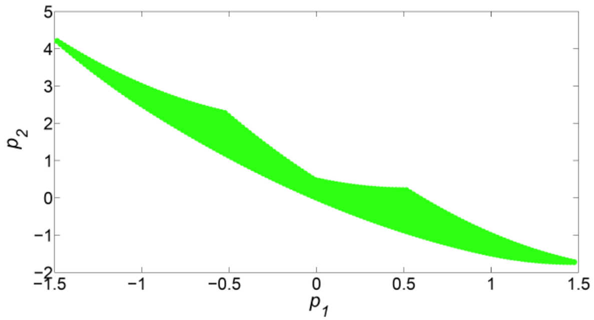
$$\min \sum_{i \in \mathcal{N}} f_i(p_i)$$

Assumption:

- $f_i(p_i)$: convex and increasing
- $f_{ij}(p_{ij})$: convex and decreasing

Convexification of GNF



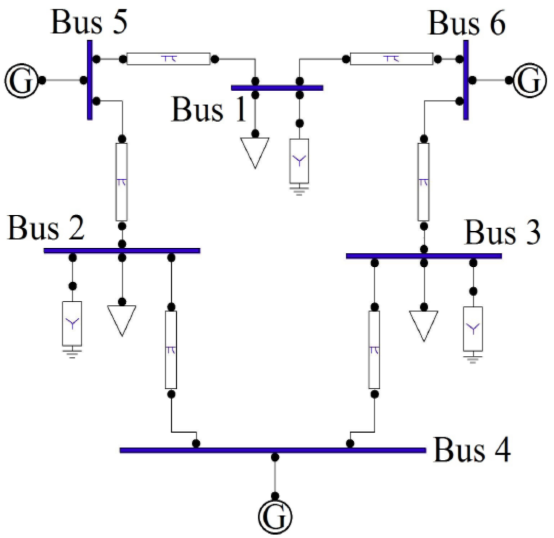


Feasible set without box constraint

❖ **Convexification:** $p_{ji} = f_{ij}(p_{ij}) \Rightarrow p_{ji} \geq f_{ij}(p_{ij})$

❖ It finds correct injection vector but not necessarily correct flow vector.

Conclusions



- **Motivation:** OPF with a 50-year history
- **Goal:** Find a good numerical algorithm

- Convexification in polar coordinates
- Convexification in rectangular coordinates
- Exact relaxation in several cases
- Some problems yet to be solved.