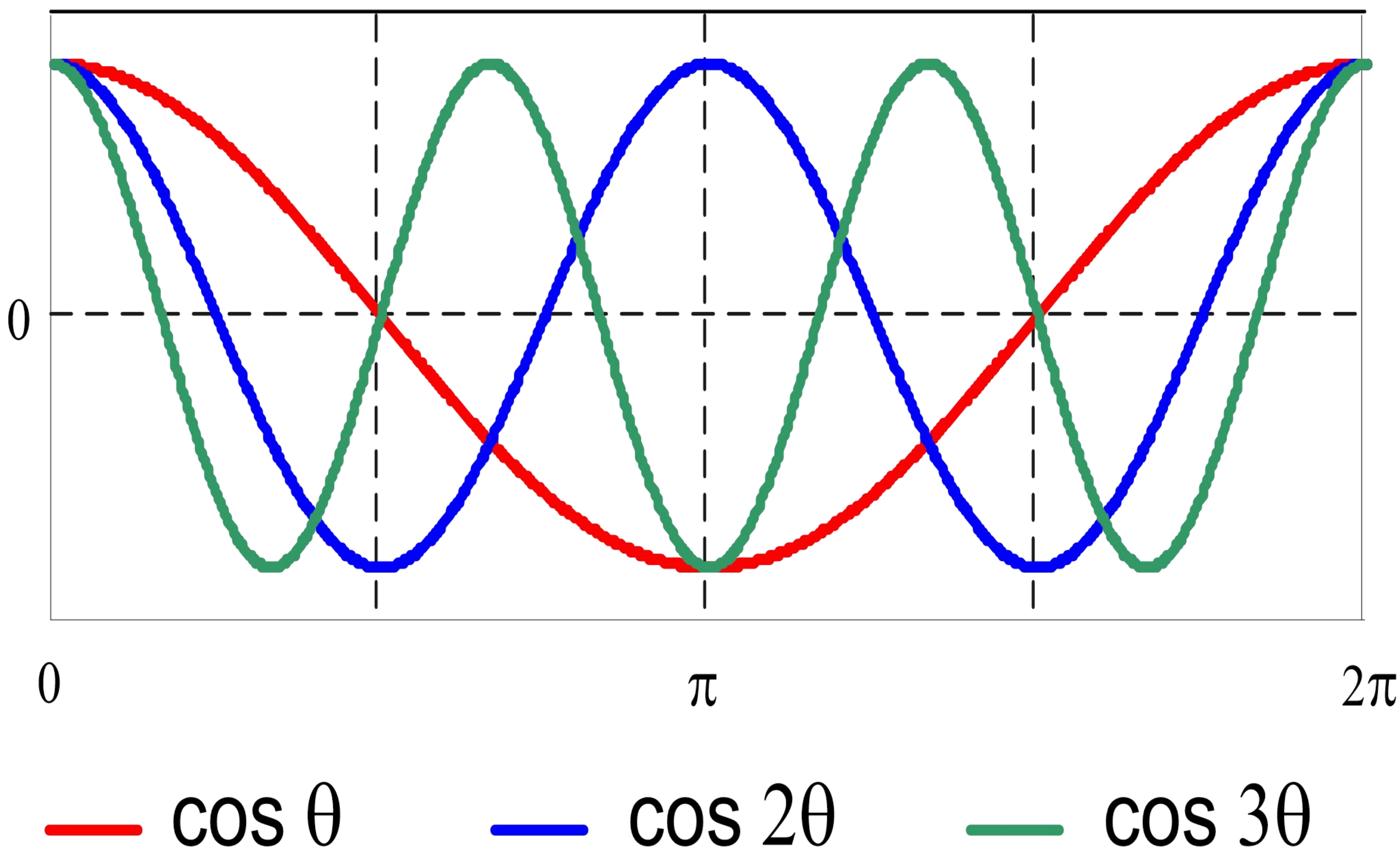
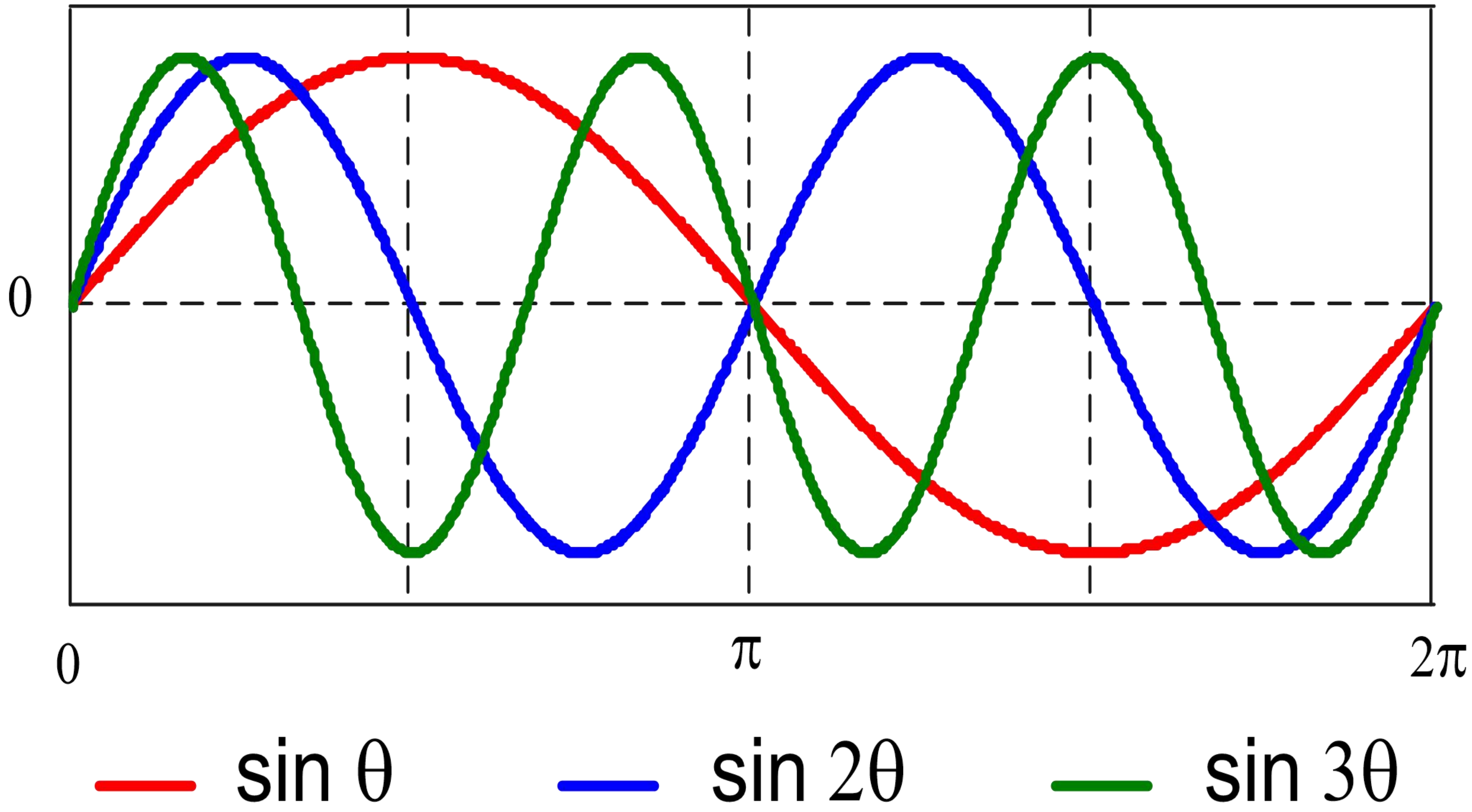
Deret Fourier

Definisi Deret Fourier

Fungsi Genap dan Ganjil


0
0
π
2π
cos θ
cos 2θ
cos 3θ


0
0
π
2π
sin θ
sin 2θ
sin 3θ

Deret Fourier periode 2π

$f(\theta)$ fungsi periodik dengan periode 2π

Fungsi tersebut dapat direpresentasikan dalam deret trigonometrik sebagai:

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta \quad n = 1, 2, \cdots$$

Integrasi dapat dilakukan pada batas

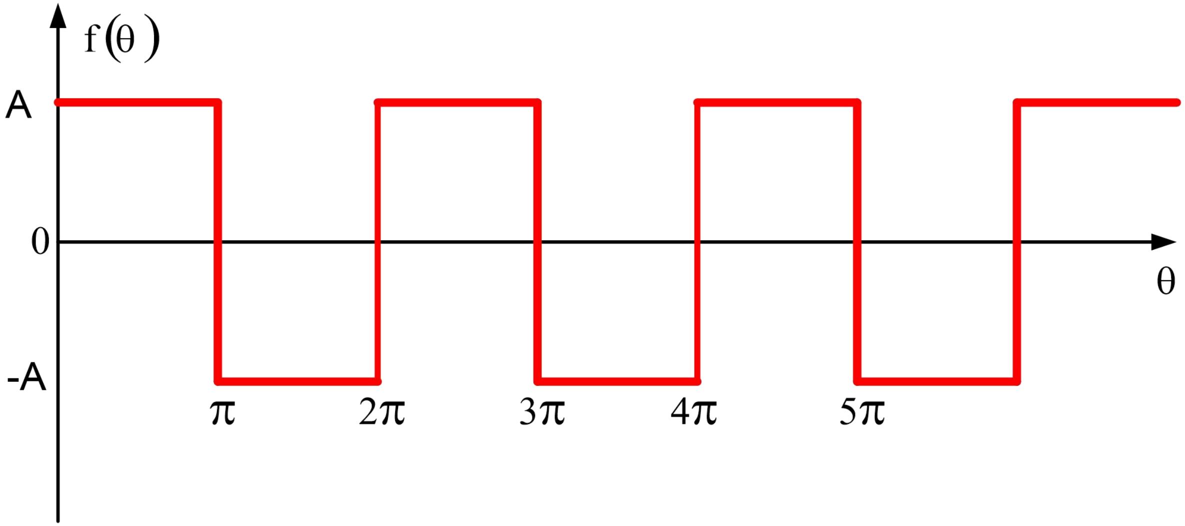
0 sampai 2π

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta \qquad n = 1, 2, \cdots$$

Contoh 1. Tentukan deret Fourier dari fungsi periodik berikut.



$$f(\theta) = A \quad jika \quad 0 < \theta < \pi$$

$$= -A \quad jika \quad \pi < \theta < 2\pi$$

$$f(\theta + 2\pi) = f(\theta)$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} f(\theta)d\theta + \int_{\pi}^{2\pi} f(\theta)d\theta\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} A\,d\theta + \int_{\pi}^{2\pi} -A\,d\theta\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} A\cos n\theta\, d\theta + \int_{\pi}^{2\pi}(-A)\cos n\theta\, d\theta\right]$$

$$= \frac{1}{\pi}\left[A\frac{\sin n\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[-A\frac{\sin n\theta}{n}\right]_{\pi}^{2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} A\sin n\theta\, d\theta + \int_{\pi}^{2\pi} (-A)\sin n\theta\, d\theta\right]$$

$$= \frac{1}{\pi}\left[-A\frac{\cos n\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[A\frac{\cos n\theta}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

$$b_n = \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

$$= \frac{A}{n\pi}\left[1+1+1+1\right]$$

$$= \frac{4A}{n\pi} \quad \text{jika n ganjil}$$

$$b_n = \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

$$= \frac{A}{n\pi}\left[-1+1+1-1\right]$$

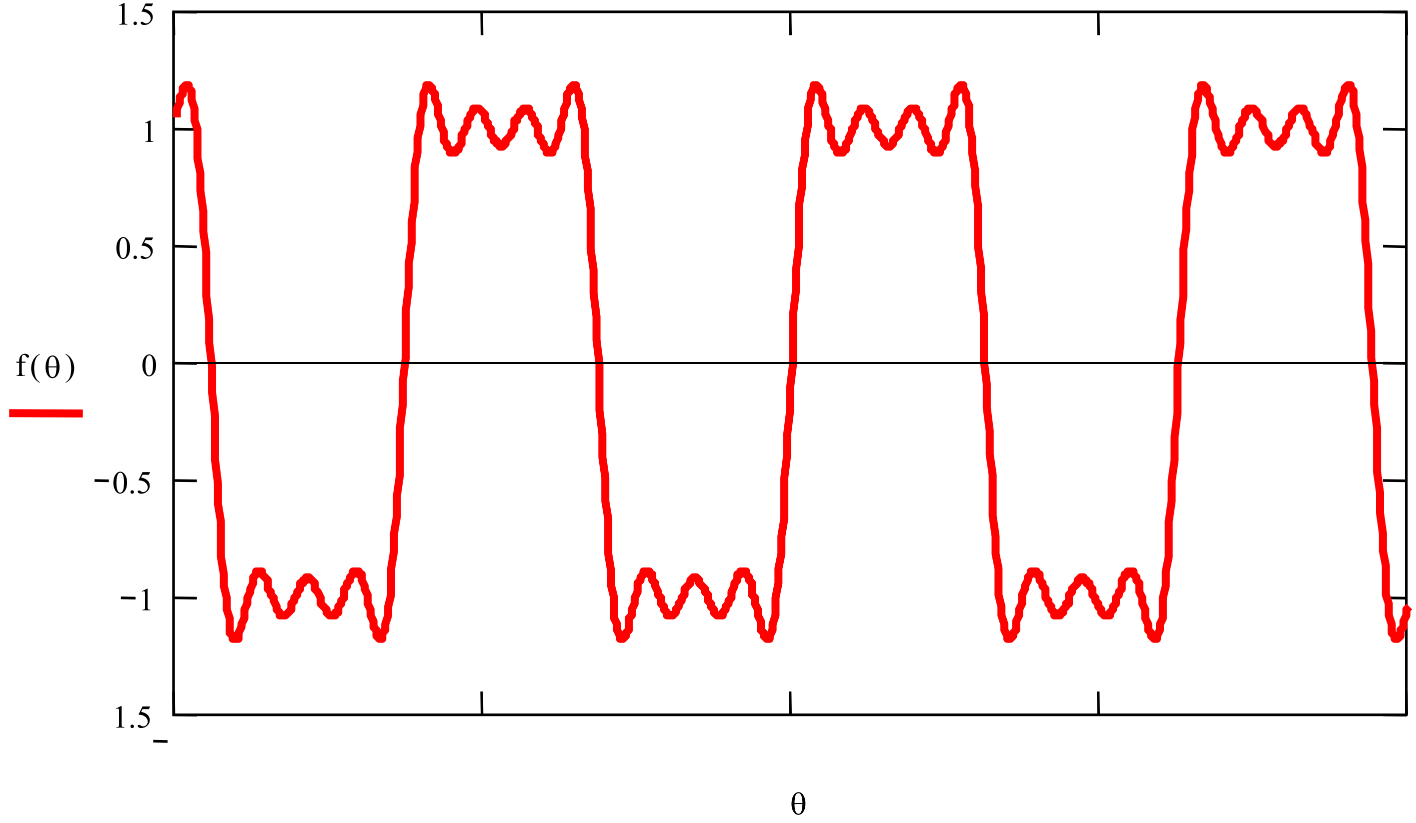
$$= 0 \quad \text{jika n genap}$$

Deret Fourier yang dimaksud adalah:

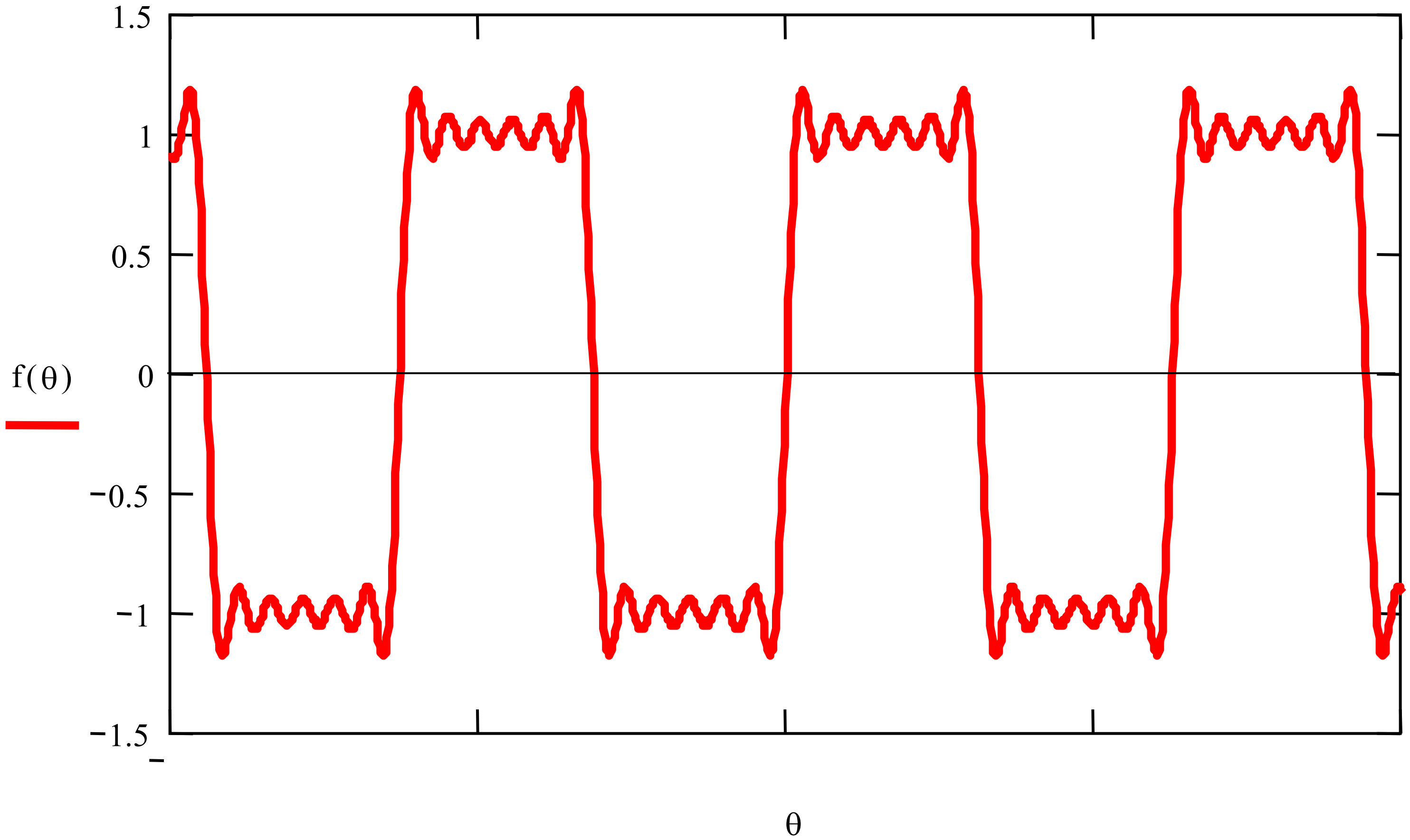
$$\frac{4A}{\pi}\left(\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \frac{1}{7}\sin 7\theta + \cdots\right)$$

Karena deret Fourier memiliki tak hingga banyaknya suku, maka timbul pertanyaan: berapa suku yang kita perlukan?

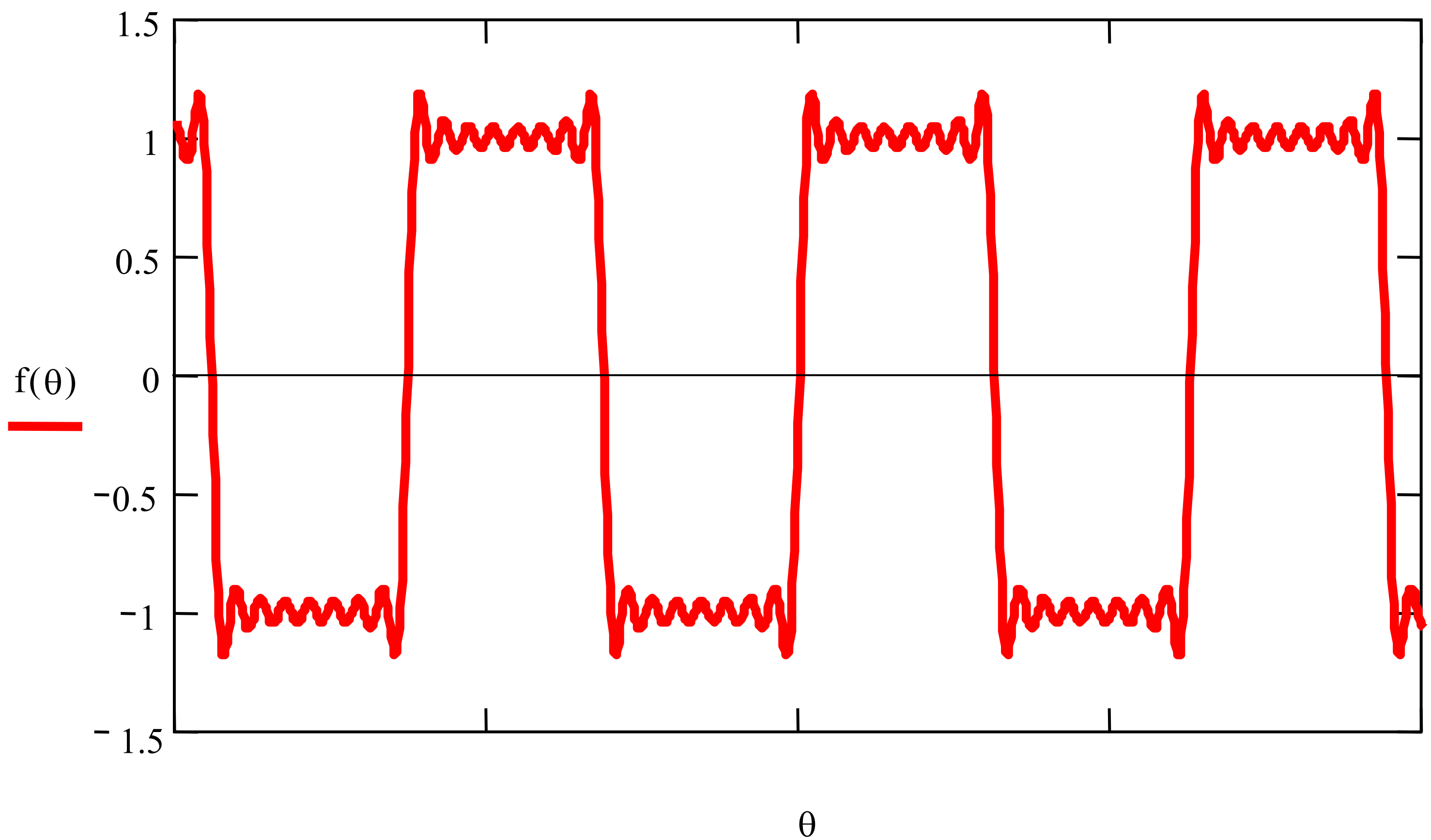
Jika kita menyertakan 4 suku pertama, maka grafik fungsinya adalah seperti berikut:



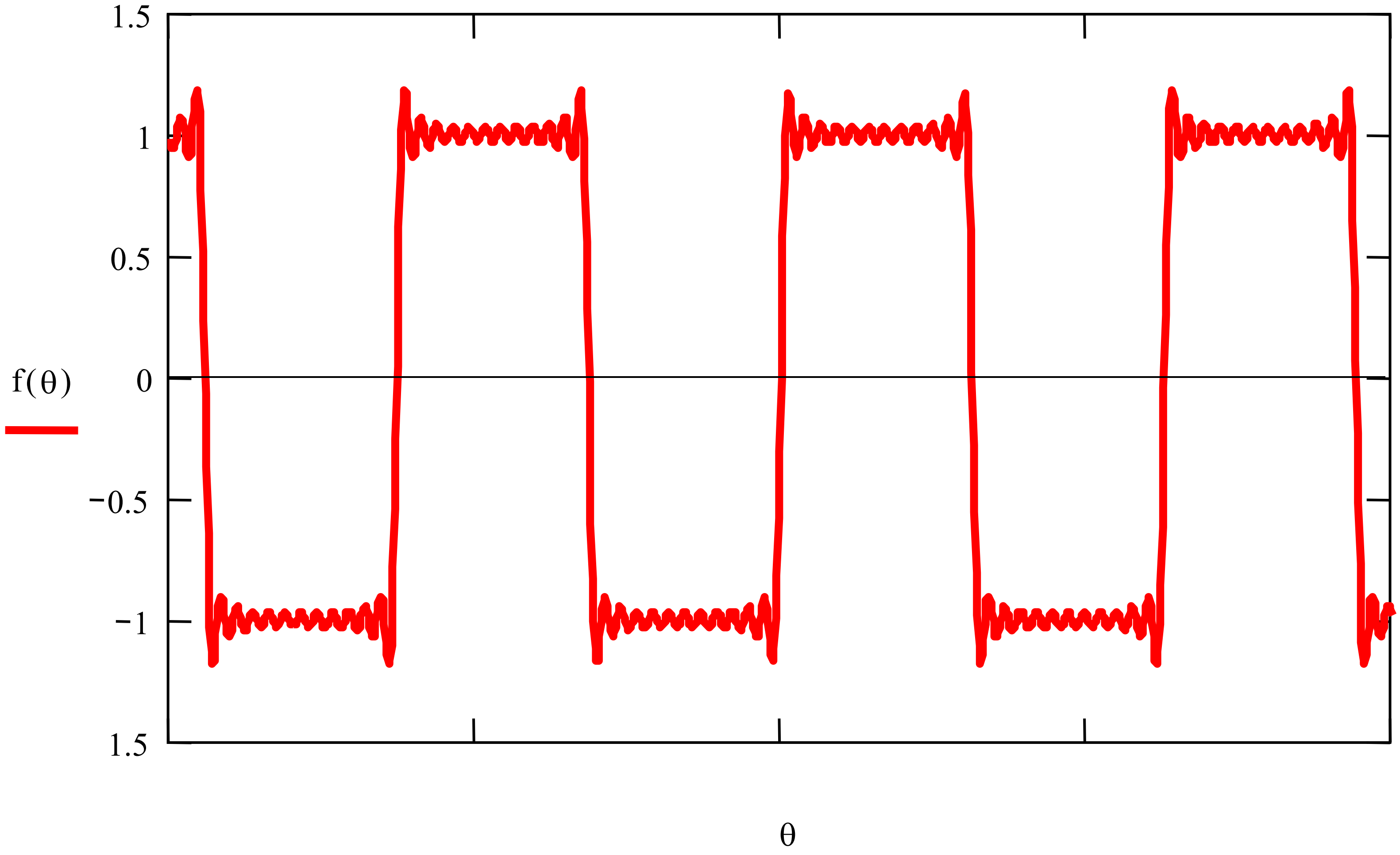
Untuk 6 suku, grafik fungsi tampak seberti ini:



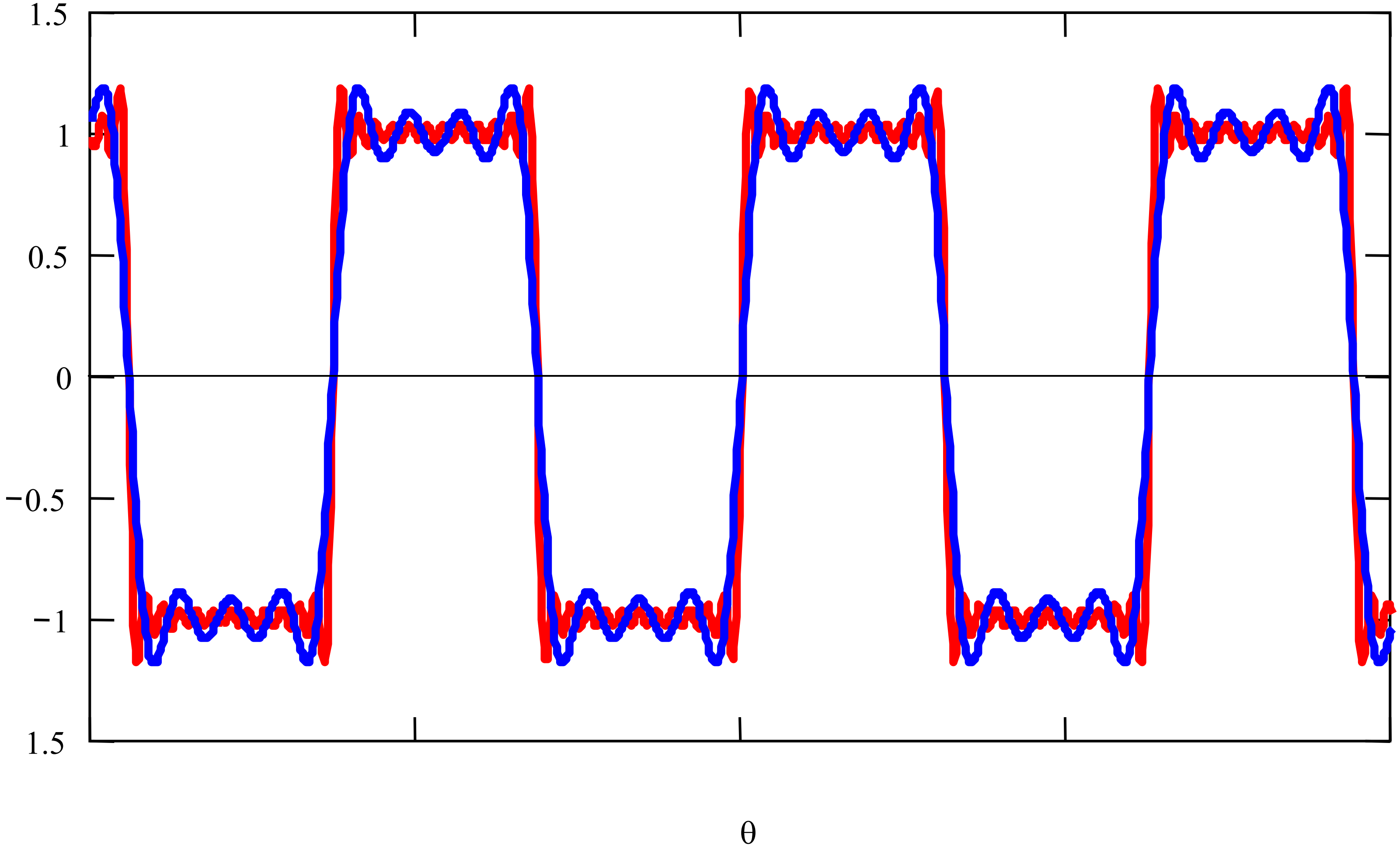
8 suku.



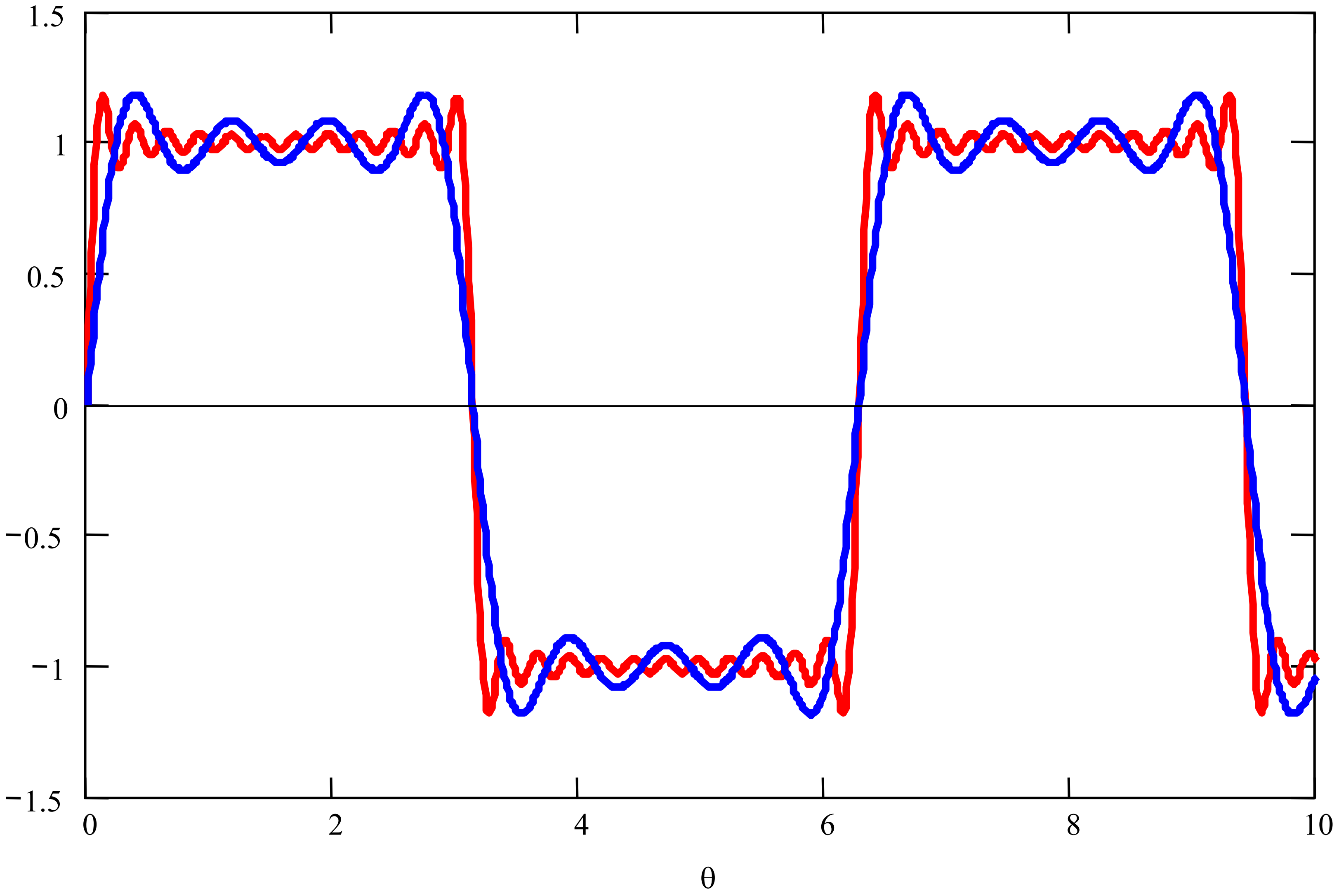
12 suku.



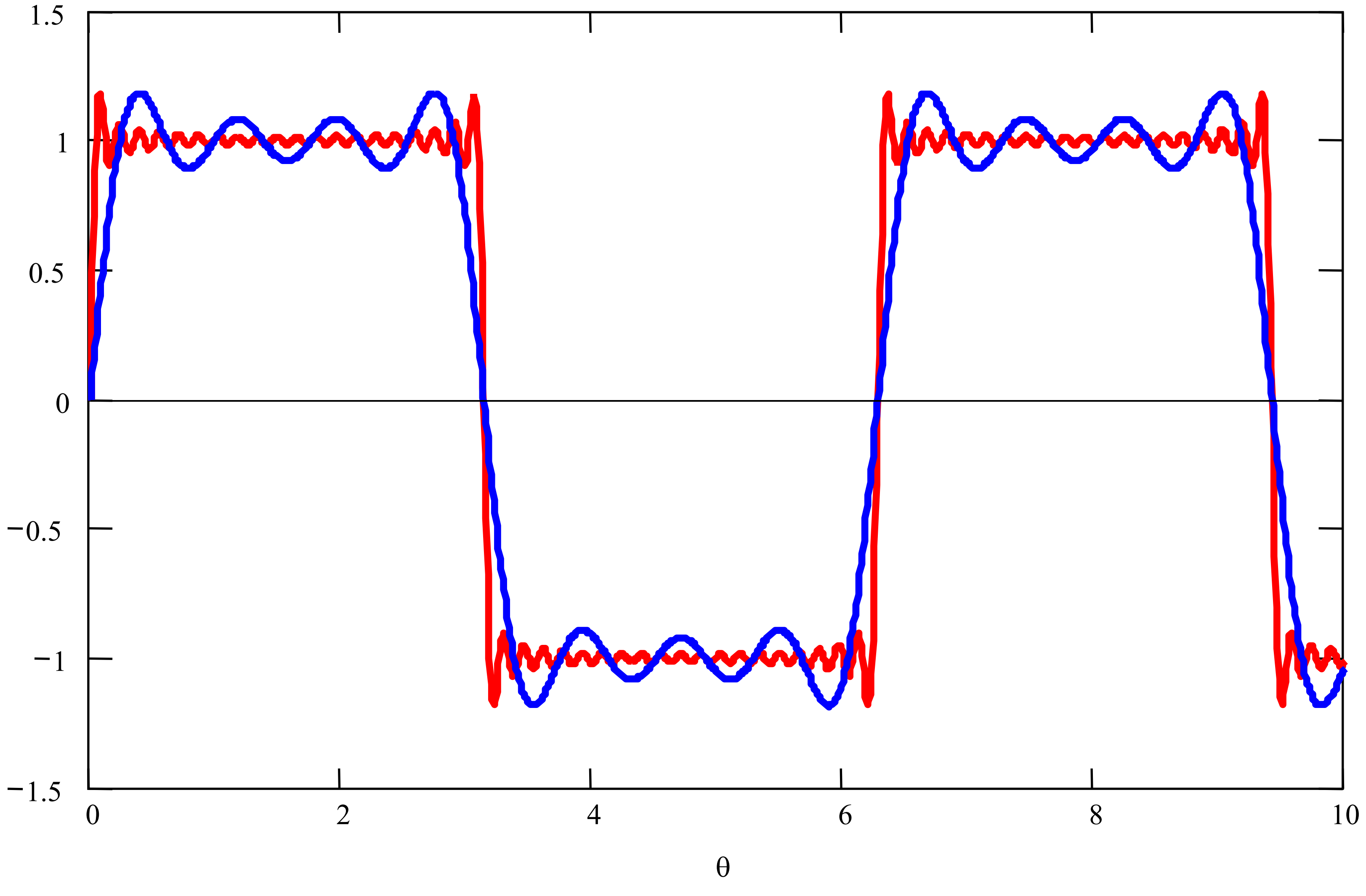
Kurva merah menyertakan 12 suku sedangkan biru 4 suku.



Kurva merah menyertakan 12 suku sedangkan biru 4 suku.



Kurva merah menyertakan 20 suku sedangkan biru 4 suku.



Deret Fourier periode 2L

$f(\theta)$ fungsi periodik dengan periode 2L

Ekspansi deret Fourier dari fungsi tersebut adalah

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

di mana:

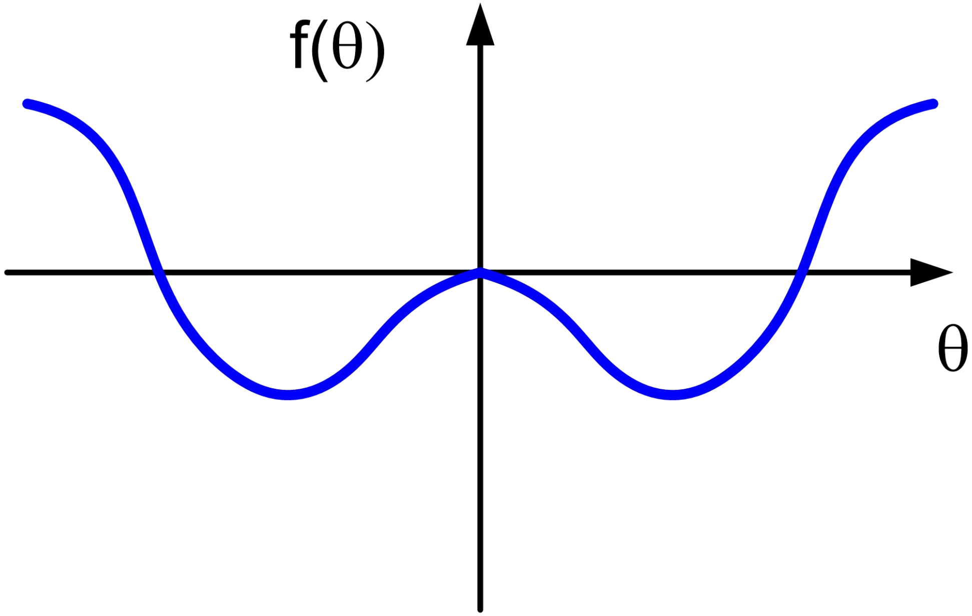
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Fungsi genap dan ganjil

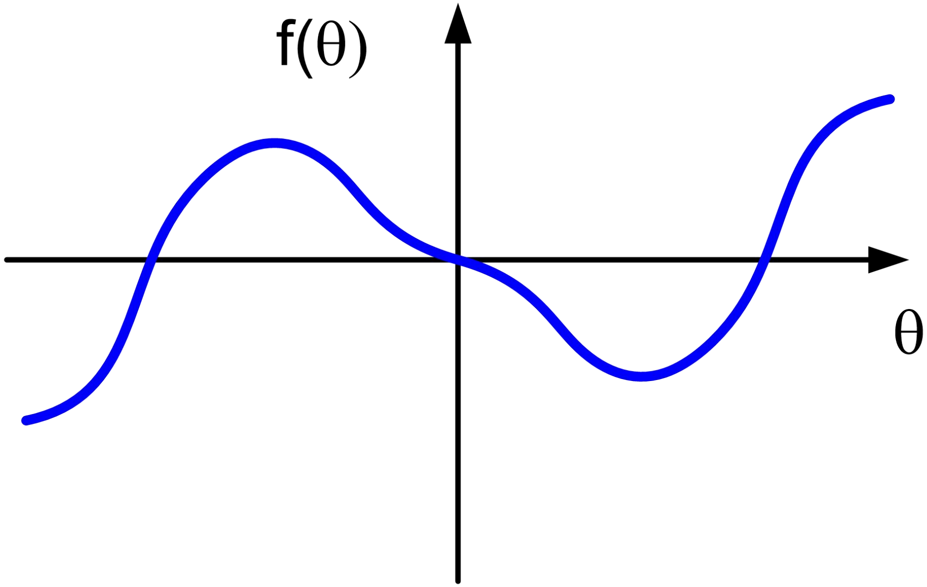
Fungsi genap



Mathematically speaking -

$$f(-\theta) = f(\theta)$$

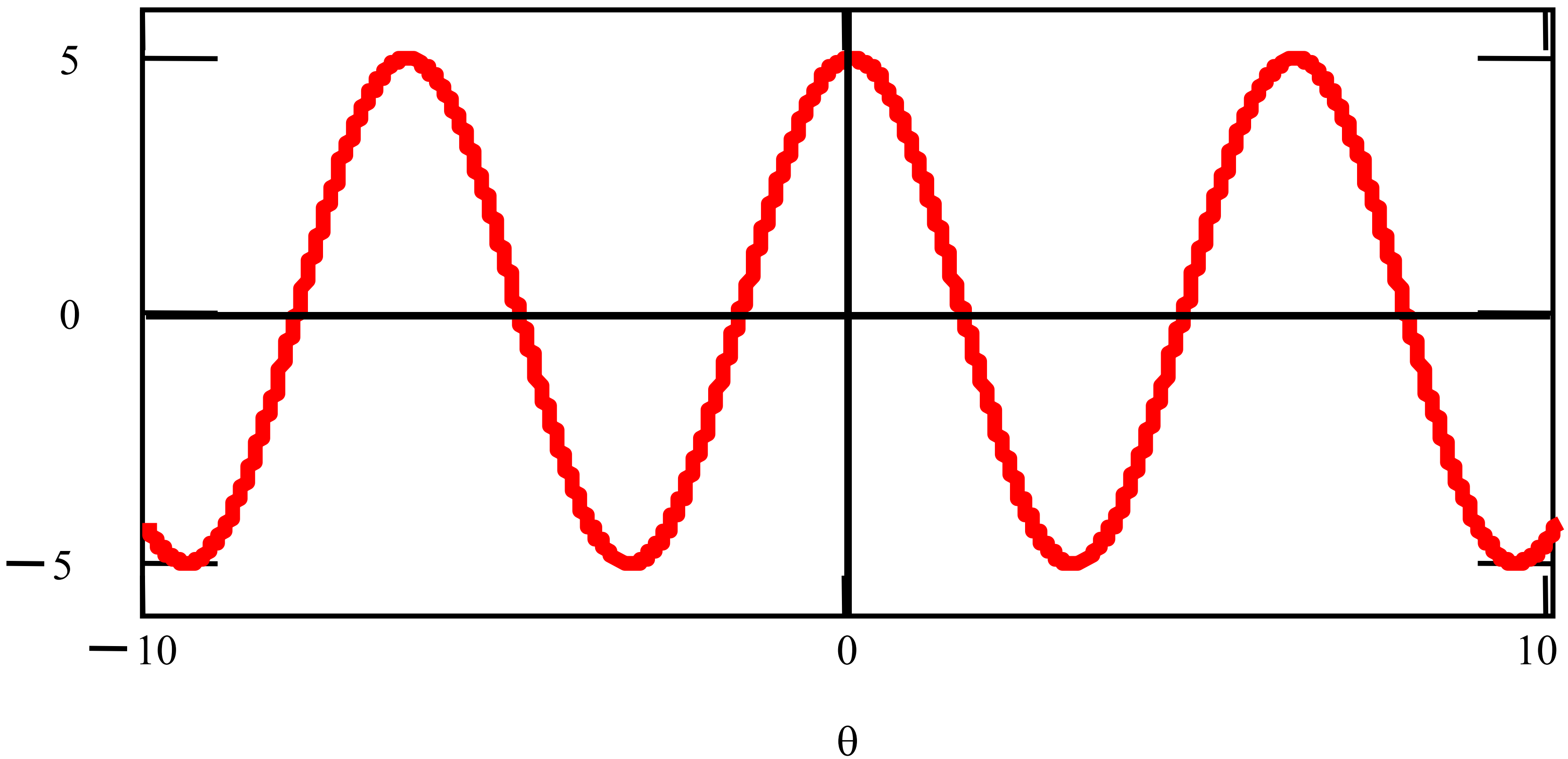
Fungsi ganjil



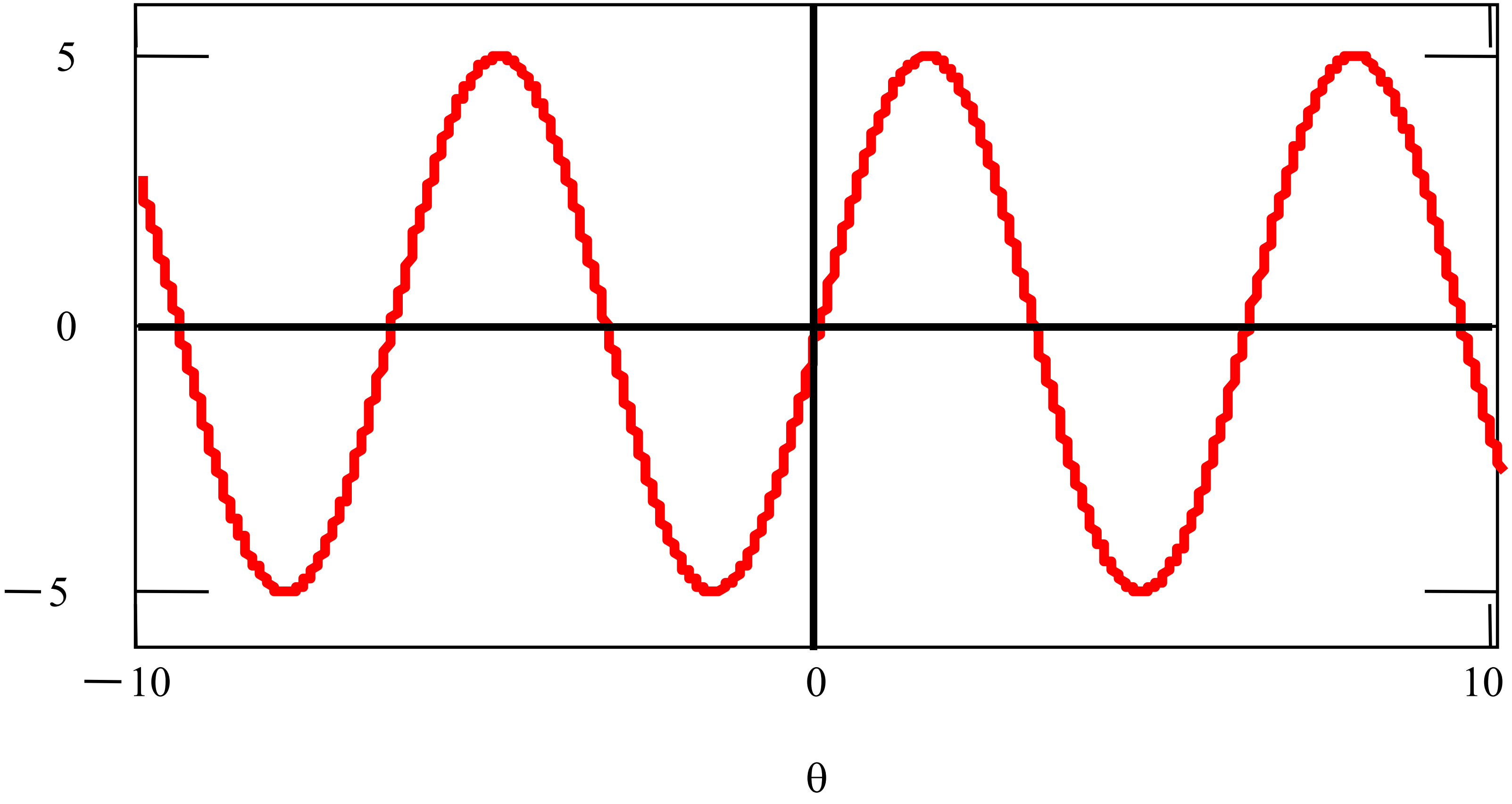
Mathematically speaking -

$$f(-\theta) = -f(\theta)$$

Fungsi genap dapat direpresentasikan oleh kurva cosinus, karena kurva cosinus adalah fungsi genap. Jumlah dari fungsi-fungsi genap adalah juga fungsi genap.



Fungsi ganjil dapat direpresentasikan oleh kurva sinus, karena kurva sinus adalah fungsi ganjil. Jumlah dari fungsi-fungsi ganjil adalah juga fungsi ganjil.



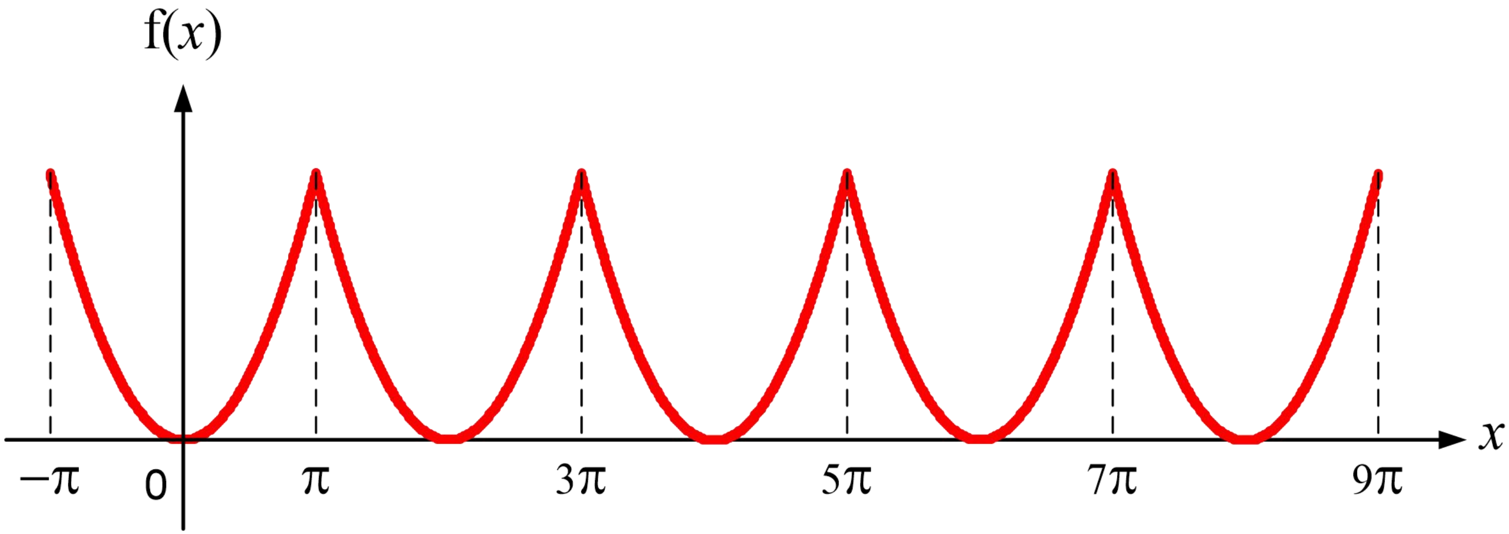
Deret Fourier dari fungsi genap $f(\theta)$ direpresentasikan dalam deret cosinus.

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta$$

Deret Fourier dari fungsi ganjil $f(\theta)$ direpresentasikan dalam deret sinus.

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$$

Contoh 2. Tentukan deret Fourier dari fungsi periodik berikut.



$$f(x) = x^2 \quad jika \quad -\pi \le x \le \pi$$

$$f(\theta + 2\pi) = f(\theta)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{x=-\pi}^{x=\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x^2 \cos nx dx\right]$$

Gunakan integral parsial.

$$a_n = \frac{4}{n^2}\cos n\pi$$

$$a_n = -\frac{4}{n^2} \quad \text{jika n ganjil}$$

$$a_n = \frac{4}{n^2} \quad \text{jika n genap}$$

Karena fungsi tersebut genap.

Maka, $b_n = 0$

Deret Fourier dari fungsi tersebut adalah

$$\frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right)$$