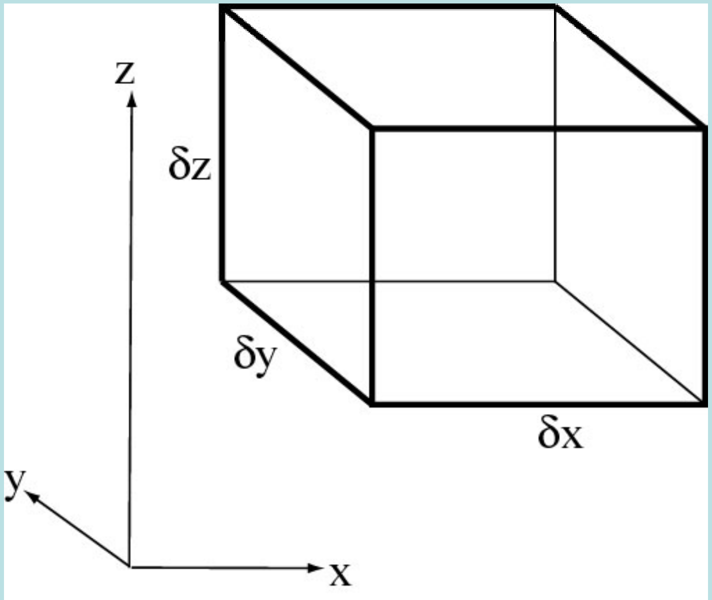
Derivation of the basic equations of *fluid* flows. No particle in the fluid at this stage (next week).

- Conservation of mass of the fluid.
- Conservation of mass of a solute (applies to non-sinking particles at low concentration).
- Conservation of momentum.
- Application of these basic equations to a turbulent fluid.

A few concepts before we get to the meat…

Tensor (Stress), Vectors (e.g. position, velocity) and scalars (e.g. T, S, CO_2).

We need to define a coordinate system, and an (infinitesimal) element of volume.



We assume a continuous fluid, and that all the fields of interest are differentiable.

The Lagrangian framework is the framework in which the laws of classical mechanics are often stated. The coordinates of a point $\vec{x}(t)$ describe the trajectory from $\vec{x}_0 = \vec{x}(t=0)$. Density, ρ, can evolve along the trajectory. By the chain rule, along a parcel trajectory:

$$\frac{d\rho}{dt} = \frac{d\vec{x}}{dt}\left(\frac{\partial\rho}{\partial\vec{x}}\right)_{t=const} + \left(\frac{\partial\rho}{\partial t}\right)_{\vec{x}=const} = \vec{u}\cdot\nabla\rho + \frac{\partial\rho}{\partial t}$$

$$\rightarrow \frac{d\rho\left(\vec{x}(t),t\right)}{dt} = \left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\rho\left(\vec{x},t\right)$$

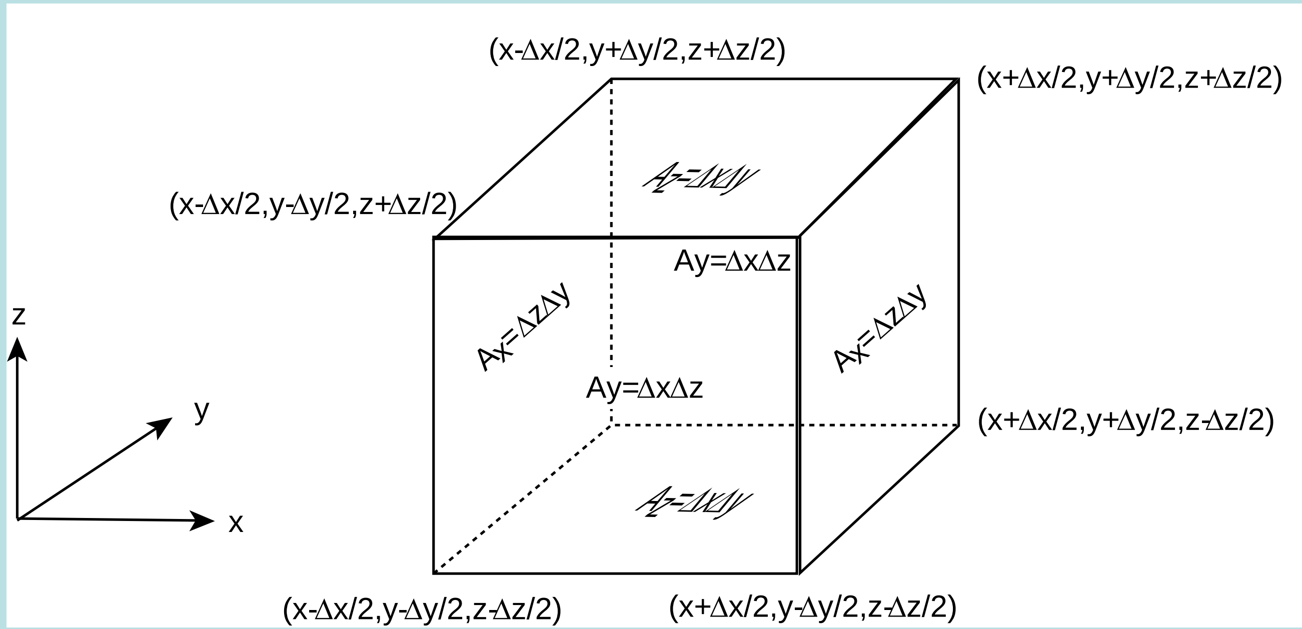
$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{u}\cdot\nabla$$ ←Conversion from Lagrangian to Eulerian

Example:
Let's assume that we are in a river that feeds on glacial melt. The water warms at a constant rate that is a function of distance from the source. If we drift down river (A la 'Huckleberry Fin'), the temperature increases with time ($DT/Dt>0$). At one point along the river, however, we may see no change in temperature with time ($\partial T/\partial t=0$), as the water arriving there is always at the same temperature. The heat flux is *advective*, ($u\partial T/\partial x>0$).

In short, the **convective** derivative is:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla$$

Mass conservation (Eulerian, differential approach):
Accounting for the change in mass inside a fixed, constant-size volume:



$$Mass = \rho V$$

$$\frac{\partial(\rho V)}{\partial t} = -A_x\left(\rho u|_{+\Delta x/2} - \rho u|_{-\Delta x/2}\right) - A_y\left(\rho v|_{+\Delta y/2} - \rho v|_{-\Delta y/2}\right) - A_z\left(\rho w|_{+\Delta z/2} - \rho w|_{-\Delta z/2}\right)$$

$$\frac{\partial \rho}{\partial t} = -\frac{1}{\Delta x}\left(\rho u|_{+\Delta x/2} - \rho u|_{-\Delta x/2}\right) - \frac{1}{\Delta y}\left(\rho v|_{+\Delta y/2} - \rho v|_{-\Delta y/2}\right) - \frac{1}{\Delta z}\left(\rho w|_{+\Delta z/2} - \rho w|_{-\Delta z/2}\right)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

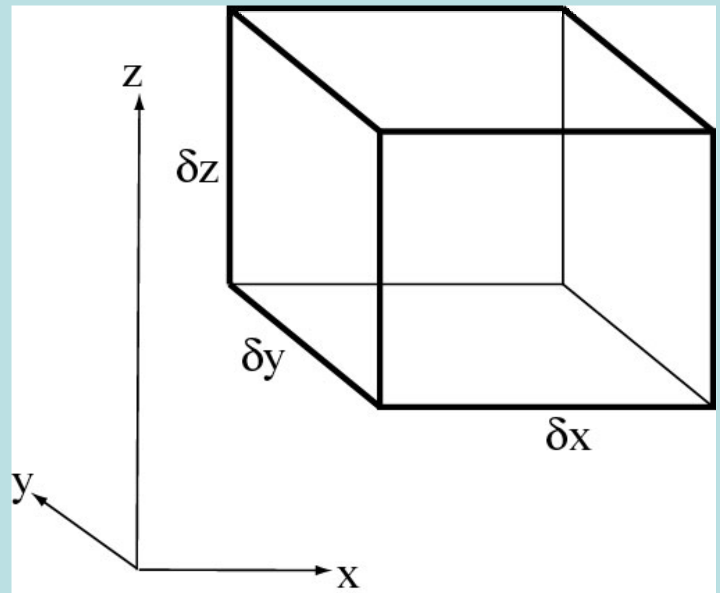
Mass conservation (Eulerian, integral approach):

Accounting for the change in mass inside a fixed, constant-volume volume (V_0):

$$\frac{d}{dt}\int_{V_0} \rho dV = -\int_{\partial V_0} \rho \vec{u} \cdot \vec{n} dS$$

$$\rightarrow \int_{V_0} \frac{\partial}{\partial t} \rho dV = -\int_{V_0} \nabla \cdot \rho \vec{u} dV$$

$$\rightarrow \frac{\partial}{\partial t}\rho + \nabla \cdot \rho \vec{u} = 0$$



Where we used the divergence theorem:

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \oiint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

It states that the volume total of all sinks and sources, the volume integral of the divergence, is equal to the net flow across the volume's boundary (WIKI).

Reiteration (no sinks/sources):

Mass conservation (Lagrangian, integral):

$$\frac{D}{Dt}\int_V \rho dV = 0$$

Mass conservation (Eulerian, integral):

$$\frac{d}{dt}\int_{V_0} \rho dV = -\int_{\partial V_0} \rho \vec{u} \cdot \vec{n} dS$$

Mass conservation:

Note that:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{u} \right) = 0$$

Can be written as:

$$\frac{1 D\rho}{\rho D t} + \nabla \cdot \vec{u} = 0$$

The 2nd term is the fluid divergence (rate of outflow of volume per unit volume). This can be nonzero only for compressible fluids. It is the rate of loss of density due to compression/expansion.

For both water and air we can assume that $\nabla \cdot \vec{u} = 0$ in terms of their dynamics (we need compressibility to pass sound…).

Mass balance for conserved scalar:

Adding molecular diffusion:

$$\int_V \frac{\partial C}{\partial t} dV = -\int_S \left(C\vec{u} - K\nabla C\right)\cdot \vec{n}\, dS$$

Where V is the volume of the control volume and S its surface, and using Fick's law. By the help of the divergence theorem:

$$\int_V \left[\frac{\partial C}{\partial t} + \nabla\cdot\left(C\vec{u} - K\nabla C\right)\right] dV = 0$$

Since the volume is arbitrary, this can be true if and only if:

$$\frac{\partial C}{\partial t} + \nabla\cdot\left(C\vec{u}\right) = \nabla\cdot\left(K\nabla C\right)$$

Momentum balance (Navier-Stokes):

Newton's 2nd law of motion states that the time rate of change of momentum of a particle is equal to the force acting on it. This law is Lagrangian, the "time rate of change" is with respect to a reference system following the particle.

$$\frac{d}{dt}\int_{V(t)} \rho\vec{u}\,dV = \int_{V(t)} \rho\vec{g}\,dV + \int_{\partial V(t)} \vec{T}\,dS$$

Where g is the body force per unit mass (e.g. gravity) and T is the surface force per unit surface area bounding V.

If the volume is small enough, the integrands can be taken out of the integral:

$$\frac{d}{dt}\int_{V(t)} \rho\vec{u}\,dV = \frac{d}{dt}\left(\rho\vec{u}\int_{V(t)} dV\right) = \frac{d}{dt}\left(\rho\vec{u}\,\delta V\right)$$

$$\frac{d}{dt}\left(\rho\vec{u}\,\delta V\right) = \rho\delta V\,\frac{d\vec{u}}{dt} + \vec{u}\,\frac{d\left(\rho\delta V\right)}{dt} = \rho\delta V\,\frac{d\vec{u}}{dt}$$

Momentum balance (Navier-Stokes):

The body force is similarly treated:

$$\int_{V(t)} \rho \vec{g}\, dV = \rho \vec{g}\, \delta V$$

Defining a stress tensor (expanded on the next slide):

$$\vec{T} = \mathrm{T} \cdot \vec{n}$$

And applying the divergence theorem:

$$\int_{\partial V(t)} \vec{T}\, dS = \int_{V(t)} \nabla \cdot \mathrm{T}\, dV = \nabla \cdot \mathrm{T}\, \delta V$$

$$\rightarrow \rho \frac{D\vec{u}}{Dt} = \rho \vec{g} + \nabla \cdot \mathrm{T}$$

Surface forcing:

For an inviscid fluid, the surface force exerted by the surrounding fluid is normal to the surface, i.e. $\vec{T} = -p \cdot \vec{n}$, and p is called the pressure force.

In general, viscous stress force, S, is also present. For viscous fluids: $\vec{T} = -p \cdot \vec{n} + \vec{S}$. By definition $\vec{T} = \mathrm{T} \cdot \vec{n}$, and we now have $\mathrm{T} = -p\mathrm{I} + \sum$, where $\vec{S} = \sum \cdot \vec{n}$ and I is the identity tensor.

For Newtonian fluids,

$$\nabla \cdot \Sigma = \mu \nabla^2 \vec{u}$$

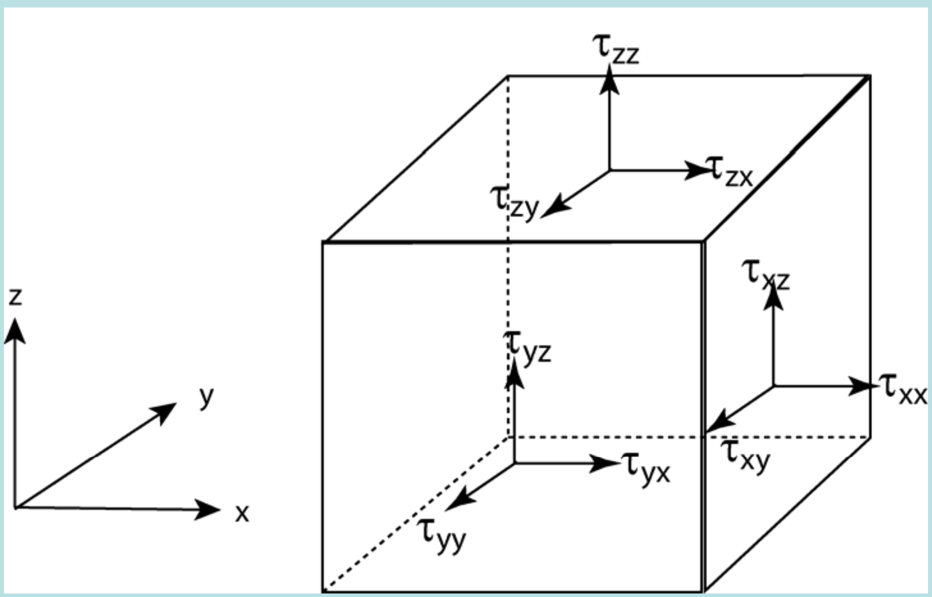
And the resultant Navier-Stokes equations for incompressible fluids are:

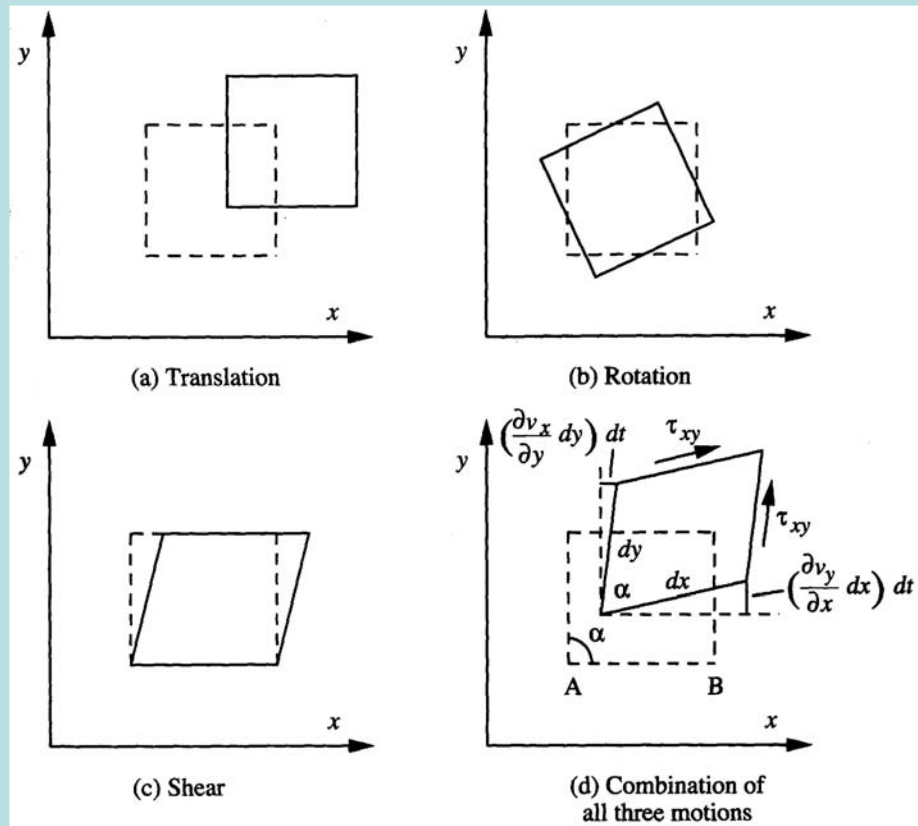
$$\rho \frac{D\vec{u}}{Dt} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{u}$$

Rotational symmetry:

$$\mathrm{T} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \tau_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{pmatrix}$$

$$-p \equiv \frac{1}{3}\left(\tau_{xx} + \tau_{yy} + \tau_{zz}\right)$$





Total stress tensor, Newtonian fluid:

$$\mathrm{T} = \begin{pmatrix} -p + 2\mu \frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & -p + 2\mu \frac{\partial v}{\partial y} & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & -p + 2\mu \frac{\partial w}{\partial z} \end{pmatrix} = -p\delta_{ij} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

Σ_{ij}

$i, j = \{1, 2, 3\} = \{x, y, z\}$

Stokes, 1845:

1. Σ_{ij} linear function of velocity gradients.
2. Σ_{ij} should vanish if there is no deformation of fluid elements.
3. Relationship between stress and shear should be isotropic.

$$\hat{i} \cdot (\nabla \cdot \mathrm{T}) = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} = -\frac{\partial p}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 v}{\partial y \partial x}$$

$$+ \mu \frac{\partial^2 u}{\partial z^2} + \mu \frac{\partial^2 w}{\partial z \partial x} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u + \frac{\partial \cdots}{\partial x} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u$$

Navier-Stokes equations:

$$\frac{D\vec{u}}{Dt} = -\frac{\nabla p}{\rho} + \frac{\mu}{\rho}\nabla^2\vec{u} + \vec{g},\ \ \nabla \cdot \vec{\mathrm{u}} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Coriolis is added when moving the framework to an accelerating framework. Have to add boundary & initial conditions.

Navier-Stokes equations (Boussinesq approximation):
Separate balance of fluid at rest from moving fluid.

$$p = p_0(z) + p'(x, y, z, t)$$

$$\rho = \rho_0 + \rho'(x, y, z, t)$$

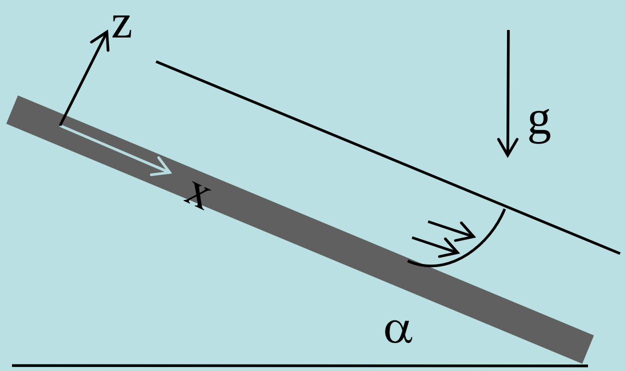
First order balance (hydrostatic):

$$0 = -\frac{1}{\rho_0}\frac{\partial p_0}{\partial z} + g$$

2nd order balance:

$$\rho_0 \frac{D\vec{u}}{Dt} = -\nabla p' + \mu \nabla^2 \vec{u} + g\rho_0, \ \nabla \cdot \vec{\mathrm{u}} = 0$$

Example: steady flow under gravity down an inclined plane.



From: Acheson, 1990

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

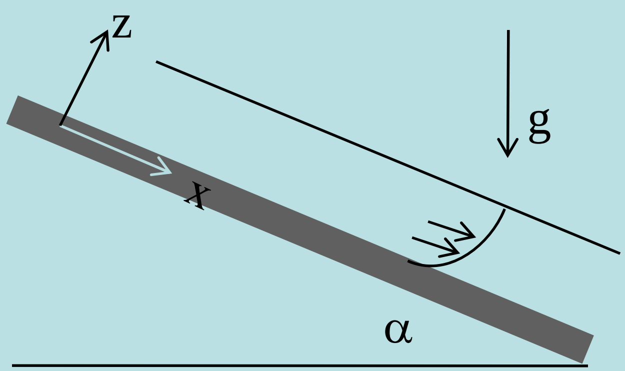
$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} + g \sin\alpha$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g \cos\alpha$$

BCs: $z = 0: \; w = u = 0$

$$z = h: \; \mu \frac{\partial u}{\partial z} = 0, \; p = p_a$$

Example: steady flow under gravity down an inclined plane.



Solution:

$$p - p_a = \rho_0 g\left(h - z\right)\cos\alpha$$

$$u = \frac{g}{2\nu} z\left(2h - z\right)\sin\alpha$$

Q: what ν should we use?

Reynolds decomposition of the N-S equations

Assume a turbulent flow. At any given point in space we separate the mean flow (mean can be in time, space, or ensemble) and deviation from the mean such that:

$$p = \overline{p} + p', u = \overline{u} + u', u = \overline{v} + v', u = \overline{w} + w'$$

$$\langle p \rangle = \overline{p}, \langle p' \rangle = 0, etc'$$

Substituting into the continuity equation (linear):

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0; \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

Substituting into x-momentum Navier-Stokes equation:

$$\frac{\partial \overline{u}}{\partial t}+\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}+\overline{w}\frac{\partial \overline{u}}{\partial z}=-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 \overline{u}}{\partial x^2}+\frac{\partial^2 \overline{u}}{\partial y^2}+\frac{\partial^2 \overline{u}}{\partial z^2}\right)$$

$$-\left(\frac{\overline{\partial(u'u')}}{\partial x}+\frac{\overline{\partial(v'u')}}{\partial y}+\frac{\overline{\partial(w'u')}}{\partial z}\right)$$

The evolution of the mean is forced by correlations of fluctuating properties. The correlation terms time density are the "Reynolds stresses".

Substituting into a scalar conservation equation:

$$\frac{\partial \overline{C}}{\partial t}+\overline{u}\frac{\partial \overline{C}}{\partial x}+\overline{v}\frac{\partial \overline{C}}{\partial y}+\overline{w}\frac{\partial \overline{C}}{\partial z}=\frac{K}{\rho}\left(\frac{\partial^2 \overline{C}}{\partial x^2}+\frac{\partial^2 \overline{C}}{\partial y^2}+\frac{\partial^2 \overline{C}}{\partial z^2}\right)$$

$$-\left(\frac{\overline{\partial(u'C')}}{\partial x}+\frac{\overline{\partial(v'C')}}{\partial y}+\frac{\overline{\partial(w'C')}}{\partial z}\right)$$

Note that the Reynolds stress tensor is symmetric (as is the viscous stress tensor):

$$\tau = -\rho_0 \begin{pmatrix} \overline{u'u'} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'v'} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'w'} \end{pmatrix}$$

<u>The closure problem:</u> to develop equations for the evolution of the Reynolds stresses theselves, higher order correlation are needed (e.g. $w'u'u'$) and so on. For this reason theories have been devised to describe τ_{ij} in terms of the mean flow.

For more, see: http://www.cfd-online.com/Wiki/Introduction_to_turbulence/Reynolds_averaged_equations

One solution to the closure problem is to link the Reynolds' stress to mean-flow Quantities. For example:

$$-\rho_0 \overline{w'u'} = K_{eddy} \frac{\partial U}{\partial z}$$

$$-\rho_0 \overline{w'T'} = K_{eddy} \frac{\partial T}{\partial z}$$

This type of formulation is appealing because it:

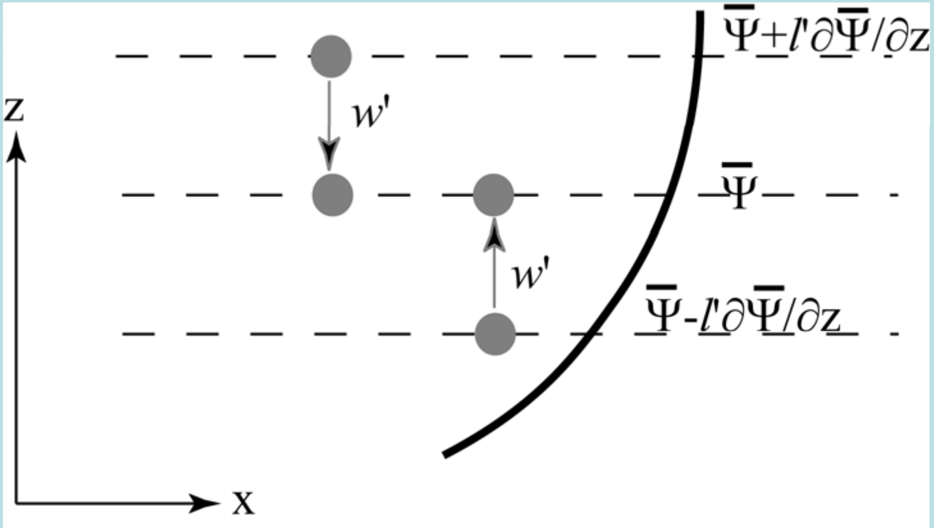
a. Provide for down-gradient flux.
b. Is reminiscent of molecular diffusion and viscosity.
c. Provide closure to the equations of the the mean fields.

This type of formulation is problematic because:

a. K_{eddy} is a property of the flow and not the fluid.
b. K_{eddy} is likely to vary with orientation, unlike molecular processes.

How is K_{eddy} related to the turbulence?

Assume a gradient in a mean property (momentum, heat, solute, etc'. Remember: no mean gradient→no flux). Assume a fluctuating velocity field:



l' is the distance a parcel travels before it loses its identity. The rate of upward vertical turbulent transfer of <Ψ> is down the mean gradient:

$$\overline{w'\left(\overline{\psi}+l'\frac{\partial\overline{\psi}}{\partial z}\right)}=\overline{w'l'}\frac{\partial\overline{\psi}}{\partial z}=-K_{eddy}\frac{\partial\overline{\psi}}{\partial z}$$

How is K_{eddy} related to the turbulence?

$$-\rho_0 \overline{w'u'} = \rho_0 K_{eddy} \frac{\partial U}{\partial z}$$

$$-\rho_0 \overline{w'T'} = \rho_0 \gamma_T \frac{\partial T}{\partial z}$$

Tennekes and Lumley (1972) approach this problem from dimensional analysis based on assuming a single length scale-l and a single velocity scale $\omega = \langle w'^2 \rangle^{1/2}$.

$$-\rho_0 \overline{w'u'} = c\rho_0 \omega^2; \; c \sim O(1)$$

The eddies involved in momentum transfer have vorticities, ω / l; this vorticity is maintained by the mean shear (l is the length scales of the eddies, e.g. the decorrelation scale).

$$\omega / l = c \frac{\partial U}{\partial z}; \; c \sim O(1)$$

It follows that:

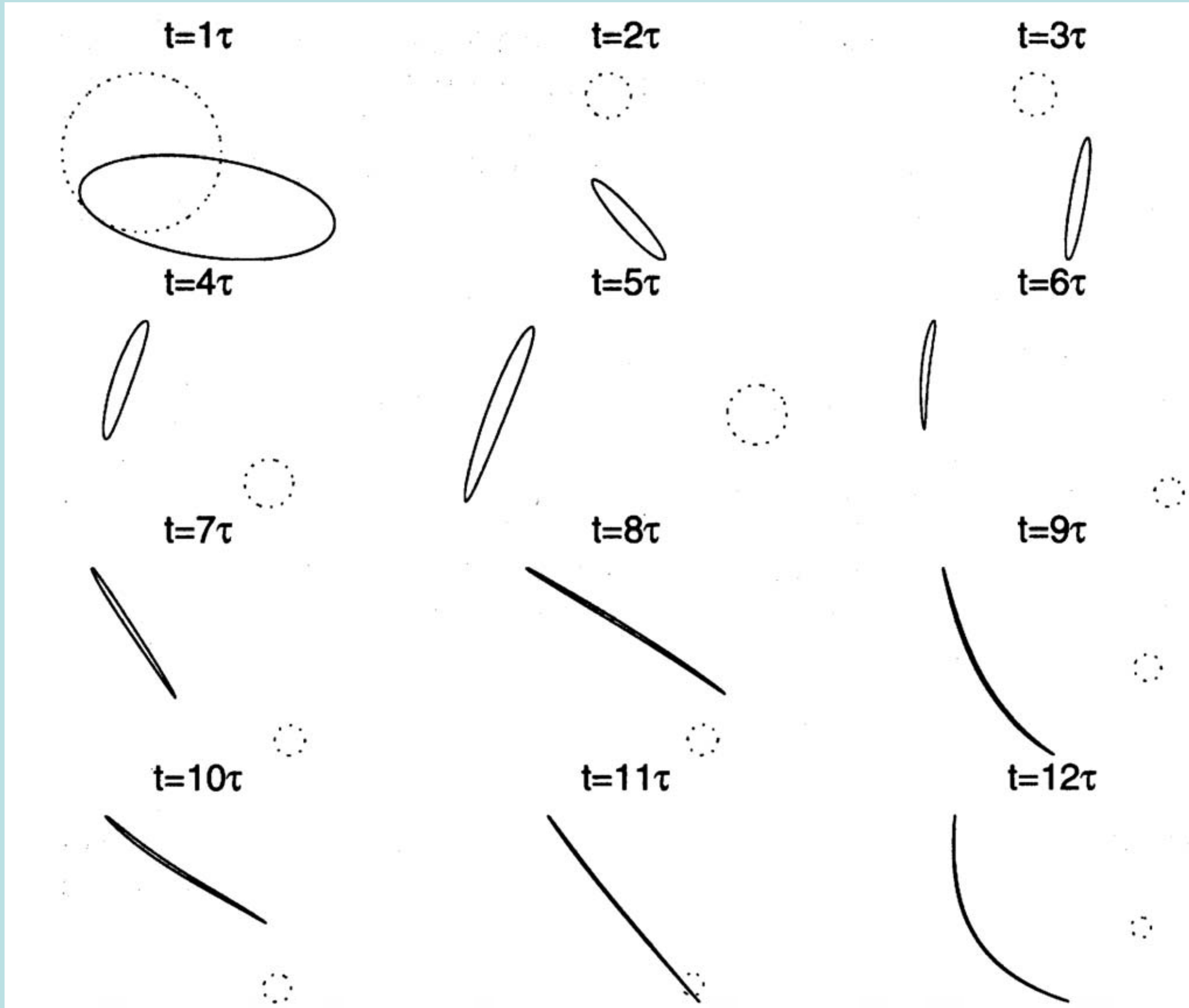
$$K_{eddy} \sim l\omega \sim l^2 \left| \frac{\partial U}{\partial z} \right|$$

In analogy with momentum flux, for heat we have:

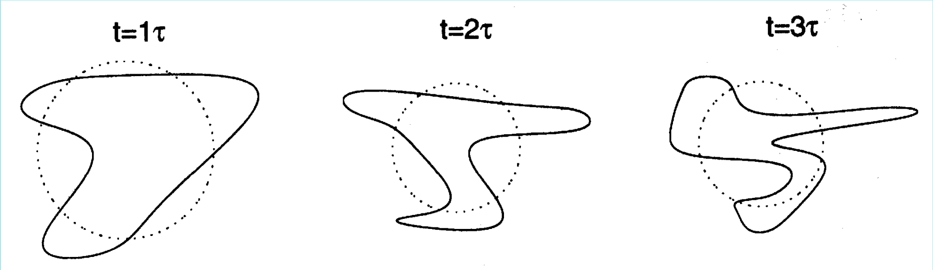
$$-\rho_0 c_p \overline{w'T'} = \rho_0 c_p \gamma_T \frac{\partial T}{\partial z}$$

It is most commonly assumed, and verified that $\gamma_T = K_{eddy}$.

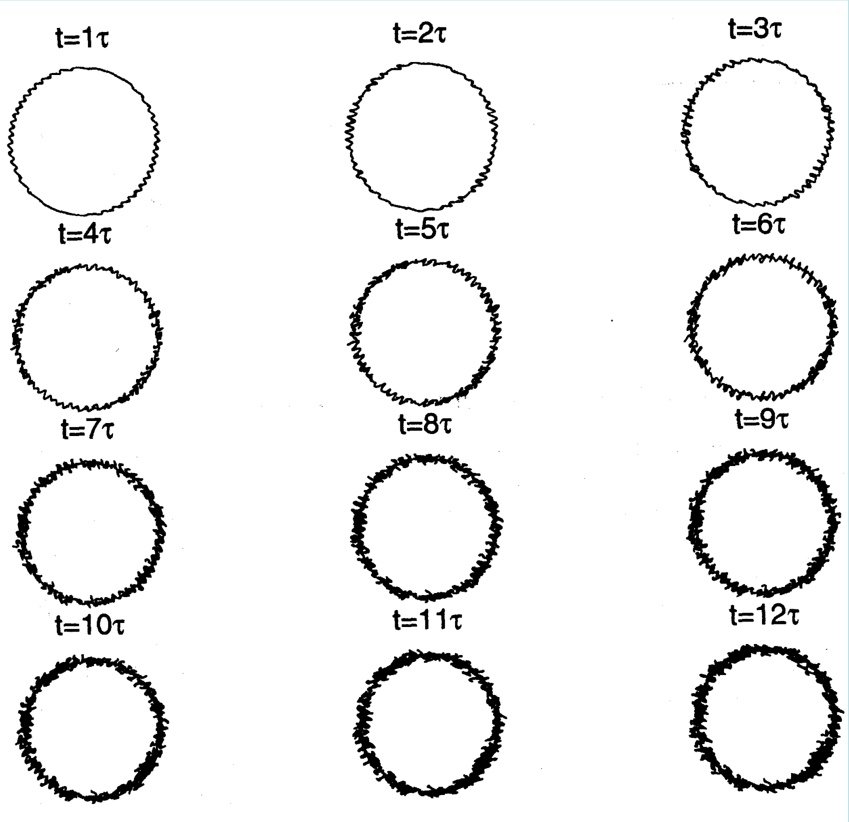
Eddy-diffusion: perspective from a dye patch (figures from lecture notes of Bill Young, UCSD)



Dye patch << dominant scale of eddies. Dashed circle denotes initial position of patch.



Dye patch ≈ dominant scale of eddies



Dye patch >> dominant scale of eddies

Cheat sheet:

1. Gradient of a scalar (a vector):

$$\nabla \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

2. Divergence of a vector (a scalar):

$$\nabla \cdot \vec{\varphi} = \frac{\partial \varphi_x}{\partial x} + \frac{\partial \varphi_y}{\partial y} + \frac{\partial \varphi_z}{\partial z}$$

3. Divergence of a Tensor, $T_{12}=T_{(face)(direction)}$:

$$\nabla \cdot \mathrm{T} = \hat{i}\left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z}\right) + \hat{j}\left(\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z}\right) +$$

$$\hat{k}\left(\frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z}\right)$$

Cheat sheet (cotinued):

4. Laplacian of a vector (a vector):

$$\nabla^2\vec{\varphi} = \nabla\cdot\left(\nabla\cdot\vec{\varphi}\right) = \hat{i}\left(\frac{\partial^2\varphi_x}{\partial x^2} + \frac{\partial^2\varphi_x}{\partial y^2} + \frac{\partial^2\varphi_x}{\partial z^2}\right) +$$

$$\hat{j}\left(\frac{\partial^2\varphi_y}{\partial x^2} + \frac{\partial^2\varphi_y}{\partial y^2} + \frac{\partial^2\varphi_y}{\partial z^2}\right) + \hat{k}\left(\frac{\partial^2\varphi_z}{\partial x^2} + \frac{\partial^2\varphi_z}{\partial y^2} + \frac{\partial^2\varphi_z}{\partial z^2}\right)$$