Derivatives and the Tangent Line Problem

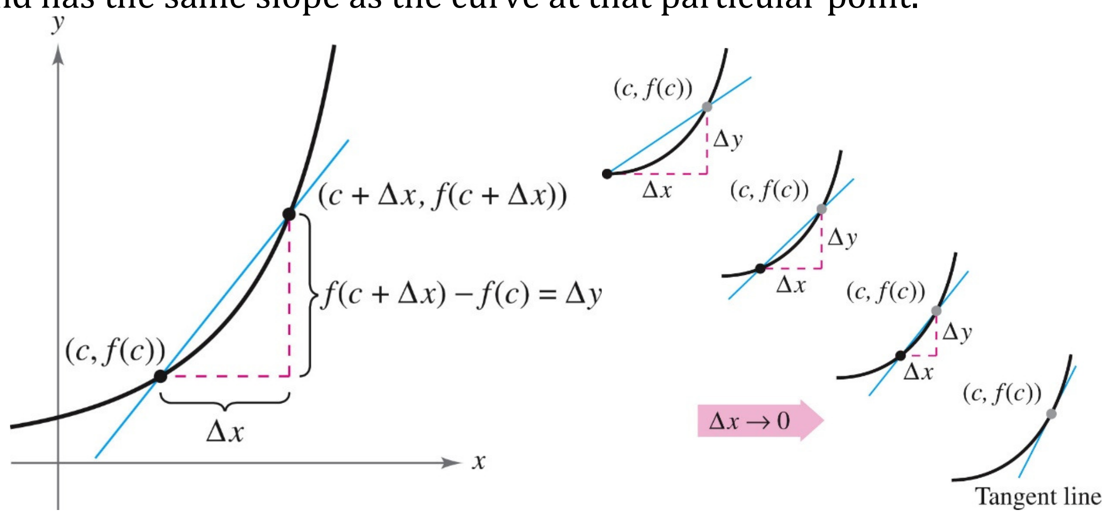
Objective: Find the slope of the tangent line to a curve at a point. Use the limit definition to find the derivative of a function. Understand the relationship between differentiability and continuity.

Calculus grew out of 4 major problems that European mathematicians were working on during the seventeenth century.

1. The Tangent Line Problem
2. The Velocity and Acceleration Problem
3. The Minimum and Maximum Problem
4. The Area Problem

"And I dare say that this is not only the most useful and general problem in geometry that I know, but even that I desire to know" Descartes

Secant line of a curve: is a line that (locally) intersects two points on the curve, this line will give you the average slope of the curve between those two points.
Tangent Line to a curve: is the line that goes through a point p on the curve and has the same slope as the curve at that particular point.



Definition of Tangent Line with Slope m:
If *f* is defined on an open interval containing *x*, and if the limit

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(c+\Delta x)-f(c)}{(c+\Delta x)-c}=\lim_{\Delta x\to 0}\frac{f(c+\Delta x)-f(c)}{\Delta x}=m_{\tan}$$

exists, then the line passing through (*c,f (c)*) with slope=$m_{\tan}$ is the tangent line to the graph of f at point (*c,f (c)*).

The slope of the tangent line to the graph of *f* at the point (*c,f(c)*) is also called the **slope of the graph** of *f* at *x = c*.

Ex: Find the slope of the graph of *f(x) = 5x - 2* at point (*2,8*).

Ex: Find the slopes of the tangent lines to the graph of $f(x) = x^2 + 2$ at points (0,2) and (-1,3).

The limit used to define the slope of a tangent line is also used to define one of the two major operations of calculus: **differentiation**.

Definition of the Derivative of a Function:

The **derivative** of *f* at *x*, denoted as $f'(x)$, is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all *x* for which this limit exists, f' function of *x*.

This "new" function gives the slope of the tangent line to the graph of *f* at the point (*x, f(x)*), provided the graph has a tangent line at this point.

The process of finding the derivative is called **differentiation**.

Notation:

$f'(x)$ is read "*f prime of x*"

Some common notations you will see used to denote the derivative of y = f(x) are

$f'(x)$	$\frac{dy}{dx}$ Leibnitz	y'	$\frac{d}{dx}[f(x)]$	$D_x[y]$ Euler	$\dot{y}$

Ex: Find the derivative of $f(x) = x^3 + 2x$

Ex: Find the derivative of $f(x)=\sqrt{x}$. Find the slope of the functions= at (1,1), (4,2), and (0,0).

Ex: Find the derivative with respect to *t* for the function $y=\frac{2}{t}$.

<u>Vertical Tangent Lines:</u>
If *f* is continuous at *c* and

$$\lim_{\Delta x\to 0}\frac{f(c+\Delta x)-f(c)}{\Delta x}=\pm\infty$$

the tangent line to the graph of *f(x)* is vertical.

<u>**Differentiability and Continuity:**</u>
An alternate form of the derivative is

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}.$$

This form is useful in the relationship between differentiability and continuity, provided this limit exists.

- The existence of this limit requires the one-sided limits exists and are equal.
- These one-sided limits are called the derivatives from the left and from the right, respectively.

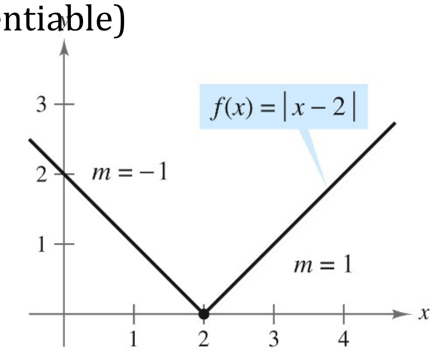
A function is **differentiable at *x*** if its derivative exists at *x* and is **differentiable on an open interval (a,b)** if it is differentiable at every point in that interval.

A function *f* is **differentiable on the closed interval [a,b]** if it is differentiable on (a,b) and if the derivative from the right at *a* and the derivative from the left at *b* both exist.

If a function is not continuous at x = c then it is also not differentiable at x = c.

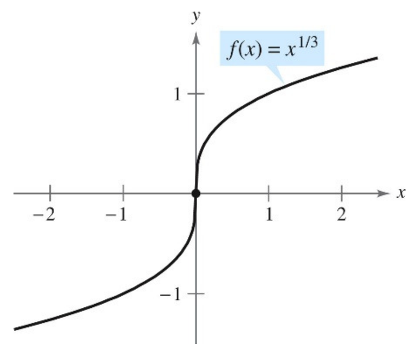
Graph with a sharp turn (continuous but not differentiable)

Ex: $f(x) = |x-2|$ is continuous at $x = 2$ but the one sided derivative limits are not equal.



Graph with a vertical Tangent Line (continuous but not differentiable)

Ex: $f(x) = x^{1/3}$ is continuous at $x = 0$ the derivative limit as *x* approaches zero in infinite.



Differentiability Implies Continuity

If f is differentiable at $x = c$, then *f* is continuous at $x = c$.

- If a function is differentiable at x = c, then it is continuous at $x = c$. So differentiability implies continuity.
- It is possible for a function to be continuous at $x = c$ and not differentiable at $x = c$. So, continuity does not imply differentiability.

Basic Differentiation Rules and Rates of Change

Objective: Find the derivative of a function using the Constant Rule. Find the derivative of a function using the Power Rule. Find the derivative of a function using the Constant Multiple Rule. Find the derivative of a function using the Sum and Difference Rules. Find the derivative of the sine function and of the cosine function. Use derivatives to find rates of change.

Using the limit definition we can see that finding derivative is very repetitive. For certain types of functions we can use the limit definition to find a "rule" that will facilitate in find the derivative faster.

The Constant Rule:

The derivative of a constant function is 0. For any constant c,

$$\frac{d}{dx}[c]=0.$$

Ex: Find dy/dx for y = 7

Ex: Find the derivative of y for $y=k\pi^2$, if k is a constant

The Power Rule:

If n is a rational number, then the function $f(x)=x^n$ is differentiable and

$$\frac{d}{dx}\left[x^n\right]=nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

Ex: Find the derivative of the following

a. $f(x)=x^4$ **b.** $g(x)=\sqrt[3]{x^2}$ **c.** $y=\frac{1}{x^3}$ **d.** $y = x$

Ex: Find the slope of the graph of $f(x)=x^3$ when x = -2.

Ex: Find the equation of the tangent line to the graph of $f(x)=x^2$ when $x = -2$

Constant Multiple Rule:

If f is a differentiable function and c is a real number, then $c\cdot f$ is also differentiable and

$$\frac{d}{dx}[c\cdot f(x)]=c\cdot\frac{d}{dx}[f(x)].$$

Ex: Find the derivative of the following functions

a. $y=\frac{3}{x^2}$ b. $f(t)=\frac{-3t}{5}$ c. $y=\frac{1}{-2\sqrt[4]{x^3}}$

<u>The Sum and Difference Rule:</u>

If *f(x)* and *g(x)* are differentiable, then so is the sum (or difference) and

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)].$$

So the derivative of *f* + *g* is the sum (or difference) of the derivatives of f and g.

Ex: Find the derivative of the following

a. $f(x) = x^3 + 2x^2 - 5x + 1$ **b.** $g(x) = -\frac{x^5}{3} + 3x^2 - 1$

<u>The Product and Quotient Rules and Higher-Order Derivatives</u>

Objective: Find the derivative of a function using the Product Rule. Find the derivative of a function using the Quotient Rule. Find the derivative of a trigonometric function. Find a higher-order derivative of a function

The Product Rule:

If *f(x)* and *g(x)* are differentiable at *x*, then so is their product *f(x)·g(x)*

$$\frac{d}{dx}\left[f(x)g(x)\right] = \frac{d}{dx}\left[f(x)\right] \cdot g(x) + \frac{d}{dx}\left[g(x)\right] \cdot f(x)$$

We could also write this in prime notation

$$\left(f \cdot g\right)' = f' \cdot g + g' \cdot f$$

Ex: Find the derivative of the following

a. $y = \left(2x + 5x^2\right)(3x - 7)$ **b.** $f(x) = -6x^3 \sin x - x + 1$ **c.** $y = (2x-1)^2$ **d.** $y = 3(5x^2 + x + 3)$

The Quotient Rule:
If *f(x)* and *g(x)* are differentiable functions at *x*, then so is the quotient *f(x)/g(x)*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{\frac{d}{dx}[f(x)]\cdot g(x)-\frac{d}{dx}[g(x)]\cdot f(x)}{g^2(x)}$$

We could also write this in prime notation

$$\left(\frac{f}{g}\right)'=\frac{g\cdot f'-f\cdot g'}{g^2}$$

Ex: Find the derivative of the following:

a. $f(x)=\dfrac{3x+2}{x^2-5}$ **b.** $g(t)=\dfrac{12x^3-2x^2+5x}{7x}$.

Ex: Find the equation of the tangent line to the graph of $f(x)=\dfrac{1+\frac{1}{x}}{x+1}$ at (1,1).

Make sure all your derivatives are SIMPLIFIED!!!!!

Higher-Order Derivatives:
If derivative of a function is another function then we can then take the derivative of that function and do that again and again and again and again and again.....

First Derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$D_x y$
Second Derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$D_x^2 y$
Third Derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\vdots$
Fourth Derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\vdots$
...	...	...	...	

nth Derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^n y}{dx^n}$	...

Ex: Find $f''(x)$ for $f(x) = \frac{x-1}{x+1}$

Rates of Change:

Since the derivative represents slope of a curve, it can be used to determine the rate of change of one variable with respect to another. Applications involving rates of change occur in many aspects of math and science. A few examples are population growth rates, production rates, water flow rates, velocity, and acceleration.

A common use of rate of change is to describe the motion of an object in a straight line, either horizontal or vertical. Upward usually positive and downward negative, right is positive and left is negative.

You should be familiar with $\text{rate} = \frac{\text{distance}}{\text{time}}$ (another term for rate is velocity)

We use a similar formula for average velocity

$$Average\ \ Velocity = \frac{\text{change in distance}}{\text{change in time}}$$

Ex: If a billiard ball is dropped from a height of 100 feet, its height *s* at time *t* is given by the position function:

$$s(t) = -16t^2 + 100$$

where *s* is measured in feet and *t* is measured in seconds. Find the average velocity over the interval where *t = 1* sec. to *t = 2* sec.

Instantaneous Velocity (Velocity)

Suppose you wanted to find instantaneous velocity (or simply velocity) of an object when *t = 1* sec. This would be the same as the approximation of the tangent line problem where we went from the average slope to the instantaneous slope. By taking limit of the average velocity as Δt approaches zero we calculate the **instantaneous velocity** as the derivative of the position function, $s(t)$.

$$s'(t) = v(t) = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t}$$

The velocity can be negative, positive, or zero, this will give information as to direction.

While speed is a component of the velocity they are not directly equal. **Speed** of an object is the absolute value of the velocity.

The position of a free falling object (neglecting air resistance) under the influence of gravity can be represented by the equation:

$$s(t) = \frac{1}{2}gt^2 + v_0 t + s_0$$

(this formula comes from physics)

Where, s_0 is the initial height of the object

v_0 is the initial velocity of the object

g is the acceleration due to gravity

On Earth the value of g is approximately -32 feet per second per second or 9.8 meters per second per second.

Ex: Ever since you started your calculus class you've suffered from an irritating rash. Nothing helps; acupuncture, meds, etc., you've tried them all, but the rash gets worse and worse. The pain is unbearable, and you decide to take action. You drive to the middle of the Golden Gate Bridge and climb over the safety rail, 400 feet above the water and decide to **get rid of that book**. So you fling your calculus text (which you carry everywhere) over the edge. Your books height in feet over the water after t seconds is given by the function

$s(t) = 400 - 16t^2$. *(Mr. B. does not condone the actions of this fictional student)*

a. How long will it take until your book hits the water?

b. What is your books velocity when it hits the shark infested waters?

Acceleration:

Acceleration is the instantaneous rate of change in the velocity with respect to time. Just as we can obtain a velocity function by differentiating a position function, you can obtain an acceleration function by differentiating a velocity function.

$s(t)$ ⇢ position function

$s'(t) = v(t)$ ⇢ velocity function

$s''(t) = v'(t) = a(t)$ ⇢ acceleration function

Ex: The position of a particle is a given by $s(t)=t^3-6t^2+9t$ where *t* is measured in seconds and *s* is measured in meters.

a. Find the velocity at time *t*.
b. What is the velocity after 2 seconds? 4 seconds?
c. When is the particle at rest?
d. Find the acceleration at time *t*.
e. What is the acceleration at 2 seconds? 4 seconds?

The Chain Rule

Objective: Find the derivative of a composite function using the Chain Rule. Find the derivative of a function using the General Power Rule. Simplify the derivative of a function using algebra. Find the derivative of a trigonometric function using the Chain Rule

The most powerful and useful differentiation rule is **THE CHAIN RULE**!!!

The Chain Rule:

If $y=f(u)$ is a differentiable function of *u* and $u=g(x)$ is a differentiable function of *x*, then the composition $y=f(g(x))$ is a differentiable function of *x* and

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

or equivalently

$$\frac{d}{dx}\left[f(g(x))\right]=f'(g(x))\cdot g'(x)$$

Ex: Find the derivative of the following

a. $y=\left(x^2+5\right)^3$

b. $y=\sqrt[3]{5x-4x^2}$

The General Power Rule:
If $y=(u(x))^n$, where *u* is a differentiable function of *x* and *n* is a rational number, then

$$\frac{dy}{dx}=n(u(x))^{n-1}\cdot u'(x).$$

Ex: Find the derivative of the following:

a. $f(x)=\dfrac{1}{(x^2+1)^3}$

b. $y=x^3\sqrt{2-x^2}$

c. $y=\left(\dfrac{x+2}{x+3}\right)^2$

Ex: Find all points on the graph of $f(x)=\sqrt[3]{(x^2-1)^2}$ for which $f'(x)=0$ or does not exist.

Ex: Find the second derivative of $f(x)=\dfrac{1}{(x^2+5)^2}$.

Implicit Differentiation

Objective: Distinguish between functions written in implicit form and explicit form. Use implicit differentiation to find the derivative of a function

Up to this point all of the functions we have differentiated have been in explicit form, example: $y=3x^2+2$. The variable *y* is explicitly written as a function of *x*.

Some functions are only implied by an equation, example: $xy=1$ is in an implicit form.

While some functions we can easily rewrite in explicit form in order to take the derivative, *dy/dx*, some we can't, example: $x^2-2y^3+4y=2$
This is when we must use implicit differentiation. In order to accomplish this we must realize that we are still differentiating with respect to *x* and treat *y* as a function of *x*. The chain rule is a vital part of this type of differentiation.

Ex: Differentiate with respect to x:

a. $\dfrac{d}{dx}\left[x^4\right]$ **b.** $\dfrac{d}{dx}\left[y^4\right]$ **c.** $\dfrac{d}{dx}\left[x^2+y^2\right]$ **d.** $\dfrac{d}{dx}\left[3xy^3\right]$

Guidelines for Implicit Differentiation:

1. Differentiate both sides of the equation with respect to *x.*
2. Collect all terms involving *dy/dx* on the left of the equation and move all other terms to the right side of the equation.
3. Factor *dy/dx* out of the left side of the equation.
4. Solve for *dy/dx*.

Ex: Find *dy/dx* given that $x^3+y^2-5x^2y=-4$.

Ex: Determine the slope of the tangent line to the graph of $x^2 + y^2 = 1$ at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Ex: Find the equation of the tangent line of $3\left(x^2 + y^2\right)^2 = 100xy$ at (3,1).

Ex: Given $x^2 + y^2 = 25$ find $\frac{d^2y}{dx^2}$.