DIFFERENTIATION OPTIMIZATION PROBLEMS

Question 1 (***)

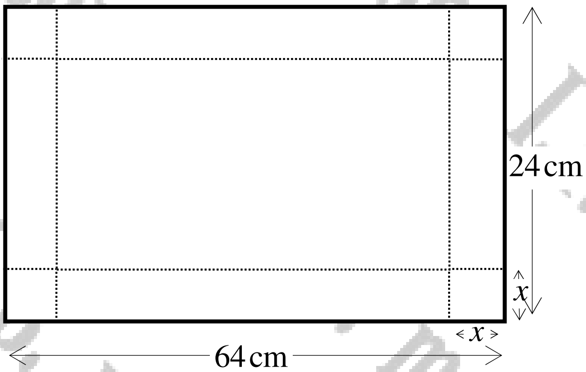


figure 1

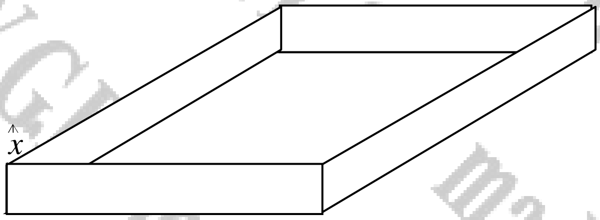


figure 2

An **open** box is to be made out of a rectangular piece of card measuring 64 cm by 24 cm. Figure 1 shows how a square of side length x cm is to be cut out of each corner so that the box can be made by folding, as shown in figure 2.

a) Show that the volume of the box, V cm^3, is given by

$$V = 4x^3 - 176x^2 + 1536x.$$

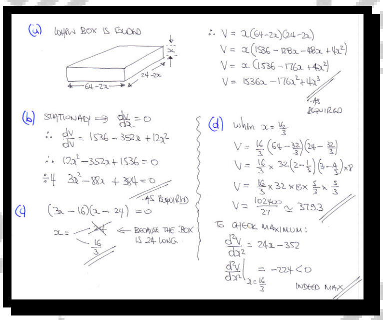
b) Show further that the stationary points of V occur when

$$3x^2 - 88x + 384 = 0.$$

c) Find the value of x for which V is stationary.
(You may find the fact $24 \times 16 = 384$ useful.)

d) Find, to the nearest cm^3, the maximum value for V, justifying that it is indeed the maximum value.

$\boxed{x = \frac{16}{3}}$, $\boxed{V_{\text{max}} \approx 3793}$


(a) WHEN BOX IS FOLDED

64 − 2x, 24 − 2x, x

$\therefore V = x(64-2x)(24-2x)$
$V = x(1536 - 128x - 48x + 4x^2)$
$V = x(1536 - 176x + 4x^2)$
$V = 1536x - 176x^2 + 4x^3$
AS REQUIRED

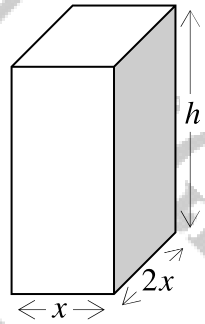
(b) STATIONARY $\Rightarrow \frac{dV}{dx} = 0$
$\therefore \frac{dV}{dx} = 1536 - 352x + 12x^2$
$\therefore 12x^2 - 352x + 1536 = 0$
$\div 4 \quad 3x^2 - 88x + 384 = 0$
AS REQUIRED

(c) $(3x-16)(x-24) = 0$
$x = \cancel{24}$ ← BECAUSE THE BOX IS 24 LONG
$x = \frac{16}{3}$

(d) when $x = \frac{16}{3}$
$V = \frac{16}{3}\left(64 - \frac{32}{3}\right)\left(24 - \frac{32}{3}\right)$
$V = \frac{16}{3} \times 32\left(2 - \frac{1}{3}\right)\left(3 - \frac{4}{3}\right) \times 8$
$V = \frac{16}{3} \times 32 \times 8 \times \frac{5}{3} \times \frac{5}{3}$
$V = \frac{102400}{27} \simeq 3793$

TO CHECK MAXIMUM:
$\frac{d^2V}{dx^2} = 24x - 352$
$\left.\frac{d^2V}{dx^2}\right|_{x=\frac{16}{3}} = -224 < 0$
INDEED MAX

Question 2 (***)



The figure above shows the design of a fruit juice carton with capacity of 1000 cm^3.

The design of the carton is that of a closed cuboid whose base measures x cm by $2x$ cm, and its height is h cm.

a) Show that the surface area of the carton, $A \text{ cm}^2$, is given by

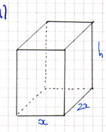
$$A = 4x^2 + \frac{3000}{x}.$$

b) Find the value of x for which A is stationary.

c) Calculate the minimum value for A, justifying fully the fact that it is indeed the minimum value of A.

$\boxed{x = \sqrt[3]{375} \approx 7.21}$, $\boxed{A_{\text{min}} \approx 624}$

a)



CONSTRAINT $V = 1000 \text{ cm}^3$

$\Rightarrow V = x(2x)h$

$\Rightarrow 1000 = 2x^2h$

$\Rightarrow x^2h = 500$

$xh = \frac{500}{x}$

$\boxed{6xh = \frac{3000}{x}}$

SURFACE AREA ($A \text{ cm}^2$)

$\Rightarrow A = 2\left[2x^2 + xh + 2xh\right]$

$\Rightarrow A = 4x^2 + 6xh$

$\Rightarrow A = 4x^2 + \frac{3000}{x}$

AS REQUIRED

b) $A = 4x^2 + 3000x^{-1}$

$\Rightarrow \frac{dA}{dx} = 8x - 3000x^{-2}$

FOR STATIONARY VALUES $\frac{dA}{dx} = 0$

$\Rightarrow 8x - \frac{3000}{x^2} = 0$

$\Rightarrow 8x = \frac{3000}{x^2}$

$\Rightarrow 8x^3 = 3000$

$\Rightarrow x^3 = 375$

$\Rightarrow x = \sqrt[3]{375} \approx 7.21 \text{ cm}$

c) $A = 4x^2 + \frac{3000}{x}$

$\Rightarrow A_{\text{MIN}} = 4(7.21...)^2 + \frac{3000}{7.21...}$

$\Rightarrow A_{\text{MIN}} \approx 624 \text{ cm}^2$

TO JUSTIFY IT IS A MIN, USE 2ND DERIVATIVE

$\Rightarrow \frac{dA}{dx} = 8x - 3000x^{-2}$

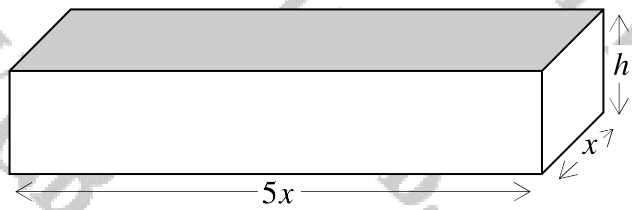
$\Rightarrow \frac{d^2A}{dx^2} = 8 + 6000x^{-3}$

$\Rightarrow \frac{d^2A}{dx^2} = 8 + \frac{6000}{x^3}$

$\Rightarrow \left.\frac{d^2A}{dx^2}\right|_{x=7.21...} = 8 + \frac{6000}{(7.21...)^3} = 24 > 0$

INDEED A MINIMUM

Question 3 (***)



The figure above shows a **solid** brick, in the shape of a cuboid, measuring $5x$ cm by x cm by h cm. The total surface area of the brick is 720 cm^2.

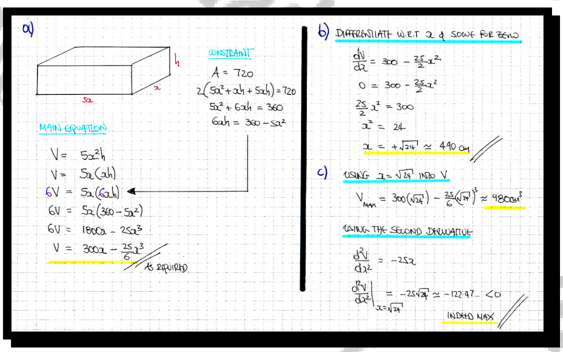
a) Show that the volume of the brick, $V \text{ cm}^3$, is given by

$$V = 300x - \frac{25}{6}x^3.$$

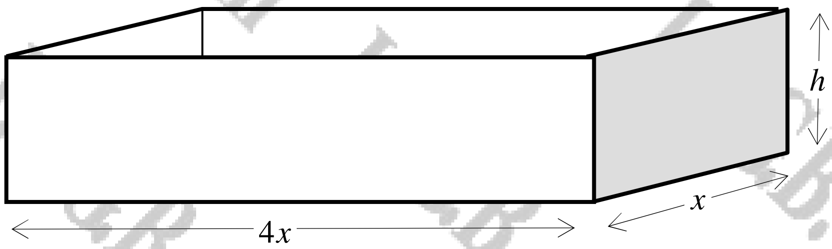
b) Find the value of x for which V is stationary.

c) Calculate the maximum value for V, fully justifying the fact that it is indeed the maximum value.

$\boxed{x = 2\sqrt{6} \approx 4.90}$, $\boxed{V_{\text{max}} = 400\sqrt{6} \approx 980}$



Question 4 (***)



The figure above shows a box in the shape of a cuboid with a rectangular base x cm by $4x$ cm and **no top**. The height of the box is h cm.

It is given that the surface area of the box is 1728 cm^2.

a) Show clearly that

$$h = \frac{864 - 2x^2}{5x}.$$

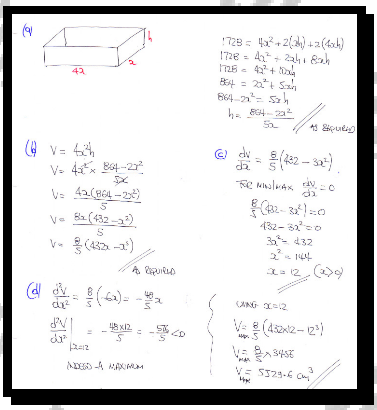
b) Use part **(a)** to show that the volume of the box, $V \text{ cm}^3$, is given by

$$V = \frac{8}{5}\left(432x - x^3\right).$$

c) Find the value of x for which V is stationary.

d) Find the maximum value for V, fully justifying the fact that it is the maximum.

$\boxed{x = 12}$, $\boxed{V_{\text{max}} = 5529.6}$


(a) $1728 = 4x^2 + 2(xh) + 2(4xh)$
$1728 = 4x^2 + 2xh + 8xh$
$1728 = 4x^2 + 10xh$
$864 = 2x^2 + 5xh$
$864 - 2x^2 = 5xh$
$h = \frac{864 - 2x^2}{5x}$ AS REQUIRED

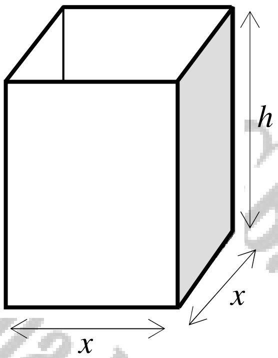
(b) $V = 4x^2h$
$V = 4x^2 \times \frac{864 - 2x^2}{5x}$
$V = \frac{4x(864 - 2x^2)}{5}$
$V = \frac{8x(432 - x^2)}{5}$
$V = \frac{8}{5}(432x - x^3)$ AS REQUIRED

(c) $\frac{dV}{dx} = \frac{8}{5}(432 - 3x^2)$
FOR MIN/MAX $\frac{dV}{dx} = 0$
$\frac{8}{5}(432 - 3x^2) = 0$
$432 - 3x^2 = 0$
$3x^2 = 432$
$x^2 = 144$
$x = 12$ $(x > 0)$

(d) $\frac{d^2V}{dx^2} = \frac{8}{5}(-6x) = -\frac{48}{5}x$
$\left.\frac{d^2V}{dx^2}\right|_{x=12} = -\frac{48 \times 12}{5} = -\frac{576}{5} < 0$
INDEED A MAXIMUM

USING $x = 12$
$V_{\text{MAX}} = \frac{8}{5}(432 \times 12 - 12^3)$
$V_{\text{MAX}} = \frac{8}{5} \times 3456$
$V_{\text{MAX}} = 5529.6 \text{ cm}^3$

Question 5 (***)



The figure above shows the design of a large water tank in the shape of a cuboid with a square base and **no top**.

The square base is of length x metres and its height is h metres.

It is given that the volume of the tank is 500 m^3.

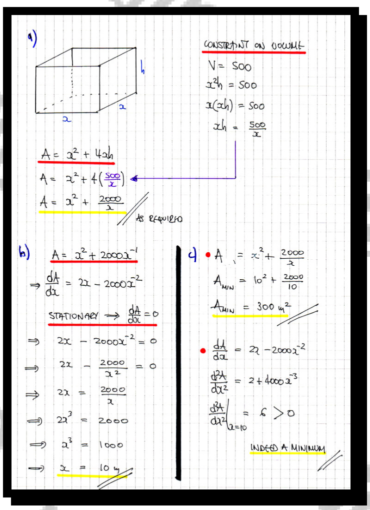
a) Show that the surface area of the tank, $A \text{ m}^2$, is given by

$$A = x^2 + \frac{2000}{x}.$$

b) Find the value of x for which A is stationary.

c) Find the minimum value for A, fully justifying the fact that it is the minimum.

$\boxed{x = 10}$, $\boxed{A_{\text{min}} = 300}$



a)

CONSTRAINT ON VOLUME

$V = 500$

$x^2h = 500$

$x(xh) = 500$

$xh = \frac{500}{x}$

$A = x^2 + 4xh$

$A = x^2 + 4\left(\frac{500}{x}\right)$

$A = x^2 + \frac{2000}{x}$ // AS REQUIRED

b)

$A = x^2 + 2000x^{-1}$

$\Rightarrow \frac{dA}{dx} = 2x - 2000x^{-2}$

STATIONARY $\Rightarrow \frac{dA}{dx} = 0$

$\Rightarrow 2x - 2000x^{-2} = 0$

$\Rightarrow 2x - \frac{2000}{x^2} = 0$

$\Rightarrow 2x = \frac{2000}{x}$

$\Rightarrow 2x^3 = 2000$

$\Rightarrow x^3 = 1000$

$\Rightarrow x = 10 \text{ m}$ //

c)

• $A = x^2 + \frac{2000}{x}$

$A_{MIN} = 10^2 + \frac{2000}{10}$

$A_{MIN} = 300 \text{ m}^2$ //

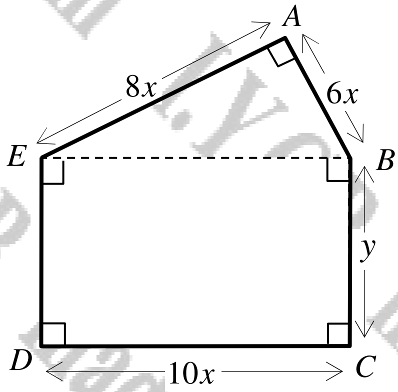
• $\frac{dA}{dx} = 2x - 2000x^{-2}$

$\frac{d^2A}{dx^2} = 2 + 4000x^{-3}$

$\left.\frac{d^2A}{dx^2}\right|_{x=10} = 6 > 0$

INDEED A MINIMUM //

Question 6 (***)



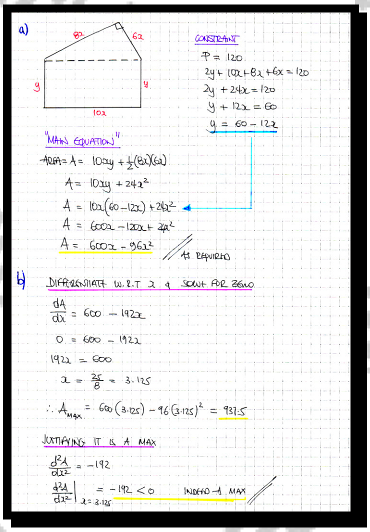
The figure above shows a pentagon $ABCDE$ whose measurements, in cm, are given in terms of x and y.

a) If the perimeter of the pentagon is 120 cm, show clearly that its area, A cm^2, is given by

$$A = 600x - 96x^2 .$$

b) Use a method based on differentiation to calculate the maximum value for A, fully justifying the fact that it is indeed the maximum value.

$$\boxed{A_{\max} = 937.5}$$



a)

CONSTRAINT

$P = 120$

$2y + 10x + 8x + 6x = 120$

$2y + 24x = 120$

$y + 12x = 60$

$y = 60 - 12x$

"MAIN EQUATION"

AREA $= A = 10xy + \frac{1}{2}(8x)(6x)$

$A = 10xy + 24x^2$

$A = 10x(60 - 12x) + 24x^2$

$A = 600x - 120x^2 + 24x^2$

$A = 600x - 96x^2$ // AS REQUIRED

b) DIFFERENTIATE W.R.T x & SOLVE FOR ZERO

$\frac{dA}{dx} = 600 - 192x$

$0 = 600 - 192x$

$192x = 600$

$x = \frac{25}{8} = 3.125$

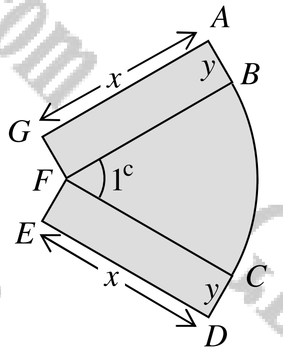
$\therefore A_{MAX} = 600(3.125) - 96(3.125)^2 = 937.5$

JUSTIFYING IT IS A MAX

$\frac{d^2A}{dx^2} = -192$

$\left.\frac{d^2A}{dx^2}\right|_{x=3.125} = -192 < 0$ INDEED A MAX //

Question 7 (***)



The figure above shows a clothes design consisting of two identical rectangles attached to each of the straight sides of a circular sector of radius x cm.

The rectangles measure x cm by y cm and the circular sector subtends an angle of one radian at the centre.

The perimeter of the design is 40 cm.

a) Show that the area of the design, A cm^2, is given by

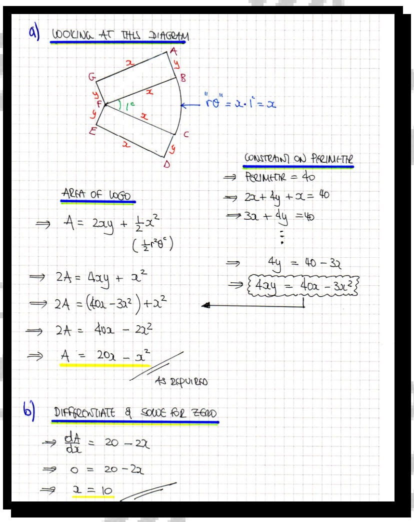
$$A = 20x - x^2 .$$

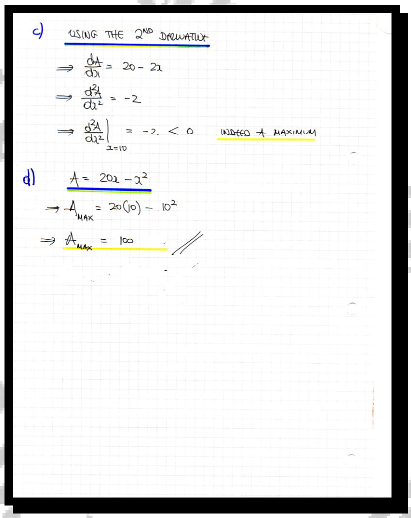
b) Determine **by differentiation** the value of x for which A is stationary.

c) Show that the value of x found in part **(b)** gives the maximum value for A.

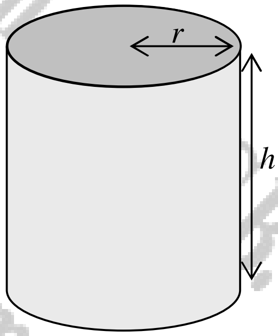
d) Find the maximum area of the design.

$\boxed{x = 10}$, $\boxed{A_{\text{max}} = 100}$





Question 8 (***+)



The figure above shows a **closed** cylindrical can of radius r cm and height h cm.

a) Given that the surface area of the can is 192π cm^2, show that the volume of the can, V cm^3, is given by

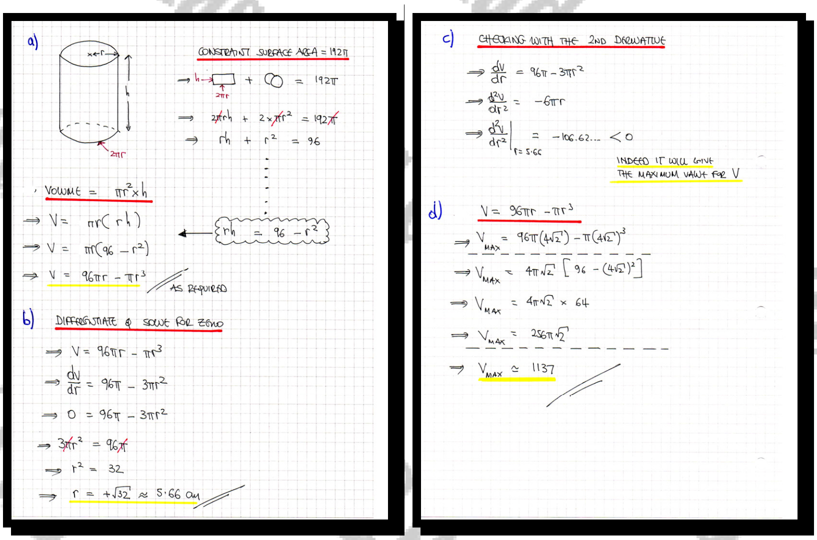
$$V = 96\pi r - \pi r^3 .$$

b) Find the value of r for which V is stationary.

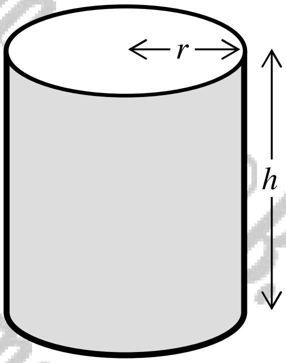
c) Justify that the value of r found in part **(b)** gives the maximum value for V.

d) Calculate the maximum value of V.

$\boxed{r = 4\sqrt{2} \approx 5.66}$, $\boxed{V_{\max} = 256\pi\sqrt{2} \approx 1137}$



Question 9 (***+)



A pencil holder is in the shape of a right circular cylinder, which is **open** at one of its circular ends.

The cylinder has radius r cm and height h cm and the total surface area of the cylinder, including its base, is 360 cm^2.

a) Show that the volume, $V \text{ cm}^3$, of the cylinder is given by

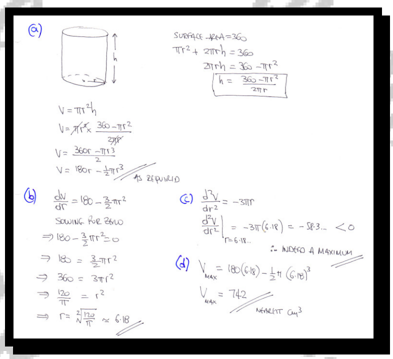
$$V = 180r - \frac{1}{2}\pi r^3 .$$

b) Determine by differentiation the value of r for which V has a stationary value.

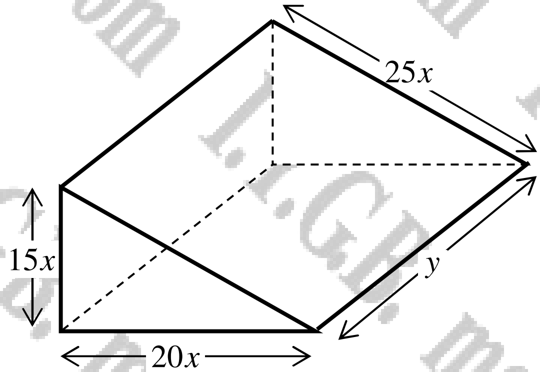
c) Show that the value of r found in part (b) gives the maximum value for V.

d) Calculate, to the nearest cm^3, the maximum volume of the pencil holder.

$\boxed{r = \sqrt{\frac{120}{\pi}} \approx 6.18}$, $\boxed{V_{\text{max}} \approx 742}$



Question 10 (***+)



The figure above shows a solid triangular prism with a **total** surface area of 3600 cm^2.

The triangular faces of the prism are right angled with a base of $20x$ cm and a height of $15x$ cm. The length of the prism is y cm.

a) Show that the volume of the prism, $V \text{ cm}^3$, is given by

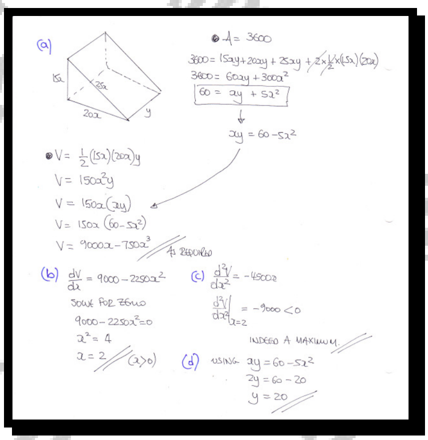
$$V = 9000x - 750x^3.$$

b) Find the value of x for which V is stationary.

c) Show that the value of x found in part (b) gives the maximum value for V.

d) Determine the value of y when V becomes maximum.

$\boxed{x=2}$, $\boxed{y=20}$



(a) 15x, 25x, 20x, y

• A = 3600

$3600 = 15xy + 20xy + 25xy + \cancel{2} \times \cancel{\tfrac{1}{2}} \times (15x)(20x)$

$3600 = 60xy + 300x^2$

$\boxed{60 = xy + 5x^2}$

$xy = 60 - 5x^2$

• $V = \frac{1}{2}(15x)(20x)y$

$V = 150x^2 y$

$V = 150x(xy)$

$V = 150x(60 - 5x^2)$

$V = 9000x - 750x^3$ AS REQUIRED

(b) $\frac{dV}{dx} = 9000 - 2250x^2$

SOLVE FOR ZERO

$9000 - 2250x^2 = 0$

$x^2 = 4$

$x = 2$ $(x>0)$

(c) $\frac{d^2V}{dx^2} = -4500x$

$\left.\frac{d^2V}{dx^2}\right|_{x=2} = -9000 < 0$

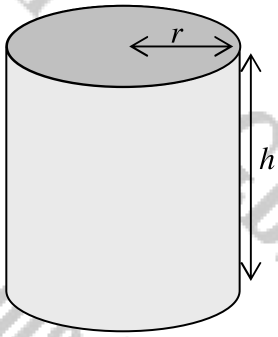
INDEED A MAXIMUM.

(d) USING $xy = 60 - 5x^2$

$2y = 60 - 20$

$y = 20$

Question 11 (*+)**



The figure above shows a **closed** cylindrical can, of radius r cm and height h cm.

a) If the volume of the can is 330 cm^3, show that surface area of the can, $A \text{ cm}^2$, is given by

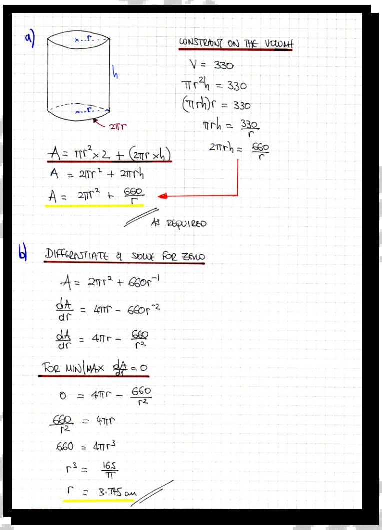
$$A = 2\pi r^2 + \frac{660}{r}.$$

b) Find the value of r for which A is stationary.

c) Justify that the value of r found in part **(b)** gives the minimum value for A.

d) Hence calculate the minimum value of A.

$\boxed{r \approx 3.745}$, $\boxed{A_{\min} \approx 264}$



a)

CONSTRAINT ON THE VOLUME

$V = 330$

$\pi r^2 h = 330$

$(\pi r h) r = 330$

$\pi r h = \frac{330}{r}$

$2\pi r h = \frac{660}{r}$

$A = \pi r^2 \times 2 + (2\pi r \times h)$

$A = 2\pi r^2 + 2\pi r h$

$A = 2\pi r^2 + \frac{660}{r}$

AS REQUIRED

b) DIFFERENTIATE & SOLVE FOR ZERO

$A = 2\pi r^2 + 660 r^{-1}$

$\frac{dA}{dr} = 4\pi r - 660 r^{-2}$

$\frac{dA}{dr} = 4\pi r - \frac{660}{r^2}$

FOR MIN/MAX $\frac{dA}{dr} = 0$

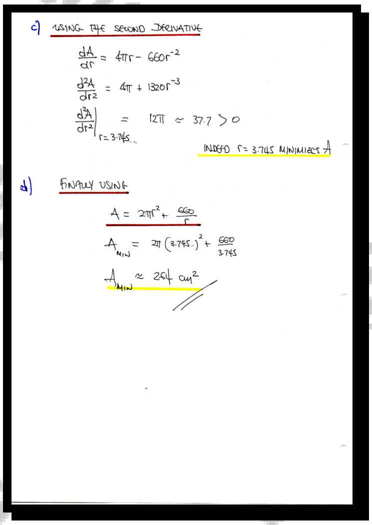
$0 = 4\pi r - \frac{660}{r^2}$

$\frac{660}{r^2} = 4\pi r$

$660 = 4\pi r^3$

$r^3 = \frac{165}{\pi}$

$r = 3.745 \text{ cm}$



c) USING THE SECOND DERIVATIVE

$\frac{dA}{dr} = 4\pi r - 660 r^{-2}$

$\frac{d^2A}{dr^2} = 4\pi + 1320 r^{-3}$

$\left.\frac{d^2A}{dr^2}\right|_{r = 3.745\ldots} = 12\pi \approx 37.7 > 0$

INDEED $r = 3.745$ MINIMIZES A

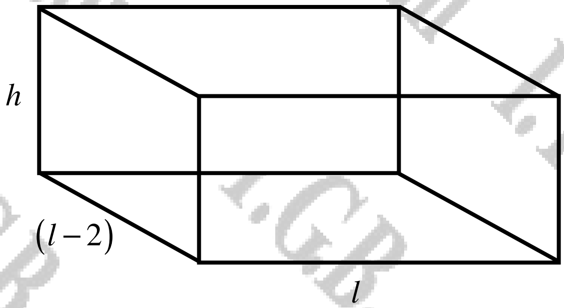
d) FINALLY USING

$A = 2\pi r^2 + \frac{660}{r}$

$A_{\text{MIN}} = 2\pi (3.745\ldots)^2 + \frac{660}{3.745}$

$A_{\text{MIN}} \approx 264 \text{ cm}^2$

Question 12 (***+)



The figure above shows 12 rigid rods, joined together to form the framework of a storage container, which in the shape of a cuboid.

Each of the four upright rods has height h m. Each of the longer horizontal rods has length l m and each of the shorter horizontal rods have length $(l-2)$ m.

a) Given that the total length of the 12 rods is 36 m show that the volume, V m^3, of the container satisfies

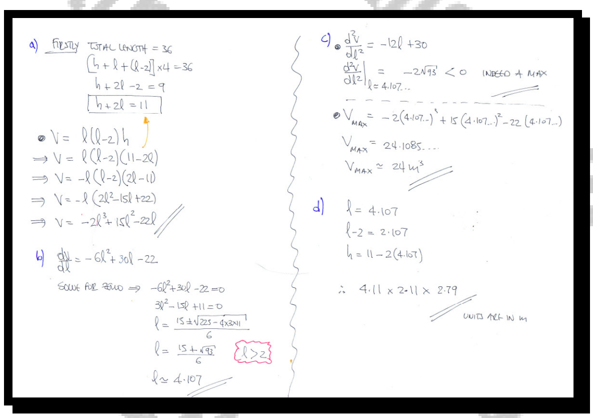
$$V = -2l^3 + 15l^2 - 22l .$$

b) Find, correct to 3 decimal places, the value of l which make V stationary.

c) Justify that the value of l found in part (b) maximizes the value of V, and find this maximum value of V, correct to the nearest m^3.

d) State the three measurements of the container when its volume is maximum.

$\boxed{l = 4.107}$, $\boxed{V_{\max} \approx 24}$, $\boxed{4.11 \times 2.11 \times 2.79}$



a) FIRSTLY TOTAL LENGTH = 36

$[h + l + (l-2)] \times 4 = 36$

$h + 2l - 2 = 9$

$\boxed{h + 2l = 11}$

- $V = l(l-2)h$

$\Rightarrow V = l(l-2)(11-2l)$

$\Rightarrow V = -l(l-2)(2l-11)$

$\Rightarrow V = -l(2l^2 - 15l + 22)$

$\Rightarrow V = -2l^3 + 15l^2 - 22l$

b) $\frac{dV}{dl} = -6l^2 + 30l - 22$

SOLVE FOR ZERO $\Rightarrow$ $-6l^2 + 30l - 22 = 0$

$3l^2 - 15l + 11 = 0$

$l = \frac{15 \pm \sqrt{225 - 4 \times 3 \times 11}}{6}$

$l = \frac{15 + \sqrt{93}}{6}$ $l > 2$

$l \simeq 4.107$

c) - $\frac{d^2V}{dl^2} = -12l + 30$

$\left.\frac{d^2V}{dl^2}\right|_{l \simeq 4.107...} = -2\sqrt{93} < 0$ INDEED A MAX

- $V_{MAX} = -2(4.107...)^3 + 15(4.107...)^2 - 22(4.107...)$

$V_{MAX} = 24.1085....$

$V_{MAX} \simeq 24 \text{ m}^3$

d) $l = 4.107$

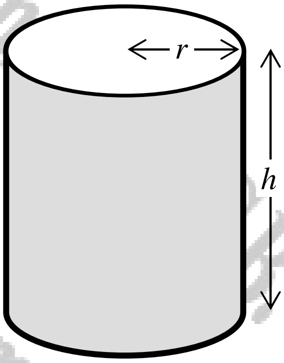
$l - 2 = 2.107$

$h = 11 - 2(4.107)$

$\therefore$ $4.11 \times 2.11 \times 2.79$

UNITS ARE IN m

Question 13 (***+)



A hollow container, made of thin sheet metal, is in the shape of a right circular cylinder, which is **open** at one of its circular ends.

The container has radius r cm, height h cm and a capacity of 1500 cm^3.

a) Show that the surface area, A cm^2, of the container is given by

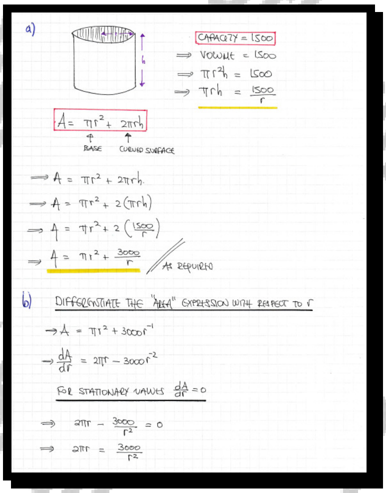
$$A = \pi r^2 + \frac{3000}{r}.$$

b) Determine the value of r for which A has a stationary value.

c) Show that the value of r found in part (**b**) gives the minimum value for A.

d) Calculate, to the nearest cm^2, the minimum surface area of the container.

$\boxed{r \approx 7.816}$, $\boxed{A_{\text{min}} \approx 576}$



a)

$\boxed{\text{CAPACITY} = 1500}$

$\Rightarrow$ VOLUME $= 1500$

$\Rightarrow \pi r^2 h = 1500$

$\Rightarrow \pi r h = \frac{1500}{r}$

$\boxed{A = \pi r^2 + 2\pi r h}$

BASE CURVED SURFACE

$\Rightarrow A = \pi r^2 + 2\pi r h$

$\Rightarrow A = \pi r^2 + 2(\pi r h)$

$\Rightarrow A = \pi r^2 + 2\left(\frac{1500}{r}\right)$

$\Rightarrow A = \pi r^2 + \frac{3000}{r}$ AS REQUIRED

b) DIFFERENTIATE THE "AREA" EXPRESSION WITH RESPECT TO r

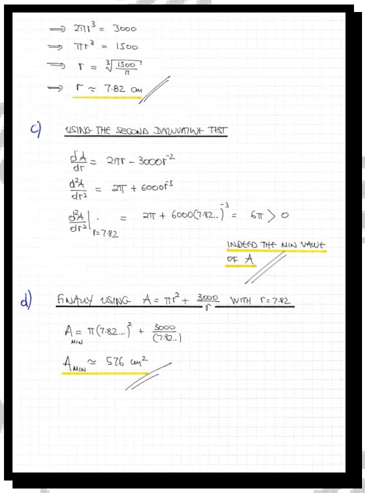
$\Rightarrow A = \pi r^2 + 3000 r^{-1}$

$\Rightarrow \frac{dA}{dr} = 2\pi r - 3000 r^{-2}$

FOR STATIONARY VALUES $\frac{dA}{dr} = 0$

$\Rightarrow 2\pi r - \frac{3000}{r^2} = 0$

$\Rightarrow 2\pi r = \frac{3000}{r^2}$



$\Rightarrow 2\pi r^3 = 3000$

$\Rightarrow \pi r^3 = 1500$

$\Rightarrow r = \sqrt[3]{\frac{1500}{\pi}}$

$\Rightarrow r \approx 7.82$ cm

c) USING THE SECOND DERIVATIVE TEST

$\frac{dA}{dr} = 2\pi r - 3000 r^{-2}$

$\frac{d^2A}{dr^2} = 2\pi + 6000 r^{-3}$

$\left.\frac{d^2A}{dr^2}\right|_{r=7.82} = 2\pi + 6000(7.82...)^{-3} = 6\pi > 0$

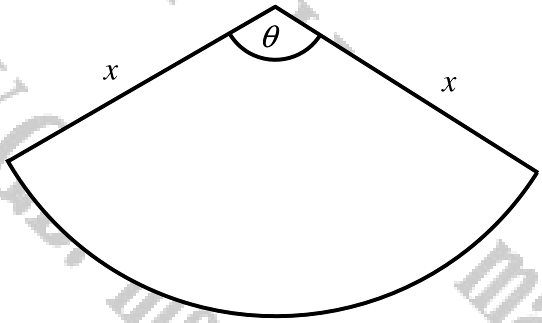
INDEED THE MIN VALUE OF A

d) FINALLY USING $A = \pi r^2 + \frac{3000}{r}$ WITH $r = 7.82$

$A_{\text{MIN}} = \pi(7.82...)^2 + \frac{3000}{(7.82..)}$

$A_{\text{MIN}} \approx 576$ cm^2

Question 14 (***+)



A circular sector of radius x cm subtends an angle of θ radians at the centre.

The area of the sector is 36 cm^2 and its perimeter is P cm.

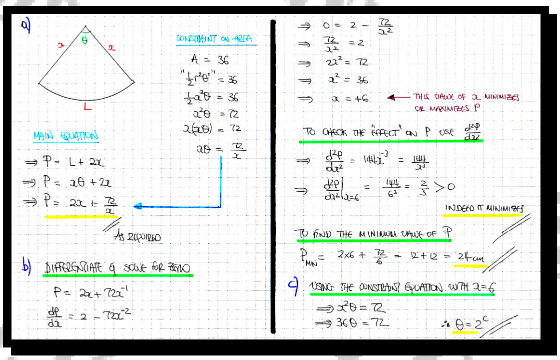
a) Show clearly that

$$P = 2x + \frac{72}{x}.$$

b) Find the minimum value of P, fully justifying the fact that it is a minimum.

c) Deduce the value of θ when P is minimum.

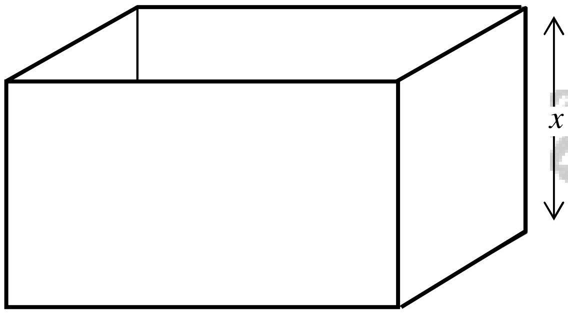
$\boxed{\quad}$, $\boxed{P_{\text{min}} = 24}$, $\boxed{\theta = 2^{\text{c}}}$



Question 15 (***+)

The figure below shows a large tank in the shape of a cuboid with a **rectangular** base and **no top**.

Two of the vertical opposite faces of the cuboid are square and the height of the cuboid is x metres.

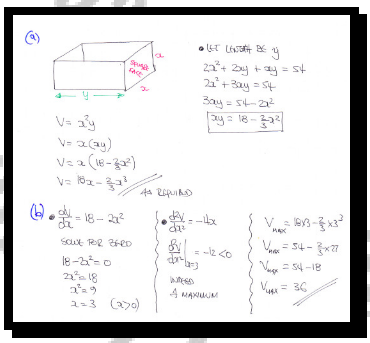


a) Given that the surface area of the tank is 54 m^2, show that the capacity of the tank, $V \text{ m}^3$, is given by

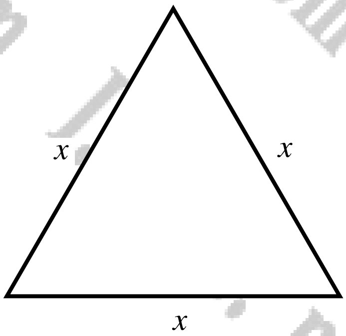
$$V = 18x - \frac{2}{3}x^3 .$$

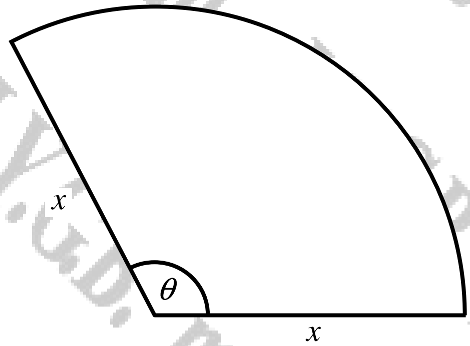
b) Find the maximum value for V, fully justifying the fact that it is indeed the maximum value.

$$\boxed{V_{\max} = 36}$$



Question 16 (***+)





A wire of total length 60 cm is to be cut into two pieces. The first piece is bent to form an equilateral triangle of side length x cm and the second piece is bent to form a circular sector of radius x cm. The circular sector subtends an angle of θ radians at the centre.

a) Show that

$$x\theta = 60 - 5x .$$

The **total** area of the **two** shapes is A cm^2.

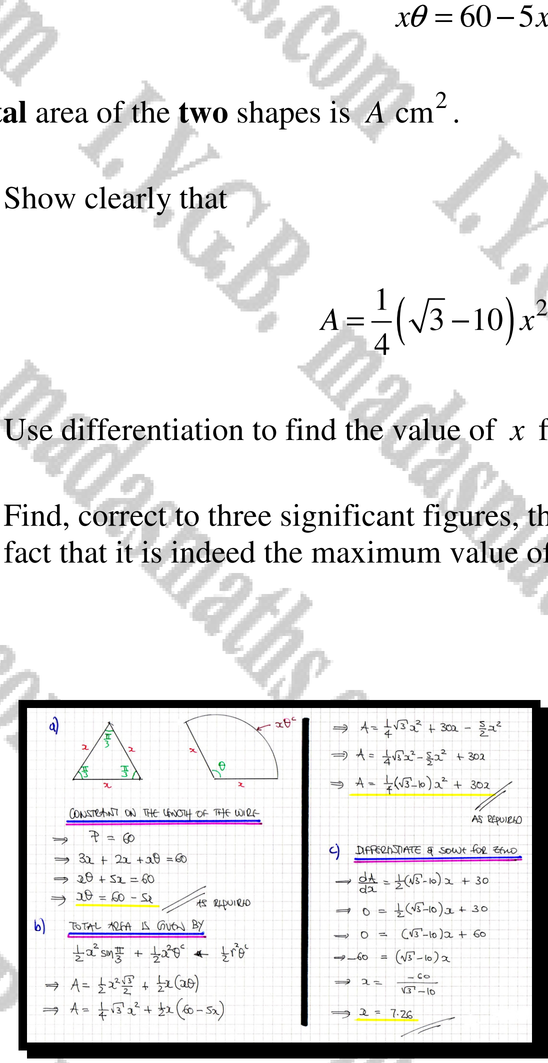
b) Show clearly that

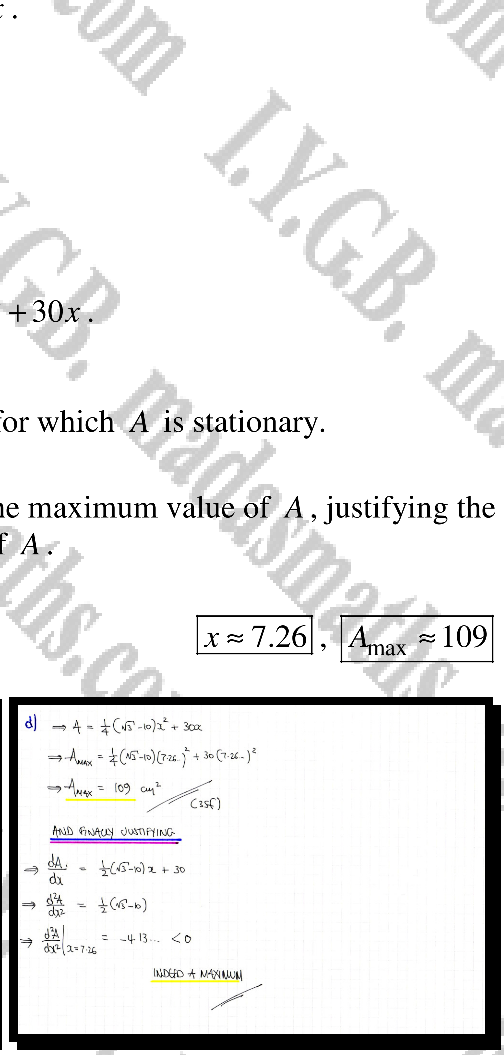
$$A = \frac{1}{4}\left(\sqrt{3} - 10\right)x^2 + 30x .$$

c) Use differentiation to find the value of x for which A is stationary.

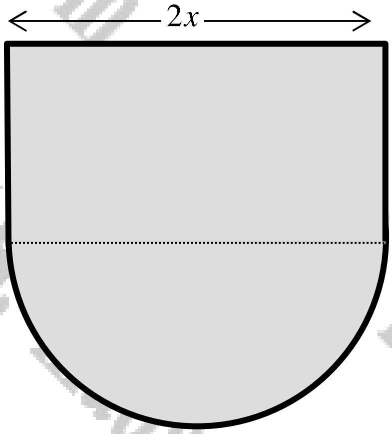
d) Find, correct to three significant figures, the maximum value of A, justifying the fact that it is indeed the maximum value of A.

$\boxed{x \approx 7.26}$, $\boxed{A_{\max} \approx 109}$





Question 17 (****)



The figure above shows the design of a theatre stage which is the shape of a semicircle attached to rectangle. The diameter of the semicircle is $2x$ m and is attached to one side of the rectangle also measuring $2x$ m. The other side of the rectangle is y m.

The perimeter of the stage is 60 m.

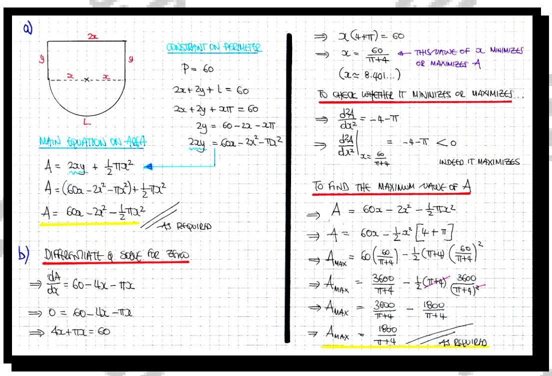
a) Show that the total area of the stage, A m^2, is given by

$$A = 60x - 2x^2 - \frac{1}{2}\pi x^2.$$

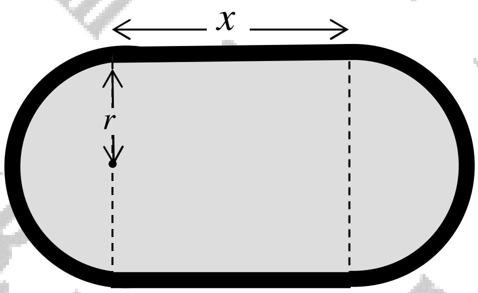
b) Show further, by using a **differentiation** method, that the maximum area of the stage is

$$\frac{1800}{\pi + 4} \text{ m}^2.$$

proof



Question 18 (****)



The figure above shows the design of an athletics track inside a stadium.

The track consists of two semicircles, each of radius r m, joined up to a rectangular section of length x metres.

The total length of the track is 400 m and encloses an area of A m^2.

a) By obtaining and manipulating expressions for the total length of the track and the area enclosed by the track, show that

$$A = 400r - \pi r^2 .$$

In order to hold field events safely, it is required for the area enclosed by the track to be as large as possible.

b) Determine by **differentiation** an exact value of r for which A is stationary.

c) Show that the value of r found in part **(b)** gives the maximum value for A.

d) Show further that the maximum area the area enclosed by the track is

$$\frac{40000}{\pi} \text{ m}^2 .$$

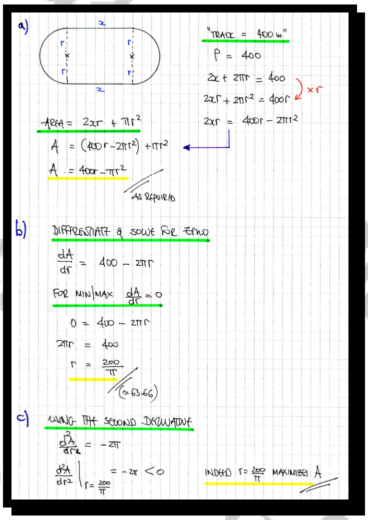
[continues overleaf]

[continued from overleaf]

The calculations for maximizing the area of the field within the track are shown to a mathematician. The mathematician agrees that the calculations are correct but he feels the resulting shape of the track might not be suitable.

e) Explain, by calculations, the mathematician's reasoning.

$$\boxed{r = \frac{200}{\pi} \approx 63.66}$$



a)

"TRACK = 400 m"

$P = 400$

$2x + 2\pi r = 400$ $\times r$

$2xr + 2\pi r^2 = 400r$

$2xr = 400r - 2\pi r^2$

AREA $= 2xr + \pi r^2$

$A = (400r - 2\pi r^2) + \pi r^2$

$A = 400r - \pi r^2$ // AS REQUIRED

b) DIFFERENTIATE & SOLVE FOR ZERO

$\frac{dA}{dr} = 400 - 2\pi r$

FOR MIN/MAX $\frac{dA}{dr} = 0$

$0 = 400 - 2\pi r$

$2\pi r = 400$

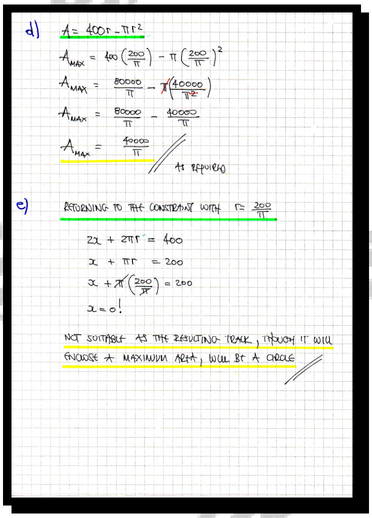
$r = \frac{200}{\pi}$ // (≈ 63.66)

c) USING THE SECOND DERIVATIVE

$\frac{d^2A}{dr^2} = -2\pi$

$\left.\frac{d^2A}{dr^2}\right|_{r=\frac{200}{\pi}} = -2\pi < 0$

INDEED $r = \frac{200}{\pi}$ MAXIMIZES A //



d) $A = 400r - \pi r^2$

$A_{MAX} = 400\left(\frac{200}{\pi}\right) - \pi\left(\frac{200}{\pi}\right)^2$

$A_{MAX} = \frac{80000}{\pi} - \cancel{\pi}\left(\frac{40000}{\pi^{\cancel{2}}}\right)$

$A_{MAX} = \frac{80000}{\pi} - \frac{40000}{\pi}$

$A_{MAX} = \frac{40000}{\pi}$ // AS REQUIRED

e) RETURNING TO THE CONSTRAINT WITH $r = \frac{200}{\pi}$

$2x + 2\pi r = 400$

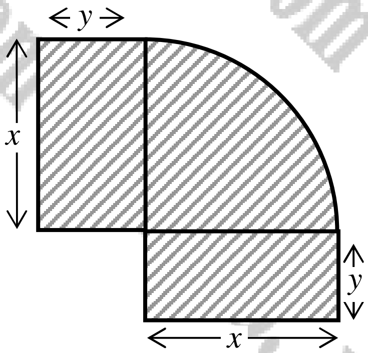
$x + \pi r = 200$

$x + \cancel{\pi}\left(\frac{200}{\cancel{\pi}}\right) = 200$

$x = 0!$

NOT SUITABLE AS THE RESULTING TRACK, THOUGH IT WILL ENCLOSE A MAXIMUM AREA, WILL BE A CIRCLE //

Question 19 (****)



The figure above shows the design for an earring consisting of a quarter circle with two identical rectangles attached to either straight edge of the quarter circle. The quarter circle has radius x cm and the each of the rectangles measure x cm by y cm.

The earring is assumed to have negligible thickness and treated as a two dimensional object with area 12.25 cm^2.

a) Show that the perimeter, P cm, of the earring is given by

$$P = 2x + \frac{49}{2x}.$$

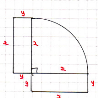
b) Find the value of x that makes the perimeter of the earring minimum, fully justifying that this value of x produces a minimum perimeter.

c) Show that for the value of x found in part **(b)**, the corresponding value of y is

$$\frac{7}{16}(4-\pi).$$

$\boxed{x = 3.5}$

a)



CONSTRAINT ON AREA

$A = 12.25$

$2xy + \frac{1}{4} \times \pi x^2 = 12.25$ $\times 4$

$8xy + \pi x^2 = 49$

$\frac{8xy}{2x} + \frac{\pi x^2}{2x} = \frac{49}{2x}$ $\div 2x$

$4y + \frac{\pi}{2}x = \frac{49}{2x}$

$4y = \frac{49}{2x} - \frac{1}{2}\pi x$

PERIMETER $= 2x + 4y + \frac{1}{4}(2\pi x)$

$\Rightarrow P = 2x + 4y + \frac{1}{2}\pi x$

$\Rightarrow P = 2x + \left(\frac{49}{2x} - \cancel{\frac{1}{2}\pi x}\right) + \cancel{\frac{1}{2}\pi x}$

$\Rightarrow P = 2x + \frac{49}{2x}$ // AS REQUIRED

b) DIFFERENTIATE & SOLVE FOR ZERO

$\Rightarrow P = 2x + \frac{49}{2}x^{-1}$

$\Rightarrow \frac{dP}{dx} = 2 - \frac{49}{2}x^{-2}$

FOR MIN/MAX $\frac{dP}{dx} = 0$

$\Rightarrow 2 - \frac{49}{2x^2} = 0$

$\Rightarrow 2 = \frac{49}{2x^2}$

$\Rightarrow 4x^2 = 49$

$\Rightarrow x^2 = 12.25$

$\Rightarrow x = 3.5 \quad (x > 0)$

USE 2ND DERIVATIVE TO JUSTIFY MINIMUM.

$\Rightarrow \frac{dP}{dx} = 2 - \frac{49}{2}x^{-2}$

$\Rightarrow \frac{d^2P}{dx^2} = 49x^{-3} = \frac{49}{x^3}$

$\Rightarrow \left.\frac{d^2P}{dx^2}\right|_{x=3.5} = \frac{8}{7} > 0$

INDEED $x = 3.5$ MINIMIZES P

c) USING THE CONSTRAINT EQUATION

$\Rightarrow 8xy + \pi x^2 = 49$

$\Rightarrow 8(3.5)y + \pi(3.5)^2 = 49$

$\Rightarrow 28y + \frac{49}{4}\pi = 49$ $\div 7$

$\Rightarrow 4y + \frac{7}{4}\pi = 7$ $\times 4$

$\Rightarrow 16y + 7\pi = 28$

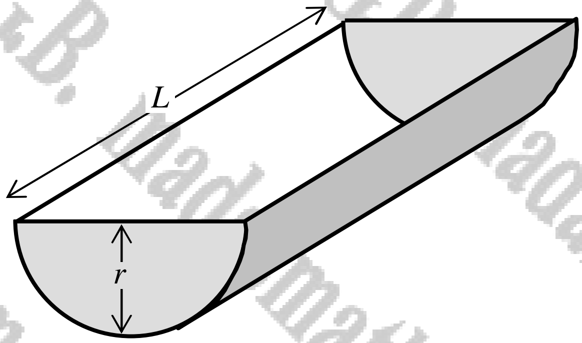
$\Rightarrow 16y = 28 - 7\pi$

$\Rightarrow y = \frac{7(4-\pi)}{16}$ // AS REQUIRED

Question 20 (****)

The figure below shows the design of an animal feeder which in the shape of a hollow, open topped half cylinder, made of thin sheet metal. The radius of the semicircular ends is r cm and the length of the feeder is L cm.

The metal used in the construction of the feeder is $600\pi \text{ cm}^2$.



a) Show that the capacity, $V \text{ cm}^3$, of the feeder is given by

$$V = 300\pi r - \frac{1}{2}\pi r^3 .$$

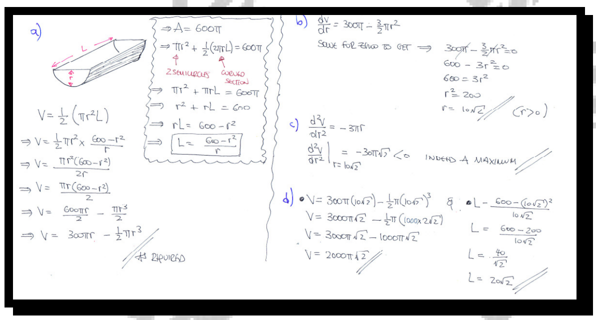
The design of the feeder is such so its capacity is maximum.

b) Determine the exact value of r for which V is stationary.

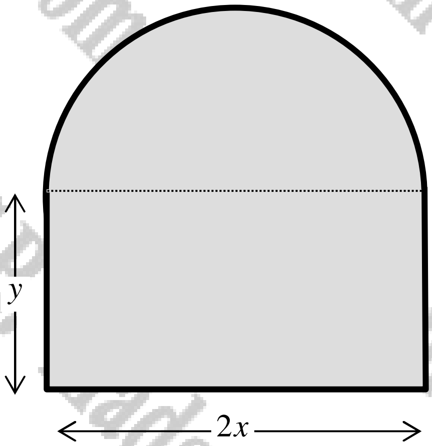
c) Show that the value of r found in part (b) gives the maximum value for V.

d) Find, in exact form, the capacity and the length of the feeder.

$\boxed{r = 10\sqrt{2} \approx 14.14}$, $\boxed{L = 20\sqrt{2} \approx 28.28}$, $\boxed{V_{\text{max}} = 2000\pi\sqrt{2} \approx 8886}$



Question 21 (****)



The figure above shows the design of a window which is the shape of a semicircle attached to rectangle. The diameter of the semicircle is $2x$ m and is attached to one side of the rectangle also measuring $2x$ m. The other side of the rectangle is y m.

The **perimeter** of the window is 6 m.

a) Show that the total area of the window, A m^2, is given by

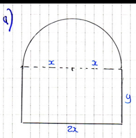
$$A = 6x - \tfrac{1}{2}(4+\pi)x^2 .$$

b) Given that the measurements of the window are such so that A is maximum, show by a method involving differentiation that this maximum value of A is

$$\frac{18}{4+\pi} .$$

proof

a)



CONSTRAINT ON PERIMETER

$P = 6$

$2x + 2y + \frac{1}{2}(2\pi x) = 6$

$2x + 2y + \pi x = 6$

$2y = 6 - 2x - \pi x \quad \times x$

$2xy = 6x - 2x^2 - \pi x^2$

$A = 2xy + \frac{1}{2}\pi x^2$

$A = (6x - 2x^2 - \pi x^2) + \frac{1}{2}\pi x^2$

$A = 6x - 2x^2 - \frac{1}{2}\pi x^2$

$A = 6x - \frac{1}{2}(4+\pi)x^2$ // AS REQUIRED

b) TO MAXIMIZE USE DIFFERENTIATION

$\frac{dA}{dx} = 6 - (4+\pi)x$

FOR MIN/MAX $\frac{dA}{dx} = 0$

$6 - (4+\pi)x = 0$

$6 = (4+\pi)x$

$x = \frac{6}{4+\pi} \approx 0.84$ m

- TO JUSTIFY THAT THIS VALUE OF x YIELDS A MAX

$\frac{dA}{dx} = 6 - (4+\pi)x$

$\frac{d^2A}{dx^2} = -(4+\pi)$

$\left.\frac{d^2A}{dx^2}\right|_{x=\frac{6}{4+\pi}} = -(4+\pi) < 0$

INDEED $x = \frac{6}{4+\pi}$ MAXIMIZES A //

- $A = 6x - \frac{1}{2}(4+\pi)x^2$

$A_{MAX} = 6\left(\frac{6}{4+\pi}\right) - \frac{1}{2}(4+\pi)\left(\frac{6}{4+\pi}\right)^2$

$A_{MAX} = \frac{36}{4+\pi} - \frac{1}{2}\cancel{(4+\pi)} \times \frac{36}{(4+\pi)^{\cancel{2}}}$

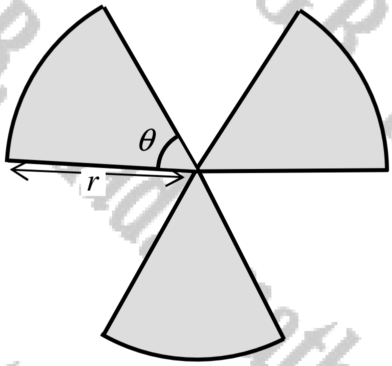
$A_{MAX} = \frac{36}{4+\pi} - \frac{18}{4+\pi}$

$A_{MAX} = \frac{18}{4+\pi}$ // AS REQUIRED

Question 22 (****)

The figure below shows the design of a hazard warning logo which consists of three identical sectors of radius r cm, joined together at the centre.

Each sector subtends an angle θ radians at the centre and the sectors are equally spaced so that the logo has rotational symmetry of order 3.



The area of the logo is 75 cm^2.

a) Show that the perimeter P cm of the logo is given by

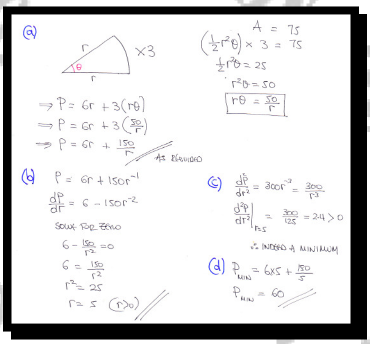
$$P = 6r + \frac{150}{r}.$$

b) Determine by differentiation the value of r for which P is stationary.

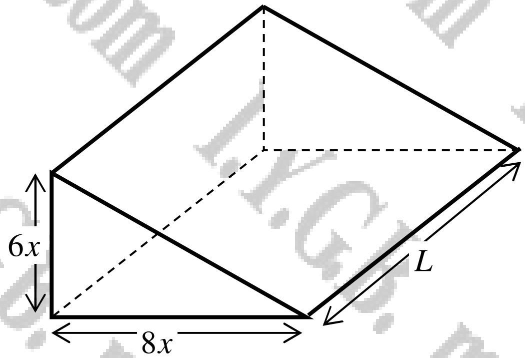
c) Show that the value of r found in part (b) gives the minimum value for P.

d) Find the minimum perimeter of the feeder.

$\boxed{r = 5}$, $\boxed{P_{\text{min}} = 60}$



Question 23 (****)



The figure above shows a triangular prism with a volume of 960 cm^3.

The triangular faces of the prism are right angled with a base $8x$ cm and a height of $6x$ cm. The length of the prism is L cm.

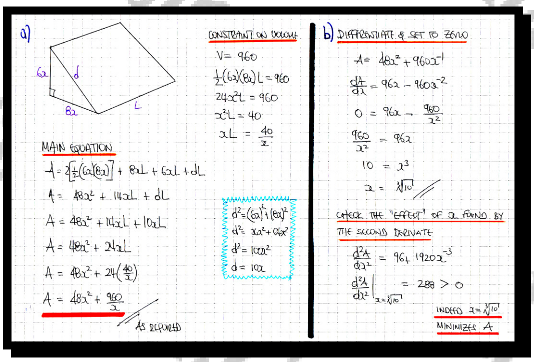
a) Show that the surface area of the prism, $A \text{ cm}^2$, is given by

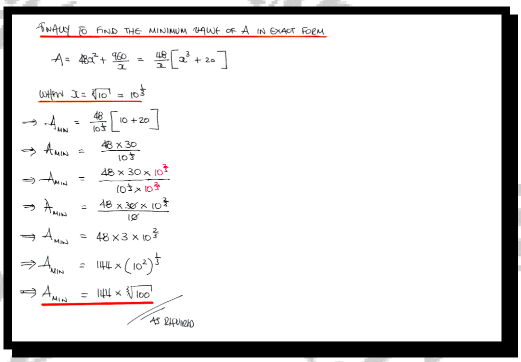
$$A = 48x^2 + \frac{960}{x}.$$

b) Determine an exact value of x for which A is stationary and show that this value of x minimizes A.

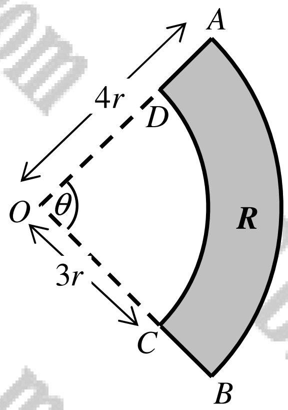
c) Show further that the minimum surface area of the prism is $144\sqrt[3]{100} \text{ cm}^2$.

$\boxed{x = \sqrt[3]{10} \approx 2.15}$





Question 24 (****)



The figure above shows a circular sector OAB of radius $4r$ subtending an angle θ radians at the centre O. Another circular sector OCD of radius $3r$ also subtending an angle θ radians at the centre O is removed from the first sector leaving the shaded region R.

It is given that R has an area of 50 square units.

a) Show that the perimeter P, of the region R, is given by

$$P = 2r + \frac{100}{r}.$$

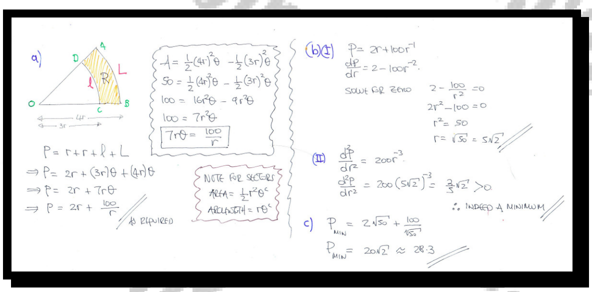
b) Given that the value of r can vary, …

i. … find an exact value of r for which P is stationary.

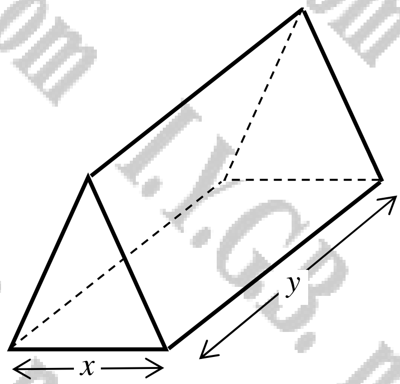
ii. … show that the value of r found above gives the minimum value for P.

c) Calculate the minimum value of P.

$\boxed{r = 5\sqrt{2} \approx 7.07}$, $\boxed{P_{\min} = 20\sqrt{2} \approx 28.28}$



Question 25 (****)



The figure above shows a triangular prism whose triangular faces are parallel to each other and are in the shape of **equilateral** triangles of side length x cm.

The length of the prism is y.

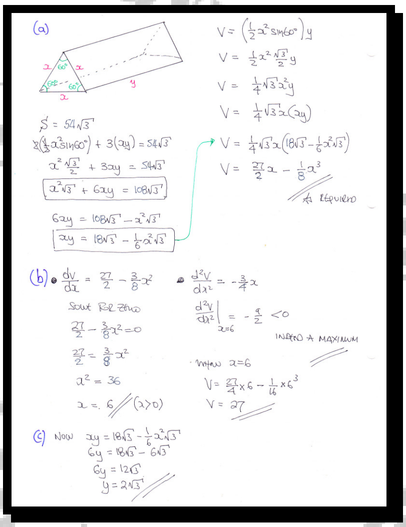
a) Given that total surface area of the prism is exactly $54\sqrt{3}$ cm^2, show clearly that the volume of the prism, V cm^3, is given by

$$V = \frac{27}{2}x - \frac{1}{8}x^3.$$

b) Find the maximum value of V, fully justifying the fact that it is indeed the maximum value.

c) Determine the value of y when V takes this maximum value.

$\boxed{V_{max} = 27}$, $\boxed{y = 2\sqrt{3}}$



(a)

x, $60°$, x, $60°$, $60°$, x, y

$S = 54\sqrt{3}$

$2\left(\frac{1}{2}x^2 \sin 60°\right) + 3(xy) = 54\sqrt{3}$

$x^2\frac{\sqrt{3}}{2} + 3xy = 54\sqrt{3}$

$\boxed{x^2\sqrt{3} + 6xy = 108\sqrt{3}}$

$6xy = 108\sqrt{3} - x^2\sqrt{3}$

$\boxed{xy = 18\sqrt{3} - \frac{1}{6}x^2\sqrt{3}}$

$V = \left(\frac{1}{2}x^2 \sin 60°\right)y$

$V = \frac{1}{2}x^2\frac{\sqrt{3}}{2}y$

$V = \frac{1}{4}\sqrt{3}x^2y$

$V = \frac{1}{4}\sqrt{3}x(xy)$

$V = \frac{1}{4}\sqrt{3}x\left(18\sqrt{3} - \frac{1}{6}x^2\sqrt{3}\right)$

$V = \frac{27}{2}x - \frac{1}{8}x^3$

AS REQUIRED

(b) • $\frac{dV}{dx} = \frac{27}{2} - \frac{3}{8}x^2$

SOLVE FOR ZERO

$\frac{27}{2} - \frac{3}{8}x^2 = 0$

$\frac{27}{2} = \frac{3}{8}x^2$

$x^2 = 36$

$x = 6$ $(x>0)$

• $\frac{d^2V}{dx^2} = -\frac{3}{4}x$

$\left.\frac{d^2V}{dx^2}\right|_{x=6} = -\frac{9}{2} < 0$

INDEED A MAXIMUM

WHEN $x=6$

$V = \frac{27}{4}\times 6 - \frac{1}{16}\times 6^3$

$V = 27$

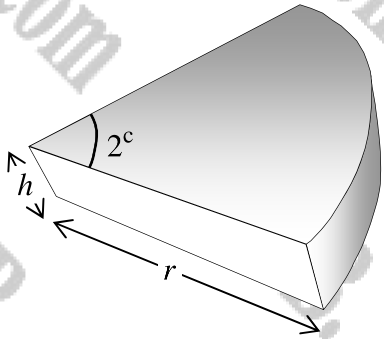
(c) NOW $xy = 18\sqrt{3} - \frac{1}{6}x^2\sqrt{3}$

$6y = 18\sqrt{3} - 6\sqrt{3}$

$6y = 12\sqrt{3}$

$y = 2\sqrt{3}$

Question 26 (****)



The figure above shows solid right prism of height h cm.

The cross section of the prism is a circular sector of radius r cm, subtending an angle of 2 radians at the centre.

a) Given that the volume of the prism is 1000 cm^3, show clearly that

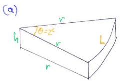
$$S = 2r^2 + \frac{4000}{r},$$

where $S \text{ cm}^2$ is the total surface area of the prism.

b) Hence determine the value of r and the value of h which make S **least**, fully justifying your answer.

$\boxed{r = 10}$, $\boxed{h = 10}$

(a)



$V = 1000$

$\left(\frac{1}{2}r^2 \times 2\right)h = 1000$ $\quad A = \frac{1}{2}r^2\theta^c$

$r^2h = 1000$

$\boxed{h = \frac{1000}{r^2}}$

THUS $S = 2(rh) + 2\left(\frac{1}{2}r^2\theta\right) + Lh$

$\Rightarrow S = 2rh + r^2\theta + Lh$

$\Rightarrow S = 2rh + 2r^2 + (r\theta)h$

$\Rightarrow S = 2rh + 2r^2 + 2rh$

$\Rightarrow S = 4rh + 2r^2$

$\Rightarrow S = 2r^2 + 4r\left(\frac{1000}{r^2}\right)$

$\Rightarrow S = 2r^2 + \frac{4000}{r}$ AS REQUIRED

(b)

$S = 2r^2 + 4000r^{-1}$

$\frac{dS}{dr} = 4r - 4000r^{-2}$

$\frac{dS}{dr} = 4r - \frac{4000}{r^2}$

SOLVE FOR ZERO

$4r - \frac{4000}{r^2} = 0$

$4r = \frac{4000}{r^2}$

$4r^3 = 4000$

$r^3 = 1000$

$r = 10$

& $h = \frac{1000}{r^2}$ $\therefore h = 10$

$\frac{d^2S}{dr^2} = 4 + 8000r^{-3}$

$\frac{d^2S}{dr^2} = 4 + \frac{8000}{r^3}$

$\left.\frac{d^2S}{dr^2}\right|_{r=10} = 4 + 8 = 12 > 0$

$\therefore$ INDEED LEAST

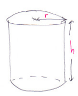
Question 27 (****)

A tank is in the shape of a closed right circular cylinder of radius r m and height h m.

The tank has a volume of 16π m^3 and is made of thin sheet metal.

Given the surface area of the tank is a minimum, determine the value of r and the value of h.

$\boxed{r=2}$, $\boxed{h=4}$



$V = 16\pi$

$\Rightarrow \pi r^2 h = 16\pi$

$\Rightarrow r^2 h = 16$

$\Rightarrow \boxed{h = \frac{16}{r^2}}$

$S = 2\pi r^2 + 2\pi r h$

$S = 2\pi r^2 + 2\pi r\left(\frac{16}{r^2}\right)$

$S = 2\pi r^2 + \frac{32\pi}{r}$

$S = 2\pi r^2 + 32\pi r^{-1}$

$\frac{dS}{dr} = 4\pi r - 32\pi r^{-2}$

$\boxed{\frac{dS}{dr} = 4\pi r - \frac{32\pi}{r^2}}$

GIVEN MINIMUM (NOT NEED TO CHECK)

$4\pi r - \frac{32\pi}{r^2} = 0$

$4\pi r^3 - 32\pi = 0$

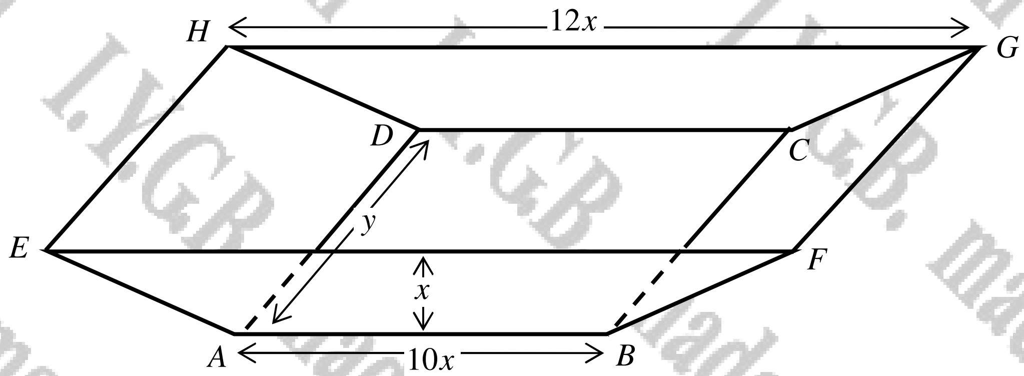
$r^3 = 8$

$r = 2$

& $h = \frac{16}{r^2}$

$h = 4$

Question 28 (****+)



The figure above shows the design of a baking tray with a horizontal rectangular base $ABCD$, measuring $10x$ cm by y cm.

The faces $ABFE$ and $DCGH$ are isosceles trapeziums, parallel to each other.

The lengths of the edges EF and HG are $12x$ cm.

The faces $ADHE$ and $BCGF$ are identical rectangles.

The height of the tray is x cm.

The capacity of the tray is 1980 cm^3.

a) Show that the surface area, A cm^2, of the tray is given by

$$A = 22x^2 + \frac{360}{x}\left(5+\sqrt{2}\right).$$

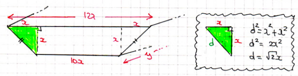
b) Determine the value of x for which A is stationary, showing that this value of x minimizes the value for A.

c) Calculate the minimum surface area of the tray.

$\boxed{x \approx 3.744}$, $\boxed{A_{\text{min}} \approx 925}$

[solution overleaf]

a)



<u>THE TOTAL SURFACE AREA</u>

$$\Rightarrow A = 10xy + 2\left[\frac{12x+10x}{2}x\right] + 2dy$$

FLOOR — TRAPEZOIDAL SIDES — SLOPING SIDES

$$\Rightarrow A = 10xy + 22x^2 + 2y\left(\sqrt{2}x\right)$$

$$\Rightarrow A = 22x^2 + 10xy + 2\sqrt{2}xy$$

$$\Rightarrow A = 22x^2 + \left(10 + 2\sqrt{2}\right)xy$$

$$\Rightarrow A = 22x^2 + \left(10 + 2\sqrt{2}\right)\left(\frac{180}{x}\right)$$

$$\Rightarrow A = 22x^2 + \frac{360}{x}\left(5 + \sqrt{2}\right)$$

AS REQUIRED

<u>CONSTRAINT ON CAPACITY</u>

$$V = 1980$$

$$1980 = \underbrace{\left(\frac{12x+10x}{2}\right)x}_{\text{AREA OF TRAPEZIUM ABFE}} \times y$$

$$11x^2y = 1980$$

$$x^2y = 180$$

$$x(xy) = 180$$

$$xy = \frac{180}{x}$$

b) <u>DIFFERENTIATE THE ABOVE EQUATION & SOLVING FOR ZERO</u>

$$\Rightarrow A = 22x^2 + 360\left(5+\sqrt{2}\right)x^{-1}$$

$$\Rightarrow \frac{dA}{dx} = 44x - 360\left(5+\sqrt{2}\right)x^{-2}$$

$$\Rightarrow \frac{dA}{dx} = 44x - \frac{360\left(5+\sqrt{2}\right)}{x^2}$$

<u>SOLVING FOR ZERO</u>

$$\Rightarrow 0 = 44x - \frac{360\left(5+\sqrt{2}\right)}{x^2}$$

$$\Rightarrow 0 = 44x^3 - 360\left(5+\sqrt{2}\right)$$

$$\Rightarrow x^3 = \frac{90\left(5+\sqrt{2}\right)}{11}$$

$$\Rightarrow x \approx 3.744$$

<u>USING THE SECOND DERIVATIVE</u>

$$\Rightarrow \frac{d^2A}{dx^2} = 44 + 720\left(5+\sqrt{2}\right)x^{-3}$$

$$\Rightarrow \frac{d^2A}{dx^2} = 132 > 0$$

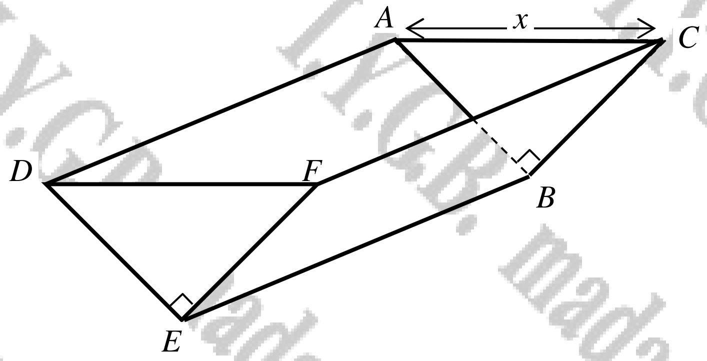
INDEED $x \approx 3.744$ MINIMIZES A

c) $A = 22x^2 + \frac{360}{x}\left(5+\sqrt{2}\right)$

$$\Rightarrow A_{MIN} = 22(3.744)^2 + \frac{360}{3.744}\left(5+\sqrt{2}\right)$$

$$\Rightarrow A_{MIN} \simeq 925 \text{ cm}^3$$

Question 29 (****+)



The figure above shows the design of a horse feeder which in the shape of a hollow, open topped triangular prism.

The triangular faces at the two ends of the feeder are isosceles and right angled, so that $AB = BC = DE = EF$ and $\widehat{ABC} = \widehat{DEF} = 90°$.

The triangular faces are vertical, and the edges AD, BE and CF are horizontal.

The capacity of the feeder is 4 m^3.

a) Show that the surface area, A m^2, of the feeder is given by

$$A = \frac{1}{2}x^2 + \frac{16\sqrt{2}}{x},$$

where x is the length of AC.

b) Determine by differentiation the value of x for which A is stationary, giving the answer in the form $k\sqrt{2}$, where k is an integer.

c) Show that the value of x found in part (**b**) gives the minimum value for A.

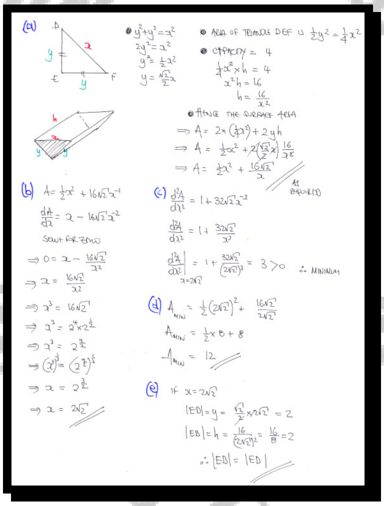
[continues overleaf]

[continued from overleaf]

d) Show, by exact calculations, that the minimum surface area of the feeder is 12 m^2.

e) Show further that the length ED is equal to the length EB.

$\boxed{x = 2\sqrt{2} \approx 2.82}$, $\boxed{ED = EB = 2}$



(a)

- $y^2 + y^2 = x^2$
  $2y^2 = x^2$
  $y^2 = \frac{1}{2}x^2$
  $y = \frac{\sqrt{2}}{2}x$
- AREA OF TRIANGLE DEF IS $\frac{1}{2}y^2 = \frac{1}{4}x^2$
- CAPACITY $= 4$
  $\frac{1}{4}x^2 \times h = 4$
  $x^2 h = 16$
  $h = \frac{16}{x^2}$
- HENCE THE SURFACE AREA
  $\Rightarrow A = 2 \times \left(\frac{1}{4}x^2\right) + 2yh$
  $\Rightarrow A = \frac{1}{2}x^2 + 2\left(\frac{\sqrt{2}}{2}x\right)\frac{16}{x^2}$
  $\Rightarrow A = \frac{1}{2}x^2 + \frac{16\sqrt{2}}{x}$ AS REQUIRED

(b) $A = \frac{1}{2}x^2 + 16\sqrt{2}x^{-1}$

$\frac{dA}{dx} = x - 16\sqrt{2}x^{-2}$

SOLVE FOR ZERO

$\Rightarrow 0 = x - \frac{16\sqrt{2}}{x^2}$

$\Rightarrow x = \frac{16\sqrt{2}}{x^2}$

$\Rightarrow x^3 = 16\sqrt{2}$

$\Rightarrow x^3 = 2^4 \times 2^{\frac{1}{2}}$

$\Rightarrow x^3 = 2^{\frac{9}{2}}$

$\Rightarrow (x^3)^{\frac{1}{3}} = \left(2^{\frac{9}{2}}\right)^{\frac{1}{3}}$

$\Rightarrow x = 2^{\frac{3}{2}}$

$\Rightarrow x = 2\sqrt{2}$

(c) $\frac{d^2A}{dx^2} = 1 + 32\sqrt{2}x^{-3}$

$\frac{d^2A}{dx^2} = 1 + \frac{32\sqrt{2}}{x^3}$

$\left.\frac{d^2A}{dx^2}\right|_{x=2\sqrt{2}} = 1 + \frac{32\sqrt{2}}{(2\sqrt{2})^3} = 3 > 0$ $\therefore$ MINIMUM

(d) $A_{MIN} = \frac{1}{2}(2\sqrt{2})^2 + \frac{16\sqrt{2}}{2\sqrt{2}}$

$A_{MIN} = \frac{1}{2} \times 8 + 8$

$A_{MIN} = 12$

(e) IF $x = 2\sqrt{2}$

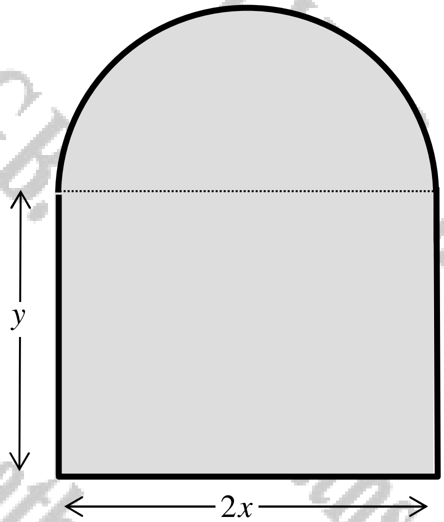
$|ED| = y = \frac{\sqrt{2}}{2} \times 2\sqrt{2} = 2$

$|EB| = h = \frac{16}{(2\sqrt{2})^2} = \frac{16}{8} = 2$

$\therefore |EB| = |ED|$

Question 30 **(****+)**

The figure below shows the design of a window which is the shape of a semicircle attached to rectangle.



The diameter of the semicircle is $2x$ metres and is attached to one side of the rectangle also measuring $2x$ meters. The other side of the rectangle is y metres.

The total area of the window is 2 m^2.

a) Show that perimeter, P m, is given by

$$P = \frac{1}{2}(4+\pi)x + \frac{2}{x}.$$

b) Determine by differentiation an exact value of x for which P is stationary.

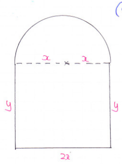
[continues overleaf]

[continued from overleaf]

c) Show that the value of x found in part (b) gives the minimum value for P.

d) Show that when P takes a minimum value $x = y$.

$$\boxed{x = \frac{2}{\sqrt{\pi + 4}} \approx 0.748}$$



(a)

- $A = 2$

$\Rightarrow 2xy + \frac{1}{2} \times \pi x^2 = 2$

$\Rightarrow 4xy + \pi x^2 = 4$

$\Rightarrow \boxed{y = \frac{4 - \pi x^2}{4x}}$

- $P = 2y + 2x + \frac{1}{2}(2\pi x)$

$\Rightarrow P = 2y + 2x + \pi x$

$\Rightarrow P = 2 \times \left(\frac{4 - \pi x^2}{4x}\right) + 2x + \pi x$

$\Rightarrow P = \frac{4 - \pi x^2}{2x} + 2x + \pi x$

$\Rightarrow P = \frac{2}{x} - \frac{\pi x}{2} + 2x + \pi x$

$\Rightarrow P = \frac{\pi x}{2} + 2x + \frac{2}{x}$

$\Rightarrow P = \frac{1}{2}x(\pi + 4) + \frac{2}{x}$ // AS REQUIRED

(b) $P = \frac{1}{2}(\pi + 4)x + 2x^{-1}$

$\boxed{\frac{dP}{dx} = \frac{1}{2}(\pi + 4) - 2x^{-2}}$

SOLVE FOR ZERO

$\frac{1}{2}(\pi + 4) - \frac{2}{x^2} = 0$

$\frac{1}{2}(\pi + 4) = \frac{2}{x^2}$

$x^2 = \frac{2}{\frac{1}{2}(\pi + 4)}$

$x^2 = \frac{4}{\pi + 4}$

$x = \frac{2}{\sqrt{\pi + 4}}$ // $(x > 0)$

(c) $\frac{d^2P}{dx^2} = 4x^{-3} = \frac{4}{x^3}$

$\left.\frac{d^2P}{dx^2}\right|_{x = \frac{2}{\sqrt{\pi + 4}}} = \frac{4}{\left(\frac{2}{\sqrt{\pi + 4}}\right)^3} > 0$

$\therefore$ INDEED A MINIMUM //

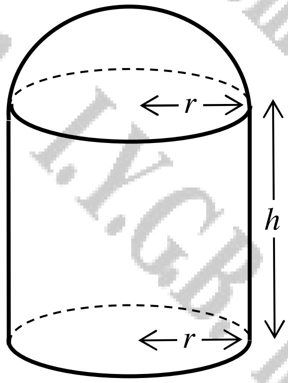
(d) $y = \frac{4 - \pi x^2}{4x}$

$y = \frac{4 - \pi\left(\frac{4}{\pi + 4}\right)}{4 \times \left(\frac{2}{\sqrt{\pi + 4}}\right)} = \frac{4 - \frac{4\pi}{\pi + 4}}{\frac{8}{(\pi + 4)^{\frac{1}{2}}}}$

MULTIPLY TOP & BOTTOM OF THE FRACTION BY $\pi + 4$

$y = \frac{4(\pi + 4) - 4\pi}{8(\pi + 4)^{\frac{1}{2}}} = \frac{\cancel{4\pi} + 16 - \cancel{4\pi}}{8(\pi + 4)^{\frac{1}{2}}} = \frac{2}{(\pi + 4)^{\frac{1}{2}}} = x$ // AS REQUIRED

Question 31 (****+)



The figure above shows a hollow container consisting of a right circular cylinder of radius r cm and of height h cm joined to a hemisphere of radius r cm.

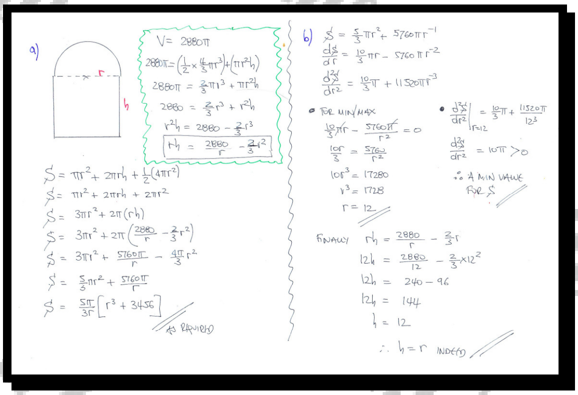
The cylinder is open on one of the circular ends and the hemisphere is also open on its circular base. The cylinder is joined to the hemisphere at their open ends so that the resulting object is completely sealed.

a) Given that volume of the container is exactly $2880\pi \text{ cm}^3$, show clearly that the total surface area of the container, $S \text{ cm}^2$, is given by

$$S = \frac{5\pi}{3r}\left(r^3 + 3456\right).$$

b) Show further than when S is minimum, $r = h$.

proof



a)

$V = 2880\pi$

$2880\pi = \left(\frac{1}{2} \times \frac{4}{3}\pi r^3\right) + \left(\pi r^2 h\right)$

$2880\pi = \frac{2}{3}\pi r^3 + \pi r^2 h$

$2880 = \frac{2}{3}r^3 + r^2 h$

$r^2 h = 2880 - \frac{2}{3}r^3$

$\boxed{rh = \frac{2880}{r} - \frac{2}{3}r^2}$

$S = \pi r^2 + 2\pi rh + \frac{1}{2}(4\pi r^2)$

$S = \pi r^2 + 2\pi rh + 2\pi r^2$

$S = 3\pi r^2 + 2\pi(rh)$

$S = 3\pi r^2 + 2\pi\left(\frac{2880}{r} - \frac{2}{3}r^2\right)$

$S = 3\pi r^2 + \frac{5760\pi}{r} - \frac{4\pi}{3}r^2$

$S = \frac{5}{3}\pi r^2 + \frac{5760\pi}{r}$

$S = \frac{5\pi}{3r}\left[r^3 + 3456\right]$ AS REQUIRED

b)

$S = \frac{5}{3}\pi r^2 + 5760\pi r^{-1}$

$\frac{dS}{dr} = \frac{10}{3}\pi r - 5760\pi r^{-2}$

$\frac{d^2S}{dr^2} = \frac{10}{3}\pi + 11520\pi r^{-3}$

• FOR MIN/MAX

$\frac{10}{3}\cancel{\pi} r - \frac{5760\cancel{\pi}}{r^2} = 0$

$\frac{10r}{3} = \frac{5760}{r^2}$

$10r^3 = 17280$

$r^3 = 1728$

$r = 12$

• $\left.\frac{d^2S}{dr^2}\right|_{r=12} = \frac{10}{3}\pi + \frac{11520\pi}{12^3}$

$\frac{d^2S}{dr^2} = 10\pi > 0$

∴ A MIN VALUE FOR S

FINALLY $rh = \frac{2880}{r} - \frac{2}{3}r$

$12h = \frac{2880}{12} - \frac{2}{3} \times 12^2$

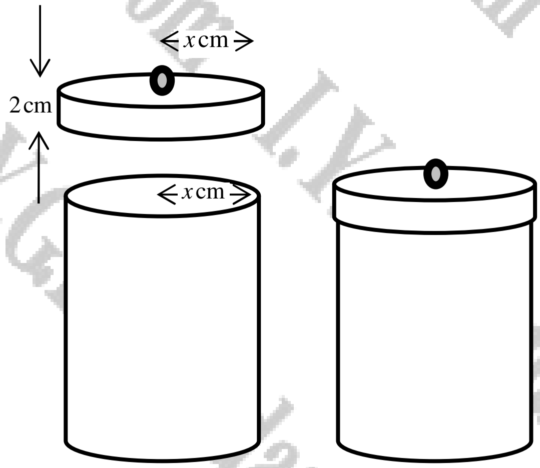
$12h = 240 - 96$

$12h = 144$

$h = 12$

$\therefore h = r$ INDEED

Question 32 (****+)



The figure above shows the design of coffee jar with a "push on" lid.

The jar is in the shape of a right circular cylinder of radius x cm. It is fitted with a lid of width 2 cm, which fits tightly on the top of the jar, so it may be assumed that it has the same radius as the jar.

The jar and its lid is made of sheet metal and there is no wastage.

The total metal used to make the jar and its lid is 190π cm^2.

(*This figure does not include the handle of the lid which is made of different material.*)

a) Show that volume of the jar, V cm^3, is given by

$$V = \pi\left(95x - 2x^2 - x^3\right).$$

b) Determine by differentiation the value of x for which V is stationary.

[continues overleaf]

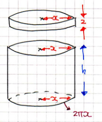
[continued from overleaf]

c) Show that the value of x found in part **(b)** gives the maximum value for V.

d) Hence determine the maximum volume of the jar.

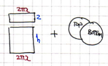
$\boxed{x = 5 \text{ cm}}$, $\boxed{V_{\max} = 300\pi \approx 942 \text{ cm}^3}$

a)



CONSTRAINT ON SURFACE AREA

$\Rightarrow A = 190\pi$



$\Rightarrow$ TOP + BOTTOM $= 190\pi$

$\Rightarrow \cancel{2\pi}x(h+2) + 2(\cancel{\pi}x^2) = 190\cancel{\pi}$

$\Rightarrow 2x(h+2) + 2x^2 = 190$

$\Rightarrow x(h+2) + x^2 = 95$

$\Rightarrow xh + 2x + x^2 = 95$

$\Rightarrow xh = 95 - 2x - x^2$

VOLUME OF THE JAR

$\Rightarrow V = \pi r^2 h$

$\Rightarrow V = \pi x^2 h$

$\Rightarrow V = \pi x(xh)$

$\Rightarrow V = \pi x(95 - 2x - x^2)$

$\Rightarrow V = \pi(95x - 2x^2 - x^3)$

AS REQUIRED

b) DIFFERENTIATE & SOLVE FOR ZERO

$\Rightarrow V = \pi(95x - 2x^2 - x^3)$

$\Rightarrow \frac{dV}{dx} = \pi(95 - 4x - 3x^2)$

$\Rightarrow 0 = \cancel{\pi}(95 - 4x - 3x^2)$

$\Rightarrow 3x^2 - 4x - 95 = 0$

$\Rightarrow (3x + 19)(x - 5) = 0$

$\Rightarrow x = \left\langle \begin{array}{l} 5 \\ \cancel{-\frac{19}{3}} \end{array} \right.$

c) USING THE 2ND DERIVATIVE TEST

$\frac{dV}{dx} = \pi(95 - 4x - 3x^2)$

$\frac{d^2V}{dx^2} = \pi(-4 - 6x)$

$\left.\frac{d^2V}{dx^2}\right|_{x=5} = -34\pi < 0$ INDEED A MAXIMUM

d) $V = \pi(95x - 2x^2 - x^3)$

$V_{MAX} = \pi(95 \times 5 - 2 \times 5^2 - 5^3)$

$V_{MAX} = 300\pi \approx 943 \text{ cm}^3$

Question 33 (****+)

The profit of a small business, £P is modelled by the equation

$$P = \frac{\left(54x + 6y - xy - 324\right)^2}{3x},$$

where x and y are positive variables associated with the running of the company.

It is further known that x and y constrained by the relation

$$3x + y = 54.$$

a) Show clearly that

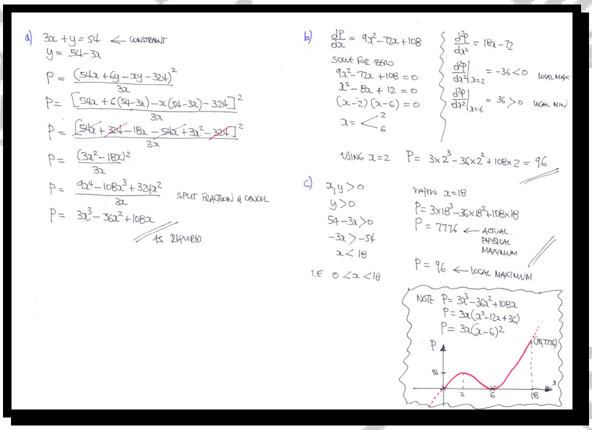
$$P = 108x - 36x^2 + 3x^3.$$

b) Hence show that the stationary value of P produces a maximum value of £96.

The owner is very concerned about the very small profit and shows the calculations to a mathematician. The mathematician agrees that the calculations are correct but he asserts that the profit is substantially higher.

c) Explain, by calculations, the mathematician's reasoning.

proof



a) $3x + y = 54$ ← CONSTRAINT

$y = 54 - 3x$

$P = \frac{(54x + 6y - xy - 324)^2}{3x}$

$P = \frac{\left[54x + 6(54-3x) - x(54-3x) - 324\right]^2}{3x}$

$P = \frac{\left[\cancel{54x} + \cancel{324} - 18x - \cancel{54x} + 3x^2 - \cancel{324}\right]^2}{3x}$

$P = \frac{(3x^2 - 18x)^2}{3x}$

$P = \frac{9x^4 - 108x^3 + 324x^2}{3x}$ SPLIT FRACTION & CANCEL

$P = 3x^3 - 36x^2 + 108x$

AS REQUIRED

b) $\frac{dP}{dx} = 9x^2 - 72x + 108$

SOLVE FOR ZERO

$9x^2 - 72x + 108 = 0$

$x^2 - 8x + 12 = 0$

$(x-2)(x-6) = 0$

$x = 2, 6$

$\frac{d^2P}{dx^2} = 18x - 72$

$\left.\frac{d^2P}{dx^2}\right|_{x=2} = -36 < 0$ LOCAL MAX

$\left.\frac{d^2P}{dx^2}\right|_{x=6} = 36 > 0$ LOCAL MIN

USING $x = 2$ $P = 3 \times 2^3 - 36 \times 2^2 + 108 \times 2 = 96$

c) $x, y > 0$

$y > 0$

$54 - 3x > 0$

$-3x > -54$

$x < 18$

I.E $0 < x < 18$

TAKING $x = 18$

$P = 3 \times 18^3 - 36 \times 18^2 + 108 \times 18$

$P = 7776$ ← ACTUAL PHYSICAL MAXIMUM

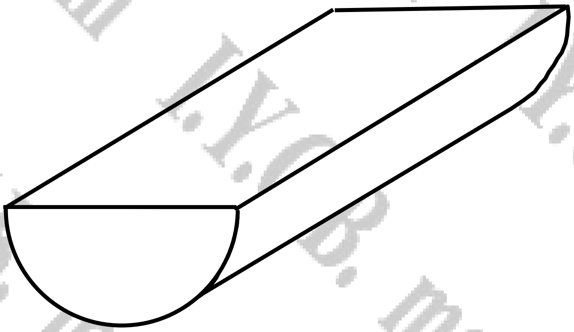
$P = 96$ ← LOCAL MAXIMUM

NOTE $P = 3x^3 - 36x^2 + 108x$

$P = 3x(x^2 - 12x + 36)$

$P = 3x(x-6)^2$

Question 34 (****+)

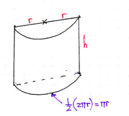


The figure above shows a solid prism, which is in the shape a right semi-circular cylinder.

The total surface area of the 4 faces of the prism is $\sqrt[3]{27\pi}$.

Given that the measurements of the prism are such so that its volume is maximized, find in exact simplified form the volume of the prism.

$$\boxed{V_{\max} = \frac{\pi}{\pi + 2}}$$



- LET THE RADIUS BE r & THE VERTICAL HEIGHT h
- CONSTRAINT ON SURFACE AREA

$$\Rightarrow \pi r^2 + \pi r h + 2rh = \sqrt[3]{27\pi}$$
$$\Rightarrow \pi r h + 2rh = \sqrt[3]{27\pi} - \pi r^2$$
$$\Rightarrow rh(\pi + 2) = 3\pi^{\frac{1}{3}} - \pi r^2$$
$$\Rightarrow \boxed{rh = \frac{3\pi^{\frac{1}{3}} - \pi r^2}{\pi + 2}}$$

- NOW LOOKING AT THE VOLUME

$$\Rightarrow V = \frac{1}{2}(\pi r^2 h) = \frac{1}{2}\pi r(rh) = \frac{1}{2}\pi r\left(\frac{3\pi^{\frac{1}{3}} - \pi r^2}{\pi + 2}\right)$$
$$\Rightarrow V = \frac{\pi r(3\pi^{\frac{1}{3}} - \pi r^2)}{2(\pi + 2)}$$
$$\Rightarrow \boxed{V = \frac{\pi}{2(\pi+2)}\left[3\pi^{\frac{1}{3}} r - \pi r^3\right]}$$

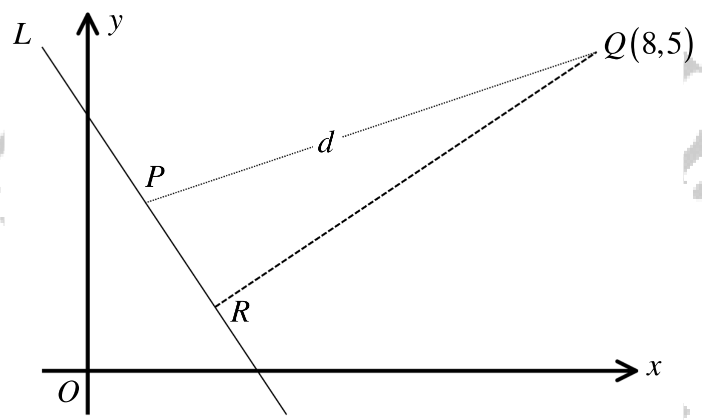
- DIFFERENTIATE & SOLVE FOR ZERO

$$\Rightarrow \frac{dV}{dr} = \frac{\pi}{2(\pi+2)}\left[3\pi^{\frac{1}{3}} - 3\pi r^2\right]$$
$$\Rightarrow 0 = \cancel{\frac{\pi}{2(\pi+2)}}\left[3\pi^{\frac{1}{3}} - 3\pi r^2\right]$$
$$\Rightarrow \cancel{3}\pi^{\frac{1}{3}} - \cancel{3}\pi r^2 = 0$$
$$\Rightarrow \pi^{\frac{1}{3}} = \pi r^2$$
$$\Rightarrow r^2 = \pi^{-\frac{2}{3}}$$
$$\Rightarrow r^2 = \frac{1}{\pi^{2/3}}$$
$$\Rightarrow r = +\frac{1}{\pi^{1/3}}$$

FINALLY TO OBTAIN THE MAXIMUM VOLUME

$$\Rightarrow V = \frac{\pi}{2(\pi+2)}\left[3\pi^{\frac{1}{3}} r - \pi r^3\right]$$
$$\Rightarrow V = \frac{\pi r}{2(\pi+2)}\left[3\pi^{\frac{1}{3}} - \pi r^2\right]$$
$$\Rightarrow V_{MAX} = \frac{\pi}{2(\pi+2)}\left(\frac{1}{\pi^{1/3}}\right)\left[3\pi^{\frac{1}{3}} - \pi \times \frac{1}{\pi^{2/3}}\right]$$
$$\Rightarrow V_{MAX} = \frac{\pi^{\frac{2}{3}}}{2(\pi+2)}\left[3\pi^{\frac{1}{3}} - \pi^{\frac{1}{3}}\right]$$
$$\Rightarrow V_{MAX} = \frac{\pi^{\frac{2}{3}}}{2(\pi+2)} \times 2\pi^{\frac{1}{3}}$$
$$\Rightarrow V_{MAX} = \frac{2\pi}{2(\pi+2)}$$
$$\Rightarrow V_{MAX} = \frac{\pi}{\pi+2}$$

Question 35 (*****)



The straight line L has equation $3x+2y=8$.

The point $P(x,y)$ lies on L and the point $Q(8,5)$ lies outside L. The point R lies on L so that QR is perpendicular to L. The length PQ is denoted by d.

a) Show clearly that

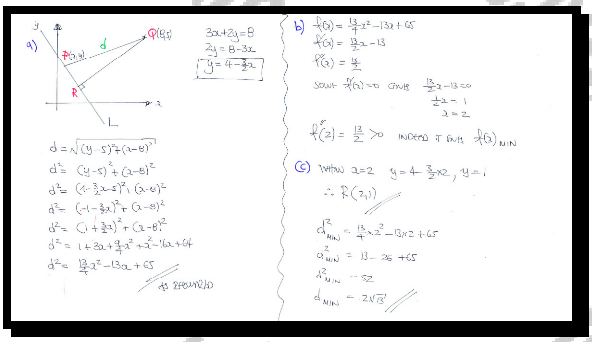
$$d^2 = 65 - 13x + \frac{13}{4}x^2 .$$

Let $f(x) = 65 - 13x + \frac{13}{4}x^2$.

b) Use **differentiation** to find the stationary value of $f(x)$, fully justifying that this value of x minimizes the value of $f(x)$.

c) State the coordinates of R and find, as an exact surd, the shortest distance of the point Q from L

$\boxed{x=2}$, $\boxed{R(2,1)}$, $\boxed{\sqrt{52}}$



Question 36 (*****)

An **open** box is to be made of thin sheet metal, in the shape of a cuboid with a square base of length x and height h.

The box is to have a **fixed** volume.

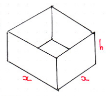
Determine the value of x, in terms of h, when the surface area of the box is minimum.

proof

THE BOX HAS TO HAVE A FIXED VOLUME

$x^2h = \text{constant volume}$

$x^2h = V$



THE SURFACE AREA OF THE BOX IS GIVEN BY

$$A = x^2 + 4xh$$
$$A = x^2 + \frac{4x^2h}{x}$$
$$A = x^2 + \frac{4V}{x}$$
$$A = x^2 + 4Vx^{-1}$$

DIFFERENTIATING & SETTING TO ZERO

$$\frac{dA}{dx} = 2x - 4Vx^{-2}$$
$$0 = 2x - \frac{4V}{x^2}$$
$$\frac{4V}{x^2} = 2x$$
$$2x^3 = 4V$$
$$x^3 = 2V$$
$$x = \sqrt[3]{2V}$$

CHECK WHETHER THIS VALUE OF x PRODUCES A MINIMUM OR MAXIMUM

$$\frac{d^2A}{dx^2} = 2 + 8Vx^{-3} = 2 + \frac{8V}{x^3}$$
$$\left.\frac{d^2A}{dx^2}\right|_{x=\sqrt[3]{2V}} = 2 + \frac{8V}{2V} = 6 > 0$$

INDEED THIS VALUE OF x MINIMIZES

USING $x^2h = V$

$$\Rightarrow h = \frac{V}{x^2}$$
$$\Rightarrow h = \frac{V}{(\sqrt[3]{2V})^2} = \frac{V}{\left[(2V)^{\frac{1}{3}}\right]^2} = \frac{V}{(2V)^{\frac{2}{3}}}$$
$$\Rightarrow h = \frac{V}{2^{\frac{2}{3}}V^{\frac{2}{3}}} = \frac{V^{\frac{1}{3}}}{2^{\frac{2}{3}}} = \frac{V^{\frac{1}{3}} \times 2^{\frac{1}{3}}}{2^{\frac{2}{3}} \times 2^{\frac{1}{3}}}$$
$$\Rightarrow h = \frac{2^{\frac{1}{3}}V^{\frac{1}{3}}}{2} = \frac{(2V)^{\frac{1}{3}}}{2}$$

HENCE $h = \frac{x}{2}$

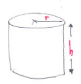
$\therefore$ $x = 2h$ WHEN SURFACE IS MINIMUM

Question 37 (*****)

A solid right circular cylinder of fixed volume has radius r and height h.

Show clearly that when the surface area of the cylinder is minimum $h : r = 2 : 1$.

proof



$\pi r^2 h = V$ (CONSTANT)

$\boxed{\pi r h = \frac{V}{r}}$

$$A = 2\pi r^2 + 2\pi r h$$
$$\Rightarrow A = 2\pi r^2 + \frac{2V}{r}$$
$$\Rightarrow \frac{dA}{dr} = 4\pi r - \frac{2V}{r^2}$$
$$\Rightarrow \frac{d^2A}{dr^2} = 4\pi + \frac{4V}{r^3}$$

Now $\frac{dA}{dr} = 0$

$$\Rightarrow 4\pi r - \frac{2V}{r^2} = 0$$
$$\Rightarrow 4\pi r^3 - 2V = 0$$
$$\Rightarrow 4\pi r^3 = 2V$$
$$\Rightarrow r^3 = \frac{V}{2\pi}$$
$$\boxed{r = \left(\frac{V}{2\pi}\right)^{\frac{1}{3}}}$$

- FIRST CHECK WHETHER IT PRODUCES MIN/MAX

$$\left.\frac{d^2A}{dr^2}\right|_{r=\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}} = 4\pi + \frac{4V}{\frac{V}{2\pi}} = 12\pi > 0$$

INDEED A MINIMUM

- $h = \frac{V}{\pi r^2}$

$$\frac{h}{r} = \frac{V}{\pi r^3}$$
$$\frac{h}{r} = \frac{V}{\pi\left(\frac{V}{2\pi}\right)}$$
$$\frac{h}{r} = 2$$

$\therefore h : r = 2 : 1$

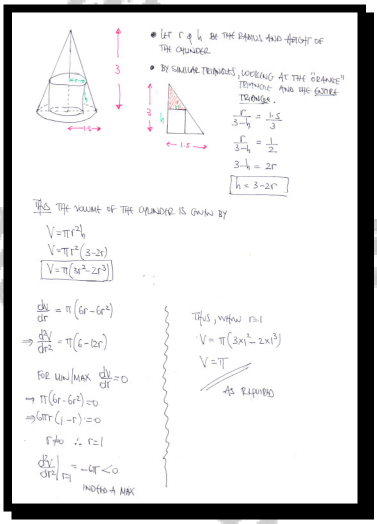
Question 38 (*****)

A solid right circular cylinder is to be cut out of a solid right circular cone, whose radius is 1.5 m and its height is 3 m.

The axis of symmetry of the cone coincides with the axis of symmetry of the cylinder which passes though its circular ends. The circumference of one end of the cylinder is in contact with the curved surface of the cone and the other end of the cylinder lies on the base of the cone.

Show that the maximum volume of the cylinder to be cut out is $\pi \text{ m}^3$.

proof

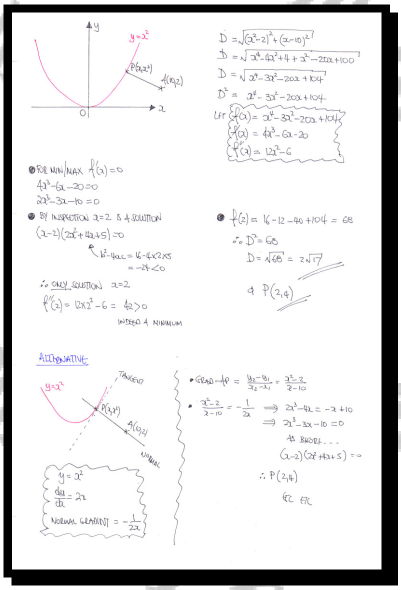


Question 39 (*****)

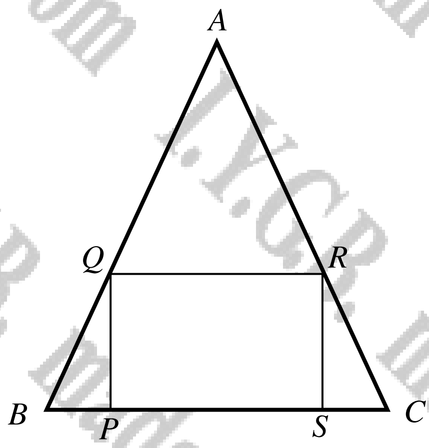
The point P lies on the curve with equation $y = x^2$, so that its distance from the point $A(10,2)$ is **least**.

Determine the coordinates of P and the distance AP.

$\boxed{P(2,4)}$, $\boxed{d_{\text{min}} = 2\sqrt{17}}$



Question 40 (*****)

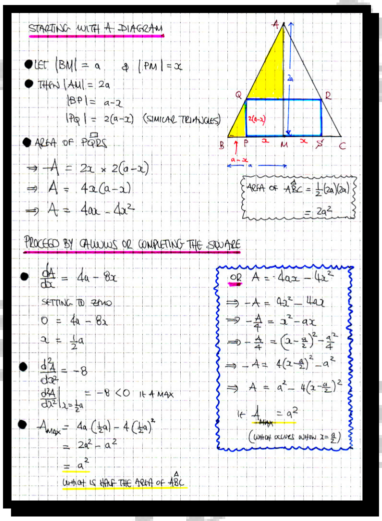


The figure above shows an isosceles triangle ABC, where $|AB| = |AC|$ and a rectangle $PQRS$ drawn inside the triangle.

The points P and S lie on BC, the point Q lies on AB and the point R lies on AC.

Given that the base of the triangle BC is equal in length to its height, show clearly that the largest area that the rectangle $PQRS$ can achieve is $\frac{1}{2}$ the area of the triangle ABC.

proof



Question 41 (*****)

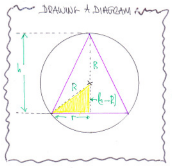
A right circular cone of radius r and height h is to be cut out of a sphere of radius R.

It is a requirement that the circumference of the base of the cone and its vertex lie on the surface of the sphere.

Determine, in exact form in terms of R, and with full justification, the maximum volume of the cone that can be cut out of this sphere.

$$\boxed{V_{\max} = \frac{32\pi R^3}{81}}$$

DRAWING A DIAGRAM



- LOOKING AT THE YELLOW TRIANGLE

$\Rightarrow r^2 + (h-R)^2 = R^2$

$\Rightarrow r^2 + h^2 - 2hR + \cancel{R^2} = \cancel{R^2}$

$\Rightarrow r^2 = 2hR - h^2$

- VOLUME OF THE CONE

$\Rightarrow V = \frac{1}{3}\pi r^2 h$

$\Rightarrow V = \frac{1}{3}\pi (2hR - h^2)h$

$\Rightarrow V = \frac{1}{3}\pi (2Rh^2 - h^3)$

- BY DIFFERENTIATION

$\boxed{\frac{dV}{dh} = \frac{1}{3}\pi(4Rh - 3h^2)}$

- SOLVING FOR ZERO

$\Rightarrow 4Rh - 3h^2 = 0$

$\Rightarrow h(4R - 3h) = 0$

$\Rightarrow h = \begin{cases} 0 \\ \frac{4}{3}R \end{cases}$

- $\boxed{\frac{d^2V}{dh^2} = \frac{1}{3}\pi(4R - 6h)}$

$\left.\frac{d^2V}{dh^2}\right|_{h=0} = \frac{4}{3}\pi R > 0$ (LOCAL MINIMUM)

$\left.\frac{d^2V}{dh^2}\right|_{h=\frac{4}{3}R} = \frac{1}{3}\pi\left(4R - 6\times\frac{4}{3}h\right) = -\frac{4}{3}\pi R < 0$

LOCAL MAXIMUM

- $V_{MAX} = \frac{1}{3}\pi\left[2R\times\frac{16}{9}R^2 - \frac{64}{27}R^3\right] = \frac{1}{3}\pi\left[\frac{32}{9}R^3 - \frac{64}{27}R^3\right]$

$= \frac{1}{3}\pi\left[\frac{96}{27}R^3 - \frac{64}{27}R^3\right] = \frac{1}{3}\pi \times \frac{32}{27}R^3 = \frac{32}{81}\pi R^3$

Question 42 (*****)

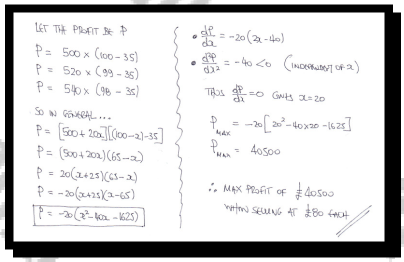
A mobile phone wholesaler buys a certain brand of phone for £35 a unit and sells it to shops for £100 a unit.

In a typical week the wholesaler expects to sell 500 of these phones.

Research however showed that on a typical week for every £1 reduced of the selling price of this phone, an extra 20 sales can be achieved.

Determine the **selling** price for this phone if the weekly profit is to be maximized, and find this maximum weekly profit.

$\boxed{\text{£80, maximum profit £40500}}$



LET THE PROFIT BE P

$P = 500 \times (100 - 35)$

$P = 520 \times (99 - 35)$

$P = 540 \times (98 - 35)$

SO IN GENERAL...

$P = [500 + 20x][(100 - x) - 35]$

$P = (500 + 20x)(65 - x)$

$P = 20(x + 25)(65 - x)$

$P = -20(x + 25)(x - 65)$

$\boxed{P = -20(x^2 - 40x - 1625)}$

• $\frac{dP}{dx} = -20(2x - 40)$

• $\frac{d^2P}{dx^2} = -40 < 0$ (INDEPENDENT OF x)

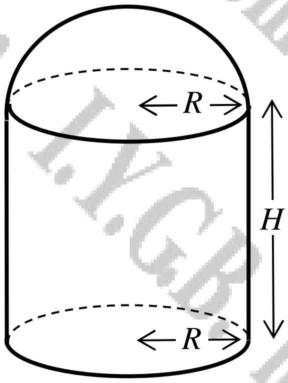
THUS $\frac{dP}{dx} = 0$ GIVES $x = 20$

$P_{MAX} = -20[20^2 - 40 \times 20 - 1625]$

$P_{MAX} = 40500$

$\therefore$ MAX PROFIT OF £40500 WHEN SELLING AT £80 EACH

Question 43 (*****)



The figure above shows a hollow container consisting of a right circular cylinder of radius R and of height H joined to a hemisphere of radius R.

The cylinder is open on one of the circular ends and the hemisphere is also open on its circular base. The cylinder is joined to the hemisphere at their open ends so that the resulting object is completely sealed.

Given that volume of the container is V, show the surface area of the container is minimised when $R = H$, and hence show further that this minimum surface area is

$$\sqrt[3]{45\pi V^2}.$$

proof

START WITH AN EXPRESSION FOR THE VOLUME OF THE COMPOSITE

$$V = \frac{1}{2}\left(\frac{4}{3}\pi R^3\right) + \pi R^2 H$$
$$V = \frac{2}{3}\pi R^3 + \pi R^2 H$$

NEXT AN EXPRESSION FOR THE VARIABLE SURFACE AREA

$$A = \pi R^2 + 2\pi RH + \frac{1}{2}(4\pi R^2)$$
$$A = \pi R^2 + 2\pi RH + 2\pi R^2$$
$$A = 3\pi R^2 + 2\pi RH$$

MANIPULATE THE VOLUME EXPRESSION AS FOLLOWS

$$V = \frac{2}{3}\pi R^3 + \pi R^2 H \qquad \times 2$$
$$2V = \frac{4}{3}\pi R^3 + 2\pi R^2 H \qquad \div R$$
$$\frac{2V}{R} = \frac{4}{3}\pi R^2 + 2\pi RH$$
$$2\pi RH = \frac{2V}{R} - \frac{4}{3}\pi R^2$$

FINALLY SUBSTITUTE THE ABOVE INTO THE SURFACE AREA EXPRESSION

$$A = 3\pi R^2 + 2\pi RH$$
$$A = 3\pi R^2 + \frac{2V}{R} - \frac{4}{3}\pi R^2$$
$$A = \frac{5}{3}\pi R^2 + \frac{2V}{R}$$

(NOTE THAT R MAY VARY BUT V IS A CONSTANT)

NEXT DIFFERENTIATE A W.R.T R & SOLVE FOR ZERO

$$\frac{dA}{dR} = \frac{10}{3}\pi R - \frac{2V}{R^2}$$
$$0 = \frac{10}{3}\pi R - \frac{2V}{R^2}$$
$$\frac{10}{3}\pi R = \frac{2V}{R^2}$$
$$10\pi R^3 = 6V$$
$$R^3 = \frac{3V}{5\pi} \qquad \text{IE} \quad R = \left(\frac{3V}{5\pi}\right)^{\frac{1}{3}}$$

JUSTIFY THE NATURE

$$\frac{d^2A}{dR^2} = \frac{10}{3}\pi + \frac{4V}{R^3}$$
$$\left.\frac{d^2A}{dR^2}\right|_{R^3 = \frac{3V}{5\pi}} = \frac{10}{3}\pi + \frac{4V}{\frac{3V}{5\pi}} = \frac{10}{3}\pi + \frac{20}{3}\pi = 10\pi > 0$$

INDEED MINIMIZES

NOW RETURNING TO THE CONSTRAINT

$$2\pi RH = \frac{2V}{R} - \frac{4}{3}\pi R^2 \qquad \text{WITH } R^3 = \frac{3V}{5\pi}$$
$$2\pi R^2 H = 2V - \frac{4}{3}\pi R^3$$
$$6\pi R^2 H = 6V - 4\pi R^3$$
$$3\pi R^2 H = 3V - 2\pi R^3 \qquad \text{NOW } V = \frac{5\pi R^3}{3}$$
$$3\pi R^2 H = 3\left(\frac{5\pi R^3}{3}\right) - 2\pi R^3$$
$$3\pi R^2 H = 5\pi R^3 - 2\pi R^3$$
$$3\pi R^2 H = 3\pi R^3$$
$$H = R$$

AS REQUIRED

FINALLY TO FIND THE MINIMUM SURFACE AREA

$$A = \frac{5}{3}\pi R^2 + \frac{2V}{R} = \frac{1}{R}\left[\frac{5}{3}\pi R^3 + 2V\right]$$
$$A_{MIN} = \frac{1}{R}\left[\frac{5}{3}\pi\left(\frac{3V}{5\pi}\right) + 2V\right]$$
$$A_{MIN} = \frac{1}{R}\left[V + 2V\right]$$
$$A_{MIN} = \frac{3V}{R}$$
$$A_{MIN} = 3V \times R^{-1}$$
$$A_{MIN} = 3V \times \left(\frac{5\pi}{3V}\right)^{\frac{1}{3}}$$
$$A_{MIN} = (27V^3)^{\frac{1}{3}} \times \left(\frac{5\pi}{3V}\right)^{\frac{1}{3}}$$
$$A_{MIN} = \left(27V^3 \times \frac{5\pi}{3V}\right)^{\frac{1}{3}}$$
$$A_{MIN} = (45\pi V^2)^{\frac{1}{3}}$$
$$A_{MIN} = \sqrt[3]{45\pi V^2}$$

AS REQUIRED

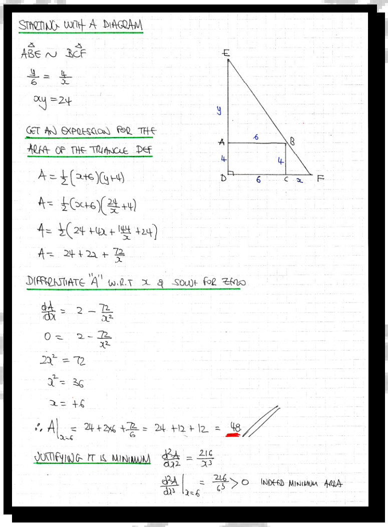
Question 44 (*****)

A rectangle $ABCD$ is such so that $|DC| = 6$ and $|DA| = 4$.

The side DA is extended to the point E and the side DC is extended to the point F so that EBF is a straight line.

Determine, with full justification, the minimum area of the triangle EDF.

$\boxed{A_{\text{min}} = 48}$



STARTING WITH A DIAGRAM

$\hat{ABE} \sim \hat{BCF}$

$\frac{y}{6} = \frac{4}{x}$

$xy = 24$

GET AN EXPRESSION FOR THE AREA OF THE TRIANGLE DEF

$A = \frac{1}{2}(x+6)(y+4)$

$A = \frac{1}{2}(x+6)\left(\frac{24}{x}+4\right)$

$A = \frac{1}{2}\left(24 + 4x + \frac{144}{x} + 24\right)$

$A = 24 + 2x + \frac{72}{x}$

DIFFERENTIATE "A" W.R.T x & SOLVE FOR ZERO

$\frac{dA}{dx} = 2 - \frac{72}{x^2}$

$0 = 2 - \frac{72}{x^2}$

$2x^2 = 72$

$x^2 = 36$

$x = +6$

$\therefore A\big|_{x=6} = 24 + 2 \times 6 + \frac{72}{6} = 24 + 12 + 12 = 48$

JUSTIFYING IT IS MINIMUM $\frac{d^2A}{dx^2} = \frac{216}{x^3}$

$\frac{d^2A}{dx^2}\bigg|_{x=6} = \frac{216}{6^3} > 0$ INDEED MINIMUM AREA