Chapter 5

DEVELOPING FRACTION CONCEPTS

For students in the upper elementary grades and even middle school, fractions present a considerable challenge. The area of fractions is where students often give up trying to understand and resort instead to rules. This lack of understanding is then translated into untold difficulties with fraction computation, decimal and percent concepts, the use of fractions in measurement, and ratio and proportion concepts.

Traditional programs for primary grades typically offer students limited exposure to fractions, with most of the work on fraction development occurring in the third grade. Few if any programs provide students with adequate time or experiences to help them with this complex area of the curriculum. This chapter will explore a conceptual development of fraction concepts that can help students construct a firm foundation, preparing them for the skills that are later built on these ideas.

big ideas

1. **Fractional parts are equal shares or equal-sized portions of a whole or unit. A unit can be an object or a collection of things. More abstractly, the unit is counted as 1. On the number line, the distance from 0 to 1 is the unit.**
2. **Fractional parts have special names that tell how many parts of that size are needed to make the whole. For example, *thirds* require three parts to make a whole.**
3. **The more fractional parts used to make a whole, the smaller the parts. For example, eighths are smaller than fifths.**
4. **The denominator of a fraction indicates by what number the whole has been divided in order to produce the type of part under consideration. Thus, the denominator is a divisor. In practical terms, the denominator names the kind of fractional part that is under consideration. The numerator of a fraction counts or tells how many of the fractional parts (of the type indicated by the denominator) are under consideration. Therefore, the numerator is a multiplier—it indicates a multiple of the given fractional part.**
5. **Two equivalent fractions are two ways of describing the same amount by using different-sized fractional parts. For example, in the fraction $\frac{6}{8}$, if the eighths are taken in twos, then each pair of eighths is a fourth. The six-eighths then can be seen to be three-fourths.**

Sharing and the Concept of Fractional Parts

The first goal in the development of fractions should be to help children construct the idea of *fractional parts of the* *whole*—the parts that result when the whole or unit has been partitioned into *equal-sized portions* or *fair shares.*

Children seem to understand the idea of separating a quantity into two or more parts to be shared fairly among friends. They eventually make connections between the idea of fair shares and fractional parts. Sharing tasks are, therefore, good places to begin the development of fractions.

Sharing Tasks

Considerable research has been done with children from first through eighth grades to determine how they go about the process of forming fair shares and how the tasks posed to students influence their responses (e.g., Empson, 2002; Lamon, 1996; Mack, 2001; Pothier & Sawada, 1983).

Sharing tasks are generally posed in the form of a simple story problem. *Suppose there are four square brownies to be shared among three children so that each child gets the same amount. How much (or show how much) will each child get?* Task difficulty changes with the numbers involved, the types of things to be shared (regions such as brownies, discrete objects such as pieces of chewing gum), and the presence or use of a model.

Students initially perform sharing tasks (division) by distributing items one at a time. When this process leaves leftover pieces, it is much easier to think of sharing them fairly if the items can be subdivided. Typical “regions” to share are brownies (rectangles), sandwiches, pizzas, crackers, cake, candy bars, and so on. The problems and variations that follow are adapted from Empson (2002).

Four children are sharing 10 brownies so that each one will get the same amount. How much can each child have?

Problem difficulty is determined by the relationship between the number of things to be shared and the number of sharers. Because students’ initial strategies for sharing involve halving, a good place to begin is with two, four, or even eight sharers. For ten brownies and four sharers, many children will deal out two to each child and then halve each of the remaining brownies. (See Figure 5.1.)

Consider these variations in numbers:

- 5 brownies shared with 2 children
- 2 brownies shared with 4 children
- 5 brownies shared with 4 children
- 4 brownies shared with 8 children
- 3 brownies shared with 4 children

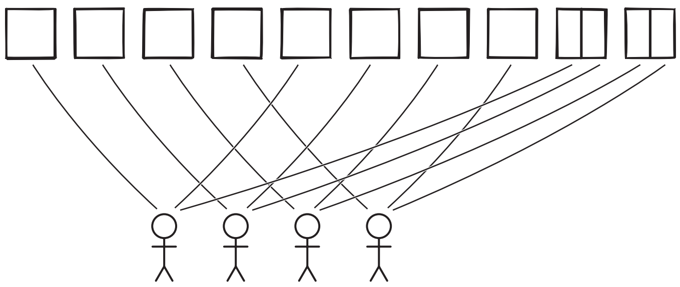


FIGURE 5.1 Ten brownies shared with four students.

Try drawing pictures for each of the preceding sharing tasks. Which do you think is most difficult? Which of these represents essentially the same degree of difficulty? What other tasks involving two, four, or eight sharers would you consider as similar, easier, or more difficult than these?

When the numbers allow for some items to be distributed whole (five shared with two), some students will first share whole items and then cut up the leftovers. Others will slice every piece in half and then distribute the halves. When there are more sharers than items, some partitioning must happen at the beginning of the solution process.

When students who are still using a halving strategy try to share five things among four children, they will eventually get down to two halves to give to four children. For some, the solution is to cut each half in half; that is, "each child gets a whole (or two halves) and a half of a half."

It is a progression to move to three or six sharers because this will force students to confront their halving strategies.

Try solving the following variations using drawings. Can you do them in different ways?

4 pizzas shared with 6 children

7 pizzas shared with 6 children

5 pizzas shared with 3 children

To subdivide a region into a number of parts other than a power of two (four, eight, etc.) requires an odd subdivision at some point. Several types of sharing solutions might be observed. Figure 5.2 shows some different approaches.

Use a variety of representations for these problems. The items to be shared can be drawn on worksheets as rectangles or circles along with a statement of the problem. Another possibility is to cut out construction paper circles or squares. Some students may need to cut and physically distribute the pieces. Students can use connecting cubes to make bars that they can separate into pieces. Or they can use more traditional fraction models such as circular "pie" pieces.

Models for Fractions

There is substantial evidence to suggest that the use of models in fraction tasks is important. Unfortunately, many teachers in the upper grades, where manipulative materials are not as common, fail to use models for fraction development. Models can help

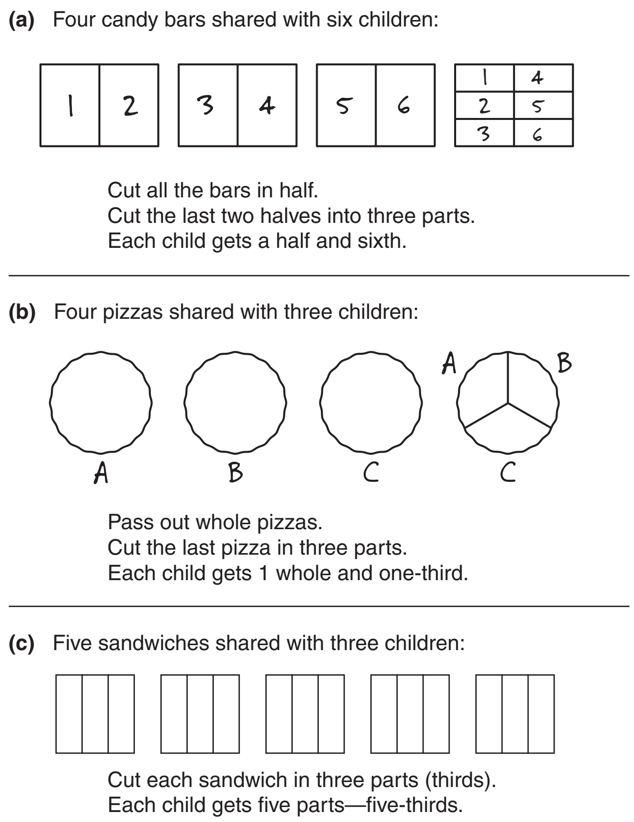


FIGURE 5.2

Three different sharing processes.

students clarify ideas that are often confused in a purely symbolic mode. Sometimes it is useful to do the same activity with two quite different models; from the viewpoint of the students, the activity is quite different. In this chapter we will distinguish among three types of models: area or region models, length models, and set models.

Region or Area Models

In the discussion of sharing, all of the tasks involved sharing something that could be cut into smaller parts. The fractions are based on parts of an area or region. This is a good place to begin and is almost essential when doing sharing tasks. There are many good region models, as shown in Figure 5.3.

Circular "pie" piece models are by far the most commonly used area model. (See the Blackline Masters for masters of pie models.) The main advantage of the circular region is that it emphasizes the amount that is remaining to make up a whole. The other models in Figure 5.3 are more flexible and allow for different-sized units or wholes. Paper grids, several of which can be found in the Blackline Masters, are especially flexible and do not require management of materials.

Length or Measurement Models

With measurement models, lengths are compared instead of areas. Either lines are drawn and subdivided or physical materials are compared on the basis of length, as shown in Figure 5.4. Manipulative versions provide more opportunity for trial and error and for exploration.

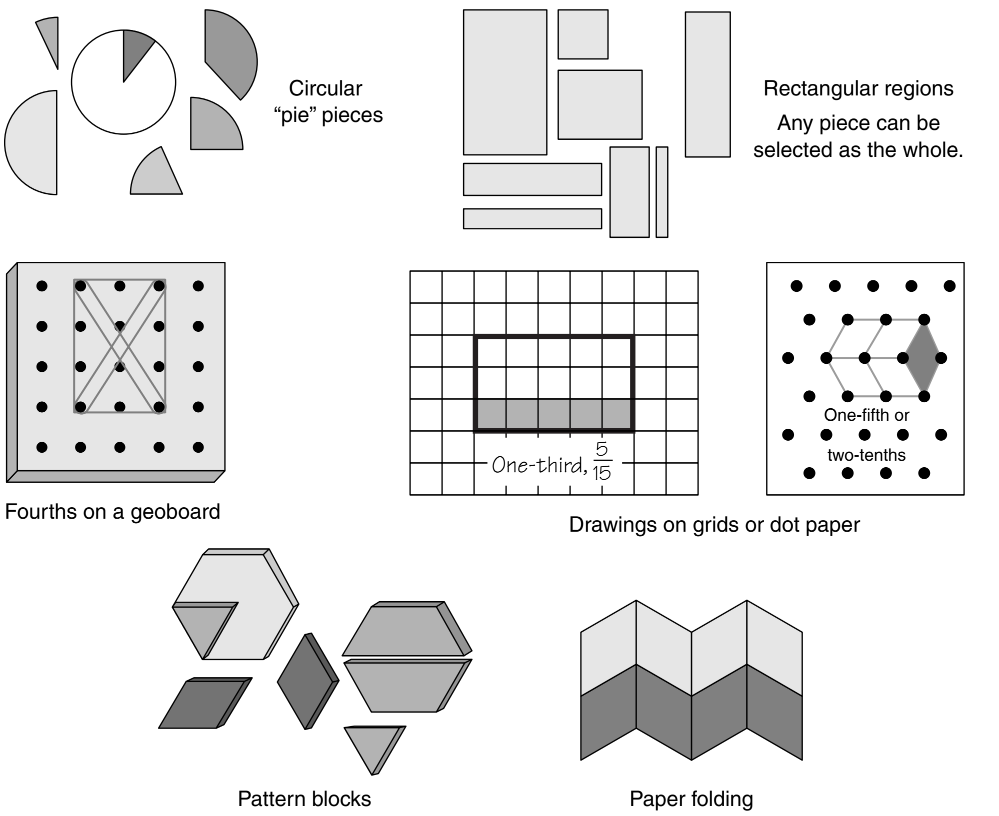


FIGURE 5.3

Area or region models for fractions.

Fraction strips are a teacher-made version of Cuisenaire rods. Both the strips and the rods have pieces that are in lengths of 1 to 10 measured in terms of the smallest strip or rod. Each length is a different color for ease of identification. Strips of construction paper or adding-machine tape can be folded to produce equal-sized subparts.

The rod or strip model provides the most flexibility while still having separate pieces for comparisons. To make fraction strips, cut 11 different colors of poster board into strips 2 cm wide. Cut the smallest strips into 2-cm squares. Other strips are then 4, 6, 8, . . . , 20 cm, producing lengths 1 to 10 in terms of the smallest strip. Cut the last color into strips 24 cm long to produce a 12 strip. If you are using Cuisenaire rods, tape a red 2 rod to an orange 10 rod to make a 12 rod. In this chapter's illustrations, the colors of the strips will be the same as the corresponding lengths of the Cuisenaire rods:

1 White	7 Black
2 Red	8 Brown
3 Light green	9 Blue
4 Purple	10 Orange
5 Yellow	12 Pink or red-orange
6 Dark green	

The number line is a significantly more sophisticated measurement model. From a student's vantage point, there is a real difference between putting a number on a number line and comparing one length to another. Each number on a line denotes the distance of the labeled point from zero, not the point itself. This distinction is often difficult for students.

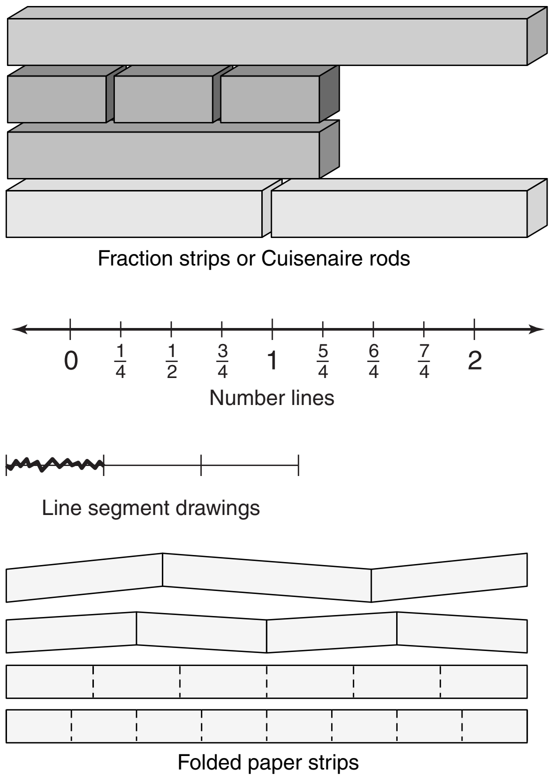


FIGURE 5.4

Length or measurement models for fractions.

Set Models

In set models, the whole is understood to be a set of objects, and subsets of the whole make up fractional parts. For example, three objects are one-fourth of a set of 12 objects. The set of 12, in this example, represents the whole or 1. It is the idea of referring to a collection of counters as a single entity that makes set models difficult for some students. However, the set model helps establish important connections with many real-world uses of fractions and with ratio concepts. Figure 5.5 illustrates several set models for fractions.

Counters in two colors on opposite sides are frequently used. They can easily be flipped to change their color to model various fractional parts of a whole set.

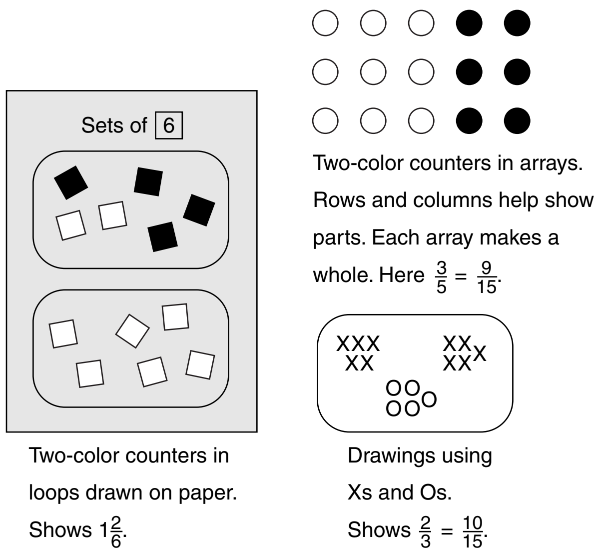


FIGURE 5.5

Set models for fractions.

From Fractional Parts to Fraction Symbols

During the discussions of students' solutions (and discussions are essential!) is a good time to emphasize the vocabulary of fractional parts. Students need to be aware of two aspects or components of fractional parts: (1) the number of parts and (2) the equality of the parts (in size, not necessarily in shape). Emphasize that the number of equal parts or fair shares that make up a whole determines the name of the fractional parts or shares. One of the best ways to introduce the concept of fractional parts is through sharing tasks. However, the idea of fractional parts is so fundamental to a strong development of fraction concepts that it should be explored further with additional tasks.

Fractional Parts and Words

In addition to helping students use the words *halves, thirds, fourths, fifths,* and so on, be sure to make regular comparison of fractional parts to the whole. Make it a point to use the terms *whole,* or *one whole,* or simply *one* so that students have a language that they can use regardless of the model involved.

The following activity is a simple extension of the sharing tasks. It is important that students can tell when a region has been separated into a particular type of fractional part.

ACTIVITY 5.1

Correct Shares

As in Figure 5.6, show examples and nonexamples of specified fractional parts. Have students identify the wholes that are correctly divided into requested fractional parts and those that are not. For each response, have students explain their reasoning. The activity should be done with a variety of models, including length and set models.

In the "Correct Shares" activity, the most important part is the discussion of the nonexamples. The wholes are already partitioned either correctly or incorrectly, and the students were not involved in the partitioning. It is also useful for students to create designated equal shares given a whole, as they are asked to do in the next activity.

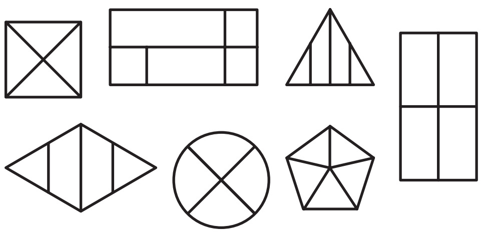


FIGURE 5.6

Students learning about fractional parts should be able to tell which of these figures are correctly partitioned in fourths. They should also be able to explain why the other figures are not showing fourths.

ACTIVITY 5.2

Finding Fair Shares

Give students models, and have them find fifths or eighths or other fractional parts using the models. (The models should never have fractions written on them.) The activity is especially interesting when different wholes can be designated in the same model. That way, a given fractional part does not get identified with a special shape or color but with the relationship of the part to the designated whole. Some ideas are suggested in Figure 5.7.

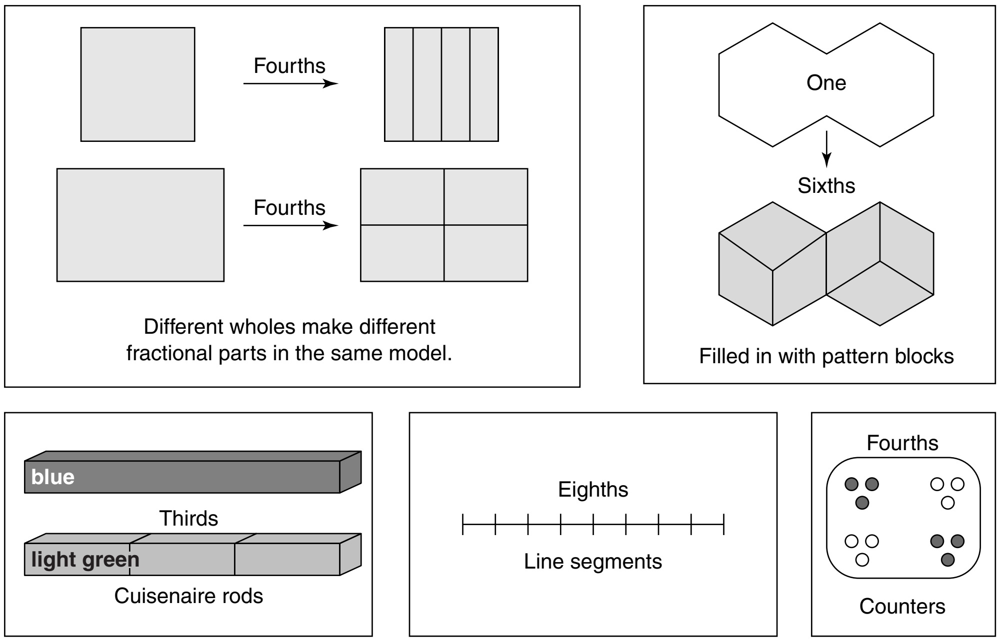


FIGURE 5.7

Given a whole, find fractional parts.

Notice when partitioning sets that children frequently confuse the number of counters in a share with the name of the share. In the example in Figure 5.7, the 12 counters are partitioned into four sets—*fourths*. Each share or part has three counters, but it is the number of shares that makes the partition show *fourths*.

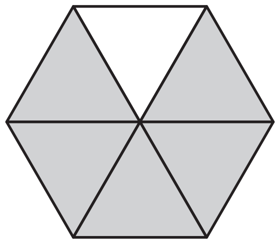
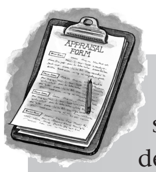


FIGURE 5.8

A student represents $\frac{5}{6}$ using pattern blocks.



Assessment Note

Many of the activities in this chapter suggest the use of various models such as fraction strips, pie pieces, and pattern blocks. Suppose that a student represents $\frac{5}{6}$ using pattern blocks as in Figure 5.8. Based on this representation, you might be inclined to think that this student has a good grasp of the two components of fractional parts (number of parts and equality of parts). Now consider this same student's work in which he was asked to draw a picture of $\frac{5}{6}$ and $\frac{5}{9}$. (See Figure 5.9.) Now what are you inclined to think? The student appears to understand the component about the number of parts; however, he does not seem to understand the necessity for the parts to be equal. With the pattern blocks, the idea of equal parts is never an issue. It's not until the student is asked to *draw* a fraction representation that it becomes apparent that the notion of equal parts has not been incorporated into his emerging understanding of fractions.

Although their own drawings can sometimes mislead students, drawings provide opportunities for you to assess students' understanding. At the very least, drawings provide opportunities for you to ask students questions about their ideas to gain insight into how they are making sense of fractions. Be careful to distinguish between incorrect drawings that are due to weak drawing skills and those that are the result of mistaken ideas. Help make students' ideas public by encouraging the use of a variety of models—various physical materials as well as student-made drawings.

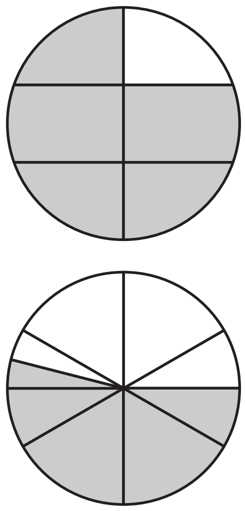


FIGURE 5.9

A student represents $\frac{5}{6}$ and $\frac{5}{9}$ by partitioning circles.

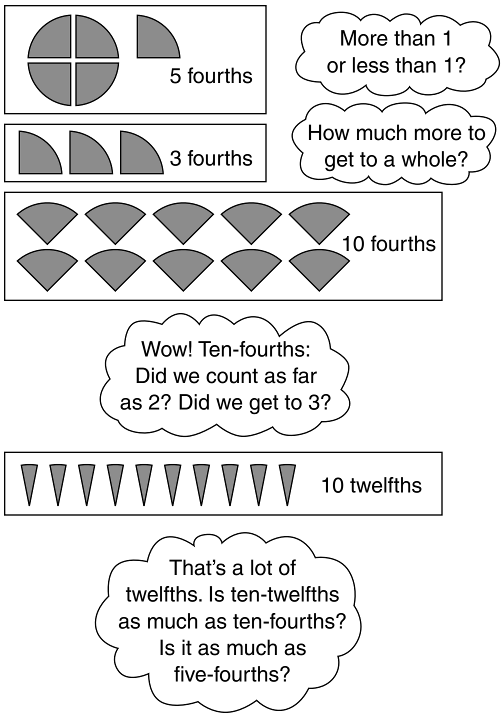


FIGURE 5.10

Counting fractional parts.

Understanding Fraction Symbols

Fraction symbolism represents a fairly complex convention that is often misleading to children. It is well worth your time to help students develop a strong understanding of what the top and bottom numbers of a fraction tell us.

Fractional-Parts Counting

Counting fractional parts to see how multiple parts compare to the whole creates a foundation for the two parts of a fraction. Students should come to think of counting fractional parts in much the same way as they might count apples or any other objects. If you know the kind of part you are counting, you can tell when you get to one, when you get to two, and so on. Students who understand factional parts should not need to arrange pie pieces into a circle to know that four-fourths make a whole.

Display some pie-piece fraction parts in groups as shown in Figure 5.10. For each collection, tell students what type of piece is being shown and simply count them together: "*one*-fourth, *two*-fourths, *three*-fourths, *four*-fourths, *five*-fourths." Ask, "If we have five-fourths, is that more than one whole, less than one whole, or the same as one whole?"

As students count each collection of parts, discuss the relationship to one whole. Make informal comparisons between different collections. "Why did we get almost two wholes with seven-fourths, and yet we don't even have one whole with ten-twelfths?"

Also take this opportunity to lay verbal groundwork for mixed fractions. "What is another way that we could say seven-thirds?" (Two wholes and one more third or one whole and four-thirds.)

With this introduction, students are ready for the following task.

ACTIVITY 5.3

More, Less, or Equal to One Whole

Give students a collection of fractional parts (all the same type) and indicate the kind of fractional part they have. Parts can be drawn on a worksheet or physical models can be placed in plastic baggies with an identifying card. For example, if done with Cuisenaire rods or fraction strips, the collection might have seven light green rods/strips with a caption or note indicating "these are eighths." The task is to decide if the collection is less than one whole, equal to one whole, or more than one whole. Students must draw pictures and/or use numbers to explain their answer. They can also tell how close the set is to a complete whole. Several collections constitute a reasonable task.

Try Activity 5.3 with several different fraction models (although pie pieces are too much of a giveaway). Pattern blocks make a good manipulative format and are also easily drawn with a template. The same is true of Cuisenaire rods. A set model may cause students some initial difficulty but it is especially important, even if they have been successful with region or length models. For example, show a collection of 15 counters (dots or actual counters) and indicate that a set of 5 counters is one-fourth. How much is the set of 15 counters?

Top and Bottom Numbers

The way that we write fractions with a top and a bottom number and a bar between is a convention—an arbitrary agreement for how to represent fractions. (By the way, always write fractions with a horizontal bar, not a slanted one. Write $\frac{3}{4}$, not 3/4.) As a convention, it falls in the category of things that you simply tell students. However, a good idea is to make the convention so clear by way of demonstration that students will tell *you* what the top and bottom numbers stand for. The following procedure is recommended even if your students have been "using" symbolic fractions for several years.

Display several collections of fractional parts in a manner similar to those in Figure 5.10. Have students count the parts together. After each count, write the correct fraction, indicating that this is how it is written as a symbol. Include sets that are more than one but write them as simple or "improper" fractions and not as mixed numbers. Include at least two pairs of sets with the same top numbers such as $\frac{4}{8}$ and $\frac{4}{3}$. Likewise, include sets with the same bottom numbers. After the class has counted and you have written the fraction for at least six sets of fractional parts, pose the following questions:

- What does the bottom number in a fraction tell us?
- What does the top number in a fraction tell us?

STOP **Before reading further, answer these two questions in your own words. Don't rely on formulations you've heard before. Think in terms of what we have been talking about—namely, fractional parts and counting fractional parts. Imagine counting a set of 5 eighths and a set of 5 fourths and writing the fractions for these sets. Use children's language in your formulations and try to come up with a way to explain these meanings that has nothing to do with the type of model involved.**

Here are some reasonable explanations for the top and bottom numbers.

- *Top number:* This is the counting number. It tells how many shares or parts we have. It tells how many have been counted. It tells how many parts we are talking about. It counts the parts or shares.
- *Bottom number:* This tells what is being counted. It tells what fractional part is being counted. If it is a 4, it means we are counting *fourths;* if it is a 6, we are counting *sixths;* and so on.

This formulation of the meanings of the top and bottom numbers may seem unusual to you. It is often said that the top number tells "how many." (This phrase seems unfinished. How many *what?*) And the bottom tells "how many parts it takes to make a whole." This may be correct but can be misleading. For example, a $\frac{1}{6}$ piece is often cut from a cake without making any slices in the remaining $\frac{5}{6}$ of the cake. That the cake is only in two pieces does not change the fact that the piece taken is $\frac{1}{6}$. Or if a pizza is cut in 12 pieces, two pieces still make $\frac{1}{6}$ of the pizza. In neither of these instances does the bottom number tell how many pieces make a whole.

There is evidence that an iterative notion of fractions, one that views a fraction such as $\frac{3}{4}$ as a count of three things called *fourths,* is an important idea for children to develop. The iterative concept is most clear when focusing on these two ideas about fraction symbols:

- The top number *counts.*
- The bottom number tells *what is being counted.*

The *what* of fractions are the fractional parts. They can be counted. Fraction symbols are just a shorthand for saying *how many* and *what.*

Smith (2002) points out a slightly more "mathematical" definition of the top and bottom numbers that is completely in accord with the one we've just discussed. For Smith, it is important to see the bottom number as the divisor and the top as the multiplier. That is, $\frac{3}{4}$ is three *times* what you get when you *divide* a whole into four parts. This multiplier and divisor idea is especially useful when students are asked later to think of fractions as an indicated division; that is, $\frac{3}{4}$ also means $3 \div 4$.

Numerator and Denominator

To count a set is to *enumerate* it. The common name for the top number in a fraction is the *numerator.*

A denomination is the name of a class or type of thing. A $1 bill, a $5 bill, and a $10 bill are said to be bills of different *denominations.* The common name for the bottom number in a fraction is the *denominator.*

The words *numerator* and *denominator* have no common reference for children. Whether these words are used or not, the words themselves will not help young students understand the meanings.

Mixed Numbers and Improper Fractions

If you have counted fractional parts beyond a whole, your students already know how to write $\frac{13}{16}$ or $\frac{13}{3}$. Ask, "What is another way that you could say 13 *sixths?*" Students may suggest "two wholes and one-sixth more," or "two plus one-sixth." Explain that these are correct and that $2 + \frac{1}{6}$ is usually written as $2\frac{1}{6}$ and is called a *mixed number.* Note that this is a symbolism convention and must be explained to students. What is not at all necessary is to teach a rule for converting mixed numbers to common fractions and the reverse. Rather, consider the following task.

ACTIVITY 5.4

Mixed-Number Names

Give students a mixed number such as $3\frac{2}{5}$. Their task is to find a single fraction that names the same amount. They may use any familiar materials or make drawings, but they must be able to give an explanation for their result. Similarly, have students start with a fraction greater than 1, such as $\frac{17}{4}$, and have them determine the mixed number and provide a justification for their result.

Repeat the "Mixed-Number Names" task several times with different fractions. After a while, challenge students to figure out the new fraction name without the use of models. A good explanation for $3\frac{1}{4}$ might be that there are 4 fourths in one whole, so there are 8 fourths in two wholes and 12 fourths in three wholes. The extra fourth makes 13 fourths in all, or $\frac{13}{4}$. (Note the iteration concept playing a role.)

There is absolutely no reason ever to provide a rule about multiplying the whole number by the bottom number and adding the top number. Nor should students need a rule about dividing the bottom number into the top to convert fractions to mixed numbers. These rules will readily be developed by the students but in their own words and with complete understanding.

ACTIVITY 5.5

Calculator Fraction Counting

Calculators that permit fraction entries and displays are now quite common in schools. Many, like the TI-15, now display fractions in correct fraction format and offer a choice of showing results as mixed numbers or simple fractions. Counting by fourths with the TI-15 is done by first storing $\frac{1}{4}$ in one of the two operation keys: [Op1] [+] 1 [n] 4 [d] [Op1]. To count, press 0 [Op1] [Op1] [Op1] . . . . The display will show the counts by fourths and also the number of times that the [Op1] key has been pressed. Students should coordinate their counts with fraction models, adding a new fourths piece to the pile with each count. At any time the display can be shifted from mixed form to simple fractions with a press of a key. The TI-15 can be set so that it will not simplify fractions automatically, the appropriate setting prior to the introduction of equivalent fractions.

Fraction calculators provide a powerful way to help children develop fractional symbolism. A variation on Activity 5.5 is to show students a mixed number such as $3\frac{1}{8}$ and ask how many counts of $\frac{1}{8}$ on the calculator it will take to count that high. The students should try to stop at the correct number $\frac{25}{8}$ before pressing the mixed number key.

Parts-and-Whole Tasks

The exercises presented here can help students develop their understanding of fractional parts as well as the meanings of the top and bottom numbers in a fraction. Models are used to represent wholes and parts of wholes. Written or oral fraction names represent the relationship between the parts and wholes. Given any two of these—whole, part, and fraction—the students can use their models to determine the third.

Any type of model can be used as long as different sizes can represent the whole. Traditional pie pieces do not work because the whole is always the circle, and all the pieces are *unit fractions*. (A *unit fraction* is a single fractional part. The fractions $\frac{1}{3}$ and $\frac{1}{8}$ are unit fractions.)

Examples of each type of exercise are provided in Figure 5.11, Figure 5.12, and Figure 5.13. Each figure

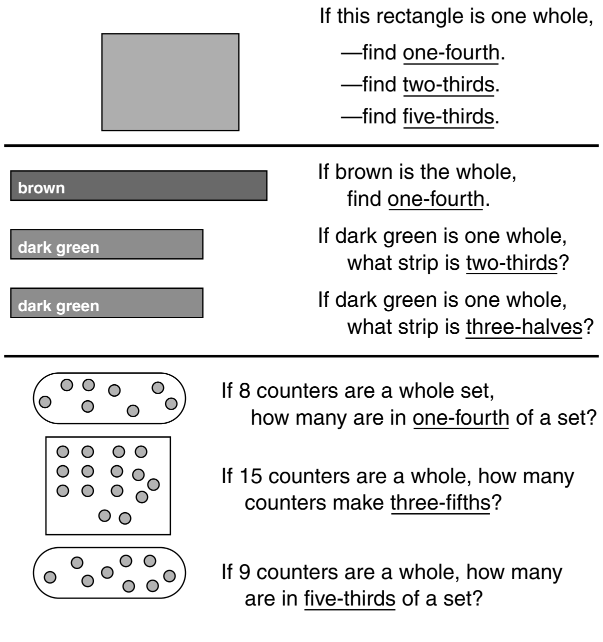


FIGURE 5.11

Given the whole and the fraction, find the part.

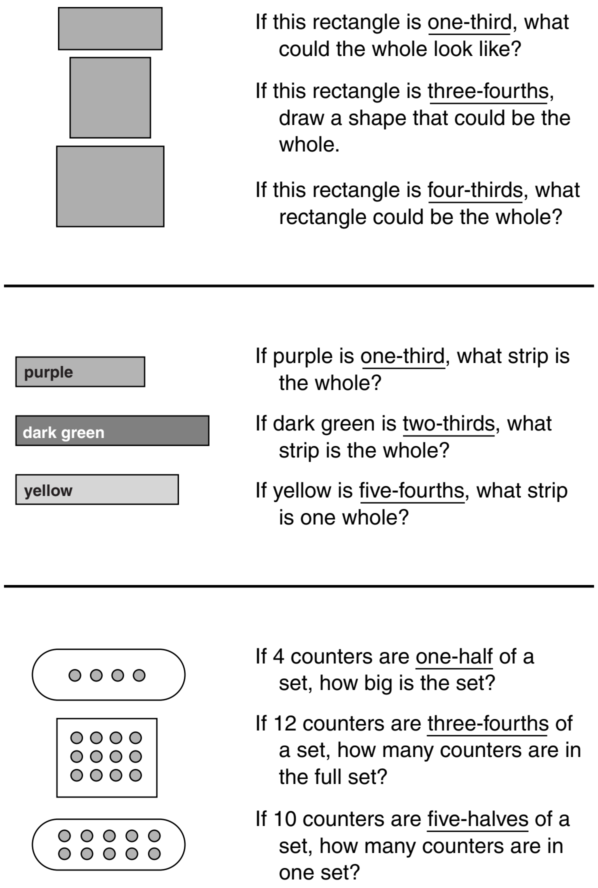


FIGURE 5.12

Given the part and the fraction, find the whole.

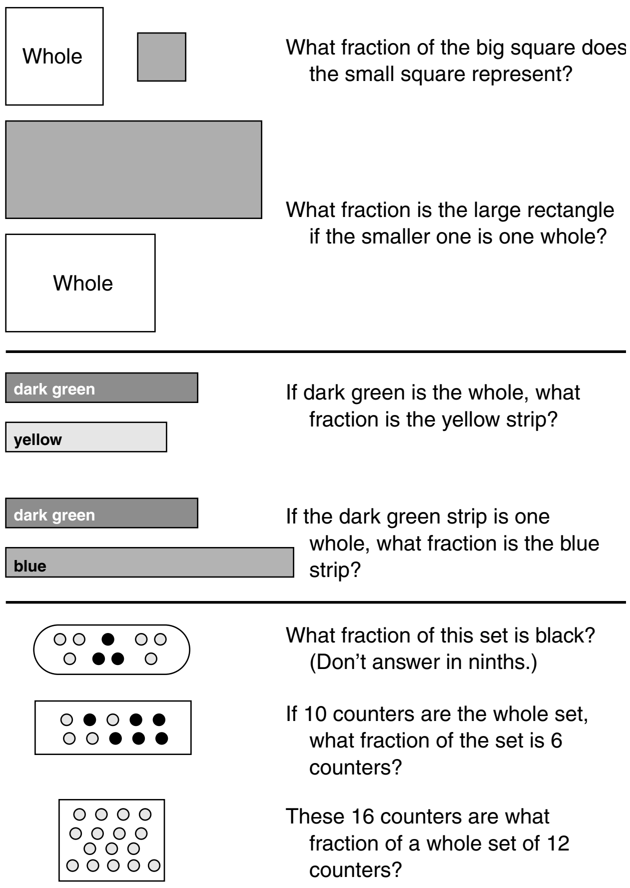


FIGURE 5.13

Given the whole and the part, find the fraction.

includes examples with a region model (freely drawn rectangles), a length model (Cuisenaire rods or fraction strips), and set models.

STOP **It would be a good idea to work through these exercises before reading on. For the rectangle models, simply sketch a similar rectangle on paper. For the rod or strip models, use Cuisenaire rods or make fraction strips. The colors used correspond to the actual rod colors. Lengths are not given in the figures so that you will not be tempted to use an adult-type numeric approach. If you do not have access to rods or strips, just draw lines on paper. The process you use with lines will correspond to what is done with rods.**

These three types of problems vary in difficulty as well as in what they can help students learn. The first type, in which students find the part given the whole and fraction (Figure 5.11), is commonly encountered in textbooks. What may make it different is that the given whole is not partitioned at all. Students must know that the denominator will tell them how to partition the whole—it is the divisor. The numerator counts. Therefore, once partitioned, they count the necessary number of fractional parts. Notice that you can ask for students to show a fraction that is more than a whole even though only one whole is provided. Usually students will create a second whole and partition that as well.

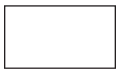
In the second type of task, students are asked to find or create the whole given a part of the whole. (See Figure 5.12.) Students will find this task a bit more difficult than the first. The struggle and discussion among students will be worth the effort. This exercise emphasizes that a fraction is not an absolute quantity; rather it is a relationship between the part and the whole. If the white strip is given as $\frac{1}{4}$, then the purple strip is the whole (see the lengths of the Cuisenaire rods on p. 135). However, if the red strip is given as $\frac{1}{4}$, then the brown strip is the whole. Furthermore, if the white strip is given as $\frac{1}{5}$, then the yellow is the whole. When the given part is not a unit fraction, the task is considerably more difficult. For the second example in Figure 5.12, students first must realize that the given rectangle is three of something called *fourths*. Therefore, if that given piece is subdivided into three parts, then one of those parts will be a fourth. From the unit fraction, counting produces the whole—four of the one-fourth pieces make a whole. Notice again how the task forces students to think of counting unit fractional parts.

The third type of exercise will likely involve some estimation, especially if drawings are used. Different estimates can prompt excellent discussion. With Cuisenaire rods or sets, one specific answer is always correct.

Two or three challenging parts-and-whole questions can make an excellent lesson. The tasks should be presented to the class in just the same form as in the figures. Physical models are often the best way to present the tasks so that students can use a trial-and-error approach to determine their results. As with all tasks, it should be clear that an explanation is required to justify each answer. For each task, let several students supply answers and explanations.

Sometimes it is a good idea to create simple story problems that ask the same questions.

Mr. Samuels has finished $\frac{2}{5}$ of his patio. It looks like this:



Draw a picture that might be the shape of the finished patio.

The problems can also involve numbers instead of models:

If the swim team sold 400 raffle tickets, it would have enough money to pay for new team shirts. So far the swimmers have $\frac{5}{8}$ of the necessary raffle tickets sold. How many more tickets do they need to sell?

With some models, it is necessary to be certain that the answer exists within the model. For example, if you were using fraction strips, you could ask, "If the blue strip (9) is the whole, what strip is two-thirds?" The answer is the 6 strip, or dark green. You could not ask students to find "three-fourths of the blue strip" because each fourth of 9 would be $2\frac{1}{4}$ units, and no strip has that length. Similar caution must be taken with rectangular pieces.

Questions involving unit fractions are generally the easiest. The hardest questions usually involve fractions greater than 1. For example, *If 15 chips are five-thirds of one* *whole set, how many chips are in a whole?* However, in every question, the unit fraction plays a significant role. If you have $\frac{5}{3}$ and want the whole, you first need to find $\frac{1}{3}$.

Avoid being the answer book for your students. Make students responsible for determining the validity of their own answers. In these exercises, the results can always be confirmed in terms of what is given.

It is good to periodically place fraction activities into a context. Context encourages students to explore ideas in a more open and informal manner and not to overly depend on rules. The way that children approach fraction concepts in these contexts may surprise you. The following activity uses literature to provide an excellent context for discussing fractional parts of sets and how fractional parts change as the whole changes.

ACTIVITY 5.6

Sharing Camels

As a class, read the story "Beasts of Burden" in the book *The Man Who Counted: A Collection of Mathematical Adventures* (Tahan, 1993). This story is about a wise mathematician, Beremiz, and the narrator, who are traveling together on one camel. They are asked by three brothers to solve an argument. Their father has left them 35 camels to divide among them in this way: one-half to one brother, one-third to another, and one-ninth to the third. Have students grapple with this situation to try to come up with a solution. Make sure to discuss students' approaches and their conjectures before changing the number of camels. Try to choose the number of camels based on students' conjectures so they have an opportunity to test their hunches. For example, if students claim that they cannot divide an odd number of camels (e.g., 35) in half, they may state that the starting number has to be even. So start with an even number of camels, say 34 or 36. Or students may claim that the starting number must be divisible by three because there are three brothers. In this case, start with a number such as 33. Students should share what they think they have discovered as each number is tested. No matter how many camels are involved, the problem of the indicated shares cannot be resolved. (Why does this happen?)

STOP **Before going further, try the preceding activity. What do you discover as you test various numbers? Why can you not find a number that will work?**

The problem of the indicated shares cannot be resolved because the sum of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{9}$ will never be one whole. No matter how many camels are involved, there will always be some "left over." Bresser (1995) describes three full days of wonderful discussions with his fifth graders, who proposed a wide range of solutions. Bresser's suggestions are worth considering.

Fraction Number Sense

The focus on fractional parts is an important beginning. But number sense with fractions demands more—it requires that students have some intuitive feel for fractions. They should know "about" how big a particular fraction is and be able to tell easily which of two fractions is larger.

Benchmarks of Zero, One-Half, and One

The most important reference points or benchmarks for fractions are 0, $\frac{1}{2}$, and 1. For fractions less than 1, simply comparing them to these three numbers gives quite a lot of information. For example, $\frac{3}{20}$ is small, close to 0, whereas $\frac{3}{4}$ is between $\frac{1}{2}$ and 1. The fraction $\frac{9}{10}$ is quite close to 1. Since any fraction greater than 1 is a whole number plus an amount less than 1, the same reference points are just as helpful: $3\frac{3}{7}$ is almost $3\frac{1}{2}$.

ACTIVITY 5.7

Zero, One-Half, or One

On the board or overhead, write a collection of 10 to 15 fractions. A few should be greater than 1 ($\frac{9}{8}$ or $\frac{11}{10}$), with the others ranging from 0 to 1. Let students sort the fractions into three groups: those close to 0, close to $\frac{1}{2}$, and close to 1. For those close to $\frac{1}{2}$, have them decide if the fraction is more or less than $\frac{1}{2}$. The difficulty of this task largely depends on the fractions. The first time you try this, use fractions such as $\frac{1}{20}$, $\frac{53}{100}$, or $\frac{9}{10}$ that are very close to the three benchmarks. On subsequent days, use fractions with most of the denominators less than 20. You might include one or two fractions such as $\frac{2}{8}$ or $\frac{3}{4}$ that are exactly in between the benchmarks. As usual, require explanations for each fraction.

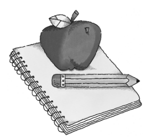
The next activity is also aimed at developing the same three reference points for fractions. In "Close Fractions," however, the students must come up with the fractions rather than sort them.

ACTIVITY 5.8

Close Fractions

Have your students name a fraction that is close to 1 but not more than 1. Next have them name another fraction that is even closer to 1 than that. For the second response, they have to explain why they believe the fraction is closer to 1 than the previous fraction. Continue for several fractions in the same manner, each one being closer to 1 than the previous fraction. Similarly, try close to 0 or close to $\frac{1}{2}$ (either under or over). The first several times you try this activity, let the students use models to help with their thinking. Later, see how well their explanations work when they cannot use models or drawings. Focus discussions on the relative size of fractional parts.

Understanding why a fraction is close to 0, $\frac{1}{2}$, or 1 is a good beginning for fraction number sense. It begins to focus on the size of fractions in an important yet simple manner. The next activity also helps students reflect on fraction size.



EXPANDED LESSON

(pages 158–159)

A complete lesson plan based on "About How Much?" can be found at the end of this chapter.

ACTIVITY 5.9

About How Much?

Draw a picture like one of those in Figure 5.14 (or prepare some ahead of time for the overhead). Have each student write down a fraction that he or

(continued)

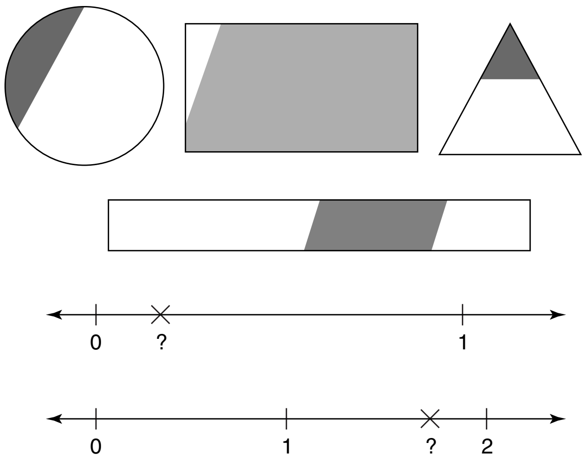


FIGURE 5.14

About how much? Name a fraction for each drawing and explain why you chose that fraction.

she thinks is a good estimate of the amount shown (or the indicated mark on the number line). Listen without judgment to the ideas of several students and discuss with them why any particular estimate might be a good one. There is no single correct answer, but estimates should be "in the ballpark." If students have difficulty coming up with an estimate, ask if they think the amount is closer to 0, $\frac{1}{2}$, or 1.

Thinking About Which Is More

The ability to tell which of two fractions is greater is another aspect of number sense with fractions. That ability is built around concepts of fractions, not on an algorithmic skill or symbolic tricks.

Concepts, Not Rules

Students have a tremendously strong mind-set about numbers that causes them difficulties with the relative size of fractions. In their experience, larger numbers mean "more." The tendency is to transfer this whole-number concept to fractions: Seven is more than four, so sevenths should be bigger than fourths. The inverse relationship between number of parts and size of parts cannot be told but must be a creation of each student's own thought process.

ACTIVITY 5.10

Ordering Unit Fractions

List a set of unit fractions such as $\frac{1}{3}$, $\frac{1}{8}$, $\frac{1}{5}$, and $\frac{1}{10}$. Ask students to put the fractions in order from least to most. Challenge students to defend the way they ordered the fractions. The first few times you do this activity, have them explain their ideas by using models.

This idea is so basic to the understanding of fractions that arbitrary rules ("larger bottom numbers mean smaller fractions") are not only inappropriate but also dangerous. Come back to this basic idea periodically. Students will seem to understand one day and revert to their more comfortable ideas about big numbers a day or two later. Repeat Activity 5.10 with all numerators equal to 4. See how students' ideas change.

You have probably learned rules or algorithms for comparing two fractions. The usual approach is to find a common denominator. This rule can be effective in getting correct answers but requires no thought about the size of the fractions. If students are taught the common denominator rule before they have had the opportunity to think about the relative size of various fractions, there is little chance that they will develop any familiarity with or number sense about fraction size. Comparison activities (which fraction is more?) can play a significant role in helping students develop concepts of relative fraction sizes. But keep in mind that reflective thought is the goal, not an algorithmic method of choosing the correct answer.

Before reading further, try the following exercise. Assume for a moment that you know nothing about equivalent fractions or common denominators or cross-multiplication. Assume that you are a fourth- or fifth-grade student who was never taught these procedures. Now examine the pairs of fractions in Figure 5.15 and select the larger of each pair. Write down or explain one or more reasons for your choice in each case.

Which fraction in each pair is greater?
Give one or more reasons. Try not to use drawings or models. Do not use common denominators or cross-multiplication. Rely on concepts.

A. $\frac{4}{5}$ or $\frac{4}{9}$

B. $\frac{4}{7}$ or $\frac{5}{7}$

C. $\frac{3}{8}$ or $\frac{4}{10}$

D. $\frac{5}{3}$ or $\frac{5}{8}$

E. $\frac{3}{4}$ or $\frac{9}{10}$

F. $\frac{3}{8}$ or $\frac{4}{7}$

G. $\frac{7}{12}$ or $\frac{5}{12}$

H. $\frac{3}{5}$ or $\frac{3}{7}$

I. $\frac{5}{8}$ or $\frac{6}{10}$

J. $\frac{9}{8}$ or $\frac{4}{3}$

K. $\frac{4}{6}$ or $\frac{7}{12}$

L. $\frac{8}{9}$ or $\frac{7}{8}$

FIGURE 5.15

Comparing fractions using concepts.

Conceptual Thought Patterns for Comparison

The first two comparison schemes listed here rely on the meanings of the top and bottom numbers in fractions and on the relative sizes of unit fractional parts. The third and fourth ideas use the additional ideas of 0, $\frac{1}{2}$, and 1 as convenient anchors or benchmarks for thinking about the size of fractions.

1. *More of the same-size parts.* To compare $\frac{3}{8}$ and $\frac{5}{8}$, it is easy to think about having 3 of something and also 5 of the same thing. It is common for children to choose $\frac{5}{8}$ as larger simply because 5 is more than 3 and the other numbers are the same. Right choice, wrong reason. Comparing $\frac{3}{8}$ and $\frac{5}{8}$ should be like comparing 3 apples and 5 apples.
2. *Same number of parts but parts of different sizes.* Consider the case of $\frac{3}{4}$ and $\frac{3}{7}$. If a whole is divided into 7 parts, the parts will certainly be smaller than if divided into only 4 parts. Many children will select $\frac{3}{7}$ as larger because 7 is more than 4 and the top numbers are the same. That approach yields correct choices when the parts are the same size, but it causes problems in this case. This is like comparing 3 apples with 3 melons. You have the same number of things, but melons are larger.
3. *More and less than one-half or one whole.* The fraction pairs $\frac{3}{7}$ versus $\frac{5}{8}$ and $\frac{5}{8}$ versus $\frac{7}{8}$ do not lend themselves to either of the previous thought processes. In the first pair, $\frac{3}{7}$ is less than half of the number of sevenths needed to make a whole, and so $\frac{3}{7}$ is less than a half. Similarly, $\frac{5}{8}$ is more than a half. Therefore, $\frac{5}{8}$ is the larger fraction. The second pair is determined by noting that one fraction is less than 1 and the other is greater than 1.
4. *Distance from one-half or one whole.* Why is $\frac{9}{10}$ greater than $\frac{3}{4}$? Not because the 9 and 10 are big numbers, although you will find that to be a common student response. Each is one fractional part away from one whole, and tenths are smaller than fourths. Similarly, notice that $\frac{5}{8}$ is smaller than $\frac{4}{6}$ because it is only one-eighth more than a half, while $\frac{4}{6}$ is a sixth more than a half. Can you use this basic idea to compare $\frac{3}{5}$ and $\frac{5}{9}$? (*Hint:* Each is half of a fractional part more than $\frac{1}{2}$.) Also try $\frac{5}{7}$ and $\frac{7}{9}$.

How did your reasons for choosing fractions in Figure 5.15 compare to these ideas? It is important that you are comfortable with these informal comparison strategies as a major component of your own number sense as well as for helping students develop theirs.

Tasks you design for your students should assist them in developing these and possibly other methods of comparing two fractions. It is important that the ideas come from your students and their discussions. To teach "the four ways to compare fractions" would be adding four more mysterious rules and would be defeating for many students.

ACTIVITY 5.11

Choose, Explain, Test

Present two or three pairs of fractions to students. The students' task is to decide which fraction is greater (choose), to explain why they think this is so (explain), and then to test their choice using any model that they wish to use. They should write a description of how they made their test and whether or not it agreed with their choice. If their choice was incorrect, they should try to say what they would change in their thinking. In the student explanations, rule out drawing as an option. Explain that it is difficult to draw fraction pictures accurately and for this activity, pictures may cause them to make mistakes.

Rather than directly teach the different possible methods for comparing fractions, select pairs that will likely elicit desired comparison strategies. On one day, for example, you might have two pairs with the same denominators and one with the same numerators. On another day, you might pick fraction pairs in which each fraction is exactly one part away from a whole. Try to build strategies over several days by the appropriate choice of fraction pairs.

The use of a model in Activity 5.11 is an important part of students' development of strategies as long as the model is helping students create the strategy. However, after several experiences, change the activity so that the testing portion with a model is omitted. Place greater emphasis on students' reasoning. If class discussions yield different choices, allow students to use their own arguments for their choices in order to make a decision about which fraction is greater.

The next activity extends the comparison task a bit more.

ACTIVITY 5.12

Line 'Em Up

Select four or five fractions for students to put in order from least to most. Have them indicate approximately where each fraction belongs on a number line labeled only with the points 0, $\frac{1}{2}$, and 1. Students should include a description of how they decided on the order for the fractions. To place the fractions on the number line, students must also make estimates of fraction size in addition to simply ordering the fractions.

Including Equivalent Fractions

The discussion to this point has somewhat artificially ignored the idea that students might use equivalent fraction concepts in making comparisons. Equivalent fraction concepts are such an important idea that we have devoted a separate section to the development of that idea. However, equivalent fraction concepts need not be put off until last and certainly should be allowed in the discussions of which fraction is more.

Smith (2002) thinks that it is essential that the comparison question is asked as follows: "Which of the following two (or more) fractions is greater, or *are they equal?*" (p. 9, emphasis added). He points out that this question leaves open the possibility that two fractions that may look different can, in fact, be equal.

In addition to this point, with equivalent fraction concepts, students can adjust how a fraction looks so that they can use ideas that make sense to them. Burns (1999) told of fifth graders who were comparing $\frac{6}{8}$ to $\frac{4}{5}$. (You might want to stop for a moment and think how you would compare these two.) One child changed the $\frac{4}{5}$ to $\frac{8}{10}$ so that both fractions would be two parts away from the whole and he reasoned from there. Another changed both fractions to a common *numerator* of 12.

Be absolutely certain to revisit the comparison activities and include pairs such as $\frac{8}{12}$ and $\frac{2}{3}$ in which the fractions are equal but do not appear to be. Also include fractions that are not in lowest terms.

Only One Size for the Whole

A key idea about fractions that students must come to understand is that a fraction does not say anything about the size of the whole or the size of the parts. A fraction tells us only about the *relationship between* the part and the whole. Consider the following situation.

Mark is offered the choice of a third of a pizza or a half of a pizza. Since he is hungry and likes pizza, he chooses the half. His friend Jane gets a third of a pizza but ends up with more than Mark. How can that be? Figure 5.16 illustrates how Mark got misdirected in his choice. The point of the "pizza fallacy" is that whenever two or more fractions are discussed in the same context, the correct assumption (the one Mark made in choosing a half of the pizza) is that the fractions are all parts of the same size whole.

Comparisons with any model can be made only if both fractions are parts of the same whole. For example, $\frac{2}{3}$ of a light green strip cannot be compared to $\frac{2}{5}$ of an orange strip.

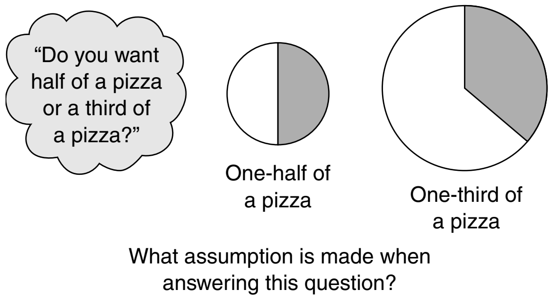


FIGURE 5.16

The "pizza fallacy."

Assessment Note

It is a good idea to periodically pose tasks in which students are required to draw representations of the fractions they are comparing. Consider the drawing in Figure 5.17, in which a student is comparing $\frac{4}{5}$ to $\frac{2}{3}$. What insights into his understanding does this student's work provide?

This student came to the correct conclusion but in an erroneous way. It appears that the student used the same size "fractional piece" to build his representations, resulting in inequivalent wholes. His representations suggest that he may not realize that the wholes should be the same size when comparing fractions.

Physical models, such as pie pieces and fraction bars, can mask this misconception because the size of the whole circle or whole rectangle is fixed. This eliminates the need for the student to think about the relative sizes of the associated wholes. Providing outlines of the wholes for students also diverts the focus from the size of the wholes. In short, if the representations are always provided for the students, a teacher may not realize a student has this misconception.

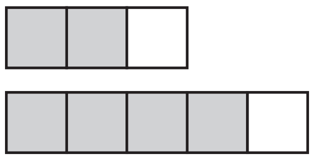


FIGURE 5.17

A student compares $\frac{2}{3}$ and $\frac{4}{5}$.

It is probably best to postpone fraction computation until at least fourth grade so that students can have an adequate amount of time to develop a firm foundation of fraction concepts. Having said that, consider the following: For addition and subtraction of fractions, a surprising number of problems found on standardized tests can be solved with simple number sense without knowledge of an algorithm. For example, $\frac{3}{4} + \frac{1}{2}$ requires only that students can think of $\frac{3}{4}$ as $\frac{1}{2}$ and $\frac{1}{4}$ more, or alternatively, think of $\frac{1}{2}$ as $\frac{1}{4}$ and $\frac{1}{4}$. This sort of thinking is a result of a focus on fraction meanings, not on algorithms.

The development of fraction number sense, even at grade 3, should certainly involve estimation of sums and differences of fractions. Estimation focuses on the size of the fractions and encourages students to use a variety of strategies.

The following activity can be used as a regular short warm-up for any fraction lesson.

ACTIVITY 5.13

First Estimates

Tell students that they are going to estimate a sum or difference of two fractions. They are to decide only if the exact answer is more or less than one. On the overhead projector show, for no more than about 10 seconds, a fraction addition or subtraction problem involving two proper fractions. Keep all denominators to 12 or less. Students write down on paper their choice of more or less than one. Do several problems in a row. Then return to each problem and discuss how students decided on their estimate.

Restricting Activity 5.13 to proper fractions keeps the difficulty to a minimum. When students are ready for a tougher challenge, choose from the following variations:

- Use fractions that are less than one. Estimate to the nearest half (0, $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2).
- Use both proper and mixed fractions. Estimate to the nearest half.
- Use proper and mixed fractions. Estimate the best answer you can.

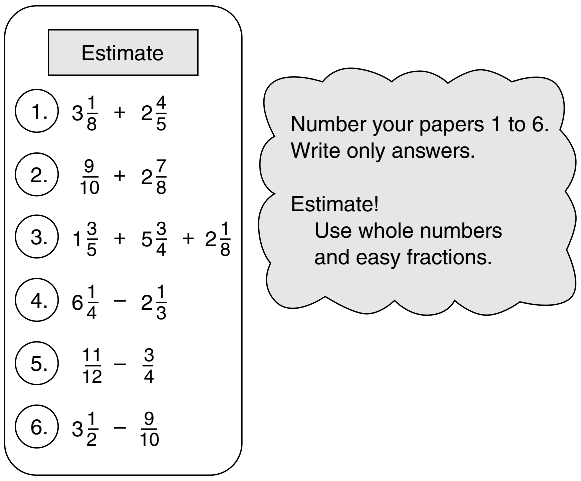


FIGURE 5.18

Fraction estimation drill.

In the discussions following these estimation exercises, ask students if they think that the exact answer is more or less than the estimate that they gave. What is their reasoning?

Figure 5.18 shows six sample sums and differences that might be used in a "First Estimates" activity.

STOP **Test your own estimation skills with the sample problems in Figure 5.18. Look at each computation for only about 10 seconds and write down an estimate. After writing down all six of your estimates, look at the problems and decide if your estimate is higher or lower than the actual computation. Don't guess! Have a good reason.**

In most cases students' estimates should not be much more than $\frac{1}{2}$ away from the exact sum or difference.

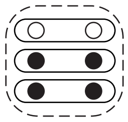
Equivalent-Fraction Concepts

How do you know that $\frac{4}{6} = \frac{2}{3}$? Before reading further, think of at least two different explanations.

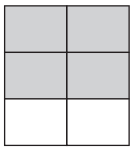
Concepts Versus Rules

Here are some possible answers to the question just posed:

1. They are the same because you can reduce $\frac{4}{6}$ and get $\frac{2}{3}$.
2. If you have a set of 6 things and you take 4 of them, that would be $\frac{4}{6}$. But you can make the 6 into groups of 2. So then there would be 3 groups, and the 4 would be 2 groups out of the 3 groups. That means it's $\frac{2}{3}$.



3. If you start with $\frac{2}{3}$, you can multiply the top and the bottom numbers by 2, and that will give you $\frac{4}{6}$, so they are equal.
4. If you had a square cut into 3 parts and you shaded 2, that would be $\frac{2}{3}$ shaded. If you cut all 3 of these parts in half, that would be 4 parts shaded and 6 parts in all. That's $\frac{4}{6}$, and it would be the same amount.



All of these answers are correct. But let's think about what they tell us. Responses 2 and 4 are very conceptual, although not very efficient. The procedural responses, 1 and 3, are quite efficient but indicate no conceptual knowledge. All students should eventually be able to write an equivalent fraction for a given fraction. At the same time, the rules should never be taught or used until the students understand what the result means. Consider how different the algorithm and the concept appear to be.

Concept: Two fractions are equivalent if they are representations for the same amount or quantity—if they are the same number.

Algorithm: To get an equivalent fraction, multiply (or divide) the top and bottom numbers by the same nonzero number.

In a problem-based classroom, students can develop an understanding of equivalent fractions and also develop from that understanding a conceptually based algorithm. As with most algorithms, a serious instructional error is to rush too quickly to the rule. Be patient! Intuitive methods are always best at first.

Equivalent-Fraction Concepts

The general approach to helping students create an understanding of equivalent fractions is to have them use models to find different names for a fraction. Consider that this is the first time in their experience that a fixed quantity can have multiple names (actually an infinite number of names). The following activities are possible starting places.

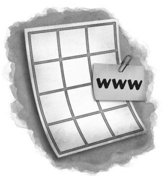
ACTIVITY 5.14

Different Fillers

Using an area model for fractions that is familiar to your students, prepare a worksheet with two or at most three outlines of different fractions. Do not limit yourself to unit fractions. For example, if the model is circular pie pieces, you might draw an outline for $\frac{2}{3}$, $\frac{1}{2}$, and $\frac{3}{4}$. The students' task is to use their own fraction pieces to find as many single-fraction names for the region as possible. After completing the three examples, have students write about the ideas or patterns they may have noticed in finding the names. Follow the activity with a class discussion.

In the class discussion following the "Different Fillers" activity, a good question to ask involves what names could be found if students had any size pieces that they wanted. For example, ask students, "What names could you find if we had sixteenths in our fraction kit? What names could you find if you could have any piece at all?" The idea is to push beyond filling in the region in a pure trial-and-error approach.

The following activity is just a variation of "Different Fillers." Instead of a manipulative model, the task is constructed on dot paper.



BLMs 10–13

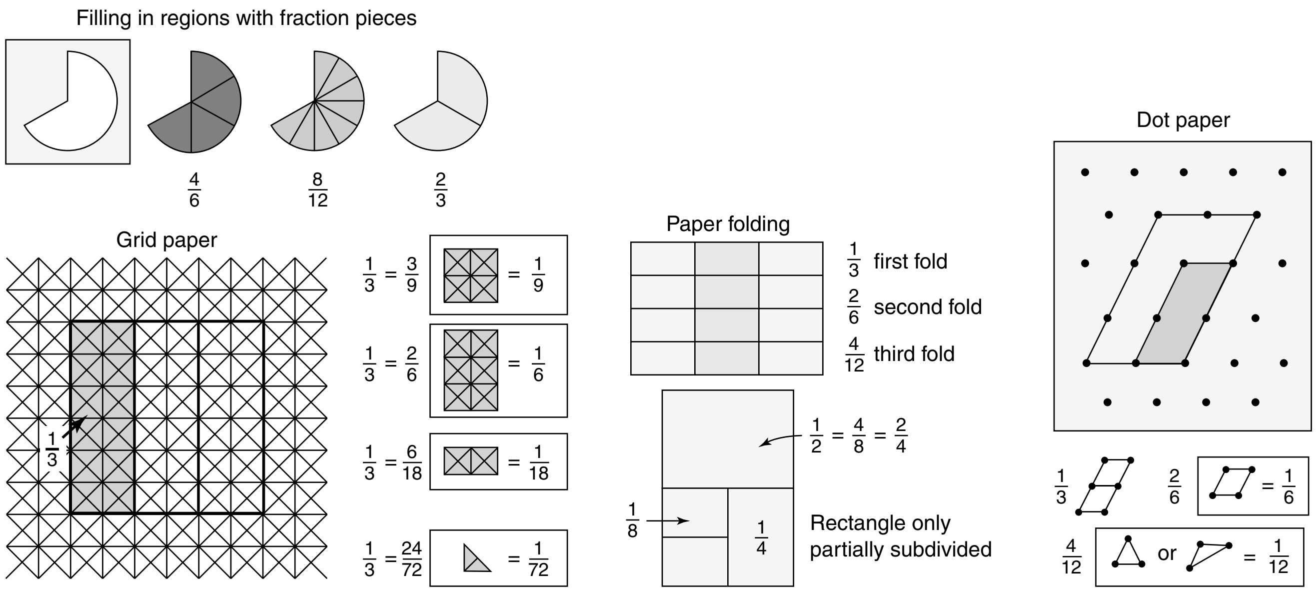
ACTIVITY 5.15

Dot Paper Equivalencies

Create a worksheet using a portion of either isometric or rectangular dot grid paper. (These can be found in the Blackline Masters.) On the grid, draw the outline of a region and designate it as one whole. Draw and lightly shade a part of the region within the whole. The task is to use different parts of the whole determined by the grid to find names for the part. Figure 5.19 includes an example drawn on an isometric grid. Students should draw a picture of the unit fractional part that they use for each fraction name. The larger the size of the whole, the more names the activity will generate.

FIGURE 5.19

Area models for equivalent fractions.



The "Dot Paper Equivalencies" activity is a form of what Lamon (2002) calls "unitizing," that is, given a quantity, finding different ways to chunk the quantity into parts in order to name it.

Length models can be used to create activities similar to the "Different Fillers" task. For example, as shown in Figure 5.20, rods or strips can be used to designate both a whole and a part. Students use smaller rods to find fraction names for the given part. To have larger wholes and, thus, more possible parts, use a train of two or three rods for the whole and the part. Folding paper strips is another method of creating fraction names. In the example shown in Figure 5.20, one-half is subdivided by successive folding in half. Other folds would produce other names and these possibilities should be discussed if no one tries to fold the strip in an odd number of parts.

The following activity is also a unitizing activity in which students look for different units or chunks of the whole in order to name a part of the whole in different ways. This activity is significant because it utilizes a set model.

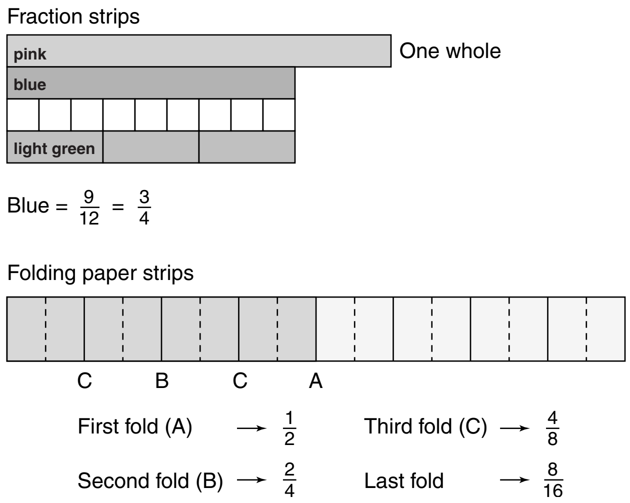


FIGURE 5.20

Length models for equivalent fractions.

ACTIVITY 5.16

Group the Counters, Find the Names

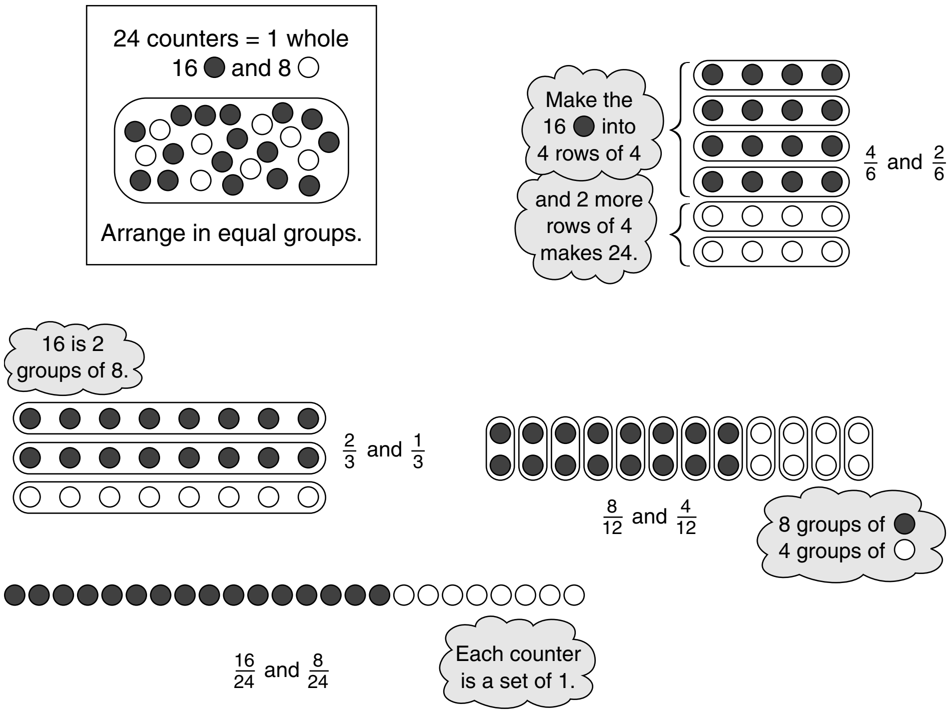
Have students set out a specific number of counters in two colors—for example, 24 counters, 16 of them black and 8 white. The 24 make up the whole. The task is to group the counters into different fractional parts of the whole and use the parts to create fraction names for the black and the white counters. In Figure 5.21, 24 counters are arranged in different array patterns. You might want to suggest arrays or allow students to arrange them in any way they wish. Students should record their different groupings and explain how they found the fraction names. They can simply use Xs and Os for the counters.

In Lamon's version of the last activity, she prompts students with questions such as, "If we make groups of four, what part of the set is black?" With these prompts you can suggest fraction names that students are unlikely to think of. For our example in Figure 5.21, if we make groups of one-half counters, what would the white set be called? Suppose we made groups of six? (Groups of six result in a fractional numerator. Why not?)

A challenging exploration of finding equivalent fractions is presented in *Gator Pie* (Mathews, 1979), a delightful book about Alice, Alvin, and other alligators sharing a pie they found in the woods. At first glance this book seems too juvenile for students in the upper grades. However, students can enjoy this story when it is a springboard into an interesting problem-solving situation.

FIGURE 5.21

Set models for equivalent fractions.



ACTIVITY 5.17

Divide and Divide Again

In the book *Gator Pie,* Alvin and Alice find a pie in the woods. However, before they can cut it, another gator appears and demands a share of the pie. As the story continues, more and more gators arrive until there are 100 gators who want a piece of the pie. Finally, Alice painstakingly cuts the pie into hundredths. In the story, Alvin and Alice are prevented from cutting the pie each time because more gators show up. An interesting twist is to change the story so that the pie is cut before more gators appear. The problem is how to share it among a larger number once it is already cut. To illustrate, cut a circle (or rectangle) into halves or thirds, and then ask students to decide how to share it among a larger number once it is already cut. You may want to start going from halves to sixths. This is reasonably easy but may surprise you. After students have shared their approaches, progress into more difficult divisions. For example, what if the pie is cut in thirds and we want to share it in tenths? Students should be expected to identify the fractional parts they used and explain how and why they used those particular fractional parts.

As students work through "Divide and Divide Again," they have to think about the part-whole meaning of fractions: how to divide an amount into equal-sized portions or fair shares. What is challenging is that oftentimes the equal-sized portions may not be the same shape or may be pieced together from smaller pieces. Placing a challenging task in the familiar context of fair sharing makes the task seem possible to students.

In the activities so far, there has only been a hint of a rule for finding equivalent fractions. The following activity moves a bit closer but should still be done before development of a rule.

ACTIVITY 5.18

Missing-Number Equivalencies

Give students an equation expressing an equivalence between two fractions but with one of the numbers missing. Here are four different examples:

$$\frac{5}{3} = \frac{\square}{6} \qquad \frac{2}{3} = \frac{6}{\square} \qquad \frac{8}{12} = \frac{\square}{3} \qquad \frac{9}{12} = \frac{3}{\square}$$

The missing number can be either a numerator or a denominator. Furthermore, the missing number can either be larger or smaller than the corresponding part of the equivalent fraction. (All four of these possibilities are represented in the examples.) The task is to find the missing number and to explain your solution.

When doing "Missing-Number Equivalencies" you may want to specify a particular model, such as sets or pie pieces. Alternatively, you can allow students to select whatever methods they wish to solve these problems. One or two equivalencies followed by a discussion is sufficient for a good lesson. This activity is surprisingly challenging, especially if students are required to use a set model.

Before continuing with development of an algorithm for equivalent fractions with your class, you should revisit the comparison tasks as students begin to realize that they can change the names of fractions in order to help reason about which fraction is greater.

Developing an Equivalent-Fraction Algorithm

Kamii and Clark (1995) argue that undue reliance on physical models does not help students construct equivalence schemes. When students understand that fractions can have different names, they should be challenged to develop a method for finding equivalent names. It might also be argued that students who are experienced at looking for patterns and developing schemes for doing things can invent an algorithm for equivalent fractions without further assistance. However, the following approach will certainly improve the chances of that happening.

An Area Model Approach

Your goal is to help students see that if they multiply both the top and bottom numbers by the same number, they will always get an equivalent fraction. The approach suggested here is to look for a pattern in the way that the fractional parts in both the part as well as the whole are counted. Activity 5.19 is a good beginning, but a good class discussion following the activity will also be required.

ACTIVITY 5.19

Slicing Squares

Give students a worksheet with four squares in a row, each approximately 3 cm on a side. Have them shade in the same fraction in each square using vertical dividing lines. For example, slice each square in fourths and shade three-fourths as in Figure 5.22. Next, tell students to slice each square into an

(continued)

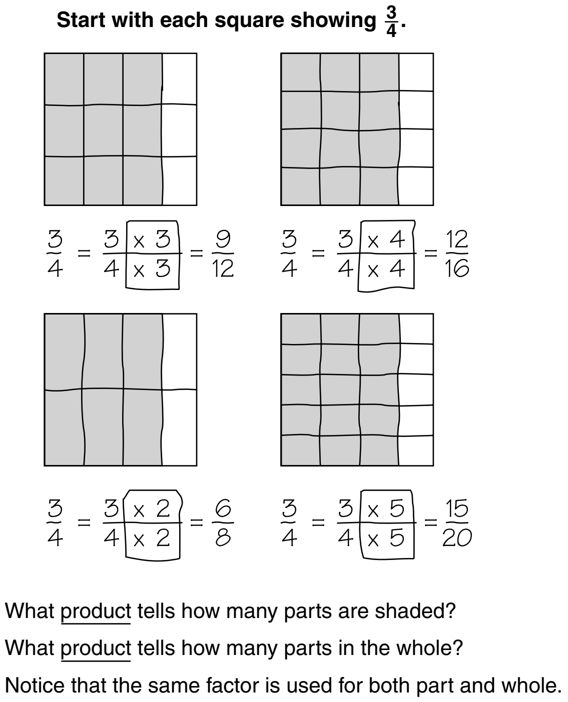


FIGURE 5.22

A model for the equivalent-fraction algorithm.

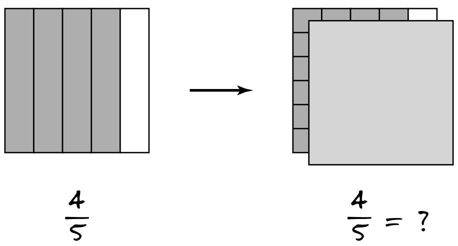


FIGURE 5.23

How can you count the fractional parts if you cannot see them all?

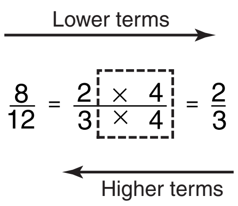


FIGURE 5.24

Using the equivalent-fraction algorithm to write fractions in simplest terms.

equal number of horizontal slices. Each square is sliced with a different number of slices, using anywhere from one to eight slices. For each sliced square, they should write an equation showing the equivalent fraction. Have them examine their four equations and the drawings and challenge them to discover any patterns in what they have done. You may want them to repeat this with four more squares and a different fraction.

Following this activity, write on the board the equations for four or five different fraction names found by the students. Discuss any patterns they found. To focus the discussion, show on the overhead a square illustrating $\frac{4}{5}$ made with vertical slices as in Figure 5.23. Turn off the overhead and slice the square into six parts in the opposite direction. Cover all but two edges of the square as shown in the figure. Ask, "What is the new name for my $\frac{4}{5}$?"

The reason for this exercise is that many students simply count the small regions and never think to use multiplication. With the covered square, students can see that there are four columns and six rows to the shaded part, so there must be 4 × 6 parts shaded. Similarly, there must be 5 × 6 parts in the whole. Therefore, the new name for $\frac{4}{5}$ is $\frac{4}{5 \times 6}$.

Using this idea, have students return to the fractions on their worksheet to see if the pattern works for other fractions.

Examine examples of equivalent fractions that have been generated with other models, and see if the rule of multiplying top and bottom numbers by the same number holds there also. If the rule is correct, how can $\frac{6}{8}$ and $\frac{9}{12}$ be equivalent? What about fractions like $2\frac{1}{4}$? How could it be demonstrated that $\frac{9}{4}$ is the same as $\frac{2}{12}$?

Writing Fractions in Simplest Terms

The multiplication scheme for equivalent fractions produces fractions with larger denominators. To write a fraction in *simplest terms* means to write it so that numerator and denominator have no common whole number factors. (Some texts use the name *lowest terms* instead of *simplest terms*.) One meaningful approach to this task of finding simplest terms is to reverse the earlier process, as illustrated in Figure 5.24. Try to devise a problem-based task that will help students develop this reverse idea.

Of course, finding and eliminating a common factor is the same as dividing both top and bottom by the same number. The search for a common factor keeps the process of writing an equivalent fraction to one rule: Top and bottom numbers of a fraction can be multiplied by the same nonzero number. There is no need for a different rule for rewriting fractions in lowest terms.

Two additional notes:

1. Notice that the phrase *reducing fractions* was not used. This unfortunate terminology implies making a fraction smaller and is rarely used anymore in textbooks.
2. Many teachers seem to believe that fraction answers are incorrect if not in simplest or lowest terms. This is also unfortunate. When students add $\frac{1}{6} + \frac{1}{2}$ and get $\frac{4}{6}$, they have added correctly and have found the answer. Rewriting $\frac{4}{6}$ as $\frac{2}{3}$ is a separate issue.

Multiplying by One

A strictly symbolic approach to equivalent fractions is based on the multiplicative property that says that any number multiplied by 1 remains unchanged. Any fraction of the form $\frac{n}{n}$ can be used as the identity element. Therefore, $\frac{3}{4} = \frac{3}{4} \times 1 = \frac{3}{4} \times \frac{2}{2} = \frac{6}{8}$. Furthermore, the numerator and denominator of the identity element can also be fractions. In this way, $\frac{6}{12} = \frac{6}{12} \times \left(\frac{1/6}{1/6}\right) = \frac{1}{2}$.

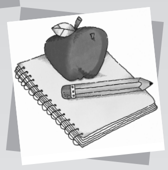
This explanation relies on an understanding of the multiplicative identity property, which most students in grades 4 to 6 do not fully appreciate. It also relies on the procedure for multiplying two fractions. Finally, the argument uses solely deductive reasoning based on an axiom of the rational number system. It does not lend itself to intuitive modeling. A reasonable conclusion is to delay this important explanation until at least seventh or eighth grade in an appropriate prealgebra context and not as a method or a rationale for producing equivalent fractions.

Technology Note

In the NCTM e-examples (www.nctm.org), there is a very nice fraction game for two players (*Applet 5.1, Communicating About Mathematics Using Games*). The game uses a number-line model, and knowledge of equivalent fractions plays a significant role.

The NLVM website (http://matti.usu.edu/nlvm/nav/vlibrary.html) has a limited applet tool for exploring equivalent fractions, *Fractions—Equivalent.* Proper fractions are presented randomly in either square or circular formats. Students can slice the model in as many parts as they wish to see which slicings create equivalent fractions. For squares, the new slices go in the same direction as the original slices. For circles, it is a bit hard to distinguish new slices from old. Students enter an equivalent fraction and then click a button to check their response.

EXPANDED LESSON



About How Much

Based on: Activity 5.9, p. 145

GRADE LEVEL: Third or fourth grade.

MATHEMATICS GOALS

- To develop a concept of the size of fractions.
- To develop the reference points or benchmarks of 0, $\frac{1}{2}$, and 1 for fractions.

THINKING ABOUT THE STUDENTS

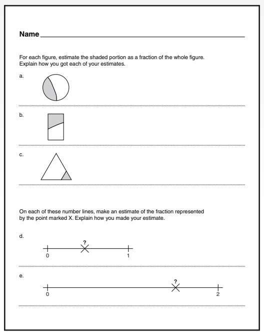
Students understand that in the context of part-whole fractions, the whole is divided into equivalent parts. They also understand the symbolic notation of fractions; that is, they know what the top number in the fraction means (the number of parts) and what the bottom number in the fraction means (the kind of parts we are counting). Equivalent fractions have not been explored fully.

MATERIALS AND PREPARATION

- Make copies of Blackline Master L-1 for each student and also a transparency of L-1.
- Prepare a transparency with a rectangle divided into 6 equal pieces as shown here.

- A colored overhead pen.

lesson



BLM L-1

BEFORE

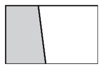
Begin with a simpler version of the task:

- Show the transparency with the rectangle divided into six equal pieces. Shade in three of the six sections. Ask students to tell how much of the rectangle is shaded. Be sure that answers include $\frac{1}{2}$ as well as $\frac{3}{6}$.
- Now shade in just a bit more of the rectangle as shown here. Ask students what an estimate is. Negotiate an appropriate definition of an estimate. Ask students what a good estimate might be for the shaded amount. Most students will say $\frac{3}{6}$ or $\frac{1}{2}$. Discuss why they think their answer is a good estimate.

- Add to the shaded portion so that half of the fourth piece in the rectangle is now shaded. Ask whether their estimate would change if this is the amount that they want to estimate. If students still want to use $\frac{3}{6}$ as an estimate, ask what they could do if they wanted a closer estimate (e.g., divide the pieces in half to form twelfths).

Brainstorm

- Now draw a rectangle on the transparency with no partitions. Shade in about $\frac{1}{3}$ of the rectangle.



- Ask students to determine how they would get an estimate for the shaded amount. Give them a minute to think about it individually and then share their ideas with a partner. Come together as a class to hear different strategies. Two possible strategies: (1) Divide the rectangle into equal parts and use those parts to determine the estimate; and (2) decide whether the amount is closer to 0, $\frac{1}{2}$, or 1. Then you might divide the parts further to decide if the amount is closer to 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, or 1. It is a good idea to list these ideas on the board for students to refer to when doing the task.

The Task

Each student is to determine a fraction that he or she thinks is a good estimate of the amount shown in each picture.

Establish Expectations

Students should be ready to share their estimates and the ways that they determined the estimates for each picture. They should use the handout to draw on the pictures and write words and numbers to explain their thinking.

DURING

- Without evaluating the ideas of several students, listen to why any particular estimate might be a good one.
- If students have difficulty coming up with an estimate, ask if they think the amount is closer to 0, $\frac{1}{2}$, or 1.

AFTER

- Ask various students to come forward to share their estimates and their ways of determining the estimates. Ask the class to comment or ask questions about the estimate and/or the strategy.
- If there are students who divided shapes into unequal parts, have them share their approaches. Do not evaluate but instead encourage the class to discuss their approaches.
- When discussing the pictures with number lines, focus on the interval between 0 and 1 as the whole, not the point where the number 1 is.

ASSESSMENT NOTES

- How are students finding equivalent pieces, in particular for circles and triangles? Are they dividing the shapes into equivalent pieces? How do they know they are equivalent?
- Are estimates "in the ballpark"?
- When working with number lines, are students counting intervals between numbers or are they counting the numbers?

next steps

- If students have difficulty with this task, continue helping them develop the reference points of 0, $\frac{1}{2}$, and 1 for fractions by doing activities such as "Zero, One-Half, or One" (Activity 5.7) and "Close Fractions" (Activity 5.8).
- If students have difficulty understanding the problem with unequal parts in circles and triangles, provide them with circles and triangles from which they can cut various pieces to lay over each other to show that they are not the same area.
- Ordering unit fractions and making comparisons are appropriate next activities. (See Activities 5.10 and 5.11.) It is also reasonable to begin exploring equivalent fractions.