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# Some critical metrics on 3-dimensional trans-Sasakian manifolds

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**Abstract** The object of the present paper is to characterize 3-dimensional trans-Sasakian manifolds satisfying the Miao-Tam critical equation and Fischer-Marsden conjecture. It is shown that a 3-dimensional trans-Sasakian manifold satisfying the Miao-Tam critical equation or the Fischer-Marsden conjecture is either $\beta$-Kenmotsu or a space of constant curvature. Also, we proved that a complete 3-dimensional trans-Sasakian manifold satisfying the Miao-Tam critical equation or the Fischer-Marsden conjecture is isometric to a sphere. As a corollary of the Fischer-Marsden conjecture, it is shown that the solution space of the Fischer-Marsden conjecture on a complete, non-compact 3-dimensional cosymplectic manifold is a linear space of harmonic functions over the field of real numbers.

## 1 Introduction

Let $M$ be an $n$-dimensional compact orientable manifold together with the Riemannian metric $g$ and let $\mathcal{M}$ denote the set of all smooth Riemannian metrics on $(M, g)$ and $g^*$ be any symmetric bilinear form on $M$. Then the linearization of the scalar curvature $\mathcal{L}_g(g^*)$ is given by

$$\mathcal{L}_g(g^*) = -\Delta_g(Tr_g g^*) + div(div(g^*)) - g(g^*, S_g),$$

where $\Delta_g$ is the negative Laplacian of $g$ and $S_g$ is its $(0,2)$ Ricci tensor. The formal $L^2$-adjoint $\mathcal{L}_g^*$ of the linearized scalar curvature $\mathcal{L}_g$ is defined as

$$\mathcal{L}_g^*(\lambda) = -(\Delta_g\lambda)g + \nabla_g^2\lambda - \lambda S_g,$$

where $\nabla_g^2$ is the Hessian operator with respect to the metric $g$ defines by $(\nabla_g^2\lambda)(X, Y) = \text{Hess}_\lambda(X, Y) = g(\nabla_X D\lambda, Y)$, $D$ is the gradient operator. Throughout the paper we refer the equation

$$\mathcal{L}_g^*(\lambda) = g \tag{1.1}$$

as the Miao-Tam critical equation and

$$\mathcal{L}_g^*(\lambda) = 0 \tag{1.2}$$

as Fischer-Marsden equation. Obviously, if the potential function $\lambda$ is a non-zero constant, then (1.1) becomes an Einstein metric. In [19], Miao-Tam proved that any Riemannian metric $g$ satisfying (1.1) must have constant scalar curvature. In [19], the authors proved that any connected, compact, Einstein manifold with smooth boundary satisfying Miao-Tam critical condition is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{H}^n$ or $S^n$. Recently Patra and Ghosh [23] studied the Miao-Tam critical equation on contact metric manifolds. In [25], Wang studied the critical condition (1.1) on certain class of Riemannian manifolds. In [9], Fischer-Marsden conjectured that a compact Riemannian manifold $(M, g)$ that admits a non-trivial solution of (1.2) is necessarily an Einstein manifold. From this it follows that a critical metric $g$ always has constant scalar curvature. A counter example of this conjecture was obtained by Kobayshi [16] and Lafontaine [17] when $g$ is conformally flat. In [4], Cerena and Guan proved that if a Riemannian manifold $(M, g)$ is closed, homogeneous and admits a Fischer-Marsden solution, then $(M, g)$ must be of the form $S^m \times N$, where $S^m$ and $N$ are respectively Euclidean sphere and Einstein manifold. Recently Patra and Ghosh [22] proved that if a non-Sasakian $(k, \mu)$-contact metric manifold satisfies (1.2), then for $n = 1$, $M^3$ is flat and for $n > 1$, $M^{2n+1}$ is locally isometric to the product of an Euclidean space $E^{n+1}$ and a sphere $S^n(4)$ of constant curvature 4.

A new class of almost contact metric manifolds, namely trans-Sasakian manifolds, has been introduced by Oubina [21]. It is known that there are sixteen different types of structures on the almost Hermitian manifold $(\overline{M}, J, G)$ [10], and recently, using the structure in the class $\mathcal{W}_4$ on $(\overline{M}, J, G)$ a structure $(\phi, \xi, \eta, g, \alpha, \beta)$ on $M$ called trans-Sasakian structure is introduced [20], which generalizes Sasakian structure and Kenmotsu structure on almost contact metric manifolds ([2], [13]), where $\alpha$, $\beta$ are smooth functions defined on $M$. In general, a trans-Sasakian manifold $(M, \phi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type $(\alpha, \beta)$ and trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are called a cosymplectic, a $\alpha$-Sasakian and a $\beta$-Kenmotsu manifolds, respectively, provided $\alpha$, $\beta \in \mathbb{R}$ [12]. After the introduction of trans-Sasakian manifolds, Blair and Oubi$\tilde{n}$a [2] and Marrero [18] studied the geometry of trans-Sasakian manifolds. Marrero [18] has shown that a trans-Sasakian manifold of dimension $\geq 5$ is either a cosymplectic manifold, a $\alpha$-Sasakian manifold or a $\beta$-Kenmotsu manifold. Since then there is an attention on studying geometry of 3-dimensional trans-Sasakian manifolds only. In ([5]-[8],[14],[15], [26]), authors have studied 3-dimensional trans-Sasakian manifolds with some restrictions on the smooth functions $\alpha$, $\beta$ appearing in the definition of trans-Sasakian manifolds.
Throughout the paper we assume that the smooth functions $\alpha$ and $\beta$ satisfy the condition

$$\phi \,\mathrm{grad}\, \alpha = \mathrm{grad}\, \beta, \tag{1.3}$$

which implies that

$$X\beta + (\phi X)\alpha = 0 \tag{1.4}$$

and hence $\xi\beta = 0$.

Motivated by the above studies, in this paper, we consider the Miao-Tam critical equation and the Fischer-Marsden conjecture in the framework of 3-dimensional trans-Sasakian manifolds. Precisely we have shown that a 3-dimensional trans-Sasakian manifold satisfying the Miao-Tam critical equation or the Fischer-Marsden conjecture is either $\beta$-Kenmotsu or a space of constant curvature. In addition, we prove that a complete 3-dimensional trans-Sasakian manifold satisfying the Miao-Tam critical equation or the Fischer-Marsden conjecture is either $\beta$-Kenmotsu or isometric to a sphere. As a corollary of the Fischer-Marsden conjecture we obtain that the solution space of the Fischer-Marsden conjecture on a complete, non-compact 3-dimensional cosymplectic manifold is a linear space of harmonic functions over $\mathbb{R}$. Also we obtain several fruitful corollaries.

## 2 Preliminaries

Let $(M, \phi, \xi, \eta, g)$ be a 3-dimensional almost contact metric manifold, where $\phi$ being a $(1,1)$-tensor field, $\xi$ a unit vector field and $\eta$ smooth 1-form dual to $\xi$ with respect to the Riemanian metric $g$ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0 \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$X, Y \in \chi(M)$, where $\chi(M)$ being the Lie algebra of smooth vector fields on $M$ [1]. If there are smooth functions $\alpha$, $\beta$ on an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ satisfying

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.3}$$

$X, Y \in \chi(M)$, then it is said to be a trans-Sasakian manifold, where $\nabla$ is the Levi-Civita connection with respect to the metric $g$ ([2], [18], [21]). We shall denote the trans-Sasakian manifold by $(M, \phi, \xi, \eta, g, \alpha, \beta)$ and it is called trans-Sasakian manifold of type $(\alpha, \beta)$. From (2.3) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.4}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.5}$$

A trans-Sasakian manifold is said to be

- cosymplectic or co-Kaehler if $\alpha = \beta = 0$,
- quasi-Sasakian manifold if $\beta = 0$ and $\xi(\alpha) = 0$,
- $\alpha$-Sasakian manifold if $\alpha$ is a non-zero constant and $\beta = 0$,
- $\beta$-Kenmotsu manifold if $\alpha = 0$ and $\beta$ is a non-zero constant.

Therefore, trans-Sasakian manifold generalizes a large class of almost contact manifolds. From [5] we know that for a 3-dimensional trans-Sasakian manifold

$$2\alpha\beta + \xi\alpha = 0. \tag{2.6}$$

The Ricci operator $Q$ satisfies [5]

$$Q(\xi) = \phi(\nabla\alpha) - \nabla\beta + 2(\alpha^2 - \beta^2)\xi - g(\nabla\beta, \xi)\xi. \tag{2.7}$$

$$\begin{aligned} S(X,Y) &= (\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2))g(X,Y) \\ &\quad -(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\eta(Y) \\ &\quad -(Y\beta + (\phi Y)\alpha)\eta(X) - (X\beta + (\phi X)\alpha)\eta(Y) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))(g(Y,Z)X - g(X,Z)Y) \\ &\quad -g(Y,Z)((\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\xi \\ &\quad -\eta(X)(\phi\, \mathrm{grad}\, \alpha - \mathrm{grad}\, \beta) + (X\beta + (\phi X)\alpha)\xi) \\ &\quad +g(X,Z)((\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\xi \\ &\quad -\eta(Y)(\phi\, \mathrm{grad}\, \alpha - \mathrm{grad}\, \beta) + (Y\beta + (\phi Y)\alpha)\xi) \\ &\quad -((Z\beta + (\phi Z)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(Z) \\ &\quad +(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\eta(Z))X \\ &\quad ((Z\beta + (\phi Z)\alpha)\eta(X) + (X\beta + (\phi X)\alpha)\eta(Z) \\ &\quad +(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\eta(Z))Y \end{aligned} \tag{2.9}$$

hold, where $S$ is the Ricci tensor of type $(0, 2)$, $R$ is the Riemannian curvature tensor of type $(1, 3)$ and $r$ is the scalar curvature of the manifold $M$.
If $M$ satisfies the condition (1.3), then the equations (2.8) and (2.9) reduces to

$$S(X,Y) = (\frac{r}{2} - (\alpha^2 - \beta^2))g(X,Y) - (\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(X)\eta(Y), \tag{2.10}$$

$$\begin{aligned}R(X,Y)Z &= (\frac{r}{2}-2(\alpha^2-\beta^2))(g(Y,Z)X-g(X,Z)Y)\\ &-g(Y,Z)((\frac{r}{2}-3(\alpha^2-\beta^2))\eta(X)\xi\\ &+g(X,Z)((\frac{r}{2}-3(\alpha^2-\beta^2))\eta(Y)\xi)\\ &-(\frac{r}{2}-3(\alpha^2-\beta^2))\eta(Y)\eta(Z)X\\ &+(\frac{r}{2}-3(\alpha^2-\beta^2))\eta(X)\eta(Z)Y. \end{aligned} \tag{2.11}$$

From (2.10) we get

$$S(X,\xi) = 2(\alpha^2-\beta^2)\eta(X) \tag{2.12}$$

and from (2.11) it follows that

$$R(X,Y)\xi = (\alpha^2-\beta^2)(\eta(Y)X-\eta(X)Y), \tag{2.13}$$

$$R(\xi,X)Y = (\alpha^2-\beta^2)(g(X,Y)\xi-\eta(Y)X). \tag{2.14}$$

## 3 Critical metrics on 3-dimensional trans-Sasakian manifolds

To prove our main results we first state the followings:

**Lemma 3.1.** *(Lemma 3.4 of [11]) Let a Riemannian manifold $(M^n, g)$ satisfies the Miao-Tam equation. Then the curvature tensor $R$ can be expressed*

$$\begin{aligned}R(X,Y)D\lambda &= (X\lambda)QY-(Y\lambda)QX+\lambda\{(\nabla_X Q)Y-(\nabla_Y Q)X\}\\ &+(Xf)Y-(Yf)X, \end{aligned} \tag{3.1}$$

*where $D$ is the gradient operator and $f = -\frac{r\lambda+1}{n-1}$.*

**Lemma 3.2.** *(Lemma 3.1 of [22]) If $(g, \lambda)$ is a non-trivial solution of the Fischer-Marsden equation (1.2) on a $(2n+1)$-dimensional contact metric manifold $M$, then the curvature tensor $R$ can be expressed*

$$\begin{aligned}R(X,Y)D\lambda &= (X\lambda)QY-(Y\lambda)QX+\lambda\{(\nabla_X Q)Y-(\nabla_Y Q)X\}\\ &+(Xf)Y-(Yf)X, \end{aligned} \tag{3.2}$$

*for any vector fields $X$, $Y$ on $M$ and $f = -\frac{r\lambda}{2n}$.*

**Lemma 3.3.** *(Theorem 1 of [3]) For a trans-Sasakian manifold $M^n$, $n > 1$, under the condition $\phi\,\mathrm{grad}\,\alpha = (n-2)\,\mathrm{grad}\,\beta$, we have*

$$\begin{aligned}&[(\nabla_\xi S)(Y,Z)-(\nabla_Y S)(\xi,Z)]\\ &= \beta S(Y,Z)-(n-1)(\alpha^2-\beta^2)\beta g(Y,Z)\\ &-(n-1)(\alpha^2-\beta^2)\alpha g(Y,\phi Z)+\alpha S(Y,\phi Z). \end{aligned}$$

Now for a 3-dimensional trans-Sasakian manifold under the condition (1.3), we can write from Lemma 3.3

$$\begin{aligned}[(\nabla_\xi Q)Y-(\nabla_Y Q)\xi] &= \beta QY-2(\alpha^2-\beta^2)\beta Y\\ &+2(\alpha^2-\beta^2)\alpha\phi Y-\alpha\phi QY. \end{aligned} \tag{3.3}$$

We now prove our main results.

**Theorem 3.4.** *Let $M$ be a 3-dimensional trans-Sasakian manifold fulfilling the condition (1.3). If there is a non-constant function $\lambda$ on $M$ satisfying the Miao-Tam critical equation (1.1), then the manifold $M$ is either $\beta$-Kenmotsu or of constant curvature.*

**Proof.** Substituting $X = \xi$ in (3.1) and using (3.2) we get

$$\begin{aligned} R(\xi, Y)D\lambda &= (\xi\lambda)QY - 2(\alpha^2 - \beta^2)(Y\lambda)\xi + \lambda\{\beta QY - 2(\alpha^2 - \beta^2)\beta Y \\ &\quad +2(\alpha^2 - \beta^2)\alpha\phi Y - \alpha\phi QY\} - \frac{r}{2}(\xi\lambda)Y + \frac{r}{2}(Y\lambda)\xi. \end{aligned} \tag{3.4}$$

Taking inner product of (3.4) with $X$ we obtain

$$\begin{aligned} g(R(\xi, Y)D\lambda, X) &= (\xi\lambda)S(X, Y) - 2(\alpha^2 - \beta^2)(Y\lambda)\eta(X) + \lambda\{\beta S(X, Y) \\ &\quad -2(\alpha^2 - \beta^2)\beta g(X, Y) + 2(\alpha^2 - \beta^2)\alpha g(X, \phi Y) \\ &\quad -\alpha g(X, \phi QY)\} - \frac{r}{2}(\xi\lambda)g(X, Y) + \frac{r}{2}(Y\lambda)\eta(X). \end{aligned} \tag{3.5}$$

Again,

$$g(R(\xi, Y)D\lambda, X) = -g(R(\xi, Y)X, D\lambda).$$

Making use of (2.14), the above equation yields

$$g(R(\xi, Y)D\lambda, X) = -(\alpha^2 - \beta^2)g(X, Y)(\xi\lambda) + (\alpha^2 - \beta^2)\eta(X)(Y\lambda). \tag{3.6}$$

From (3.5) and (3.6) we get

$$\begin{aligned} &-(\alpha^2 - \beta^2)g(X, Y)(\xi\lambda) + (\alpha^2 - \beta^2)\eta(X)(Y\lambda) \\ =\ &(\xi\lambda)S(X, Y) - 2(\alpha^2 - \beta^2)(Y\lambda)\eta(X) + \lambda\{\beta S(X, Y) \\ &-2(\alpha^2 - \beta^2)\beta g(X, Y) + 2(\alpha^2 - \beta^2)\alpha g(X, \phi Y) \\ &-\alpha g(X, \phi QY)\} - \frac{r}{2}(\xi\lambda)g(X, Y) + \frac{r}{2}(Y\lambda)\eta(X). \end{aligned} \tag{3.7}$$

Now interchanging $X$ and $Y$ in (3.7) and subtracting the resulting equation from (3.7) gives

$$\begin{aligned} &(\frac{r}{2} - 3(\alpha^2 - \beta^2))\{\eta(X)(Y\lambda) - \eta(Y)(X\lambda)\} \\ =\ &2\alpha(\alpha^2 - \beta^2)\lambda\{g(\phi X, Y) - g(X, \phi Y)\} \\ &-\alpha\lambda\{g(\phi QX, Y) - g(\phi QY, X)\}. \end{aligned} \tag{3.8}$$

From (2.10), we can see that $Q\phi = \phi Q$ on $M$ under the condition (1.3). Using this relation, the foregoing equation yields

$$\begin{aligned} (\frac{r}{2} - 3(\alpha^2 - \beta^2))\{\eta(X)(Y\lambda) - \eta(Y)(X\lambda)\} &= 4\alpha(\alpha^2 - \beta^2)\lambda g(\phi X, Y) \\ &\quad -2\alpha\lambda g(\phi QX, Y). \end{aligned} \tag{3.9}$$

Putting $X = \phi X$ and $Y = \phi Y$ in (3.9) and using (2.2) we get

$$\alpha\lambda\{2(\alpha^2 - \beta^2)g(\phi X, Y) - g(Q\phi X, Y)\} = 0. \tag{3.10}$$

Since $\lambda$ is a non-constant function, replacing $X$ by $\phi X$ in (2.10) and using (2.1) we have

$$\alpha\{S(X, Y) - 2(\alpha^2 - \beta^2)g(X, Y)\} = 0, \tag{3.11}$$

which implies that
either $\alpha = 0$ or $S(X, Y) = 2(\alpha^2 - \beta^2)g(X, Y)$.

Case 1: If $\alpha = 0$, then from (1.4), we have $\beta = constant$. This implies that the manifold $M$ is $\beta$-Kenmotsu.

Case 2: If $S(X, Y) = 2(\alpha^2 - \beta^2)g(X, Y)$, then the manifold is Einstein. Since the dimension of the manifold is three, then it becomes a space of constant curvature. This complees the proof.□

**Remark 3.5.** Since $\beta$ is constant if we change the metric $g$ by $\beta g$ by homothetic transformation, this homothetic transformation gives the homothety between $\beta$-Kenmotsu manifold and the Kenmotsu manifold.

**Corollary 3.6.** *Let $M$ be a 3-dimensional trans-Sasakian manifold fulfilling the condition (1.3). If there is a non-constant function $\lambda$ on $M$ satisfying the Miao-Tam critical equation (1.1), then the manifold $M$ is homothetic to a Kenmotsu manifold, provided $M$ is not of constant curvature.*

**Remark 3.7.** Since $\alpha = 0$ and $\beta =$ constant, from (2.7) it follows that

$$S(X,\xi) = -2\beta^2 g(X,\xi),$$

which implies that the characteristic vector field $\xi$ is an eigen vector of the Ricci operator $Q$ corresponding to the eigen value $-2\beta^2$.

**Corollary 3.8.** *Let $M$ be a complete 3-dimensional trans-Sasakian manifold fulfilling the condition (1.3) with $\alpha^2 > \beta^2$. If there is a non-constant function $\lambda$ on $M$ satisfying the Miao-Tam critical equation (1.1), then the manifold $M$ is isometric to a spherical space of curvature $(\alpha^2 - \beta^2)$.*

**Proof.** Considering the dimension of the manifold and tracing (1.1), yields $\Delta\lambda = -\frac{r\lambda+3}{2}$. From (3.11), we obtained that the manifold satisfies

$$S(X,Y) = 2(\alpha^2 - \beta^2)g(X,Y),$$

which implies $r = 6(\alpha^2 - \beta^2)$. Substituting this value of $S$ and $r$ in (1.1) we obtain

$$\nabla^2\lambda = \{-(\alpha^2 - \beta^2)\lambda - \frac{1}{2}\}g. \tag{3.12}$$

Now we apply Tashiro's theorem [24]: " If a complete Riemannian manifold $M^n$ of dimension $\geq 2$ admits a special concircular field $\rho$ satisfying $\nabla\nabla\rho = (-c^2\rho + b)g$, then it is isometric to a spherical space of curvature $c^2$ " to conclude that the manifold $M$ is isometric to a spherical space of curvature $(\alpha^2 - \beta^2)$. This complete the proof. □

Now if we take $\alpha$ a non-zero constant and $\beta = 0$, then the manifold becomes a 3-dimensional $\alpha$-Sasakian manifold and we can state the above corollary as given below.

**Corollary 3.9.** *Let $M$ be a complete 3-dimensional $\alpha$-Sasakian manifold. If there is a non-constant function $\lambda$ on $M$ satisfying the Miao-Tam critical equation (1.1), then the manifold $M$ is isometric to a spherical space of curvature $\alpha^2$.*

Again if we take $\alpha = 1$ and $\beta = 0$, then the trans-Sasakian manifold reduces to a Sasakian manifold. Thus we can state the following:

**Corollary 3.10.** *Let $M$ be a complete 3-dimensional Sasakian manifold. If there is a non-constant function $\lambda$ on $M$ satisfying the Miao-Tam critical equation (1.1), then the manifold $M$ is isometric to a unit sphere.*

**Remark 3.11.** Since 3-dimensional $K$-contact manifold reduces to a Sasakian manifold, the above corollary recovers the theorem of Patra and Ghosh [23].

**Theorem 3.12.** *Let $M$ be a 3-dimensional trans-Sasakian manifold fulfilling the condition (1.3). If there is a non-constant function $\lambda$ on $M$ satisfying the Fischer-Marsden equation (1.2), then the manifold $M$ is either $\beta$-Kenmotsu or of constant curvature.*

The proof of the Theorem 3.12 is similar to that of Theorem 3.4. Here we have to initialize the proof with Lemma 3.2. So we do not mention the proof here. We only obtain the consequences of this theorem.

**Corollary 3.13.** *Let $M$ be a 3-dimensional $\alpha$-Sasakian manifold. If there is a non-constant function $\lambda$ on $M$ satisfying the Fischer-Marsden equation (1.2), then the manifold $M$ is a space of contant curvature.*

**Corollary 3.14.** *Let $M$ be a complete 3-dimensional trans-Sasakian manifold fulfilling the condition (1.3) with $\alpha^2 > \beta^2$. If there is a non-constant function $\lambda$ on $M$ satisfying the Fischer-Marsden equation (1.2), then the manifold $M$ is isometric to the sphere $S^3(\sqrt{(\alpha^2-\beta^2)})$ of radius $\frac{1}{\sqrt{(\alpha^2-\beta^2)}}$.*

**Proof.** Considering the dimension $n = 3$ and tracing the Fischer-Marsden equation (1.2) yields $\Delta\lambda = -\frac{r\lambda}{2}$.
In a similar manner as in (3.11), we can easily obtain that the manifold satisfies

$$S(X,Y) = 2(\alpha^2 - \beta^2)g(X,Y),$$

which implies $r = 6(\alpha^2 - \beta^2)$. Substituting this value of $S$ and $r$ in (1.2) we obtain

$$\nabla^2\lambda = -(\alpha^2 - \beta^2)\lambda g. \tag{3.13}$$

The foregoing equation reveals that

$$\nabla_X D\lambda = -(\alpha^2 - \beta^2)\lambda X. \tag{3.14}$$

We now apply Obata's theorem [20]: " In order for a complete Riemannian manifold of dimension $n \geq 2$ to admit a non-constant function $\lambda$ with $\nabla_X D\lambda = -c^2\lambda X$ for any vector $X$, it is necessary and sufficient that the manifold is isometric with a sphere $S^n(c)$ of radius $\frac{1}{c}$ " to conclude that the manifold is isometric to the sphere $S^3(\sqrt{(\alpha^2-\beta^2)})$ of radius $\frac{1}{\sqrt{(\alpha^2-\beta^2)}}$. This completes the proof. □

Now if we take $\alpha = 1$ and $\beta = 0$, then the trans-Sasakian manifold reduces to a Sasakian manifold. Thus we can state the following:

**Corollary 3.15.** *Let $M$ be a complete 3-dimensional Sasakian manifold. If there is a non-constant function $\lambda$ on $M$ satisfying the Fischer-Marsden equation (1.2), then the manifold $M$ is isometric to a unit sphere.*

**Remark 3.16.** Since 3-dimensional $K$-contact manifold reduces to a Sasakian manifold, the above corollary recovers the theorem of Patra and Ghosh [22].

Again if we take $\alpha = \beta = 0$, then the trans-Sasakian manifold reduces to a cosymplectic manifold. Hence from (3.13) we have $\nabla^2\lambda = 0$, which implies that the function $\lambda$ is harmonic. Moreover if $\lambda_1$, $\lambda_2$ be two harmonic functions satisfying (1.2) then their sum $(\lambda_1 + \lambda_2)$ and scalar prodauct $k\lambda$, $k \in \mathbb{R}$ is again a harmonic function. This shows that the solution space $\mathcal{S}(\lambda)$ of (1.2) of harmonic functions is a linear space over $\mathbb{R}$. Thus we arrive to the following:

**Corollary 3.17.** *Let $M$ be a complete, non-compact 3-dimensional cosymplectic manifold. Then the solution space $\mathcal{S}(\lambda)$ of the Fischer-Marsden equation (1.2) is a linear space of harmonic functions over the set of real numbers.*

Again if the cosymplectic manifold is compact, then the function $\lambda$ satisfying $\nabla^2\lambda = 0$ becomes constant. Hence we can state the following:

**Corollary 3.18.** *If $M$ be a compact 3-dimensional cosymplectic manifold, then Fischer-Marsden equation (1.2) admits only trivial solution on $M$.*

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