Design and Estimation of FX Models for Derivative Pricing

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Overview

- The FX market is huge, and it is inherently **_linked_** with the fixed income market:

  - Covered interest-rate parity: $f_{t,\tau} - s_t = r_{t,\tau}^d - r_{t,\tau}^f$.

- FX exhibits large **_independent_** movements, with **_time-varying risk premiums_**.

  - Uncovered interest-rate parity: $\mathbb{E}_t[s_{t,t+\tau}] - s_t = (r_t^d(\tau) - r_t^f(\tau))$.

  - Evidence: $s_{t,t+\tau} - s_t = a + b(r_t^d(\tau) - r_t^f(\tau)) + e_t \Rightarrow \widehat{a} \neq 0, \widehat{b} < 0$. R^2 is small.

- Compared to equities, interest rates, FX also possesses **_unique characteristics_**:

  - Stochastic skew in currency options.

  - Inherent linkages between primary rates and cross rates.

  - Inherent linkages to interest rates.

Outline

- Modeling unique features in FX for currency option pricing

  – Stochastic skew

  – Inherent linkages across different currency pairs

  – Linkages to stochastic interest rates

  – A new angle on FX between asymmetric economies: Money as stock.

  – Linkages between FX and sovereign credit risk.

- Option pricing under time-changed Lévy processes

- Dynamically consistent model estimation

Well-known Features: Smiles and Stochastic Volatility
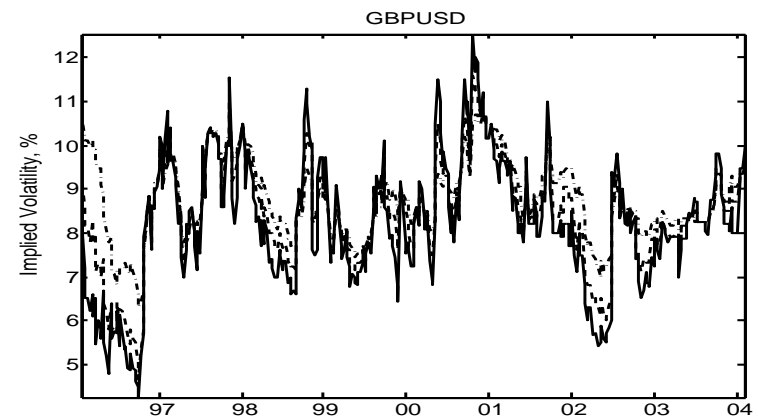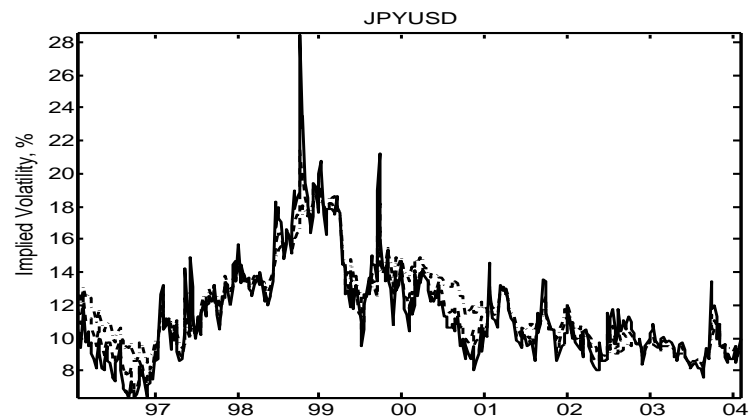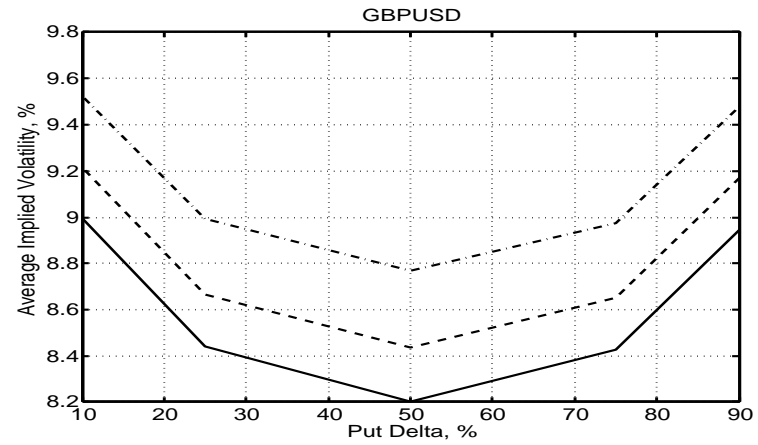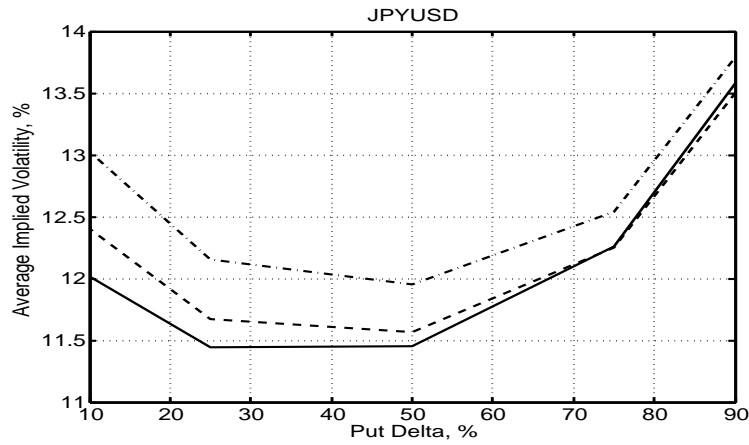
Unique Feature I: Stochastic Skew
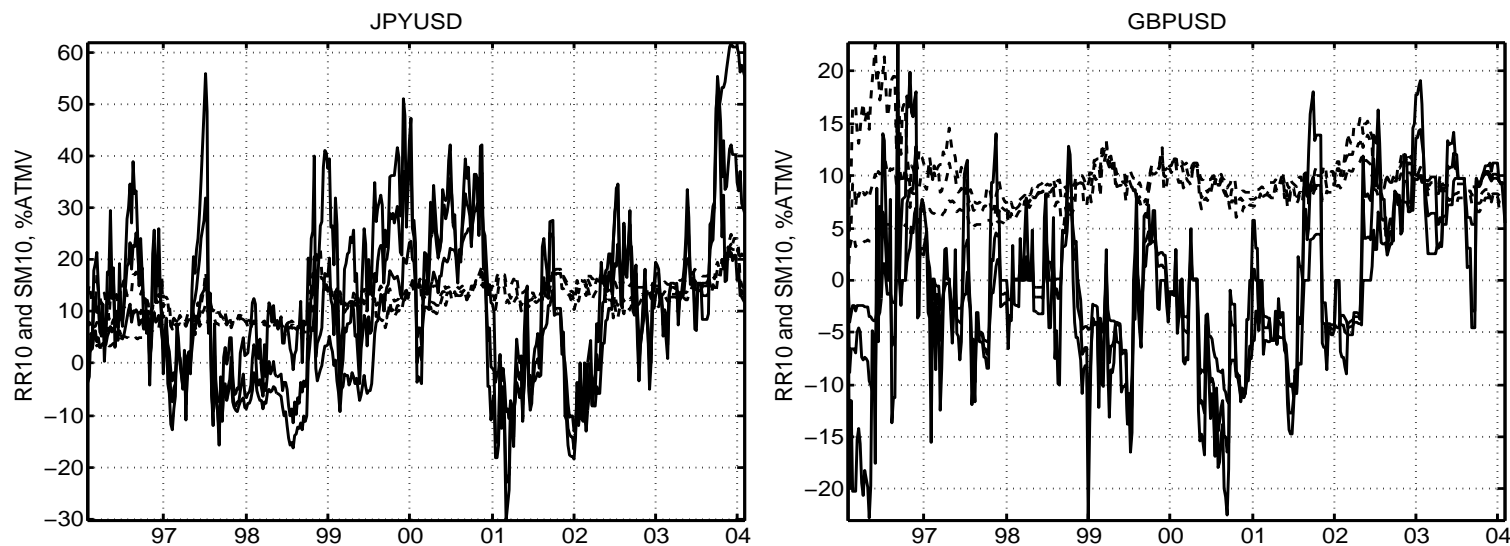
- Dashed lines: 10-delta Butterfly Spread (BF10)=(IV(10c)+IV(10p))/2-ATMV
  Measure of risk-neutral return kurtosis: relatively stable over time.

- Solid lines: 10-delta Risk Reversal (RR10)=IV(10c)-IV(10p)
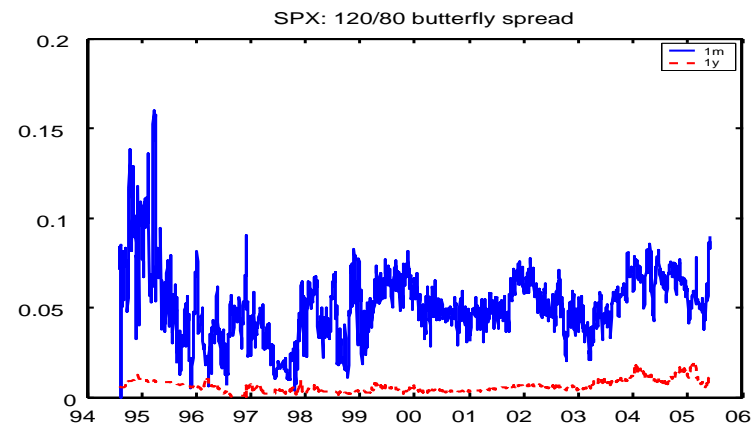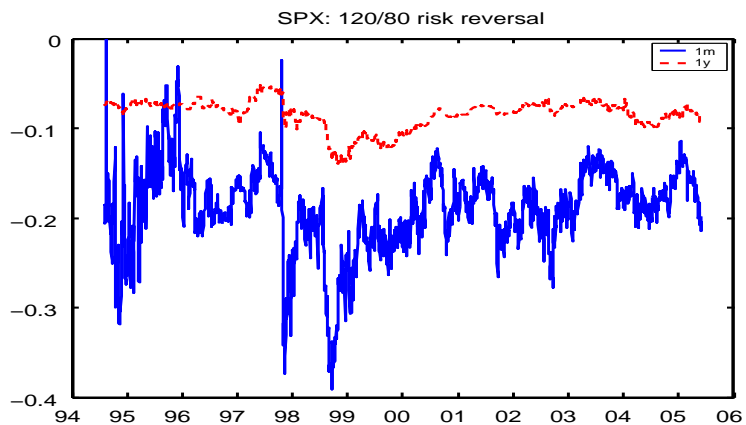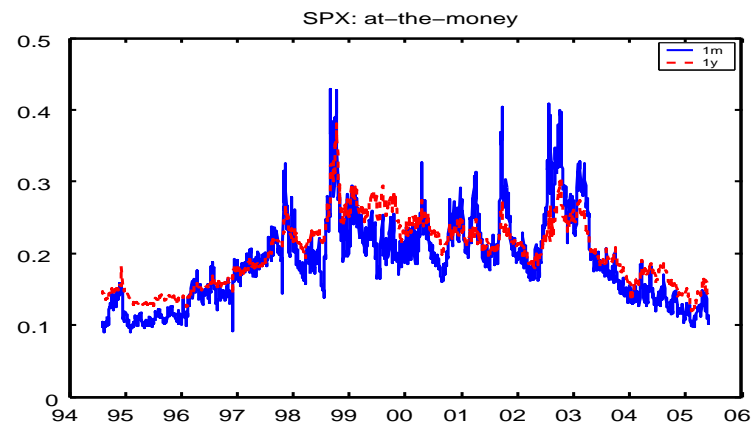 Skewness measure: *vary greatly over time* — switching signs.

- Contrast: equity options — always negatively skewed.

Negative Skew in Equity Index Options
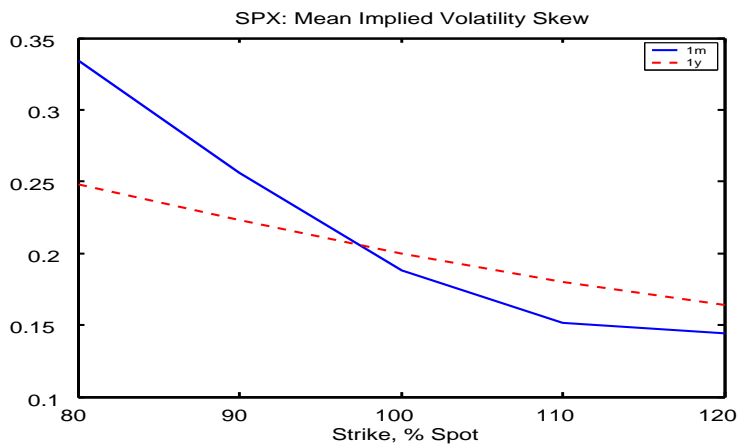
How Does the Literature Capture Smiles/Skews?

Two ways to generate a smile or skew:

1. **Add jumps**: Merton (1976)'s jump-diffusion model

$$dS_t/S_{t-} = (r_d - r_f)dt + \sigma dW_t + \int_{-\infty}^{\infty} (e^x - 1)[\mu(dx,dt) - \lambda n(\mu_j, \sigma_j)dxdt]$$

- The arrival of jumps is controlled by a Poisson process with arrival rate λ.
- Conditional on one jump occurring, the percentage jump size x is normally distributed, with density $n(\mu_j, \sigma_j^2)$.
- Uncertainty around the jump size (σ_j^2) generates smiles at short horizons.
- Nonzero mean jump size μ_j generates asymmetry (skew) at short horizons.

$\Rightarrow$ Stochastic skew would require μ_j to be stochastic ...

Not tractable!

How Does the Literature Capture Smiles/Skews?

Two ways to generate a smile or skew:

1. **Add jumps**

2. **Stochastic volatility**: Heston (1993)

$$
\begin{aligned}
dS_t/S_t &= (r_d - r_f)dt + \sqrt{v_t}dW_t, \\
dv_t &= \kappa(\theta - v_t)dt + \sigma_v\sqrt{v_t}dZ_t, \quad \rho dt = \mathbb{E}[dW_t dZ_t]
\end{aligned}
$$

   - Vol of vol (σ_v) generates smiles,
   - Correlation (ρ) generates skew at longer horizons.

   $\Rightarrow$ Stochastic skew would require correlation ρ to be stochastic ...

   Not tractable!

3. **Bates (1996)** combines 1&2 to generate stochastic volatility and **static** smiles/skews at both short and long horizons ... but NOT **stochastic skew**.

The Carr-Wu Stochastic Skew Model (SSM)

In the Language of Time-Changed Lévy Processes

$$\ln S_t/S_0 = (r_d - r_f)t + \left(L^R_{T^R_t} - \xi^R T^R_t \right) + \left(L^L_{T^L_t} - \xi^L T^L_t \right), \tag{1}$$

- L^R_t is a Lévy process that generates positive skewness (diffusion + positive jumps)

- L^L_t is a Lévy process that generates negative skewness (diffusion + negative jumps)

- $[T^R_t \equiv \int_0^t v^R_s ds, T^L_t \equiv \int_0^t v^L_s ds]$ randomize the clock underlying the two Lévy processes so that

    - $[T^R_t + T^L_t]$ determines total volatility: stochastic
    - $[T^R_t - T^L_t]$ determines skewness (risk reversal): ALSO stochastic

    $\Rightarrow$ **Stochastic Skew Model (SSM)**

- (r_d, r_f) assumed deterministic, hence interaction with interest rates ignored.

SSM In the Language of Merton and Heston

$$dS_t/S_{t-} = (r_d - r_f)dt \leftarrow \text{risk-neutral drift}$$

$$+ \sigma\sqrt{v_t^R}dW_t^R + \int_0^\infty (e^x - 1)[\mu^R(dx,dt) - k^R(x)dx v_t^R dt] \leftarrow \text{right skew}$$

$$+ \sigma\sqrt{v_t^L}dW_t^L + \int_{-\infty}^0 (e^x - 1)[\mu^L(dx,dt) - k^L(x)dx v_t^L dt] \leftarrow \text{left skew}$$

$$dv_t^j = \kappa(1 - v_t)dt + \sigma_v\sqrt{v_t}dZ_t^j, \quad \rho^j dt = E[dW_t^j dZ_t^j], \quad j = R, L \leftarrow \text{activity rates}$$

- At short term, the Lévy density $k^R(x)$ has support on $x \in (0, \infty) \mapsto$ Positive skew. The Lévy density $k^L(x)$ has support on $x \in (-\infty, 0) \mapsto$ Negative skew.

- At long term, $\rho^R > 0 \mapsto$ Positive skew. $\rho^L < 0 \mapsto$ Negative skew.

- **Stochastic skew** is generated via the randomness in $[v_t^R, v_t^L]$, which randomizes the contribution from the two jumps and from the two correlations.

- Bloomberg terminology: **Random Risk Reversal**, or R^3.

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Specifications of the Lévy Jumps

- The arrival rates of upside and downside jumps (**Lévy density**) follow exponential dampened power law (**DPL**):

$$k^R(x) = \begin{cases} \lambda e^{-\frac{|x|}{v_j}}|x|^{-\alpha-1}, & x > 0, \\ 0, & x < 0. \end{cases} \quad , \quad k^L(x) = \begin{cases} 0, & x > 0, \\ \lambda e^{-\frac{|x|}{v_j}}|x|^{-\alpha-1}, & x < 0. \end{cases} \quad (2)$$

  – The specification originates in Carr, Géman, Madan, Yor (2002), and captures much of the stylized evidence on both equities and currencies (Wu, 2004).

  – A general and intuitive specification with many interesting special cases:

   * $\alpha = -1$: Kou's double exponential model (KJ), **finite activity**.

   * $\alpha = 0$: Madan's VG model, **infinite activity, finite variation**.

   * $\alpha = 1$: Cauchy dampened by exponential functions (CJ), **infinite variation.**

- DPL jumps & square-root dynamics lead to tractable solns for option pricing.

Model Performance Comparison

"Stochastic Skew in Currency Options," Carr & Wu, Journal of Financial Economics, forthcoming.

	Heston	Bates	SSM with different jumps			
JPYUSD: rmse	1.099	1.065	0.865	0.865	0.866	0.865
$\mathcal{L}, \times 10^3$	-9.430	-9.021	-6.416	-6.402	-6.384	-6.336
GBPUSD: rmse	0.464	0.442	0.387	0.387	0.387	0.388
$\mathcal{L}, \times 10^3$	4.356	4.960	6.501	6.502	6.497	6.521

- SSM models with different jumps perform similarly.

- All SSM models perform much better than MJDSV, which is better than HSTSV.

Theory and Evidence on the Stochastic Skew
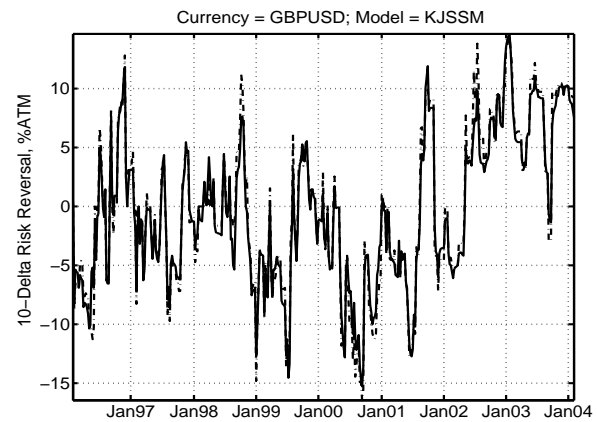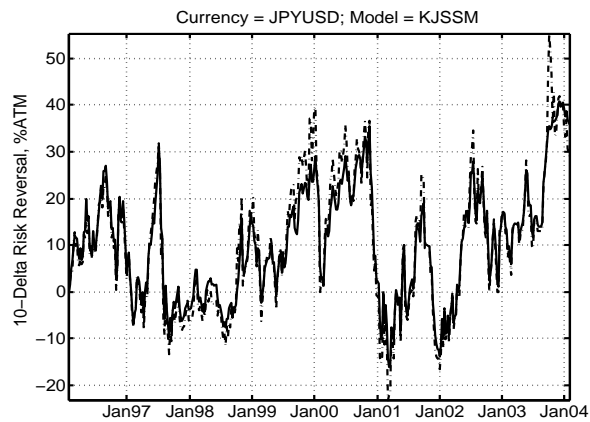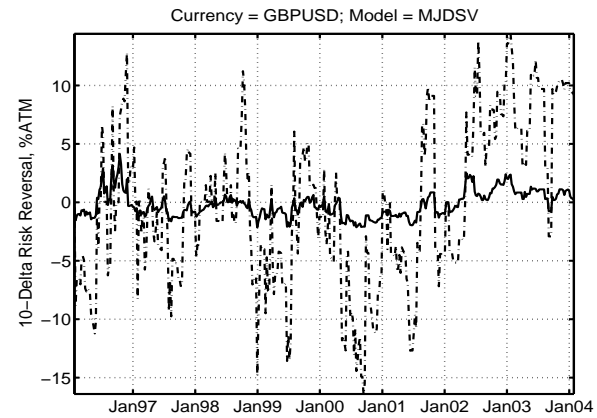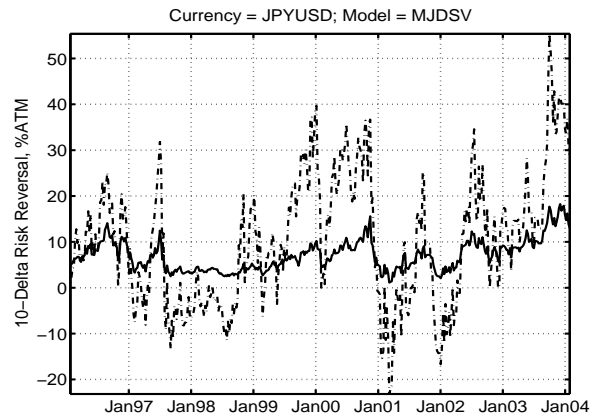
II. From Primary Rates to Cross Rates

- The SSM captures the stochastic skew behavior of currency options well.

- But SSM models each currency pair separately, ignoring their potential linkages.

- In reality, cross rates and primary rates are inherently linked through pairing
  - The triangular relation: $s_t^{AC} = s_t^{AB} - s_t^{BC}$.
  - If we model the dynamics of dollar-pound and dollar-yen, the dynamics of pound-yen is determined from the triangular relation.

- Implications:
  - Where to start matters.
  - If we start with modeling dollar-pound and dollar-yen each with two jump-diffusions as in SSM, we would have four jump-diffusion for pound-yen!

- Bottom line: *We need a more internally consistent way to modeling exchange rates across different pairs of economies.*

Start with the Pricing Kernel

- No arbitrage guarantees the existence of (at least) one *stochastic discount factor* (pricing kernel), $\mathcal{M}_t^h$, for each economy h such that

$$p_0 = \mathbb{E}_0 \left[\int_0^\infty \mathcal{M}_s^h \pi_s ds \right]$$

  - p_0: today's value of an asset in country h.

  - π_t: the payoff (denominated in country h's currency) of the asset at time t.

- In a Lucas-type representative agent economy,

$$\mathcal{M}_t = \beta \frac{u'(C_t)}{u'(C_0)} \quad \text{time-additive utility}$$
$$= \beta \exp(-\gamma R_t) \quad \text{CRRA}$$

  with

  - $R_t = \ln C_t / C_0$ return on aggregate consumption (wealth).
  - γ: relative risk aversion of the representative investor.

Linking the Pricing Kernel to Bond and FX Pricing

- Generically, we can write the pricing kernel for economy h as

$$\mathcal{M}_t^h = \exp\left(-\int_0^t r_s^h \, ds\right) \mathcal{E}\left(-\int_0^t \gamma_s^h \cdot dX_s^h\right)$$

  - r_t^h: the instantaneous interest rate in economy h.
  - X_t: the uncertainty of the economy (**return on aggregate wealth/consumption**).
  - γ_t: the market price of the risk (**relative risk aversion**).
  - $\mathcal{E}(\cdot)$: the exponential martingale operator:
    The exponential martingale defines the measure change from the objective/statistical measure $\mathbb{P}$ to the country-h risk-neutral measure $\mathbb{Q}^h$:

$$\left.\frac{d\mathbb{Q}^h}{d\mathbb{P}}\right|_t \equiv \mathcal{E}\left(-\int_0^t \gamma_s^h \, dX_s^h\right).$$

  - Bond pricing — The time-0 value of a zero-coupon bond with maturity t:
    $B(0,t)^h = \mathbb{E}_0^{\mathbb{P}}\left[\mathcal{M}_t^h\right] = \mathbb{E}_0^{\mathbb{Q}}\left[\exp\left(-\int_0^t r_s^h \, ds\right)\right]$
  - Currency pricing — Return on the h-currency (home) price of f-currency:
    $s_t^{hf} \equiv \ln S_t^{hf}/S_0^{hf} = \ln \mathcal{M}_t^f - \ln \mathcal{M}_t^h.$

Model Design: Bakshi, Carr, and Wu

"*Stochastic Risk Premiums, Stochastic Skewness in Currency Options, and Stochastic Discount Factors in International Economies.*" JFE, forthcoming.

$$\mathcal{M}_t^h = \exp\left(-r_t^h t\right) \exp\left(-W_{\Pi_t^h}^g - \frac{1}{2}\Pi_t^h\right) \exp\left(-\left(W_{\Lambda_t^h}^h + J_{\Lambda_t^h}^h\right) - \left(\frac{1}{2} + k_{J^h}[-1]\right)\Lambda_t^h\right),$$

- r^h: Deterministic interest rates.

- W_t^g: a diffusion **global** risk factor

- $W_t^h + J_t^h$: a jump-diffusion **country-specific** risk factor

- **Stochastic risk premium rates** for both: $\Pi_t^h = \xi^h \int_0^t Z_s ds, \quad \Lambda_t^h = \xi^h \int_0^t Y_s^h ds$

  – Z_t is the common stochastic risk premium rate on the global risk factor.
  – Y_t^h is the stochastic risk premium rate on the country-specific risk factors.
  – ξ^h is a constant scalar, captures the average difference in risk premium rates.

- Skewness on the log pricing kernel can be generated from (1) correlation between W_t^g and Z_t, (2) correlation between W_t^h and Y_t^h, and (3) J^h.

- **Stochastic risk premium rates $\Rightarrow$ stochastic volatility & skewness**.

Simpler Models Do Not Work

Everything should be made as simple as possible, but not simpler. — Albert Einstein

- Constant risk premium with diffusion risk: $\mathcal{M}_t^h = \exp\left(-r^h t\right) \exp\left(-\gamma^h \sigma^h W_t^h - \frac{1}{2}(\gamma^h \sigma^h)^2 t\right)$.

  $\Rightarrow$ Currency return would be normally distributed with constant volatility.

  No skew or smile. Let alone stochastic skew!

- Constant risk premium with jump-diffusion risk:
  (Set the time changes Π, Γ to constants, or Lévy without time change).

  $\Rightarrow$ Currency return distribution would be non-normal, but iid over time.

  No *stochastic* skew or stochastic volatility!

- *Stochastic skew in currency options warrants stochastic risk premium in international economies.*

- Details: DPL for J_t and square-root processes for Y^h, Z to generate tractable solns.

Currency Return Dynamics

- Currency return dynamics under the statistical measure ($\mathbb{P}$):

$$s_t^{hf} = \left(r^h - r^f\right)t + \left(\sqrt{\xi^h} - \sqrt{\xi^f}\right)W_{\Pi_t}^g + \left(W_{\xi^h \Lambda_t}^h + J_{\xi^h \Lambda_t}^h\right) - \left(W_{\xi^f \Lambda_t}^f + J_{\xi^f \Lambda_t}^f\right) + \cdots$$

  - **One** diffusion global risk factor (W^g),
  - **Two** jump-diffusion country-specific factors ($W^h + J^h, W^f + J^f$),
  - **Three** stochastic risk premium rates (Z, Y_t^h, Y_t^f).

- Return dynamics under the home-country risk-neutral measure ($\mathbb{Q}^h$)

$$s_t^{hf} = \left(r^h - r^f\right)t + \left(\sqrt{\xi^h} - \sqrt{\xi^f}\right)W_{\Pi_t^Q}^{gQ} + \left(W_{\xi^h \Lambda_t^Q}^{hQ} + J_{\xi^{hQ} \Lambda_t^Q}^h\right) - \left(W_{\xi^f \Lambda_t}^f + J_{\xi^f \Lambda_t}^f\right) + \cdots$$

- Exchange rates between two strictly symmetric economies: $\xi^h = \xi^f = 1$:

$$s_t^{hf} = \left(r^h - r^f\right)t + \left(W_{\Lambda_t}^h + J_{\Lambda_t}^h\right) - \left(W_{\Lambda_t}^f + J_{\Lambda_t}^f\right) + \cdots$$

The global risk factor cannot be identified.

Identification via the Currency Triangle

- With one pricing kernel per economy, all exchange rates can be modeled simultaneously and consistently.

- Bakshi, Carr, & Wu: Exploit the information in time series returns and option prices on three currency pairs that form a triangular relation: dollar-yen, dollar-pound, yen-pound, to identify the pricing kernel in the US, UK, and Japan.

- Results:

  - Asymmetry: The average risk premium rate in Japan is 50% higher than in US or UK: $\xi^{JPY} \approx 1.5, \xi^{UK} \approx \xi^{US} = 1$.

  - Global risk premium rates (Z) are **more persistent** and **more volatile** than country-specific rp rates (Y)
    $\Rightarrow$ *A high degree of int'l integration, especially over long horizon.*

  - Aggregate wealth jumps frequently, but only negative jumps are priced:
    $\Rightarrow$ *Negative skew for equity index options; stochastic skew for currencies.*

Go Beyond the Triangle to Price Cross rates

- With one pricing kernel per economy, all exchange rates can be modeled simultaneously and consistently.

- Mo & Wu: Go beyond the triangle to price illiquid cross rate based on one liquid triangle and liquid primary rates:

- Example: (dollar-yen-pound)+dollar/peso ⇒ yen-peso, pound-peso.

- Example: (dollar-yen-pound)+dollar/peso+dollar/real ⇒ peso-real.
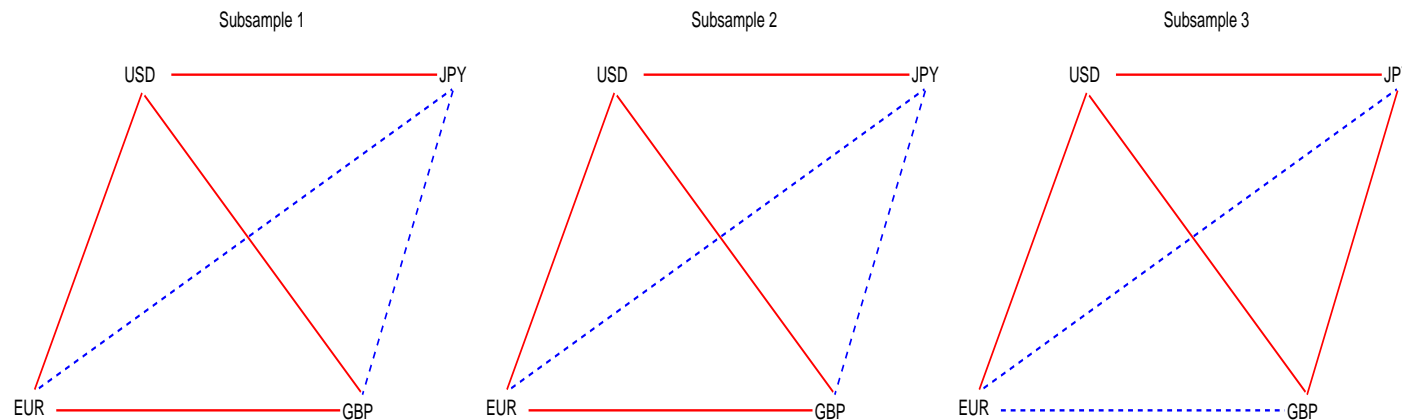
In-Sample Performance: RMSE

Contract	ATMV						25-delta call			25-delta put		
Maturity	1w	1m	3m	6m	12m	2y	1m	3m	12m	1m	3m	12m
						Average rmse=0.48						
EURUSD	0.88	0.46	0.18	0.22	0.30	0.43	0.46	0.38	0.41	0.45	0.20	0.43
GBPUSD	0.75	0.41	0.25	0.29	0.31	0.35	0.35	0.21	0.29	0.42	0.36	0.42
USDJPY	1.04	0.58	0.31	0.32	0.41	0.62	0.35	0.44	0.82	0.79	0.55	0.44
EURGBP	0.73	0.46	0.29	0.35	0.39	0.62	0.44	0.38	0.43	0.54	0.41	0.45
EURJPY	1.30	0.85	0.39	0.22	0.34	0.56	0.55	0.26	0.60	0.98	0.59	0.32
GBPJPY	1.21	0.74	0.31	0.28	0.35	0.52	0.52	0.32	0.57	0.84	0.48	0.32

Out-Of-Sample Performance: RMSE

Contract	ATMV						25-delta call			25-delta put		
Maturity	1w	1m	3m	6m	12m	2y	1m	3m	12m	1m	3m	12m
Subsample 1: average rmse=1.11												
EURJPY	2.19	1.72	1.11	0.71	0.57	0.66	1.41	0.88	0.69	1.82	1.28	0.62
GBPJPY	1.95	1.41	0.88	0.77	0.85	0.86	1.14	0.73	0.94	1.56	1.05	0.79
Subsample 2: average rmse=1.12												
EURGBP	1.97	1.72	1.34	1.06	0.92	1.24	1.70	1.32	0.90	1.72	1.38	1.01
GBPJPY	1.42	1.03	0.81	0.81	0.77	0.68	1.01	0.92	0.89	1.05	0.70	0.58
Subsample 3: average rmse=1.46												
EURJPY	1.90	1.63	1.30	1.13	1.23	1.54	1.42	1.25	1.37	1.59	1.29	1.07
EURGBP	1.93	1.78	1.48	1.28	1.26	1.68	1.71	1.44	1.33	1.71	1.43	1.22

III. Interactions between Stochastic Interest Rates and FX

- The pricing kernel is also a good place to start to capture the interactions between stochastic interest rates and exchange rates.

- Recall : Generically, we can write the pricing kernel for economy h as

$$\mathcal{M}_t^h = \exp\left(-\int_0^t r_s^h \, ds\right) \mathcal{E}\left(-\int_0^t \gamma_s^h \cdot dX_s^h\right)$$

  - Bond pricing — The time-0 value of a zero-coupon bond with maturity t:
  $B(0,t)^h = \mathbb{E}_0^{\mathbb{P}}\left[\mathcal{M}_t^h\right] = \mathbb{E}_0^{\mathbb{Q}}\left[\exp\left(-\int_0^t r_s^h \, ds\right)\right]$

  - Currency pricing — Return on the h-currency (home) price of f-currency:
  $s_t^{hf} \equiv \ln S_t^{hf}/S_0^{hf} = \ln \mathcal{M}_t^f - \ln \mathcal{M}_t^h$.

- Earlier academic literature: Specify r_t^h as driven by $X_t^h \Rightarrow$ Exchange rates are completely driven by interest rate dynamics. *The link is too strong!*

Independent Currency Movements

- Leippold & Wu: Allow independent currency movements via a multiplicative orthogonal decomposition:

$$\mathcal{M}_t^h = \exp\left(-\int_0^t r_s^h \, ds\right) \mathcal{E}\left(-\int_0^t \gamma_s^h \cdot dX_s^h\right) \mathcal{E}\left(-\int_0^t \eta_s^h \cdot dY_s^h\right)$$

where r is driven by X, but orthogonal to Y.

- Bond pricing: The second exponential martingale drops out:

$$B(0,t)^h = \mathbb{E}_0^{\mathbb{P}}\left[\mathcal{M}_t^h\right] = \mathbb{E}_0^{\mathbb{P}}\left[\exp\left(-\int_0^t r_s^h \, ds\right) \mathcal{E}\left(-\int_0^t \gamma_s^h \cdot dX_s^h\right)\right].$$

- Currency pricing: Depends on both X and Y:
  $$s_t^{hf} \equiv \ln S_t^{hf}/S_0^{hf} = \ln \mathcal{M}_t^f - \ln \mathcal{M}_t^h.$$

- Assuming independence between interest rates and FX amounts to assume that interest-rate risk is not priced ($\gamma^h = 0$).

From Deterministic to Stochastic Interest Rates in SSMs

- Bakshi, Carr, & Wu: Deterministic interest rates (r)

$$\mathcal{M}_t^h = \exp\left(-\int_0^t r_s^h ds\right) e^{\left(-W_{\Pi_t^h}^g - \frac{1}{2}\Pi_t^h\right)} e^{\left(-\left(W_{\Lambda_t^h}^h + J_{\Lambda_t^h}^h\right) - \left(\frac{1}{2} + k_{Jh}[-1]\right)\Lambda_t^h\right)}.$$

- A one-factor stochastic interest rate extension:

$$\mathcal{M}_t^h = \exp\left(-\int_0^t r_s^h ds\right) \exp\left(-\int_0^t \gamma^{hr}\sigma(s)^h dW_s^{hr}\right) e^{\left(-W_{\Pi_t^h}^g - \frac{1}{2}\Pi_t^h\right)} e^{\left(-\left(W_{\Lambda_t^h}^h + J_{\Lambda_t^h}^h\right) - \left(\frac{1}{2} + k_{Jh}[-1]\right)\Lambda_t^h\right)}$$

$$dr_t^h = \kappa\left(\theta(t)^h - r_t^h\right) dt + \sigma(t)^h dW_t^{hr}.$$

  - W_t^{hr} captures interest-rate risk, with γ^{hr} capturing its market price.

  - (W^g, W^h) capture the portion of currency risk *independent* of interest rates.

  - We use deterministic drift function $\theta(t)$ to match the observed yield curve, and volatility function $\sigma(t)$ to match the cap implied volatility term structure.

  - Correlations across term structures: $\rho^{hfr} dt = \mathbb{E}[dW_t^{hr} dW_t^{fr}]$

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Variance Contribution from Stochastic Interest Rates

- Currency return variance under deterministic interest rates (BCW):

$$\text{Var}(s_t^{hf}) = \left(\sqrt{\xi^h} - \sqrt{\xi^f}\right)^2 Z_t + \left(1 + \lambda\Gamma[2-\alpha]\left(\beta_+^{\alpha-2} + \beta_-^{\alpha-2}\right)\right)\left(\xi^h Y_t^h + \xi^f Y_t^f\right).$$

- Instantaneous variance contribution from the one-factor stochastic interest rate:

$$\begin{aligned}
\text{Var}(s_t^{hf}) &= \left(\gamma^{hr}\sigma(t)^h\right)^2 + \left(\gamma^{fr}\sigma(t)^f\right)^2 - \left(\gamma^{hr}\sigma(t)^h\gamma^{fr}\sigma(t)^f\rho^{hfr}\right) \\
&\quad + \left(\sqrt{\xi^h} - \sqrt{\xi^f}\right)^2 Z_t + \left(1 + \lambda\Gamma[2-\alpha]\left(\beta_+^{\alpha-2} + \beta_-^{\alpha-2}\right)\right)\left(\xi^h Y_t^h + \xi^f Y_t^f\right).
\end{aligned}$$

  - At short maturities, the contribution is usually small as currencies are much more volatile than interest rates.

  - But as interest rates are more persistent than currency volatilities, the impact of interest rate variation increases as option maturity increases.

  - Example: Power Reverse Dual Currency Notes (PRDC), with maturity about 30 years: $\text{Notitional} \times \left(C_f \frac{S_T}{S_0} - C_h\right)^+.$

A Two-Factor Stochastic Interest Rate Model

- Bloomberg has built a two-factor stochastic interest rate model for MBS pricing:

$$dX_t = -\kappa X_t dt + \Sigma(t) dW_t, \quad r_t = \theta(t) + X_{1t} + X_{2t}$$

where

$$\kappa = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}, \quad \Sigma\Sigma' = \begin{bmatrix} \sigma(t)_1^2 & \rho\sigma(t)_1\sigma(t)_2 \\ \rho\sigma(t)_1\sigma(t)_2 & \sigma(t)_2^2 \end{bmatrix}.$$

  – $\theta(t)$ matches the yield curve.

  – $\Sigma(t)$ matches to two volatility term structure (one cap term structure, and a swaption term structure).

- It can be easily superimposed onto a currency option pricing model with sequential estimation.

IV. A New Angle on FX Between Asymmetric Economies

- The pricing kernel approach can be used to model any FX pair.

- For FX between highly asymmetric economies such as dollar-peso and dollar-real, an alternative angle is to view peso/real just like a US stock.

- The analogy:

  - Government budget constraint for Mexico:
    Foreign(Dollar) Debt $+$ (Domestic Debt+ Money) $\times$ Dollar price of Peso
    $=$ Present value of future government surplus in dollars.

  - Corporate finance equality for a US firm:
    Corporate Debt $+$ (Shares) $\times$ Stock Price $=$ Present value of future earnings.

- *(1) Foreign debt is like corporate debt.*
  (2) Money is like stock.
  (3) FX is like stock price.

- Apply a stock model (with negative skew) to FX between asymmetric economies.

V. Linkages to Credit Spreads

- *Capital structure linkages (Merton(74))*: Corporate bond credit spread increases as stock prices fall and stock volatility increases.

- *Extension to sovereign credit*: Sovereign credit spread increases as the FX (dollar price of peso) depreciates and the FX volatility increases.

- Evidence:

  - Stock option implied volatility and corporate CDS are positively correlated.
  - Currency option implied volatility and sovereign CDS are positively correlated.

- Model design (that incorporates linkages) (Carr and Wu):

  - Stock price drops to zero when corporate defaults. Prior to default, stock volatility and default arrival are stochastic and have positive co-movements.
  - FX price drops by a large amount when sovereign defaults on its foreign debt. Prior to default, FX volatility and default arrival are stochastic and have positive co-movements.

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Co-movements between CDS and option implieds
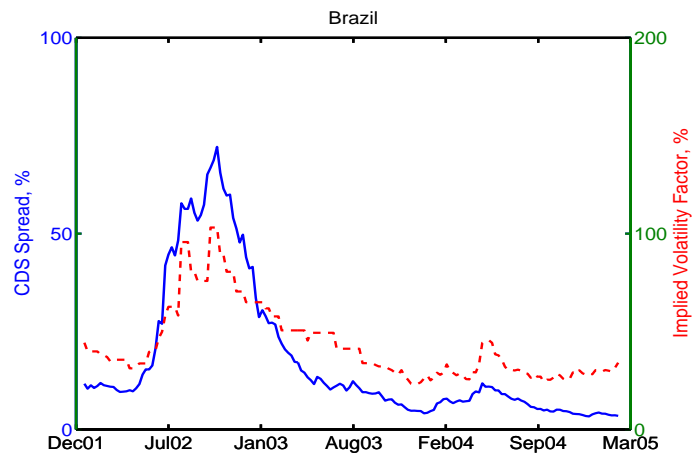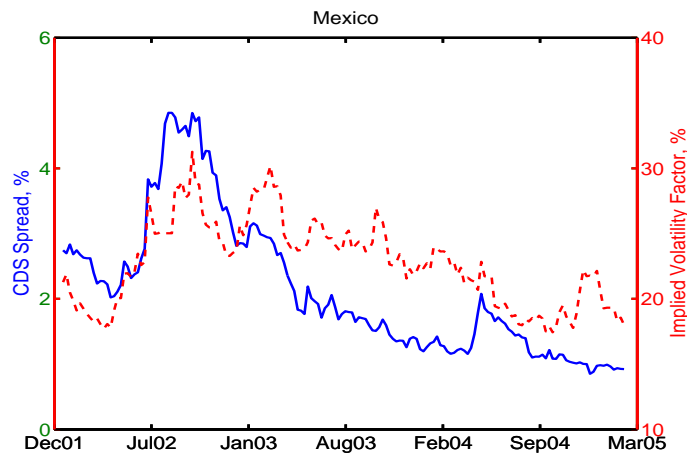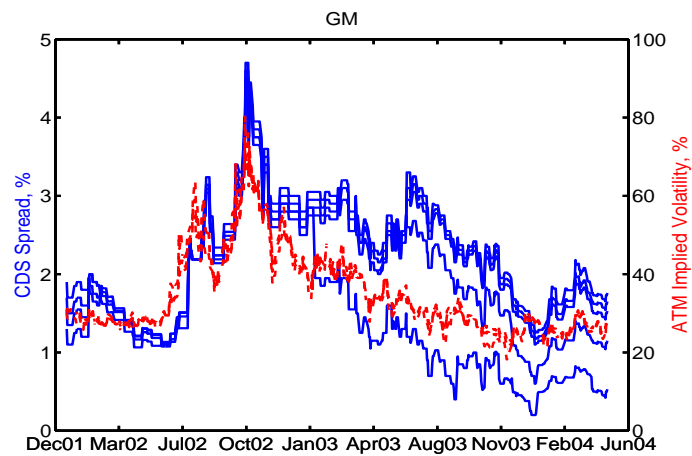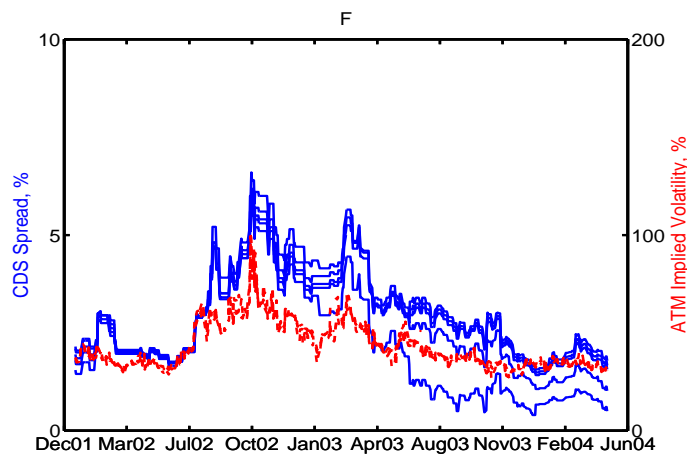
Joint Model of FX and Sovereign Default

- Currency price (dollar price of one peso) under $\mathbb{Q}$:

$$dP_t/P_t = (r(t) - r^*(t))\,dt + \sqrt{v_t}\,dW_t^s + \left(e^{-q}dJ(\lambda_t) + \zeta\lambda_t dt\right),$$

  - Normal-time market movements are captured by a diffusion component W^s with stochastic variance rate $v(t)$.

  - Default arrives via a Poisson process with stochastic arrival rate $\lambda(t)$.

  - Sovereign default causes currency price to drop by e^{-q}, with $q \sim N(\mu_j, v_j)$.

- Joint (*but separate*) dynamics of FX variance and default arrival rates:

$$dv_t = (\theta_v - \kappa_v v_t)\,dt + \sigma_v\sqrt{v_t}\,dW_t^v, \leftarrow \text{\textit{stochastic stock return variance} } (v_t)$$
$$\lambda_t = \beta v_t + z_t, \leftarrow \text{\textit{default arrival comoves} } (\beta) \text{ \textit{with stock variance}}$$
$$dz_t = (\theta_z - \kappa_z z_t)\,dt + \sigma_z\sqrt{z_t}\,dW_t^z, \leftarrow \text{\textit{independent credit risk movements} } (z_t)$$
$$\rho = \mathbb{E}\left[dW^P dW^v\right]/dt, \leftarrow \rho < 0 \Leftrightarrow \text{\textit{``leverage effect.''}}$$

- On Bloomberg soon as the *CDFX model.*

Option Pricing under Time-Changed Lévy Processes

- In all the models discussed, the currency returns can all be written as a sum of several time-changed Lévy processes

$$\ln S_t/S_0 = \int_0^t (r_s^d - d_s^f)ds + \sum_{k=1}^K \left(b^k X_{\mathcal{T}_t^k}^k - \varphi_{x^k}(b^k)\mathcal{T}_t^k \right).$$

  – Each Lévy component X captures one source of economic shock.

  – The stochastic time change on X captures the time-varying business activity or intensity on that economic shock.

- Carr and Wu (2004, JFE) show that the Fourier transform of each time-changed Lévy process can be turned into the Laplace transform of the stochastic time:

$$\phi_Y(u) \equiv \mathbb{E}^{\mathbb{Q}} \left[e^{iuX_{\mathcal{T}_t}} \right] = \mathbb{E}^{\mathbb{M}} \left[e^{-\psi_x(u)\mathcal{T}_t} \right], \quad u \in \mathcal{D} \in \mathbb{C},$$

where the new measure $\mathbb{M}$ is defined by the exponential martingale: $\left. \frac{d\mathbb{M}}{d\mathbb{Q}} \right|_t = \exp\left(iuX_{\mathcal{T}_t} + \mathcal{T}_t\psi_x(u) \right).$

Tractable Option Pricing under Time-Changed Lévy Processes

$$\phi_Y(u) \; \equiv \; \mathbb{E}^{\mathbb{Q}}\left[e^{iuX_{\mathcal{T}_t}}\right] = \mathbb{E}^{\mathbb{M}}\left[e^{-\psi_x(u)\mathcal{T}_t}\right]$$

- Tractability of the transform $\phi(u)$ depends on the tractability of
  (i) $\psi_x(u)$, and (ii) the Laplace transform of $\mathcal{T}_t$ under $\mathbb{M}$.

  – Tractable $\psi_x(u)$ comes from the Lévy specification: diffusion, compound Poisson, DPL, NIG,...

  – Tractable Laplace comes from activity rate dynamics: affine, quadratic, 3/2.

  – The two $(X, \mathcal{T}_t)$ can be chosen separately as building blocks, for different purposes.

- Given tractable solutions for the transform $\phi(u)$, option prices can be computed using fast Fourier inversion (FFT).

Model Estimation: Static v. Dynamic Consistency

- **The role of a no-arbitrage model**:

  – The prices generated from the same model on different derivative contracts are mutually consistent

  – You cannot buy one and sell the other based on the model prices to make arbitrage money.

- **Caveat:**
  The "same" model with different parameters becomes *different* models.

- **Static consistency**: Prices are consistent cross-sectionally at a fixed point in time ⇒ Sufficient for market makers and short-term investors who do not hold overnight positions.

- **Dynamic consistency**: Prices are also consistent over time ⇒ Important for hedge funds who bet on long-term convergence.

- **Bottom line:** *Everyone wants dynamic consistency if the cost is not too high*:

  – Pricing performance may deteriorate over time; estimation/computation burden may be too high.

Dynamically Consistent Estimation

- Cast the model into a state-space form:

  – State propagation — Euler approximation of the $\mathbb{P}$-dynamics of risk factors.

  – Measurement equations are on *vanilla* options quotes.

- Given model parameters, use unscented Kalman filter (UKF) to generate efficient forecasts and updates on conditional mean and covariance of states and measurements sequentially.

- Choose model parameters to maximize the likelihood on forecasting errors.

- The demand for tractability: Thousands of iterations on several years of data, with potentially hundreds of prices per day.

- Exotic options: Given parameter estimates, price exotic options via Monte Carlo simulation.

Concluding Remarks

- The FX market is inherently linked to the fixed income market.

- Different FX rates are also inherently linked through pairing.

- Starting from the pricing kernel is a nice way to model different FX rates and the interest rates consistently.

  – One pricing kernel for each economy.

  – Taking expectation on the pricing kernel generates bond pricing.

  – The ratio of two pricing kernels determines the FX move between the two economies.

- A new angle for highly asymmetric economies: One can use the "money as stock" analogy and model FX just like modeling stock prices.

- Combining different time-changed Lévy processes is a tractable way to generate stochastic skewness in currency options.

- The tractability facilitates dynamically consistent model estimation.