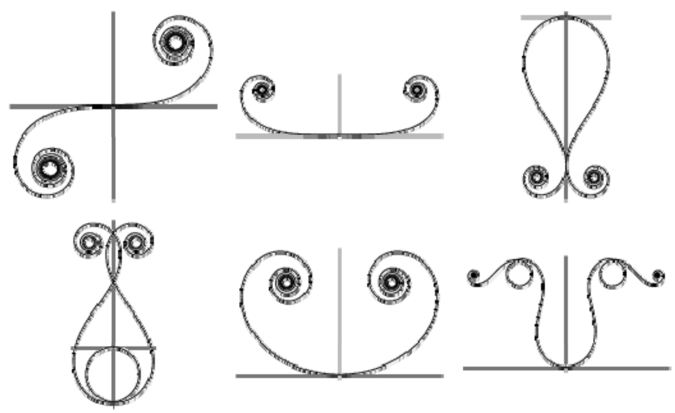
Differential Geometry of Curves



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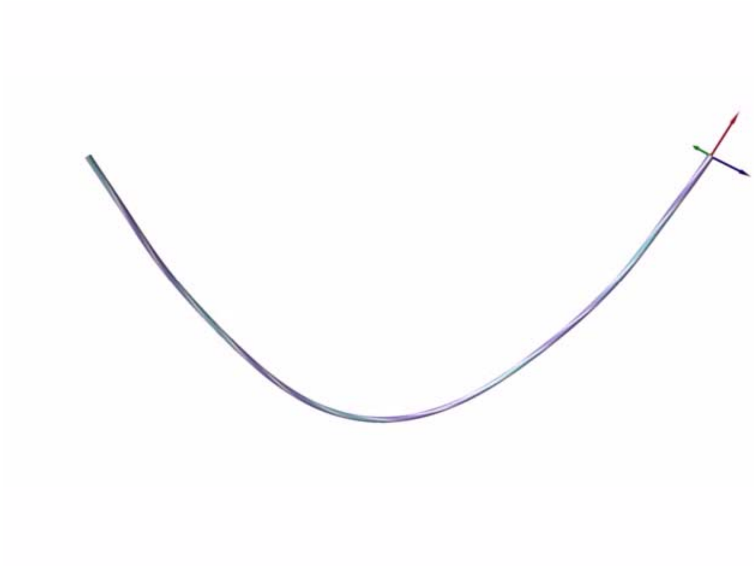
Motivation

- Applications

From "Discrete Elastic Rods" by Bergou et al.



- Good intro to differential geometry on surfaces

- Nice theorems

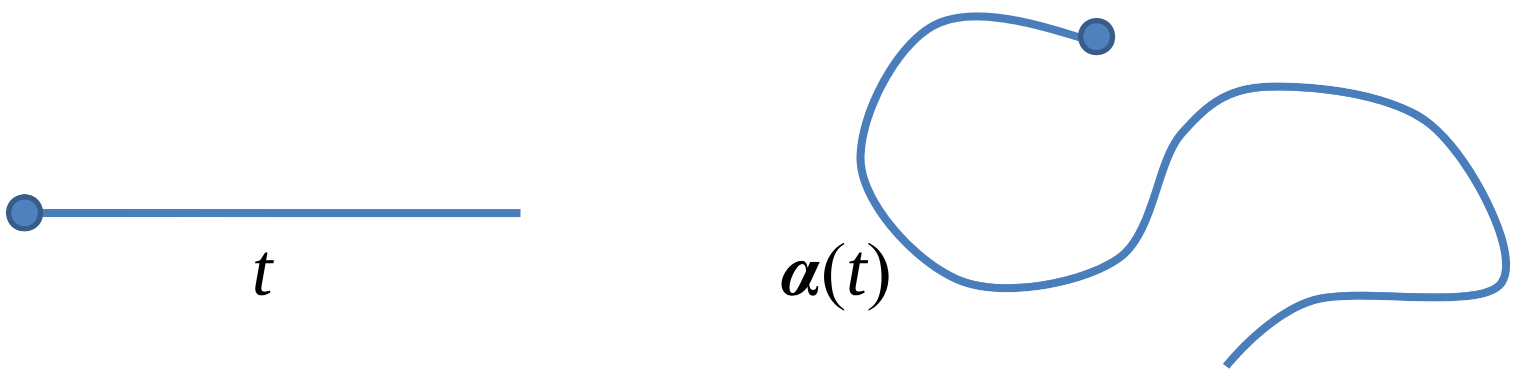
Parameterized Curves

Intuition

A particle is moving in space

At time t its position is given by

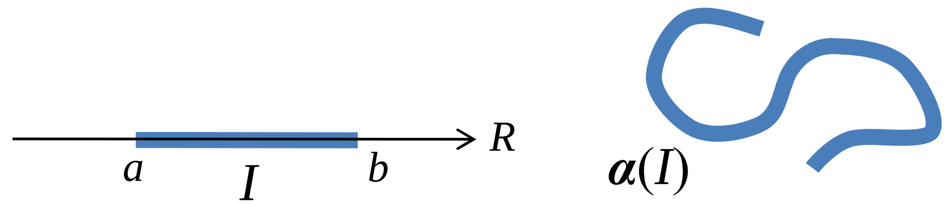
$$\boldsymbol{\alpha}(t) = (x(t), y(t), z(t))$$



Parameterized Curves

Definition

A *parameterized differentiable curve* is a differentiable map $\boldsymbol{\alpha}: I \to R^3$ of an interval $I = (\mathrm{a,b})$ of the real line R into R^3

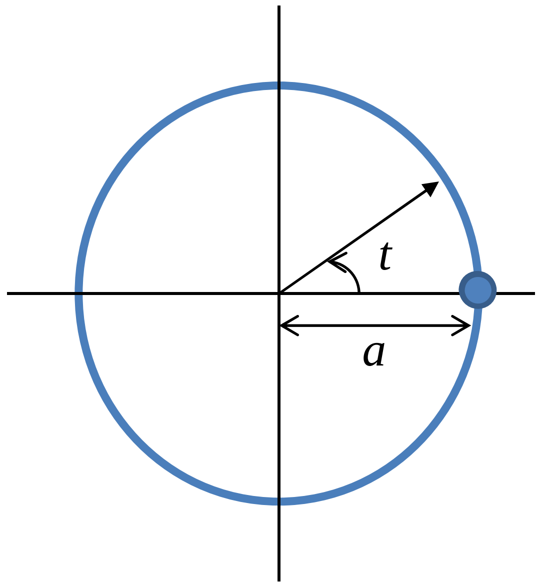


$\boldsymbol{\alpha}$ maps $t \in I$ into a point $\boldsymbol{\alpha}(t) = (x(t), y(t), z(t)) \in R^3$ such that $x(t), y(t), z(t)$ are *differentiable*

A function is *differentiable* if it has, at all points, derivatives of all orders

Parameterized Curves

A Simple Example



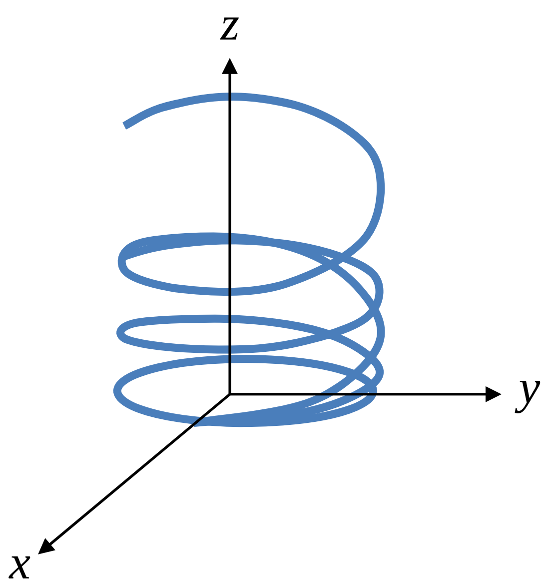
$$\boldsymbol{\alpha}_1(t) = (a\cos(t), a\sin(t))$$
$$t \in [0, 2\pi] = I$$
$$\boldsymbol{\alpha}_2(t) = (a\cos(2t), a\sin(2t))$$
$$t \in [0, \pi] = I$$

$\boldsymbol{\alpha}(I) \subset R^3$ is the *trace* of $\boldsymbol{\alpha}$

→ Different curves can have same trace

More Examples

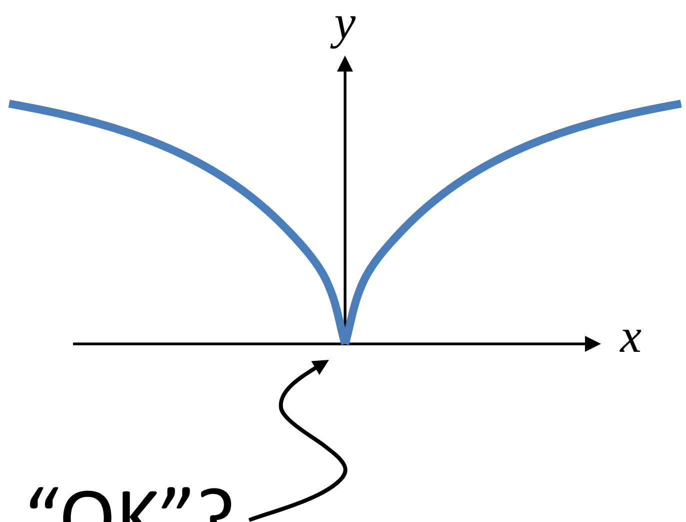
$$\boldsymbol{\alpha}(t) = (a\cos(t), a\sin(t), bt),\ \ t \in R$$



$b = 0$

More Examples

$$\boldsymbol{\alpha}(t) = (t^3, t^2),\ \ t \in R$$



Is this "OK"?

The Tangent Vector

Let

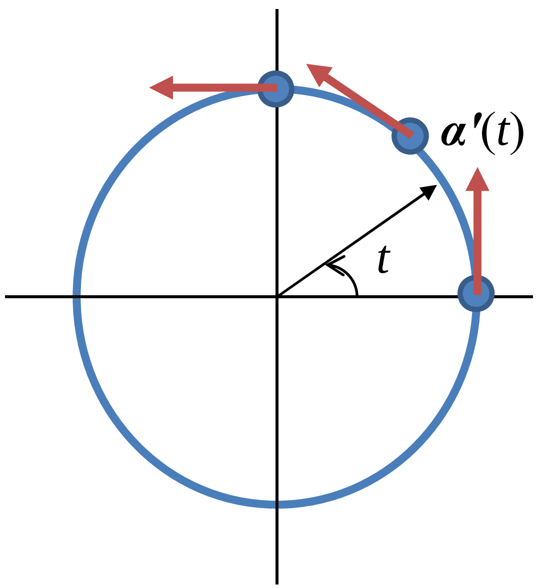
$$\boldsymbol{\alpha}(t) = (x(t), y(t), z(t)) \in R^3$$

Then

$$\boldsymbol{\alpha}'(t) = (x'(t), y'(t), z'(t)) \in R^3$$

is called the *tangent vector* (or *velocity vector*) of the curve $\boldsymbol{\alpha}$ at t

Back to the Circle



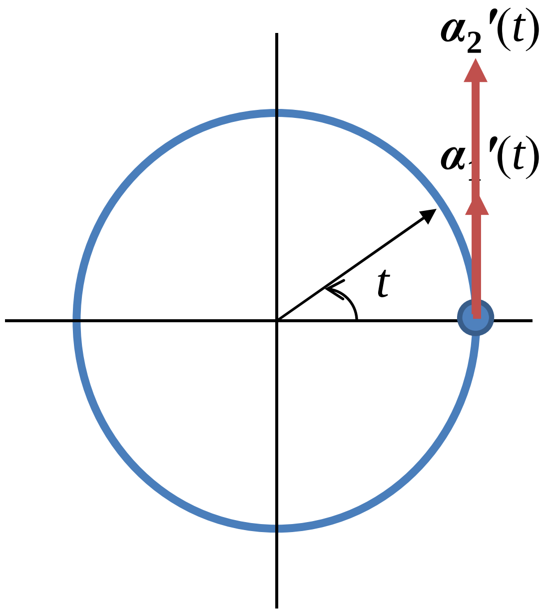
$$\boldsymbol{\alpha}(t) = (\cos(t), \sin(t))$$

$$\boldsymbol{\alpha}'(t) = (-\sin(t), \cos(t))$$

$\boldsymbol{\alpha}'(t)$ - direction of movement

$|\boldsymbol{\alpha}'(t)|$ - speed of movement

Back to the Circle



$$\alpha_1(t) = (\cos(t), \sin(t))$$

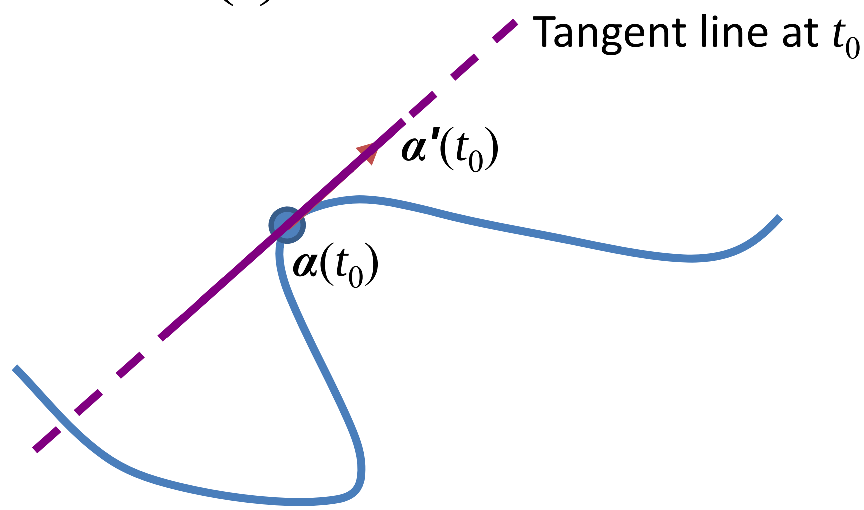
$$\alpha_2(t) = (\cos(2t), \sin(2t))$$

Same direction, different speed

The Tangent Line

Let $\boldsymbol{\alpha}: I \longrightarrow R^3$ be a parameterized differentiable curve.

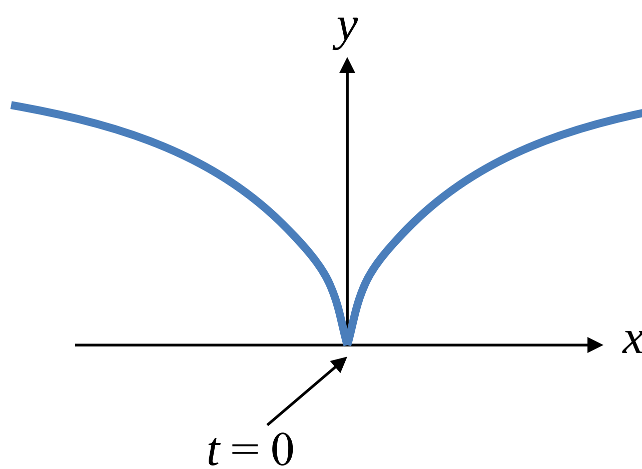
For each $t \in I$ s.t. $\boldsymbol{\alpha}'(t) \neq \mathbf{0}$ the *tangent line* to $\boldsymbol{\alpha}$ at t is the line which contains the point $\boldsymbol{\alpha}(t)$ and the vector $\boldsymbol{\alpha}'(t)$



Regular Curves

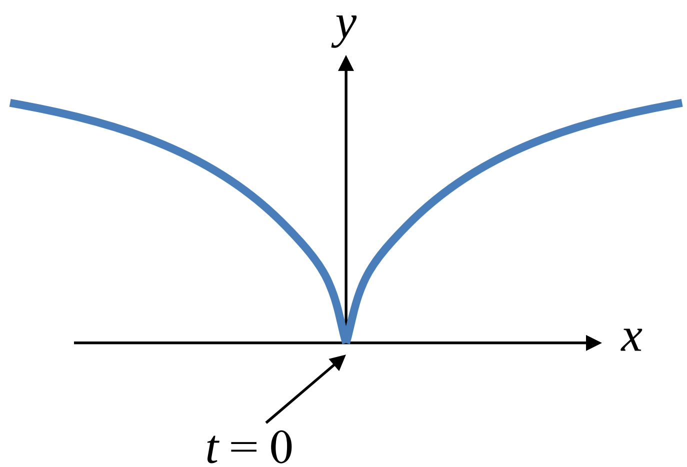
If $\boldsymbol{\alpha}'(t) = \mathbf{0}$, then t is a *singular point* of $\boldsymbol{\alpha}$.

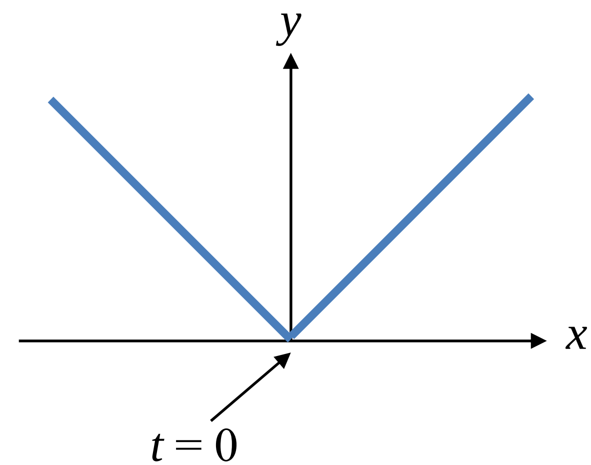
$\boldsymbol{\alpha}(t) = (t^3, t^2),\ \ t \in R$



A parameterized differentiable curve $\boldsymbol{\alpha}: I \longrightarrow R^3$ is *regular* if $\boldsymbol{\alpha}'(t) \neq \mathbf{0}$ for all $t \in I$

Spot the Difference





$\boldsymbol{\alpha}_1(t) = (t^3, t^2)$

Differentiable

Not regular

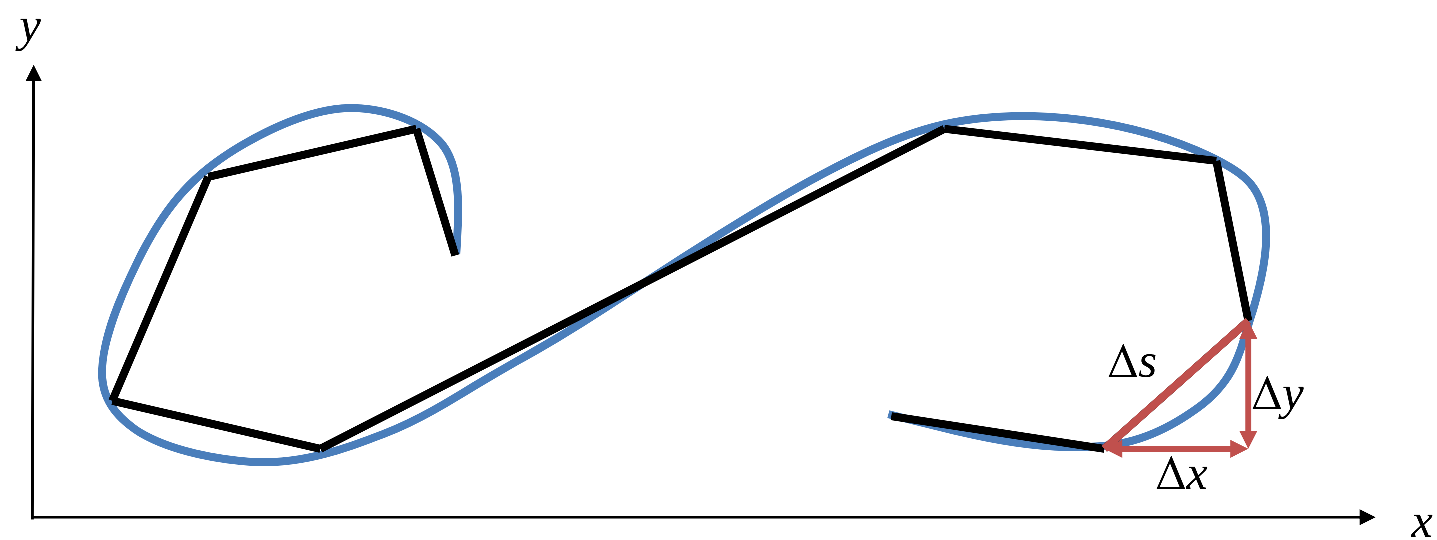
$\boldsymbol{\alpha}_2(t) = (t, |t|)$

Not differentiable

Which differentiable curve has the same trace as $\boldsymbol{\alpha}_2$?

Arc Length of a Curve

How long is this curve?



Approximate with straight lines

Sum lengths of lines: $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Arc Length

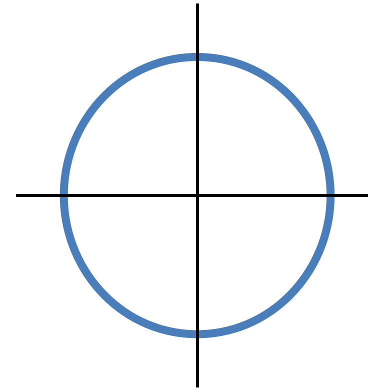
Let $\alpha: I \longrightarrow R^3$ be a parameterized differentiable curve. The *arc length* of α from the point t_0 is:

$$s(t) = \int_{t_0}^{t} |\alpha'(t)|\, dt$$

$$= \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

The arc length is an *intrinsic* property of the curve – does not depend on choice of parameterization

Examples

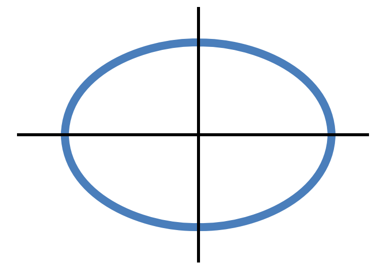


$$\boldsymbol{\alpha}(t) = (a\cos(t), a\sin(t)),\ t \in [0, 2\pi]$$

$$\boldsymbol{\alpha}'(t) = (-a\sin(t), a\cos(t))$$

$$\begin{aligned} L(\alpha) &= \int_0^{2\pi} |\alpha'(t)|\,dt \\ &= \int_0^{2\pi} \sqrt{a^2\sin^2(t) + a^2\cos^2(t)}\,dt \\ &= a\int_0^{2\pi} dt = 2\pi a \end{aligned}$$

Examples



$$\boldsymbol{\alpha}(t) = (a\cos(t), b\sin(t)),\ t \in [0,2\pi]$$

$$\boldsymbol{\alpha}'(t) = (-a\sin(t), b\cos(t))$$

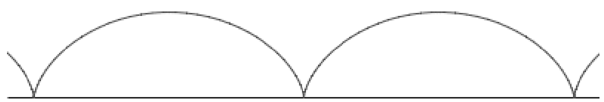
$$L(\alpha) = \int_0^{2\pi} |\alpha'(t)|\,dt$$

$$= \int_0^{2\pi} \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\,dt$$

$$= ??$$

No closed form expression for an ellipse

Closed-Form Arc Length Gallery



Cycloid

$\boldsymbol{\alpha}(t) = (at - a\sin(t), a - a\cos(t))$

$L(\boldsymbol{\alpha}) = 8a$

Logarithmic Spiral

$\boldsymbol{\alpha}(t) = (ae^{bt}\cos(t), ae^{bt}\sin(t))$

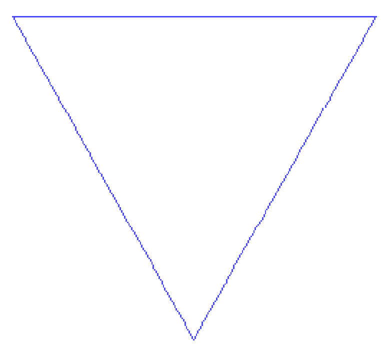
Catenary

$\boldsymbol{\alpha}(t) = (t, a/2\ (e^{t/a} + e^{-t/a}))$

Curves with Infinite Length

The integral $s(t)=\int_{t_0}^{t}|\alpha'(t)|dt$ does not always converge

→ Some curves have infinite length



Koch Snowflake

Arc Length Parameterization

A curve $\alpha: I \rightarrow R^3$ is *parameterized by arc length* if $|\alpha'(t)| = 1$, for all t

For such curves we have

$$s(t) = \int_{t_0}^{t} dt = t - t_0$$

Arc Length *Re*-Parameterization

Let $\boldsymbol{\alpha}: I \rightarrow R^3$ be a regular parameterized curve, and $s(t)$ its arc length.

Then the inverse function $t(s)$ exists, and

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(t(s))$$

is parameterized by arc length.

<u>Proof:</u>

$\boldsymbol{\alpha}$ is regular → $s'(t) = |\boldsymbol{\alpha}'(t)| > 0$

→ $s(t)$ is a monotonic increasing function

→ the inverse function $t(s)$ *exists*

→ $\beta'(s) = \alpha'(t(s))t'(s) = \alpha'(t(s)) / s'(t(s)) = \alpha'(t(s)) / |\alpha'(t(s))|$

→ $|\boldsymbol{\beta}'(s)| = 1$

The Local Theory of Curves

Defines local properties of curves

Local = properties which depend only on behavior in neighborhood of point

We will consider only curves parameterized by arc length

Curvature

Let $\boldsymbol{\alpha}: I \longrightarrow R^3$ be a curve parameterized by arc length s. The *curvature* of $\boldsymbol{\alpha}$ at s is defined by:

$$|\boldsymbol{\alpha}''(s)| = \kappa(s)$$

$\boldsymbol{\alpha}'(s)$ – the tangent vector at s

$\boldsymbol{\alpha}''(s)$ – the *change* in the tangent vector at s

$R(s) = 1/\kappa(s)$ is called the *radius of curvature* at s.

Examples

Straight line

$\boldsymbol{\alpha}(s) = us + v, \; u,v \in R^2$

$\boldsymbol{\alpha}'(s) = u$

$\boldsymbol{\alpha}''(s) = \mathbf{0} \quad \rightarrow \quad |\boldsymbol{\alpha}''(s)| = 0$

Circle

$\boldsymbol{\alpha}(s) = (a \cos(s/a), a \sin(s/a)), \, s \in [0,2\pi a]$

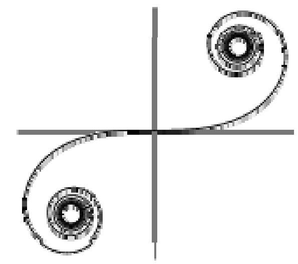
$\boldsymbol{\alpha}'(s) = (-\sin(s/a), \cos(s/a))$

$\boldsymbol{\alpha}''(s) = (-\cos(s/a)/a, -\sin(s/a)/a) \rightarrow \quad |\boldsymbol{\alpha}''(s)| = 1/a$

Examples

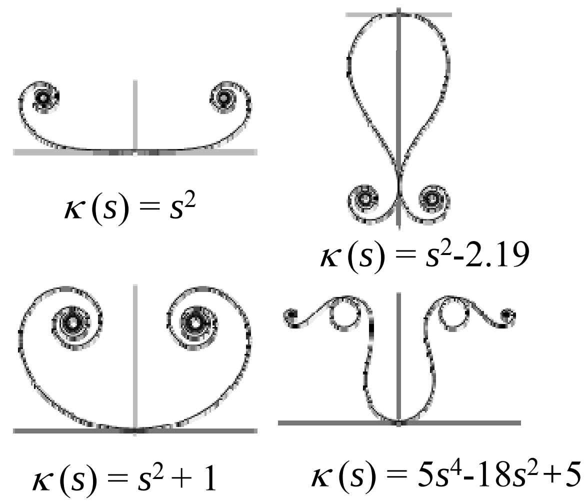
<u>Cornu Spiral</u>

A curve for which $\kappa(s) = s$



<u>Generalized Cornu Spiral</u>

A curve for which $\kappa(s)$ is a polynomial function of s



The Normal Vector

$|\boldsymbol{\alpha}'(s)|$ is the arc length

$\boldsymbol{\alpha}'(s)$ is the tangent vector

$|\boldsymbol{\alpha}''(s)|$ is the curvature

$\boldsymbol{\alpha}''(s)$ is ?

Detour to Vector Calculus

Lemma:

Let $\boldsymbol{f},\boldsymbol{g}: I \rightarrow R^3$ be differentiable maps which satisfy $\boldsymbol{f}(t)\cdot\boldsymbol{g}(t)$ = $const$ for all t.

Then:

$$\boldsymbol{f}'(t) \cdot \boldsymbol{g}(t) = -\boldsymbol{f}(t) \cdot \boldsymbol{g}'(t)$$

And in particular:

$\left|\boldsymbol{f}(t)\right| = const$ if and only if $\boldsymbol{f}(t) \cdot \boldsymbol{f}'(t) = 0$ for all t

Detour to Vector Calculus

Proof:

If $\boldsymbol{f} \cdot \boldsymbol{g}$ is constant for all t, then $(\boldsymbol{f} \cdot \boldsymbol{g})' = 0$.

From the product rule we have:

$$(\boldsymbol{f} \cdot \boldsymbol{g})'(t) = \boldsymbol{f}(t)' \cdot \boldsymbol{g}(t) + \boldsymbol{f}(t) \cdot \boldsymbol{g}'(t) = 0$$

$$\rightarrow \quad \boxed{\boldsymbol{f}'(t) \cdot \boldsymbol{g}(t) = -\boldsymbol{f}(t) \cdot \boldsymbol{g}'(t)}$$

Taking $\boldsymbol{f} = \boldsymbol{g}$ we get:

$$\boldsymbol{f}'(t) \cdot \boldsymbol{f}(t) = -\boldsymbol{f}(t) \cdot \boldsymbol{f}'(t)$$

$$\rightarrow \quad \boxed{\boldsymbol{f}'(t) \cdot \boldsymbol{f}(t) = 0}$$

Back to Curves

$\boldsymbol{\alpha}$ is parameterized by arc length

→ $$\boldsymbol{\alpha}'(s)\cdot\boldsymbol{\alpha}'(s) = 1$$

Applying the Lemma

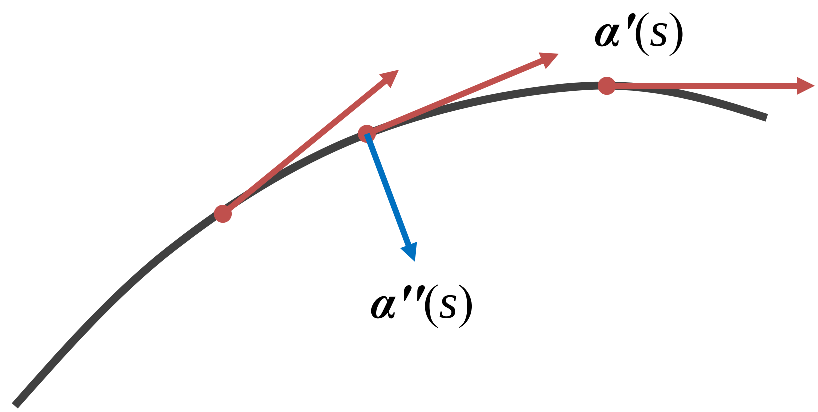
→ $$\boldsymbol{\alpha}''(s)\cdot\boldsymbol{\alpha}'(s) = 0$$

→ The tangent vector is orthogonal to $\boldsymbol{\alpha}''(s)$

The Normal Vector

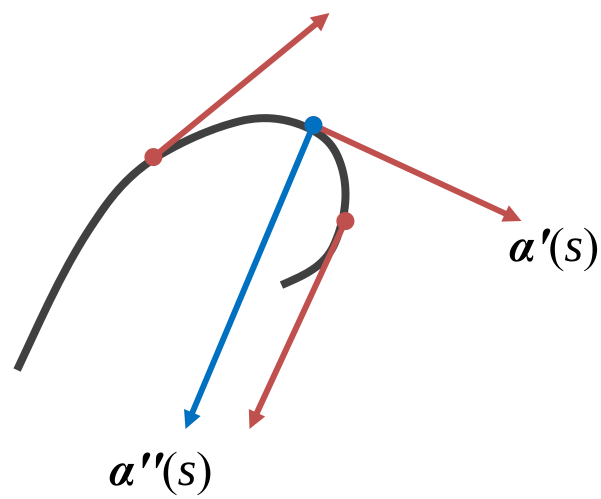
$\boldsymbol{\alpha}'(s) = \boldsymbol{T}(s)$ - tangent vector

$|\boldsymbol{\alpha}'(s)|$ - arc length



$\boldsymbol{\alpha}''(s) = \boldsymbol{T}'(s)$ - normal direction

$|\boldsymbol{\alpha}''(s)|$ - curvature

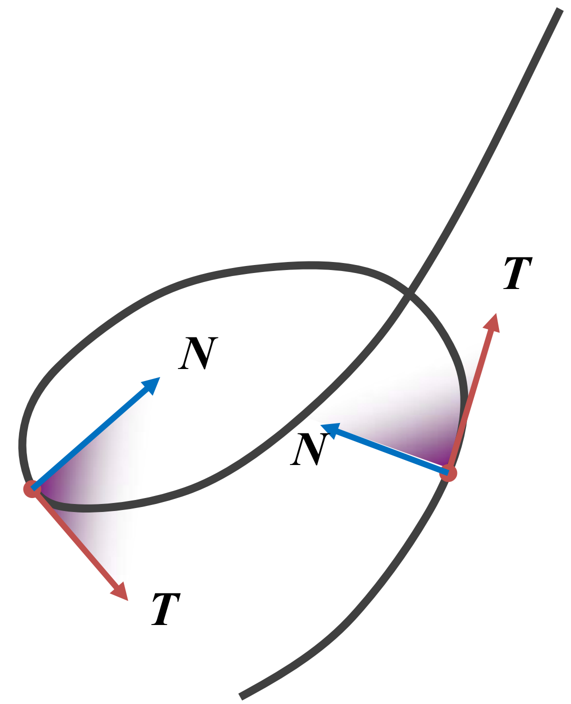


If $|\boldsymbol{\alpha}''(s)| \neq \mathbf{0}$, define $\boldsymbol{N}(s) = \boldsymbol{T}'(s)/|\boldsymbol{T}'(s)|$

Then $\boldsymbol{\alpha}''(s) = \boldsymbol{T}'(s) = \kappa(s)\boldsymbol{N}(s)$

The Osculating Plane

The plane determined by the unit tangent and normal vectors $T(s)$ and $N(s)$ is called the *osculating plane* at s

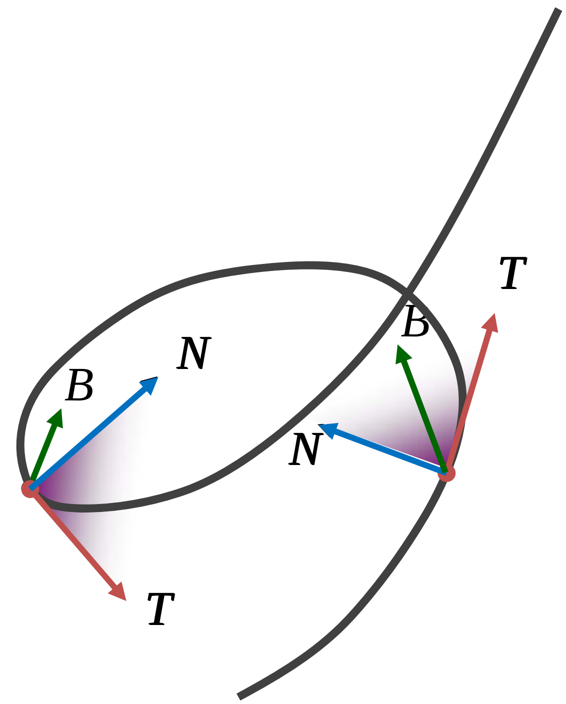


The Binormal Vector

For points s, s.t. $\kappa(s) \neq 0$, the *binormal vector* $\boldsymbol{B}(s)$ is defined as:

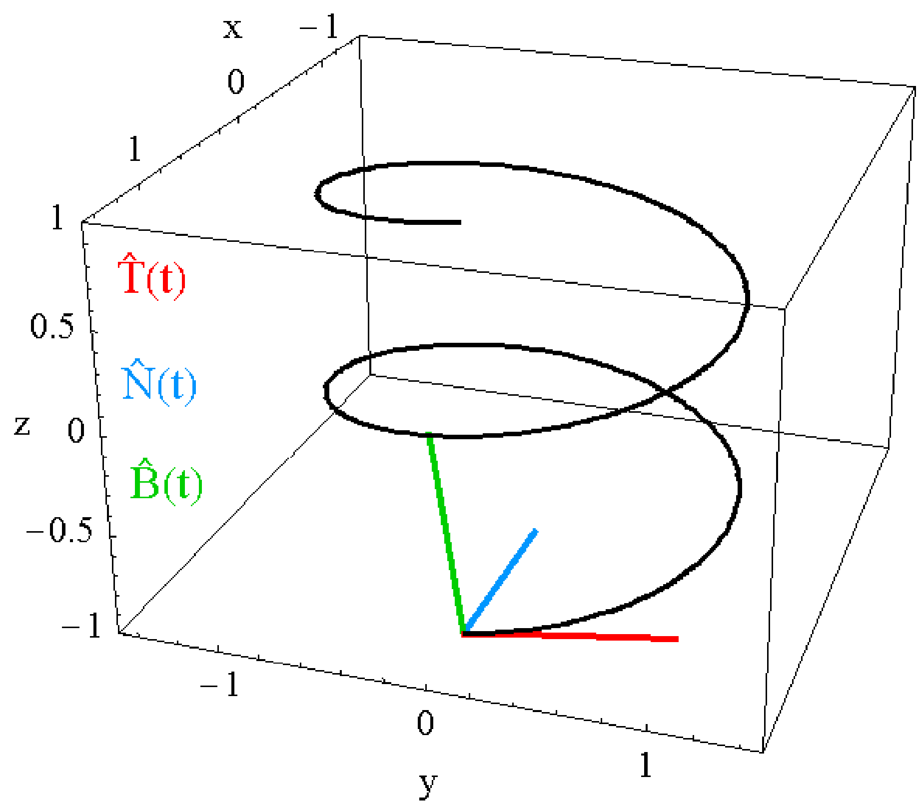
$$\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$$

The binormal vector defines the osculating plane



The Frenet Frame

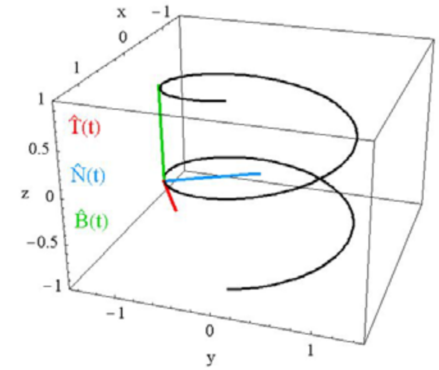
$\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)\}$ form an orthonormal basis for R^3 called the *Frenet frame*



How does the frame change when the particle moves?

What are $\boldsymbol{T}', \boldsymbol{N}', \boldsymbol{B}'$ in terms of $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$?

$\boldsymbol{T'(s)}$

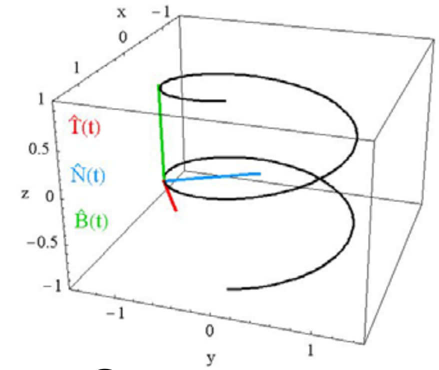


Already used it to define the curvature:

$$\boldsymbol{T}'(s) = \kappa(s)\boldsymbol{N}(s)$$

Since in the direction of the normal, its orthogonal to $\boldsymbol{B}$ and $\boldsymbol{T}$

$N'(s)$



What is $N'(s)$ as a combination of N, T, B ?

We know: $N(s) \cdot N(s) = 1$

From the lemma $\rightarrow N'(s) \cdot N(s) = 0$

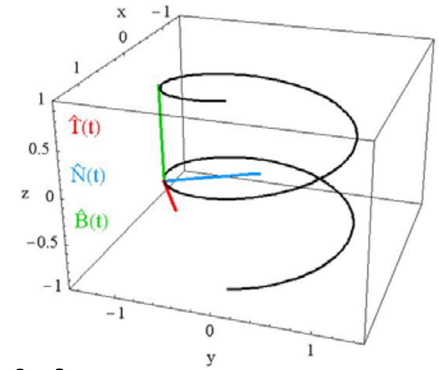
We know: $N(s) \cdot T(s) = 0$

From the lemma $\rightarrow N'(s) \cdot T(s) = -N(s) \cdot T'(s)$

From the definition $\rightarrow \kappa(s) = N(s)\ T'(s)$

$\rightarrow N'(s) \cdot T(s) = -\kappa(s)$

The Torsion



Let $\boldsymbol{\alpha}: I \longrightarrow R^3$ be a curve parameterized by arc length s. The *torsion* of $\boldsymbol{\alpha}$ at s is defined by:

$$\tau(s) = \boldsymbol{N}'(s) \cdot \boldsymbol{B}(s)$$

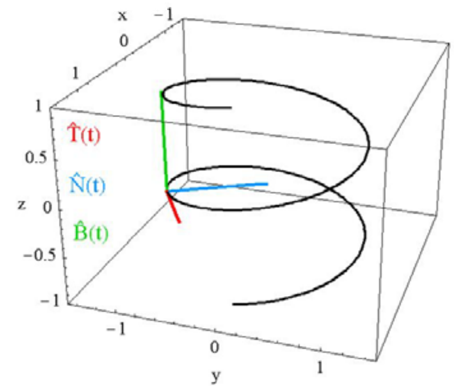
Now we can express $\boldsymbol{N}'(s)$ as:

$$\boldsymbol{N}'(s) = -\kappa(s)\, \boldsymbol{T}(s) + \tau(s)\, \boldsymbol{B}(s)$$

$N'(s) = -\kappa(s)\, T(s) + \tau(s)\, B(s)$

Curvature vs. Torsion

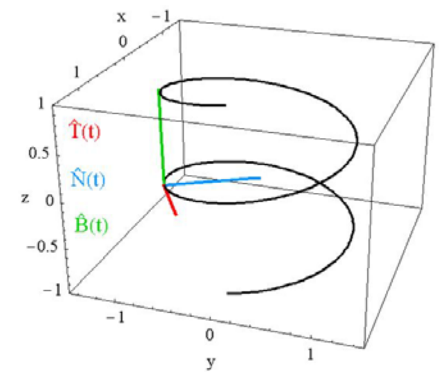
The *curvature* indicates how much the normal changes, in the direction tangent to the curve



The *torsion* indicates how much the normal changes, in the direction orthogonal to the osculating plane of the curve

The curvature is always positive, the torsion can be negative

Both properties *do not* depend on the choice of parameterization



$B'(s)$

What is $B'(s)$ as a combination of N, T, B ?

We know: $B(s) \cdot B(s) = 1$

From the lemma $\rightarrow B'(s) \cdot B(s) = 0$

We know: $B(s) \cdot T(s) = 0,\ B(s) \cdot N(s) = 0$

From the lemma $\rightarrow$

$$B'(s) \cdot T(s) = -B(s) \cdot T'(s) = -B(s) \cdot \kappa(s) N(s) = 0$$

From the lemma $\rightarrow$

$$B'(s) \cdot N(s) = -B(s) \cdot N'(s) = -\tau(s)$$

Now we can express $B'(s)$ as:

$$\boxed{B'(s) = -\tau(s)\, N(s)}$$

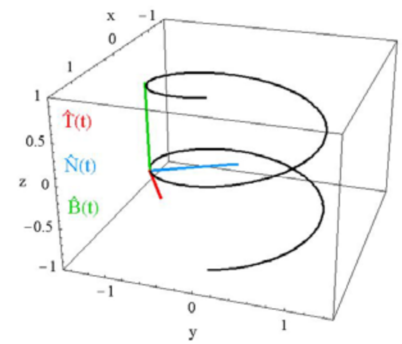
The Frenet Formulas

$$
\begin{aligned}
\boldsymbol{T}'(s) &= \qquad\qquad \kappa(s)\boldsymbol{N}(s) \\
\boldsymbol{N}'(s) &= -\kappa(s)\boldsymbol{T}(s) \qquad\qquad +\tau(s)\boldsymbol{B}(s) \\
\boldsymbol{B}'(s) &= \qquad\qquad -\tau(s)\boldsymbol{N}(s)
\end{aligned}
$$

In matrix form:

$$
\begin{bmatrix} | & | & | \\ \boldsymbol{T}'(s) & \boldsymbol{N}'(s) & \boldsymbol{B}'(s) \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \boldsymbol{T}(s) & \boldsymbol{N}(s) & \boldsymbol{B}(s) \\ | & | & | \end{bmatrix} \begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix}
$$

An Example – The Helix



$\boldsymbol{\alpha}(t) = (a \cos(t), a \sin(t), bt)$

In arc length parameterization:

$\boldsymbol{\alpha}(s) = (a \cos(s/c), a \sin(s/c), bs/c)$, where $c = \sqrt{a^2 + b^2}$

Curvature: $\kappa(s) = \dfrac{a}{a^2 + b^2}$ Torsion: $\tau(s) = \dfrac{b}{a^2 + b^2}$

Note that both the curvature and torsion are constants

A Thought Experiment

Take a straight line

Bend it to add curvature

Twist it to add torsion

$\rightarrow$ You got a curve in R^3

Can we define a curve in R^3 by specifying its curvature and torsion at every point?

The Fundamental Theorem of the Local Theory of Curves

Given differentiable functions $\kappa(s) > 0$ and $\tau(s)$, $s \in I$, there exists a regular parameterized curve $\boldsymbol{\alpha}: I \to R^3$ such that s is the arc length, $\kappa(s)$ is the curvature, and $\tau(s)$ is the torsion of $\boldsymbol{\alpha}$. Moreover, any other curve $\boldsymbol{\beta}$, satisfying the same conditions, differs from $\boldsymbol{\alpha}$ only by a rigid motion.