Differential Geometry

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Contents

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Administrative Stuff

There are some Lecture Notes online. They have some stuff that we won't cover. The best book is Spivak.

Manifolds and Vector Spaces

Smooth Manifolds

Definition 1. If $U \subset \mathbb{R}^m$ and $\delta\colon U \to \mathbb{R}$, we say that δ is **smooth** or C^∞ if has continuous partial derivatives of all orders.

Definition 2. A topological space X is called **second countable** if there exists a countable collection $\mathcal{B}$ of open subsets of X such that any open subset of X may be written as the union of sets of $\mathcal{B}$.

Definition 3. A Hausdorff, second countable topological space X is called a **topological manifold** of dimension d if each point has an open neighborhood (nbhd) homeomorphic to an open subset U of $\mathbb{R}^d$ by a homeomorphism $\phi\colon U \xrightarrow{\sim} \phi(U) \subset \mathbb{R}^d$.

The pair (U, ϕ) of a homeomorphism and open subset of M is called a **chart**: given open subsets U and V of X with $U \cap V \neq \varnothing$, and charts (U, ϕ_U) and (V, ϕ_V), with $\phi_U\colon U \to \phi(U) \subset \mathbb{R}^d$ and $\phi_V\colon V \to \phi(V) \subset \mathbb{R}^d$, we have a homeomorphism $\phi_{VU} = \phi_V \circ \phi_U^{-1}\colon \phi_U(U \cap V) \to \phi_V(U \cap V)$ of open subsets of $\mathbb{R}^d$.

Given a chart (U, ϕ_U) and a point $p \in U$, we call U a **coordinate neighborhood** of p and we call the functions $x_i\colon U \to \mathbb{R}$ given by $\pi_i \circ \phi_U$ (where π_i is the projection onto the i-th coordinate) **coordinates** of U.

Definition 4. A **smooth structure** on a topological manifold is a collection $\mathcal{A}$ of charts (U_α, ϕ_α) for $\alpha \in A$, such that

(i) $\{U_\alpha \mid \alpha \in A\}$ is an open cover of M;

(ii) for any $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$, the **transition function** $\phi_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1}$ is smooth. The charts ϕ_α and ϕ_β are said to be **compatible**;

(iii) the collection of charts ϕ_α is maximal with respect to (ii). In particular, this means that if a chart ϕ is compatible with all the ϕ_α, then it's already in the collection.

Remark 5. Since $\phi_{\alpha\beta} = \phi_{\beta\alpha}^{-1}\colon \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$, both $\phi_{\beta\alpha}$ and $\phi_{\alpha\beta}$ are in fact diffeomorphisms (since by assumption, they are smooth).

This remark shows that item (ii) in Definition 4 implies that transition functions are diffeomorphisms.

For notation, we sometimes write $U_{\alpha\beta} = U_\alpha \cap U_\beta$.

Definition 6. A collection of charts $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ satisfying items (i) and (ii) in Definition 4 is called an **atlas**.

Claim 7. Any atlas $\mathcal{A}$ is contained in a unique maximal atlas and so defines a unique smooth structure on the manifold.

Proof. If $\mathcal{A} = \{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ is an atlas, we define a new atlas $\mathcal{A}^*$ of all charts on M compatible with every chart in A. To be compatible with every chart in $\mathcal{A}$ means that if $(U, \phi) \in \mathcal{A}^*$, $\phi_{UU_\alpha} = \phi \circ \phi_\alpha^{-1}$ is smooth for all $\alpha \in A$.

We should justify that $\mathcal{A}^*$ is an atlas. This involves checking conditions (i) and (ii) in Definition 4.

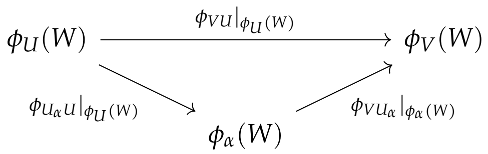
Clearly (i) is satisfied, because $\mathcal{A}^*$ contains $\mathcal{A}$ and $\mathcal{A}$ covers M.

For (ii), we suppose (U, ϕ_U) and (V, ϕ_V) are elements of $\mathcal{A}^*$. We show that the homeomorphism ϕ_{VU} is smooth. It suffices to show that ϕ_{VU} is smooth in a neighborhood of each point $\phi_\alpha(p)$ for $\phi_\alpha \in \mathcal{A}$. To that end, consider the neighborhood $\phi_U(U_\alpha \cap U \cap V)$ of $\phi_\alpha(p)$ in $\phi_U(U \cap V)$. It suffices to show that ϕ_{VU} is smooth when restricted to this neighborhood; that is, we want to show that

$$\phi_{VU} \mid_{\phi_U(U \cap V \cap U_\alpha)} : \phi_U(U \cap V \cap U_\alpha) \to \phi_V(U \cap V \cap U_\alpha)$$

is smooth. Let $W = U \cap V \cap U_\alpha$. $\phi_{VU} \mid_{\phi_U(W)}$ can be realized as the composition of two smooth transition functions as follows:

$$\phi_{VU} \mid_{\phi_U(W)} = \phi_V \circ \phi_\alpha^{-1} \circ \phi_\alpha \circ \phi_U^{-1} \mid_{\phi_U(W)} = (\phi_V \circ \phi_\alpha^{-1}) \mid_{\phi_\alpha(W)} \circ (\phi_\alpha \circ \phi_U^{-1}) \mid_{\phi_U(W)}$$



Since each of $\phi_{U_\alpha U}$ and ϕ_{VU_α} is smooth by assumption, then so is their composite and so ϕ_{VU} is smooth at $\phi_\alpha(p)$. Therefore, it is smooth.

Now finally, we need to justify that $\mathcal{A}^*$ is maximal. Clearly any atlas containing $\mathcal{A}$ must consist of elements of $\mathcal{A}^*$. So $\mathcal{A}^*$ is maximal and unique. $\square$

Definition 8. A topological manifold M with a smooth structure is called a **smooth manifold** of dimension d. Sometimes we use M^d to denote dimension d.

Remark 9. We can also talk about C^k manifolds for $k > 0$. But this course is about smooth manifolds.

Example 10.

(i) $\mathbb{R}^d$ with the chart consisting of one element, the identity, is a smooth manifold.

(ii) $S^d \subseteq \mathbb{R}^{d+1}$ is clearly a Hausdorff, second-countable topological space. Let $U_i^+ = \{\vec{x} \in S^d \mid x_i > 0\}$ and let $U_i^- = \{\vec{x} \in S^d \mid x_i < 0\}$. We have charts $\phi_i \colon U_i^+ \to \mathbb{R}^d$ and $\psi_i \colon U_i^- \to \mathbb{R}^d$ given by just forgetting the i-th coordinate. Note that $\phi_2 \circ \phi_1^{-1}$ (and $\psi_2 \circ \phi_1^{-1}$) are both maps defined by

$$(y_2, \ldots, y_{d+1}) \to \left(\sqrt{1 - y_2^2 - \ldots - y_{d+1}^2}\,,\ y_3, \ldots, y_{d+1} \right).$$

This is smooth on an appropriate subset of

$$\phi_1(U_1^+) = \left\{ (y_2, \ldots, y_{d+1}) \mid y_2^2 + \ldots + y_{d+1}^2 < 1 \right\}$$

given by $y_2 > 0$ (resp. $y_2 < 0$). The reason that $y_2 > 0$ is the appropriate subset is because $U_1^+ \cap U_2^+ = \{\vec{x} \in S^d \mid x_1 > 0 \text{ and } x_2 > 0\}$, and we want $\phi_1^{-1}(y_2, \ldots, y_{d+1})$ to be in U_2^+ so that it's in the domain of ϕ_2.

From this it follows that S^d is a smooth manifold. We should be careful to note that each $\vec{x} \in S^d$ has some $x_i \neq 0$, so lies in one of the sets U_i^+ or U_i^-.

(iii) Similarly the **real projective space** $\mathbb{RP}^d = S^d/\{\pm 1\}$ identifying antipodal points is a smooth manifold.

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Example 11. Further examples. Continued from last time.

(iv) Consider the equivalence relation on $\mathbb{R}^2$ given by $\vec{x} \sim \vec{y}$ if and only if $x_1 - y_1 \in \mathbb{Z}$, $x_2 - y_2 \in \mathbb{Z}$. Let T denote the quotient topological space the 2-dimensional torus. Any unit square Q in $\mathbb{R}^2$ with vertices at (a,b), $(a+1,b)$, $(a,b+1)$, and $(a+1,b+1)$ determines a homeomorphism $\pi \colon \operatorname{int} Q \xrightarrow{\sim} U(Q) \subset T$, with $U(Q) = \pi(\operatorname{int} Q)$ open in T. The inverse is a chart. Given two different unit squares Q_1, Q_2, we get the coordinate transform ϕ_{21} which is locally (but not globally) just given by translation. This gives a smooth structure on T. Similarly define the n**-torus** $T^n = \mathbb{R}^n/\mathbb{Z}^n$ as a smooth manifold.

Definition 12. Let M^m, N^n be smooth manifolds with given smooth structures. A continuous map $f \colon M \to N$ is **smooth** if for each $p \in M$, there are charts (U, ϕ_U), (V, ψ_V) with $p \in U$, $f(p) \in V$, such that $\overline{f} = \psi_V \circ f \circ \phi_U^{-1}$ is smooth.

$$\begin{array}{ccc}
p & \longmapsto & f(p) \\
\cap & & \cap \\
U \cap f^{-1}(V) & \xrightarrow{\quad f \quad} & V \\
\downarrow{\scriptstyle \phi_U} & & \downarrow{\scriptstyle \psi_V} \\
\phi_U(U \cap f^{-1}(V)) & \xrightarrow[\quad \overline{f} \quad]{} & \psi_V(V)
\end{array}$$

Note that since the coordinate transforms for different charts are diffeomorphisms, this implies that the condition that $\overline{f}$ is smooth holds for all charts $(U', \phi_{U'})$, $(V', \psi_{V'})$ with $p \in U'$, $f(p) \in V'$.

Definition 13. A **smooth function** f on an open $U \subseteq M$ is just a smooth map $f \colon U \to \mathbb{R}$ where $\mathbb{R}$ has its natural structure.

Definition 14. A homeomorphism $f \colon M \to N$ of smooth manifolds is called a **diffeomorphism** if both f and f^{-1} are smooth maps.

Tangent Spaces

Definition 15. Suppose $p \in M$. Smooth functions f, g defined on open neighborhoods of p are said to have the same **germ** if they agree on some open neighborhood. More precisely, a germ is an equivalence class on the set $\{(U, f) \mid p \in U, f \colon U \to \mathbb{R}\}$ under the relation $\sim$ where $(U, f) \sim (V, g)$ if and only if there is an open $W \subseteq U \cap V$ such that $f|_W = g|_W$.

Denote the set of germs of smooth functions at p by $\mathcal{A}_p = \mathcal{A}_{M,p}$. We can add, subtract, multiply germs without problems. Hence, $\mathcal{A}_p$ is a ring. There is a natural inclusion $\mathbb{R} \hookrightarrow \mathcal{A}_p$ of constant germs. So $\mathcal{A}_p$ is an $\mathbb{R}$-module. This is the **ring of germs at** p.

A germ has a well-defined value at p. We set $\mathcal{F}_p \subset \mathcal{A}_p$ to be the ideal of germs vanishing at p. We can also say that this is the kernel of the evaluation map $\mathcal{F}_p = \ker(f \mapsto f(p))$. This is the unique maximal ideal of $\mathcal{A}_p$ (and so $\mathcal{A}_p$ is a local ring) because any germ which doesn't vanish at p has an inverse in $\mathcal{A}_p$ (after an appropriate shrinking of the neighborhood of p) and so cannot lie in any maximal ideal.

Definition 16. A **tangent vector** v at $p \in M$ is a linear derivation of the algebra $\mathcal{A}_p$. In particular, this means that $v(fg) = f(p)v(g) + v(f)g(p)$ for all $f, g \in \mathcal{A}_p$.

Definition 17. The tangent vectors form an $\mathbb{R}$-vector space: given tangent vectors v, w and $\lambda \in \mathbb{R}$, we define $(v + w)(f) = v(f) + w(f)$ and $\lambda v(f) = v(\lambda f)$. The **tangent space to** M **at** p is this vector space, denoted by M_p or T_pM or $(TM)_p$.

If c denotes the constant germ at p for $c \in \mathbb{R}$, then for any tangent vector v, $v(c) = cv(1)$. What's $v(1)$? Well, $v(1) = v(1 \cdot 1) = v(1) + v(1) = 2v(1)$, so $v(c) = 0$ for all $c \in \mathbb{R}$.

Let M be a manifold and let $p \in M$. Let $\overline{\mathcal{A}_0} = \mathcal{A}_{\mathbb{R}^d, \vec{0}}$ denote the germs of smooth functions at $\vec{0}$ in $\mathbb{R}^d$ and (U, ϕ) be a chart with $p \in U$ and $\phi(p) = 0$. By definition of smooth functions on an open subset of M, we have an isomorphism of $\mathbb{R}$-algebras $\phi^* \colon \overline{\mathcal{A}_0} \xrightarrow{\sim} \mathcal{A}_p$ given locally at p by $\overline{f} \mapsto \delta = \overline{f} \circ \phi$. The inverse of ϕ^* is given locally by $f \mapsto \overline{f} = f \circ \phi^{-1}$.

A tangent vector v at $p \in M$ determines a tangent vector $\phi_*(v)$ at zero in $\mathbb{R}^d$.

$$\phi_*(v)\left(\overline{f}\right) = v\left(\overline{f} \circ \phi\right)$$

So the chart ϕ determines an identification $\phi_* \colon T_pM \to T_{\vec{0}}\mathbb{R}^d$.

Therefore, to understand the tangent space, it suffices to understand the tangent space $T_{\vec{0}}\mathbb{R}^d$. This is just the linear derivations of $\mathcal{A}_{\mathbb{R}^d, \vec{0}}$ If $\mathbb{R}^d$ has standard coordinates $r_1, \dots, r_d$, then ${}^{\partial}\!/_{\partial r_1}\big|_{\vec{0}}, \dots, {}^{\partial}\!/_{\partial r_d}\big|_{\vec{0}}$ are linear derivations of $\overline{\mathcal{A}}_0$.

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More tangent spaces

If $\mathbb{R}^d$ has standard coordinates $r_1, \ldots, r_d$, then ${}^{\partial}/_{\partial r_1}\big|_{\vec{0}}, \ldots, {}^{\partial}/_{\partial r_d}\big|_{\vec{0}}$ are linear derivations on $\overline{\mathcal{A}}_{\vec{0}}$.

Let (U, ϕ) be a chart with $p \in U$. Denoting $\phi\colon U \to \mathbb{R}^d$ on M by $\phi = (x_1, \ldots, x_d)$, set ${}^{\partial}/_{\partial x_i}\big|_p$ to be the linear derivation on $\mathcal{A}_p$ defined by

$$f \mapsto \frac{\partial(f \circ \phi^{-1})}{\partial r_i}(\vec{0})$$

Note that

$$\left.\frac{\partial}{\partial x_i}\right|_p (x_j) = \delta_{ij}.$$

Claim 18. The linear derivations

$${}^{\partial}/_{\partial r_1}\big|_{\vec{0}}, \ldots, {}^{\partial}/_{\partial r_d}\big|_{\vec{0}}$$

form a basis for $T_{\vec{0}}\mathbb{R}^d$ and so $\dim T_pM = d$ with basis

$$\left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_d}\right|_p$$

Proof. Since

$$\left(\sum_i a_i \left.\frac{\partial}{\partial r_i}\right|_{\vec{0}}\right)(r_j) = a_j,$$

it is clear they are linearly independent.

Now we need to show spanning. Given a linear derivation $v\colon \overline{\mathcal{A}}_{\vec{0}} \to \mathbb{R}$, set $a_i = v(r_i)$ and

$$r_0 = \sum_i a_i \left.\frac{\partial}{\partial r_i}\right|_{\vec{0}}.$$

Given any smooth germ (V, f) in $\overline{\mathcal{A}}_{\vec{0}}$ represented by a smooth function f on $V \ni \vec{0}$, a standard result from analysis says that we can, on some $B(\vec{0}, \varepsilon) \subset V$, write f as

$$f(\vec{r}) = f(\vec{0}) + \sum_i r_i \frac{\partial f}{\partial r_i}(\vec{0}) + \sum_{i,j} r_i r_j g_{ij}(\vec{r})$$

for some smooth functions g_{ij} on $B(\vec{0}, \varepsilon)$. Hence

$$v(f) = 0 + \sum a_i \frac{\partial f}{\partial r_i}(\vec{0}) + 0 = r_0(f)$$

for all germs f. Hence, $v = r_0$ and so the ${}^{\partial}/_{\partial r_i}$ span as well. □

Remark 19. Above proof shows that for any tangent vector v at p on M,

$$v = \sum_{i=1}^{d} v(x_i) \frac{\partial}{\partial x_i}\bigg|_p .$$

In particular, given local coordinate charts at p, $\phi = (x_1, \dots, x_d)$ and $\psi = (y_1, \dots, y_d)$, then

$$\frac{\partial}{\partial y_j}\bigg|_p = \sum_{i=1}^{d} \frac{\partial x_i}{\partial y_j}\bigg|_p \frac{\partial}{\partial x_i}\bigg|_p$$

Where

$$\frac{\partial x_i}{\partial y_j}\bigg|_p = \frac{\partial}{\partial y_j}\bigg|_p (x_i).$$

Applying this to a germ at p, this is a just local version of the chain rule.

Remark 20. There's a dangerous bend here! Even if $y_1 = x_1$, it's not in general true that

$$\frac{\partial}{\partial x_1}\bigg|_p = \frac{\partial}{\partial y_1}\bigg|_p$$

It depends on the charts!

Let $F = \phi \circ \psi^{-1}$ be a local coordinate transform and let the coordinates on $\operatorname{im} \psi$ be $s_1, \dots, s_d$,

$$\frac{\partial x_i}{\partial y_j}\bigg|_p = \frac{\partial}{\partial y_j}\bigg|_p (x_i) = \frac{\partial (x_i \circ \psi^{-1})}{\partial s_j}(\vec{0}),$$

where $x_i = r_i \circ \phi$. This implies that

$$\frac{\partial x_i}{\partial y_j} = \frac{\partial r_i \circ F}{\partial s_j}(\vec{0}) = \frac{\partial F_i}{\partial s_j}(\vec{0}).$$

Where F_i is the i-th coordinate of the transition function F. Therefore, the matrix

$$\left(\frac{\partial x_i}{\partial y_j}\bigg|_p \right)_{1 \leqslant i,j \leqslant d}$$

is just the Jacobian matrix of the coordinate transformation F, evaluated at $\vec{0}$.

Example 21. Let $M = \mathbb{R}^d$, the tangent space of $p \in M$ has natural basis

$$\left\{ \frac{\partial}{\partial x_i}\bigg|_p \right\}_{1 \leqslant i \leqslant d},$$

and so there exists a natural identification

$$T_p M \xrightarrow{\sim} T_{\vec{0}} \mathbb{R}^d \xrightarrow{\sim} \mathbb{R}^d$$

which identifies

$$\frac{\partial}{\partial x_i}\bigg|_p \leftrightarrow \frac{\partial}{\partial r_i}\bigg|_{\vec{0}} \leftrightarrow e_i$$

Maps between smooth manifolds

Given a map $f\colon M \to N$ of smooth manifolds with $f(p) = q$, we have an induced map $f^*\colon \mathcal{A}_{N,q} \to \mathcal{A}_{M,p}$ via $h \mapsto h \circ f$.

Definition 22. The **derivative** or **differential** of f is

$$d_p f = (df)_p\colon T_pM \to T_qN$$

for $v \in T_pM$, we define

$$(d_p f)(v)(h) = v(h \circ f)$$

for all $h \in \mathcal{A}_{N,q}$.

Claim 23. The **chain rule** is now easy. If $g\colon N \to X$ is a smooth map of manifolds with $g(q) = r$, then

$$d_p(g \circ f) = d_q g \circ d_p f\colon T_pM \to T_rX$$

Proof. For $v \in T_pM, h \in \mathcal{A}_{X,r}$, we compute the left hand side:

$$d_p(g \circ f)(v)(h) = v(h \circ g \circ f)$$

and the right hand side:

$$(d_q(g) \circ d_p f)(v)(h) = (d_p f)(v)(h \circ g) = v(h \circ g \circ f)$$

Hey look, they're equal! □

Example 24. If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and we write $f = (f_1, \dots, f_m)$, then

$$d_p f\colon T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m.$$

We give $T_p\mathbb{R}^n$ the basis

$$\left.\frac{\partial}{\partial x_1}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p$$

and give $T_{f(p)}\mathbb{R}^m$ the basis

$$\left.\frac{\partial}{\partial y_1}\right|_{f(p)}, \dots, \left.\frac{\partial}{\partial y_m}\right|_{f(p)},$$

then $d_p f$ corresponds to the map $\mathbb{R}^n \to \mathbb{R}^m$ given by the Jacobian matrix of f, since

$$(df)_p\left(\left.\frac{\partial}{\partial x_j}\right|_p\right)(y_i) = \left.\frac{\partial}{\partial x_j}\right|_p (y_i \circ f) = \left.\frac{\partial f_i}{\partial x_j}\right|_p$$

This then implies that

$$(df)_p\left(\left.\frac{\partial}{\partial x_1}\right|_p\right) = \sum_{i=1}^{m} \left.\frac{\partial f_i}{\partial x_j}\right|_p \left.\frac{\partial}{\partial y_i}\right|_{f(p)}.$$

More generally, given any coordinate chart $\phi = (x_1, \ldots, x_d)\colon U \to \mathbb{R}^d$, we can define

$$\left.\frac{\partial}{\partial x_i}\right|_p$$

by

$$\left.\frac{\partial}{\partial x_i}\right|_p (f) = \left.\frac{\partial(f \circ \phi^{-1})}{\partial r_i}\right|_{\phi(p)}$$

at *all* $p \in U$ where $f \in \mathcal{A}_p$.

If $\phi(p) = \vec{c} \in \mathbb{R}^d$, we may translate by $\vec{c}$, taking the chart $\psi = (y_1, \ldots, y_d)$ with $y_i = x_i - c_i$.

Thus for $f \in \mathcal{A}_p$, the previous definition implies that

$$\left.\frac{\partial f}{\partial y_i}\right|_p = \left.\frac{\partial(f \circ \psi^{-1})}{\partial r_i}\right|_{\vec{0}} = \left.\frac{\partial(f \circ \phi^{-1})}{\partial r_i}\right|_{\vec{c}} = \left.\frac{\partial}{\partial x_i}\right|_p.$$

Thus any coordinate system ϕ gives rise to tangent vectors $^{\partial}\!/_{\partial x_i}$ for *all* $p \in U$. Moreover, if f is a smooth function on U, then

$$\frac{\partial f}{\partial x_i} = \frac{\partial(f \circ \phi^{-1})}{\partial r_i}(\phi(p))$$

is the composition of two smooth functions on U, with

$$\frac{\partial f}{\partial x_i}(p) = \left.\frac{\partial}{\partial x_i}\right|_p (f)$$

for all p.

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A different way to think about tangent spaces

Definition 25. A **smooth curve** on M is a smooth map $\sigma\colon (a,b) \to M$. For $t \in (a,b)$, the tangent to the curve at $\sigma(t)$ is

$$(d\sigma)_t \left(\left.\frac{d}{dr}\right|_t \right) \in T_{\sigma(t)}M.$$

We denote this $\dot{\sigma}(t)$.

Example 26. If $\sigma\colon (a,b) \to \mathbb{R}^n$, and $\mathbb{R}^n$ has coordinates $x_1, \ldots, x_n$, say $\sigma(t) = (\sigma_1(t), \ldots, \sigma_n(t))$, then

$$(d\sigma)_t \left(\left.\frac{d}{dr}\right|_t \right)(x_i) = \left.\frac{d}{dr}\right|_t \sigma_i = \left.\frac{d\sigma_i}{dr}\right|_t = \dot{\sigma}_i(t).$$

Therefore,

$$\dot{\sigma}(t) = \sum_i \dot{\sigma}_i(t) \left.\frac{\partial}{\partial x_i}\right|_{\sigma(t)}$$

That is, in terms of natural identifications of $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ with basis

$$\left.\frac{\partial}{\partial x_1}\right|_{\sigma(t)}, \dots, \left.\frac{\partial}{\partial x_n}\right|_{\sigma(t)},$$

we have that $\dot{\sigma}(t)$ corresponds to $(\dot{\sigma}_1(t), \dots, \dot{\sigma}_n(t))$.

We say that a smooth curve $\sigma\colon (-\varepsilon, \varepsilon) \to M$ with $\sigma(0) = p$ defines a tangent vector $\dot{\sigma}(0) \in T_pM$. Informally, if σ is a germ of a smooth curve (i.e. has a small domain like $(-\varepsilon, \varepsilon)$), we call it a **short curve.**

If ϕ is a chart around p with $\phi(p) = 0$, then two such curves σ_1, σ_2 define the same tangent vector if and only if $\phi \circ \sigma_1$ and $\phi \circ \sigma_2$ have the same tangent vector at $\vec{0} \in \mathbb{R}^n$. We say that two short curves are **equivalent** if they define the same tangent vector.

Conversely, given a tangent vector

$$v = \sum a_i \left.\frac{\partial}{\partial x_i}\right|_p \in T_pM$$

with a coordinate chart $\phi = (x_1, \dots, x_n)$ such that $\phi(p) = \vec{0}$, then

$$\phi_* v = \sum a_i \left.\frac{\partial}{\partial r_i}\right|_{\vec{0}}$$

By a linear change of coordinates, we may assume this is just $\left.\partial/\partial r_1\right|_{\vec{0}}$, that is, $v = \left.\partial/\partial x_1\right|_p$.

Set $\sigma(r) = \phi^{-1}(r, 0, 0, \dots, 0) = \phi^{-1} \circ i_1$, where i_1 is inclusion into the first coordinate. Then compute

$$\begin{aligned}
\dot{\sigma}(0)(h) &= (d\sigma)_0 \left(\left.\frac{d}{dr}\right|_0 \right)(h) \\
&= \left.\frac{d}{dr}\right|_0 \left(h \circ \phi^{-1} \circ i_1 \right) \\
&= \left.\frac{\partial}{\partial r_1}\right|_{\vec{0}} \left(h \circ \phi^{-1} \right) = \left.\frac{\partial}{\partial x_1}\right|_p (h) = v(h)
\end{aligned}$$

Therefore, we can represent $v \in T_pM$ by an equivalence class of germs of smooth curves $\sigma\colon (-\varepsilon, \varepsilon) \to M$ with $\sigma(0) = p$.

Vector Fields

Definition 27. Let M be a smooth manifold. The **tangent bundle** of M is

$$TM = \bigsqcup_{p \in M} T_pM,$$

with a natural projection $\pi\colon TM \to M$.

Claim 28. *TM is naturally a smooth manifold of dimension $2n$, where n is the dimension of M.*

Proof Sketch. For any chart $\phi = (x_1, \ldots, x_n)\colon U \to \mathbb{R}^n$, T_pM has basis

$$\left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p$$

for any $p \in U$. We can then identify $\pi^{-1}(U)$ with $U \times \mathbb{R}^n$ via a map $\widetilde{\phi}$.

Given $p \in U$ and

$$v = \sum a_i \left.\frac{\partial}{\partial x_i}\right|_p \in T_pM,$$

define the image of (p, v) under $\widetilde{\phi}$ to be $(p, a_1, \ldots, a_n)$.

But this looks chart-dependent, so what happens if we take another chart? Given $\psi = (y_1, \ldots, y_n)$ on U, we can do the same. We write in these coordinates

$$v = \sum b_j \left.\frac{\partial}{\partial y_j}\right|_p$$

and the image of (p, v) under $\widetilde{\psi}$ is $(p, b_1, \ldots, b_n)$.

The map $\widetilde{\psi} \circ \widetilde{\psi}^{-1}$ is determined by

$$a_i = \sum_j \left.\frac{\partial x_i}{\partial y_j}\right|_p b_j,$$

where as in last lecture, $\left(\frac{\partial x_i}{\partial y_j}\right)$ corresponds to the Jacobian matrix of the coordinate transform.

We claimed that TM was a smooth manifold, so we should say what the topology on it is. The natural topology on $\pi^{-1}(U)$ is given by identification with $U \times \mathbb{R}^n$. We define a topology on TM whereby $W \subset TM$ is open if and only if $W \cap \pi^{-1}(U)$ is open for all charts patches (U, ϕ) of M.

We can also define a smooth atlas on TM by taking charts $\left(\pi^{-1}(U), (\phi \times \mathrm{id}) \circ \widetilde{\phi}\right)$ for chart (U, ϕ). We justify the coordinate transforms being smooth with the Jacobian matrix stuff from above.

The fact that $\widetilde{\psi} \circ \widetilde{\phi}^{-1}$ is linear on the fibers (given by the Jacobian matrix acting on $\mathbb{R}^n$) is the statement that TM is a vector bundle (which we'll talk about later). ☐

Exercise 29. A smooth map $f\colon M \to N$ induces a smooth map $df\colon TM \to TN$.

Definition 30. A **vector field** X on M is given by a **smooth section** $X\colon M \to TM$. (X being a smooth section means that $\pi \circ X = \mathrm{id}_M$). This says

$$X\colon M \to \bigsqcup_{p \in M} T_pM$$

with property that, for any coordinate chart $\phi = (x_1, \ldots, x_n)\colon U \to \mathbb{R}^n$, writing

$$X_p := X(P) = \sum_i a_i(p) \left.\frac{\partial}{\partial x_i}\right|_p,$$

the a_i are smooth functions on U (equivalently, $X(x_i)$ is smooth for all i).

Definition 31. Vector fields $X^{(1)}, \dots, X^{(n)}$ on M are **independent** if $X^{(1)}(p), \dots, X^{(n)}(p)$ form a basis for T_pM for any $p \in M$.

Theorem 32. Suppose M is a smooth manifold of dimension n on which there exist n independent vector fields $X^{(1)}, \dots, X^{(n)}$. Then TM is isomorphic to $M \times \mathbb{R}^n$ as a vector bundle (there is a diffeomorphism $TM \to M \times \mathbb{R}^n$ and for any $p \in M$, the restriction to T_pM is an isomorphism $T_pM \xrightarrow{\sim} \mathbb{R}^n$).

Proof. An element of TM is given by some $v \in T_pM$. Write

$$v = \sum_i a_i X^{(i)}(p),$$

and define a map $\Psi \colon TM \to M \times \mathbb{R}^n$ by

$$\Psi \colon (P, v) \mapsto (P, a_1, \dots, a_n)$$

with obvious inverse.

A mechanical check verifies that for a coordinate chart $\phi = (x_1, \dots, x_n) \colon U \to \mathbb{R}$, the corresponding map

$$U \times \mathbb{R}^n \xrightarrow{\sim} \pi^{-1}(U) \xrightarrow{\Psi|_U} U \times \mathbb{R}^n$$

is a diffeomorphism of smooth manifolds and an isomorphism on fibers $\mathbb{R}^n$. □

Example 33. TS^1 is isomorphic to $S^1 \times \mathbb{R}$ because there is a nowhere vanishing vector field $\partial/\partial\theta$. But TS^2 is not isomorphic to $S^2 \times \mathbb{R}^2$ by the Hairy Ball Theorem.

Lecture 5 — 17 October 2015

Let's begin with a little lemma that's often useful in calculating derivatives of maps. This is really just reinterpreting something we already know from calculus in the language of tangent spaces.

Lemma 34. Suppose $\psi \colon U \to \mathbb{R}^m$ is smooth, and

$$v = \sum_{i=1}^{n} h_i \left.\frac{\partial}{\partial x_i}\right|_{\vec{a}} \in T_{\vec{a}}(U) \cong \mathbb{R}^n,$$

then if $\kappa \colon T_{\psi(\vec{a})}\mathbb{R}^m \to \mathbb{R}^m$ gives $T_{\psi(\vec{a})}\mathbb{R}^m$ the canonical identification with $\mathbb{R}^m$ with the basis

$$\left\{ \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\},$$

we have

$$\kappa(d_{\vec{a}}\psi(v)) = \left.\frac{d}{dt}\right|_0 \psi(\vec{a} + t\vec{h}) \tag{1}$$

Proof. Set $\gamma\colon (-\varepsilon,\varepsilon) \to U$ given by $\gamma(t) = \vec{a} + t\vec{h}$.

$$\dot{\gamma}(0)(x_i) = d_0\gamma\left(\left.\frac{d}{dt}\right|_0\right) x_i = \left.\frac{\partial}{\partial t}\right|_0 (\vec{a} + t\vec{h})_i = h_i$$

Therefore, $\dot{\gamma}(0) = v$.

$$d_{\vec{a}}\psi(v)(y_j) = d_{\vec{a}}\psi d_0\gamma\left(\left.\frac{d}{dt}\right|_0\right)(y_j)$$

But now the chain rule is staring us in the face. So this becomes

$$d_0(\psi \circ \gamma)\left(\left.\frac{d}{dt}\right|_0\right)(y_j).$$

Now using the definition of derivative,

$$\left.\frac{d}{dt}\right|_0 \psi(\vec{a} + t\vec{h})_j$$

□

To show that this lemma is useful, consider the following example.

Example 35. Let $\psi\colon M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ given by $\psi(A) = AA^T$, where A^T is the transpose and $\vec{a} = I$. Then for $H \in M_{n\times n}(\mathbb{R})$, the right hand side of (1) is

$$\left.\frac{d}{dt}\right|_0 (I + tH)\,(I + tH)^T = H + H^T = \kappa d_I\psi\left(\sum_{p,q} H_{pq}\frac{d}{dx_{pq}}\right).$$

Vector Fields

Recall that if $\phi = (x_1, \dots, x_n)\colon U \to \mathbb{R}^n$ is a coordinate chart, then for f smooth on U,

$$\frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i}(f) = \frac{\partial(f \circ \phi^{-1})}{\partial r_i} \circ \phi$$

is smooth on U.

Definition 36. Given now $X\colon M \to TM$ a smooth vector field and $f\colon M \to \mathbb{R}$ a smooth function, we can define the function $X(f)\colon M \to \mathbb{R}$ by $X(f)(p) = X_p(f)$.

So there are two ways to think about X. Either as a map $M \to TM$, or as a map $C^\infty(M) \to C^\infty(M)$.

If locally for some chart (U, ϕ), with $\phi = (x_1, \dots, x_n)$,

$$X = \sum_i X_i \frac{\partial}{\partial x_i}$$

with X_i smooth, then

$$X(f) = \sum_i X_i \frac{\partial f}{\partial x_i} \tag{2}$$

is also smooth.

For X, Y smooth vector fields on M, we might hope that XY is a vector field by $(XY)(f) = X(Y(f))$ is a vector field. But it's not, because looking at (2) and multiplying it out or something,

$$(XY)(fg) = X(f)Y(g) + X(g)Y(f) + f(XY)(g) + g(XY)(f)$$

contains terms $X(f)Y(g) + X(g)Y(f)$ which are extra. We want XY to obey the Leibniz rule so that it's a tangent vector, but this clearly does not! Instead, we can get around this by using the **Lie bracket** which will cause the mixed terms to cancel. This is to say,

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

is a vector field. In particular, the Lie bracket is a bilinear form on vector fields.

Locally in a coordinate chart (U, ϕ), there are local vector fields ${}^{\partial}\!/\!{}_{\partial x_i} : U \to TU$. Note that $\left[{}^{\partial}\!/\!{}_{\partial x_i}, {}^{\partial}\!/\!{}_{\partial x_j}\right] = 0$, so mixed partials commute.

Exercise 37. Properties of the Lie Bracket (check these!)

(a) $[Y, X] = -[X, Y]$;

(b) $[fX, gY] = fg[X, Y] + f \cdot (X(g))Y - g \cdot (Y(f))X$ for all smooth f, g;

(c) $[X, [Y, Z]] + [Y, [X, Z]] + [Z, [X, Y]] = 0$, **(Jacobi Identity)**.

For (c), we need only check for $X = f{}^{\partial}\!/\!{}_{\partial x_i}$, $Y = g{}^{\partial}\!/\!{}_{\partial x_j}$, $Z = h{}^{\partial}\!/\!{}_{\partial x_k}$. Use (b) and the vanishing of the bracket for fields of the form ${}^{\partial}\!/\!{}_{\partial x_i}$.

Definition 38. A real vector space (perhaps infinite-dimensional) equipped with a bracket $[-, -]$ which is bilinear, antisymmetric, and satisfies the Jacobi identity is called a **Lie algebra**.

The case we're interested in is the space of smooth vector fields on M, which we denote $\Theta(M)$.

Given a diffeomorphism of manifolds $F : M \to N$ and a smooth vector field X on M, we have a vector field $F_* X$ on N defined by $(F_* X)(h) = X(h \circ F) \circ F^{-1}$. For a particular point $p \in M$,

$$(F_* X)_{F(p)}(h) = X_p(h \circ F) \circ F^{-1} = ((d_p F)(X_p))(h).$$

Exercise 39. On the first example sheet, show that

$$F_*[X, Y] = [F_* X, F_* Y]$$

Recall that a smooth curve $\sigma : (a, b) \to M$ determines a tangent vector

$$\dot\sigma(t) = d_t\sigma \left(\left.\frac{d}{dr}\right|_t \right) \in T_{\sigma(t)}M.$$

$$\dot\sigma(t)(f) = \left(d_t\sigma \left(\left.\frac{d}{dr}\right|_t \right) \right)(f) = (f \circ \sigma)'(t)$$

Definition 40. If X is a smooth vector field on M, a smooth curve $\sigma\colon (a,b) \to M$ is called an **integral curve** for X if $\dot{\sigma}(t) = X(\sigma(t))$ for all $t \in (a,b)$.

Theorem 41. Given a smooth vector field X on M, and $p \in M$, then exist $a, b \in \mathbb{R} \cup \{\pm\infty\}$ depending on p and a smooth curve $\gamma\colon (a,b) \to M$ such that

(i) $0 \in (a,b)$ and $\gamma(0) = p$;

(ii) γ is an integral curve of X;

(iii) if $\mu\colon (c,d) \to M$ is a smooth curve satisfying (i) and (ii), then $(c,d) \subseteq (a,b)$ and $\mu = \gamma|_{(c,d)}$.

Proof. To see this, work in local coordinates and reduce to a question about differential equations in $\mathbb{R}^n$. We want $d_\gamma \left({}^{d}\!/\!_{dr}\big|_t \right) = X(\gamma(t))$ for $t \in (a,b)$. We may assume that $0 \in (a,b)$ and $\gamma(0) = p$. Choose coordinates $x_1, \ldots, x_d$ around p (that is, a chart $\phi\colon U \to \mathbb{R}^d$). In these coordinates, write

$$X|_U = \sum_{i=1}^{d} f_i \frac{\partial}{\partial x_i}$$

for some f_i smooth functions on U.

Moreover, if $\gamma(t) \in U$, then

$$d_t\gamma \left(\frac{\partial}{\partial r}\bigg|_t \right) = \sum_{i=1}^{d} \frac{d(x_i \circ \gamma)}{dr}\bigg|_t \frac{\partial}{\partial x_i}\bigg|_{\gamma(t)},$$

since for any tangent vector v,

$$v = \sum v(x_i) \frac{\partial}{\partial x_i}\bigg|_p.$$

So if $\gamma_i = x_i \circ \gamma$, we wish to solve the first order system of ODE's

$$\frac{d\gamma_i}{dt} = f_i(\gamma(t)) = f_i \circ \phi^{-1}(\gamma_1(t), \ldots, \gamma_d(t)) = g_i(\gamma_1(t), \ldots, \gamma_d(t)).$$

For $g_i = f_i \circ \phi^{-1}$. The standard theory of ODE's implies that there is a solution. □

Remark 42. If we also vary p, and set $\phi_t(p) = \gamma_p(t)$, where $\gamma_p(t)$ is just the integral curve we discovered for X through p, we obtain what's called a **local flow**. A local flow is an open $U \ni p$, for $\varepsilon > 0$ and diffeomorphisms $\phi_t\colon U \to \phi_t(U) \subseteq M$ for $|t| < \varepsilon$ such that $\gamma_p(t)$ is smooth in both t and p.

Lecture 6 — 20 October 2015

Submanifolds

Definition 43. Suppose that $F\colon M \to N$ is a smooth map of manifolds. We have several concepts:

(i) F is an **immersion** if $(dF)_p = d_pF$ is an injection for each $p \in M$;

(ii) (M, F) is a **submanifold** of N if F is an injective immersion;

(iii) F is an **embedding** if (M, F) is a submanifold of N and F is a homeomorphism onto its image (with the subspace topology).

Example 44. Note that an immersion may not have a manifold as its image. For example, the embedding of the real line in $\mathbb{R}^2$ as the nodal cubic.

An example of a submanifold that is not an immersion is as follows: a line with irrational slope in $\mathbb{R}^2$ gives rise to a submanifold of the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ whose image is dense in T, and therefore not an embedded submanifold.

From now on, I'll take the word "submanifold" to mean "embedded submanifold." Usually we identify M with its image in N and take F to be the inclusion map.

Definition 45. Given a smooth map $F\colon M \to N$ of manifolds, a point $q \in N$ is called a **regular value** if, for any $p \in M$ such that $F(p) = q$, we have $d_pF\colon T_pM \to T_qN$ is surjective.

Theorem 46. If $F\colon M \to N$ is smooth, q is a regular value in $F(M)$, then the fiber $F^{-1}(q)$ is an embedded submanifold of M of dimension $\dim M - \dim N$, and for any point $p \in F^{-1}(q)$,

$$T_p(F^{-1}(q)) = \ker(d_pF\colon T_pM \to T_qN).$$

Proof. This is easily seen as just an application (in local coordinates) of the inverse/implicit function theorem – see the part II course or Warner, Theorem 1.38. □

Example 47. The group $\mathrm{GL}(n, \mathbb{R})$ is an open submanifold of $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$. The symmetric $n \times n$ matrices S may be identified with $\mathbb{R}^{n(n+1)/2}$. Define $\psi\colon \mathrm{GL}(n, \mathbb{R}) \to S$ by $A \mapsto AA^T$. Note that $\psi^{-1}(I) = O(n)$ is the orthogonal group ($A \in O(n) \iff AA^T = I$). Since $A \in O(n)$ if and only if its columns are orthogonal, we see that $O(n)$ is compact.

For any A in $\mathrm{GL}(n, \mathbb{R})$, we can define a linear map $R_A\colon M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ given by right multiplication by A, inducing a diffeomorphism $R_A\colon \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$. Observe that for $A \in O(n)$, the $\psi \circ R_A = \psi$. The extended version of the chain rule implies that when $A \in O(n)$, $d_A\psi \circ R_A = d_I\psi$, and hence $d\psi$ has the same rank at all points of $O(n)$.

But $d_I\psi(H)$ for $H \in M_{n\times n}(\mathbb{R})$ was identified as $H + H^T$, and a general symmetric matrix is of this form, so the map $d_I\psi$ is surjective. This implies by the previous theorem that $O(n)$ is an embedded submanifold of $\mathrm{GL}(n, \mathbb{R})$ of dimension ${}^{n(n-1)}/_2$.

Since $A \in O(n)$ has $\det(A) = \pm 1$, then $O(n)$ has two connected components. $\mathrm{SO}(n)$ is the component with $\det(A) = +1$, containing the identity.

Now, the tangent space of $O(n)$ at the identity is just

$$\ker(d_I\psi\colon T_I\mathrm{GL}(n, \mathbb{R}) \to T_{\psi(I)=I}S).$$

But $d_I\psi$ is the map $H \mapsto H + H^T$, so $T_I O(n) = \{H \in M_{n\times n}(\mathbb{R}) \mid H + H^T = 0\}$.

If now $M \hookrightarrow N$ is an embedded submanifold, then $T_pM \hookrightarrow T_pN$ in a natural way: $v \in T_pM$ acts on $\mathcal{A}_p(N)$ by $f \mapsto v(f|_M)$. Furthermore, $TM \hookrightarrow TN$ as an embedded submanifold (easiest to see by quoting example sheet 1, question 9).

Definition 48. Given a smooth manifold N, an r-dimensional **distribution** $\mathcal{D}$ is a choice of r-dimensional subspaces $\mathcal{D}(p)$ of T_pN for each $p \in N$. Such a distribution is a **smooth distribution** if for each point $p \in N$, there is an open neighborhood $U \ni p$ and smooth vector fields $X_1, \dots, X_r$ on U spanning $\mathcal{D}(p)$.

Definition 49. A **smooth distribution** is called **involutive** or **completely integrable** if for all smooth vector fields X, Y belonging to $\mathcal{D}$, (i.e. $X(q), Y(q) \in \mathcal{D}(q)$ for all q), the Lie bracket $[X, Y]$ also belongs to $\mathcal{D}$.

Definition 50. A **local integrable submanifold** M of $\mathcal{D}$ through p is a local embedded submanifold, $(M \hookrightarrow U \ni p)$ with $T_qM = \mathcal{D}(q) \subseteq T_qN$ for all $q \in M$. If $\mathcal{D}$ is r-dimensional, it must be the case that M is also r-dimensional.

Remark 51. If there is a local integrable submanifold through each point N, then it's easy to check that if $\mathcal{D}$ satisfies $\mathcal{D}(q) = \operatorname{im}(T_qM \to T_qU)$, then it is an **involution**.

The following (in red) is possibly wrong, or at least misleading.

Given an embedded submanifold $M \subseteq N$, there are local coordinates $x_1, \dots, x_n$ on N such that M is given by $x_{m+1}, \dots, x_n = 0$ and $x_1, \dots, x_m$ are local coordinates on M. Then

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$$

is a local involutive distribution.

Theorem 52 (Frobenius Corrected)**.** The converse of this statement is also true! If $X_1, \dots, X_m$ is an involutive distribution, then locally there is a submanifold $M \subseteq N$ and $x_1, \dots, x_n$ on N such that M is given by $x_{m+1}, \dots, x_n = 0$ and $x_1, \dots, x_m$ are coordinates on M, with $X_i = \partial/\partial x_i$ for $1 \leqslant i \leqslant m$.

I won't prove it because it takes up four pages in Warner's book (pg. 42-46). The proof proceeds by induction on the dimension of the distribution, and depends heavily on the involutive property.

Remark 53 (Final word on conditions for involutive distributions)**.** $\mathcal{D}$ is an involution $\iff$ there are local integrable manifolds $(M \subset U \subset N)$ such that $T_qM = \mathcal{D}(q)$.

Remark 54. A (hard!) theorem of Whitney says that any smooth manifold of dimension m may be embedded in $\mathbb{R}^{2m}$. In the compact case, there is an easy proof that it embeds in $\mathbb{R}^N$ for some large N. (The proof is in Thomas & Barden Section 1.4).

Lie Groups

Definition 55. A group G is called a **Lie groups** if it is also a smooth manifold and the group operations $\mu\colon G \times G \to G$ and $i\colon G \to G$ are smooth maps. (It suffices to requires that the map $G \times G \to G\colon (g,h) \mapsto gh^{-1}$ is smooth.)

Lecture 7 22 October 2015

Example 56. Some examples of Lie groups.

(1) The matrix groups $\mathrm{GL}(n,\mathbb{R})$, $O(n)$, $\mathrm{SL}(n)$.

(2) The n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is a Lie group, with group operation inherited from addition on $\mathbb{R}$. It's abelian.

(3) $(\mathbb{R}^3, \cdot)$ with $(a_1,a_2,a_3)\cdot(b_1,b_2,b_3) = (a_1+b_1, a_2+b_2+a_1b_3, a_3+b_3)$. This can be identified with the subgroup of $\mathrm{GL}(3,\mathbb{R})$ consisting of matrices of the form

$$\begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{bmatrix}$$

So some manifolds may be Lie groups in two different ways.

Recall that tangent space at I to $\mathrm{GL}(n,\mathbb{R}) \subset \mathbb{R}^{n^2}$ is identified with the $n \times n$ matrices $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$. $O(n)$ is an embedded submanifold of $\mathrm{GL}(n,\mathbb{R})$ of dimension $n(n-1)/2$; the tangent space $T_I O(n)$ at the identity is identified as a subspace of $n \times n$ matrices which are antisymmetric.

Remark 57.

(1) The projection $A \mapsto \frac{1}{2}(A - A^T)$ yields a chart on some neighborhood of $I \in \mathrm{SO}(n)$ to an open neighborhood of $\vec{0}$ in the Lie algebra $T_I\mathrm{SO}(n)$.

(2) If H is an antisymmetric matrix, we can define a curve σ on $O(n)$ by $\sigma(t) = \exp(tH) = I + tH + 1/2\, t^2H^2 + 1/3!\, t^3H^3 + \ldots$. This is absolutely convergent with $\sigma(t) \in O(n)$ and $\dot\sigma(t) = \sigma(t)H$.

(3) Similar arguments work for other subgroups of $\mathrm{GL}(n,\mathbb{R})$.

Construction of left-invariant vector fields

Suppose that G is a Lie group and $e \in G$. We denote by $\mathfrak{g}$ the tangent space T_eG. Denote multiplication $L\colon G \times G \to G$, and for a given $g \in G$, the left-translation diffeomorphism $L_g\colon G \to G$ is given by $h \mapsto gh$.

A note on notation: sometimes we've used d_pF and sometimes dF_p. In this section, we'll be very careful to use d_pF because otherwise there's a risk of becoming confused.

For $\preceq \in \mathfrak{g}$, define $X = X_{(\preceq)}\colon G \to TG$ by $X_{(\preceq)}(g) = (d_eL_g)(\preceq) \in T_gG$. Clearly $X_{(\preceq)} \neq 0$ at any given point $g \in G$ for $\preceq \neq 0$, since d_eL_g is an isomorphism.

Claim 58. $X_{(\preceq)}$ is a smooth vector field on G.

Proof. Take charts $\phi_e = (x_1, \ldots, x_n)\colon U'_e \to \mathbb{R}^n$, and $\phi_g = (y_1, \ldots, y_n)\colon U'_g \to \mathbb{R}^n$, with say $\phi_e(e) = 0$, $U'_g = gU'_e = L_g(U'_e)$, and finally $\phi_g = \phi_e \circ L_g^{-1} = \phi_e \circ L_{g^{-1}}$.

Why have we put primes on U'_e and U'_g? Well, we can find smaller open neighborhoods $U_e \subset U'_e$ and $U_g \subset U'_g$ such that $U_g \times U_e \subset (U'_g \times U'_e) \cap L^{-1}(U'_g)$ in the product manifold $G \times G$. In particular, this means that $L\colon U_g \times U_e \to U'_g$.

$$\begin{array}{ccc} U_g \times U_e & \xrightarrow{\quad L \quad} & U'_g \\ {\scriptstyle \phi_g \times \phi_e}\big\downarrow & & \big\downarrow \\ \mathbb{R}^{2n} \supset V_g \times V_e & \xrightarrow{\ F\ } & V'_g \subset \mathbb{R}^n \end{array}$$

where $F(\vec{r}, \vec{s}) = (F_1, \ldots, F_n)$ and given $a \in U_g$, $d_e L_a\colon T_eG \to T_aG$ is given by the Jacobian matrix

$$\left(\frac{\partial F_i}{\partial s_j}\right)(\phi_g(a), 0).$$

This is basically just saying that

$$\left.\frac{d}{dx_j}\right|_e \longmapsto \sum_i \frac{dF_i}{ds_j}(\phi_g(a), 0)\left.\frac{d}{dy_i}\right|_a.$$

Since the entries are smooth functions on U_g (since $F(\vec{r}, \vec{s})$ smooth in $\vec{r}$), it follows that for a fixed $\preceq = \sum a_j {}^{d}\!/_{dx_j}\big|_e$ some tangent vector, $X_{(\preceq)}(a) = (d_e L_a)(\preceq)$ defines a smooth vector field on U_g. □

Definition 59. A vector field X is **left-invariant** if $(L_g)_* X = X$ for all $g \in G$.

Proposition 60. If X is left invariant, then $X = X_{(\preceq)}$ where $\preceq = X(e)$.

Proof. First let X be a left-invariant vector field. Recall that for any diffeomorphism $F\colon M \to N$ of smooth manifolds and X a smooth vector field on M, we defined a vector field $F_* X$ by $(F_* X)(F(p)) = (d_p F)(X(p))$. For h smooth, $(F_* X)(h) = X(h \circ F) \circ F^{-1}$.

Apply this to $F = L_g\colon G \times G \to G$. So

$$((L_g)_* X)(g) = d_e L_g(X(e)) = X_{(\preceq)}(g),$$

where $\preceq = X(e)$.

It remains to show that any vector field of the form $X_{(\preceq)}$ is left-invariant. This is just a simple calculation.

$$\left((L_g)_* X_{(\preceq)}\right)(ga) = d_a L_g X_{(\preceq)}(a) = (d_a L_g)(d_e L_a)(\preceq) = (d_e L_{ga})(\preceq) = X_{(\preceq)}(ga)$$

□

Definition 61. In general, for a diffeomorphism $F\colon M\ to M$, we say that a vector field X is **invariant under** F if $F_* X = X$.

Following the previous proposition, $\mathfrak{g} = T_e G$ may be embedded as the space of left-invariant vector fields in the space $\Theta(G)$ of all smooth vector fields via $\preceq \mapsto X_{(\preceq)}$. We know that there's a bracket operation on $\Theta(G)$. The hope is that this induces a bracket operation on $\mathfrak{g}$, thereby making it a Lie algebra.

Proposition 62. The bracket operation on $\Theta(G)$ induces a bracket operation on $\mathfrak{g}$, thereby making $\mathfrak{g}$ into a Lie algebra (the Lie algebra of G).

Proof. We have to show that the bracket of two left-invariant vector fields is left invariant. By a question on example sheet 1,

$$[(L_g)_* X, (L_g) * Y] = (L_g)_*[X, Y].$$

Because X, Y are left-invariant, then

$$[X, Y] = [(L_g)_* X, (L_g) * Y] = (L_g)_*[X, Y].$$

So the Lie bracket of two left-invariant vector fields is also left-invariant. □

To sum it all up, for $\preceq \in \mathfrak{g}$, we have a left-invariant vector field $X_{(\preceq)}$ and a curve $\theta \colon (-\varepsilon, \varepsilon) \to G$ with $\theta(0) = e$ and $\dot{\theta}(t) = X_{(\preceq)}(\theta(t))$ for all $t \in (-\varepsilon, \varepsilon)$

Lemma 63. For s, t such that $|s|, |t| < \varepsilon/2$, we have that $\theta(s + t) = \theta(s)\theta(t)$ (multiplication in the Lie group G).

Lecture 8 — 24 October 2015

Last time, we defined for $\preceq \in \mathfrak{g}$ a left-invariant vector field $X_{(\preceq)}$ and a curve $\theta \colon (-\varepsilon, \varepsilon) \to G$ such that $\theta(0) = \varepsilon$, $\dot{\theta}(t) = X_{(\preceq)}(\theta(t))$ for all $t \in (-\varepsilon, \varepsilon)$.

Lemma 64. For s, t with $|s|, |t| < \varepsilon/2$, we have that $\theta(s + t) = \theta(s)\theta(t)$.

Proof. For fixed s, we show that the curves $\theta(s + t)$ and $\theta(s)\theta(t)$ are solutions to the differential equation $\phi \colon (-\varepsilon/2, \varepsilon/2) \to G$ with $\phi(0) = \theta(s)$, $\dot{\phi}(t) = X_{(\preceq)}(\theta(t))$ and so we must have equality. We show that both $\theta(s + t)$ and $\theta(s)\theta(t)$ are solutions to the same differential equation, which by uniqueness of solutions must give us that they are equal.

(a) $\phi(t) = \theta(s + t)$ is a composition locally of maps $\mathbb{R} \to \mathbb{R} \xrightarrow{\theta} G$, where the first map is $t \mapsto s + t$. Therefore,

$$\dot{\phi}(t) = (d_t \phi)\left(\frac{\partial}{\partial r}\right) = (d_{s+t}\theta)\left(\frac{\partial}{\partial r}\right) = \dot{\theta}(s + t) = X_{(\preceq)}(\phi(t)) = \theta(s + t).$$

(b) Let $g = \theta(s)$. Set $\phi(t) = g\theta(t) = L_g\theta(t)$. Then we use the chain rule:

$$\begin{aligned}
\dot{\phi}(t) &= d_t(L_g \circ \theta)\left(\frac{\partial}{\partial r}\right) \\
&= (d_{\theta(t)}L_g)(d_t\theta)\left(\frac{\partial}{\partial r}\right) \\
&= (d_{\theta(t)}L_g)\dot{\theta}(t) \\
&= (d_{\theta(t)}L_g)X_{(\preceq)}(\theta(t)) \\
&= X_{(\preceq)}\left(L_g\theta(t)\right) && \text{by left-invariance of } X_{(\preceq)} \\
&= X_{(\preceq)}(g\theta(t)) = X_{(\preceq)}(\phi(t))
\end{aligned}$$

□

This enables us to define a **1-parameter subroup** as a homomorphism of Lie groups $\psi\colon \mathbb{R} \to G$ such that $\dot{\psi}(t) = X_{(\preceq)}(\psi(t))$ for all $t \in \mathbb{R}$ by recipe.

For given t, choose N such that ${}^t\!/_N \in (-\varepsilon, \varepsilon)$ and define $\psi(t) := \theta\left({}^t\!/_N\right)^N$. Let's check that this is well-defined. If M is another such integer,

$$\left(\theta\left({}^t\!/_{MN}\right)\right)^N = \theta\left({}^t\!/_M\right)$$

and so

$$\theta\left({}^t\!/_N\right)^N = \left(\theta\left({}^t\!/_{MN}\right)\right)^{MN} = \theta\left({}^t\!/_M\right)^M.$$

Example 65. For $G = \mathrm{GL}(N, \mathbb{R}) \subset M_{n\times n}(\mathbb{R})$, with tangent space at I being $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$, then for any $A \in M_{n\times n}(\mathbb{R})$ corresponding to the tangent vector

$$\sum a_{ij}\frac{\partial}{\partial x_{ij}},$$

the corresponding 1-parameter subgroup ψ is just

$$\psi(t) = \exp(tA) = I + tA + \tfrac{1}{2!}(tA)^2 + \tfrac{1}{3!}(tA)^3 + \dots$$

A standard check shows that

$$\dot{\psi}(t) = \psi(t)A = L_{\psi(t)}A = d_eL_{\psi(t)}A = X_{(A)}(\psi(t)),$$

which is as required to define a one-parameter subgroup.

Remark 66. In general, given a 1-parameter subgroup $\psi(t) = \psi(\preceq, t)$ defined by $\preceq \in T_eG$, we one can show that $\psi(\preceq, t) = \psi(t\preceq, 1)$. In this way we can define in general a map $\exp\colon T_eG \to G$ such that $\preceq \mapsto \psi(\preceq, 1)$. This is a smooth map and a local diffeomorphism.

Example 67. $G = \mathrm{GL}(n, \mathbb{R}) \subset M_{n\times n}(\mathbb{R})$. We have $T_eG \cong M_{n\times n}(\mathbb{R})$ with the basis ${}^{\partial}\!/_{\partial x_{pq}}$. Suppose

$$\preceq = \sum a_{pq}\frac{\partial}{\partial x_{pq}}\bigg|_e$$

corresponds to some matrix $A \in M_{n\times n}(\mathbb{R})$. Then if $g = (x_{rp})$ (so $g_{pq}(e) = \delta_{pq}$)

$$X_{(\preceq)}(g) = d_e L_g(\preceq) = L_g(\preceq) = \sum x_{rp} a_{pq} \frac{\partial}{\partial x_{rq}}\bigg|_g$$

Given also

$$\eta = \sum b_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_e$$

corresponding to a matrix $B \in M_{n\times n}(\mathbb{R})$, then

$$X_{(\eta)}(g) = d_e L_g(\eta) = L_g(\eta) = \sum x_{ki} b_{ij} \frac{\partial}{\partial x_{kj}}\bigg|_g$$

Now we can work out explicitly what the Lie bracket of these vector fields is.

$$\begin{aligned}
[X_{(\preceq)}, X_{(\eta)}]_e &= \sum_{p,q,k,i,j} \delta_{rp} a_{pq} \delta_{rk} \delta_{qi} b_{ij} \frac{\partial}{\partial x_{kj}} - \sum_{p,q,k,i,j} \delta_{rp} b_{pq} \delta_{rk} \delta_{qi} a_{ij} \frac{\partial}{\partial x_{kj}} \\
&= \sum_{i,j,k} a_{ki} b_{ij} \frac{\partial}{\partial x_{kj}} - \sum_{i,j,k} b_{ki} a_{ij} \frac{\partial}{\partial x_{kj}} \\
&= \sum_{k,j} [A, B]_{kj} \frac{\partial}{\partial x_{kj}} ,
\end{aligned}$$

where $[A, B] = AB - BA \in M_{n\times n}(\mathbb{R})$. So the Lie algebra of left-invariant vector fields on G is just the Lie algebra of $n \times n$ matrices under the natural bracket.

Remark 68. If $G \subseteq \mathrm{GL}(n, \mathbb{R})$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$, then for a tangent vector $\preceq \in T_e G \subseteq M_{n\times n}(\mathbb{R})$ there is a left-invariant vector field $X_{(\preceq)}$ on $\mathrm{GL}(n, \mathbb{R})$ restricting to a left-invariant vector field $X_{(\preceq)}|_G$ on G. And moreover

$$[X_{(\preceq)}|_G, X_{(\eta)}|_G] = [X_{(\preceq)}, X_{(\eta)}]|_G,$$

so the induced bracket on $T_e G$ is just the restriction of the natural bracket on $M_{n\times n}(\mathbb{R})$.

Example 69. If $G = \mathrm{SO}(n)$, then $\mathfrak{g}$ is just the antisymmetric matrices and the Lie bracket on $\mathrm{SO}(n)$ is just given by $[A, B] = AB - BA$.

Forms and Tensors on Manifolds

Differential Forms

In many ways, vector fields are important objects to study on manifolds, but differential forms are quite possibly even more important.

Given a smooth manifold M and $U \subseteq M$ open, a smooth function $f\colon U \to \mathbb{R}$ gives rise to the differential $df\colon TU \to T\mathbb{R}$ consisting of the linear forms $d_p f\colon T_p U \to \mathbb{R}$ for $p \in U$. Here, we identify $T_{f(p)}\mathbb{R}$ with $\mathbb{R}^{\partial}/_{\partial r}$ via $v \mapsto v(r)$.

Given $g\colon U \to \mathbb{R}$ smooth, we have the family of linear forms $g(p) d_p f\colon T_p U \to \mathbb{R}$. Note that $d_p(fg) = f(p) d_p g + g(p) d_p f$.

If we have coordinates for U given by $x_1, \dots, x_n$, then we have

$$(d_p f)\left(\frac{\partial}{\partial x_j}\bigg|_p\right)(r) = \frac{\partial}{\partial x_j}\bigg|_p (r \circ f) = \frac{\partial f}{\partial x_j}\bigg|_p$$

for all $p \in U$. So therefore,

$$d_p f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(p) d_p x_j.$$

In particular,

$$(d_p x_i)\left(\frac{\partial}{\partial x_j}\bigg|_p\right) = \left(\frac{\partial x_i}{\partial x_j}\right)_p = \delta_{ij}$$

So $d_p x_1, \dots, d_p x_n$ gives a basis of the dual space $T_p^* M$ dual to the basis ${}^{\partial}/_{\partial x_1}|_p, \dots, {}^{\partial}/_{\partial x_n}|_p$ of $T_p M$.

Lecture 9 27 October 2015

Last time someone asked me what facts we were using when we computed the Lie bracket of matrices, and I forgot to mention some details. We defined the composition of vector fields X, Y as $XY(h) = X(Y(h))$. If $X = {}^{\partial}/_{\partial x_i}$ and $Y = {}^{\partial}/_{\partial x_j}$, then $XY(h) = {}^{\partial^2 h}/_{\partial x_i \partial x_j} = YX(h)$, so in this case $[X, Y] = 0$.

Okay, so last time we were talking about differential forms. Let's make this definition formal.

Definition 70. A **smooth 1-form** on M is a map $\omega\colon M \to \bigsqcup_{p \in M} T_p^* M$ with $\omega(p) \in T_p^* M$ for all p, which can locally be written in the form $\sum_i f_i dg_i$ with f_i, g_i (locally) smooth. Equivalently, for any coordinate system $x_1, \dots, x_n$ on $U \subseteq M$, it may be written as $\sum_i f_i dx_i$ with f_i smooth functions.

We denote the collection of smooth 1-forms on M by $\Omega^1(M)$

When we talked about vector fields, we were using the tangent bundle. The definition above uses something that looks very similar, which we call the **cotangent bundle**.

Definition 71. The **cotangent bundle** on M is the set $T^* M = \bigsqcup_{p \in M} T_p^* M$, with $\pi\colon T^* M \to M$ the projection map.

Just as for the tangent bundle, $T^* M$ is naturally a smooth manifold of dimension $2n$. How do we see this? Given a chart with $\phi(x_1, \dots, x_n)\colon U \to \mathbb{R}^n$, $T_p^* M$ has basis $d_p x_1, \dots, d_p x_n$ for all $p \in U$. We then identify $\pi^{-1}(U) = TU$ with $U \times \mathbb{R}^n$ via the map

$$\omega_p = \sum_i a_i d_p x_i \mapsto (p; a_1, \dots, a_n).$$

In this case, if $\sum_k a_k dx_k = \sum_j b_j dy_j$, then

$$a_i = \left(\sum_k a_k dx_k\right) \frac{\partial}{\partial x_i}\bigg|_p = \sum_j b_j d_p y_j \left(\frac{\partial}{\partial x_i}\bigg|_p\right) = \sum_j \left(\frac{\partial y_j}{\partial x_i}\right)_p b_j.$$

The matrix here $\left({}^{\partial y_j}/_{\partial x_i}\right)_p$ is the inverse transpose of the one we had for the tangent bundle we saw in Claim 28.

Warning: this is backwards from the way that you transform coordinates for tangent vectors!

We can also say that the projection $\pi\colon T^*M \to M$ is smooth, and by construction the fiber over p is the cotangent space at p. In equations, this reads $\pi^{-1}(p) = T_p^*M$.

Our definition of a smooth 1-form ω could therefore have been a **smooth section** $\omega\colon M \to T^*M$ such that $\pi \circ \omega = \mathrm{id}_M$.

Vector Bundles

Now that we've seen how both the tangent bundle TM and the cotangent bundle T^*M are smooth manifolds of dimension $2n$, we should set up the language of general vector bundles. Note that I'll probably stop saying "smooth" soon, but you should know that we're working in categories of smooth maps.

Definition 72. Let B be a smooth manifold. A manifold E together with a surjective smooth map $\pi\colon E \to B$ is called a **vector bundle of rank** k over B if the following conditions hold.

(i) There is a k-dimensional real vector space F such that for any $p \in B$, the **fiber** $E_p = \pi^{-1}(p)$ is a vector space isomorphic to F.

(ii) Any point $p \in B$ has a neighborhood U such that there is a diffeomorphism $\Phi_U\colon \pi^{-1}(U) \to U \times F$ such that the diagram below commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi_U} & U \times F \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \mathrm{pr}} \\ U & \xrightarrow{\sim} & U \end{array}$$

Here, pr is projection onto the first factor $U \times F \to U$. Φ_U is called a **trivialization** of E.

(iii) $\Phi_U|_{E_q} \to F$ is an isomorphism on vector spaces for all $q \in U$.

B is called the **base space** and E is the **total space** of the bundle. If $k = 1$, we call it a **line bundle**.

Definition 73. A smooth map $s\colon B \to E$ such that $\pi \circ s = \mathrm{id}_B$ is called a **section** of E. Denote the sections of E by $\Gamma(E)$ or $\Omega(E)$.

Example 74. $\Gamma(T^*M) = \Omega^1(M)$.

If we have two trivializations $\Phi_V : \pi^{-1}(V) \to V \times F$ and $\Phi_U : \pi^{-1}(U) \to U \times F$, then we compute the diffeomorphism

$$\Phi_V \circ \Phi_U^{-1} : (U \cap V) \times F \to (U \cap V) \times F.$$

For $p \in U \cap V$, we have an isomorphism of vector spaces $f_{VU}(p) : F \to F$.

Choosing a basis for F identifies $\mathrm{GL}(F)$ with $\mathrm{GL}(k, \mathbb{R})$ and then the f_{VU} can be thought of as matrices $f_{VU} : U \cap V \to \mathrm{GL}(k, \mathbb{R})$, whose entries are smooth functions on $U \cap V$. These functions f_{VU} are called **transition functions**.

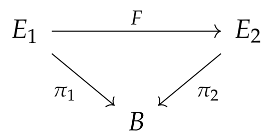
Fact 75. There are some pretty obvious properties satisfied by these f_{VU}.

(i) $f_{UU} = \mathrm{id}$ is the identity matrix;

(ii) $f_{VU} = f_{UV}^{-1}$ on $U \cap V$;

(iii) $f_{WV} \circ f_{VU} = f_{WU}$ on $U \cap V \cap W$.

Definition 76. Now given vector bundles E_1, E_2 over the same base space B, a smooth map $F : E_1 \to E_2$ such that $\pi_2 \circ F = \pi_1$



is called a **morphism of vector bundles** if the induced maps on fibers are linear maps of vector spaces. Morphisms with inverses are isomorphisms, and a subbundle is defined in the obvious way.

So we've seen that if we have a fiber bundle, then we have transition functions f_{ij}. Now what if we have an open cover of a manifold with transition functions as in Fact 75? It turns out we can construct a fiber bundle that these come from. This is what we sketch below.

Theorem 77. Suppose B is a smooth manifold with an open ocver $\mathcal{U} = \{U_i \mid i \in I\}$, $\bigcup_i U_i = B$, and smooth functions $f_{ij} : U_i \cap U_j \to \mathrm{GL}(k, \mathbb{R})$ such that

1. $f_{ii} = I_k$;
2. $f_{ji}(p) = f_{ij}(p)^{-1}$ for all $p \in U_i \cap U_j$;
3. $f_{kj}(p) f_{ji}(p) = f_{ki}(p)$ for all $p \in U_i \cap U_j \cap U_k$. (matrix multiplication)

Then there exists a rank k vector bundle (unique up to isomorphism) $\pi : E \to B$ for which $\mathcal{U}$ is a trivializing cover of B and the transition functions are the f_{ij}.

Proof sketch. (See Darling Chapter 6 for full details.)

As a topological space, set $E = \bigsqcup_{i \in I} U_i \times \mathbb{R}^k / \sim$, where $\sim$ is the equivalence relation $(p, \vec{a}) \sim (q, \vec{b}) \iff p = q$ and $\vec{a} = f_{ij}(q)\vec{b}$.

Define $\pi : E \to B$ by the projection onto the first factor.

To put a manifold structure on E, we notice that for each j, the inclusion

$$U_j \times \mathbb{R}^k \hookrightarrow \bigsqcup_{i \in I} U_i \times \mathbb{R}^k \to \pi^{-1}(U_j)$$

is a homeomorphism. This gives the trivializations required with transition functions f_{ij} defining smooth maps $(U_i \cap U_j) \times \mathbb{R}^k \to (U_i \cap U_j) \times \mathbb{R}^k$.

Hence, E is a smooth manifold with fibers E_p isomorphic to $\mathbb{R}^k$, and the above maps restrict vector space isomorphisms on the fibers via the $f_{ij}(p)$.

If however two vector bundles $\pi_1 \colon E_1 \to B$ and $\pi_2 \colon E_2 \to B$ have the same trivializing cover and the same transition functions, then we can define a vector bundle isomorphism between them. The point is that we know the isomorphism F locally, and then by the definition of the equivalence relation $\sim$ they should be compatible.

Locally, this isomorphism F is given by the diagram

$$\begin{array}{ccc} \pi_1^{-1}(U_i) & \xrightarrow{F} & \pi_2^{-1}(U_i) \\ \downarrow \Phi & & \downarrow \Psi \\ U_i \times \mathbb{R}^k & \xrightarrow{\text{id}} & U_i \times \mathbb{R}^k \end{array}$$

Why doesn't this depend on the choice of coordinates? It doeesn't depend on the U_i we choose here because the transition functions are the same. Hence, this is a well-defined isomorphism. □

Lecture 10 — 29 October 2015

Last time, we wrote down a map $U_j \times \mathbb{R}^k \xrightarrow{\sim} \pi^{-1}(U_j)$, but this requires a bit of interpretation. To clarify, I meant that for each j, the inclusion $U_j \times \mathbb{R}^k \hookrightarrow \bigsqcup_i U_i \times R^k$ induces a homeomorphism $U_j \times \mathbb{R}^k \xrightarrow{\sim} \pi^{-1}(U_j)$.

This should take care of all of the boring stuff about vector bundles, so now let's see some examples.

Example 78.

(1) The trivial bundle $E = M \times \mathbb{R}^k \to M$ with $\Gamma(E) = C^\infty(M)^k$.

(2) The tangent and cotangent bundles are examples of vector bundles $MTM \to M$ and $T^*M \to M$, with $\Gamma(M) = \Theta(M)$ and $\Gamma(T^*M) = \Omega^1(M)$.

(3) The **tautological bundle** or **Hopf bundle** on $\mathbb{CP}^n$ is a *complex* line bundle, that is, a bundle of rank 1 over $\mathbb{C}$. This means that it's a rank 2 bundle over $\mathbb{R}$. Each point of $\mathbb{CP}^n$ corresponds to a line through the origin in $\mathbb{C}^{n+1}$ and hence to an equivalence class of points in $\mathbb{C}^{n+1} \backslash \{0\}$ where $\vec{x} \sim \vec{y} \iff \exists \lambda \in \mathbb{C}^*$ s.t. $\vec{x} = \lambda \vec{y}$. So points of $\mathbb{CP}^n$ are represented by homogeneous coordinates $(x_0 \colon x_1 \colon \dots \colon x_n)$ with x_i not all zero, and only the ratios matter.

$\mathbb{CP}^n$ has an open cover by open sets $U_i \cong \mathbb{C}^n$ where $U_i = \{\vec{x} \mid x_i \neq 0\}$ and the chart on this open set U_i is given by

$$(x_0 \colon x_1 \colon \dots \colon x_n) \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)$$

note that we omit the i-th coordinate.

Define the tautological bundle or Hopf bundle $E \to \mathbb{CP}^n$ to have fiber E_p being the line in $\mathbb{C}^{n+1}$ corresponding to a point $P \in \mathbb{CP}^n$, $\left(E = \bigsqcup_{p \in \mathbb{CP}^n} E_p\right)$. This is in fact a sub-bundle of the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$.

Let's try to understand the trivializations and the transition functions on this bundle. For simplicity, let's take the case $n = 1$. There are here two open sets, U_0 and U_1, with charts

$$\begin{array}{ccc} U_0 & \xrightarrow{\sim} & \mathbb{C} \\ (1\colon z) & \mapsto & z \end{array} \qquad\qquad \begin{array}{ccc} U_1 & \xrightarrow{\sim} & \mathbb{C} \\ (\zeta\colon 1) & \mapsto & \zeta \end{array}$$

We also have a coordinate transformation $U_0 \to U_1$ given by $z \mapsto {}^1\!/\!_z = \zeta$. There is an obvious trivialization of $E = \bigsqcup_{p \in \mathbb{CP}^1} E_p$ over U_0 given by

$$E_{(1\colon z)} \ni (w, wz) \longmapsto ((1\colon z), w) \in U_0 \times \mathbb{C}$$

and over U_1,

$$E_{(\zeta\colon 1)} \ni (v\zeta, v) \longmapsto ((\zeta\colon 1), v) \in U_1 \times \mathbb{C}$$

So $(w, wz) = (v\zeta, v) \iff v = wz$, where $\zeta = {}^1\!/\!_z$. Therefore, the transition functions are the 1×1 matrices $f_{10} = z$ and $f_{01} = \zeta$.

Another choice of trivialization is given by Φ_0 on U_0,

$$E_{(1\colon z)} \ni (w, wz) \overset{\Phi_0}{\longmapsto} \left((1\colon z), w\sqrt{1 + |z|^2}\right) \in U_0 \times \mathbb{C}$$

Let's set $t = w\sqrt{1 + |z|^2}$. This has the property $|t| = 1$ if and only if the corresponding point (w, wz) lies on the appropriate unit sphere $S^3 \subseteq \mathbb{C}^2$.

We also have a similar trivialization Φ_1 on U_1 given by

$$E_{(\zeta\colon 1)} \ni (v\zeta, v) \overset{\Phi_1}{\longmapsto} \left((\zeta\colon 1), v\sqrt{1 + |\zeta|^2}\right) \in U_1 \times \mathbb{C},$$

and we call $s = v\sqrt{1 + |\zeta|^2}$.

There is a transition function

$$\begin{aligned} \Phi_1 \circ \Phi_0^{-1}((1\colon z), t) &= \Phi_1\left(\frac{t}{\sqrt{1 + |z|^2}}, \frac{tz}{\sqrt{1 + |z|^2}}\right) \\ &= \Phi_1\left(\frac{t|\zeta|}{\sqrt{1 + |\zeta|^2}}, \frac{{}^{t|\zeta|}\!/\!_{\zeta}}{\sqrt{1 + |\zeta|^2}}\right) \\ &= \left((\zeta\colon 1), {}^{|\zeta|}\!/\!_{\zeta} t\right) \end{aligned}$$

and the transition function is gven by $s = {}^{z}\!/_{|z|} t$. So the transition function ρ_{10} is just multiplication by ${}^{z}\!/_{|z|} \in U(1)$. This means that E is what we call a **unitary** bundle over $\mathbb{CP}^1 = S^2$.

So E is a smooth rank 1 complex vector bundle over $\mathbb{CP}^1 = S^2$. Finally, note that

$$|w|^2 + |wz|^2 = |t|^2,$$
$$|v\zeta|^2 + |v|^2 = |s|^2,$$

so lengths on trivializations corresponds to taking a standard (Hermitian) length of vectors in $\mathbb{C}^2$.

Definition 79. If the transition functions of a vector bundle with respect to some trivialization all lie in a subgroup $G \subset \mathrm{GL}(k, \mathbb{R})$, we say that the **structure group** of E is G.

Example 80.

(1) Let $G = \mathrm{GL}^+(k, \mathbb{R})$ be the matrices with positive determinant. A vector bundle with structure group E is called **orientable.** If the tangent space of a manifold M is orientable, then M is an **orientable manifold**.

(2) If $G = O(k) = \{$ matrices preserving the standard inner product on $\mathbb{R}^k\}$, this means that we have a well-defined family of inner products on the fibers vary smoothly over the base. This is just the concept of a **metric** on E in Riemannian geometry.

On the Hopf bundle, this metric corresponds to the standard one on $\mathbb{C}^2$. Example sheet 2, question 9 says that we can always find such a metric.

New Bundles from Old

Given vector bundles E and E' on M, of ranks k and ℓ, respectively, we can always find a common trivializing cover $\mathcal{U} = \{U_i\}$. We can define the (Whitney) sum $E \oplus E' \to M$ by

$$\bigsqcup_{p \in M} E_p \oplus E'_p \xrightarrow{\widetilde{\pi}} M.$$

Given $U \in \mathcal{U}$ and trivializations $\Phi_U \colon \pi^{-1}(U) \to U \times F$ and $\Phi' \colon (\pi')^{-1}(U) \to U \times F'$, we have a natural structure on $\widetilde{\pi}^{-1}(U)$, namely $U \times (F \oplus F')$. The identification is given by

$$E_p \oplus E'_p \ni (s_p, s'_p) \longmapsto (p, (\Phi_U(s_p), \Phi'_U(s'_p))) \in U \times (F \oplus F')$$

If Φ_U and Φ'_U are determined by frames (a collection of smooth sections over U) $s_1, \ldots, s_k$ and $\sigma_1, \ldots, \sigma_\ell$, then $\widetilde{\Phi}_U$ is determined by the frame

$$(s_1, 0), (s_2, 0), \ldots, (s_k, 0), (0, \sigma_1), \ldots, (0, \sigma_n).$$

As for the tangent bundle, this determines a topological space structure on $E \oplus E'$ Namely a subset is open if and only if all its intersections with such subsets $\tilde{\pi}^{-1}(U)$ are open, where $\tilde{\pi}^{-1}(U)$ has been identified as this product $U \times (F \oplus F')$. There are natural charts on $E \oplus E'$.

With $\mathcal{U}$ as above, with transition functions $\{f_{ij}\}$ on E and $\{g_{ij}\}$ on E' then the vector bundle $E \oplus E'$ has transition functions given by the block diagonal matrices $f_{ij} \oplus g_{ij} \colon U_i \cap U_j \to \mathrm{GL}(k + \ell, \mathbb{R})$.

$$\left[\begin{array}{c|c} f_{ij} & 0 \\ \hline 0 & g_{ij} \end{array}\right]$$

Recall from Theorem 77 last time that this in any case determines the bundle up to isomorphism.

Lecture 11 31 October 2015

Last time we defined the Whitney sum of two vector bundles. There are many other operations on vector bundles that are analogous to those on vector spaces.

In a similar way to the dual space of a vector space, we can define the **dual bundle** $E^* \to M$ with transition functions $\left(f_{ij}^T\right)^{-1} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{R})$.

In the case of a line bundle, this is just a nowhere vanishing function ${}^1/_{f_{ij}}$.

Note that a metric on E gives rise to an isomorphism $E \xrightarrow{\sim} E^*$; this is on example sheet 2 as question 9. But this isomorphism isn't natural – it depends on the choice of metric.

Similarly, there is a **tensor product** of two bundles $E \otimes E' \to M$ with transition functions given by $f_{ij} \otimes f'_{ij} \colon U_i \cap U_j \to \mathrm{GL}(k\ell, \mathbb{R}) = \mathrm{GL}(\mathbb{R}^k \otimes \mathbb{R}^\ell)$.

There is also a bundle $\mathrm{Hom}(E, E')$ such that for each $p \in B$, we have that

$$\begin{array}{ccc} E_p & \longrightarrow & E'_p \\ \downarrow{\scriptstyle \Phi_p} & & \downarrow{\scriptstyle \Phi'_p} \\ F & \longrightarrow & F' \end{array}$$

The bundle $\mathrm{Hom}(E, E')$ is isomorphic to $E^* \otimes E'$.

There is also an **exterior power** bundle $\bigwedge^r E \to M$ for $0 \leqslant r \leqslant k$ with transition functions $\bigwedge f_{ij} \colon U_i \cap U_j \to \mathrm{GL}\left(\binom{n}{r}, \mathbb{R}\right)$. To be more precise, if $\alpha \colon F \to F$, then

$$\begin{array}{rcl} \alpha \otimes \alpha \otimes \cdots \otimes \alpha \colon F \otimes F \otimes \cdots \otimes F & \longrightarrow & F \otimes F \otimes \cdots \otimes F \\ \downarrow & & \downarrow \\ \bigwedge^r \alpha \colon \bigwedge^r F & \dashrightarrow & \bigwedge^r F \end{array}$$

Definition 81. A (mixed) **tensor** of type (r, s) on a manifold M is a smooth section of the bundle

$$\underbrace{TM \otimes \cdots \otimes TM}_{r} \otimes \underbrace{T^*M \otimes T^*M \otimes \cdots \otimes T^*M}_{s}$$

It has r contravariant factors and s covariant factors.

If we have local coordinates $x_1, \ldots, x_n$ on $U \subseteq M$, then teh tensor can locally be written in the form

$$\sum_{i_1,\ldots,i_r} \sum_{j_1,\ldots,j_s} T^{i_1,i_2,\ldots,i_r}_{j_1,j_2,\ldots,j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}$$

Remark 82. If one employs the **Einstein summation convention**, one would write coordinates as $x^1, \ldots, x^n$, and the sum is over all repeated indices with one up and one down. For example $a^i b_i = \sum_i a_i b_i$.

Interlude – a little multilinear algebra

You may not think you need this, but you probably do.

Definition 83. Recall that given vector spaces $V_1, \ldots, V_r$, the **tensor product** of $V_1, \ldots, V_r$ is the universal multilinear object, meaning that there is a map $\otimes\colon V_1 \times V_2 \times \cdots \times V_r \to V_1 \otimes V_2 \otimes \cdots \otimes V_r$ such that, given a multilinear form $f\colon V_1 \times V_2 \times \cdots \times V_r \to \mathbb{R}$, then there is a unique map $g\colon V_1 \otimes V_2 \otimes \cdots \otimes V_r$ such that $f = g \circ \otimes$.

$$\begin{array}{ccc} \prod_{i=1}^r V_i & \xrightarrow{\quad f \quad} & \mathbb{R} \\ & \searrow^{\otimes} & \uparrow g \\ & & \bigotimes_{i=1}^r V_i \end{array}$$

Definition 84. A **perfect pairing** between finite dimensional vector spaces V, W is a bilinear map $(-,-)\colon V \times W \to \mathbb{R}$ for which

$$w \in W \backslash \{0\} \implies \exists\, v \text{ s.t. } (v, w) \neq 0$$

$$v \in V \backslash \{0\} \implies \exists\, w \text{ s.t. } (v, w) \neq 0$$

Therefore, a perfect pairing induces isomorphisms $V \to W^*$ and $W \to V^*$.

Example 85. There is a natural perfect pairing

$$(V_1^* \otimes V_2^* \otimes \cdots \otimes V_r^*) \times (V_1 \otimes V_2 \otimes \cdots V_r) \to \mathbb{R}$$

given (on the elementary tensors) by

$$(v_1^* \otimes \cdots \otimes v_r^*)(v_1 \otimes \cdots \otimes v_r) = v_1^*(v_1) v_2^*(v_2) \cdots v_r^*(v_r)$$

and extended linearly to the whole space.

This gives a natural isomorphism $V_1^* \otimes \cdots \otimes V_r^* \xrightarrow{\sim} (V_1 \otimes \cdots \otimes V_r)^*$.

So for a fixed vector space V, we may identify the multilinear forms on V with $V^* \otimes \cdots \otimes V^*$.

Definition 86. The **exterior power** $\Lambda^r V$ is a quotient subspace of $V^{\otimes r}$ that is universal among all alternating multilinear forms $f\colon V^r \to \mathbb{R}$.

$$\begin{array}{ccc} V^r & \xrightarrow{f} & \mathbb{R} \\ & \searrow & \uparrow \exists!\ \text{linear map} \\ & & \bigwedge^r V \end{array}$$

We denote the image of $v_1 \otimes \cdots \otimes v_r$ in $\bigwedge^r V$ by $v_1 \wedge \cdots \wedge v_r$.

We can identify the alternating forms on V, $\mathrm{Alt}^r(V)$, with $\left(\bigwedge^r V\right)^*$. Now $\mathrm{Alt}^r(V) \hookrightarrow (V*)^{\otimes r} \rightarrow \bigwedge^r V^*$ is an isomorphisms, whose image is determined by

$$f_1 \wedge \ldots \wedge f_r \mapsto \frac{1}{r!} \sum_{\pi \in S_r} \operatorname{sgn}(\pi) f_{\pi(1)} \otimes \cdots \otimes f_{\pi(r)}$$

We call this the **logical convention**.

Definition 87. Unfortunately, most books *do not* adopt this convention. So, to be consistent with all of the books, we'll therefore *define*

$$(f_1 \wedge \ldots \wedge f_r)(v_1, \ldots, v_r) := \det\left(\left[f_i(v_j)\right]_{i,j=1}^r\right)$$

This differs from the usual definition because we drop the factor of $1/r!$. Under this definition, we identify $f_1 \wedge \ldots \wedge f_r$ with

$$\sum_{\pi \in S_r} \operatorname{sgn}(\pi) f_{\pi(1)} \otimes \cdots \otimes f_{\pi(r)}$$

and $f_1 \wedge f_2 \leftrightarrow f_1 \otimes f_2 - f_2 \otimes f_1$.

Remark 88 (WARNING!)**.** The composite of this map with projection is *not* the identity, but instead multiplication by $r!$ (r factorial).

$$\bigwedge^r V^* \to \mathrm{Alt}^r(V) \hookrightarrow (V^*)^r \to \bigwedge^r V^*$$

With this identification of $\bigwedge^r V^*$ with $\mathrm{Alt}^r(V)$, the natural map

$$\bigwedge^p V^* \times \bigwedge^q V^* \xrightarrow{\wedge} \bigwedge^{p+q} V^*$$

induces a wedge product on alternating forms

$$\begin{array}{ccc} \mathrm{Alt}^r(V) \times \mathrm{Alt}^q(V) & \to & \mathrm{Alt}^{p+q}(V) \\ (f, g) & \longmapsto & f \wedge g \end{array}$$

where

$$\begin{aligned}(f \wedge g)(v_1, \ldots, v_{p+q}) \\ = \frac{1}{p!q!} \sum_{\pi \in S_{p+q}} \operatorname{sgn}(\pi) f(v_{\pi(1)}, \ldots, v_{\pi(p)}) g(v_{\pi(p+1)}, \ldots, v_{\pi(p+q)}). \quad (3)\end{aligned}$$

Remark 89. Above in (3) we use the convention given in Definition 87. The "logical identification" would have a factor of $^1/_{(p+q)!}$.

Definition 90. In defining $f \wedge g$ in (3), we form an algebra of alternating forms

$$\mathrm{Alt}(V) := \bigoplus_{r \geqslant 0} \mathrm{Alt}^r(V)$$

where $\dim \mathrm{Alt}^r(V) = \binom{n}{r}$, for $n = \dim V$.

Differential forms on manifolds

Now after that interlude, we can go back to doing geometry.

Definition 91. A (smooth) r**-form** ω on a manifold M is a smooth section of $\bigwedge^r T^*M$ for some r, $0 \leqslant r \leqslant \dim M$.

Using the identification above, we may alternatively regard this as a family of alternating forms on tangent spaces. If $x_1, \ldots, x_n$ are local coordinates on $U \subseteq M$, then we write

$$\omega = \sum_{i_1 < i_2 < \ldots < i_r} f_{i_1,\ldots,i_r} dx_{i_1} \wedge \ldots \wedge dx_{i_r}$$

locally and uniquely since $dx_1, \ldots, dx_n$ are a basis for T^*M at each point of U.

By convention, the zero-forms on a manifold are just the smooth functions. By convention, $\bigwedge^0 E$ is just the trivial bundle $M \times R$.

Denote the space of smooth r-forms on M by $\Omega^r := \Omega^r(M) = \Gamma(\bigwedge^r T^*M)$. r is called the **degree** of the form, and $\Omega^0(M) = C^\infty(M)$.

Lecture 12 — 3 November 2015

Theorem 92 (Orientations)**.** Let M be an n-dimensional manifold. Then the following are equivalent:

(a) there is a nowhere vanishing smooth differential n-form ω on M;

(b) $\bigwedge^n T^*M \cong M \times \mathbb{R}$;

(c) there is a family of charts $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ in the differential structure on M such that the U_α cover M and the Jacobian matrices $\left[{}^{\partial y_j}/_{\partial x_i}(p)\right]$ for the change in coordinates have positive determinant for $p \in U_\alpha \cap U_\beta$ for each α, β.

Proof sketch. $(a) \iff (b)$ is easy, and is similar to the criterion for the parallelizability of manifolds.

$(a) \implies (c)$: Given two charts $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ we have

$$dx_1 \wedge \ldots \wedge dx_n = \det\left[\frac{\partial x_i}{\partial y_j}\right] dy_1 \wedge \ldots \wedge y_n$$

on the overlap. Now cover M by such (connected) coordinate charts (U, ϕ) with $\phi = (x_1, \ldots, x_n)$, choosing the order of coordinates so that on U,

$$\omega = f\, dx_1 \wedge \ldots \wedge dx_n$$

with $f(p) > 0$ for all $p \in U$.

$(c) \implies (a)$. For this, we need the next theorem. □

This theorem will be the first time we've used the condition that manifolds are second countable in this course.

Theorem 93 (Partitions of Unity exist). For any open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of M, there is a countable collection of functions $\rho_i \in C^\infty(M)$ for $i \in \mathbb{N}$, such that

(i) for any i the **support** $\text{supp}(\rho_i) :=$ closure of $\{x \in M \colon \rho_i(x) \neq 0\}$ is compact and contained in U_α for some $\alpha \in A$;

(ii) the collection is **locally finite**: each $p \in M$ has an open neighborhood $W(p)$ such that ρ_i is identically zero on $W(p)$ except for finitely many i;

(iii) $\rho_i \geqslant 0$ on M for all i, and for each $p \in M$,

$$\sum_i \rho_i(p) = 1.$$

Definition 94. The collection $\{\rho_i \mid i \in \mathbb{N}\}$ as in Theorem 93 is called a **partition of unity subordinate to** $\{U_\alpha \mid \alpha \in A\}$.

The proof of the existence of partitions of unity comes down to standard general topology and the existence of smooth bump functions. This proof is in Warner, Theorem 1.1 or Bott & Tu, Theorem 1.5.2 or Spivak Chapter 2.

Now we can return to the proof of Theorem 92.

Proof of Theorem 92, continued. $(c) \implies (a)$. Given a family of coordinate neighborhoods as in (c), $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$, choose a partition of unity suboordinate to $\mathcal{U}$. For each i, we set $\omega_i = dx_1^{(\alpha)} \wedge \ldots \wedge dx_n^{(\alpha)}$ where $\alpha = \alpha(i)$ with $\text{supp}(\rho_i) \subseteq U_{\alpha(i)}$, and order the coordinates choses so that the Jacobian matrices have positive determinant. Then $\rho_i \omega_i$ is a well-defined smooth n-form on M. Define

$$\omega = \sum_i \rho_i \omega_i.$$

This is the required nowhere vanishing global form, because the Jacobian condition rules out any possible cancellations. □

Definition 95. A connected manifold M satisfying one of the above conditions is called **orientable**. If M is orientable, there are two possible global choices of sign, which are called **orientations**.

Example 96. $\mathbb{RP}^n = S^n / \{\pm 1\}$ is orientable for n odd (on example sheet 2) and non-orientable for n even (Spivak pages 87-88).

Exterior Differentiation

The approach we take here is the sheaf-theoretic version of the definition of exterior derivative, which is different to most books. We'll also take the sheaf-theoretic definition of connections, later.

Theorem 97. Given M, $r \geqslant 0$, there exists a unique linear operator $d\colon \Omega^r(U) \to \Omega^{r+1}(U)$ for all $U \subseteq M$ open, such that for open $V \subseteq U$,

$$\begin{array}{ccc} \Omega^r(U) & \xrightarrow{d} & \Omega^{r+1}(U) \\ \downarrow & & \downarrow \\ \Omega^r(V) & \xrightarrow{d} & \Omega^{r+1}(V) \end{array}$$

commutes. Furthermore,

(i) if $f \in \Omega^0(U)$, then df is the 1-form defined previously;

(ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega}\omega \wedge d\eta$ for any forms ω, η on open $U \subseteq M$;

(iii) $d(d\omega) = 0$ for any form ω on an open subset U of M.

Proof. In local coordinates on some chart U, the three conditions above mean that we must have, if d exists,

$$\begin{aligned} d(f dx_{i_1} \wedge \ldots \wedge dx_{i_r}) &= df \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_r} \\ &= \sum_j \left(\frac{\partial f}{\partial x_j}\right) dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r} \end{aligned}$$

So we define d this way locally, and extend linearly to all of $\Omega^r(U)$. The conditions (i), (ii), and (iii) follow from this recipe by direct calculation. For example,

$$\begin{aligned} d^2(f dx_{i_1} \wedge \ldots \wedge dx_{i_r}) &= d\left(\sum_j \left(\frac{\partial f}{\partial x_j}\right) dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r}\right) \\ &= \sum_j d\left(\frac{\partial f}{\partial x_j}\right) dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r} \\ &= \sum_{j,k} \left(\frac{\partial^2 f}{\partial x_j \partial x_k}\right) dx_j \wedge dx_k \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r} \end{aligned}$$

Now because the second derivative is symmetric, terms cancel.

If d exists, then it has to be given locally by the above formula, and that $(d\omega)_p$ depends only on the value of ω locally.

To show existence, we need to prove that the above recipe doesn't depend on the choice of local coordinates. Suppose d' is defined with respect to other local coordinates, $y_1, \ldots, y_n$. Then by the above, d' also satisfies (i), (ii), and (iii). So let's consider

$$d'(f dx_{i_1} \wedge \ldots \wedge dx_{i_n}) = d'f \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r} + \sum_{j=1}^{r} (-1)^{j-1} f dx_{i_1} \wedge \ldots \wedge d'(dx_{i_j}) \wedge \ldots \wedge dx_{i_n} \tag{4}$$

But $d'f = df$ and since x_k is a function on U, we have that $dx_k = d'x_k$. The definition of df is just the old definition of df we had before. Therefore, $d'(dx_k) = d'(d'x_k) = 0$. Hence, the terms on the second line in (4) vanish and therefore,

$$d'(f dx_{i_1} \wedge \ldots \wedge dx_{i_r}) = df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r} \qquad \square$$

De Rahm Cohomology

Definition 98. $\omega \in \Omega^r(M)$ is **closed** if $d\omega = 0$, and **exact** if $\omega = d\eta$ for some $\eta \in \Omega^{r-1}(M)$.

The quotient space

$$H^r_{DR}(M) := \frac{\text{closed } r\text{-forms on } M}{\text{exact } r\text{-forms on } M} = {}^{\ker d}\!/_{\operatorname{im} d}$$

is the r-th **de Rahm cohomology group** of M.

Lecture 13 — 5 November 2015

Last time we introduced de Rahm cohomology.

Definition 99. Any smooth map $F\colon M \to N$ of smooth manifolds induces a map

$$F^* := (d_pF)^* \colon T^*_{F(p)}M \to T^*_pM$$

for all $p \in M$. For $\alpha \in T^*_{F(p)}M$ and $v \in T_pM$,

$$F^*(\alpha)(v) = \alpha(d_pF(v))$$

This is called the **pullback** of F. Notice for any $g\colon N \to \mathbb{R}$,

$$F^*(dg)(v) = dg(d_pF(v)) = (d_pF)(v)(g) = v(g \circ F) = v(F^*g) = d(F^*g)(v)$$

This defines a **pullback map** $F^*\colon \Omega^r(N) \to \Omega^r(M)$ given by

$$(F^*\omega)_p(v_1, \ldots, v_r) = \omega_{F(p)}(d_pF(v_1), \ldots, d_pF(v_r))$$

for some tangent vectors $v_i \in T_pM$.

Given this definition, what's the pullback of the wedge of two forms ω and η in $\Omega(N)$? Well, the definition above implies that

$$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$$

because it is true pointwise. This also includes the case when ω is a 0-form, that is, a smooth function.

The fact that $F^*\omega$ is smooth (that is, in $\Omega^r(N)$) when ω is smooth follows from a local calculation. For any local coordinate system $x_1, \ldots, x_n$ on $U \subseteq N$,

$$F^*(g dx_{i_1} \wedge \ldots \wedge dx_{i_r}) = F^*(g) dh_{i_1} \wedge \ldots \wedge dh_{i_r}$$

where $h_{i_j} = F^*(x_{i_j}) = x_{i_j} \circ F$ are smooth functions on $F^{-1}U \subseteq M$ and hence the above form is locally smooth on M. Linearity implies that $F^*\omega \in \Omega^r(M)$ for any ω.

Fact 100.

(a) Following from the definition, we also observe that for $F\colon M \to N$ and $G\colon P \to M$, $(F \circ G)^* = G^* \circ F^*$ by the chain rule.

(b) $F^* d\omega = d(F^*\omega)$ follows straight from calculations with the definition of exterior derivative.

(c) From item (b), we can see that the pullback of a closed form is closed, and the pullback of an exact form is exact.

Therefore, by item (c) above, any smooth map $F\colon M \to N$ induces a linear map $F^*\colon H^r_{\mathrm{DR}}(N) \to H^r_{DR}(M)$. If F is a diffeomorphism with inverse G, then F^* on de Rahm cohomology is an isomorphism with inverse G^*.

Remark 101. This is kind of a weak statement. de Rahm Cohommology is a topological invariant, not just a smooth invariant.

Lemma 102 (Poincaré Lemma)**.** $H^k(D) = 0$ for any $k > 0$ and open ball D in $\mathbb{R}^n$.

Proof Sketch. One constructs linear maps $h_k\colon \Omega^k(D) \to \Omega^{k-1}(D)$ such that

$$h_{k+1} \circ d + d \circ h_k = \mathrm{id}_{\Omega^k(D)}.$$

(See Warner pg 155-156 for the construction). Then given $\omega \in \Omega^r(D)$ closed, apply the identity to see that

$$\omega = h_{k+1}(d\omega) + d(h_k\omega) = h_{k+1}(0) + d(h_k\omega)$$

and therefore ω is exact. Therefore, $H^k_{\mathrm{DR}}(D) = 0$. □

Exercise 103. For any **connected** manifold M, $H^0_{\mathrm{DR}}(M) = \mathbb{R}$ is just the constant function.

Integration on Manifolds

Let M be an n-dimensional oriented manifold. Let $\omega \in \Omega^n(M)$ such that the support of ω

$$\mathrm{supp}(\omega) := \text{ closure of } \{p \in M \mid \omega_p \neq 0\}$$

is compact. We say ω is **compactly supported**. If M is itself compact, then this is a silly consideration because $\operatorname{supp}(\omega)$ is closed anyway, and hence compact.

Suppose we have a coordinate chart $\phi = (x_1, \ldots, x_n)\colon U \to \mathbb{R}^n$ with U connected and $\phi(U)$ is bounded. Let $\omega \in \Omega^n(U)$. Since $\Omega^n T^*M$ is 1 dimensional and generated over U by $dx_1 \wedge \ldots \wedge dx_n$, we can write $\omega = f dx_1 \wedge \ldots \wedge dx_n$ on U with f smooth on U. Without loss of generality, assume $f \circ \phi^{-1}$ extends continuously to $\overline{\phi(U)} \subseteq \mathbb{R}^n$. Assume that the order of the coordinates has been chosen such that $dx_1 \wedge \ldots \wedge dx_n$ is in the given orientation.

Definition 104. We define

$$\int_U \omega = \int_U f dx_1 \wedge \ldots \wedge dx_n := \int_{\phi(U)} (f \circ \phi^{-1})\, dr_1 \wedge dr_2 \wedge \ldots \wedge dr_n$$

where the rightmost integral is as in ordinary multivariable calculus.

Claim 105. This definition doesn't depend on the choice of chart.

Proof. Suppose $\psi = (y_1, \ldots, y_n)$ on U is a chart in the same orientation. Then

$$f dx_1 \wedge \ldots \wedge dx_n = f \det \left(\frac{\partial x_i}{\partial x_j} \right) dy_1 \wedge \ldots \wedge dy_n,$$

where because ϕ, ψ are in the same orientation, $\det \left({}^{\partial x_i}/_{\partial y_j} \right) > 0$. Recall that

$$\left(\frac{\partial x_i}{\partial y_j} \right) = J \circ \psi$$

where J is the Jacobian matrix of the coordinate transformation

$$F = \phi \circ \psi^{-1}\colon V = \psi(U) \to \phi(U) = F(V).$$

Change of variable formula for multivariable calculus says that

$$\int_{F(V)} h dr_1 \wedge \ldots \wedge dr_n = \int_V (h \circ F) |\det J| ds_1 \wedge \cdots \wedge ds_n,$$

where s_i are the coordinates on $\psi(U) \subseteq \mathbb{R}^n$. When $h = f \circ \phi^{-1}$, we see that

$$\begin{aligned}
\int_{\phi(U)} f dx_1 \wedge \ldots \wedge dx_n &= \int_{\phi(U)} (f \circ \phi^{-1}) dr_1 \wedge \ldots \wedge dr_n \\
&= \int_{\psi(U)} (f \circ \psi^{-1}) |\det J| ds_1 \wedge \ldots \wedge ds_n \\
&= \int_{\psi(U)} (f \circ \psi^{-1}) \det J ds_1 \wedge \ldots \wedge ds_n \\
&= \int_{\psi(U)} f \det \left(\frac{\partial x_i}{\partial y_j} \right) dy_1 \wedge \ldots \wedge dy_n
\end{aligned}$$

Therefore, $\int_U \omega$ is well-defined. □

We can make our integrations more general. Given $\mathrm{supp}(\omega)$ compact, there is a finite collection $\phi_i \colon U_i \to \mathbb{R}^n$ of bounded coordinate charts ($i = 1, \ldots, r$) such that $\mathrm{supp}(\omega) \subseteq \bigcup_{i=1}^r U_i$. We set

$$A_i = \int_{U_i} \omega \text{ for } i = 1, \ldots, r,$$
$$A_{ij} = \int_{U_i \cap U_j} \omega \text{ for } i < j, \text{ and}$$
$$A_{ijk} = \int_{U_i \cap U_j \cap U_k} \text{ for } i < j < k.$$

These are all well-defined by the previous claim. Define

$$\int_M \omega = \sum_{i=1}^r A_i - \sum_{i<j} A_{ij} + \sum_{i<j<k} A_{ijk} + \ldots + (-1)^{r+1} A_{1,2,\ldots,n}$$

Lemma 106. This is independent of the choice of scharts. That is, if we have another collection of charts $\psi_j \colon V_j \to \mathbb{R}^n$ of charts with $j = 1, \ldots, s$ and with $\mathrm{supp}(\omega) \subseteq \bigcup_j V_j$, set B_i, B_{ij}, B_{ijk} similarly to the above. Then

$$\sum_{i=1}^r A_i - \sum_{i<j} A_{ij} + \sum_{i<j<k} A_{ijk} + \ldots = \sum_{i=1}^r B_i - \sum_{i<j} B_{ij} + \sum_{i<j<k} B_{ijk} +$$

Lecture 14 — 7 November 2015

Last time we defined integration on manifolds. There were a few hiccups with the last lecture so let's make some clarifications.

Remark 107. Clarification of the definition of $\int_M \omega$.

(1) One can assume that $f \circ \phi^{-1}$ extends continuously to the closure of $\phi(U)$ by shrinking U if required.

(2) Recall $\omega = f dx_1 \wedge \ldots \wedge dx_n = \Delta dy_1 \wedge \ldots \wedge dy_n$, where Δ is the determinant of the Jacobian. The left hand side of the change of variable formula is $\int_U \omega$ in the x_i coordinates,

$$\int_U f dx_1 \wedge \ldots \wedge dx_n$$

and the right hand side is $\int_U \omega$ in the y_j coordinates,

$$\int_U f \Delta dy_1 \wedge \ldots \wedge dy_n.$$

Theorem 108 (Stokes's Theorem without proof)**.** Suppose $\eta \in \Omega^{n-1}(M)$ has compact support. Then

$$\int_M d\eta = 0.$$

Fact 109. Stokes's Theorem produces a perfect pairing between $H_r(M, \mathbb{R}) \times H^r_{\text{DR}}(M, \mathbb{R}) \to \mathbb{R}$ by means of "integrating over cycles." Here $H_r(M, \mathbb{R})$ is singular homology.

Corollary 110 (Integration by Parts)**.** Suppose α, β are compactly supported forms on M, with $\deg \alpha + \deg \beta = \dim M - 1$. Then

$$\int_M \alpha \wedge d\beta = (-1)^{\deg \alpha + 1} \int_M (d\alpha) \wedge \beta$$

Corollary 111. If M is a compact, orientable n-manifold, then $H^n_{\text{DR}}(M) \neq 0$.

Proof. Choose an orientation $\omega \in \Omega^n(M)$. Then $\int_M \omega > 0$. But ω is clearly closed, but not exact by Stokes. Hence, $H^n_{\text{DR}}(M) \neq 0$. $\square$

Lie Derivatives

These won't play a major part in this course, but they do have an important relation with connections, which will be the major topic of this course after this lecture.

Definition 112. Given a vector field X on a manifold M and $p \in M$, and an open neighborhood $U \ni p$, a **flow on** U is a collection of functions $\phi_t \colon U \to \phi_t(U)$ for $|t| < \varepsilon$ such that

(i) $\phi \colon (-\varepsilon, \varepsilon) \times U \to M$ defined by $\phi(t, q) = \phi_t(q)$ is smooth;

(ii) if $|s|, |t|, |t + s| < \varepsilon$ and $\phi_t(q) \in U$, then $\phi_{s+t}(q) = \phi_s(\phi_t(q))$;

(iii) if $q \in U$, then $X(q)$ is the tangent vector at $t = 0$ of the curve $t \mapsto \phi_t(q)$.

So if $f \colon U \to \mathbb{R}$ is a smooth function on an appropriate neighborhood of $U \ni p$, by assumption $\gamma \colon t \to \phi_t(p)$ is an integral curve for X with $\gamma(0) = p$. Furthermore, $X(p) = \dot{\gamma}(0) = d_0\gamma \left({}^{d}/_{dr}\right)$. Therefore,

$$X(f)(p) = X(p)(f) = d_0\gamma \left(\frac{d}{dr}\right) f = (f \circ \gamma)'(0)$$

This map $f \circ \gamma$ is now a function $\mathbb{R} \to \mathbb{R}$, so we can write out the derivative in terms of limits.

$$\begin{aligned} X(f)(p) &= (f \circ \gamma)'(0) \\ &= \lim_{h \to 0} \frac{(f \circ \gamma)(h) - (f \circ \gamma)(0)}{h} \\ &= \lim_{h \to 0} \frac{f(\phi_h(p)) - f(p)}{h} \\ &= \lim_{h \to 0} \frac{\phi_h^*(f)(p) - f(p)}{h} \end{aligned}$$

Locally, we therefore have that

$$X(f) = \lim_{h \to 0} \frac{\phi_h^*(f) - f}{h}$$

This is the **Lie Derivative** on functions.

Definition 113. The **Lie Derivative** of $f\colon M \to \mathbb{R}$ is $\mathcal{L}_X(f) = X(f) \in C^\infty(U)$

Now we can extend the definition of the Lie derivative to forms. For $F\colon N \to M$ smooth and $\omega \in \bigwedge^r T^*M$ a smooth r-for, last time we defined $F^*\omega$ induced pointwise from maps $d_pF\colon T_pN \to T_{f(p)}M$. Namely,

$$F^*(\omega)(p) = (d_pF)^*(\omega_{F(p)}),$$

which we'll also denote by $F^*(\omega_{F(p)})$.

Definition 114. If ω is an r-form on M, we define the **Lie Derivative with respect to** X by

$$\mathcal{L}_X(\omega) = \lim_{h\to 0} \frac{\phi_h^*(\omega) - \omega}{h},$$

or pointwise by

$$\mathcal{L}_X(\omega)(p) = \lim_{h\to 0} \frac{\phi_h^*(\omega)(p) - \omega(p)}{h}$$

Fact 115. Some facts regarding Lie derivatives.

(a) If ω, η are smooth forms, then

$$\begin{aligned}(\phi_h^*(\omega \wedge \eta) - \omega \wedge \eta)_p &= (\phi_h^*(\omega) \wedge \phi_h^*(\eta) - \omega \wedge \eta)_p \\ &= \phi_h^*(\omega_{\phi_h(p)}) \wedge \left(\phi_h^*\eta_{\phi_h(p)} - \eta_p\right) + \left(\phi_h^*\omega_{\phi_h(p)} - \omega_p\right) \wedge \eta_p\end{aligned}$$

This implies that $\mathcal{L}_X$ is a derivation:

$$\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X(\omega) \wedge \eta + \omega \wedge \mathcal{L}_X(\eta).$$

(b) For any smooth map ϕ, we saw that $\phi^*(d\omega) = d(\phi^*\omega)$. Hence,

$$\begin{aligned}\mathcal{L}_X(d\omega)_p &= \lim_{h\to 0} \frac{1}{h} \left((\phi_h^*)(d\omega)_{\phi_h(p)} - d\omega_p\right) \\ &= \lim_{h\to 0} \frac{1}{h} d\left(\phi_h^*\omega_{\phi_h(p)} - \omega_p\right) \\ &= d\mathcal{L}_X\omega\end{aligned}$$

(c) If $X = \sum_i X_i {}^{\partial}/_{\partial x_i}$ in local coordinates and $\omega = f dx_{i_1} \wedge \ldots \wedge dx_{i_r}$, then (a) implies that

$$\mathcal{L}_X\omega = (Xf)dx_1 \wedge \ldots \wedge dx_r + f\sum_{j=1}^{r} dx_{i_1} \wedge \ldots \wedge dx_{i_{j-1}} \wedge dX_{i_j} \wedge \ldots \wedge dx_{i_r}$$

sine $\mathcal{L}_X(dx_j) = d(\mathcal{L}_X(x_j)) = d(X(x_j)) = dX_j$.

Suppose now $\phi\colon U \to V$ is a diffeomorphism and Y is a vector field on V. We can define $\phi^*(Y) := (\phi^{-1})_*Y$, which produces a vector field on M.

Definition 116. Thus for Y a vector field on M, we can define a **Lie derivative of a vector field** Y by

$$\mathcal{L}_X(Y)(p) = \lim_{h\to 0} \frac{(\phi_h)_*(Y)(p) - Y(p)}{h} = \lim_{h\to 0} \frac{(\phi_{-h})_* Y_{\phi_h(p)} - Y_p}{h}$$

Therefore,

$$\mathcal{L}_X(Y) = \lim_{h\to 0} \frac{(\phi_{-h})_* Y - Y}{h}$$

Remark 117. Setting $k = -h$, this is also

$$\lim_{k\to 0} \frac{1}{k} \left(Y_p - (\phi_k)_* Y_{\phi_k(p)} \right)$$

$$\implies \mathcal{L}_X(Y) = \lim_{k\to 0} \frac{1}{k} \left(Y - (\phi_k)_* Y \right)$$

Example sheet 2, question 11 asks you to prove that $\mathcal{L}_X(Y) = [X, Y]$.

Remark 118.

(1) $\mathcal{L}_X$ defines an operator on all tensors of a given type in exactly the same way.

(2) $(\mathcal{L}_X T)_p$ depends on X in a neighborhood of p and not just on $X(p)$. (Contrast this with connections when we talk about them next time.)

(3) In general, $(\mathcal{L}_{fX} T)_p \neq f(p)(\mathcal{L}_X T)_p$.

Lecture 15 — 10 November 2015

Connections on Vector Bundles

This is really the crux of the course. Here we're going to talk about connections on arbitrary vector bundles, and later we're going to specialize to connections on the tangent bundle. Even later, we'll introduce metrics into the equation and then there's a canonical connection called the Levi-Civita connection.

We start with vector bundle valued forms.

Definition 119. Suppose $\pi\colon E \to M$ is a smooth rank k vector bundle over M. An E**-valued** q**-form** is a smooth section of the vector bundle $E \otimes \bigwedge^q T^*M = E \otimes (\bigwedge TM)^* = \mathrm{Hom}\left(\bigwedge^q TM, E\right)$.

Denote such forms as $\Omega^q(M, E)$.

Definition 120. If $U \subseteq M$ is an open subset for which $E|_U = \pi^{-1}(U) \cong U \times \mathbb{R}^k$, then we have a **frame** of smooth sections $e_1, \ldots, e_k$ of $E|_U$ which form a basis for the fiber E_p for all $p \in U$.

Therefore,
$$E|_U \otimes \bigwedge^q T^*M|_U \cong \left(\bigwedge^q T^*M|_U\right)^k,$$
and sections of $\Omega^q(U, E)$ may be written in the form $\omega_1 e_1 + \ldots + \omega_k e_k \in \Omega^k(U)$.

If moreover U is a coordinate neighborhood in M with coordinates $x_1, \ldots, x_n$, each ω_i is of the form $\omega_i = \sum_I f_I dx_{i_1} \wedge \ldots \wedge dx_{i_q}$ and so an element of $\Omega^q(U, E)$ may be written as
$$\sum_{\substack{I \subseteq \{1,\ldots,n\} \\ \#I = q, 1 \leqslant j \leqslant k}} f_{I,j} dx_{i_1} \wedge \ldots \wedge dx_{i_q} \otimes e_j$$
This shows that the bundle $(E \otimes \bigwedge^q T^*M)|_U$ is trivial, isomorphic to $U \times \mathbb{R}^{k\binom{n}{q}}$.

Similarly, we have smooth sections of $\Omega^q(\mathrm{End}(E))$ of
$$\mathrm{Hom}(E, E) \otimes \bigwedge^q T^*M,$$
where $\mathrm{End}(E) := \mathrm{Hom}(E, E)$. And if $E|_U$ is trivial, the sections of this bundle over U may be regarded as matrix-valued q-forms.

Definition 121. If $e_1, \ldots, e_k$ is a local frame for E over U, we have the dual frame $\varepsilon_1, \ldots, \varepsilon_k$ for E^* over U and any element of $\Omega^q(U, \mathrm{End}(E))$ may be written in the form
$$\sum_{i,j} \omega_{ij} \otimes \varepsilon_i \otimes e_j$$
with $\omega_{ij} \in \Omega^q(U)$.

Fact 122. Given finite dimensional vector spaces V and W, there is a natural identification $\mathrm{Hom}(V, V) \otimes W \to \mathrm{Hom}(V, V \otimes W)$. In particular, this identifies the vector bundles
$$\mathrm{Hom}(E, E) \otimes \bigwedge^q T^*M \xrightarrow{\sim} \mathrm{Hom}\left(E, E \otimes \bigwedge^q T^*M\right)$$

Definition 123. Given a vector valued forms $\sigma_1 \in \Omega^p(M, E)$, $\sigma_2 \in \Omega^q(M, E')$, we can define a product $\sigma_1 \wedge \sigma_2 \in \Omega^{p+q}(M, E \otimes E')$. On forms, this is just taking the wedge product, and on the bundle part it's just tensoring.

Locally, with respect to a trivialization $e_1, \ldots, e_k$ of E and $e'_1, \ldots, e'_\ell$ of E', this is defined by
$$(\omega_1 \otimes e_i) \wedge (\omega_2 \otimes e'_j) \mapsto (\omega_1 \wedge \omega_2) e_1 \otimes e'_j,$$
and extending linearly. Morally, we should check that this definition makes global sense (i.e. agrees on overlaps of trivialization chosen).

Definition 124. When $E' = E^*$, we have a natural map $E \otimes E^*$ to the trivial bundle given locally by $e_i \otimes \varepsilon_j \mapsto \varepsilon_j(e_i)$. If we identify $E \otimes E^* = \mathrm{Hom}(E, E)$, then this is just given by the trace map.

This defines a product on E-valued p-forms and E^*-valued q-forms via the composition
$$\Omega^p(M, E) \times \Omega^q(M, E^*) \xrightarrow{\wedge} \Omega^{p+q}(M, E \otimes E^*) \xrightarrow{\mathrm{tr}} \Omega^{p+q}(M)$$
This is usually just denoted by $\wedge$.

Definition 125. Similarly, we have a product

$$\Omega^p(M, \text{End}(E)) \times \Omega^q(M, \text{End}(E)) \xrightarrow{\wedge} \Omega^{p+q}(M, \text{End}(E)).$$

This is just multiplying these matrices, but using the wedge product instead of multiplication.

Definition 126. Of particular importance is the product

$$\Omega^p(M, \text{End}(E)) \times \Omega^q(M, E) \longrightarrow \Omega^{p+q}(M, E)$$

given locally by

$$\left(\sum_i \omega_i \otimes \theta_i, \sum_j \eta_j \otimes s_j\right) \mapsto \sum_{i,j} \omega_i \wedge \eta_j \otimes (\theta_i(s_j))$$

This is usually just denoted by $\wedge$.

Example 127. When we define the curvature $\mathcal{R}$, it is an element of $\Omega^2(M, \text{End}(E))$ and we have an induced map

$$\begin{array}{ccc} \Omega^q(E) & \longrightarrow & \Omega^{q+2}(E) \\ \sigma & \longmapsto & \mathcal{R} \wedge \sigma \end{array}$$

Connections

Connections enable us to differentiate sections of a vector bundle of rank r.

Definition 128. A **linear connection** on the vector bundle E over M is given by, for any open $U \subseteq M$, a map $\mathcal{D} = \mathcal{D}(U) \colon \Gamma(E, U) \to \Omega^1(U, E) = \Gamma(U, E \otimes T^*M)$, such that

(i) if $U \supseteq V$ and $\sigma \in \Gamma(U, E)$, then $\mathcal{D}(\sigma|_V) = (D\sigma)|_V$;

(ii) $\mathcal{D}(f\sigma) = f\mathcal{D}(\sigma) + df \otimes \sigma$;

(iii) $\mathcal{D}(\sigma_1 + \sigma_2) = D\sigma_1 + D\sigma_2$. Where f is a smooth function on M.

Remark 129. This definition of the connection differs form almost every book on differential geometry. It's the sheaf-theoretical definition of connections. Most books define it to be a global map $\Gamma(E) \to \Omega^1(M, E)$ satisfying Definition 128(ii) and Definition 128(iii).

While in some cases we've taken the standard notation to agree with the books, defining this thing globally is just *wrong*. Many books require some illegal finesse to discuss global-to-local property.

Our definition avoids this problem because if we know $\mathcal{D}(U_\alpha)$ for some open cover $\{U_\alpha \mid \alpha \in A\}$ of M, then Definition 128(i) guarantees that we have a well-defined global map.

Definition 130. For a given $p \in M$ and $\alpha \in T_pM$, we can define a map

$$\mathcal{D}_\alpha \colon \Gamma(U, E) \to E_p$$

for any neighborhood $U \ni p$ by

$$\mathcal{D}_\alpha(\sigma) = (\mathcal{D}\sigma)(\alpha).$$

This is the **covariant derivative along** α.

Moreover, if X is a smooth vector field on $U \subseteq M$, then define the **covariant derivative along** X by

$$\mathcal{D}_X(\sigma) = (\mathcal{D}\sigma)(X) \in \Gamma(U, E)$$

Note that $\mathcal{D}_X(\sigma)(p) \in E_p$ only depends on locally on σ and X_p.

Fact 131. From the properties of $\mathcal{D}$, we see that

$$\mathcal{D}_X(\sigma_1 + \sigma_2) = \mathcal{D}_X\sigma_1 + \mathcal{D}_X\sigma_1$$

$$\mathcal{D}_X(f\sigma) = f\mathcal{D}_X\sigma + X(f)\sigma$$

$$\mathcal{D}_{fX+gY}(\sigma) = f\mathcal{D}_X\sigma + g\mathcal{D}_Y\sigma$$

Contrast the covariant derivative with the Lie derivative, on say $E = TM$. Recall that

$$\mathcal{L}_{fX+gY}(Z) \neq f\mathcal{L}_XZ + g\mathcal{L}_YZ$$

in general.

Lecture 16 — 12 November 2015

Last time we introduced the essential topic of connections on vector bundles in a sheaf-theoretic way. What dos this look like in local coordinates? This lecture is somewhat of a tangent (no pun intended) wherein we explore the alternative definition of connections that is found in most books, and compare to our definition.

Suppose now that $\{e_1, \ldots, e_r\}$ is a local frame for E over $U \subseteq M$; let us set

$$\mathcal{D}e_j = \sum_j \theta_{kj}e_k \in \Omega^1(U, E),$$

where the juxtaposition $\theta_{kj} \otimes e_k$. We also often write $\theta^k_{\ j} = \theta_{kj}$.

The matrix $\theta_e = \left[\theta_{ij}\right]_{1 \leqslant i,j \leqslant k}$ of local 1-forms is called the **connection matrix**.

If U also a coordinate neighborhood with coordinates $x_1, \ldots, x_n$, we can write entries of the connection matrix in terms of $dx_1, \ldots, dx_n$, say

$$\theta^k_{\ j} = \sum_{i=1}^n \Gamma^k_{\ ij} dx_i$$

with ${\Gamma^k}_{ij}$ smooth functions on U. Then setting

$$\mathcal{D}_i = D_{\partial/\partial x_i}$$

(this is $\mathcal{D}_X$ where X is the vector field $X = {}^{\partial}/_{\partial x_i}$). Then we have

$$D_i e_j = \sum_{k=1}^{r} {\Gamma^k}_{ij} e_k.$$

(Note that in some books, the indices i and j may be transposed.)

What happens when we change coordinates? If we change the chart, then θ^k_j and ${\Gamma^k}_{ij}$ will also change. Suppose for instance we have another frame $e'_1, \dots, e'_r$ and the transition functions between the two trivializations is given by an $r \times r$ matrix of smooth functions $\psi = \left[\psi_{ij}\right]_{1 \leqslant i,j \leqslant r}$. This means that with respect to the $\{e'_i\}$-basis

$$e'_j = \sum_k \psi_{kj} e_k.$$

Therefore,

$$De'_j = D\left(\sum_k \psi_{kj} e_k\right) = \sum_k d\psi_{kj} e_k + \sum_{k,\ell} \psi_{kj} \theta_{\ell k} e_\ell$$

We can rewrite this in terms of the $\{e'_i\}$-basis by applying ψ^{-1}:

$$= \sum_p \left(\sum_k d\psi_{kj} (\psi^{-1})_{pk} + \sum_{k,\ell} \psi_{kj} \theta_{\ell k} (\psi^{-1})_{p\ell}\right) e'_p$$

These terms in parentheses are the coordinates of $\theta_{e'}$, so

$$(\theta_{e'})_{pj} = \left(\sum_k d\psi_{kj} (\psi^{-1})_{pk} + \sum_{k,\ell} \psi_{kj} \theta_{\ell k} (\psi^{-1})_{p\ell}\right)$$

So we are left with the important equation

$$\boxed{\theta_{e'} = \psi^{-1} d\psi + \psi^{-1} \theta_e \psi}$$

Exercise 132. We could also change coordinate systems on U, say to $y_1, \dots, y_n$ and find expressions for ${(\Gamma')^k}_{ij}$ in terms of ${\Gamma^k}_{ij}$. Check that

$${(\Gamma')^k}_{pj} = (\psi^{-1})_{ik} \frac{\partial \psi_{kj}}{\partial y_p} + (\psi^{-1})_{ij} {\Gamma^k}_{q\ell} \psi_{\ell j} \left(\frac{\partial x_q}{\partial y_p}\right),$$

where we have assumed the summation convention in the expression above.

Definition 133. We say that a section $\sigma \in \Gamma(U, E)$ is **horizontal** at $p \in U$ with respect to the connection if and only if $\mathcal{D}_\alpha \sigma = 0$ for all $\alpha \in T_p M$, if and only if $(D\sigma)_p = 0$.

What does this really mean? Given a local trivialization $\sigma = \sum_j f_j e_j$ as above,

$$\mathcal{D}\left(\sum_j f_j e_j\right) = \sum_{j=1}^{r}\left(df_j \otimes e_j + \sum_{k=1}^{r} f_j \theta_{kj} e_k\right)$$
$$= \sum_{k=1}^{r}\left(df_k + \sum_{j=1}^{r} \theta_{kj} f_j\right) e_k$$

This is an equation at p. So $\mathcal{D}(\sigma) = 0$ at p if and only if the coefficients vanish,

$$df_k + \sum_{j=1}^{r} \theta_j f_j = 0,$$

at p for all k.

If moreover we have coordinates $x_1, \ldots, x_n$ on U, we may rewrite this condition as

$$df_k + \sum_{i,j} f_j \Gamma^k{}_{ij} dx_i = 0$$

for $k = 1, \ldots, r$. Plugging in x_i to this equation, we get the condition

$$\boxed{\frac{\partial f_k}{\partial x_i} + \sum_j \Gamma^k{}_{ij} f_j = 0}$$

at p for all $k = 1, \ldots, r$ and all $i = 1, \ldots, n$.

Under the above trivialization given by the frame $e_1, \ldots, e_r$ and coordinates $x_1, \ldots, x_n$ on U, we have coordinates on $E|_U \cong U \times R$ given by $(x_1, \ldots, x_n; a_1, \ldots, a_r)$. The tangent space $T_q E$ for $q \in E|_U$ has dimension $r + n$ and basis

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial a_1}, \ldots, \frac{\partial}{\partial a_r}.$$

Let σ be the section of E given by $\sigma(\vec{x}) = (\vec{x}; f_1(\vec{x}), \ldots, f_r(\vec{x}))$. The tangent space to $\sigma(U)$ at $\sigma(p)$ is generated by tangent vectors of the form

$$(d\sigma)\left(\frac{\partial}{\partial x_i}\right),$$

where

$$(d\sigma)\left(\frac{\partial}{\partial x_i}\right)(x_k) = \frac{\partial}{\partial x_i}(x_k \circ \sigma) = f_{ik}$$

and

$$(d\sigma)\left(\frac{\partial}{\partial x_i}\right)(a_j) = \frac{\partial}{\partial x_i}(a_j \circ \sigma) = \left.\frac{\partial f_j}{\partial x_i}\right|_p.$$

This means that

$$d\sigma\left(\frac{\partial}{\partial x_i}\right) = \left.\frac{\partial}{\partial x_i}\right|_{\sigma(p)} + \sum_j \frac{\partial f_j}{\partial x_i}(p) \left.\frac{\partial}{\partial a_j}\right|_{\sigma(p)}$$

What does this all have to do with what we did before? Let's evaluate the form

$$\left(da_k + \sum_{i,j} {\Gamma^k}_{ij} a_j dx_i\right)$$

on the vector

$$\left(\left.\frac{\partial}{\partial x_\ell}\right|_{\sigma(p)} + \sum_j \frac{\partial f_j}{\partial x_\ell}(p) \left.\frac{\partial}{\partial a_j}\right|_{\sigma(p)}\right),$$

then

$$\left(da_k + \sum_{i,j} {\Gamma^k}_{ij} a_j dx_i\right)\left(\left.\frac{\partial}{\partial x_\ell}\right|_{\sigma(p)} + \sum_j \frac{\partial f_j}{\partial x_\ell}(p) \left.\frac{\partial}{\partial a_j}\right|_{\sigma(p)}\right) = \left(\frac{\partial f_k}{\partial x_\ell} + \sum_j {\Gamma^k}_{ij} f_j\right)_p$$

And if σ is horizontal at p, then this is zero.

Definition 134. Note that the forms

$$da_k + \sum_{i,j} {\Gamma^k}_{ij} a_j dx_i$$

on $T_{\sigma(p)}E$ for $k = 1, \ldots, r$ are linearly independent, and when σ is horizontal at p they also span. So the tangent space to $\sigma(U)$ at $\sigma(p)$ is cut out precisely by these forms. We then say that the tangent space at $\sigma(p)$ of $\sigma(U)$ is **horizontal** with respect to the connection.

Definition 135. This yields an alternative description of the connection as a family $S_q \subseteq T_q E$ of n-dimensional subspaces (what we previously called a distribution), called the **horizontal subspaces**; the corresponding sub-bundle generated by this distribution is called a **horizontal bundle**.

In terms of any local trivialization of $\pi^{-1}(U)$ with coordinates $x_1, \ldots, x_n$, $a_1, \ldots, a_r$ as above, S_q is defined by forms of the **type**

$$da_k + \sum_{i,j} {\Gamma^k}_{ij} a_j dx_i = da_k + \sum_j \theta_{kj} a_j$$

and is independent of the trivialization.

Reversing the argument gives a connection in the sense we've defined it in the previous lecture (Definition 128).

Definition 136. A local section $\sigma\colon U \to E$ is **horizontal/parallel/covariantly constant** if it is horizontal at all points p of U.

Example 137. The standard connection on $T\mathbb{R}^n$ is given by

$$\mathcal{D}\left(\frac{\partial}{\partial x_i}\right) = 0$$

for all i. If $\sigma = \sum f_i {}^{\partial}/_{\partial x_i}$, then

$$D\sigma = \sum_i df_i \otimes \frac{\partial}{\partial x_i} = 0 \iff df_i = 0 \text{ for all } i \iff f_i \text{ constant for all } i$$

Lecture 17 14 November 2015

Lemma 138. Given a vector bundle E over M, there is a connection on E.

Proof. Locally, $E|_U \cong U \times \mathbb{R}^r$ is trivial, where r is the rank of the bundle E. There is a connection ∇ on $U \times \mathbb{R}^r$ such that $\nabla(e_k) = 0$ for all k, where $\{e_i\}$ defines a frame on U.

Now choose an open cover $\mathcal{U} = \{U_j \mid j \in J\}$ of M consisting of such open sets, and a partition of unity $\{\rho_i \mid i \in I\}$ subordinate to $\mathcal{U}$ (which means that for each $i \in I$, $\operatorname{supp}(\rho_i) \subseteq U_{j(i)}$ for some $j \in J$). Then define the connection on E by

$$\mathcal{D} = \sum_{i \in I} \rho_i \nabla^{j(i)},$$

where $\nabla^{j(i)}$ is the connection on $E|_U$. □

Homomorphisms of bundles

Recall that a homomorphism of vector bundles over M is a smooth map $\Psi \colon E \to F$ with maps on fibers $\Psi_p \colon E_p \to F_p$ for each p, commuting with the maps $E \to M$ and $F \to M$.

$$\begin{array}{ccc} E & \xrightarrow{\quad\Psi\quad} & F \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

So if $U \subseteq M$, we have an induced map $\Psi_* = \Psi(U) \colon \Gamma(U, E) \to \Gamma(U, F)$ given by $\Psi_*(\sigma) = \Psi \circ \sigma$. Note that

$$\Psi_*(f\sigma + g\tau) = f\Psi_*(\sigma) + g\Psi_*(\tau) \tag{5}$$

for all smooth $f, g \in \Omega^0(U)$.

I think I messed up the difference between capital and lowercase ψ in the following. I got confused by the lecturer's handwriting! The point is that $\psi(U)$ is the local map $\Gamma(U, E) \to \Gamma(U, F)$, while Ψ is the map of bundles globally $E \to F$.

Conversely, suppose we have maps $\psi(U) \colon \Gamma(U, E) \to \Gamma(U, F)$ compatible with restrictions (as in sheaf morphisms) such that (5) holds

$$\psi(U)(f\sigma + g\tau) = f\Psi(U)(\sigma) + g\psi(U)(\tau)$$

for all $\sigma, \tau \in \Gamma(U, E)$ and $f, g \in \Omega^0(U)$.

We have a well-defined map $\Psi \colon E \to F$ given for any section $s \in \Gamma(U, E)$, $U \ni p$, by $\Psi(s(p)) = \psi(s)(p)$.

What does this look like locally? In any open neighborhood of p, we choose a frame $e_1, \ldots, e_r$ of $E|_U$ (that is, $e_1(q), \ldots, e_r(q)$ a basis for E_q for all $q \in U$) and then any section s of $E|_U$ is of the form $s = \sum_i f_i e_i$ for some $f_i \in \Omega^0(U)$. Then (5) implies that

$$\psi(s) = \sum_i f_i s_i,$$

where $s = \psi(U)(e_i) \in \Gamma(U, F)$. So when evaluating at p, we get

$$\psi(s)(p) = \sum_i f_i(p) s_i(p),$$

and any element of E_p is of the form $\sum_i \lambda_i e_i(p)$, and so define

$$\Psi\left(\sum_i \lambda_i e_i(p)\right) := \sum_i \lambda_i s_i(p).$$

This is well-defined by the compatibility conditions we imposed. Hence, Ψ gives a homomorphism of vector bundles, and moreover for any section $\sigma \in \Gamma(V, E)$, V open in M,

$$\Psi_*(\sigma)(p) = \Psi(\sigma(p)) = \psi(\sigma)(p)$$

for all $p \in V$. This implies that $\Psi_* = \psi$ over any open set.

Lemma 139. Suppose $\mathcal{D}_1, \mathcal{D}_2$ are connections on a vector bundle E over M, then $(\mathcal{D}_1 - \mathcal{D}_2)$ corresponds to an element of $\Omega^1(\mathrm{End}(E)) \cong \Gamma(\mathrm{Hom}(E, E \otimes T^*M))$. Essentially, we can take any connection, add a 1-form over $\mathrm{End}(E)$, and get another connection.

Remark 140. For bundles E, F, $\mathrm{Hom}(E, E \otimes F) \cong E^* \otimes E \otimes F \cong \mathrm{Hom}(E, E) \otimes F$.

Proof of Lemma 139. Just note that for any open set U and sections $\sigma, \tau \in \Gamma(U, E)$ and $f, g \in \Omega^0(U)$, compute

$$(\mathcal{D}_1 - \mathcal{D}_2)(f\sigma + g\tau) = f(\mathcal{D}_1 - \mathcal{D}_2)(\sigma) + g(\mathcal{D}_1 - \mathcal{D}_2)(\tau)$$

Hence, the result follows from the discussion above. □

Following this lemma, we can see that the connections on a vector bundle are an infinite dimensional **affine space** (meaning that we have a vector space without an origin) over the vector field $\Omega^1(\mathrm{End}(E))$. The automorphism group of the vector bundle acts in a natural way on this affine space of connections.

Covariant Exterior Derivative

Definition 141. Given a connection $\mathcal{D}\colon \Omega^0(E) \to \Omega^1(E)$ (this is really shorthand for $\mathcal{D}(U)$ on sections over U for all open U, compatible with restrictions). We can define a **covariant exterior derivative** $D = d^E\colon \Omega^p(E) \to \Omega^{p+1}(E)$, satisfying the Liebniz rule, that is, for any E-valued form μ and every differential form ω,

$$d^E(\mu \wedge \omega) = d^E\mu \wedge \omega + (-1)^{\deg \mu} \mu \wedge d\omega$$
$$d^E(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^{\deg \omega} \omega \wedge d^E\mu$$

Lemma 142. Given a connection $\mathcal{D}$ on a vector bundle E, there is a unique covariant derivative d^E such that $d^E(\sigma) = \mathcal{D}\sigma$ for all $\sigma \in \Omega^0(E)$.

Proof. Suppose that we have a local frame $s_1, \dots, s_r$ for E. Then for $\sigma = \sum_i f_i s_i$,

$$\mathcal{D}\sigma = \sum_i df_i \otimes s_i + \sum_i f_i \mathcal{D} s_i = \sum_{i,k} f_i \theta_{ki} s_k$$

We extend this as follows. There's really only one choice, since we have the Leibniz rule. Given $\sum_i \omega_i \otimes s_i \in \Omega^p(U, E)$, we set

$$d^E \left(\sum_i \omega_i \otimes s_i \right) = \sum_i \left(d\omega_i \otimes s_i + (-1)^p \omega_i \wedge d^E s \right),$$

where $d^E(s_i) = Ds_i = \sum_k \theta_{ki} s_k$. Therefore,

$$d^E \left(\sum_i \omega_i \otimes s_i \right) = \sum_i d\omega_i \otimes s_i + (-1)^p \sum_{i,k} \omega_i \wedge \theta_{ki} s_k.$$

Given a change of frame $s'_j = \sum \psi_{ij} s_i$, one checks easily that this definition doesn't depend on the choice of frame.

We're forced by the Leibniz rule to make this definition the way that we did, and so d^E is defined uniquely over such an open set U. In particular, these patch together to give a well-defined and unique map $d^E \colon \Omega^p(U, E) \to \Omega^{p+1}(U, E)$ for any open U, including $U = M$. □

Definition 143. Consider now the map $\mathcal{R} = d^E \circ d^E = D^2 \colon \Omega^0(E) \to \Omega^2(E)$. This is called the **curvature operator**.

Note that

$$D^2(f\sigma) = D(df \otimes \sigma + fD\sigma) = d^2 f \otimes \sigma - df \wedge D\sigma + df \wedge D\sigma + fD^2\sigma = fD^2\sigma.$$

So even though D doesn't correspond to a homomorphism of vector bundles, $\mathcal{R}$ in fact does. Our previous discussion shows that $\mathcal{R} \in \Gamma(\mathrm{Hom}(E, \bigwedge^2 T^*M \otimes E))$, but we can in fact identify the bundle $\mathrm{Hom}(E, \bigwedge^2 T^*M \otimes E))$ with $\bigwedge^2 T^*M \otimes \mathrm{Hom}(E, E)$, and hence $\mathcal{R}$ corresponds to an element

$$R \in \Gamma \left(\bigwedge^2 T^*M \otimes \mathrm{Hom}(E, E) \right)$$

where $\mathcal{R}(\sigma) = R \wedge \sigma$, that is,

$$\mathcal{R}(\sigma)(X, Y)\sigma = R(X, Y)\sigma \in \Gamma(E)$$

for all $\sigma \in \Gamma(E)$.

Usually we denote $\mathcal{R}$ also by R, that is, we identify

$$\mathrm{Hom}(E, \bigwedge^2 T^*M \otimes E) \cong \bigwedge^2 T^*M \otimes \mathrm{Hom}(E, E)$$

Definition 144. $\Omega^2(\mathrm{End}(E)) := \Gamma(\bigwedge^2 T^*M \otimes \mathrm{Hom}(E, E))$.

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Last time we defined the curvature by setting $D^2 = \mathcal{R} \in \Gamma(\text{Hom}(E, \Lambda^2 T^*M \otimes E)) \cong \Omega^2(\text{End}(E))$. This curvature $\mathcal{R}$ corresponds to $R \in \Omega^2(\text{End}(E))$ by

$$\mathcal{R}(\sigma)(X,Y) = R(X,Y)\sigma$$

for all vector fields X, Y. With respect to a trivialization $e_1, \ldots, e_k$ of E, it's given by a matrix of 2-forms Θ_e, namely

$$D^2\left(\sum_i f_i e_i\right) = \sum_i f_i D^2(e_i)$$

where

$$\begin{aligned} D^2(e_i) &= D(\sum_k \theta_{ki} e_k) \\ &= \sum_k d\theta_{ki} e_k - \sum_{k,j} \theta_{ki} \wedge \theta_{jk} e_j \\ &= \sum_k d\theta_{ki} e_k + \sum_{k,j} \theta_{jk} \wedge \theta_{ki} e_j \end{aligned}$$

Therefore,

$$D^2(e_i) = \sum \Theta_{ki} e_k$$

where $\Theta_e = d\theta_e + \theta_e \wedge \theta_e$ is a matrix of 2-forms.

If $e'_j = \sum \psi_{ij} e_i$ is another frame, the curvature matrix changes as follows:

$$\begin{aligned} D^2 e'_j &= D^2\left(\sum_i \psi_{ij} e_i\right) \\ &= \sum_i \psi_{ij} D^2(e_i) \\ &= \sum_{i,k} \psi_{ij} \Theta_{ki} e_k \\ &= \sum_{i,k,\ell} \psi_{ij} \Theta_{ki} (\psi^{-1})_{\ell k} e'_\ell \end{aligned}$$

Therefore,

$$(\Theta_{e'})_{\ell j} = \sum_{i,k} (\psi^{-1})_{\ell k} \Theta_{ki} \psi_{ij}$$

but again this looks much neater when we write this as a matrix:

$$\Theta_{e'} = \psi^{-1} \Theta_e \psi.$$

Definition 145. A connection is called **flat** if it's curvature is zero.

For example, if $E = M \times \mathbb{R}^r$ is the trivial bundle with trivializing frame $e_1, \ldots, e_r$ such that $e_i(p) = (p, e_i)$, then we can define a flat connection on E by specifying that the e_i are parallel, that is, $D(e_i) = 0$ for all $i = 1, \ldots, r$.

Exercise 146 (Example Sheet 3, Question 7)**.** If a vector bundle E admits a flat connection, then there is a choice of local trivializations so that the transition functions are constant: $\psi_{\beta\alpha}(p) = h_{\beta\alpha}$ for all $p \in U_\alpha \cap U_\beta$. Moreover, if M is simply connected, then the vector bundle is isomorphic to a trivial bundle (trivialized by a parallel frame).

With respect to a local frame $e_1, \ldots, e_r$ for E, $R \in \Omega^2(\mathrm{End}(E))$ corresponds to a matrix Θ_e of 2-forms, and

$$\mathcal{R}(e_i) = \sum \Theta_{ki} e_k = \sum \Theta_i^k e_k,$$

where $\Theta_{ki} = \Theta_i^k$.

Therefore, $R = \sum \Theta_i^k \varepsilon_i \otimes e_k$, where $\varepsilon_1, \ldots, \varepsilon_r$ are the dual frame for E^*. Given a local coordinate system $x_1, \ldots, x_n$, we have that

$$R\left(\frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q}\right) e_i = \sum_k \Theta_i^k \left(\frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q}\right) e_k$$

Therefore,

$$R\left(\frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q}\right) e_i = \sum_k R^k_{\ ipq} e_k,$$

where the coefficients are given by $R^k_{\ ipq} = \Theta_i^k \left({}^{\partial}/_{\partial x_p}, {}^{\partial}/_{\partial x_q}\right)$.

So

$$\begin{aligned}\mathcal{R}(e_i) &= \sum_{\substack{k=1,\ldots,r \\ p<q}} R^k_{\ ipq} dx_p \wedge dx_q \otimes e_k \\ &= \sum_{\substack{k=1,\ldots,r \\ p<q}} R^k_{\ ipq} dx_p \otimes dx_q \otimes e_k\end{aligned}$$

where $R^k_{\ iqp} = -R^k_{\ ipq}$.

Exercise 147 (Example Sheet 3, Question 4)**.** If σ is a section of E, then

$$R(X, Y)\sigma = \mathcal{R}(\sigma)(X, Y) = D_X D_Y \sigma - D_Y D_X \sigma - D_{[X,Y]}\sigma$$

In essence, the curvature measures the failure of D_X and D_Y to commute.

From now on, denote the curvature map also by R rather than $\mathcal{R}$. This is a consequence of identifying $\mathrm{Hom}(E, \bigwedge^2 T^*M \otimes E)$ with $\bigwedge^2 T^*M \otimes \mathrm{Hom}(E, E)$.

Proposition 148 (General Bianchi Identity, coordinate version)**.** Having chosen a local trivialization $e_1, \ldots, e_r$ for E over U, recall that

$$D^2(e_i) = \mathcal{R}(e_i) = \sum_k \Theta_{ki} e_k,$$

with $\Theta_{ki} = \Theta_i^k$. This matrix is given by $\Theta_e = (\Theta_{ki})$, given by

$$\Theta_e = d\theta_e + \theta_e \wedge \theta_e,$$

where θ is the connection matrix. Then,

$$\begin{aligned} d\Theta &= d\theta \wedge \theta - \theta \wedge d\theta \\ &= d\theta \wedge \theta + \theta \wedge \theta \wedge \theta - \theta \wedge d\theta - \theta \wedge \theta \wedge \theta \\ &= \Theta \wedge \theta - \theta \wedge \Theta \end{aligned}$$

Consequently,

$$d\Theta_{ki} = \sum_j \left(\Theta_{kj} \wedge \theta_{ji} - \theta_{kj} \wedge \Theta_{ji}\right)$$

A coordinate free version of the Bianchi identity is on Example Sheet 3, Question 5.

Orthogonal Connections

Suppose we have an orthogonal structure on a vector bundle E over M of rank r in which all the transition functions lie in $O(r)$. In this case, the standard inner product on $\mathbb{R}^r$ yields a well-defined inner product $\langle\ ,\ \rangle_p$ on fibers E_p of E varying smoothly with p. More abstractly, this is a smooth section of $E^* \otimes E^*$ which induces the inner product on each fiber. This smooth section is symmetric and positive definite.

We call such a section of $E^* \otimes E^*$ a **smooth metric** on E, denoted by $\langle\ ,\ \rangle$.

Conversely, if we have a smooth metric on E, then we may reduce the structure group to $O(r)$. Locally, we can apply Gram-Schmidt orthonormalization to any given frame.

Lemma 149. Metrics always exist on any given vector bundle E.

Less of a proof and more of some words that vaguely justify why. Clearly, they exist locally, and then we can use a partition of unity to get a global metric. □

Definition 150. A connection D on E is **orthogonal** with respect to a given metric $\langle\ ,\ \rangle$ on E if

$$d\langle s_1, s_2\rangle = \langle Ds_1, s_2\rangle + \langle s_1, Ds_2\rangle$$

for all $s_1, s_2 \in \Gamma(E)$. And for any vector field X,

$$X\langle s_1, s_2\rangle = \langle D_X s_1, s_2\rangle + \langle s_1, D_X s_2\rangle.$$

Proposition 151. An orthogonal connection has a skew-symmetric connection matrix θ_e and skew-symmetric curvature matrix Θ_e with respect to any orthonormal frame.

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Recall that a connection D is **orthogonal** with respect to a metric $\langle\ ,\ \rangle$ on E if

$$d\langle s_1, s_2\rangle = \langle Ds_1, s_2\rangle + \langle s_1, Ds_2\rangle$$

for all $s_1, s_2 \in \Gamma(E)$.

For a vector field X, this means that

$$X\langle s_1, s_2\rangle = \langle D_X s_1, s_2\rangle + \langle s_1, D_X s_2\rangle.$$

Proposition 152. An orthogonal connection has a skew-symmmetric connection matrix θ_e and skew-symmetric Θ_e with respect to any orthonormal frame.

Proof. Suppose that $e_1, \ldots, e_n$ is a local orthonormal frame and

$$De_i = \sum_k \theta_{ki} e_k.$$

Then

$$\begin{aligned} 0 &= d\langle e_i, e_j\rangle \\ &= \left\langle \sum_k \theta_{ki} e_k, e_j \right\rangle + \left\langle {}_i, \sum_\ell \theta_{\ell j} e_\ell \right\rangle \\ &= \theta_{ji} + \theta_{ij}. \end{aligned}$$

Hence θ is skew-symmetric. Now given $\Theta_e = d\theta_e + \theta_e \wedge \theta_e$, we know that

$$\begin{aligned} \Theta_{ik} &= d\theta_{ij} + \sum_j \theta_{ij} \wedge \theta_{jk} \\ \Theta_{ki} &= d\theta_{ki} + \sum_j \theta_{kj} \wedge \theta_{ji} \\ &= -d\theta_{ik} - \sum_j \theta_{ij} \wedge \theta_{jk} = -\Theta_{ik} \end{aligned}$$

□

Connections on the Tangent Bundle

Koszul Connections

In this chapter, we now specialize to the case of connections ∇ on the tangent bundle, called **Koszul Connections**. For notational convenience, we set

$$\nabla_i = \nabla_{\partial/\partial x_i}$$

with respect to a local coordinate system $x_1, \ldots, x_n$. Therefore,

$$\nabla_i \left(\frac{\partial}{\partial x_j} \right) = \sum_{k=1}^n \Gamma^k{}_{ij} \frac{\partial}{\partial x_k},$$

and the Γ^k_{ij} are called the **Christoffel Symbols.**

The curvature $R \in \Omega^2(\mathrm{End}(TM))$ determines, for X, Y, Z vector fields, a vector field $R(X,Y)Z$ where

$$R(X,Y)Z = -R(X,Y)Z.$$

As a tensor, we can write R with respect to local coordinates $x_1, \dots, x_n$ as

$$R = \sum_{i,p,q,k} R^k_{\ ipq}\, dx_p \otimes dx_q \otimes dx_i \otimes \frac{\partial}{\partial x_k}$$

Note that $R^k_{\ ipq} = -R^k_{\ iqp}$.

Remark 153 (WARNING!)**.** You won't find consistency between any two books with how the coordinates of the curvature tensor are written. Sometimes what we write as $R^k_{\ ipq}$ is $R^k_{\ piq}$ in books or something even weirder.

This definition of R in local coordinates in particular means that

$$R\left(\frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q}\right)\left(\frac{\partial}{\partial x_i}\right) = \sum_k R^k_{\ ipq} \frac{\partial}{\partial x_k}$$

Definition 154. Let $\gamma\colon [a,b] \to M$ be a smooth curve. A **vector field** V **along** γ is a smooth function V on $[a,b]$ with $V_t = V(t) \in T_{\gamma(t)}M$. Locally we can write

$$V_t = \sum_i v_i(t) \frac{\partial}{\partial x_i}\bigg|_{\gamma(t)}$$

for smooth functions v_i on $[a,b]$.

Suppose now V is a smooth vector field in a neighborhood of $\gamma([a,b])$. Then

$$t \mapsto \nabla_{\dot{\gamma}} V$$

is a vector field along γ. This vector field is called the **covariant derivative** of V along γ, written $\frac{DV}{dt}$; this may however be generalized for any smooth vector field V along γ.

Proposition 155. There is a unique operation $V \mapsto \frac{DV}{dt}$ from smooth vector fields along γ to smooth vector fields along γ such that

(a) $\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$;

(b) $\frac{D(fV)}{dt} = \left({}^{df}/_{dt}\right) V + f\frac{DV}{dT}$ for $f\colon [a,b] \to \mathbb{R}$ smooth;

(c) If $V_s = Y_{\gamma(s)}$ for some smooth vector field Y defined on a neighborhood of $\gamma(t)$, then $\frac{DV}{dt}(s) = \nabla_{\dot{\gamma}(s)} Y$.

Proof. If $x_1, \ldots, x_n$ is a local coordinate system around $p = \gamma(t_0)$, then for t sufficiently close to t_0, we may write

$$V(t) = \sum_{j=1}^{n} v_j(t) \frac{\partial}{\partial x_j}\bigg|_{\gamma(t)}.$$

Then using (a),

$$\begin{aligned}
\frac{DV}{dt} &= \sum_{j=1}^{n} \frac{D}{dt}\left(v_j(t) \frac{\partial}{\partial x_j}\bigg|_{\gamma(t)} \right) && \text{by (a)} \\
&= \sum_{j=1}^{n} \left(\frac{dv_j}{dt} \frac{\partial}{\partial x_j}\bigg|_{\gamma(t)} + v_j(t) \frac{D}{dt} \frac{\partial}{\partial x_j}\bigg|_{\gamma(t)} \right) && \text{by (b)} \\
&= \sum_{j=1}^{n} \left(\frac{dv_j}{dt} \frac{\partial}{\partial x_j}\bigg|_{\gamma(t)} + v_j(t) \nabla_{\dot{\gamma}(t)} \frac{\partial}{\partial x_j} \right) && \text{by (c)}
\end{aligned}$$

Now as

$$\dot{\gamma}(t) = \sum_i \frac{d\gamma_i}{dt} \frac{\partial}{\partial x_i}\bigg|_{\gamma(t)}$$

where $\gamma_i(t) = x_i(\gamma(t))$, this is just

$$\begin{aligned}
\frac{DV}{dt} &= \sum_{j=1}^{n} \left(\frac{dv_j}{dt} \frac{\partial}{\partial x_j}\bigg|_{\gamma(t)} + v_j(t) \sum_{i=1}^{n} \frac{d\gamma_i}{dt} \nabla_{\partial/\partial x_i|_{\gamma(t)}} \frac{\partial}{\partial x_j} \right) \\
&= \sum_{k=1}^{n} \left(\frac{dv_k}{dt} + \sum_{i,j=1}^{n} \Gamma^k_{ij}(\gamma(t)) \frac{d\gamma_i}{dt} v_j(t) \right) \frac{\partial}{\partial x_k}\bigg|_{\gamma(t)}
\end{aligned}$$

So there is at most one such operation, and it's easy, if tedious, to check that the above formula has the required properties. □

Remark 156. This yields a value for $\frac{DV}{dt}$, even at points where $\dot{\gamma}(0) = 0$. For example, if γ is a constant curve, then a vector field along γ is just a curve in the corresponding tangent space T_pM. Moreover, in the case where γ is constant, then $\frac{DV}{dt}$ is the usual derivative of a vector-valued function.

Definition 157. A vector field V along γ is **parallel along** γ with respect to ∇ if $\frac{DV}{dt} = 0$ along γ.

This definition makes sense, because when $M = \mathbb{R}^n$ and ∇ is the directional derivative

$$\begin{aligned}
\nabla \left(\sum_i f_i e_i \right) &= \sum_i df_i \, e_i \\
\implies \nabla(f_1, \ldots, f_n) &= (df_1, \ldots, df_n) \\
\implies \nabla_X(f_1, \ldots, f_n) &= (X(f_1), \ldots, X(f_n))
\end{aligned}$$

we obtain the standard picture of a parallel vector field along γ, since the equations reduce down to $\frac{dv_k}{dt} = 0$ for all k.

Remark 158. In general, given a curve $\gamma\colon [a,b] \to M$ and a vector $V_a \in T_{\gamma(a)}M$, there is a unique vector field along γ which is parallel along γ. This is because the linear ODEs

$$\sum_{k=1}^{n} \left(\frac{dv_k}{dt} + \sum_{i,j=1}^{n} \Gamma^k{}_{ij}(\gamma(t)) \frac{d\gamma_i}{dt} v_j(t) \right) \frac{\partial}{\partial x_k}\bigg|_{\gamma(t)} = 0 \tag{6}$$

have unique solutions v_k defined on $[a,b]$ with initial data $V(\gamma(a)) = V_a$, and the required vector field is then

$$V = \sum_{j=1}^{n} v_j(t) \frac{\partial}{\partial x_j}\bigg|_{\gamma(t)}.$$

Definition 159. We say that the vector $V_t \in T_{\gamma(t)}M$ is said to be obtained from V_a by **parallel transport** or **parallel translation** along γ.

Clearly from the equations (6), the map $\tau_t\colon T_{\gamma(a)}M \to T_{\gamma(t)}M$ is a line map; it has inverse given by parallel transport along the reversed curve and so is an isomorphisms of vector spaces.

This gives us a way to connect tangent spaces at different points. Parallel translation is determined in terms of ∇, but we can reverse the process as well. This will let us define parallel connections on *any* tensor bundle, not just the tangent bundle.

Lecture 20 — 21 November 2015

Recall that given a connection ∇ on TM and a curve $\gamma\colon [a,b] \to M$, we have a parallel translation map $\tau_t\colon T_{\gamma(a)}M \to T_{\gamma(t)}M$.

Proposition 160. Let $\gamma\colon [0,1] \to M$ be a curve with $\gamma(0) = p$ and $\dot\gamma(0) = X_p$. Then for any vector field Y defined locally at p,

$$\nabla_{X_p} Y = \lim_{h\to 0} \frac{1}{h} \left(\tau_h^{-1} Y_{\gamma(h)} - Y_p \right)$$

Proof. Let $V_1, \ldots, V_n$ be parallel vector fields along γ which are independent at $\gamma(0)$, and hence at all points $\gamma(t)$. Set

$$Y(\gamma(t)) = \sum_{i=1}^{n} \alpha_i(t) V_i(t).$$

Therefore,

$$\begin{aligned}
\lim_{h\to 0}\frac{1}{h}\left(\tau_h^{-1}Y_{\gamma(h)} - Y_p\right) &= \lim_{h\to 0}\left(\sum_{i=1}^{n}\alpha_i(h)\tau_h^{-1}V_i(h) - \alpha_i(0)V_i(0)\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\sum_{i=1}^{n}\left(\alpha_i(h)-\alpha_i(0)\right)V_i(0)\right)\\
&= \sum_{i=1}^{n}\left.\frac{d\alpha_i}{dt}\right|_0 V_i(0)\\
&= \left.\frac{D}{dt}\right|_{t=0}\sum_{i=1}^{n}\alpha_i(t)V_i(t)\\
&= \nabla_{X_p}Y \qquad \text{by property (c)}
\end{aligned}$$

□

Remark 161. Let $T^k_\ell(M)$ denote the tensor bundle

$$T^k_\ell(M) := \underbrace{TM\otimes\cdots\otimes TM}_{k}\otimes\underbrace{T^*M\otimes\cdots\otimes T^*M}_{\ell}.$$

Parallel translation $\tau_t\colon T_{\gamma(0)}M \xrightarrow{\sim} T_{\gamma(t)}M$ induces isomorphisms that we call $T^k_\ell\tau_t\colon (T^k_\ell)_{\gamma(0)} \to (T^k_\ell)_{\gamma(t)}$. For any tensor $A\in\Gamma(T^k_\ell(M))$, we can define

$$\nabla_{X_p}A = \lim_{h\to 0}\frac{1}{h}\left(T^k_\ell(\tau_h^{-1})A(\gamma(h)) - A(p)\right)$$

where $X_p = \dot\gamma(0)$. Note that $T^k_\ell(\tau_h^{-1}) = T^k_\ell(\tau_h)^{-1}$.

We need to check this is a connection on the tensor bundle $T^k_\ell M$. Most conditions here are clear, for example

$$\begin{aligned}
\nabla_{X_p}(fA) &= \lim_{h\to 0}\frac{1}{h}\left[f(\gamma(h))\left(T^k_\ell\tau_h^{-1}A(\gamma(h)) - A(p)\right) + \left(f(\gamma(h)) - f(\gamma(0))\right)A(p)\right]\\
&= f(p)\nabla_{X_p}A + X_p(f)A(p)
\end{aligned}$$

where $(f\circ\gamma)'(0) = d_{\gamma(0)}f(\dot\gamma(0)) = X_p(f)$.

But it's less clear in general that the definition is independent of the choice of γ with $\dot\gamma(0) = X_p$, and moreover that

$$\nabla_{fX+gY}A = f\nabla_X A + g\nabla_Y A,$$

but in the cases we're interested in, this will follow by the formula derived below, and in general by an inductive extension of this argument (see example sheet 3, question 9).

Example 162. How does this connection act on various tensors?

(1) For $A\in C^\infty(U)$,
$$\nabla_{X_p}A = \frac{d(A\circ\gamma)}{dt}(0) = X_pA.$$

(2) Suppose $A \in \text{End}(TM)$. For a given local vector field Y, we have

$$\nabla_{X(p)}(A(Y)) = (\nabla_{X_p} A)Y + A(\nabla_{X_p} Y).$$

Example sheet 2, question 2, is the case of an arbitrary vector bundle E.

Proof. Given γ with $\dot{\gamma}(0) = X_p$, we can write down linearly independent parallel vector fields $V_1, \ldots, V_n$ along γ, and linearly independent dual 1-forms $\phi_1, \ldots, \phi_n$ along γ. Note that ϕ_i and $\phi_i \otimes V_j$ are parallel for all i, j, since they are just given by parallel translation (c.f. example sheet 3, question 8). Set

$$A(\gamma(t)) = \sum_{i=1}^{n} A_{ij}(t)\phi_i \otimes V_j.$$

So if we have a vector field

$$Y(\gamma(t)) = \sum_k Y_k(t) V_k,$$

say, then we have

$$\begin{aligned}
\nabla_{X_p}(AY) &= \sum_{i,j} \frac{d}{dt}\left(A_{ij}(t)Y_i(t)\right)\Big|_0 V_j(0) \\
&= \sum_{i,j}\left(\frac{dA_{ij}}{dt}\Big|_0 Y_i(0) + A_{ij}(0)\frac{dY_i}{dt}\Big|_0\right) V_j(0) \\
&= (\nabla_{X_p}A)Y + A(\nabla_{X_p}Y)
\end{aligned}$$

□

(3) Suppose $A \in \Gamma(T^*M \otimes T^*M)$. This is the sort of thing we'll have when we have a metric. Then

$$(\nabla_{X_p}A)(Y,Z) = X_p(A(Y,Z)) - A(\nabla_{X_p}(Y), Z) - A(Y, \nabla_{X_p} Z).$$

The proof is exactly the same as before – we write it down in terms of parallel bases and then compute.

Proof. With notation as above, we write

$$\begin{aligned}
A(\gamma(t)) &= \sum_{i,j} A_{ij}(t)\phi_i(t) \otimes \phi_j(t) \\
Y(\gamma(t)) &= \sum_j Y_j(t) V_j(t) \\
Z(\gamma(t)) &= \sum_k Z_k(t) V_k(t)
\end{aligned}$$

Therefore,

$$\begin{aligned}
X_p(A(Y,Z)) &= \nabla_{X_p}(A(Y,Z)) \\
&= \sum_{i,j} \frac{d}{dt}\left(A_{ij}Y_iZ_j\right)\Big|_{t=0} \\
&= (\nabla_{X_p}A)(Y,Z) + A(\nabla_{X_p}Y, Z) + A(Y, \nabla_{X_p}Z)
\end{aligned}$$

□

(4) This generalizes to $A \in \Omega^2(\text{End}(TM)) \subseteq \Gamma(T^*M \otimes T^*M \otimes \text{End}(TM))$. Setting

$$A(t) = \sum_{i,j} A_{ij}(t)\phi_i(t) \otimes V_j(t)$$

with $A_{ij}(t)$ now 2-forms, a similar argument implies

$$\nabla_{X_p}(A(Y,Z)) = (\nabla_{X_p}A)(Y,Z) + A(\nabla_{X_p}Y,Z) + A(Y,\nabla_{X_p}Z) \qquad (7)$$

as sections of $\text{End}(TM)$. We're particularly interested in this case because of curvature. In particular, given a connection ∇ on the tangent bundle, the formula (7) defines a connection ∇ on $\Omega^2(\text{End}(TM))$ (the fact that it is a connection is an easy exercise).

For the curvature $R \in \Omega^2(\text{End}(TM))$, this gives the formula for $(\nabla_X R)(Y,Z)$, which is needed in the proof of the second Bianchi identity.

Torsion Free Connections

Definition 163. Given a Koszul connection ∇ on TM, define for vector fields X, Y a new vector field

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

called the**torsion** T of the connection ∇.

Remark 164. It's easy to check that T is bilinear over smooth functions:

$$T(fX,Y) = fT(X,Y) = T(X,fY).$$

And so $T(X,Y)_p$ depends only on X_p, Y_p and hence it defines a tensor in $\Gamma(T^*M \otimes T^*M \otimes TM)$.

If ∇ has Christoffel symbols Γ^k_{ij} with respect to a given local coordinate system, then

$$T\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \sum_k \left(\Gamma^k_{ij} - \Gamma^k_{ji}\right)\frac{\partial}{\partial x_k}.$$

So T has components $T^k_{ij} = (\Gamma^k_{ij} - \Gamma^k_{ji})$.

Lecture 21 — 24 November 2015

Recall that last time we defined the Torsion tensor T by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

with respect to a given coordinate system, T has coordinates

$$T^k_{ij} = \left(\Gamma^k_{ij} - \Gamma^k_{ji}\right)$$

Definition 165. A connection is called **symmetric** or **torsion free** if $T = 0$.

Proposition 166. For $p \in M$, the torsion tensor T of a connection ∇ vanishes at p if and only if there is a coordinate system around p with $\Gamma^k{}_{ij}(p) = 0$ for all i, j, k.

Proof. ($\Longleftarrow$). Clearly we have $T(p) = 0$ independent of the coordinate system.

($\Longrightarrow$). Suppose we have local coordinates $x_1, \dots, x_n$ and that

$$\Gamma^k{}_{ij}(p) = \Gamma^k{}_{ji}(p)$$

for all i, j, k.

Define a new coordinate system $y_1, \dots, y_n$ by

$$y_k = (x_k - x_k(p)) + \frac{1}{2} \sum_{i,j=1}^{n} \Gamma^k{}_{ij}(p)(x_i - x_i(p))(x_j - x_j(p)).$$

Using the symmetry of ∇, we compute

$$\frac{\partial y_k}{\partial x_\ell} = \delta_{k\ell} + \sum_{i=1}^{n} \Gamma^k{}_{i\ell}(p)(x_i - x_i(p))$$

with ${}^{\partial y_K}/_{\partial x_\ell}(p) = \delta_{k\ell}$.

This shows that in a neighborhood of p, $y_1, \dots, y_n$ is also a coordinate system around p and moreover that

$$\frac{\partial^2 y_k}{\partial x_i \partial x_\ell}(p) = \Gamma^k{}_{i\ell}(p) \tag{8}$$

What are the Christoffel symbols with respect to the new coordinate system? Call them Γ'.

$$\begin{aligned}
\sum_k (\Gamma')^k{}_{ij} \frac{\partial}{\partial y_k} &= \nabla_{\partial/\partial y_i}\left(\frac{\partial}{\partial y_j}\right) \\
&= \nabla_{\partial/\partial y_i}\left(\sum_\ell \frac{\partial x_\ell}{\partial y_j}\frac{\partial}{\partial x_\ell}\right) \\
&= \sum_\ell \frac{\partial x_\ell}{\partial y_i \partial y_j}\frac{\partial}{\partial x_\ell} + \sum_{\ell,r} \frac{\partial x_\ell}{\partial y_j}\frac{\partial x_r}{\partial y_i} \nabla_{\partial/\partial x_r}\left(\frac{\partial}{\partial x_\ell}\right) \\
&= \sum_\ell \frac{\partial x_\ell}{\partial y_i \partial y_j}\frac{\partial}{\partial x_\ell} + \sum_{\ell,r} \frac{\partial x_\ell}{\partial y_j}\frac{\partial x_r}{\partial y_i} \sum_s \Gamma^s{}_{r\ell}\frac{\partial}{\partial x_s}
\end{aligned}$$

Now evaluate this whole thing on y_k to get

$$(\Gamma')^k{}_{ij}(p) = \frac{\partial^2 x_k}{\partial y_i \partial y_j}(p) + \Gamma^k{}_{ij}(p) \tag{9}$$

using ${}^{\partial y_k}/_{\partial x_\ell}(p) = \delta_{k\ell}$.

Now in a neighborhood of p,

$$\sum_\ell \frac{\partial y_k}{\partial x_\ell}\frac{\partial x_\ell}{\partial y_j} = \delta_{kj}$$

Operate by ${}^{\partial}/_{\partial x_i}$ to get (the right term is derived from throwing in an extra chain rule)

$$\sum_\ell \frac{\partial^2 y_k}{\partial x_i \partial x_\ell} \frac{\partial x_\ell}{\partial y_j} + \sum_{\ell,r} \frac{\partial y_k}{\partial x_\ell} \frac{\partial y_r}{\partial x_i} \frac{\partial^2 x_\ell}{\partial y_r \partial y_j}$$

This then implies, using (8), that

$$\Gamma^k_{ij}(p) + \frac{\partial^2 x_k}{\partial y_i \partial y_j}(p) = 0$$

Hence, we deduce from (9) that

$$\left(\Gamma'\right)^k_{ij}(p) = 0,$$

which is what we wanted to show. □

Proposition 167 (**Bianchi's Identities** for Torsion-free Koszul Connections)**.**

(i) **1st Bianchi Identity** $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

(ii) **2nd Bianchi Identity** $(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$

In coordinates, this can be written as

$$R^h{}_{ijk;\ell} + R^h{}_{ik\ell;j} + R^h{}_{i\ell j;k}$$

for all $i, j, k\ell$ where

$$\left(\nabla_{\partial/\partial x_\ell} R\right)\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)\left(\frac{\partial}{\partial x_i}\right) = \sum_h R^h{}_{ijk;\ell} \frac{\partial}{\partial x_h}$$

Proof.

(i) Use Example sheet 3 question 4:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

It suffices to verify the identity for coordinate vector fields ${}^{\partial}/_{\partial x_i}$, and so we may assume that the Lie brackets vanish. Then it's clear that the cyclic sum vanishes using the symmetry of the connection.

So $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z$. Now take the cyclic sum and use symmetry $\nabla_Y Z = \nabla_Z Y$; everything cancels.

(ii) Again, since everything in sight is a tensor (and therefore linear with respect to multiplication by smooth functions in all variables), we only need check this pointwise in local coordinates. Suppose given p, we can choose coordinates $x_1, \ldots, x_n$ so that the Christoffel symbols vanish at p (using the symmetry of ∇).

Thus using the formula for the covariant derivative of the curvature from last time,

$$\begin{aligned}\left(\nabla_{\partial/\partial x_i} R\right)\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)_p &= \nabla_{\partial/\partial x_i}\left(R\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)\right)_p \\ &\quad - R\left(\nabla_{\partial/\partial x_i}\left(\frac{\partial}{\partial x_j}\right), \frac{\partial}{\partial x_k}\right)_p \\ &\quad - R\left(\frac{\partial}{\partial x_j}, \nabla_{\partial/\partial x_i}\left(\frac{\partial}{\partial x_k}\right)\right)_p\end{aligned}$$

But $\nabla_{\partial/\partial x_a}\left({}^{\partial}/_{\partial x_b}\right) = 0$ by our choice of coordinates. Therefore,

$$\begin{aligned}\left(\nabla_{\partial/\partial x_i} R\right)\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)_p &= \nabla_{\partial/\partial x_i}\left(R\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)\right)_p \left(\frac{\partial}{\partial x_\ell}\right) \\ &= \nabla_{\partial/\partial x_i}\left(\sum_h R^h{}_{\ell jk}\frac{\partial}{\partial x_h}\right)_p - R\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)\left(\nabla_{\partial/\partial x_i}\left(\frac{\partial}{\partial x_\ell}\right)\right)_p\end{aligned}$$

Again, the second term vanishes because $\nabla_{\partial/\partial x_a}\left({}^{\partial}/_{\partial x_b}\right) = 0$ by our choice of coordinates, so we get

$$\left(\nabla_{\partial/\partial x_i} R\right)\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)_p = \sum_h \frac{\partial}{\partial x_i}\left(R^h{}_{\ell jk}\frac{\partial}{\partial x_h}\bigg|_p\right)$$

Thus, with respect to the given coordinates $x_1, \ldots, x_n$, it remains to prove that

$$\frac{\partial}{\partial x_i}R^m{}_{\ell jk} + \frac{\partial}{\partial x_j}R^m{}_{\ell ki} + \frac{\partial}{\partial x_k}R^m{}_{\ell ij} = 0$$

To that end, given the connection matrix θ_e is assumes zero at p, the general Bianchi identity we proved is

$$d\Theta = \Theta \wedge \theta - \theta \wedge \Theta$$

with $d\Theta^m_\ell = 0$ at p for all m, ℓ. Now,

$$\begin{aligned}\Theta^m_\ell &= \sum_{j<k} R^m{}_{\ell jk}\, dx_j \wedge dx_k \\ &= \frac{1}{2}\sum_{i,j} R^m{}_{\ell jk}\, dx_j \wedge dx_k\end{aligned}$$

Therefore,

$$d\Theta^m_\ell = \frac{1}{2}\sum_{i,j,k}\frac{\partial}{\partial x_i}R^m{}_{\ell jk}\, dx_i \wedge dx_j \wedge dx_k = 0$$

at p for all m, ℓ. This implies the statement required because this is valid for all p. □

Remark 168. There is a coordinate-free approach to these identities on Examples Sheet 3, Question 5. A connection ∇ on TM induces covariant exterior derivative $d^{\text{End}}\colon \Omega^2(\operatorname{End} TM) \to \Omega^3(\operatorname{End} TM)$. The curvature tensor R of ∇ lies in $\Omega^2(\operatorname{End} TM)$. The coordinate-free form of the second Bianchi identity says that

$$d^{\text{End}}(R) = 0$$

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Riemannian Manifolds

Definition 169. A **Riemannian manifold** is a smooth manifold M equipped with a **Riemannian metric**, that is, a metric $g = \langle\ ,\ \rangle$ on TM. Note that g is therefore a symmetric tensor in $\Gamma(T^*M \otimes T^*M)$. Sometimes we say "a metric on M" meaning "a metric on TM".

Remark 170. Riemannian metric always exist on any smooth M; we can write such a metric in local coordinates $x_1, \dots, x_n$ on $U \subseteq M$ as

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j$$

where for each $p \in U$, $(g_{ij}(p))$ is a positive definite symmetric matrix.

As with vector spaces, giving a metric on TM is equivalent to giving a (non-canonical) isomorphism of a vector bundle $TM \to T^*M$.

Remark 171. Given a Koszul connection ∇, we have an induced connection ∇ on $T^*M \otimes T^*M$; moreover for $X_p \in T_pM$,

$$(\nabla_{X_p} g)(Y, Z) = X_p(g(Y, Z)) - g(\nabla_{X_p} Y, Z) - g(Y, \nabla_{X_p} Z).$$

Thus the metric g is **covariantly constant** with respect to ∇, meaning that $\nabla g = 0$ if and only if for all ∇ is an orthogonal connection with respect to the metric (meaning that $dg(Y, Z) = (\nabla Y, Z) + g(Y, \nabla Z)$).

Definition 172. In this case, where $\nabla g = 0$, we say that ∇ is a **metric connection** on M.

Proposition 173. ∇ is a metric connection if and only if parallel translation τ_t along any curve $\gamma\colon [a, b] \to M$ is an isometry with respect to $\langle\ ,\ \rangle_{\gamma(a)}$ and $\langle\ ,\ \rangle_{\gamma(t)}$.

Proof. ($\Longrightarrow$). Suppose V is a parallel vector field along γ (recall parallel means $\frac{DV}{dt} = 0$). Write V locally as

$$\sum_i V_i(t) \frac{\partial}{\partial x_i}$$

and so

$$\frac{DV}{dt} = \sum_i \left(\frac{dV_i}{dt} \frac{\partial}{\partial x_i} + V_i \frac{D}{dt} \frac{\partial}{\partial x_i} \right).$$

Now

$$
\begin{aligned}
\frac{d}{dt}\langle V, V\rangle &= \frac{d}{dt}\sum_{i,j} V_i V_j \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle \\
&= 2\sum_{i,j} \frac{dV_i}{dt} V_j \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle + \sum_{i,h} V_i V_j \frac{d}{dt} \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle
\end{aligned}
$$

where, since ∇ is a metric connection,

$$
\begin{aligned}
\frac{d}{dt}\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle &= \frac{d}{dt}\left(\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle \circ \gamma\right) \\
&= \dot{\gamma}(t)\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle \\
&= \left\langle \nabla_{\dot{\gamma}(t)}\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle + \left\langle \frac{\partial}{\partial x_i}, \nabla_{\dot{\gamma}(t)}\frac{\partial}{\partial x_j} \right\rangle
\end{aligned}
$$

Substituting this in the above, we see that

$$
\begin{aligned}
\frac{d}{dt}\langle V, V\rangle &= 2\sum_{i,j} \frac{dV_i}{dt} V_j \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle + \sum_{i,h} V_i V_j \frac{d}{dt} \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle \\
&= 2\sum_{i,j} \left\langle \frac{dV_i}{dt}\frac{\partial}{\partial x_i} + V_i \frac{D}{dt}\frac{\partial}{\partial x_i},\ V_j \frac{\partial}{\partial x_j} \right\rangle \\
&= 2\left\langle \frac{DV}{dt}, V \right\rangle = 0
\end{aligned}
$$

($\Longleftarrow$). For given $p \in M$ and $X_p \in T_pM$, chose a curve γ with $\gamma(0) = p$, $\dot{\gamma}(0) = X_p$. Our assumption implies that we can choose parallel vector fields $v_1, \dots, v_n$ along γ which form an orthonormal basis for $T_{\gamma(t)}M$ for all t.

For given vector fields Y, Z in a neighborhood of p, write

$$
\begin{aligned}
Y(\gamma(t)) &= \sum Y_i(t) V_i(t), \\
Z(\gamma(t)) &= \sum Z_j(t) V_j(t).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
X_p\langle Y, Z\rangle &= \frac{d}{dt}\langle Y, Z\rangle \circ \gamma\Big|_0 \\
&= \frac{d}{dt}\sum_i Y_i(t) Z_i(t)\Big|_0 \\
&= \sum_i \left(\frac{dY_i}{dt}(0) Z_i(0) + Y_i(0)\frac{dZ_i}{dt}(0)\right) \\
&= \langle \nabla_{X_p} Y, Z\rangle_p + \langle Y, \nabla_{X_p} Z\rangle_p. \qquad \square
\end{aligned}
$$

Remark 174. Given a connection ∇ and a metric $\langle \ , \ \rangle$, we can form a $(0,4)$ tensor $R \in \Gamma(T^*M \otimes T^*M \otimes T^*M \otimes T^*M)$ where

$$R(W,Z,X,Y) = \langle R(X,Y)Z, W\rangle$$

In coordinates,

$$R = R_{ijpq} dx_i \otimes dx_j \otimes dx_p \otimes dx_q$$

where

$$R_{ijpq} = \left\langle R\left({}^{\partial}/_{\partial x_p}, {}^{\partial}/_{\partial x_q}\right) {}^{\partial}/_{\partial x_j}, {}^{\partial}/_{\partial x_i}\right\rangle = \textstyle\sum_k g_{ki} R^k{}_{jpq},$$

$R^k{}_{jpq}$ in our previous notation.

Symmetries of R

Proposition 175. If ∇ is both a metric and symmetric connection, then we have

(a) We always have $R(W,Z,Y,X) = -R(W,Z,X,Y) \implies R_{k\ell ji} = -R_{k\ell ij}$.

(b) For a metric connection, we have $R(Z,W,X,Y) = -R(W,Z,X,Y) \implies R_{k\ell ij} = -R_{\ell kij}$. Without loss of generality we may take a local orthonormal frame $v_1, \dots, v_n$, and then use that the matrix $\Theta^k_\ell(X,Y)$ is skew-symmetric.

(c) For a symmetric connection, we have the first Bianchi identity

$$R(W,Z,X,Y) + R(W,X,Y,Z) + R(W,Y,Z,X) = 0;$$

in coordinates, $R_{k\ell ij} + R_{kij\ell} + R_{kj\ell i} = 0$.

(d) $R(W,Z,X,Y) = R(X,Y,W,Z) \implies R_{\ell kij} = R_{ij\ell k}$.

Proof of (d).

$$\langle R(X,Y)Z, W\rangle = \langle R(W,Z)Y, X\rangle$$

Then by (1), the left hand side is

$$\begin{aligned} \text{LHS} &= -\langle R(Y,X)Z, W && \text{by (a)} \\ &= \langle R(X,Z)Y, W\rangle + \langle R(Z,Y)X, W\rangle && \text{by (c)} \end{aligned} \tag{10}$$

Also,

$$\begin{aligned} \text{LHS} &= -\langle R(X,Y)W, Z\rangle && \text{by (b)} \\ &= \langle R(Y,W)X, Z\rangle + \langle R(W,X)Y, Z\rangle && \text{by (c)} \end{aligned} \tag{11}$$

Now add together (10) and (11) to see that

$$2\,\text{LHS} = \langle R(X,Z)Y, W\rangle + \langle R(Z,Y)X, W\rangle + \langle R(Y,W)X, Z\rangle + \langle R(W,X)Y, Z\rangle$$

and similarly with $X \leftrightarrow W$ and $Y \leftrightarrow Z$. Likewise,

$$2\,\text{RHS} = \langle R(W,Y)Z, X\rangle + \langle R(Y,Z)W, X\rangle + \langle R(Z,X)W, Y\rangle + \langle R(X,W)Z, Y\rangle$$

Now properties (a), (b) and uniqueness imply that these are equal! $\square$

Levi-Civita Connection

Lemma 176 (Fundamental Lemma of Riemannian Geometry)**.** On a Riemannian manifold (M, g) with $g = \langle \, , \, \rangle$, there exists a unique symmetric connection compatible with the metric defined by

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle + \langle [X, Y], Z\rangle \tag{12}$$

for all vector fields X, Y, Z.

Proof. **Uniqueness:** given a symemtric metric connection, we show that it satisfies (12).

Compatability with metric implies

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Symmetric implies

$$\langle \nabla_X Y, Z\rangle - \langle \nabla_Y X, Z\rangle = \langle [X, Y], Z\rangle.$$

Therefore,

$$\begin{aligned} X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \langle \nabla_Y Z, X\rangle \\ &\quad + \langle Z, \nabla_Y X\rangle - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle \\ &= (2\langle \nabla_X Y, Z\rangle - \langle [X, Y], Z\rangle) + \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle \end{aligned}$$

And this implies equation (12). Hence we have uniqueness.

Existence: If we define $\nabla_X Y$ by (12), we then need to show what we've defined is a connection. So it remains to prove

(a) $\nabla_{fX} Y = f \nabla_X Y$;

(b) $\nabla_X (fY) = X(f)Y + f\nabla_X Y$.

So we can check these individually.

(a) From the formula, we see that

$$\begin{aligned} 2\langle \nabla_{fX} Y, Z\rangle &= 2f\langle \nabla_X Y, Z\rangle + Y(f)\langle Z, X\rangle - Z(f)\langle X, Y\rangle + Z(f)\langle X, Y\rangle - Y(f)\langle X, Z\rangle \\ &= 2f\langle \nabla_X Y, Z\rangle \end{aligned}$$

This holds for any Z, so we have established (a).

(b) From the formula, we see that

$$\begin{aligned} 2\langle \nabla_X (fY), Z\rangle &= 2f\langle \nabla_X Y, Z\rangle + X(f)\langle Y, Z\rangle - Z(f)\langle X, Y\rangle + Z(f)\langle Y, X\rangle + X(f)\langle Y, Z\rangle \\ &= 2\langle X(f)Y + f\nabla_X Y, Z\rangle \end{aligned}$$

This holds for any Z, so we have established (b).

The fact that ∇ is symmetric comes straight from (12) by inspection.

The fact that ∇ is a metric connection comes by using (12) to write down formulae for $\langle \nabla_X Y, Z\rangle$ and $\langle \nabla_X Z, Y\rangle = \langle Y, \nabla_X Z\rangle$ and adding to get $\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = X\langle Y, Z\rangle$. □

Definition 177. This is called the **Levi-Civita Connection**.

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Remark 178. Classically, the Levi-Civita connection ∇ is given in terms of its Christoffel symbols – if we have coordinates $x_1, \ldots, x_n$ on $U \subseteq M$, then

$$2\left\langle \nabla_i {}^{\partial}\!/_{\partial x_j}, {}^{\partial}\!/_{\partial x_k} \right\rangle = 2\sum_{\ell} \Gamma^{\ell}_{ij} g_{\ell k}$$

But if you look at the formula (12), this is also

$$2\left\langle \nabla_i {}^{\partial}\!/_{\partial x_j}, {}^{\partial}\!/_{\partial x_k} \right\rangle = {}^{\partial g_{jk}}\!/_{\partial x_i} + {}^{\partial g_{ki}}\!/_{\partial x_j} - {}^{\partial g_{ij}}\!/_{\partial x_k}$$

This implies a formula for the Christoffel symbols of the Levi-Civita connection.

$$\Gamma^{\ell}_{ij} = \frac{1}{2}\sum_k g^{\ell k}\left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right)$$

where $g^{\ell k} := \left(g^{-1}\right)_{\ell k}$; g^{-1} is the inverse matrix to $g = (g_{ij})$.

Definition 179. The curvature of the Levi-Civita connection is a tensor of type $(1,3)$ with components, $R^k{}_{ipq}$ as before; taking

$$\left\langle R\left({}^{\partial}\!/_{\partial x_p}, {}^{\partial}\!/_{\partial x_q}\right) {}^{\partial}\!/_{\partial x_i}, {}^{\partial}\!/_{\partial x_j} \right\rangle = R_{jipq} = \sum_k g_{jk} R^k{}_{ipq}$$

we obtain a tensor of type $(0,4)$ with all indices down; this is called the **Riemannian Curvature Tensor**.

In invariant notation, the Riemannian Curvature Tensor is a $(0,4)$ tensor R such that

$$R(X,Y,Z,W) = \langle R(Z,W)Y, X\rangle = \langle R(X,Y)W, Z\rangle$$

Definition 180. Given orthonormal tangent vectors η_1, η_2 at $p \in M$, we define the **sectional curvature** of the 2-plane W spanned by η_1, η_2 to be

$$K(W) = R(\eta_1, \eta_2, \eta_1, \eta_2) = \langle R(\eta_1, \eta_2)\eta_2, \eta_1\rangle.$$

If η_1, η_2 are not orthonormal, then we define the sectional curvature

$$K(W) = \frac{\langle R(\eta_1, \eta_2)\eta_2, \eta_1\rangle}{\langle \eta_1, \eta_1\rangle\langle \eta_2, \eta_2\rangle - \langle \eta_2, \eta_2\rangle^2}.$$

Remark 181. It's easy to check that this just depends on the 2-plane spanned by η_1 and η_2, not the actual vectors themselves. This just uses the antisymmetries of the curvature tensor and symmetries of the metric.

It turns out that you can recover the information about the curvature from just the sectional curvature!

Lemma 182. If V is an $\mathbb{R}$-vector space and $R_1, R_2 \colon V \times V \times V \times V \to \mathbb{R}$ are quadrilinear maps satisfying symmetries (a), (b), (c), (d) of Proposition 175 and such that

$$R_1(X,Y,X,Y) = R_2(X,Y,X,Y)$$

for all $X, Y \in V$, then $R_1 = R_2$.

Proof. Reduce to the case that $R_1 = R$ and $R_2 = 0$ by taking their difference. Then it remains to prove that $R(X,Y,X,Y) = 0$ for all X, Y, which will show that $R = 0$.

To that end, we calculate

$$\begin{aligned} 0 &= R(X, Y+W, X, Y+W) \\ &= R(X,Y,X,W) + R(X,W,X,Y) \\ &= 2R(X,Y,X,W) && \text{by Proposition 175(d)} \end{aligned}$$

So R is skew-symmetric in the first and third entries, and similarly in the second and fourth entries. This is in addition to all the other symmetries of Proposition 175. From the 1st Bianchi identity, we see that

$$R(X,Y,Z,W) + R(X,Z,W,Y) + R(X,W,Y,Z) = 0.$$

But then our antisymmetries imply that

$$3R(X,Y,Z,W) = 0$$

for all X, Y, Z, W. □

This lemma immediately implies the following corollary.

Corollary 183. Sectional Curvatures determine the full curvature tensor.

Definition 184. When $\dim M = 2$, the sectional curvature is usually called the **Gaussian curvature** (c.f. Part II Diff Geom, or Example Sheet 3, Question 6).

Corollary 185. Suppose that a metric $\langle\ ,\ \rangle$ on M has the property that at any point p, the sectional curvatures at p are all constant with value $K = K(p)$. Then

$$R(X_p, Y_p, Z_p, W_p) = K \cdot \big(\langle X_p, Z_p\rangle\langle Y_p, W_p\rangle - \langle X_p, W_p\rangle\langle Y_p, Z_p\rangle\big) \tag{13}$$

Proof. Essentially we've seen a proof of this already. Let $R_0(X,Y,Z,W)$ be the right hand side of (13). Then if R is the Riemannian curvature, $R = R_0$ by the previous lemma Lemma 182, so $R = R_0$ at p. □

Definition 186. Set $r(X, Y)$ to be the trace of the endomorphism of TM given by $V \mapsto R(V, X)Y$. This is called the **Ricci tensor**, and is sometimes denoted $\mathrm{Ric}(g)$ where g is the metric.

If we take *any* orthonormal basis $e_1, \ldots, e_n$ for T_pM, then

$$\begin{aligned} r(X_p, Y_p) &= \mathrm{tr}(V_p \mapsto R(V_p, X_p)Y_p) \\ &= \sum_i R(e_i, Y_p, e_i, X_p) \\ &= r(Y_p, X_p) \qquad \text{by Proposition 175(d)} \end{aligned}$$

sp r is a symmetric covariant covariant tensor of rank 2. There's another symmetric covariant tensor of rank 2 floating around, namely the metric. This motivates the next definition.

Definition 187. A metric g on M is called **Einstein** if $r = \lambda g$ for some constant λ.

Definition 188. For any $0 \neq v \in T_pM$, the **Ricci curvature** in direction v is defined by

$$r(v) := \frac{r(v, v)}{\langle v, v\rangle}.$$

If we normalize so that v has length 1 (i.e. $\langle v, v\rangle = 1$), we may extend v to an orthornomal basis $v = e_1, e_2, \ldots, e_n$ of T_pM, and then

$$r(v) = \sum_{i=2}^{n} R(e_i, v, e_i, v) = \sum_{i=1}^{n} R(e_i, e_1, e_i, e_1),$$

and ${}^{r(v)}/_{n-1}$ is the average of the sectional curvatures of the planes generated by v and e_i for $i > 1$.

Lemma 189. The Ricci curvatures at p are constant with value λ if and only if the metric is Einstein ($r = \lambda g$) at p.

Proof. ($\Longleftarrow$). Clear.

($\Longrightarrow$). If $r(v) = \lambda$ for all $v \neq 0$, then we know that $r(v, v) = \lambda\langle v, v\rangle$ for all $v \in T_pM$. Therefore,

$$r(v, w) = \lambda\langle v, w\rangle$$

for any $v, w \in T_pM$. Hence $r = \lambda g$ at p. □

Example 190. If the sectional curvatures at p all have value K, then r is also constant on $T_pM \setminus \{0\}$ given by $(n - 1)K$.

Lecture 24 — 1 December 2015

Last time we defined the Ricci Tensor and the Ricci Curvature. Today we're going to go one step further with one more contraction.

Definition 191. The Ricci tensor r and the metric determine an endomorphism $T_pM \xrightarrow{\theta} T_pM$ where $r(v,-) = \langle \theta(v), - \rangle$. The **scalar curvature** is just the trace of this endomorphism. With respect to an orthonormal basis, $e_1, \ldots, e_n$, this is just

$$\sum_{i=1}^{n} \langle \theta(e_i), e_i \rangle = \sum_{i=1}^{n} r(e_i, e_i) = \sum_{i=1}^{n} r(e_i),$$

wehere $r(e_i)$ is the Ricci curvature of e_i.

So s/n is an average of Ricci curatures.

Example 192. If the Ricci curvatures at P are constant with value λ, then $s = n\lambda$. If the sectional curvatures at P are all K, then $s = n(n-1)K$.

Definition 193. Given a metric on M, we say that a local coordinate system $x_1, \ldots, x_n$ is **normal** at p if

$$\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_p = \delta_{ij} \qquad \text{and} \qquad \frac{\partial}{\partial x_k} \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_p = 0$$

Remark 194. Choosing a set of normal coordinates greatly simplifies calculations in many cases. Once we've got existence of normal coordinates, the proofs (e.g. for the second Bianchi identity) can be much much shorter.

Lemma 195. Normal coordinates exist at any point p.

Proof. Gram-Schmidt implies we may choose the coordinates $x_1, \ldots, x_n$, orthonormal with respect to the metric; that is, $g_{ij}(p) = \delta_{ij}$. Then set $a_{ijk} = {}^{dg_{ij}}/_{dx_k}(p)$, and

$$b_{kij} = \frac{1}{2} \left(a_{kij} + a_{kji} - a_{ijk} \right).$$

Notice this is symmetric in i, j. Therefore,

$$b_{ijk} + b_{jik} = a_{ijk}.$$

Define a new coordinate system by

$$y_k = x_k + \frac{1}{2} \sum_{\ell, r} b_{k\ell r} x_\ell x_r$$

This then implies that

$$\frac{\partial y_k}{\partial x_\ell} = \delta_{\ell k} + \sum_r b_{k\ell r} x_r.$$

Now a routine check verifies the required properties. □

Corollary 196. If $x_1, \ldots, x_n$ are normal coordinates at p, then the Christoffel symbols of the Levi-Civita connection all vanish.

Proof. Straight from the formula for the Christoffel symbols Γ^k_{ij}. □

Remark 197. In particular, with respect to normal coordinates $x_1, \dots, x_n$, we have the second Bianchi identity

$$\frac{\partial}{\partial x_i} R^m{}_{\ell jk} + \frac{\partial}{\partial x_j} R^m{}_{\ell ki} + \frac{\partial}{\partial x_k} R^m{}_{\ell ij} = 0$$

at p. Now

$$\frac{\partial}{\partial x_i}\left(R_{m\ell jk}\right)_p = \left(\frac{\partial}{\partial x_i}\sum_r g_{mr} R^r{}_{\ell jk}\right)_p = \left(\frac{\partial}{\partial x_i} R^m{}_{\ell jk}\right)_p,$$

the last identity because first derivatives of the metric vanish. So the second Bianchi identity may be rewritten as

$$\boxed{\frac{\partial}{\partial x_i} R_{m\ell jk} + \frac{\partial}{\partial x_j} R_{m\ell ki} + \frac{\partial}{\partial x_k} R_{m\ell ij} = 0,} \tag{14}$$

with respect to the normal coordinates $x_1, \dots, x_p$ at p. s

An application of this is the following theorem.

Theorem 198 (Schur)**.** Let M be a connected Riemannian manifold of dimension $\geqslant 3$. Then

(i) If the sectional curvatures are pointwise constant, such that for any $p \in M$ all the sectional curvatures have value $f(p)$, then f is a constant.

(ii) If the Ricci curvatures are pointwise constant, such that for any $p \in M$ all the Ricci curvatures have value $c(p)$ at p, then c is a constant.

Proof. (i) We suppose the sectional curvatures at p are all $f(p)$. We choose normal coordinates $x_1, \dots, x_n$ in a neighborhood of p; we can write

$$R_{ijk\ell} = f \cdot (g_{ik} g_{j\ell} - g_{i\ell} g_{jk})$$

in a neighborhood of p. The Bianchi identity Equation 14 implies that

$$\frac{\partial}{\partial x_h} R_{ijk\ell} + \frac{\partial}{\partial x_k} R_{ij\ell h} + \frac{\partial}{\partial x_\ell} R_{ijhk} = 0$$

at p. Letting $\partial_h f = {}^{\partial f}\!/_{\partial x_h}$, etc., we get

$$\partial_h f(p)\left(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}\right) + \partial_k f(p)\left(\delta_{i\ell}\delta_{jh} - \delta ih \delta_{j\ell}\right) + \partial_\ell f(p)\left(\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}\right) = 0$$

Since $n \geqslant 3$, for each h, we can choose $i \neq j$ with h, i, j distinct. If we set $k = i$, $\ell = j$ in the above identity then h, i, j distinct.

If we set $k = i$, $\ell = j$ in the above identity and deduce $\partial_h f(p) = 0$ for all h, then $d_p f = 0$. Hence, f is locally constant, which implies that f is globally constant.

(ii) Similar – see example sheet 4, question 11. □

Remark 199. Constant sectional curvature is not too interesting. If simply connected and complete, just have $\mathbb{R}^n$, S^n, and H^n, where H^n is hyperbolic space as defined in Example Sheet 4, question 10.

Constant Ricci curvature, on the other hand, gives the **Einstein Manifolds**.

Constant scalar curvature is not too interesting because of the following:

Theorem 200 (Yamahi Problem)**.** If (M, g) is a **compact** connected Riemannian manifold of dimension $\geqslant 3$. Then there is a smooth function f such that the conformally equivalent metric $e^{2f}g$ has constant scalar curvature. This was finally proved by Schaen in 1984.

In the complex case, a complex compact manifold having a constant scalar curvature **Kähler metric** is an interesting condition – see recent work of Tian, Donaldson et. al.