DIFFERENTIATION PRACTICE

THE OPERATION OF DIFFERENTIATION

Question 1

Evaluate the following.

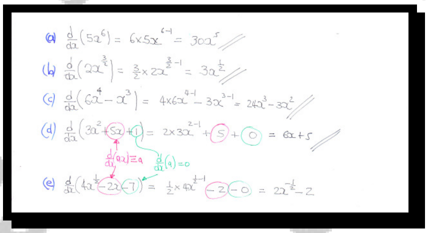
a) $\frac{d}{dx}\left(5x^6\right)$ $\quad$ $\boxed{\frac{d}{dx}\left(5x^6\right)=30x^5}$

b) $\frac{d}{dx}\left(2x^{\frac{3}{2}}\right)$ $\quad$ $\boxed{\frac{d}{dx}\left(2x^{\frac{3}{2}}\right)=3x^{\frac{1}{2}}}$

c) $\frac{d}{dx}\left(6x^4-x^3\right)$ $\quad$ $\boxed{\frac{d}{dx}\left(6x^4-x^3\right)=24x^3-3x^2}$

d) $\frac{d}{dx}\left(3x^2+5x+1\right)$ $\quad$ $\boxed{\frac{d}{dx}\left(3x^2+5x+1\right)=6x+5}$

e) $\frac{d}{dx}\left(4x^{\frac{1}{2}}-2x-7\right)$ $\quad$ $\boxed{\frac{d}{dx}\left(4x^{\frac{1}{2}}-2x-7\right)=2x^{-\frac{1}{2}}-2}$



Question 2

Evaluate the following.

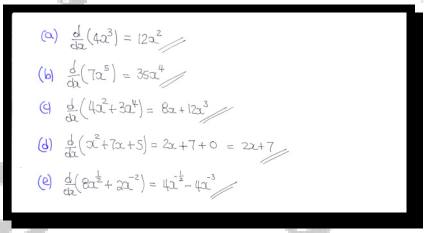
a) $\frac{d}{dx}\left(4x^3\right)$ $\boxed{\frac{d}{dx}\left(4x^3\right)=12x^2}$

b) $\frac{d}{dx}\left(7x^5\right)$ $\boxed{\frac{d}{dx}\left(7x^5\right)=35x^4}$

c) $\frac{d}{dx}\left(4x^2+3x^4\right)$ $\boxed{\frac{d}{dx}\left(4x^2+3x^4\right)=8x+12x^3}$

d) $\frac{d}{dx}\left(x^2+7x+5\right)$ $\boxed{\frac{d}{dx}\left(x^2+7x+5\right)=2x+7}$

e) $\frac{d}{dx}\left(8x^{\frac{1}{2}}+2x^{-2}\right)$ $\boxed{\frac{d}{dx}\left(8x^{\frac{1}{2}}+2x^{-2}\right)=4x^{-\frac{1}{2}}-4x^{-3}}$


(a) $\frac{d}{dx}(4x^3)=12x^2$
(b) $\frac{d}{dx}(7x^5)=35x^4$
(c) $\frac{d}{dx}(4x^2+3x^4)=8x+12x^3$
(d) $\frac{d}{dx}(x^2+7x+5)=2x+7+0=2x+7$
(e) $\frac{d}{dx}(8x^{\frac{1}{2}}+2x^{-2})=4x^{-\frac{1}{2}}-4x^{-3}$

Question 3

Differentiate the following expressions with respect to x

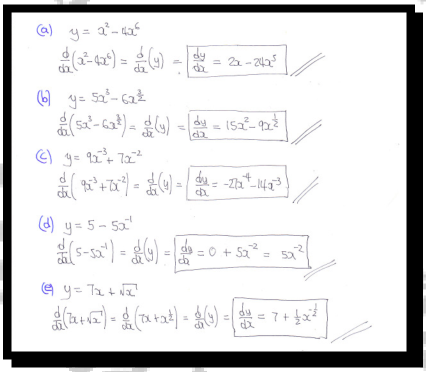
a) $y = x^2 - 4x^6$ $\boxed{\frac{dy}{dx} = 2x - 24x^5}$

b) $y = 5x^3 - 6x^{\frac{3}{2}}$ $\boxed{\frac{dy}{dx} = 15x^2 - 9x^{\frac{1}{2}}}$

c) $y = 9x^{-3} + 7x^{-2}$ $\boxed{\frac{dy}{dx} = -27x^{-4} - 14x^{-3}}$

d) $y = 5 - 5x^{-1}$ $\boxed{\frac{dy}{dx} = 5x^{-2}}$

e) $y = 7x + \sqrt{x}$ $\boxed{\frac{dy}{dx} = 7 + \frac{1}{2}x^{-\frac{1}{2}}}$



(a) $y = x^2 - 4x^6$

$\frac{d}{dx}(x^2 - 4x^6) = \frac{d}{dx}(y) \Rightarrow \boxed{\frac{dy}{dx} = 2x - 24x^5}$

(b) $y = 5x^3 - 6x^{\frac{3}{2}}$

$\frac{d}{dx}(5x^3 - 6x^{\frac{3}{2}}) = \frac{d}{dx}(y) = \boxed{\frac{dy}{dx} = 15x^2 - 9x^{\frac{1}{2}}}$

(c) $y = 9x^{-3} + 7x^{-2}$

$\frac{d}{dx}(9x^{-3} + 7x^{-2}) = \frac{d}{dx}(y) = \boxed{\frac{dy}{dx} = -27x^{-4} - 14x^{-3}}$

(d) $y = 5 - 5x^{-1}$

$\frac{d}{dx}(5 - 5x^{-1}) = \frac{d}{dx}(y) = \boxed{\frac{dy}{dx} = 0 + 5x^{-2} = 5x^{-2}}$

(e) $y = 7x + \sqrt{x}$

$\frac{d}{dx}(7x + \sqrt{x}) = \frac{d}{dx}(7x + x^{\frac{1}{2}}) = \frac{d}{dx}(y) = \boxed{\frac{dy}{dx} = 7 + \frac{1}{2}x^{-\frac{1}{2}}}$

Question 4

Differentiate the following expressions with respect to x

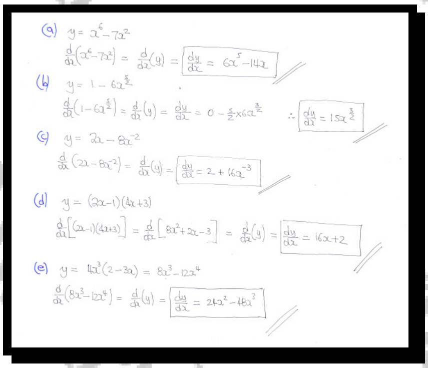
a) $y = x^6 - 7x^2$ $\boxed{\dfrac{dy}{dx} = 6x^5 - 14x}$

b) $y = 1 - 6x^{\frac{5}{2}}$ $\boxed{\dfrac{dy}{dx} = 15x^{\frac{3}{2}}}$

c) $y = 2x + 8x^{-2}$ $\boxed{\dfrac{dy}{dx} = 2 + 16x^{-3}}$

d) $y = (2x-1)(4x+3)$ $\boxed{\dfrac{dy}{dx} = 16x + 2}$

e) $y = 4x^3(2-3x)$ $\boxed{\dfrac{dy}{dx} = 24x^2 - 48x^3}$



(a) $y = x^6 - 7x^2$

$\frac{d}{dx}(x^6 - 7x^2) = \frac{d}{dx}(y) = \boxed{\frac{dy}{dx} = 6x^5 - 14x}$

(b) $y = 1 - 6x^{\frac{5}{2}}$

$\frac{d}{dx}(1 - 6x^{\frac{5}{2}}) = \frac{d}{dx}(y) = \frac{dy}{dx} = 0 - \frac{5}{2} \times 6x^{\frac{3}{2}}$ $\therefore \boxed{\frac{dy}{dx} = 15x^{\frac{3}{2}}}$

(c) $y = 2x - 8x^{-2}$

$\frac{d}{dx}(2x - 8x^{-2}) = \frac{d}{dx}(y) = \boxed{\frac{dy}{dx} = 2 + 16x^{-3}}$

(d) $y = (2x-1)(4x+3)$

$\frac{d}{dx}\left[(2x-1)(4x+3)\right] = \frac{d}{dx}\left[8x^2 + 2x - 3\right] = \frac{d}{dx}(y) = \boxed{\frac{dy}{dx} = 16x + 2}$

(e) $y = 4x^3(2-3x) = 8x^3 - 12x^4$

$\frac{d}{dx}(8x^3 - 12x^4) = \frac{d}{dx}(y) = \boxed{\frac{dy}{dx} = 24x^2 - 48x^3}$

Question 5

Find $f'(x)$ for each of the following functions.

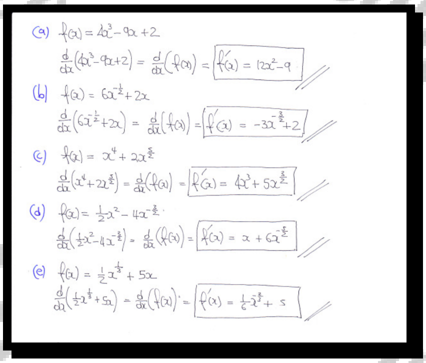
a) $f(x) = 4x^3 - 9x + 2$ $\qquad \boxed{f'(x) = 12x^2 - 9}$

b) $f(x) = 6x^{-\frac{1}{2}} + 2x$ $\qquad \boxed{f'(x) = -3x^{-\frac{3}{2}} + 2}$

c) $f(x) = x^4 + 2x^{\frac{5}{2}}$ $\qquad \boxed{f'(x) = 4x^3 + 5x^{\frac{3}{2}}}$

d) $f(x) = \frac{1}{2}x^2 - 4x^{-\frac{3}{2}}$ $\qquad \boxed{f'(x) = x + 6x^{-\frac{5}{2}}}$

e) $f(x) = \frac{1}{2}x^{\frac{1}{3}} + 5x$ $\qquad \boxed{f'(x) = \frac{1}{6}x^{-\frac{2}{3}} + 5}$


(a) $f(x) = 4x^3 - 9x + 2$
$\frac{d}{dx}(4x^3 - 9x + 2) = \frac{d}{dx}(f(x)) = \boxed{f'(x) = 12x^2 - 9}$
(b) $f(x) = 6x^{-\frac{1}{2}} + 2x$
$\frac{d}{dx}(6x^{-\frac{1}{2}} + 2x) = \frac{d}{dx}(f(x)) = \boxed{f'(x) = -3x^{-\frac{3}{2}} + 2}$
(c) $f(x) = x^4 + 2x^{\frac{5}{2}}$
$\frac{d}{dx}(x^4 + 2x^{\frac{5}{2}}) = \frac{d}{dx}(f(x)) = \boxed{f'(x) = 4x^3 + 5x^{\frac{3}{2}}}$
(d) $f(x) = \frac{1}{2}x^2 - 4x^{-\frac{3}{2}}$
$\frac{d}{dx}(\frac{1}{2}x^2 - 4x^{-\frac{3}{2}}) = \frac{d}{dx}(f(x)) = \boxed{f'(x) = x + 6x^{-\frac{5}{2}}}$
(e) $f(x) = \frac{1}{2}x^{\frac{1}{3}} + 5x$
$\frac{d}{dx}(\frac{1}{2}x^{\frac{1}{3}} + 5x) = \frac{d}{dx}(f(x)) = \boxed{f'(x) = \frac{1}{6}x^{-\frac{2}{3}} + 5}$

Question 6

Differentiate each of the following functions with respect to x.

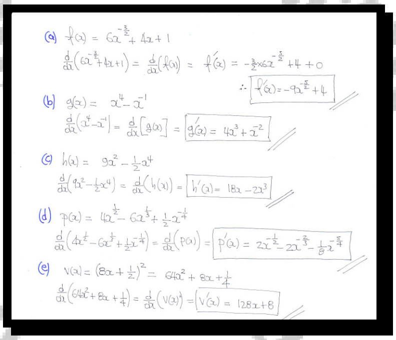
a) $f(x) = 6x^{-\frac{3}{2}} + 4x + 1$ $\quad$ $\boxed{f'(x) = -9x^{-\frac{5}{2}} + 4}$

b) $g(x) = x^4 - x^{-1}$ $\quad$ $\boxed{g'(x) = 4x^3 + x^{-2}}$

c) $h(x) = 9x^2 - \frac{1}{2}x^4$ $\quad$ $\boxed{h'(x) = 18x - 2x^3}$

d) $p(x) = 4x^{\frac{1}{2}} - 6x^{\frac{1}{3}} + \frac{1}{2}x^{-\frac{1}{4}}$ $\quad$ $\boxed{p'(x) = 2x^{-\frac{1}{2}} - 2x^{-\frac{2}{3}} - \frac{1}{8}x^{-\frac{5}{4}}}$

e) $v(x) = \left(8x + \frac{1}{2}\right)^2$ $\quad$ $\boxed{v'(x) = 128x + 8}$



Question 7

Carry out the following differentiations.

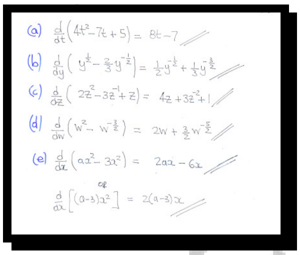
a) $\frac{d}{dt}\left(4t^2 - 7t + 5\right)$ $\qquad$ $\boxed{\frac{d}{dt}\left(4t^2 - 7t + 5\right) = 8t - 7}$

b) $\frac{d}{dy}\left(y^{\frac{1}{2}} - \frac{2}{3}y^{-\frac{1}{2}}\right)$ $\qquad$ $\boxed{\frac{d}{dy}\left(y^{\frac{1}{2}} - \frac{2}{3}y^{-\frac{1}{2}}\right) = \frac{1}{2}y^{-\frac{1}{2}} + \frac{1}{3}y^{-\frac{3}{2}}}$

c) $\frac{d}{dz}\left(2z^2 - 3z^{-1} + z\right)$ $\qquad$ $\boxed{\frac{d}{dz}\left(2z^2 - 3z^{-1} + z\right) = 4z + 3z^{-2} + 1}$

d) $\frac{d}{dw}\left(w^2 - w^{-\frac{3}{2}}\right)$ $\qquad$ $\boxed{\frac{d}{dw}\left(w^2 - w^{-\frac{3}{2}}\right) = 2w + \frac{3}{2}w^{-\frac{5}{2}}}$

e) $\frac{d}{dx}\left(ax^2 - 3x^2\right)$ $\qquad$ $\boxed{\frac{d}{dx}\left(ax^2 - 3x^2\right) = 2ax - 6x}$


(a) $\frac{d}{dt}\left(4t^2 - 7t + 5\right) = 8t - 7$

(b) $\frac{d}{dy}\left(y^{\frac{1}{2}} - \frac{2}{3}y^{-\frac{1}{2}}\right) = \frac{1}{2}y^{-\frac{1}{2}} + \frac{1}{3}y^{-\frac{3}{2}}$

(c) $\frac{d}{dz}\left(2z^2 - 3z^{-1} + z\right) = 4z + 3z^{-2} + 1$

(d) $\frac{d}{dw}\left(w^2 - w^{-\frac{3}{2}}\right) = 2w + \frac{3}{2}w^{-\frac{5}{2}}$

(e) $\frac{d}{dx}\left(ax^2 - 3x^2\right) = 2ax - 6x$

or

$\frac{d}{dx}\left[(a-3)x^2\right] = 2(a-3)x$

Question 8

Carry out the following differentiations.

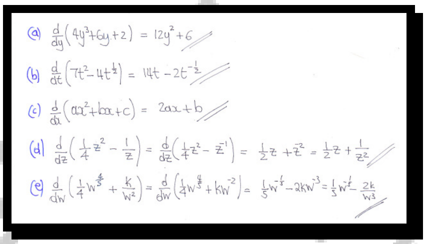
a) $\frac{d}{dy}\left(4y^3+6y+2\right)$ $\qquad$ $\boxed{\frac{d}{dy}\left(4y^3+6y+2\right)=12y^2+6}$

b) $\frac{d}{dt}\left(7t^2-4t^{\frac{1}{2}}\right)$ $\qquad$ $\boxed{\frac{d}{dt}\left(7t^2-4t^{\frac{1}{2}}\right)=14t-2t^{-\frac{1}{2}}}$

c) $\frac{d}{dx}\left(ax^2+bx+c\right)$ $\qquad$ $\boxed{\frac{d}{dx}\left(ax^2+bx+c\right)=2ax+b}$

d) $\frac{d}{dz}\left(\frac{1}{4}z^2-\frac{1}{z}\right)$ $\qquad$ $\boxed{\frac{d}{dz}\left(\frac{1}{4}z^2-\frac{1}{z}\right)=\frac{1}{2}z+\frac{1}{z^2}}$

e) $\frac{d}{dw}\left(\frac{1}{4}w^{\frac{4}{5}}+\frac{k}{w^2}\right)$ $\qquad$ $\boxed{\frac{d}{dw}\left(\frac{1}{4}w^{\frac{4}{5}}+\frac{k}{w^2}\right)=\frac{1}{5}w^{-\frac{1}{5}}-\frac{2k}{w^3}}$


(a) $\frac{d}{dy}\left(4y^3+6y+2\right)=12y^2+6$

(b) $\frac{d}{dt}\left(7t^2-4t^{\frac{1}{2}}\right)=14t-2t^{-\frac{1}{2}}$

(c) $\frac{d}{dx}\left(ax^2+bx+c\right)=2ax+b$

(d) $\frac{d}{dz}\left(\frac{1}{4}z^2-\frac{1}{z}\right)=\frac{d}{dz}\left(\frac{1}{4}z^2-z^{-1}\right)=\frac{1}{2}z+z^{-2}=\frac{1}{2}z+\frac{1}{z^2}$

(e) $\frac{d}{dw}\left(\frac{1}{4}w^{\frac{4}{5}}+\frac{k}{w^2}\right)=\frac{d}{dw}\left(\frac{1}{4}w^{\frac{4}{5}}+kw^{-2}\right)=\frac{1}{5}w^{-\frac{1}{5}}-2kw^{-3}=\frac{1}{5}w^{-\frac{1}{5}}-\frac{2k}{w^3}$

Question 9

a) If $A = \pi x^2 - 20x$, find the rate of change of A with respect to x.

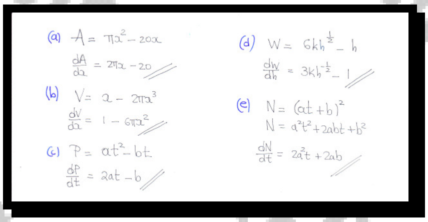
b) If $V = x - 2\pi x^3$, find the rate of change of V with respect to x.

c) If $P = at^2 - bt$, find the rate of change of P with respect to t.

d) If $W = 6kh^{\frac{1}{2}} - h$, find the rate of change of W with respect to h.

e) If $N = (at+b)^2$, find the rate of change of N with respect to t.

$\boxed{\frac{dA}{dx} = 2\pi x - 20}$, $\boxed{\frac{dV}{dx} = 1 - 6\pi x^2}$, $\boxed{\frac{dP}{dt} = 2at - b}$, $\boxed{\frac{dW}{dh} = 3kh^{-\frac{1}{2}} - 1}$, $\boxed{\frac{dN}{dt} = 2a^2 t + 2ab}$


(a) $A = \pi x^2 - 20x$
$\frac{dA}{dx} = 2\pi x - 20$
(b) $V = x - 2\pi x^3$
$\frac{dV}{dx} = 1 - 6\pi x^2$
(c) $P = at^2 - bt$
$\frac{dP}{dt} = 2at - b$
(d) $W = 6kh^{\frac{1}{2}} - h$
$\frac{dW}{dh} = 3kh^{-\frac{1}{2}} - 1$
(e) $N = (at+b)^2$
$N = a^2t^2 + 2abt + b^2$
$\frac{dN}{dt} = 2a^2t + 2ab$

DIFFERENTIATING INDICES

Question 1

Differentiate the following expressions with respect to x

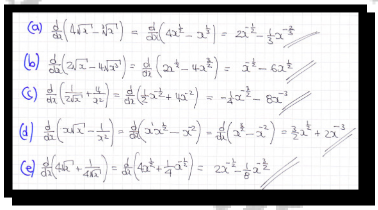
a) $y = 4\sqrt{x} - \sqrt[3]{x}$ $\qquad$ $\boxed{\dfrac{dy}{dx} = 2x^{-\frac{1}{2}} - \dfrac{1}{3}x^{-\frac{2}{3}}}$

b) $y = 2\sqrt{x} - 4\sqrt{x^3}$ $\qquad$ $\boxed{\dfrac{dy}{dx} = x^{-\frac{1}{2}} - 6x^{\frac{1}{2}}}$

c) $y = \dfrac{1}{2\sqrt{x}} + \dfrac{4}{x^2}$ $\qquad$ $\boxed{\dfrac{dy}{dx} = -\dfrac{1}{4}x^{-\frac{3}{2}} - 8x^{-3}}$

d) $y = x\sqrt{x} - \dfrac{1}{x^2}$ $\qquad$ $\boxed{\dfrac{dy}{dx} = \dfrac{3}{2}x^{\frac{1}{2}} + 2x^{-3}}$

e) $y = 4\sqrt{x} + \dfrac{1}{4\sqrt{x}}$ $\qquad$ $\boxed{\dfrac{dy}{dx} = 2x^{-\frac{1}{2}} - \dfrac{1}{8}x^{-\frac{3}{2}}}$


(a) $\frac{d}{dx}\left(4\sqrt{x} - \sqrt[3]{x}\right) = \frac{d}{dx}\left(4x^{\frac{1}{2}} - x^{\frac{1}{3}}\right) = 2x^{-\frac{1}{2}} - \frac{1}{3}x^{-\frac{2}{3}}$

(b) $\frac{d}{dx}\left(2\sqrt{x} - 4\sqrt{x^3}\right) = \frac{d}{dx}\left(2x^{\frac{1}{2}} - 4x^{\frac{3}{2}}\right) = x^{-\frac{1}{2}} - 6x^{\frac{1}{2}}$

(c) $\frac{d}{dx}\left(\frac{1}{2\sqrt{x}} + \frac{4}{x^2}\right) = \frac{d}{dx}\left(\frac{1}{2}x^{-\frac{1}{2}} + 4x^{-2}\right) = -\frac{1}{4}x^{-\frac{3}{2}} - 8x^{-3}$

(d) $\frac{d}{dx}\left(x\sqrt{x} - \frac{1}{x^2}\right) = \frac{d}{dx}\left(x^1x^{\frac{1}{2}} - x^{-2}\right) = \frac{d}{dx}\left(x^{\frac{3}{2}} - x^{-2}\right) = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-3}$

(e) $\frac{d}{dx}\left(4\sqrt{x} + \frac{1}{4\sqrt{x}}\right) = \frac{d}{dx}\left(4x^{\frac{1}{2}} + \frac{1}{4}x^{-\frac{1}{2}}\right) = 2x^{-\frac{1}{2}} - \frac{1}{8}x^{-\frac{3}{2}}$

Question 2

Find $f'(x)$ for each of the following functions.

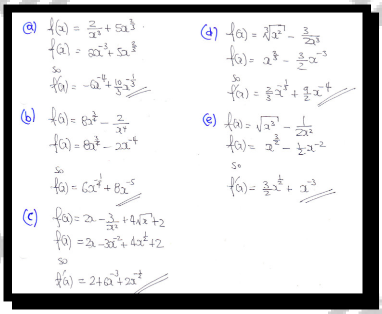
a) $f(x) = \dfrac{2}{x^3} + 5x^{\frac{2}{3}}$ $\qquad$ $\boxed{f'(x) = -6x^{-4} + \dfrac{10}{3}x^{-\frac{1}{3}}}$

b) $f(x) = 8x^{\frac{3}{4}} - \dfrac{2}{x^4}$ $\qquad$ $\boxed{f'(x) = 6x^{-\frac{1}{4}} + 8x^{-5}}$

c) $f(x) = 2x - \dfrac{3}{x^2} + 4\sqrt{x} + 2$ $\qquad$ $\boxed{f'(x) = 2 + 6x^{-3} + 2x^{-\frac{1}{2}}}$

d) $f(x) = \sqrt[3]{x^2} - \dfrac{3}{2x^3}$ $\qquad$ $\boxed{f'(x) = \dfrac{2}{3}x^{-\frac{1}{3}} + \dfrac{9}{2}x^{-4}}$

e) $f(x) = \sqrt{x^3} - \dfrac{1}{2x^2}$ $\qquad$ $\boxed{f'(x) = \dfrac{3}{2}x^{\frac{1}{2}} + x^{-3}}$


(a) $f(x) = \frac{2}{x^3} + 5x^{\frac{2}{3}}$, $f(x) = 2x^{-3} + 5x^{\frac{2}{3}}$, so $f'(x) = -6x^{-4} + \frac{10}{3}x^{-\frac{1}{3}}$

(b) $f(x) = 8x^{\frac{3}{4}} - \frac{2}{x^4}$, $f(x) = 8x^{\frac{3}{4}} - 2x^{-4}$, so $f'(x) = 6x^{-\frac{1}{4}} + 8x^{-5}$

(c) $f(x) = 2x - \frac{3}{x^2} + 4\sqrt{x} + 2$, $f(x) = 2x - 3x^{-2} + 4x^{\frac{1}{2}} + 2$, so $f'(x) = 2 + 6x^{-3} + 2x^{-\frac{1}{2}}$

(d) $f(x) = \sqrt[3]{x^2} - \frac{3}{2x^3}$, $f(x) = x^{\frac{2}{3}} - \frac{3}{2}x^{-3}$, so $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} + \frac{9}{2}x^{-4}$

(e) $f(x) = \sqrt{x^3} - \frac{1}{2x^2}$, $f(x) = x^{\frac{3}{2}} - \frac{1}{2}x^{-2}$, so $f'(x) = \frac{3}{2}x^{\frac{1}{2}} + x^{-3}$

Question 3

Differentiate the following expressions with respect to x

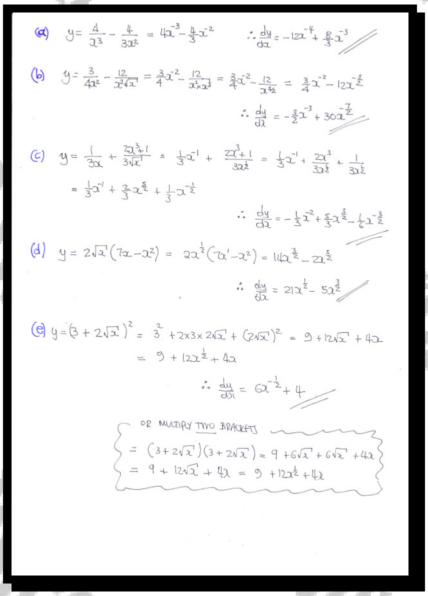
a) $y = \frac{4}{x^3} - \frac{4}{3x^2}$ $\quad \boxed{\frac{dy}{dx} = \frac{8}{3}x^{-3} - 12x^{-4}}$

b) $y = \frac{3}{4x^2} - \frac{12}{x^2\sqrt{x}}$ $\quad \boxed{\frac{dy}{dx} = 30x^{-\frac{7}{2}} - \frac{3}{2}x^{-3}}$

c) $y = \frac{1}{3x} + \frac{2x^3+1}{3\sqrt{x}}$ $\quad \boxed{\frac{dy}{dx} = -\frac{1}{3}x^{-2} + \frac{5}{3}x^{\frac{3}{2}} - \frac{1}{6}x^{-\frac{3}{2}}}$

d) $y = 2\sqrt{x}\left(7x - x^2\right)$ $\quad \boxed{\frac{dy}{dx} = 21x^{\frac{1}{2}} - 5x^{\frac{3}{2}}}$

e) $y = \left(3 + 2\sqrt{x}\right)^2$ $\quad \boxed{\frac{dy}{dx} = 6x^{-\frac{1}{2}} + 4}$


(a) $y = \frac{4}{x^3} - \frac{4}{3x^2} = 4x^{-3} - \frac{4}{3}x^{-2}$ $\quad \therefore \frac{dy}{dx} = -12x^{-4} + \frac{8}{3}x^{-3}$

(b) $y = \frac{3}{4x^2} - \frac{12}{x^2\sqrt{x}} = \frac{3}{4}x^{-2} - \frac{12}{x^2 \times x^{\frac{1}{2}}} = \frac{3}{4}x^{-2} - \frac{12}{x^{\frac{5}{2}}} = \frac{3}{4}x^{-2} - 12x^{-\frac{5}{2}}$

$\therefore \frac{dy}{dx} = -\frac{3}{2}x^{-3} + 30x^{-\frac{7}{2}}$

(c) $y = \frac{1}{3x} + \frac{2x^3+1}{3\sqrt{x}} = \frac{1}{3}x^{-1} + \frac{2x^3+1}{3x^{\frac{1}{2}}} = \frac{1}{3}x^{-1} + \frac{2x^3}{3x^{\frac{1}{2}}} + \frac{1}{3x^{\frac{1}{2}}}$

$= \frac{1}{3}x^{-1} + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{3}x^{-\frac{1}{2}}$

$\therefore \frac{dy}{dx} = -\frac{1}{3}x^{-2} + \frac{5}{3}x^{\frac{3}{2}} - \frac{1}{6}x^{-\frac{3}{2}}$

(d) $y = 2\sqrt{x}(7x - x^2) = 2x^{\frac{1}{2}}(7x^1 - x^2) = 14x^{\frac{3}{2}} - 2x^{\frac{5}{2}}$

$\therefore \frac{dy}{dx} = 21x^{\frac{1}{2}} - 5x^{\frac{3}{2}}$

(e) $y = (3 + 2\sqrt{x})^2 = 3^2 + 2 \times 3 \times 2\sqrt{x} + (2\sqrt{x})^2 = 9 + 12\sqrt{x} + 4x$

$= 9 + 12x^{\frac{1}{2}} + 4x$

$\therefore \frac{dy}{dx} = 6x^{-\frac{1}{2}} + 4$

OR MULTIPLY TWO BRACKETS

$= (3 + 2\sqrt{x})(3 + 2\sqrt{x}) = 9 + 6\sqrt{x} + 6\sqrt{x} + 4x$

$= 9 + 12\sqrt{x} + 4x = 9 + 12x^{\frac{1}{2}} + 4x$

Question 4

Evaluate the following.

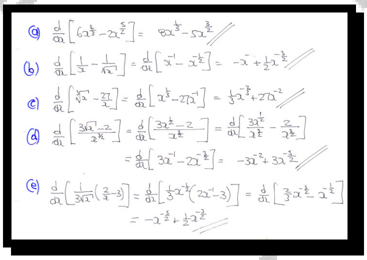
a) $\dfrac{d}{dx}\left(6x^{\frac{4}{3}}-2x^{\frac{5}{2}}\right)$ $\boxed{8x^{\frac{1}{3}}-5x^{\frac{3}{2}}}$

b) $\dfrac{d}{dx}\left(\dfrac{1}{x}-\dfrac{1}{\sqrt{x}}\right)$ $\boxed{-x^{-2}+\frac{1}{2}x^{-\frac{3}{2}}}$

c) $\dfrac{d}{dx}\left(\sqrt[3]{x}-\dfrac{27}{x}\right)$ $\boxed{\frac{1}{3}x^{-\frac{2}{3}}+27x^{-2}}$

d) $\dfrac{d}{dx}\left(\dfrac{3\sqrt{x}-2}{x^{\frac{3}{2}}}\right)$ $\boxed{-3x^{-2}+3x^{-\frac{5}{2}}}$

e) $\dfrac{d}{dx}\left[\dfrac{1}{3\sqrt{x}}\left(\dfrac{2}{x}-3\right)\right]$ $\boxed{-x^{-\frac{5}{2}}+\frac{1}{2}x^{-\frac{3}{2}}}$



Question 5

Evaluate the following.

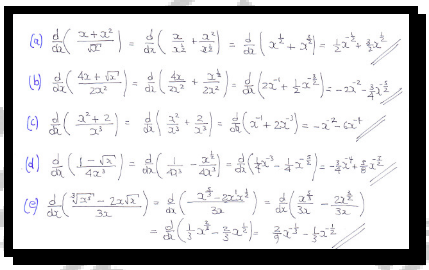
a) $\dfrac{d}{dx}\left(\dfrac{x+x^2}{\sqrt{x}}\right)$ $\boxed{\dfrac{1}{2}x^{-\frac{1}{2}}+\dfrac{3}{2}x^{\frac{1}{2}}}$

b) $\dfrac{d}{dx}\left(\dfrac{4x+\sqrt{x}}{2x^2}\right)$ $\boxed{-2x^{-2}-\dfrac{3}{4}x^{-\frac{5}{2}}}$

c) $\dfrac{d}{dx}\left(\dfrac{x^2+2}{x^3}\right)$ $\boxed{-x^{-2}-6x^{-4}}$

d) $\dfrac{d}{dx}\left(\dfrac{1-\sqrt{x}}{4x^3}\right)$ $\boxed{-\dfrac{3}{4}x^{-4}+\dfrac{5}{8}x^{-\frac{7}{2}}}$

e) $\dfrac{d}{dx}\left[\dfrac{\sqrt[3]{x^5}-2x\sqrt{x}}{3x}\right]$ $\boxed{\dfrac{2}{9}x^{-\frac{1}{3}}-\dfrac{1}{3}x^{-\frac{1}{2}}}$



(a) $\frac{d}{dx}\left(\frac{x+x^2}{\sqrt{x}}\right)=\frac{d}{dx}\left(\frac{x}{x^{\frac{1}{2}}}+\frac{x^2}{x^{\frac{1}{2}}}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}+x^{\frac{3}{2}}\right)=\frac{1}{2}x^{-\frac{1}{2}}+\frac{3}{2}x^{\frac{1}{2}}$

(b) $\frac{d}{dx}\left(\frac{4x+\sqrt{x}}{2x^2}\right)=\frac{d}{dx}\left(\frac{4x}{2x^2}+\frac{x^{\frac{1}{2}}}{2x^2}\right)=\frac{d}{dx}\left(2x^{-1}+\frac{1}{2}x^{-\frac{3}{2}}\right)=-2x^{-2}-\frac{3}{4}x^{-\frac{5}{2}}$

(c) $\frac{d}{dx}\left(\frac{x^2+2}{x^3}\right)=\frac{d}{dx}\left(\frac{x^2}{x^3}+\frac{2}{x^3}\right)=\frac{d}{dx}\left(x^{-1}+2x^{-3}\right)=-x^{-2}-6x^{-4}$

(d) $\frac{d}{dx}\left(\frac{1-\sqrt{x}}{4x^3}\right)=\frac{d}{dx}\left(\frac{1}{4x^3}-\frac{x^{\frac{1}{2}}}{4x^3}\right)=\frac{d}{dx}\left(\frac{1}{4}x^{-3}-\frac{1}{4}x^{-\frac{5}{2}}\right)=-\frac{3}{4}x^{-4}+\frac{5}{8}x^{-\frac{7}{2}}$

(e) $\frac{d}{dx}\left(\frac{\sqrt[3]{x^5}-2x\sqrt{x}}{3x}\right)=\frac{d}{dx}\left(\frac{x^{\frac{5}{3}}-2x^1x^{\frac{1}{2}}}{3x}\right)=\frac{d}{dx}\left(\frac{x^{\frac{5}{3}}}{3x}-\frac{2x^{\frac{3}{2}}}{3x}\right)$

$=\frac{d}{dx}\left(\frac{1}{3}x^{\frac{2}{3}}-\frac{2}{3}x^{\frac{1}{2}}\right)=\frac{2}{9}x^{-\frac{1}{3}}-\frac{1}{3}x^{-\frac{1}{2}}$

Question 6

Differentiate the following expressions with respect to x

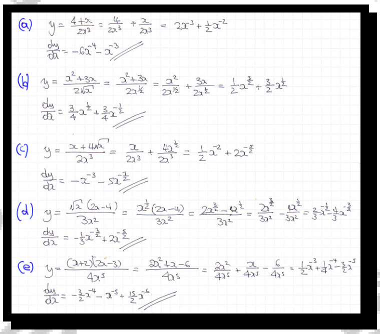
a) $y=\dfrac{4+x}{2x^3}$ $\qquad\boxed{\dfrac{dy}{dx}=-6x^{-4}-x^{-3}}$

b) $y=\dfrac{x^2+3x}{2\sqrt{x}}$ $\qquad\boxed{\dfrac{dy}{dx}=\dfrac{3}{4}x^{-\frac{1}{2}}+\dfrac{3}{4}x^{\frac{1}{2}}}$

c) $y=\dfrac{x+4\sqrt{x}}{2x^3}$ $\qquad\boxed{\dfrac{dy}{dx}=-5x^{-\frac{7}{2}}-x^{-3}}$

d) $y=\dfrac{\sqrt{x}(2x-4)}{3x^2}$ $\qquad\boxed{\dfrac{dy}{dx}=-\dfrac{1}{3}x^{-\frac{3}{2}}+2x^{-\frac{5}{2}}}$

e) $y=\dfrac{(x+2)(2x-3)}{4x^5}$ $\qquad\boxed{\dfrac{dy}{dx}=-\dfrac{3}{2}x^{-4}-x^{-5}+\dfrac{15}{2}x^{-6}}$



Question 7

Find $f'(x)$ for each of the following functions.

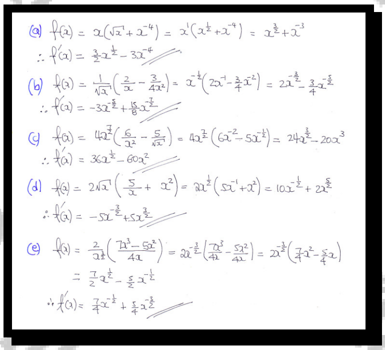
a) $f(x) = x\left(\sqrt{x} + x^{-4}\right)$ $\boxed{f'(x) = \frac{3}{2}x^{\frac{1}{2}} - 3x^{-4}}$

b) $f(x) = \frac{1}{\sqrt{x}}\left(\frac{2}{x} - \frac{3}{4x^2}\right)$ $\boxed{f'(x) = -3x^{-\frac{5}{2}} + \frac{15}{8}x^{-\frac{7}{2}}}$

c) $f(x) = 4x^{\frac{7}{2}}\left(\frac{6}{x^2} - \frac{5}{\sqrt{x}}\right)$ $\boxed{f'(x) = 36x^{\frac{1}{2}} - 60x^2}$

d) $f(x) = 2\sqrt{x}\left(\frac{5}{x} + x^2\right)$ $\boxed{f'(x) = -5x^{-\frac{3}{2}} + 5x^{\frac{3}{2}}}$

e) $f(x) = \frac{2}{x^{\frac{3}{2}}}\left(\frac{7x^3 - 5x^2}{4x}\right)$ $\boxed{f'(x) = \frac{7}{4}x^{-\frac{1}{2}} + \frac{5}{4}x^{-\frac{3}{2}}}$


(a) $f(x) = x(\sqrt{x} + x^{-4}) = x^1(x^{\frac{1}{2}} + x^{-4}) = x^{\frac{3}{2}} + x^{-3}$

$\therefore f'(x) = \frac{3}{2}x^{\frac{1}{2}} - 3x^{-4}$

(b) $f(x) = \frac{1}{\sqrt{x}}\left(\frac{2}{x} - \frac{3}{4x^2}\right) = x^{-\frac{1}{2}}\left(2x^{-1} - \frac{3}{4}x^{-2}\right) = 2x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

$\therefore f'(x) = -3x^{-\frac{5}{2}} + \frac{15}{8}x^{-\frac{7}{2}}$

(c) $f(x) = 4x^{\frac{7}{2}}\left(\frac{6}{x^2} - \frac{5}{\sqrt{x}}\right) = 4x^{\frac{7}{2}}\left(6x^{-2} - 5x^{-\frac{1}{2}}\right) = 24x^{\frac{3}{2}} - 20x^3$

$\therefore f'(x) = 36x^{\frac{1}{2}} - 60x^2$

(d) $f(x) = 2\sqrt{x}\left(\frac{5}{x} + x^2\right) = 2x^{\frac{1}{2}}\left(5x^{-1} + x^2\right) = 10x^{-\frac{1}{2}} + 2x^{\frac{5}{2}}$

$\therefore f'(x) = -5x^{-\frac{3}{2}} + 5x^{\frac{3}{2}}$

(e) $f(x) = \frac{2}{x^{\frac{3}{2}}}\left(\frac{7x^3 - 5x^2}{4x}\right) = 2x^{-\frac{3}{2}}\left(\frac{7x^3}{4x} - \frac{5x^2}{4x}\right) = 2x^{-\frac{3}{2}}\left(\frac{7}{4}x^2 - \frac{5}{4}x\right)$

$= \frac{7}{2}x^{\frac{1}{2}} - \frac{5}{2}x^{-\frac{1}{2}}$

$\therefore f'(x) = \frac{7}{4}x^{-\frac{1}{2}} + \frac{5}{4}x^{-\frac{3}{2}}$

Question 8

Differentiate the following expressions with respect to x

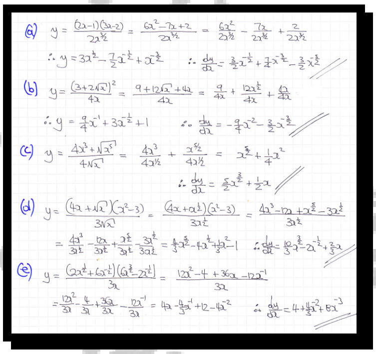
a) $y = \dfrac{(2x-1)(3x-2)}{2x^{\frac{3}{2}}}$ $\qquad$ $\boxed{\dfrac{dy}{dx} = \dfrac{3}{2}x^{-\frac{1}{2}} + \dfrac{7}{4}x^{-\frac{3}{2}} - \dfrac{3}{2}x^{-\frac{5}{2}}}$

b) $y = \dfrac{\left(3+2\sqrt{x}\right)^2}{4x}$ $\qquad$ $\boxed{\dfrac{dy}{dx} = -\dfrac{3}{2}x^{-\frac{3}{2}} - \dfrac{9}{4}x^{-2}}$

c) $y = \dfrac{4x^3 + \sqrt{x^5}}{4\sqrt{x}}$ $\qquad$ $\boxed{\dfrac{dy}{dx} = \dfrac{1}{2}x + \dfrac{5}{2}x^{\frac{3}{2}}}$

d) $y = \dfrac{\left(4x+\sqrt{x}\right)\left(x^2-3\right)}{3\sqrt{x}}$ $\qquad$ $\boxed{\dfrac{dy}{dx} = \dfrac{2}{3}x + \dfrac{10}{3}x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}}$

e) $y = \dfrac{\left(2x^{\frac{1}{2}} + 6x^{-\frac{1}{2}}\right)\left(6x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}\right)}{3x}$ $\qquad$ $\boxed{\dfrac{dy}{dx} = \dfrac{4}{3}x^{-2} + 8x^{-3} + 4}$



TANGENTS
&
NORMALS

Question 1 **(non calculator)**

For each of the following curves find an equation of the tangent to the curve at the point whose x coordinate is given.

a) $y = x^2 - 9x + 13$, where $x = 6$ $\boxed{y = 3x - 23}$

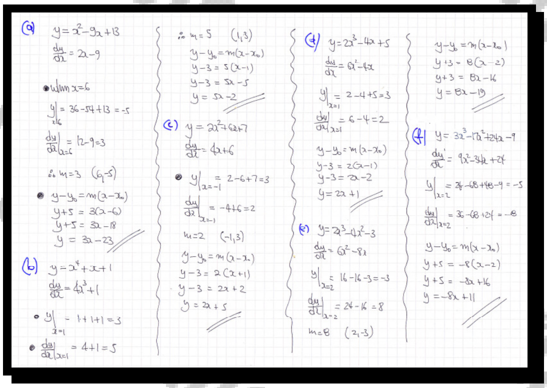
b) $y = x^4 + x + 1$, where $x = 1$ $\boxed{y = 5x - 2}$

c) $y = 2x^2 + 6x + 7$, where $x = -1$ $\boxed{y = 2x + 5}$

d) $y = 2x^3 - 4x + 5$, where $x = 1$ $\boxed{y = 2x + 1}$

e) $y = 2x^3 - 4x^2 - 3$, where $x = 2$ $\boxed{y = 8x - 19}$

f) $y = 3x^3 - 17x^2 + 24x - 9$, where $x = 2$ $\boxed{y = -8x + 11}$



Question 2 (non calculator)

For each of the following curves find an equation of the tangent to the curve at the point whose x coordinate is given.

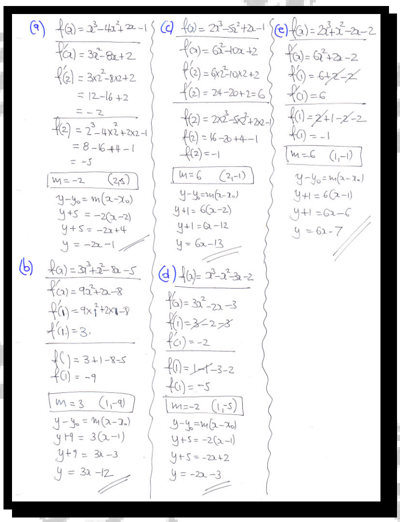
a) $f(x) = x^3 - 4x^2 + 2x - 1$, where $x = 2$ — $\boxed{y = -2x - 1}$

b) $f(x) = 3x^3 + x^2 - 8x - 5$, where $x = 1$ — $\boxed{y = 3x - 12}$

c) $f(x) = 2x^3 - 5x^2 + 2x - 1$, where $x = 2$ — $\boxed{y = 6x - 13}$

d) $f(x) = x^3 - x^2 - 3x - 2$, where $x = 1$ — $\boxed{y = -2x - 3}$

e) $f(x) = 2x^3 + x^2 - 2x - 2$, where $x = 1$ — $\boxed{y = 6x - 7}$



Question 3 (non calculator)

For each of the following curves find an equation of the tangent to the curve at the point whose x coordinate is given.

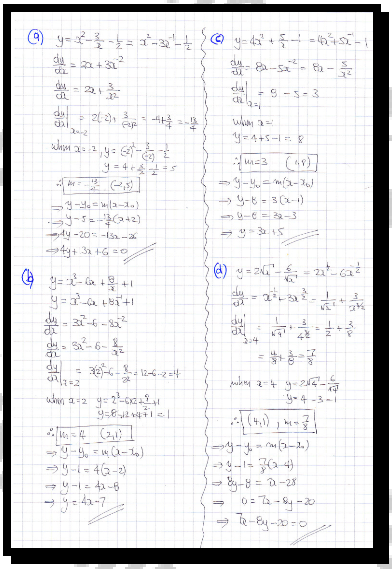
a) $y = x^2 - \frac{3}{x} - \frac{1}{2}$, where $x = -2$ $\boxed{13x + 4y + 6 = 0}$

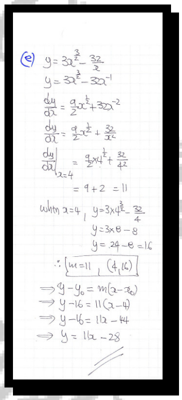
b) $y = x^3 - 6x + \frac{8}{x} + 1$, where $x = 2$ $\boxed{y = 4x - 7}$

c) $y = 4x^2 + \frac{5}{x} - 1$, where $x = 1$ $\boxed{y = 3x + 5}$

d) $y = 2\sqrt{x} - \frac{6}{\sqrt{x}}$, where $x = 4$ $\boxed{7x - 8y - 20 = 0}$

e) $y = 3x^{\frac{3}{2}} - \frac{32}{x}$, where $x = 4$ $\boxed{y = 11x - 28}$





Question 4 (non calculator)

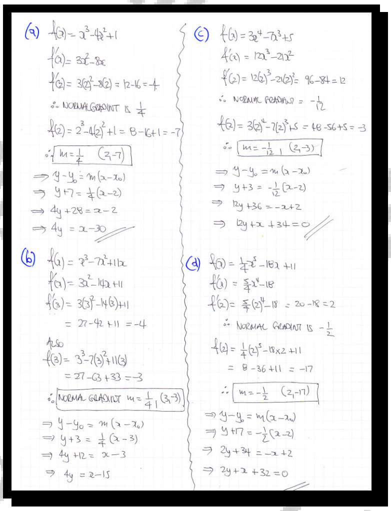
For each of the following curves find an equation of the normal to the curve at the point whose x coordinate is given.

a) $f(x) = x^3 - 4x^2 + 1$, where $x = 2$ $\boxed{4y = x - 30}$

b) $f(x) = x^3 - 7x^2 + 11x$, where $x = 3$ $\boxed{4y = x - 15}$

c) $f(x) = 3x^4 - 7x^3 + 5$ where $x = 2$ $\boxed{12y + x + 34 = 0}$

d) $f(x) = \frac{1}{4}x^5 - 18x + 11$ where $x = 2$ $\boxed{2y + x + 32 = 0}$


(a) $f(x) = x^3 - 4x^2 + 1$

$f'(x) = 3x^2 - 8x$

$f'(2) = 3(2)^2 - 8(2) = 12 - 16 = -4$

∴ NORMAL GRADIENT IS $\frac{1}{4}$

$f(2) = 2^3 - 4(2)^2 + 1 = 8 - 16 + 1 = -7$

∴ $m = \frac{1}{4}$ $(2,-7)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y + 7 = \frac{1}{4}(x - 2)$

$\Rightarrow 4y + 28 = x - 2$

$\Rightarrow 4y = x - 30$

(b) $f(x) = x^3 - 7x^2 + 11x$

$f'(x) = 3x^2 - 14x + 11$

$f'(3) = 3(3)^2 - 14(3) + 11$

$= 27 - 42 + 11 = -4$

ALSO

$f(3) = 3^3 - 7(3)^2 + 11(3)$

$= 27 - 63 + 33 = -3$

∴ NORMAL GRADIENT $m = \frac{1}{4}$, $(3,-3)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y + 3 = \frac{1}{4}(x - 3)$

$\Rightarrow 4y + 12 = x - 3$

$\Rightarrow 4y = x - 15$

(c) $f(x) = 3x^4 - 7x^3 + 5$

$f'(x) = 12x^3 - 21x^2$

$f'(2) = 12(2)^3 - 21(2)^2 = 96 - 84 = 12$

∴ NORMAL GRADIENT $= -\frac{1}{12}$

$f(2) = 3(2)^4 - 7(2)^3 + 5 = 48 - 56 + 5 = -3$

∴ $m = -\frac{1}{12}$, $(2,-3)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y + 3 = -\frac{1}{12}(x - 2)$

$\Rightarrow 12y + 36 = -x + 2$

$\Rightarrow 12y + x + 34 = 0$

(d) $f(x) = \frac{1}{4}x^5 - 18x + 11$

$f'(x) = \frac{5}{4}x^4 - 18$

$f'(2) = \frac{5}{4}(2)^4 - 18 = 20 - 18 = 2$

∴ NORMAL GRADIENT IS $-\frac{1}{2}$

$f(2) = \frac{1}{4}(2)^5 - 18 \times 2 + 11$

$= 8 - 36 + 11 = -17$

∴ $m = -\frac{1}{2}$ $(2,-17)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y + 17 = -\frac{1}{2}(x - 2)$

$\Rightarrow 2y + 34 = -x + 2$

$\Rightarrow 2y + x + 32 = 0$

Question 5 (non calculator)

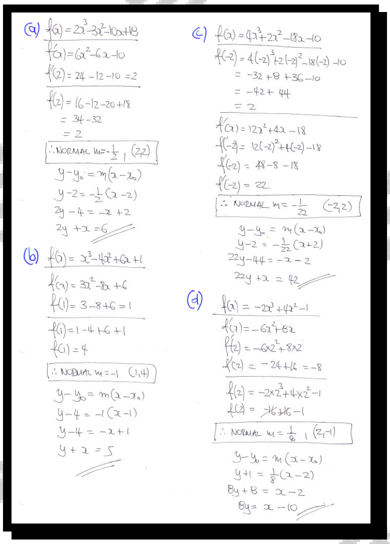
For each of the following curves find an equation of the normal to the curve at the point whose x coordinate is given.

a) $f(x) = 2x^3 - 3x^2 - 10x + 18$, where $x = 2$ $\boxed{x + 2y = 6}$

b) $f(x) = x^3 - 4x^2 + 6x + 1$, where $x = 1$ $\boxed{x + y = 5}$

c) $f(x) = 4x^3 + 2x^2 - 18x - 10$ where $x = -2$ $\boxed{22y + x = 42}$

d) $f(x) = -2x^3 + 4x^2 - 1$, where $x = 2$ $\boxed{8y = x - 10}$


(a) $f(x)=2x^3-3x^2-10x+18$
$f'(x)=6x^2-6x-10$
$f'(2)=24-12-10=2$
$f(2)=16-12-20+18$
$=34-32$
$=2$
$\therefore$ NORMAL $m=-\frac{1}{2}$, $(2,2)$
$y-y_0=m(x-x_0)$
$y-2=-\frac{1}{2}(x-2)$
$2y-4=-x+2$
$2y+x=6$

(b) $f(x)=x^3-4x^2+6x+1$
$f'(x)=3x^2-8x+6$
$f'(1)=3-8+6=1$
$f(1)=1-4+6+1$
$f(1)=4$
$\therefore$ NORMAL $m=-1$ $(1,4)$
$y-y_0=m(x-x_0)$
$y-4=-1(x-1)$
$y-4=-x+1$
$y+x=5$

(c) $f(x)=4x^3+2x^2-18x-10$
$f(-2)=4(-2)^3+2(-2)^2-18(-2)-10$
$=-32+8+36-10$
$=-42+44$
$=2$
$f'(x)=12x^2+4x-18$
$f'(-2)=12(-2)^2+4(-2)-18$
$f'(-2)=48-8-18$
$f'(-2)=22$
$\therefore$ NORMAL $m=-\frac{1}{22}$ $(-2,2)$
$y-y_0=m(x-x_0)$
$y-2=-\frac{1}{22}(x+2)$
$22y-44=-x-2$
$22y+x=42$

(d) $f(x)=-2x^3+4x^2-1$
$f'(x)=-6x^2+8x$
$f'(2)=-6\times2^2+8\times2$
$f'(2)=-24+16=-8$
$f(2)=-2\times2^3+4\times2^2-1$
$f(2)=-16+16-1$
$\therefore$ NORMAL $m=\frac{1}{8}$, $(2,-1)$
$y-y_0=m(x-x_0)$
$y+1=\frac{1}{8}(x-2)$
$8y+8=x-2$
$8y=x-10$

Question 6 (non calculator)

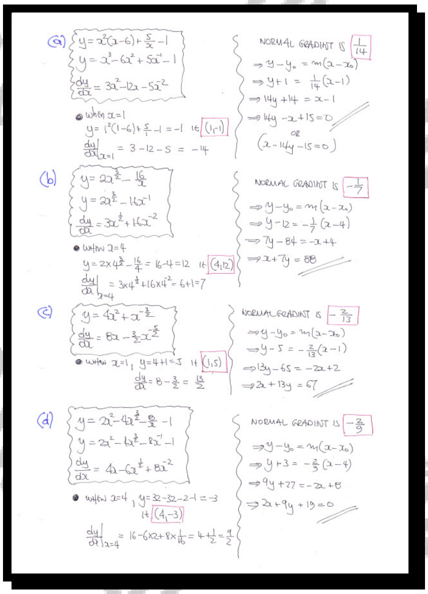
For each of the following curves find an equation of the normal to the curve at the point whose x coordinate is given.

a) $y = x^2(x-6) + \dfrac{5}{x} - 1$, where $x = 1$ $\boxed{x - 14y - 15 = 0}$

b) $y = 2x^{\frac{3}{2}} - \dfrac{16}{x}$, where $x = 4$ $\boxed{x + 7y = 88}$

c) $y = 4x^2 + x^{-\frac{3}{2}}$, where $x = 1$ $\boxed{2x + 13y = 67}$

d) $y = 2x^2 - 4x^{\frac{3}{2}} - \dfrac{8}{x} - 1$, where $x = 4$ $\boxed{2x + 9y + 19 = 0}$



STATIONARY POINTS

Question 1 (non calculator)

For each of the following cubic equations find the coordinates of their stationary points and determine their nature.

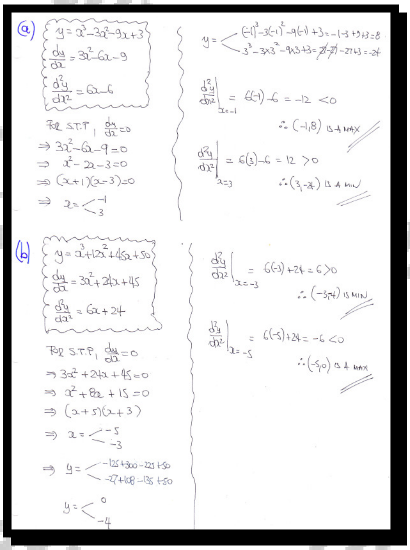
a) $y = x^3 - 3x^2 - 9x + 3$

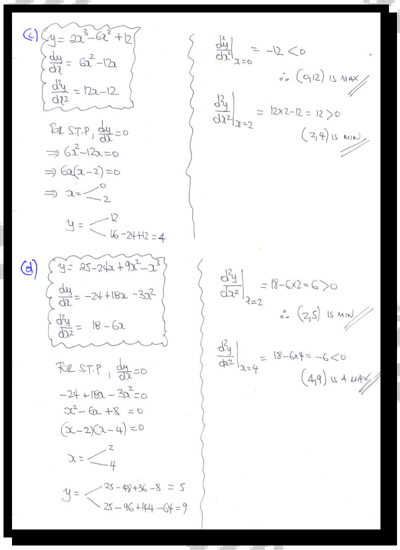
b) $y = x^3 + 12x^2 + 45x + 50$

c) $y = 2x^3 - 6x^2 + 12$

d) $y = 25 - 24x + 9x^2 - x^3$

$\boxed{\min(3,-24), \max(-1,8)}$, $\boxed{\min(-3,-4), \max(-5,0)}$, $\boxed{\min(2,4), \max(0,12)}$, $\boxed{\min(2,5), \max(4,9)}$





Question 2

For each of the following equations find the coordinates of their stationary points and determine their nature.

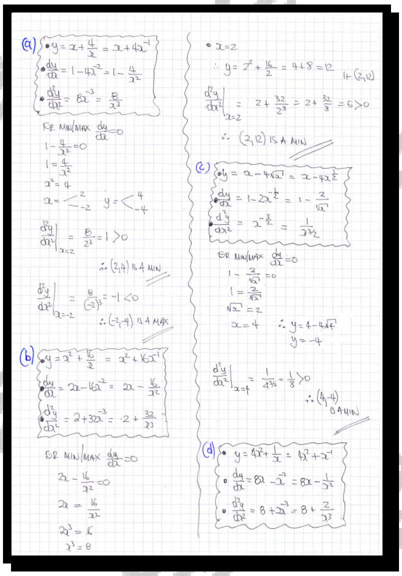
a) $y = x + \dfrac{4}{x}, \quad x \neq 0$

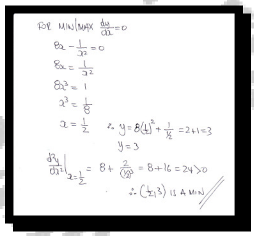
b) $y = x^2 + \dfrac{16}{x}, \quad x \neq 0$

c) $y = x - 4\sqrt{x}, \quad x > 0$

d) $y = 4x^2 + \dfrac{1}{x}, \quad x \neq 0$

$\boxed{\min(2,4), \max(-2,-4)}$, $\boxed{\min(2,12)}$, $\boxed{\min(4,-4)}$, $\boxed{\min\left(\frac{1}{2},3\right)}$





Question 3

For each of the following equations find the coordinates of their stationary points and determine their nature.

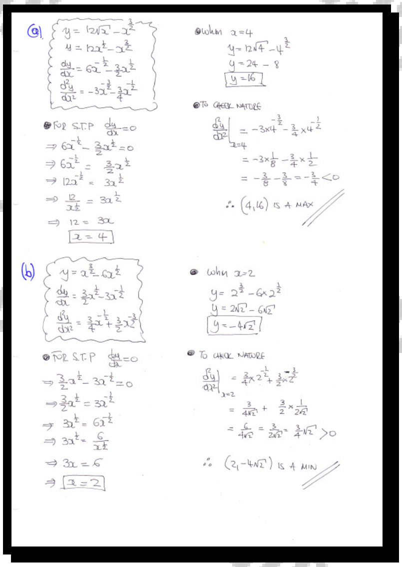
a) $y = 12\sqrt{x} - x^{\frac{3}{2}}, \quad x > 0$

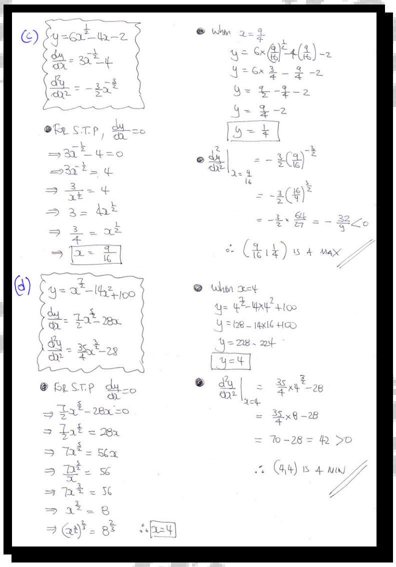
b) $y = x^{\frac{3}{2}} - 6x^{\frac{1}{2}}, \quad x > 0$

c) $y = 6x^{\frac{1}{2}} - 4x - 2, \quad x > 0$

d) $y = x^{\frac{7}{2}} - 14x^{2} + 100, \quad x > 0$

$\boxed{\max(4,16)}, \ \boxed{\min\left(2,-4\sqrt{2}\right)}, \ \boxed{\max\left(\frac{9}{16},\frac{1}{4}\right)}, \ \boxed{\min(4,4)}$





Question 4

For each of the following equations find the coordinates of their stationary points and determine their nature.

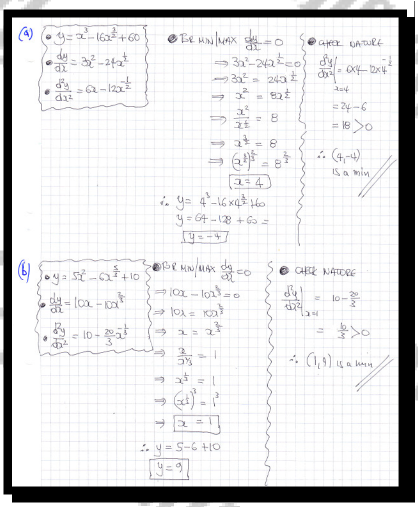
a) $y = x^3 - 16x^{\frac{3}{2}} + 60, \quad x > 0$

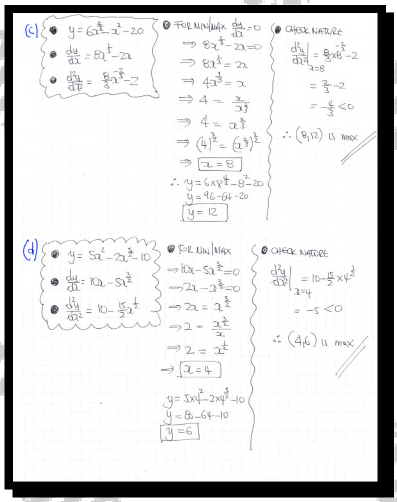
b) $y = 5x^2 - 6x^{\frac{5}{3}} + 10, \quad x > 0$

c) $y = 6x^{\frac{4}{3}} - x^2 - 20, \quad x > 0$

d) $y = 5x^2 - 2x^{\frac{5}{2}} - 10, \quad x > 0$

$\boxed{\min(4,-4)}$, $\boxed{\min(1,9)}$, $\boxed{\max(8,12)}$, $\boxed{\max(4,6)}$





Question 5

For each of the following equations find the coordinates of their stationary points and determine their nature.

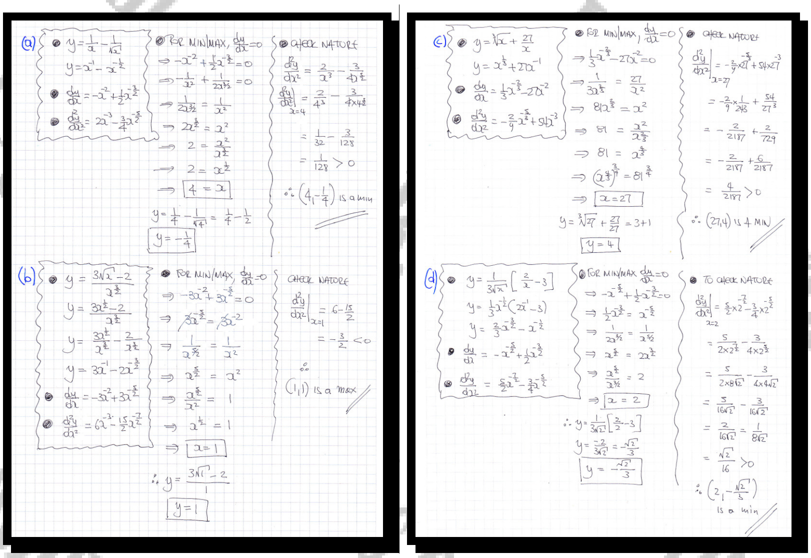
a) $y = \dfrac{1}{x} - \dfrac{1}{\sqrt{x}}, \quad x > 0$

b) $y = \dfrac{3\sqrt{x} - 2}{x^{\frac{3}{2}}}, \quad x > 0$

c) $y = \sqrt[3]{x} + \dfrac{27}{x}, \quad x > 0$

d) $y = \dfrac{1}{3\sqrt{x}}\left[\dfrac{2}{x} - 3\right], \quad x > 0$

$\boxed{\min\left(4, -\frac{1}{4}\right)}, \quad \boxed{\max(1,1)}, \quad \boxed{\min(27,4)}, \quad \boxed{\min\left(2, -\frac{\sqrt{2}}{3}\right)}$



INCREASING and DECREASING FUNCTIONS

Question 1

For each of the following equations find the range of the values of x, for which y is increasing or decreasing.

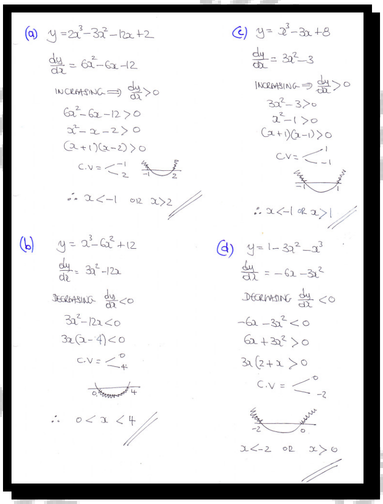
a) $y = 2x^3 - 3x^2 - 12x + 2$, increasing

b) $y = x^3 - 6x^2 + 12$, decreasing

c) $y = x^3 - 3x + 8$, increasing

d) $y = 1 - 3x^2 - x^3$, decreasing

$\boxed{x < -1 \text{ or } x > 2}$, $\boxed{0 < x < 4}$, $\boxed{x < -1 \text{ or } x > 1}$, $\boxed{x < -2 \text{ or } x > 0}$


(a) $y = 2x^3 - 3x^2 - 12x + 2$

$\frac{dy}{dx} = 6x^2 - 6x - 12$

INCREASING $\Rightarrow \frac{dy}{dx} > 0$

$6x^2 - 6x - 12 > 0$

$x^2 - x - 2 > 0$

$(x+1)(x-2) > 0$

C.V. = -1, 2

$\therefore x < -1$ OR $x > 2$

(c) $y = x^3 - 3x + 8$

$\frac{dy}{dx} = 3x^2 - 3$

INCREASING $\Rightarrow \frac{dy}{dx} > 0$

$3x^2 - 3 > 0$

$x^2 - 1 > 0$

$(x+1)(x-1) > 0$

C.V. = 1, -1

$\therefore x < -1$ OR $x > 1$

(b) $y = x^3 - 6x^2 + 12$

$\frac{dy}{dx} = 3x^2 - 12x$

DECREASING $\frac{dy}{dx} < 0$

$3x^2 - 12x < 0$

$3x(x-4) < 0$

C.V. = 0, 4

$\therefore 0 < x < 4$

(d) $y = 1 - 3x^2 - x^3$

$\frac{dy}{dx} = -6x - 3x^2$

DECREASING $\frac{dy}{dx} < 0$

$-6x - 3x^2 < 0$

$6x + 3x^2 > 0$

$3x(2 + x) > 0$

C.V. = 0, -2

$x < -2$ OR $x > 0$

Question 2

Find the range of the values of x, for which $f(x)$ is increasing or decreasing.

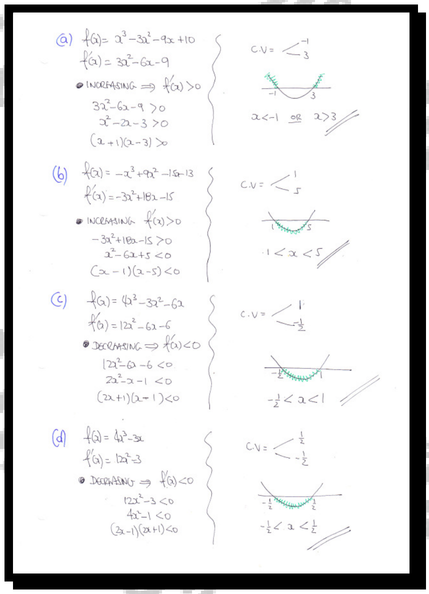
a) $f(x)=x^3-3x^2-9x+10$, increasing

b) $f(x)=-x^3+9x^2-15x-13$, increasing

c) $f(x)=4x^3-3x^2-6x$, decreasing

d) $f(x)=4x^3-3x$, decreasing

$\boxed{x<-1 \text{ or } x>3}$, $\boxed{1<x<5}$, $\boxed{-\frac{1}{2}<x<1}$, $\boxed{-\frac{1}{2}<x<\frac{1}{2}}$


(a) $f(x)=x^3-3x^2-9x+10$
$f'(x)=3x^2-6x-9$
• INCREASING $\Rightarrow f'(x)>0$
$3x^2-6x-9>0$
$x^2-2x-3>0$
$(x+1)(x-3)>0$
C.V. = -1, 3
$x<-1$ or $x>3$

(b) $f(x)=-x^3+9x^2-15x-13$
$f'(x)=-3x^2+18x-15$
• INCREASING $f'(x)>0$
$-3x^2+18x-15>0$
$x^2-6x+5<0$
$(x-1)(x-5)<0$
C.V. = 1, 5
$1<x<5$

(c) $f(x)=4x^3-3x^2-6x$
$f'(x)=12x^2-6x-6$
• DECREASING $\Rightarrow f'(x)<0$
$12x^2-6x-6<0$
$2x^2-x-1<0$
$(2x+1)(x-1)<0$
C.V. = 1, $-\frac{1}{2}$
$-\frac{1}{2}<x<1$

(d) $f(x)=4x^3-3x$
$f'(x)=12x^2-3$
• DECREASING $\Rightarrow f'(x)<0$
$12x^2-3<0$
$4x^2-1<0$
$(2x-1)(2x+1)<0$
C.V. = $\frac{1}{2}$, $-\frac{1}{2}$
$-\frac{1}{2}<x<\frac{1}{2}$

DIFFERENTIATION PRACTICE IN CONTEXT

Question 1

The curve C has equation

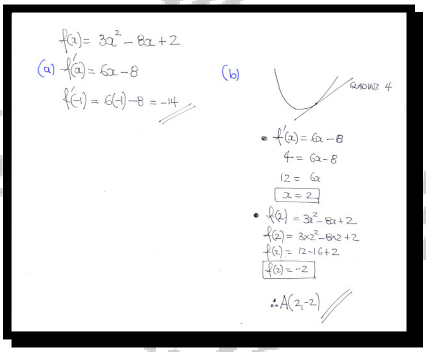
$$f(x) = 3x^2 - 8x + 2.$$

a) Find the gradient at the point on C, where $x = -1$.

The point A lies on C and the gradient at that point is 4.

b) Find the coordinates of A.

$\boxed{-14}$, $\boxed{A(2,-2)}$



Question 2

The curve C has equation

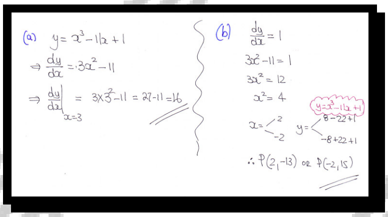
$$y = x^3 - 11x + 1.$$

a) Find the gradient at the point on C, where $x = 3$.

The point P lies on C and the gradient at that point is 1.

b) Find the possible coordinates of P.

$\boxed{16}$, $\boxed{P(2,-13) \text{ or } P(-2,15)}$



Question 3

The curve C has equation

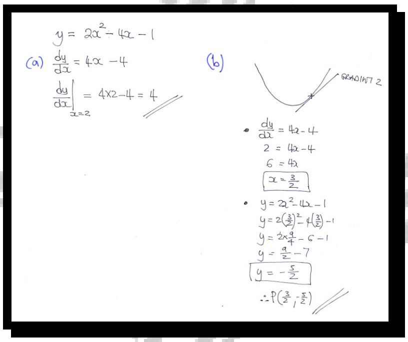
$$y = 2x^2 - 4x - 1.$$

a) Find the gradient at the point on C, where $x = 2$.

The point P lies on C and the gradient at that point is 2.

b) Find the coordinates of P.

$\boxed{4}$, $\boxed{P\left(\frac{3}{2}, -\frac{5}{2}\right)}$


$y = 2x^2 - 4x - 1$

(a) $\frac{dy}{dx} = 4x - 4$

$\left.\frac{dy}{dx}\right|_{x=2} = 4\times2 - 4 = 4$

(b)

- $\frac{dy}{dx} = 4x - 4$
  $2 = 4x - 4$
  $6 = 4x$
  $\boxed{x = \frac{3}{2}}$
- $y = 2x^2 - 4x - 1$
  $y = 2\left(\frac{3}{2}\right)^2 - 4\left(\frac{3}{2}\right) - 1$
  $y = 2\times\frac{9}{4} - 6 - 1$
  $y = \frac{9}{2} - 7$
  $\boxed{y = -\frac{5}{2}}$

$\therefore P\left(\frac{3}{2}, -\frac{5}{2}\right)$

Question 4

The curve C has equation

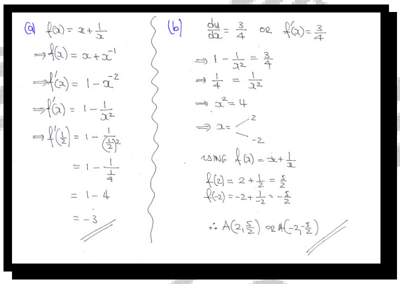
$$f(x) = x + \frac{1}{x}, \ x \neq 0.$$

a) Find the gradient at the point on C, where $x = \frac{1}{2}$.

The point A lies on C and the gradient at that point is $\frac{3}{4}$.

b) Find the possible coordinates of A.

$\boxed{-3}$, $\boxed{A\left(2, \frac{5}{2}\right) \text{ or } A\left(-2, -\frac{5}{2}\right)}$



(a) $f(x) = x + \frac{1}{x}$

$\Rightarrow f(x) = x + x^{-1}$

$\Rightarrow f'(x) = 1 - x^{-2}$

$\Rightarrow f'(x) = 1 - \frac{1}{x^2}$

$\Rightarrow f'\left(\frac{1}{2}\right) = 1 - \frac{1}{\left(\frac{1}{2}\right)^2}$

$= 1 - \frac{1}{\frac{1}{4}}$

$= 1 - 4$

$= -3$

(b) $\frac{dy}{dx} = \frac{3}{4}$ OR $f'(x) = \frac{3}{4}$

$\Rightarrow 1 - \frac{1}{x^2} = \frac{3}{4}$

$\Rightarrow \frac{1}{4} = \frac{1}{x^2}$

$\Rightarrow x^2 = 4$

$\Rightarrow x = 2, -2$

USING $f(x) = x + \frac{1}{x}$

$f(2) = 2 + \frac{1}{2} = \frac{5}{2}$

$f(-2) = -2 + \frac{1}{-2} = -\frac{5}{2}$

$\therefore A\left(2, \frac{5}{2}\right)$ OR $A\left(-2, -\frac{5}{2}\right)$

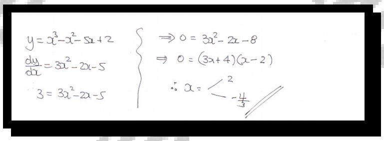
Question 5

The curve C has equation

$$y = x^3 - x^2 - 5x + 2 .$$

Find the x coordinates of the points on C with gradient 3.

$$\boxed{x = -\frac{4}{3}, 2}$$


$y = x^3 - x^2 - 5x + 2$
$\frac{dy}{dx} = 3x^2 - 2x - 5$
$3 = 3x^2 - 2x - 5$
$\Rightarrow 0 = 3x^2 - 2x - 8$
$\Rightarrow 0 = (3x+4)(x-2)$
$\therefore x = 2, -\frac{4}{3}$

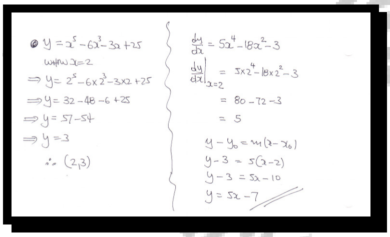
Question 6

The curve C has equation

$$y = x^5 - 6x^3 - 3x + 25 .$$

Find an equation of the tangent to C at the point where $x = 2$.

$$\boxed{y = 5x - 7}$$


$y = x^5 - 6x^3 - 3x + 25$
when $x = 2$
$\Rightarrow y = 2^5 - 6 \times 2^3 - 3 \times 2 + 25$
$\Rightarrow y = 32 - 48 - 6 + 25$
$\Rightarrow y = 57 - 54$
$\Rightarrow y = 3$
$\therefore (2,3)$
$\frac{dy}{dx} = 5x^4 - 18x^2 - 3$
$\left.\frac{dy}{dx}\right|_{x=2} = 5 \times 2^4 - 18 \times 2^2 - 3$
$= 80 - 72 - 3$
$= 5$
$y - y_0 = m(x - x_0)$
$y - 3 = 5(x - 2)$
$y - 3 = 5x - 10$
$y = 5x - 7$

Question 7

The curve C has equation

$$y = -x^2(x+1), \ x \in \mathbb{R}.$$

The curve meets the coordinate axes at the origin O and at the point A.

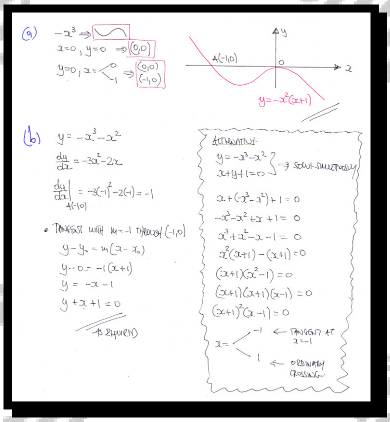
a) Sketch the graph of C, indicating clearly the coordinates of A.

b) Show that the straight line with equation

$$x + y + 1 = 0,$$

is a tangent to C at A.

$\boxed{A(-1,0)}$



Question 8

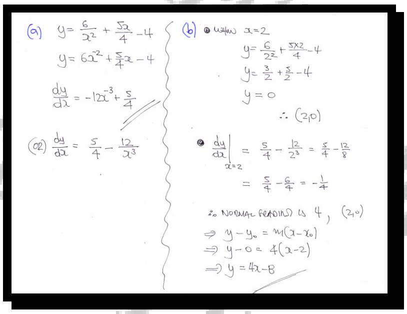
The curve C has equation

$$y = \frac{6}{x^2} + \frac{5x}{4} - 4, \quad x \neq 0.$$

a) Find an expression for $\frac{dy}{dx}$.

b) Determine an equation of the normal to the curve at the point where $x = 2$.

$$\boxed{\frac{dy}{dx} = \frac{5}{4} - \frac{12}{x^3}}, \quad \boxed{y = 4x - 8}$$



(a) $y = \frac{6}{x^2} + \frac{5x}{4} - 4$

$y = 6x^{-2} + \frac{5}{4}x - 4$

$\frac{dy}{dx} = -12x^{-3} + \frac{5}{4}$

(OR) $\frac{dy}{dx} = \frac{5}{4} - \frac{12}{x^3}$

(b) • WHEN $x = 2$

$y = \frac{6}{2^2} + \frac{5 \times 2}{4} - 4$

$y = \frac{3}{2} + \frac{5}{2} - 4$

$y = 0$

$\therefore (2,0)$

• $\left.\frac{dy}{dx}\right|_{x=2} = \frac{5}{4} - \frac{12}{2^3} = \frac{5}{4} - \frac{12}{8}$

$= \frac{5}{4} - \frac{6}{4} = -\frac{1}{4}$

$\therefore$ NORMAL GRADIENT IS 4, $(2,0)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 0 = 4(x - 2)$

$\Rightarrow y = 4x - 8$

Question 9

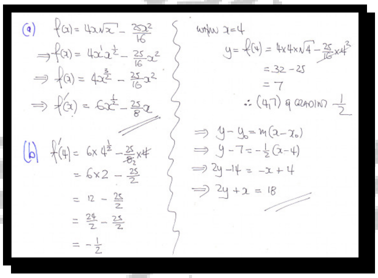
The curve C has equation

$$f(x) = 4x\sqrt{x} - \frac{25x^2}{16}, \quad x \geq 0.$$

a) Find a simplified expression for $f'(x)$.

b) Determine an equation of the tangent to C at the point where $x = 4$, giving the answer in the form $ax + by = c$, where a, b and c are integers.

$\boxed{f'(x) = 6x^{\frac{1}{2}} - \frac{25}{8}x}$, $\boxed{x + 2y = 18}$



(a) $f(x) = 4x\sqrt{x} - \frac{25x^2}{16}$

$\Rightarrow f(x) = 4x^1x^{\frac{1}{2}} - \frac{25}{16}x^2$

$\Rightarrow f(x) = 4x^{\frac{3}{2}} - \frac{25}{16}x^2$

$\Rightarrow f'(x) = 6x^{\frac{1}{2}} - \frac{25}{8}x$

(b) $f'(4) = 6 \times 4^{\frac{1}{2}} - \frac{25}{8} \times 4$

$= 6 \times 2 - \frac{25}{2}$

$= 12 - \frac{25}{2}$

$= \frac{24}{2} - \frac{25}{2}$

$= -\frac{1}{2}$

when $x = 4$

$y = f(4) = 4 \times 4 \times \sqrt{4} - \frac{25}{16} \times 4^2$

$= 32 - 25$

$= 7$

$\therefore (4,7)$ & gradient $-\frac{1}{2}$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 7 = -\frac{1}{2}(x - 4)$

$\Rightarrow 2y - 14 = -x + 4$

$\Rightarrow 2y + x = 18$

Question 10

A curve has the following equation

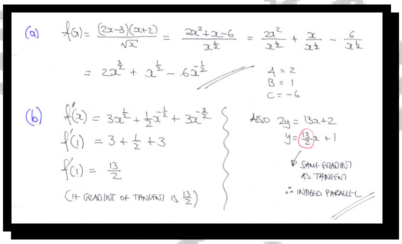
$$f(x) = \frac{(2x-3)(x+2)}{\sqrt{x}}, \quad x > 0.$$

a) Express $f(x)$ in the form $Ax^{\frac{3}{2}} + Bx^{\frac{1}{2}} + Cx^{-\frac{1}{2}}$, where A, B and C are constants to be found.

b) Show that the tangent to the curve at the point where $x = 1$ is parallel to the line with equation

$$2y = 13x + 2.$$

$\boxed{A = 2}$, $\boxed{B = 1}$, $\boxed{C = -6}$


(a) $f(x) = \frac{(2x-3)(x+2)}{\sqrt{x}} = \frac{2x^2+x-6}{x^{\frac{1}{2}}} = \frac{2x^2}{x^{\frac{1}{2}}} + \frac{x}{x^{\frac{1}{2}}} - \frac{6}{x^{\frac{1}{2}}}$

$= 2x^{\frac{3}{2}} + x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}$

A = 2
B = 1
C = −6

(b) $f'(x) = 3x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} + 3x^{-\frac{3}{2}}$

$f'(1) = 3 + \frac{1}{2} + 3$

$f'(1) = \frac{13}{2}$

(IF GRADIENT OF TANGENT IS $\frac{13}{2}$)

ALSO $2y = 13x + 2$

$y = \frac{13}{2}x + 1$

SAME GRADIENT AS TANGENT

∴ INDEED PARALLEL

Question 11

A cubic curve has equation

$$f(x) = 2x^3 - 7x^2 + 6x + 1.$$

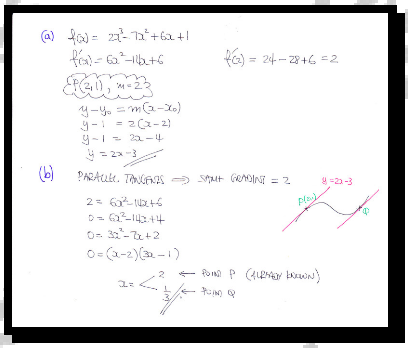
The point $P(2,1)$ lies on the curve.

a) Find an equation of the tangent to the curve at P.

The point Q lies on the curve so that the tangent to the curve at Q is parallel to the tangent to the curve at P.

b) Determine the x coordinate of Q.

$\boxed{y = 2x - 3}$, $\boxed{x_Q = \frac{1}{3}}$


(a) $f(x) = 2x^3 - 7x^2 + 6x + 1$

$f'(x) = 6x^2 - 14x + 6$ $\quad f'(2) = 24 - 28 + 6 = 2$

$P(2,1)$, $m = 2$

$y - y_0 = m(x - x_0)$

$y - 1 = 2(x - 2)$

$y - 1 = 2x - 4$

$y = 2x - 3$

(b) PARALLEL TANGENTS $\Longrightarrow$ SAME GRADIENT $= 2$

$2 = 6x^2 - 14x + 6$

$0 = 6x^2 - 14x + 4$

$0 = 3x^2 - 7x + 2$

$0 = (x - 2)(3x - 1)$

$x = 2$ ← POINT P (ALREADY KNOWN)

$x = \frac{1}{3}$ ← POINT Q

Question 12

The curve C has equation

$$y = 2x^3 - 9x^2 + 12x - 10 .$$

a) Find the coordinates of the two points on the curve where the gradient is zero.

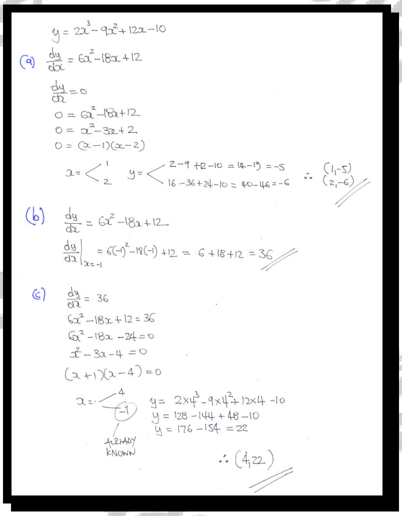
The point P lies on C and its x coordinate is -1.

b) Determine the gradient of C at the point P.

The point Q lies on C so that the gradient at Q is the same as the gradient at P.

c) Find the coordinates of Q.

$\boxed{(1,-5),(2,-6)}$, $\boxed{36}$, $\boxed{Q(4,22)}$


$y = 2x^3 - 9x^2 + 12x - 10$

(a) $\frac{dy}{dx} = 6x^2 - 18x + 12$

$\frac{dy}{dx} = 0$

$0 = 6x^2 - 18x + 12$

$0 = x^2 - 3x + 2$

$0 = (x-1)(x-2)$

$x = 1, 2$ $\quad y = 2 - 9 + 12 - 10 = 14 - 19 = -5$; $\quad 16 - 36 + 24 - 10 = 40 - 46 = -6$ $\quad \therefore (1,-5), (2,-6)$

(b) $\frac{dy}{dx} = 6x^2 - 18x + 12$

$\left.\frac{dy}{dx}\right|_{x=-1} = 6(-1)^2 - 18(-1) + 12 = 6 + 18 + 12 = 36$

(c) $\frac{dy}{dx} = 36$

$6x^2 - 18x + 12 = 36$

$6x^2 - 18x - 24 = 0$

$x^2 - 3x - 4 = 0$

$(x+1)(x-4) = 0$

$x = 4, -1$ (ALREADY KNOWN)

$y = 2 \times 4^3 - 9 \times 4^2 + 12 \times 4 - 10$

$y = 128 - 144 + 48 - 10$

$y = 176 - 154 = 22$

$\therefore (4, 22)$

Question 13

The curve C has equation

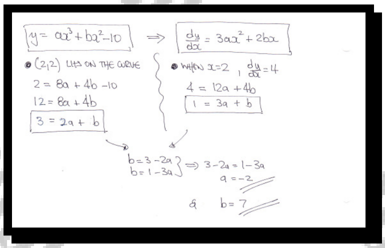
$$y = ax^3 + bx^2 - 10,$$

where a and b are constants.

The point $A(2,2)$ lies on C.

Given that the gradient at A is 4, determine the value of a and the value of b.

$\boxed{a = -2}$, $\boxed{b = 7}$



Question 14

The curve C has equation

$$y = x^3 - 4x^2 + 6x - 3.$$

The point $P(2,1)$ lies on C and the straight line L_1 is the tangent to C at P.

a) Find an equation of L_1.

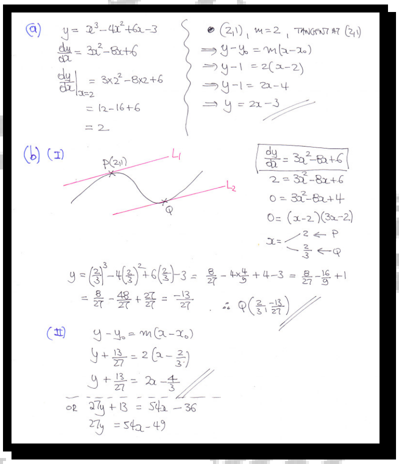
The straight line L_2 is a tangent to C at the point Q.

b) Given that L_2 is parallel to L_1, determine …

i. … the exact coordinates of Q.

ii. … an equation of L_2.

$\boxed{y = 2x - 3}$, $\boxed{Q\left(\frac{2}{3}, -\frac{13}{27}\right)}$, $\boxed{27y = 54x - 49}$



Question 15

A curve C and a straight line L have respective equations

$$y = 2x^2 - 6x + 5 \quad \text{and} \quad 2y + x = 4 .$$

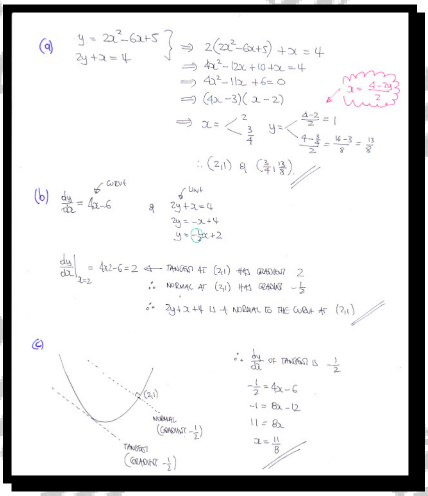
a) Find the coordinates of the points of intersection between C and L.

b) Show that L is a normal to C.

The tangent to C at the point P is parallel to L.

c) Determine the x coordinate of P.

$$\boxed{(2,1), \left(\frac{3}{4}, \frac{13}{8}\right)}, \quad \boxed{x_P = \frac{11}{8}}$$



Question 16

The curve C has equation

$$y = 2x^3 - 6x^2 + 3x + 5.$$

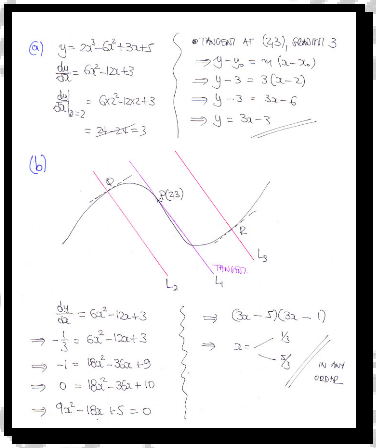
The point $P(2,3)$ lies on C and the straight line L_1 is the tangent to C at P.

a) Find an equation of L_1.

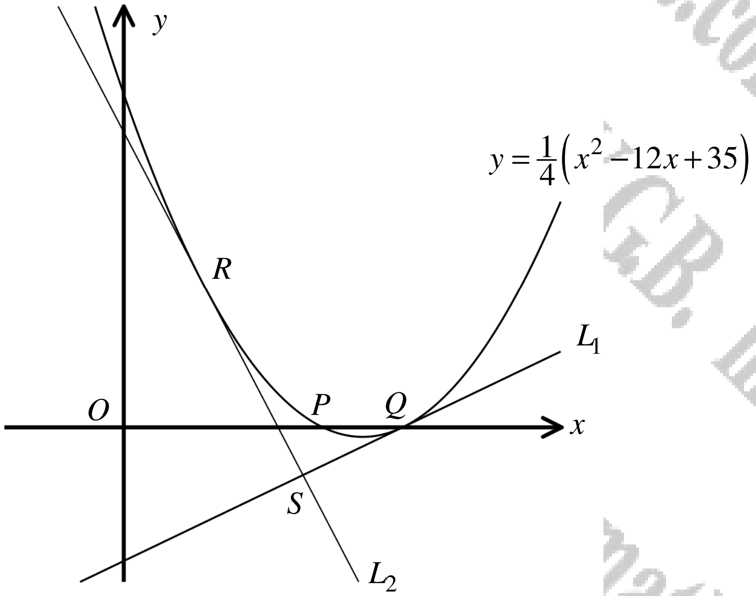
The straight lines L_2 and L_3 are parallel to L_1, and they are the respective normals to C at the points Q and R.

b) Determine the x coordinate of Q and the x coordinate of R.

$\boxed{y = 3x - 3}$, $\boxed{x = \frac{1}{3}, \frac{5}{3}}$



Question 17



The figure above shows the curve with equation

$$y=\frac{1}{4}\left(x^2-12x+35\right).$$

The curve crosses the x axis at the points $P(x_1,0)$ and $Q(x_2,0)$, where $x_2 > x_1$.

The tangent to the curve at Q is the straight line L_1.

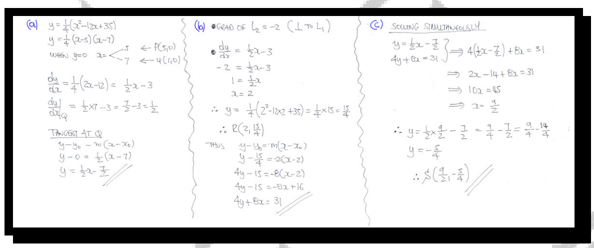
a) Find an equation of L_1.

The tangent to the curve at the point R is denoted by L_2. It is further given that L_2 meets L_1 at right angles, at the point S.

b) Find an equation of L_2.

c) Determine the exact coordinates of S.

[ ], $y=\frac{1}{2}x-\frac{7}{2}$, $4y+8x=31$, $S\left(\frac{9}{2},-\frac{5}{4}\right)$



Question 18

The point $P(1,0)$ lies on the curve C with equation

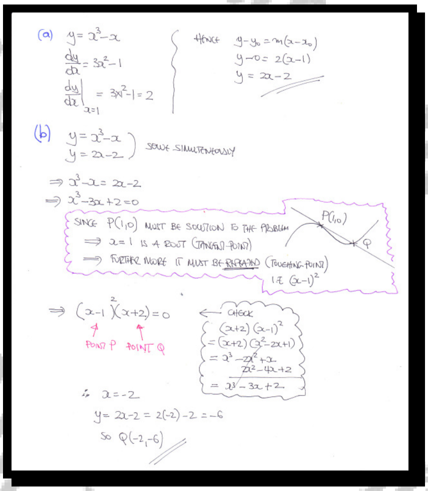
$$y = x^3 - x, \quad x \in \mathbb{R}.$$

a) Find an equation of the tangent to C at P, giving the answer in the form $y = mx + c$, where m and c are constants.

The tangent to C at P meets C again at the point Q.

b) Determine the coordinates of Q.

$\boxed{y = 2x - 2}$, $\boxed{Q(-2,-6)}$



Question 19

A curve C with equation

$$y = 4x^3 + 7x^2 + x + 11, \quad x \in \mathbb{R}.$$

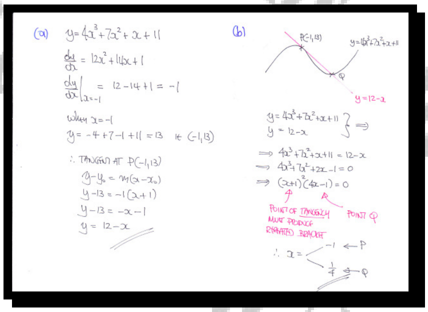
The point P lies on C, where $x = -1$.

a) Find an equation of the tangent to C at P.

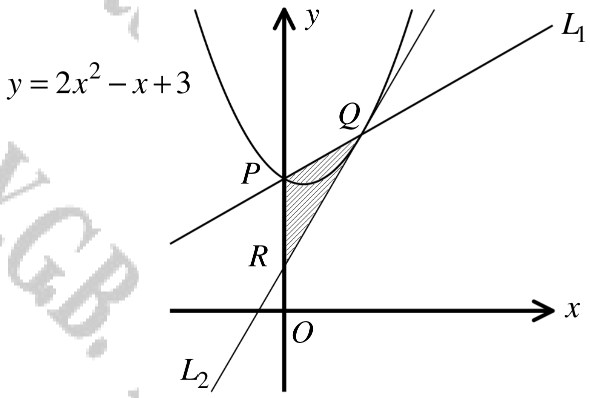
The tangent to C at P meets C again at the point Q.

b) Determine the x coordinate of Q.

$\boxed{y = 12 - x}$, $\boxed{x_Q = \frac{1}{4}}$



Question 20



The figure above shows the curve C with equation

$$y = 2x^2 - x + 3.$$

C crosses the y axis at the point P. The normal to C at P is the straight line L_1.

a) Find an equation of L_1.

L_1 meets the curve again at the point Q.

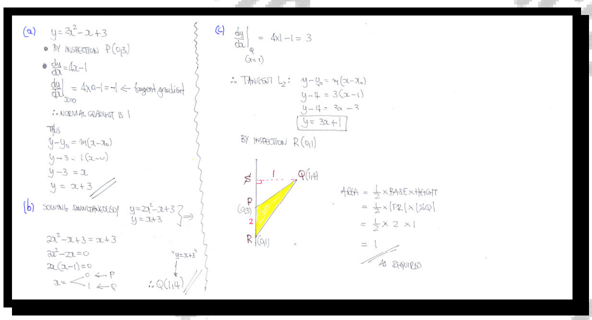
b) Determine the coordinates of Q.

The tangent to C at Q is the straight line L_2.

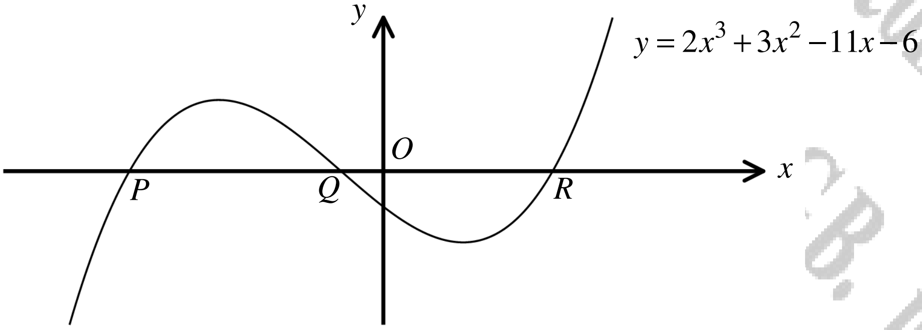
L_2 meets the y axis at the point R.

c) Show that the area of the triangle PQR is one square unit.

$\boxed{y = x + 3}$, $\boxed{Q(1,4)}$



Question 21



The figure above shows the curve C with equation

$$y = 2x^3 + 3x^2 - 11x - 6.$$

The curve crosses the x axis at the points P, Q and $R(2,0)$.

The tangent to C at R is the straight line L_1.

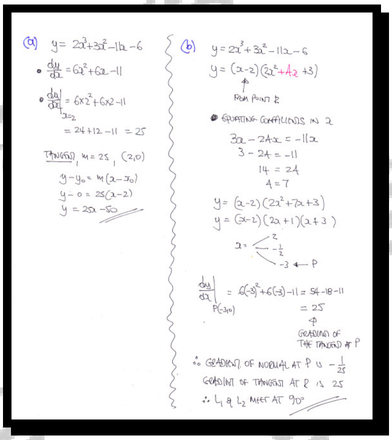
a) Find an equation of L_1.

The normal to C at P is the straight line L_2.

The straight lines L_1 and L_2 meet at the point S.

b) Show that $\measuredangle PSR = 90°$.

$\boxed{y = 25x - 50}$



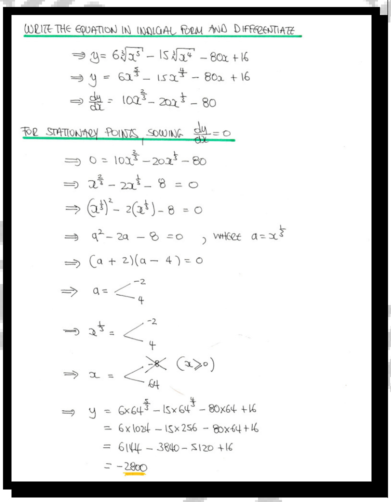
Question 22

A curve has equation

$$y = 6\sqrt[3]{x^5} - 15\sqrt[3]{x^4} - 80x + 16, \qquad x \in \mathbb{R}, \qquad x \geq 0.$$

Find the coordinates of the stationary point of the curve and determine whether it is a local maximum, a local minimum or a point of inflexion.

local minimum at $(16, -2800)$


WRITE THE EQUATION IN INDICIAL FORM AND DIFFERENTIATE

$\Rightarrow y = 6\sqrt[3]{x^5} - 15\sqrt[3]{x^4} - 80x + 16$

$\Rightarrow y = 6x^{\frac{5}{3}} - 15x^{\frac{4}{3}} - 80x + 16$

$\Rightarrow \frac{dy}{dx} = 10x^{\frac{2}{3}} - 20x^{\frac{1}{3}} - 80$

FOR STATIONARY POINTS, SOLVING $\frac{dy}{dx} = 0$

$\Rightarrow 0 = 10x^{\frac{2}{3}} - 20x^{\frac{1}{3}} - 80$

$\Rightarrow x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 8 = 0$

$\Rightarrow (x^{\frac{1}{3}})^2 - 2(x^{\frac{1}{3}}) - 8 = 0$

$\Rightarrow a^2 - 2a - 8 = 0$, WHERE $a = x^{\frac{1}{3}}$

$\Rightarrow (a+2)(a-4) = 0$

$\Rightarrow a = -2, 4$

$\Rightarrow x^{\frac{1}{3}} = -2, 4$

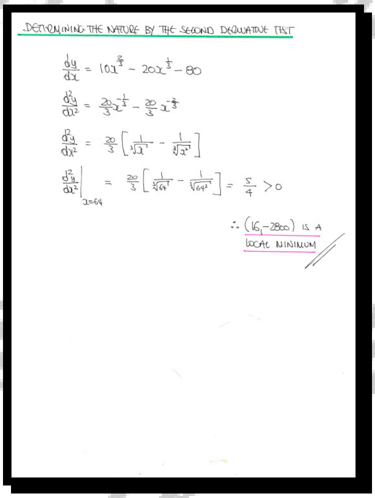
$\Rightarrow x = \cancel{-8}\ (x \geq 0),\ 64$

$\Rightarrow y = 6 \times 64^{\frac{5}{3}} - 15 \times 64^{\frac{4}{3}} - 80 \times 64 + 16$

$= 6 \times 1024 - 15 \times 256 - 80 \times 64 + 16$

$= 6144 - 3840 - 5120 + 16$

$= -2800$


DETERMINING THE NATURE BY THE SECOND DERIVATIVE TEST

$\frac{dy}{dx} = 10x^{\frac{2}{3}} - 20x^{\frac{1}{3}} - 80$

$\frac{d^2y}{dx^2} = \frac{20}{3}x^{-\frac{1}{3}} - \frac{20}{3}x^{-\frac{2}{3}}$

$\frac{d^2y}{dx^2} = \frac{20}{3}\left[\frac{1}{\sqrt[3]{x}} - \frac{1}{\sqrt[3]{x^2}}\right]$

$\left.\frac{d^2y}{dx^2}\right|_{x=64} = \frac{20}{3}\left[\frac{1}{\sqrt[3]{64}} - \frac{1}{\sqrt[3]{64^2}}\right] = \frac{5}{4} > 0$

$\therefore (16, -2800)$ IS A LOCAL MINIMUM

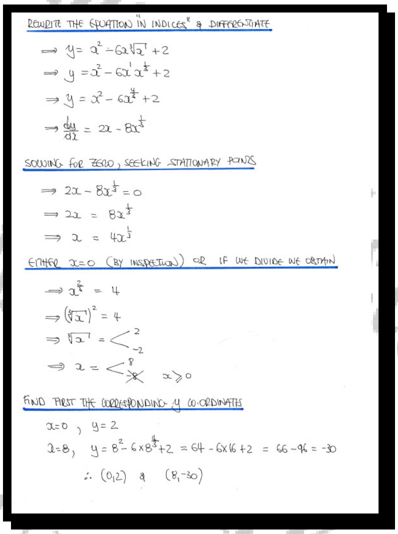
Question 23

A curve has equation

$$y = x^2 - 6x\sqrt[3]{x} + 2, \qquad x \in \mathbb{R}, \qquad x \geq 0.$$

Find the coordinates of the stationary points of the curve and classify them as local maxima, local minima or a points of inflexion.

local minimum at $(8,-30)$, local maximum at $(0,2)$



REWRITE THE EQUATION "IN INDICES" & DIFFERENTIATE

$\Rightarrow y = x^2 - 6x\sqrt[3]{x} + 2$

$\Rightarrow y = x^2 - 6x^1 x^{\frac{1}{3}} + 2$

$\Rightarrow y = x^2 - 6x^{\frac{4}{3}} + 2$

$\Rightarrow \frac{dy}{dx} = 2x - 8x^{\frac{1}{3}}$

SOLVING FOR ZERO, SEEKING STATIONARY POINTS

$\Rightarrow 2x - 8x^{\frac{1}{3}} = 0$

$\Rightarrow 2x = 8x^{\frac{1}{3}}$

$\Rightarrow x = 4x^{\frac{1}{3}}$

EITHER $x=0$ (BY INSPECTION) OR IF WE DIVIDE WE OBTAIN

$\Rightarrow x^{\frac{2}{3}} = 4$

$\Rightarrow (\sqrt[3]{x})^2 = 4$

$\Rightarrow \sqrt[3]{x} = 2$ or -2

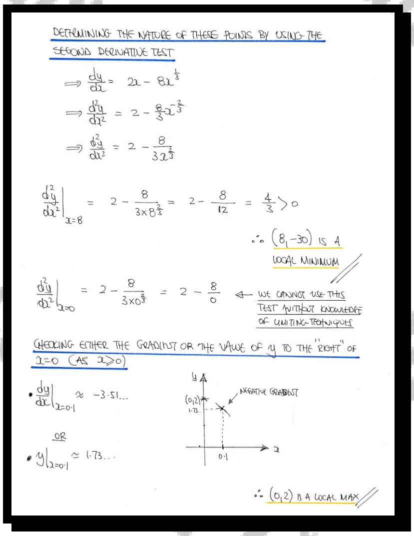
$\Rightarrow x = 8$ or ~~-8~~ $\quad x \geq 0$

FIND FIRST THE CORRESPONDING y CO-ORDINATES

$x=0, \; y=2$

$x=8, \; y = 8^2 - 6 \times 8^{\frac{4}{3}} + 2 = 64 - 6 \times 16 + 2 = 66 - 96 = -30$

$\therefore (0,2)$ & $(8,-30)$



DETERMINING THE NATURE OF THESE POINTS BY USING THE SECOND DERIVATIVE TEST

$\Rightarrow \frac{dy}{dx} = 2x - 8x^{\frac{1}{3}}$

$\Rightarrow \frac{d^2y}{dx^2} = 2 - \frac{8}{3}x^{-\frac{2}{3}}$

$\Rightarrow \frac{d^2y}{dx^2} = 2 - \frac{8}{3x^{\frac{2}{3}}}$

$\left.\frac{d^2y}{dx^2}\right|_{x=8} = 2 - \frac{8}{3 \times 8^{\frac{2}{3}}} = 2 - \frac{8}{12} = \frac{4}{3} > 0$

$\therefore (8,-30)$ IS A LOCAL MINIMUM

$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 2 - \frac{8}{3 \times 0^{\frac{2}{3}}} = 2 - \frac{8}{0}$ ← WE CANNOT USE THIS TEST WITHOUT KNOWLEDGE OF LIMITING TECHNIQUES

CHECKING EITHER THE GRADIENT OR THE VALUE OF y TO THE "RIGHT" OF $x=0$ (AS $x \geq 0$)

- $\left.\frac{dy}{dx}\right|_{x=0.1} \approx -3.51\ldots$

OR

- $\left. y \right|_{x=0.1} \simeq 1.73\ldots$

$\therefore (0,2)$ IS A LOCAL MAX

Question 24

A curve has equation

$$y = x\left(x^2 - 128\sqrt{x}\right), \qquad x \in \mathbb{R}, \qquad x > 0.$$

The curve has a single stationary point with coordinates $\left(2^\alpha, -2^\beta\right)$, where α and β are positive integers.

Find the value of β and justify that the stationary point is a local minimum.

$\boxed{\beta = 12}$

REWRITE THE EQUATION IN INDICIAL FORM & DIFFERENTIATE

$\Rightarrow y = x\left(x^2 - 128\sqrt{x}\right)$

$\Rightarrow y = x\left(x^2 - 128x^{\frac{1}{2}}\right)$

$\Rightarrow y = x^3 - 128x^{\frac{3}{2}}$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 192x^{\frac{1}{2}}$

FOR STATIONARY POINTS SET $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 192x^{\frac{1}{2}} = 0$

$\Rightarrow x^2 - 64x^{\frac{1}{2}} = 0$

$\Rightarrow x^2 = 64x^{\frac{1}{2}}$

$\Rightarrow \frac{x^2}{x^{\frac{1}{2}}} = 64$ (WE ARE NOT CONCERNED WITH $x = 0$)

$\Rightarrow x^{\frac{3}{2}} = 64$

$\Rightarrow \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = (64)^{\frac{2}{3}}$

$\Rightarrow x^1 = \left(\sqrt[3]{64}\right)^2$

$\Rightarrow x = 16$

CHECK THE NATURE OF THE POINT BY THE SECOND DERIVATIVE TEST

$\Rightarrow \frac{d^2y}{dx^2} = 6x - 96x^{-\frac{1}{2}}$

$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=16} = 6 \times 16 - 96 \times 16^{-\frac{1}{2}} = 96 - 96 \times \frac{1}{4} = 96 - 24 = 72 > 0$

LOCAL MINIMUM

FINALLY TO FIND THE y CO-ORDINATE IN THE REQUIRED FORM

$y = x\left(x^2 - 128\sqrt{x}\right)$

$y = 16\left(16^2 - 128\sqrt{16}\right)$

$y = -4096$

$y = -2^{12}$ (TRIAL & ERROR OF POWER OF 2)

$\therefore$ LOCAL MINIMUM AT $\left(16, -2^{12}\right)$

Question 25

The point P, whose x coordinate is $\frac{1}{4}$, lies on the curve with equation

$$y = \frac{k + 4x\sqrt{x}}{7x}, \quad x \in \mathbb{R}, \quad x > 0,$$

where k is a non zero constant.

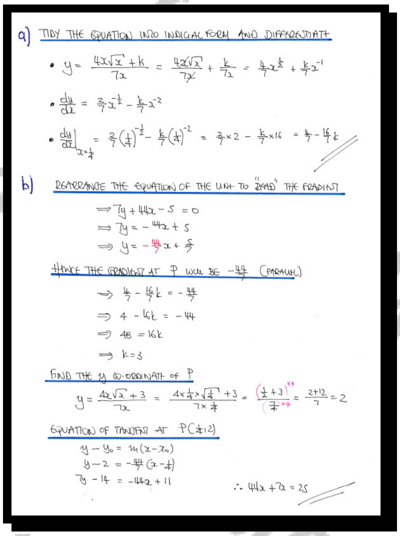
a) Determine, in terms of k, the gradient of the curve at P.

The tangent to the curve at P is parallel to the straight line with equation

$$44x + 7y - 5 = 0.$$

b) Find an equation of the tangent to the curve at P.

$$\boxed{\left.\frac{dy}{dx}\right|_{x=\frac{1}{4}} = \frac{4-16k}{7}}, \quad \boxed{44x + 7y = 25}$$



a) TIDY THE EQUATION INTO INDICIAL FORM AND DIFFERENTIATE

- $y = \frac{4x\sqrt{x}+k}{7x} = \frac{4x\sqrt{x}}{7x} + \frac{k}{7x} = \frac{4}{7}x^{\frac{1}{2}} + \frac{k}{7}x^{-1}$
- $\frac{dy}{dx} = \frac{2}{7}x^{-\frac{1}{2}} - \frac{k}{7}x^{-2}$
- $\left.\frac{dy}{dx}\right|_{x=\frac{1}{4}} = \frac{2}{7}\left(\frac{1}{4}\right)^{-\frac{1}{2}} - \frac{k}{7}\left(\frac{1}{4}\right)^{-2} = \frac{2}{7}\times 2 - \frac{k}{7}\times 16 = \frac{4}{7} - \frac{16}{7}k$

b) REARRANGE THE EQUATION OF THE LINE TO "READ" THE GRADIENT

$\Rightarrow 7y + 44x - 5 = 0$
$\Rightarrow 7y = -44x + 5$
$\Rightarrow y = -\frac{44}{7}x + \frac{5}{7}$

HENCE THE GRADIENT AT P WILL BE $-\frac{44}{7}$ (PARALLEL)

$\Rightarrow \frac{4}{7} - \frac{16}{7}k = -\frac{44}{7}$
$\Rightarrow 4 - 16k = -44$
$\Rightarrow 48 = 16k$
$\Rightarrow k = 3$

FIND THE y CO-ORDINATE OF P

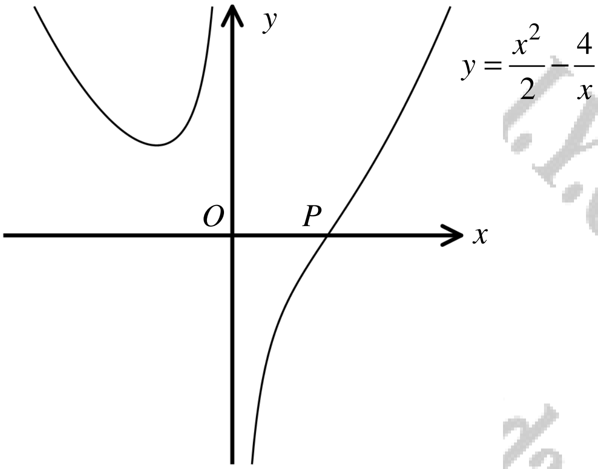
$y = \frac{4x\sqrt{x}+3}{7x} = \frac{4\times\frac{1}{4}\times\sqrt{\frac{1}{4}}+3}{7\times\frac{1}{4}} = \frac{\left(\frac{1}{2}+3\right)^{\times 4}}{\left(\frac{7}{4}\right)^{\times 4}} = \frac{2+12}{7} = 2$

EQUATION OF TANGENT AT $P\left(\frac{1}{4}, 2\right)$

$y - y_0 = m(x - x_0)$
$y - 2 = -\frac{44}{7}\left(x - \frac{1}{4}\right)$
$7y - 14 = -44x + 11$

$\therefore 44x + 7y = 25$

Question 26



The figure above shows the curve C with equation

$$y = \frac{x^2}{2} - \frac{4}{x}, \quad x \neq 0.$$

The curve crosses the x axis at the point P.

The straight line L is the normal to C at P.

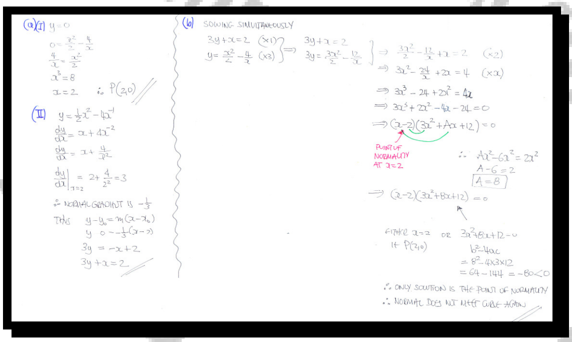
a) Find …

i. … the coordinates of P.

ii. … an equation of L.

b) Show that L does not meet C again.

$\boxed{P(2,0)}$, $\boxed{x + 3y = 2}$



Question 27

The curve C has equation

$$y=(x-1)\left(x^2+4x+5\right),\ x\in\mathbb{R}.$$

a) Show that C meets the x axis at only one point.

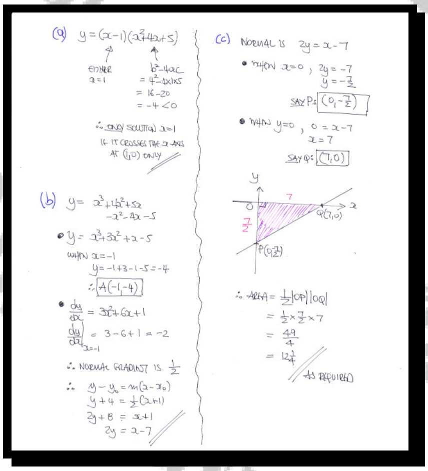
The point A, where $x=-1$, lies on C.

b) Find an equation of the normal to C at A.

The normal to C at A meets the coordinate axes at the points P and Q.

c) Show further that the area of the triangle OPQ, where O is the origin, is $12\frac{1}{4}$ square units.

$\boxed{2y=x-7}$


(a) $y=(x-1)(x^2+4x+5)$

EITHER $x=1$

$b^2-4ac = 4^2-4\times1\times5 = 16-20 = -4<0$

∴ ONLY SOLUTION $x=1$

IE IT CROSSES THE x-AXIS AT $(1,0)$ ONLY

(b) $y = x^3+4x^2+5x-x^2-4x-5$

• $y=x^3+3x^2+x-5$

WHEN $x=-1$

$y=-1+3-1-5=-4$

∴ $A(-1,-4)$

• $\frac{dy}{dx}=3x^2+6x+1$

$\left.\frac{dy}{dx}\right|_{x=-1} = 3-6+1=-2$

∴ NORMAL GRADIENT IS $\frac{1}{2}$

∴ $y-y_0=m(x-x_0)$

$y+4=\frac{1}{2}(x+1)$

$2y+8=x+1$

$2y=x-7$

(c) NORMAL IS $2y=x-7$

• WHEN $x=0$, $2y=-7$, $y=-\frac{7}{2}$

SAY $P: (0,-\frac{7}{2})$

• WHEN $y=0$, $0=x-7$, $x=7$

SAY $Q: (7,0)$

∴ AREA $=\frac{1}{2}|OP||OQ|$

$=\frac{1}{2}\times\frac{7}{2}\times7$

$=\frac{49}{4}$

$=12\frac{1}{4}$

AS REQUIRED

Question 28

A curve has equation

$$y = x - 8\sqrt{x}, \; x \in \mathbb{R}, \; x \ge 0 .$$

The curve meets the coordinate axes at the origin and at the point P.

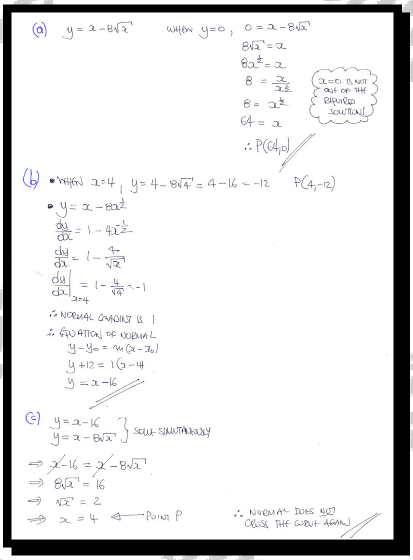
a) Determine the coordinates of P.

The point Q, where $x = 4$, lies on the curve.

b) Find an equation of the normal to curve at Q.

c) Show clearly that the normal to the curve at Q does not meet the curve again.

$\boxed{P(64,0)}$, $\boxed{y = x - 16}$



Question 29

The curve C has equation

$$y = x^3 - 9x^2 + 24x - 19, \; x \in \mathbb{R}.$$

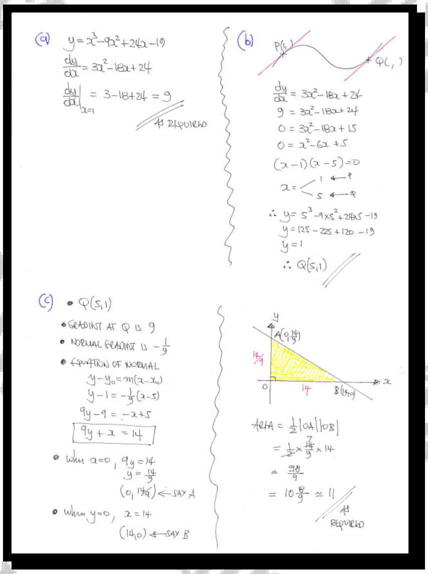
a) Show that the tangent to C at the point P, where $x = 1$, has gradient 9.

b) Find the coordinates of another point Q on C at which the tangent also has gradient 9.

The normal to C at Q meets the coordinate axes at the points A and B.

c) Show further that the **approximate** area of the triangle OAB, where O is the origin, is 11 square units.

$\boxed{Q(5,1)}$



Question 30

The point $A(2,1)$ lies on the curve with equation

$$y = \frac{(x-1)(x+2)}{2x}, \quad x \in \mathbb{R}, \; x \neq 0 .$$

a) Find the gradient of the curve at A.

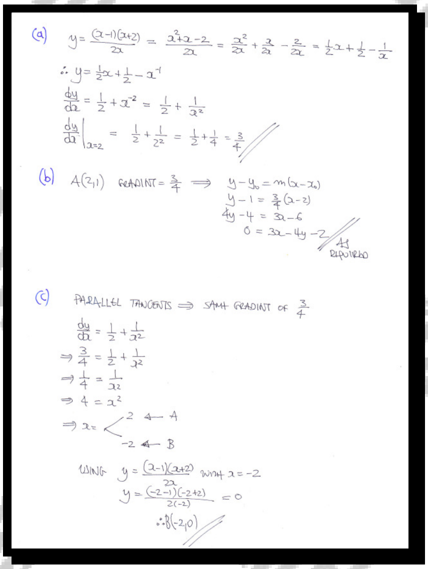
b) Show that the tangent to the curve at A has equation

$$3x - 4y - 2 = 0 .$$

The tangent to the curve at the point B is parallel to the tangent to the curve at A.

c) Determine the coordinates of B.

$\boxed{\text{gradient at } A = \frac{3}{4}}$, $\boxed{B(-2,0)}$


(a) $y = \frac{(x-1)(x+2)}{2x} = \frac{x^2+x-2}{2x} = \frac{x^2}{2x} + \frac{x}{2x} - \frac{2}{2x} = \frac{1}{2}x + \frac{1}{2} - \frac{1}{x}$

$\therefore y = \frac{1}{2}x + \frac{1}{2} - x^{-1}$

$\frac{dy}{dx} = \frac{1}{2} + x^{-2} = \frac{1}{2} + \frac{1}{x^2}$

$\left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{2} + \frac{1}{2^2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

(b) $A(2,1)$ GRADIENT $= \frac{3}{4} \Longrightarrow$ $y - y_0 = m(x - x_0)$

$y - 1 = \frac{3}{4}(x-2)$

$4y - 4 = 3x - 6$

$0 = 3x - 4y - 2$ AS REQUIRED

(c) PARALLEL TANGENTS $\Rightarrow$ SAME GRADIENT OF $\frac{3}{4}$

$\frac{dy}{dx} = \frac{1}{2} + \frac{1}{x^2}$

$\Rightarrow \frac{3}{4} = \frac{1}{2} + \frac{1}{x^2}$

$\Rightarrow \frac{1}{4} = \frac{1}{x^2}$

$\Rightarrow 4 = x^2$

$\Rightarrow x = 2 \leftarrow A$, $-2 \leftarrow B$

USING $y = \frac{(x-1)(x+2)}{2x}$ WITH $x = -2$

$y = \frac{(-2-1)(-2+2)}{2(-2)} = 0$

$\therefore B(-2,0)$

Question 31

The curve C has equation $y = f(x)$ given by

$$f(x) = 2(x-2)^3, \quad x \in \mathbb{R}.$$

a) Sketch the graph of $f(x)$.

b) Find an expression for $f'(x)$.

The point $P(3,2)$ lies on C and the straight line l_1 is the tangent to C at P.

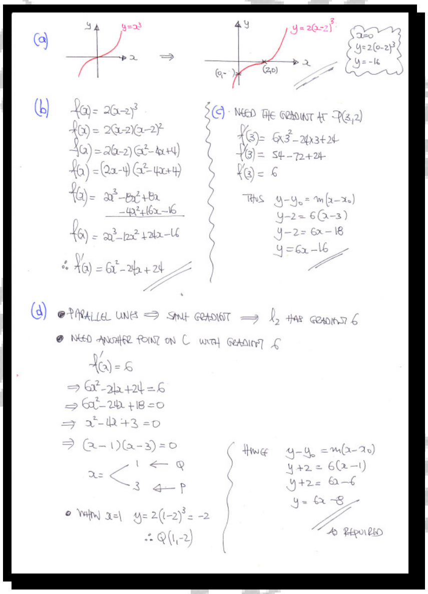
c) Find an equation of l_1.

The straight line l_2 is another tangent at a different point Q on C.

d) Given that l_1 is parallel to l_2 show that an equation of l_2 is

$$y = 6x - 8.$$

$\boxed{f'(x) = 6x^2 - 24x + 24}$, $\boxed{y = 6x - 16}$



Question 32

The point $P(2,9)$ lies on the curve C with equation

$$y = x^3 - 3x^2 + 2x + 9, \; x \in \mathbb{R}, \; x \geq 1.$$

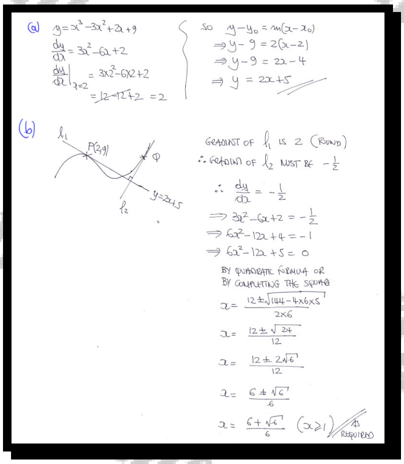
a) Find an equation of the tangent to C at P, giving the answer in the form $y = mx + c$, where m and c are constants.

The point Q also lies on C so that the tangent to C at Q is perpendicular to the tangent to C at P.

b) Show that the x coordinate of Q is

$$\frac{6+\sqrt{6}}{6}.$$

$\boxed{y = 2x + 5}$



Question 33

The volume, V cm^3, of a soap bubble is modelled by the formula

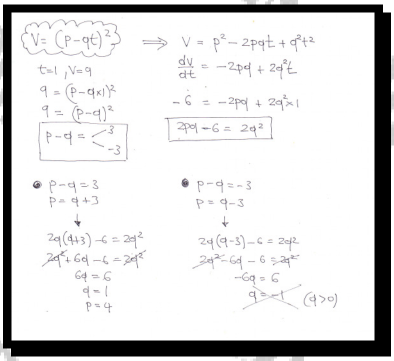
$$V = (p - qt)^2, \ t \geq 0,$$

where p and q are positive constants, and t is the time in seconds, measured after a certain instant.

When $t = 1$ the volume of a soap bubble is 9 cm^3 and at that instant its volume is decreasing at the rate of 6 cm^3 per second.

Determine the value of p and the value of q.

$\boxed{p = 4, \quad q = 1}$



Question 34

A curve C has equation

$$y = 2x^3 - 5x^2 + a, \ x \in \mathbb{R},$$

where a is a constant.

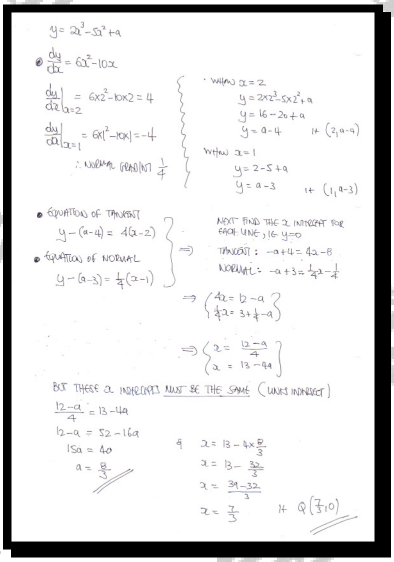
The tangent to C at the point where $x = 2$ and the normal to C at the point where $x = 1$, meet at the point Q.

Given that Q lies on the x axis, determine in any order …

a) … the value of a.

b) … the coordinates of Q.

$\boxed{a = \frac{8}{3}}$, $\boxed{Q\left(\frac{7}{3}, 0\right)}$



$y = 2x^3 - 5x^2 + a$

- $\frac{dy}{dx} = 6x^2 - 10x$

$\left.\frac{dy}{dx}\right|_{x=2} = 6 \times 2^2 - 10 \times 2 = 4$

$\left.\frac{dy}{dx}\right|_{x=1} = 6 \times 1^2 - 10 \times 1 = -4$

$\therefore$ NORMAL GRADIENT $\frac{1}{4}$

- WHEN $x = 2$
  $y = 2 \times 2^3 - 5 \times 2^2 + a$
  $y = 16 - 20 + a$
  $y = a - 4$ i.e. $(2, a-4)$

  WHEN $x = 1$
  $y = 2 - 5 + a$
  $y = a - 3$ i.e. $(1, a-3)$

- EQUATION OF TANGENT
  $y - (a-4) = 4(x-2)$
- EQUATION OF NORMAL
  $y - (a-3) = \frac{1}{4}(x-1)$

$\Rightarrow$ NEXT FIND THE x INTERCEPT FOR EACH LINE, i.e. $y = 0$

TANGENT: $-a + 4 = 4x - 8$

NORMAL: $-a + 3 = \frac{1}{4}x - \frac{1}{4}$

$\Rightarrow \begin{cases} 4x = 12 - a \\ \frac{1}{4}x = 3 + \frac{1}{4} - a \end{cases}$

$\Rightarrow \begin{cases} x = \frac{12 - a}{4} \\ x = 13 - 4a \end{cases}$

BUT THESE x INTERCEPTS MUST BE THE SAME (LINES INTERSECT)

$\frac{12 - a}{4} = 13 - 4a$

$12 - a = 52 - 16a$

$15a = 40$

$a = \frac{8}{3}$

& $x = 13 - 4 \times \frac{8}{3}$

$x = 13 - \frac{32}{3}$

$x = \frac{39 - 32}{3}$

$x = \frac{7}{3}$ i.e. $Q\left(\frac{7}{3}, 0\right)$

Question 35

The curve C has equation

$$y = \frac{x^3\left(5x\sqrt{x} - 128\right)}{\sqrt{x}}, \quad x \in \mathbb{R}, \quad x > 0.$$

a) Determine expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

b) Show that the y coordinate of the stationary point of C is $-k\sqrt[3]{4}$, where k is a positive integer.

c) Evaluate $\frac{d^2y}{dx^2}$ at the stationary point of C.

Give the answer in terms of $\sqrt[3]{2}$.

d) Find the value of $\frac{d^3y}{dx^3}$ at the point on C, where $\frac{d^2y}{dx^2} = 0$.

$\boxed{}$, $\boxed{\frac{dy}{dx} = 20x^3 - 320x^{\frac{3}{2}}}$, $\boxed{\frac{d^2y}{dx^2} = 60x^2 - 480x^{\frac{1}{2}}}$, $\boxed{\frac{d^3y}{dx^3} = 120x - 240x^{-\frac{1}{2}}}$,

$\boxed{k = 3072}$, $\boxed{960\sqrt[3]{2}}$, $\boxed{360}$