

Design space optimization using a numerical design continuation method

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SUMMARY

A generalized optimization problem in which design space is also a design to be found is defined and a numerical implementation method is proposed. In conventional optimization, only a portion of structural parameters is designated as design variables while the remaining set of other parameters related to the design space are often taken for granted. A design space is specified by the number of design variables, and the layout or configuration. To solve this type of design space problems, a simple initial design space is selected and gradually improved while the usual design variables are being optimized. To make the design space evolve into a better one, one may increase the number of design variables, but, in this transition, there are discontinuities in the objective and constraint functions. Accordingly, the sensitivity analysis methods based on continuity will not apply to this discontinuous stage. To overcome the difficulties, a numerical continuation scheme is proposed based on a new concept of a pivot phase design space. Two new categories of structural optimization problems are formulated and concrete examples shown. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: design space optimization; numerical design continuation method; shape optimization; topology optimization; sensitivity analysis; pivot phase design space

1. INTRODUCTION

The aim of optimization is to determine the values of design variables in order to extremize an objective function while satisfying given constraints. Structural optimization refers to size, shape and topology. Size optimization is determining optimum section properties, such as thickness or diameter. The aim of a shape optimization is to determine the shape of a system, in which design variables represent the shape, whereas in a topology optimization, the topology of the system is to be selected.

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Since Schmit [1] proposed a general approach to structural optimization using finite element analysis and non-linear mathematical programming in 1960, sizing problems have been routine [2, 3]. Francavilla *et al.* [4] formulated a fillet shape optimization problem to minimize stress concentration based on discretized forms. The theory of shape design sensitivity analysis is now well established by Cea [5], Zolesio [6], Reousselet [7], and Haug *et al.* [8]. Other *ad hoc* methods based on more or less intuitive concepts are also developed [9, 10]. Extensive reviews on structural shape optimization and shape sensitivity analysis can be found in Haftka [11] and Kwak [12].

Bendsoe and Kikuchi [13–15] suggested the homogenization method for topology optimization of a general 2-D linear elasticity. They used the method to obtain equivalent material properties of infinitely many micro cells with voids. Square and rectangular shaped holes were used to represent the microstructures. Cells can have different void sizes and orientations. The compliance of this material was optimized through a redistribution of the porosity using an optimality criterion procedure. Suzuki and Kikuchi [16] applied this method to the problems of anisotropic material. Kikuchi and his group extended the idea to various other problems [16–18]. Tenek and Hagiwara [19] used linear programming and compared the results with those of the optimality criterion method for static problems and vibration problems. They investigated the material distribution for plate problems [20]. Bendsoe *et al.* [21] suggested a method to restrict the checkerboard which often appears during their optimization process. The homogenization method is well-summarized by Hassani and Hinton [22]. Bendsoe [23], Kirsch [24] and Rozvany *et al.* [25] gave extensive reviews on analytical and approximation methods for topology optimization. Yang and Chuang [26] used artificial material and employed linear programming instead of the optimality criterion method. Xie and Steven [27] developed an evolutionary method for topology optimization in which elements at low stress level are removed. This method like the previous one [10] is very simple in its idea and use, but sometimes it occurs that an element once removed cannot be recovered when necessary. They also proposed a shape optimization method where a new grid is added near an element at high stress level [28], but this method cannot provide the quantitative effect of grid addition on the objective function or constraints.

In all the previous researches on topology optimization, the effect of design domain change has not been considered: that is, the shape of design domain is kept fixed during optimization process. In some cases when the shape of a design domain is not predefined, fixing the design domain can be a significant restriction in obtaining the optimum topology that would be obtained otherwise.

In the combined shape and topology optimization, up to now, it is achieved by iteratively performing topology optimization and shape optimization one after another. Maute and Ramm [29] have proposed an adaptive topology optimization using an integrated model. They perform shape optimization and topology optimization separately and map the results to each other using a 'background mesh'. They change the design patch for topology optimization using an adaptive mesh refinement strategy of FEM. In this method, a refining indicator was proposed to refine unclear domain boundaries. Although the clearness of the structural layout can be improved by this more or less *ad hoc* methodology, it cannot be assured that the refinement is properly done in order to minimize the objective function, because the refining indicator does not provide any information of the refinement effect on the objective or constraint functions.

DeRose and Diaz [30–32] proposed a meshless, wavelet-based scheme for a layout optimization. They adopted a fictitious domain and wavelet-Galerkin technique to overcome the problems of mesh degradation in convergence for large-scale layout optimization problems. Kim and Yoon [33] proposed a multi-resolution topology optimization in which the design optimization is performed progressively from low to high resolution. Although the scheme [30–32] may be used effectively for a large-scale problem and the design domain resolution is increased automatically in Reference [33], the refinement level is the same throughout the domain at a given stage, because the refinement is not done adaptively based on refinement sensitivities of the objective function. In all of the above methods except the adaptive topology optimization, the features of design variables are fixed during optimization process, and only values of design variables are optimized. In this sense, they are finding only a limited version of optimum topology using conventional size optimization methods.

In this paper, a design space optimization problem is proposed, in which the feature of a design in relation to topology as well as the usual design variables for shape and size is to be optimized. In order to change the feature of a design, the number of design variables is increased progressively from a small to a large number, and this corresponds to the evolution of design space from a simple to a sophisticated state. The capability of enlarging a design domain is a significant improvement over the conventional topology design methods, where it is not possible. The present method provides an efficient tool for this capability based on an analytical sensitivity analysis using the concept of a pivot phase design space.

The new formulation is applied to two categories of structural optimization problems: topology optimization and plate thickness distribution problems. In the proposed topology optimization, the design domain fixed up to now is relaxed to obtain an improved domain. To change the domain shape, sensitivity analyses of the objective and constraint functions with respect to design pixels are performed using the proposed numerical scheme. For a prescribed amount of design domain area, an optimum design domain shape as well as corresponding optimum topology inside the design domain can be found. Also, it is shown that these optimum shapes cannot always be obtained even when started with a large design domain.

As a second example, the optimum layout problem of design patches is taken to minimize the objective function of plate problems. The advantages of the proposed approach shown in this example are that (1) the effect of a patch refinement on the objective and constraint functions is calculated quantitatively by the proposed sensitivity analysis, and (2) the refinement is performed adaptively based on the obtained sensitivities. The evolution of design patch layout resembles a mesh adaptation of FEM. However, the objective and formulation are different: the layout adaptation is done to minimize the objective function by reconstructing design patches, whereas the aim of mesh adaptation is to improve analysis accuracy by remeshing finite elements.

2. DESIGN SPACE OPTIMIZATION

In a structural optimal design dealt up to now in the literature, various design variables for size and shape are first selected, and a prescribed performance function of the structure is optimized with respect to these variables, while satisfying given constraints. In this case, the design space is fixed either as a finite dimensional vector space or as an infinite dimensional

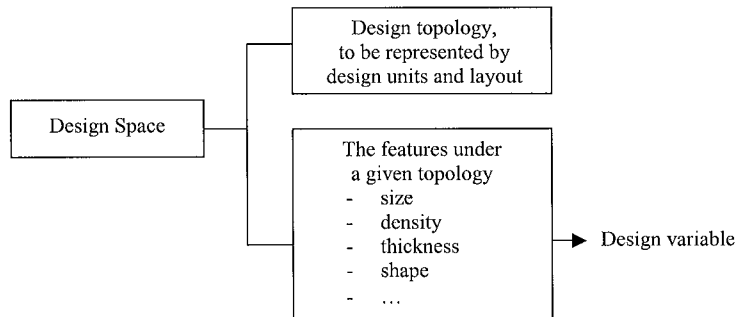


Figure 1. Design space including topology.

vector space. In the proposed design space problem, the design space is also unknown that is to be found in the optimization process. The design space to be considered is shown in Figure 1 where, as indicated, topology is included, in addition to the usual design variables.

In topology optimization of pixel-like approach, a pixel is a design unit, and the number of design units is selected before optimization process. A design unit has features, such as size, density, thickness, and shape. Usually, one feature is used as a design variable. Examples are the thickness of a plate patch for a plate thickness optimization and the density of a pixel for a topology optimization.

A conventional standard structural optimization problem is stated as follows:

$$\begin{aligned}
 &\text{extremize} && f(\mathbf{b}) \\
 &\text{subject to} && g(\mathbf{b}) \leq 0 \\
 & && h(\mathbf{b}) = 0 \\
 & && \mathbf{b} \in S \\
 & && (S \text{ is fixed})
 \end{aligned}$$

where $f(\mathbf{b})$ is an objective function, $g(\mathbf{b})$ denotes inequality constraints, $h(\mathbf{b})$ equality constraints, and \mathbf{b} design variables. The topology is assumed prescribed. A topology in this paper is loosely defined as the layout of structural members or elements, or in general, of design units as defined earlier. The design space S is defined as

$$S \equiv \{N, T_N, \{b_1, b_2, \dots, b_N\}\}$$

where T_N denotes a set of topologies for the N design units. This design space is selected before optimization, and only the design variables $\{b_1, b_2, \dots, b_N\}$ are optimized while the remaining parameters are fixed during the optimization process.

The proposed generalized structural optimization problem, to be called here the design space optimization problem, is defined as

$$\begin{aligned}
 &\text{extremize} && f(\mathbf{b}, S) \\
 &\text{subject to} && g(\mathbf{b}, S) \leq 0
 \end{aligned}$$

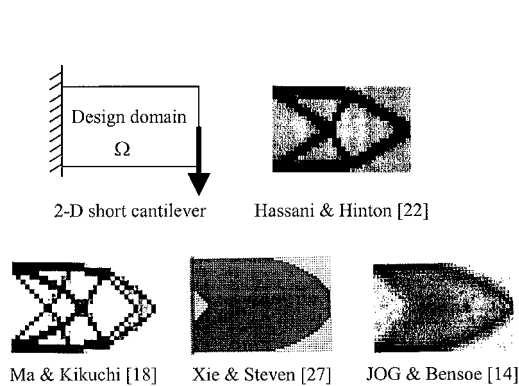


Figure 2. Results of previous studies for 2-D short cantilever topology optimization.

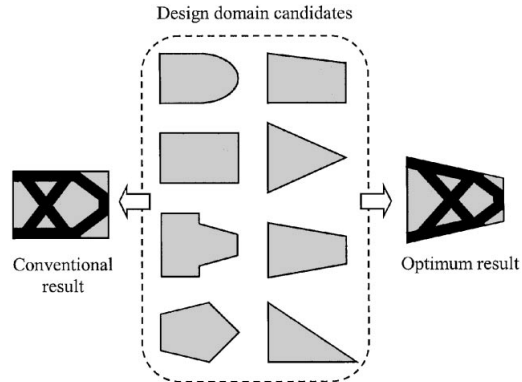


Figure 3. Candidates of design domain for given volume.

$$\begin{aligned}
 h(\mathbf{b}, S) &= 0 \\
 G(S) &\leq 0 \\
 H(S) &= 0 \\
 \mathbf{b} &\in S \\
 (S \text{ is variable})
 \end{aligned}$$

In this optimization, the design space as well as design variables is to be determined. The dimensionality N of the design space is an important unknown to be found. In other words, the number of design variables and their topologies change during the optimization process. The symbols $G(S)$ and $H(S)$ are design space constraints: $G(S) \leq 0$ and $H(S) = 0$ indicate that the spatial layout of the design be within or equal to some specified region, respectively. In the following, this formulation is specified for two categories of problems, and solution methods proposed.

2.1. Category 1. Topology optimization

Figure 2 shows the usual topology optimization problem for 2-D short cantilever, minimizing compliance under a prescribed amount of material. The shape of the design domain is rectangular and a vertical load is applied at the tip. The problem statement is

$$\begin{aligned}
 &\text{minimize} && f(\rho) \\
 &\text{subject to} && g(\rho) \leq 0 \\
 &&& (\Omega \text{ is given})
 \end{aligned}$$

where $f(\rho)$ is compliance to be minimized, $g(\rho)$ denote constraint functions, and density ρ of each design pixel is a design variable in the domain Ω . The volume of the design domain, that is the amount of material to be used, is given.

The figure shows some results of previous studies. Even though the kind of material used and optimization methods are different, the overall layouts of the optimums have some similarity: the upper- and lower-edge on the left carry loads. In these results, it is seen that the

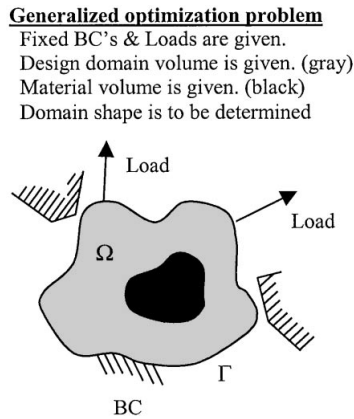


Figure 4. Generalized topology optimization.

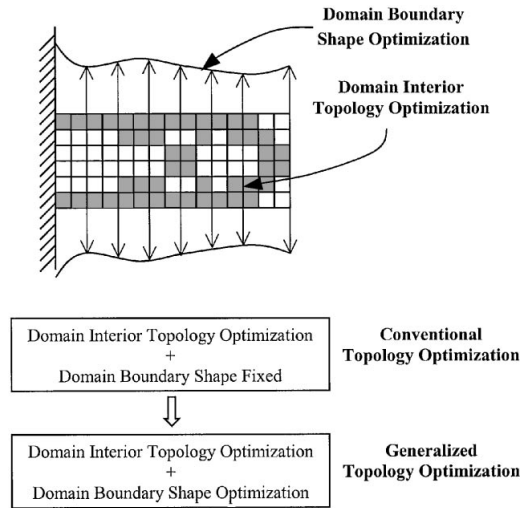


Figure 5. Comparison of conventional optimization and proposed generalized optimization.

shape of the selected design domain affects the optimum results, and it was noted by Bendsoe and Kikuchi [13] that a different choice of a design domain will generate a different topology. This means that by not allowing the change of design domain, one restricts the design to a narrow scope and thus cannot obtain the optimum topology for more general setting. Figure 3 shows that there are many candidates of a design domain for a given volume of domain and material to be used. Sometimes there may be cases when only a rectangular design domain must be used due to geometric constraints. However, in many cases, this rectangular shape is just an assumed one; and may not be the optimum for the given set of loads, boundary conditions, and amount of material. We are making a method available to expand the design space in some optimization sense.

In our problem, design variables are densities of cells, and the shape of the design domain is a feature of the design space. A design space optimization problem or a generalized optimization problem in this case can be defined as

$$\begin{aligned} &\text{minimize} && f(\rho; S(\Omega)) \\ &\text{subject to} && g(\rho; S(\Omega)) \leq 0 \end{aligned}$$

where both densities and the shape of the domain are to be determined during optimization process. The set of domain shapes is denoted by $S(\Omega)$. In the present problem, a shape is represented as an assemblage of cells. Figure 4 shows the definition of this generalized topology optimization problem, and this is a combination of interior topology optimization and domain boundary shape optimization, as shown in Figure 5. In this problem, cell densities of the domain are design variables $\{b_1, b_2, \dots, b_N\}$, and the shape of the domain is unknown, that is, the design space S is composed of varying number of cells denoted by N which is dependent on the domain Ω , and can be expressed as $S = \{N, T_N, \{b_1, b_2, \dots, b_N\}\}$.

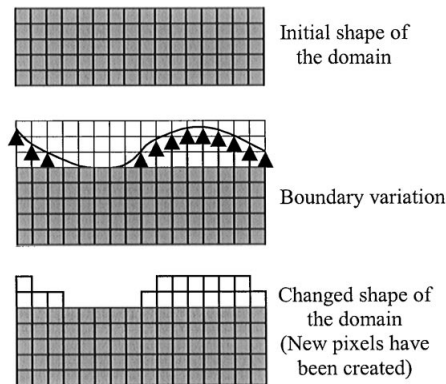


Figure 6. Procedure of shape change or updating design variable domain.

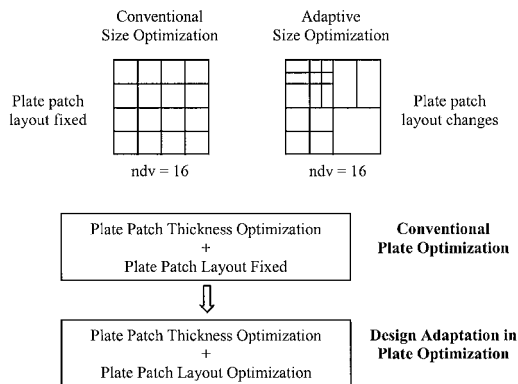


Figure 7. Comparison of conventional optimization and generalized optimization.

For the domain interior topology optimization, there are several schemes, such as homogenization method, artificial material method, and evolutionary method as already noted in the literature survey. To integrate the topology and shape optimization, a new scheme for the domain boundary shape optimization is necessary. Not like the usual shape optimization where the topology of finite element mesh remains the same, in our formulation, new design pixels or elements are added to represent the change of the boundary of the design domain. Figure 6 shows such a change. For this stage, it has been impossible to obtain sensitivities because the addition of a design variable is a discrete process. In the following, a design continuation concept is proposed and utilized.

2.2. Category 2. Plate thickness optimization

The aim of the plate thickness optimization in this paper is to optimize thickness of a plate for a given set of loading and boundary conditions to minimize compliance. Before doing optimization, the plate is discretized into design patches and each design patch is then a design unit. A design variable is thickness of a design patch. The problem statement is

$$\begin{aligned} &\text{minimize} && f(t) \\ &\text{subject to} && g(t) \leq 0 \end{aligned}$$

where t denotes thickness of design patches.

Like the previous example, the proper layout of design patches is unknown. If one changes the layout of design patches adaptively while performing a design patch thickness optimization, one can obtain better results. Figure 7 explains this idea: the resulting layout of design patches may be different from the fixed one of a conventional optimization. The proposed design adaptation is an integration of plate patch thickness optimization and plate patch layout optimization.

Mesh adaptation in FEM is performed to obtain more accurate results. On the other hand, the aim of the design adaptation is to minimize an objective function or to improve the

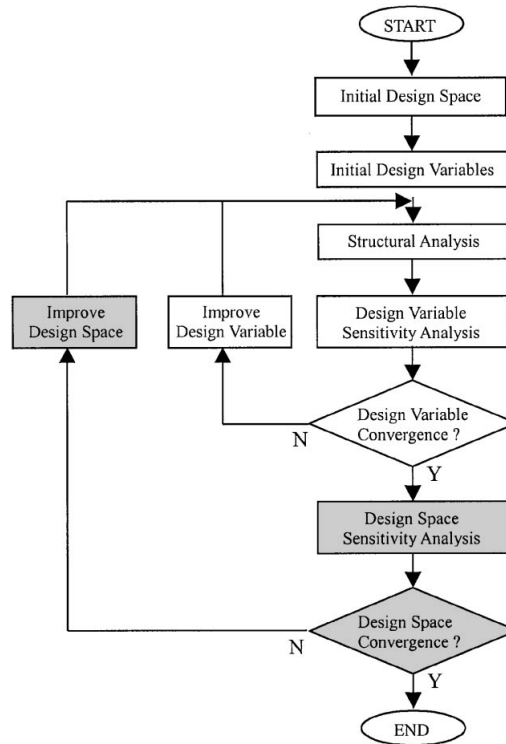


Figure 8. Procedure of design space optimization.

performance of a structure. An improved layout of design patches is obtained in the design adaptation. The design adaptation problem for plate thickness optimization is stated as follows

$$\begin{aligned} &\text{minimize} && f(t; S) \\ &\text{subject to} && g(t; S) \leq 0 \end{aligned}$$

where S is the design space composed of the set of all layouts of design patches. In order to change the layout of design patches during optimization process, a new scheme is needed, and this is the main topic of the next section.

3. NUMERICAL DESIGN CONTINUATION METHOD

When a design space is improved, it evolves into more complex one starting from an initially simple one. The procedure is equal to constructing a new set of design variables. Figure 8 shows a scheme of design space optimization procedure. Conventional design variable value optimization is drawn in the white boxes: an initial design space and design variable values are selected before iteration. According to the information obtained from a sensitivity analysis, a convergence check is made and if not converged, the design is improved. To do the design space optimization, an outer loop is added, which is drawn in the gray boxes. When the

design variable value optimization is converged in the current design space, design space convergence is checked. If it is not converged, the design space is improved and, in this stage, the number of design variables is increased.

The sensitivity of a functional $\Psi(b, z)$ with respect to a usual design variable is expressed as

$$\frac{d\Psi}{db} = \frac{\partial\Psi}{\partial b} + \frac{\partial\Psi}{\partial z} \frac{dz}{db}$$

where b is a design variable, and z is a state variable. When the number of design variables is increased, the sensitivity can be expressed conceptually as

$$\frac{d\Psi}{dn} = \frac{\partial\Psi}{\partial n} + \frac{\partial\Psi}{\partial z} \frac{dz}{dn}, \quad \text{or more properly,} \quad \frac{\Delta\Psi}{\Delta n} = \left. \frac{\Delta\Psi}{\Delta n} \right|_z + \frac{\partial\Psi}{\partial z} \frac{\Delta z}{\Delta n}$$

where n represents the number of design variables. However, the real phenomena of the objective function change are not simple: the sensitivities depend not only on the number of added design variables but also on their initial features, such as layout, size, and shape. Since n is a discrete variable, the functional is not continuous, and this means it is impossible to obtain derivatives when new design variables are created, in usual sense.

For example, when a pixel is added in a topology optimization, the objective function—compliance in this paper—changes discontinuously. And adding a design variable is equivalent to increasing the dimension of design variable space by one. This is because we assume that the value of the added design unit is the same as an already existing one. If one introduce a pivot phase as defined in the following, however, it is possible to have the changed design space with added pixels connected continuously to the current design space. In this figure, the pivot phase is $(N + 1)$ dimensional even though it has the same design as that of the present n -dimensional design space.

The process proposed is a numerical design continuation which is made possible by the introduction of the pivot phase. This has a design space different from the original one, but it has the same design as the original one by appropriately setting the values of the new design variables. The mechanical states are the same and corresponding functionals have the same values as those at the original design. Figure 9 illustrates relations among the three phases for creating new design variables in topology optimization and plate optimization. The phases on the left hand are the original ones before the design space is improved, and those on the right represents a design with an improved design space. Both the state and the design space are different from each other. The intermediate pivot phase introduced has the same design space as the improved one, but the values of the new design variables added are set at a level so that this pivot phase gives the same state and design as the original design. In this way the continuity of a functional of mechanical state is established between designs of different dimensionality.

For the topology optimization, the pivot phase is obtained by setting the densities of newly added design variables at zero. The thickness of new refined or partitioned plate patches is set at the same level as that of the original plate patch for the plate optimization problem. The sensitivities for the new design variables can then be obtained at these pivot phases.

The total sensitivities (Ψ') are defined as the sum of contributions of old design variable sensitivities (Ψ'_O) and new design variable sensitivities (Ψ'_N).

$$\Psi' = \Psi'_O + \Psi'_N$$

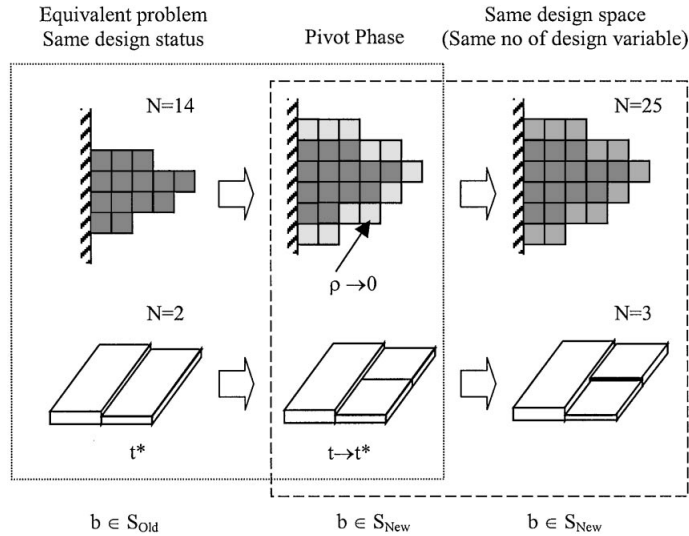


Figure 9. Establishing continuity of functionals through pivot phase.

Sensitivities from the new design variable candidates are obtained as $\Psi'_N = \Psi'_{+\delta u, u \rightarrow u^*}$ at the pivot phase. The subscript $+$ denotes directional derivatives; for the topology optimization example, we have to use directional derivatives, even though we may not need to use directional derivatives for the plate optimization example. Subscript u^* represents the value of design variables set for the pivot phase. That is, u^* is zero for the topology optimization example, and the thickness of the original plate patch for the plate optimization example.

The directional derivatives at the pivot phase of a bilinear functional $a_u(z, \bar{z})$, load linear form $l_u(\bar{z})$ and a state variable z are defined

$$a'_{+\delta u, u \rightarrow u^*}(z, \bar{z}) \equiv \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \frac{1}{\tau} [a_{u+\tau\delta u}(z, \bar{z}) - a_u(z, \bar{z})]_{u \rightarrow u^*}$$

$$l'_{+\delta u, u \rightarrow u^*}(\bar{z}) \equiv \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \frac{1}{\tau} [l_{u+\tau\delta u}(\bar{z}) - l_u(\bar{z})]_{u \rightarrow u^*}$$

$$z'_{+, u \rightarrow u^*} = z'_{+, u \rightarrow u^*}(x; u) \equiv \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \frac{1}{\tau} [z(x; u + \tau\delta u) - z(x, u)]_{u \rightarrow u^*}$$

where $+$ or $+\delta u$ denotes that directional derivatives are taken, z displacement function, and \bar{z} a variational displacement.

The variational form of the state equation is

$$a_u(z, \bar{z}) = l_u(\bar{z}) \quad \forall \bar{z} \in Z_{\text{adm}}$$

where the design space is the new one at this pivot phase. By taking the variations of both sides

$$a_u(z_{+,u \rightarrow u^*}, \bar{z}) = l'_{+\delta u, u \rightarrow u^*}(\bar{z}) - a'_{+\delta u, u \rightarrow u^*}(z, \bar{z}) \quad \forall \bar{z} \in Z_{\text{adm}}$$

The measure of structural performance may be written as

$$\psi = \int_{\Omega} g(z, \nabla z, u) \, d\Omega$$

where Ω is the state variable domain of the pivot phase, which can be different from that of the original phase. For the sensitivity analysis, take variation of this functional

$$\psi'_{+,u \rightarrow u^*} = \int_{\Omega} (g_z z'_{+,u \rightarrow u^*} + g_{\nabla z} \nabla z'_{+,u \rightarrow u^*} + g_u|_{+,u \rightarrow u^*} \delta u) \, d\Omega$$

and introduce an adjoint equation

$$a_u(\lambda, \bar{\lambda}) = \int_{\Omega} (g_z \bar{\lambda} + g_{\nabla z} \nabla \bar{\lambda}) \, d\Omega \quad \forall \bar{\lambda} \in Z_{\text{adm}}$$

Following the same procedure as in Reference [8] and using the symmetry of the bilinear form, and from the variations of the state equation, performance functional, and the adjoint equation, one can obtain the sensitivities with respect to new design variables as

$$\psi'_{+,u \rightarrow u^*} = \int_{\Omega} g_u|_{+,u \rightarrow u^*} \delta u \, d\Omega + l'_{+\delta u, u \rightarrow u^*}(\lambda) - a'_{+\delta u, u \rightarrow u^*}(z, \lambda)$$

The method is illustrated by deriving the formulas for the two categories introduced earlier. At this point, it is noted that the above approach and formulas are useful only when design variables are newly added. No similar formulas are possible, however, for the case when a design variable is removed.

3.1. Topology optimization

As a concrete case, a compliance optimization problem is taken

$$\begin{aligned} &\text{Minimize} && \int_{\Gamma} F^i z^i \, d\Gamma \\ &\text{Subject to} && \int_{\Omega} \rho(x) \, d\Omega \leq M_0 \\ &&& 0 \leq \rho(x) \leq 1 \end{aligned}$$

where the objective function is compliance, and design variables are densities $\rho(x)$. The design variable domain Ω is also to be found. Figure 10 shows an original phase and a pivot phase. The pivot phase has a design space (S) with design variable domain (Ω) different from the original one. However, the mechanical state of the pivot phase is made the same as the original one by taking the densities of newly added design variables to be zero. An artificial material approach [26] is used with the following relationship between Young's moduli

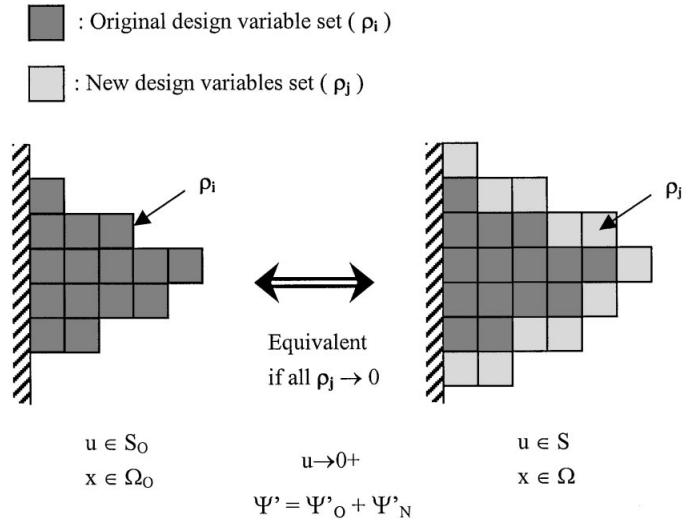


Figure 10. Pivot phase for topology optimization.

and the density:

$$\frac{E_i}{E_0} = \rho^n$$

where E_i is intermediate Young's modulus, E_0 a reference one, and n an exponent. It is known that the solution is much dependent on the choice of n . In subsequent numerical examples, n is taken as 2.

The state equation is given by

$$\int_{\Omega} \sigma^{ij}(z) \varepsilon^{ij}(\bar{z}) d\Omega = \int_{\Gamma} F^i \bar{z}^i d\Gamma \quad \forall \bar{z} \in Z_{\text{adm}}.$$

By using an adjoint equation, the sensitivity of the objective function for a new design variable is obtained

$$\psi'_{+, \rho \rightarrow 0} = - \int_{\Omega} \varepsilon^{ij}(z) D_{+, \rho \rightarrow 0}^{ijkl'} \varepsilon^{kl}(z) d\Omega$$

at the pivot phase. The design variable domain Ω is that of the new design space. Starting from a simple initial design, the shape of design domain or the design space changes are obtained while a usual topology optimization is performed in an inner loop. At each stage, sensitivities with respect to the new design variable candidates are obtained to evaluate the effect of the new addition on the objective function. Among these candidates, the candidates whose sensitivities belong to upper certain percentage are added as new design variables in the next step. It is important to note that only one additional FEM evaluation is needed for the sensitivity analysis of a stage. In contrast, the number of FEM analysis is as many as that of boundary design variables when forward differencing is used, as explained with the numerical examples.

3.2. Plate optimization

The plate problem to be considered is

$$\begin{aligned} &\text{Minimize} \quad \int_{\Omega} (F + \gamma t) z \, d\Omega \\ &\text{Subject to} \quad \int_{\Omega} t(x) \, d\Omega \leq V_0 \\ &\quad t_{\text{lower}} \leq t(x) \leq t_{\text{upper}} \end{aligned}$$

where the objective function is compliance, and design variables are thickness $t(x)$. For this example, although the design variable domain Ω is fixed as a region, the design space S defined on Ω is unknown and to be determined.

As the case of the topology optimization, the new design variable thickness at pivot phase is set to make the state of the pivot phase becomes equivalent to the original one. This thickness is written as t^* as shown in Figure 9.

The variational form of the plate equation is

$$a_t(z, \bar{z}) = l_t(\bar{z}) \quad \forall \bar{z} \in Z_{\text{adm}}$$

where

$$\begin{aligned} a_t(z, \bar{z}) &= \int_{\Omega} \hat{D}(t) [z_{11}\bar{z}_{11} + z_{22}\bar{z}_{22} + \nu(z_{22}\bar{z}_{11} + z_{11}\bar{z}_{22}) + 2(1 - \nu)z_{12}\bar{z}_{12}] \, d\Omega \\ l_t(\bar{z}) &= \int_{\Omega} [F + \gamma t] \bar{z} \, d\Omega \end{aligned}$$

and $\hat{D}(t) = Et^3/[12(1 - \nu^2)]$.

The corresponding adjoint equation is defined as

$$a_t(\lambda, \bar{\lambda}) = \int_{\Omega} (F + \gamma t) \bar{\lambda} \, d\Omega \quad \forall \bar{\lambda} \in Z_{\text{adm}}$$

The sensitivity of a new design patch at the pivot phase is obtained as follows:

$$\psi'_{+, t \rightarrow t^*} = \int_{\Omega} \left\{ 2\gamma z - Et^2 \frac{z_{11}^2 + z_{22}^2 + 2\nu z_{11}z_{22} + 2(1 - \nu)z_{12}^2}{4(1 - \nu^2)} \right\} \delta t \, d\Omega \Big|_{t \rightarrow t^*}$$

Starting with an initial design patch layout, it becomes more sophisticated as the design space optimization or design patch layout optimization is progressed as an outer loop. At each stage, after plate thickness optimization is completed in an inner loop, the sensitivity analysis is performed to evaluate the patch refinement effect on the objective function. Based on the obtained sensitivity information, the design patch is refined adaptively. Again, note that only one additional FEM calculation is needed to get all the sensitivity information for refined patches.

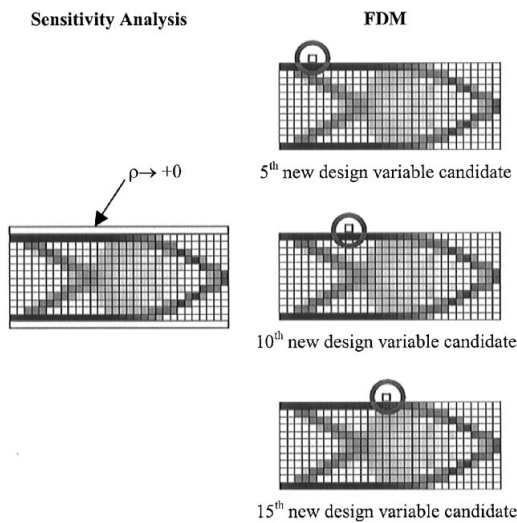


Figure 11. Comparison of proposed sensitivity analysis and FDM for sensitivity evaluations of the i th new design variable candidate.

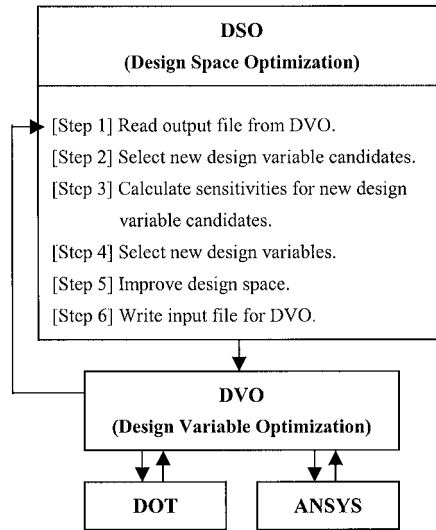


Figure 12. Schematic of computer programs.

4. NUMERICAL RESULTS AND DISCUSSIONS

4.1. Topology optimization

Figure 11 compares results of the proposed sensitivity analysis and FDM for sensitivity evaluations of the i th new design variable candidate. In the proposed sensitivity analysis, an outer band whose density is nearly zero is added to the original domain, and the sensitivities are obtained here. The band represents the candidates of new design variables. After calculating sensitivities for all the candidates, new design variables are selected according to the sensitivities or contributions to the objective function. In this example, upper 35 per cent of the candidates are selected as new design variables to be added. Through many numerical experiments, it has been found that 20–40 per cent of new design variable selection usually give good results. While we need just one FEM evaluation for this sensitivity analysis, more than 40 evaluations are necessary if FDM is used to evaluate the sensitivities. Young's modulus is $210 \times 10^9 \text{ N m}^{-2}$, and Poisson's ratio of 0.3 is used. Because bilinear elements sometimes cause the problems of checkerboards, eight node isoparametric plane elements are used in this study. The sensitivities obtained using the formula is compared with those by FDM with a difference of $\Delta b = 0.01$ in Table I. Both results match each other very well.

The computer program has a hierarchical structure as shown in Figure 12. Conventional optimization program consists of a minimization routine, FEA tool, and an interfacing program. In this study, DOT [34] and ANSYS have been used for minimization and FEA, respectively. Sequential quadratic programming (SQP) method was selected in DOT, and the numerical efficiency of this method in DOT was satisfactory even for more than 1000 design variables. A design variable optimization routine (DVO) controls sub-iterations for design variable optimization. In order to control the number of design variables, a design variable number

Table I. Sensitivities of new design variables candidates.

New design variable candidate number	FDM $\Delta\Psi_i/\delta b$	Analytic $\Psi'_i/\delta b$	$(\Psi'_i/\Delta\Psi_i \times 100)\%$
1	-116.6826	-116.6470	99.97
2	-8.5387	-8.5367	99.98
3	-1.6238	-1.6235	99.98
4	-2.3714	-2.3706	99.97
5	-3.6479	-3.6474	99.99
6	-3.9573	-3.9564	99.98
7	-3.4781	-3.4776	99.99
8	-3.2490	-3.2481	99.97
9	-3.0011	-3.0005	99.98
10	-2.7633	-2.7626	99.97
11	-2.5634	-2.5628	99.98
12	-2.2540	-2.2534	99.97
13	-2.2088	-2.2081	99.97
14	-3.5688	-3.5680	99.98
15	-5.8834	-5.8819	99.97
16	-6.7057	-6.7040	99.97
17	-5.8702	-5.8683	99.97
18	-3.7185	-3.7169	99.96
19	-2.0841	-2.0830	99.95
20	-1.1828	-1.1819	99.92
21	-1.6386	-1.6361	99.85

control program is added as an outer routine. A design space optimization routine (DSO) controls the number of design variables in this main iteration. The data are communicated among the four routines through ASCII files, and the computational costs for these communications are negligibly low compared to that of FEM evaluations and sensitivity analyses.

A case of conventional topology optimization is compared with that of the proposed optimization routine in Figure 13. For both problems, 480 design pixels and the same number of finite elements are used. The design domain shape is fixed as 30×16 in the conventional optimization, but the shape can change in the proposed design space optimization. The initial design domain taken is 30×12 . The domain shape changes with the main iterations, and the optimum shape is different from that of a conventional optimization. Figure 14 shows the change of the number of design variables along the number of main iterations and the corresponding optimum objective function changes. The dashed line represents the optimum objective function of the fixed domain problem. The objective function of the proposed scheme is less than that of the conventional problem. As another example, a geometric restriction on design variable domain is imposed as shown in Figure 15. The convergence of the procedure is also well illustrated by this.

To study the effect of domain size and aspect ratio, three types of design domains are compared in Figures 16(a) and 16(b), respectively. When a sufficiently large domain is used at the beginning, the optimum becomes a two-bar structure. When one restricts the domain area as usual, however, optimum topology and corresponding domain shape depends on the size or the aspect ratio of the domain started. As the aspect ratio defined as the height divided by the width is higher, the topology resembles more a two-bar structure. Figure 16(a) shows

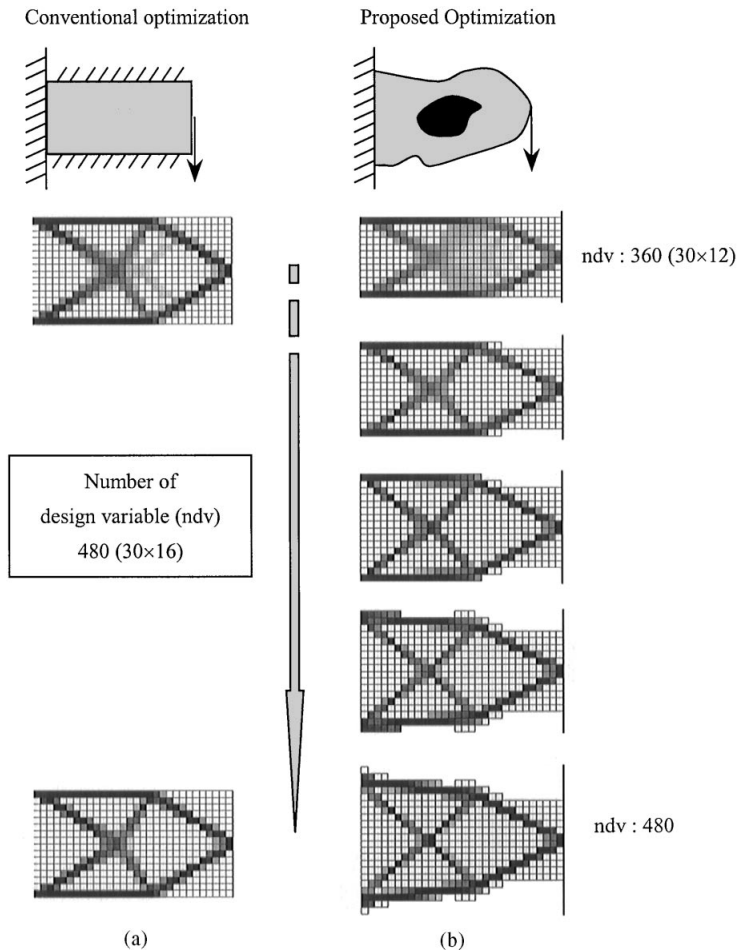


Figure 13. Change of design domain for design space optimization: (a) design space is fixed; (b) design space is varying.

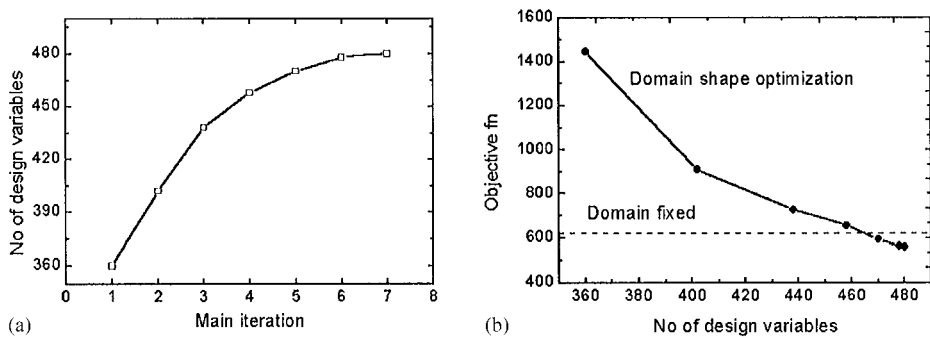


Figure 14. Number of design variables and history of optimum objective function: (a) number of design variables; (b) objective function history.

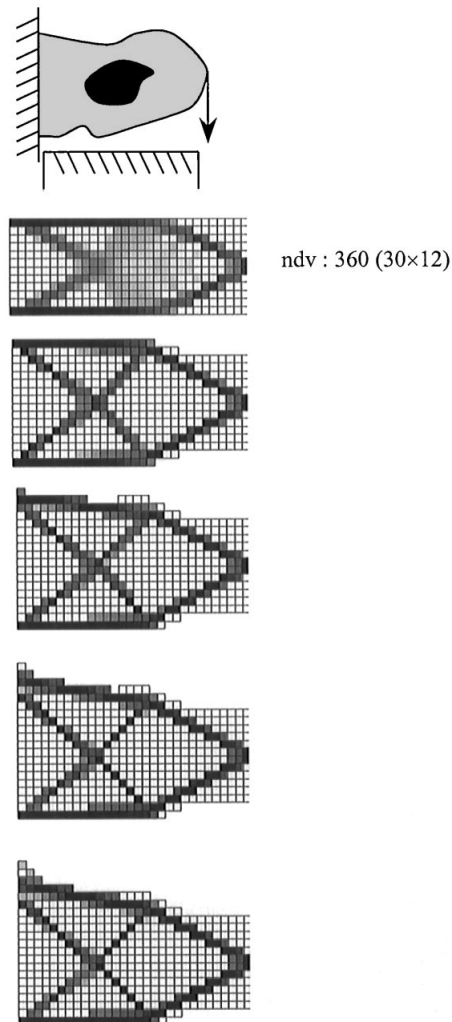


Figure 15. Design space constrained problem.

that the optimum of the variable domain problem is not the same as that of the fixed domain problem even when the same amount of domain and material is used. Also, for the same aspect ratio, the optimum of the fixed domain is different from that of the variable domain, as shown in the Figure 16(b). This figure illustrates that the proposed method enables us to find new, substantially better optimums, which are not found using the conventional fixed domain method. It is also interesting to observe that the present solutions are much clearer than those on the left side in the figure. In addition, the proposed method may have a better chance to obtain the global optimum because it starts from a simple design space and evolves gradually to more sophisticated design domains. The number of design variables and comparison of the optimum objective functions are shown in Figure 17. The speed of improvement is linear but very steady.

It is noted that a cell removal strategy as studied in the literature [27] can be utilized but no such attempt is made here to focus on the idea of the newly developed method. One may argue that the same optimum can be obtained more efficiently by using the conventional fixed domain method, if one starts with a large design domain. Usually, however, how large is large enough as the proper design domain is not known, and so if one starts with too large a domain, the topology optimization may cost much more time than by the systematic approach proposed here. The selection of the initial design domain is up to the user, and the computational time depends heavily on this choice. Another important point is that it is yet

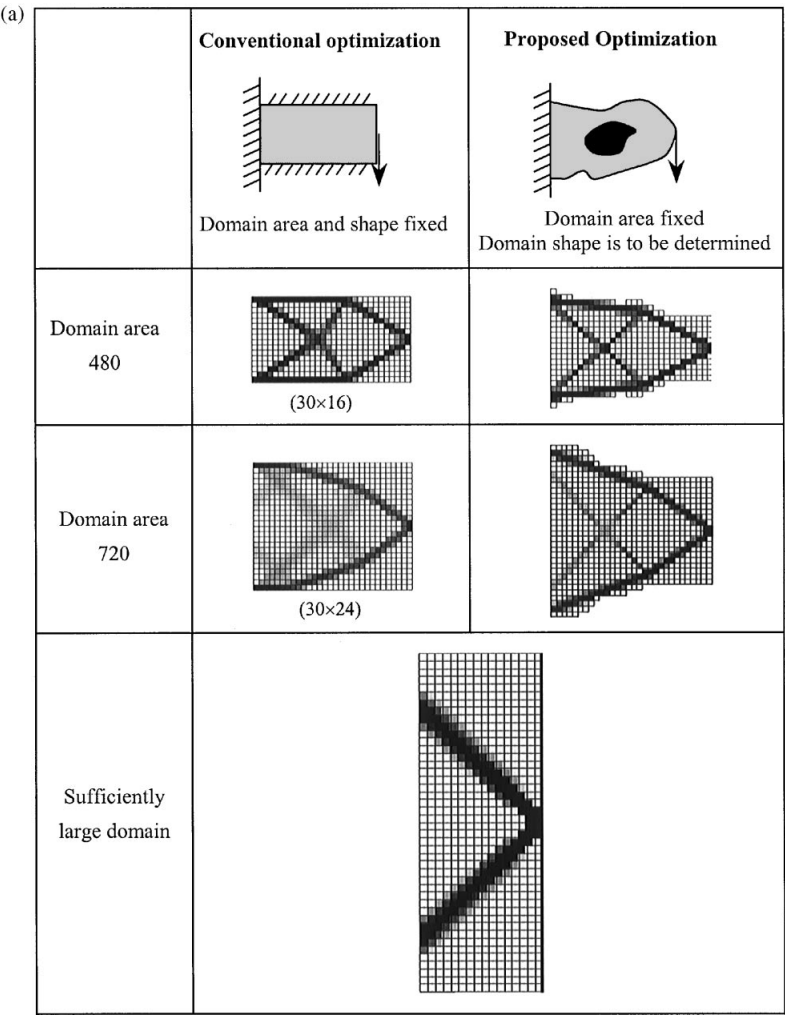


Figure 16. (a) Comparison of conventional optimization and proposed optimization according to design domain size; (b) comparison of conventional optimization and proposed optimization according to design domain aspect ratio.

(b)

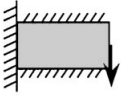
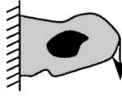
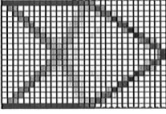
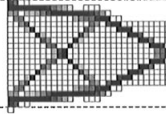
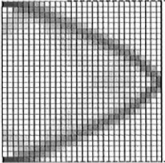
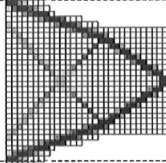
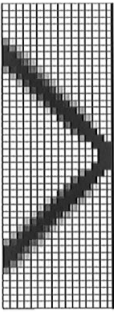
	Conventional optimization  Domain area and shape fixed	Proposed Optimization  Domain area fixed Domain shape is to be determined
Aspect ratio 30×20	 Obj = 779 domain area: 600 (30x20)	 Obj = 560 starting domain area: 360 (30x12) final domain area: 480
Aspect ratio 30×30	 Obj = 594 domain area: 900 (30x30)	 Obj = 369 starting domain area: 600 (30x20) final domain area: 720
Sufficiently large aspect ratio		

Figure 16. *Continued.*

not proven whether there is only one optimum topology for a given design domain or aspect ratio for the example. We have provided another method which can generate other optimum topology as illustrated by the example in Figure 16.

4.2. Plate thickness optimization

Figure 18 shows the geometry of the problem and the loads to be treated. The loading condition is a triangularly shaped pressure of 0.1 MPa. Young's modulus is $210 \times 10^9 \text{ N m}^{-2}$ and Poisson's ratio 0.3. Maximum thickness allowed is taken 0.012 m and minimum 0.005 m. The design patch used is 10×10 . The results for this fixed design patch layout are shown in Figure 19. Here, black patches denote the maximum thickness, and white patches the

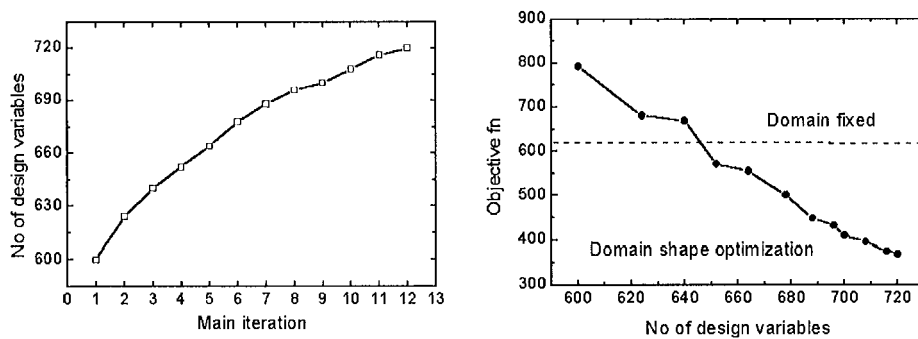


Figure 17. Number of design variables and history of optimum objective function for 30×24 design domain.

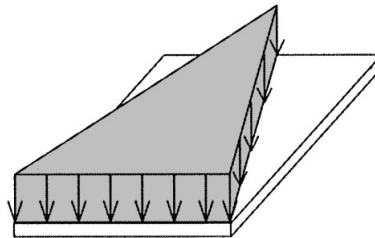


Figure 18. Loading condition of a plate (four edges are clamped).

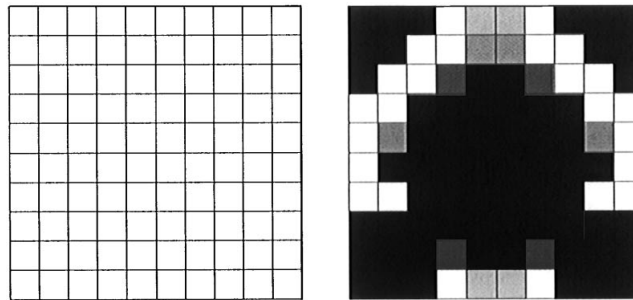


Figure 19. Optimum plate thickness distribution for fixed design patch layout.

minimum thickness. Figure 20 shows the change of design patch layout and corresponding thickness distribution for this loading case. Starting from a simple design patch layout of 5×5 , it becomes more elaborated. The final result is similar to the case of the fixed layout, but more refined in the region of thick structure boundary. The fixed domain consists of 100 design patches, but in this design adaptation, 79 design patches are used obtaining much more refined result. Figure 21(a) shows the increase of design patch numbers during main iterations: starting from 25 design patches, it reaches 79 design patches in 6 main iterations. The history of the objective function is shown and compared with that of the fixed patch layout problem in Figure 21(b). Because the design space optimization problem starts from

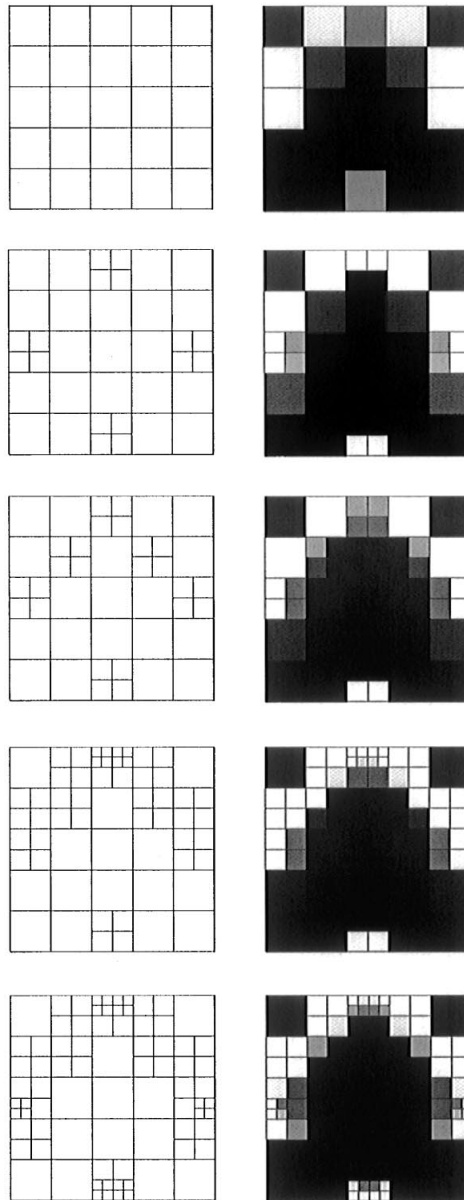


Figure 20. Results for design adaptation.

25 design patches, the objective function value is the same with that of the fixed patch layout problem with 25 patches. However, the final objective function value is lower even though the number of design patches used is less than 100.

Figure 22 shows optimum solutions for other loading conditions. For each case, optimum layout is obtained and compared with that of the fixed layout design with 100 design patches.

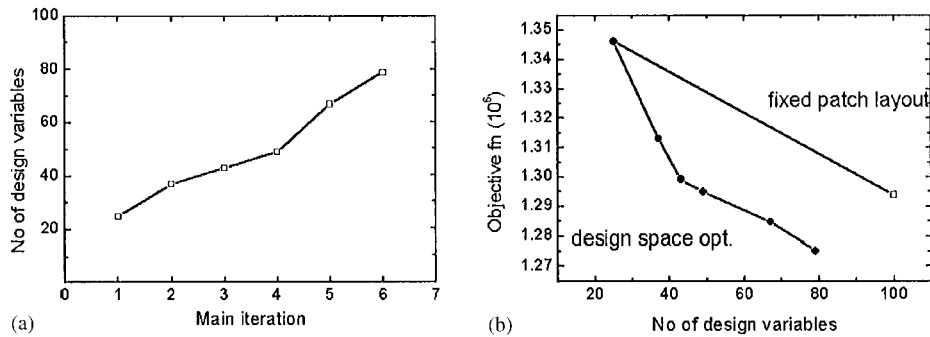


Figure 21. Number of design variables and history of optimum objective function: (a) number of design variables; (b) objective function history.

As expected, for every case, the value of the final objective function is less than that of the fixed layout problem.

In these examples, if sufficiently fine design patches are used, the same results will be obtained. But in this case, much larger number of design variables as well as finite elements must be used, which increases computational costs in two ways: high computational cost due to optimization and FEA. For example, if we refine the domain to a resolution the same as that of the proposed adaptation, the required memory is too large to deal with in a desktop computer due to the extremely large number of design variables. In addition, mesh adaptation of FEM can be used in the proposed method, which enables efficient use of finite element meshes. But it requires fully refined meshes of FEM if fixed fine design patches are to be used. Also, as in the topology problems, it may give a better chance of finding global optimum in the proposed method than with the fixed design patch layout.

5. CONCLUSIONS

A design space optimization problem is proposed and a solution method developed. The dimension of design variable space is unknown in this problem and to be obtained. The objective and constraint functionals are discrete functions in terms of the number of design variables. By introducing a pivot phase between an initial design and a new design with a different design space, continuity of the functionals has been established, and sensitivity analysis for the new design variables possible using directional derivatives.

This method is verified with two important categories of problems. In the general topology optimization, compared to the conventional fixed design space optimization where the optimum topology can only be a restricted one, the proposed design space optimization provides us with a new capability in obtaining better optimums for topology. The second category of application is plate thickness optimization, where the layout of design patches is adaptively optimized and the optimum thickness distribution elaborated.

Although the design space is enlarged in both of the two examples, their concepts are different: in topology optimization, the domain shape changes whereas design patch is elaborated in the plate optimization. It is possible to combine the two different schemes for both the categories. That is, in addition to shape change, design patch adaptation can be applied to the topology optimization. However, this is left as a future topic.

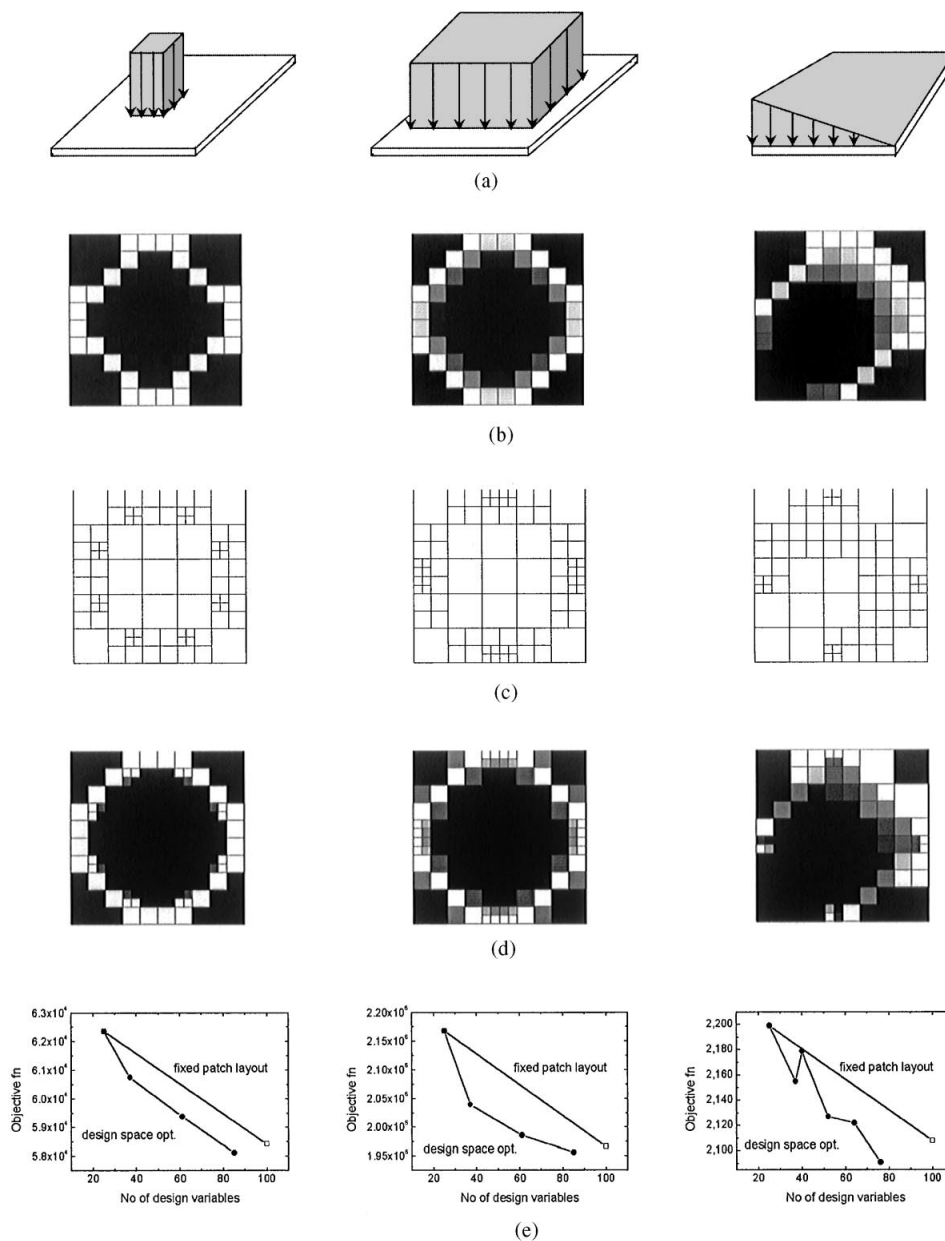


Figure 22. Several examples for plate optimization problem: (a) loading conditions; (b) results for fixed design patch layout (10×10); (c) optimum design patch layout for design adaptation; (d) results for design adaptation; (e) objective function history.

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