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DIFFERENTIATION PRACTICE

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THE OPERATION OF DIFFERENTIATION

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Question 1

Evaluate the following.

a) $\frac{d}{dx}(5x^6)$

$$\frac{d}{dx}(5x^6) = 30x^5$$

b) $\frac{d}{dx}(2x^{\frac{3}{2}})$

$$\frac{d}{dx}(2x^{\frac{3}{2}}) = 3x^{\frac{1}{2}}$$

c) $\frac{d}{dx}(6x^4 - x^3)$

$$\frac{d}{dx}(6x^4 - x^3) = 24x^3 - 3x^2$$

d) $\frac{d}{dx}(3x^2 + 5x + 1)$

$$\frac{d}{dx}(3x^2 + 5x + 1) = 6x + 5$$

e) $\frac{d}{dx}(4x^{\frac{1}{2}} - 2x - 7)$

$$\frac{d}{dx}(4x^{\frac{1}{2}} - 2x - 7) = 2x^{-\frac{1}{2}} - 2$$

Handwritten solutions for Question 1:

- a) $\frac{d}{dx}(5x^6) = 6 \times 5x^{6-1} = 30x^5$
- b) $\frac{d}{dx}(2x^{\frac{3}{2}}) = \frac{3}{2} \times 2x^{\frac{3}{2}-1} = 3x^{\frac{1}{2}}$
- c) $\frac{d}{dx}(6x^4 - x^3) = 4 \times 6x^{4-1} - 3x^{3-1} = 24x^3 - 3x^2$
- d) $\frac{d}{dx}(3x^2 + 5x + 1) = 2 \times 3x^{2-1} + 5(1) + 0 = 6x + 5$
- e) $\frac{d}{dx}(4x^{\frac{1}{2}} - 2x - 7) = \frac{1}{2} \times 4x^{\frac{1}{2}-1} - 2(1) - 0 = 2x^{-\frac{1}{2}} - 2$

Question 2

Evaluate the following.

a) $\frac{d}{dx}(4x^3)$

$$\frac{d}{dx}(4x^3) = 12x^2$$

b) $\frac{d}{dx}(7x^5)$

$$\frac{d}{dx}(7x^5) = 35x^4$$

c) $\frac{d}{dx}(4x^2 + 3x^4)$

$$\frac{d}{dx}(4x^2 + 3x^4) = 8x + 12x^3$$

d) $\frac{d}{dx}(x^2 + 7x + 5)$

$$\frac{d}{dx}(x^2 + 7x + 5) = 2x + 7$$

e) $\frac{d}{dx}(8x^{\frac{1}{2}} + 2x^{-2})$

$$\frac{d}{dx}(8x^{\frac{1}{2}} + 2x^{-2}) = 4x^{-\frac{1}{2}} - 4x^{-3}$$

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(4x^3) &= 12x^2 \\ \text{(b)} \quad \frac{d}{dx}(7x^5) &= 35x^4 \\ \text{(c)} \quad \frac{d}{dx}(4x^2 + 3x^4) &= 8x + 12x^3 \\ \text{(d)} \quad \frac{d}{dx}(x^2 + 7x + 5) &= 2x + 7 + 0 = 2x + 7 \\ \text{(e)} \quad \frac{d}{dx}(8x^{\frac{1}{2}} + 2x^{-2}) &= 4x^{-\frac{1}{2}} - 4x^{-3} \end{aligned}$$

Question 3

Differentiate the following expressions with respect to x

a) $y = x^2 - 4x^6$

$$\frac{dy}{dx} = 2x - 24x^5$$

b) $y = 5x^3 - 6x^{\frac{3}{2}}$

$$\frac{dy}{dx} = 15x^2 - 9x^{\frac{1}{2}}$$

c) $y = 9x^{-3} + 7x^{-2}$

$$\frac{dy}{dx} = -27x^{-4} - 14x^{-3}$$

d) $y = 5 - 5x^{-1}$

$$\frac{dy}{dx} = 5x^{-2}$$

e) $y = 7x + \sqrt{x}$

$$\frac{dy}{dx} = 7 + \frac{1}{2}x^{-\frac{1}{2}}$$

Handwritten solutions for Question 3:

- a) $y = x^2 - 4x^6$
 $\frac{d}{dx}(x^2 - 4x^6) = \frac{d}{dx}(y) = \frac{dy}{dx} = 2x - 24x^5$
- b) $y = 5x^3 - 6x^{\frac{3}{2}}$
 $\frac{d}{dx}(5x^3 - 6x^{\frac{3}{2}}) = \frac{d}{dx}(y) = \frac{dy}{dx} = 15x^2 - 9x^{\frac{1}{2}}$
- c) $y = 9x^{-3} + 7x^{-2}$
 $\frac{d}{dx}(9x^{-3} + 7x^{-2}) = \frac{d}{dx}(y) = \frac{dy}{dx} = -27x^{-4} - 14x^{-3}$
- d) $y = 5 - 5x^{-1}$
 $\frac{d}{dx}(5 - 5x^{-1}) = \frac{d}{dx}(y) = \frac{dy}{dx} = 0 + 5x^{-2} = 5x^{-2}$
- e) $y = 7x + \sqrt{x}$
 $\frac{d}{dx}(7x + \sqrt{x}) = \frac{d}{dx}(7x + x^{\frac{1}{2}}) = \frac{d}{dx}(y) = \frac{dy}{dx} = 7 + \frac{1}{2}x^{-\frac{1}{2}}$

Question 4

Differentiate the following expressions with respect to x

a) $y = x^6 - 7x^2$

$$\frac{dy}{dx} = 6x^5 - 14x$$

b) $y = 1 - 6x^{\frac{5}{2}}$

$$\frac{dy}{dx} = 15x^{\frac{3}{2}}$$

c) $y = 2x + 8x^{-2}$

$$\frac{dy}{dx} = 2 + 16x^{-3}$$

d) $y = (2x-1)(4x+3)$

$$\frac{dy}{dx} = 16x + 2$$

e) $y = 4x^3(2-3x)$

$$\frac{dy}{dx} = 24x^2 - 48x^3$$

Handwritten solutions for Question 4:

- a) $y = x^6 - 7x^2$
 $\frac{d}{dx}(x^6 - 7x^2) = \frac{dy}{dx} = \frac{dx^6}{dx} - \frac{d7x^2}{dx} = 6x^5 - 14x$
- b) $y = 1 - 6x^{\frac{5}{2}}$
 $\frac{d}{dx}(1 - 6x^{\frac{5}{2}}) = \frac{dy}{dx} = \frac{d1}{dx} - \frac{d6x^{\frac{5}{2}}}{dx} = 0 - \frac{5}{2} \times 6x^{\frac{3}{2}} = 15x^{\frac{3}{2}}$
- c) $y = 2x - 8x^{-2}$
 $\frac{d}{dx}(2x - 8x^{-2}) = \frac{dy}{dx} = \frac{d2x}{dx} - \frac{d8x^{-2}}{dx} = 2 + 16x^{-3}$
- d) $y = (2x-1)(4x+3)$
 $\frac{d}{dx}[(2x-1)(4x+3)] = \frac{dy}{dx} = \frac{d}{dx}[8x^2 + 2x - 3] = \frac{d8x^2}{dx} + \frac{d2x}{dx} - \frac{d3}{dx} = 16x + 2$
- e) $y = 4x^3(2-3x)$
 $\frac{d}{dx}[4x^3(2-3x)] = \frac{dy}{dx} = \frac{d}{dx}[8x^3 - 12x^4] = \frac{d8x^3}{dx} - \frac{d12x^4}{dx} = 24x^2 - 48x^3$

Question 5

Find $f'(x)$ for each of the following functions.

a) $f(x) = 4x^3 - 9x + 2$

$$f'(x) = 12x^2 - 9$$

b) $f(x) = 6x^{-\frac{1}{2}} + 2x$

$$f'(x) = -3x^{-\frac{3}{2}} + 2$$

c) $f(x) = x^4 + 2x^{\frac{5}{2}}$

$$f'(x) = 4x^3 + 5x^{\frac{3}{2}}$$

d) $f(x) = \frac{1}{2}x^2 - 4x^{-\frac{3}{2}}$

$$f'(x) = x + 6x^{-\frac{5}{2}}$$

e) $f(x) = \frac{1}{2}x^{\frac{1}{3}} + 5x$

$$f'(x) = \frac{1}{6}x^{-\frac{2}{3}} + 5$$

$$\begin{aligned}
 \text{a) } f(x) &= 4x^3 - 9x + 2 \\
 \frac{d}{dx}(4x^3 - 9x + 2) &= \frac{d}{dx}(f(x)) = f'(x) = 12x^2 - 9 \\
 \text{b) } f(x) &= 6x^{-\frac{1}{2}} + 2x \\
 \frac{d}{dx}(6x^{-\frac{1}{2}} + 2x) &= \frac{d}{dx}(f(x)) = f'(x) = -3x^{-\frac{3}{2}} + 2 \\
 \text{c) } f(x) &= x^4 + 2x^{\frac{5}{2}} \\
 \frac{d}{dx}(x^4 + 2x^{\frac{5}{2}}) &= \frac{d}{dx}(f(x)) = f'(x) = 4x^3 + 5x^{\frac{3}{2}} \\
 \text{d) } f(x) &= \frac{1}{2}x^2 - 4x^{-\frac{3}{2}} \\
 \frac{d}{dx}(\frac{1}{2}x^2 - 4x^{-\frac{3}{2}}) &= \frac{d}{dx}(f(x)) = f'(x) = x + 6x^{-\frac{5}{2}} \\
 \text{e) } f(x) &= \frac{1}{2}x^{\frac{1}{3}} + 5x \\
 \frac{d}{dx}(\frac{1}{2}x^{\frac{1}{3}} + 5x) &= \frac{d}{dx}(f(x)) = f'(x) = \frac{1}{6}x^{-\frac{2}{3}} + 5
 \end{aligned}$$

Question 6

Differentiate each of the following functions with respect to x .

a) $f(x) = 6x^{-\frac{3}{2}} + 4x + 1$

$$f'(x) = -9x^{-\frac{5}{2}} + 4$$

b) $g(x) = x^4 - x^{-1}$

$$g'(x) = 4x^3 + x^{-2}$$

c) $h(x) = 9x^2 - \frac{1}{2}x^4$

$$h'(x) = 18x - 2x^3$$

d) $p(x) = 4x^{\frac{1}{2}} - 6x^{\frac{1}{3}} + \frac{1}{2}x^{-\frac{1}{4}}$

$$p'(x) = 2x^{-\frac{1}{2}} - 2x^{-\frac{2}{3}} - \frac{1}{8}x^{-\frac{5}{4}}$$

e) $v(x) = \left(8x + \frac{1}{2}\right)^2$

$$v'(x) = 128x + 8$$

Handwritten solutions for Question 6:

- a) $f(x) = 6x^{-\frac{3}{2}} + 4x + 1$
 $\frac{d}{dx}(6x^{-\frac{3}{2}} + 4x + 1) = \frac{d}{dx}(6x^{-\frac{3}{2}}) + \frac{d}{dx}(4x) + \frac{d}{dx}(1) = -\frac{3 \times 6x^{-\frac{5}{2}}}{2} + 4 + 0$
 $\therefore f'(x) = -9x^{-\frac{5}{2}} + 4$
- b) $g(x) = x^4 - x^{-1}$
 $\frac{d}{dx}(x^4 - x^{-1}) = \frac{d}{dx}(x^4) - \frac{d}{dx}(x^{-1}) = 4x^3 - (-1)x^{-2}$
 $\therefore g'(x) = 4x^3 + x^{-2}$
- c) $h(x) = 9x^2 - \frac{1}{2}x^4$
 $\frac{d}{dx}(9x^2 - \frac{1}{2}x^4) = \frac{d}{dx}(9x^2) - \frac{d}{dx}(\frac{1}{2}x^4) = 18x - 2x^3$
- d) $p(x) = 4x^{\frac{1}{2}} - 6x^{\frac{1}{3}} + \frac{1}{2}x^{-\frac{1}{4}}$
 $\frac{d}{dx}(4x^{\frac{1}{2}} - 6x^{\frac{1}{3}} + \frac{1}{2}x^{-\frac{1}{4}}) = \frac{d}{dx}(4x^{\frac{1}{2}}) - \frac{d}{dx}(6x^{\frac{1}{3}}) + \frac{d}{dx}(\frac{1}{2}x^{-\frac{1}{4}})$
 $= 2x^{-\frac{1}{2}} - 2x^{-\frac{2}{3}} - \frac{1}{8}x^{-\frac{5}{4}}$
- e) $v(x) = (8x + \frac{1}{2})^2$
 $\frac{d}{dx}(8x + \frac{1}{2})^2 = 2(8x + \frac{1}{2}) \times \frac{d}{dx}(8x + \frac{1}{2}) = 2(8x + \frac{1}{2}) \times 8 = 128x + 8$

Question 7

Carry out the following differentiations.

a) $\frac{d}{dt}(4t^2 - 7t + 5)$

$$\frac{d}{dt}(4t^2 - 7t + 5) = 8t - 7$$

b) $\frac{d}{dy}\left(y^{\frac{1}{2}} - \frac{2}{3}y^{-\frac{1}{2}}\right)$

$$\frac{d}{dy}\left(y^{\frac{1}{2}} - \frac{2}{3}y^{-\frac{1}{2}}\right) = \frac{1}{2}y^{-\frac{1}{2}} + \frac{1}{3}y^{-\frac{3}{2}}$$

c) $\frac{d}{dz}(2z^2 - 3z^{-1} + z)$

$$\frac{d}{dz}(2z^2 - 3z^{-1} + z) = 4z + 3z^{-2} + 1$$

d) $\frac{d}{dw}\left(w^2 - w^{-\frac{3}{2}}\right)$

$$\frac{d}{dw}\left(w^2 - w^{-\frac{3}{2}}\right) = 2w + \frac{3}{2}w^{-\frac{5}{2}}$$

e) $\frac{d}{dx}(ax^2 - 3x^2)$

$$\frac{d}{dx}(ax^2 - 3x^2) = 2ax - 6x$$

$$\begin{array}{l}
 \text{(a)} \frac{d}{dt}(4t^2 - 7t + 5) = 8t - 7 \\
 \text{(b)} \frac{d}{dy}\left(y^{\frac{1}{2}} - \frac{2}{3}y^{-\frac{1}{2}}\right) = \frac{1}{2}y^{-\frac{1}{2}} + \frac{1}{3}y^{-\frac{3}{2}} \\
 \text{(c)} \frac{d}{dz}(2z^2 - 3z^{-1} + z) = 4z + 3z^{-2} + 1 \\
 \text{(d)} \frac{d}{dw}\left(w^2 - w^{-\frac{3}{2}}\right) = 2w + \frac{3}{2}w^{-\frac{5}{2}} \\
 \text{(e)} \frac{d}{dx}(ax^2 - 3x^2) = 2ax - 6x \\
 \frac{d}{dx}[(0-3)x^2] = 2(0-3)x
 \end{array}$$

Question 8

Carry out the following differentiations.

a) $\frac{d}{dy}(4y^3 + 6y + 2)$

$$\frac{d}{dy}(4y^3 + 6y + 2) = 12y^2 + 6$$

b) $\frac{d}{dt}(7t^2 - 4t^{\frac{1}{2}})$

$$\frac{d}{dt}(7t^2 - 4t^{\frac{1}{2}}) = 14t - 2t^{-\frac{1}{2}}$$

c) $\frac{d}{dx}(ax^2 + bx + c)$

$$\frac{d}{dx}(ax^2 + bx + c) = 2ax + b$$

d) $\frac{d}{dz}\left(\frac{1}{4}z^2 - \frac{1}{z}\right)$

$$\frac{d}{dz}\left(\frac{1}{4}z^2 - \frac{1}{z}\right) = \frac{1}{2}z + \frac{1}{z^2}$$

e) $\frac{d}{dw}\left(\frac{1}{4}w^{\frac{4}{5}} + \frac{k}{w^2}\right)$

$$\frac{d}{dw}\left(\frac{1}{4}w^{\frac{4}{5}} + \frac{k}{w^2}\right) = \frac{1}{5}w^{-\frac{1}{5}} - \frac{2k}{w^3}$$

$$\begin{aligned}
 \text{a) } \frac{d}{dy}(4y^3 + 6y + 2) &= 12y^2 + 6 \\
 \text{b) } \frac{d}{dt}(7t^2 - 4t^{\frac{1}{2}}) &= 14t - 2t^{-\frac{1}{2}} \\
 \text{c) } \frac{d}{dx}(ax^2 + bx + c) &= 2ax + b \\
 \text{d) } \frac{d}{dz}\left(\frac{1}{4}z^2 - \frac{1}{z}\right) &= \frac{d}{dz}\left(\frac{1}{4}z^2 - z^{-1}\right) = \frac{1}{2}z + z^{-2} = \frac{1}{2}z + \frac{1}{z^2} \\
 \text{e) } \frac{d}{dw}\left(\frac{1}{4}w^{\frac{4}{5}} + \frac{k}{w^2}\right) &= \frac{d}{dw}\left(\frac{1}{4}w^{\frac{4}{5}} + kw^{-2}\right) = \frac{1}{5}w^{-\frac{1}{5}} - 2kw^{-3} = \frac{1}{5}w^{-\frac{1}{5}} - \frac{2k}{w^3}
 \end{aligned}$$

Question 9

- a) If $A = \pi x^2 - 20x$, find the rate of change of A with respect to x .
- b) If $V = x - 2\pi x^3$, find the rate of change of V with respect to x .
- c) If $P = at^2 - bt$, find the rate of change of P with respect to t .
- d) If $W = 6kh^{\frac{1}{2}} - h$, find the rate of change of W with respect to h .
- e) If $N = (at + b)^2$, find the rate of change of N with respect to t .

$$\frac{dA}{dx} = 2\pi x - 20, \quad \frac{dV}{dx} = 1 - 6\pi x^2, \quad \frac{dP}{dt} = 2at - b, \quad \frac{dW}{dh} = 3kh^{-\frac{1}{2}} - 1,$$

$$\frac{dN}{dt} = 2a^2t + 2ab$$

(a) $A = \pi x^2 - 20x$ $\frac{dA}{dx} = 2\pi x - 20$	(d) $W = 6kh^{\frac{1}{2}} - h$ $\frac{dW}{dh} = 3k\frac{1}{2}h^{-\frac{1}{2}} - 1$
(b) $V = x - 2\pi x^3$ $\frac{dV}{dx} = 1 - 6\pi x^2$	(e) $N = (at + b)^2$ $N = a^2t^2 + 2abt + b^2$ $\frac{dN}{dt} = 2a^2t + 2ab$
(c) $P = at^2 - bt$ $\frac{dP}{dt} = 2at - b$	

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DIFFERENTIATING INDICES

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Question 1

Differentiate the following expressions with respect to x

a) $y = 4\sqrt{x} - \sqrt[3]{x}$

$$\frac{dy}{dx} = 2x^{-\frac{1}{2}} - \frac{1}{3}x^{-\frac{2}{3}}$$

b) $y = 2\sqrt{x} - 4\sqrt{x^3}$

$$\frac{dy}{dx} = x^{-\frac{1}{2}} - 6x^{\frac{1}{2}}$$

c) $y = \frac{1}{2\sqrt{x}} + \frac{4}{x^2}$

$$\frac{dy}{dx} = -\frac{1}{4}x^{-\frac{3}{2}} - 8x^{-3}$$

d) $y = x\sqrt{x} - \frac{1}{x^2}$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-3}$$

e) $y = 4\sqrt{x} + \frac{1}{4\sqrt{x}}$

$$\frac{dy}{dx} = 2x^{-\frac{1}{2}} - \frac{1}{8}x^{-\frac{3}{2}}$$

(a) $\frac{d}{dx}(4x^{\frac{1}{2}} - x^{\frac{1}{3}}) = \frac{d}{dx}(4x^{\frac{1}{2}} - x^{\frac{1}{3}}) = 2x^{-\frac{1}{2}} - \frac{1}{3}x^{-\frac{2}{3}}$
 (b) $\frac{d}{dx}(2x^{\frac{1}{2}} - 4x^{\frac{3}{2}}) = \frac{d}{dx}(2x^{\frac{1}{2}} - 4x^{\frac{3}{2}}) = x^{-\frac{1}{2}} - 6x^{\frac{1}{2}}$
 (c) $\frac{d}{dx}(\frac{1}{2x^{\frac{1}{2}}} + \frac{4}{x^2}) = \frac{d}{dx}(\frac{1}{2}x^{-\frac{1}{2}} + 4x^{-2}) = -\frac{1}{4}x^{-\frac{3}{2}} - 8x^{-3}$
 (d) $\frac{d}{dx}(x^{\frac{3}{2}} - x^{-2}) = \frac{d}{dx}(x^{\frac{3}{2}} - x^{-2}) = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-3}$
 (e) $\frac{d}{dx}(4x^{\frac{1}{2}} + \frac{1}{4x^{\frac{1}{2}}}) = \frac{d}{dx}(4x^{\frac{1}{2}} + \frac{1}{4}x^{-\frac{1}{2}}) = 2x^{-\frac{1}{2}} - \frac{1}{8}x^{-\frac{3}{2}}$

Question 2

Find $f'(x)$ for each of the following functions.

a) $f(x) = \frac{2}{x^3} + 5x^{\frac{2}{3}}$

$$f'(x) = -6x^{-4} + \frac{10}{3}x^{-\frac{1}{3}}$$

b) $f(x) = 8x^{\frac{3}{4}} - \frac{2}{x^4}$

$$f'(x) = 6x^{-\frac{1}{4}} + 8x^{-5}$$

c) $f(x) = 2x - \frac{3}{x^2} + 4\sqrt{x} + 2$

$$f'(x) = 2 + 6x^{-3} + 2x^{-\frac{1}{2}}$$

d) $f(x) = \sqrt[3]{x^2} - \frac{3}{2x^3}$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} + \frac{9}{2}x^{-4}$$

e) $f(x) = \sqrt{x^3} - \frac{1}{2x^2}$

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}} + x^{-3}$$

<p>a) $f(x) = \frac{2}{x^3} + 5x^{\frac{2}{3}}$ $f(x) = 2x^{-3} + 5x^{\frac{2}{3}}$ $f'(x) = -6x^{-4} + \frac{10}{3}x^{-\frac{1}{3}}$</p>	<p>c) $f(x) = 2x - \frac{3}{x^2} + 4\sqrt{x} + 2$ $f(x) = 2x - 3x^{-2} + 4x^{\frac{1}{2}} + 2$ $f'(x) = 2 + 6x^{-3} + 2x^{-\frac{1}{2}}$</p>
<p>b) $f(x) = 8x^{\frac{3}{4}} - \frac{2}{x^4}$ $f(x) = 8x^{\frac{3}{4}} - 2x^{-4}$ $f'(x) = 6x^{-\frac{1}{4}} + 8x^{-5}$</p>	<p>e) $f(x) = \sqrt{x^3} - \frac{1}{2x^2}$ $f(x) = x^{\frac{3}{2}} - \frac{1}{2}x^{-2}$ $f'(x) = \frac{3}{2}x^{\frac{1}{2}} + x^{-3}$</p>
<p>d) $f(x) = \sqrt[3]{x^2} - \frac{3}{2x^3}$ $f(x) = x^{\frac{2}{3}} - \frac{3}{2}x^{-3}$ $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} + \frac{9}{2}x^{-4}$</p>	

Question 3

Differentiate the following expressions with respect to x

a) $y = \frac{4}{x^3} - \frac{4}{3x^2}$

$$\frac{dy}{dx} = \frac{8}{3}x^{-3} - 12x^{-4}$$

b) $y = \frac{3}{4x^2} - \frac{12}{x^2\sqrt{x}}$

$$\frac{dy}{dx} = 30x^{-\frac{7}{2}} - \frac{3}{2}x^{-3}$$

c) $y = \frac{1}{3x} + \frac{2x^3+1}{3\sqrt{x}}$

$$\frac{dy}{dx} = -\frac{1}{3}x^{-2} + \frac{5}{3}x^{\frac{3}{2}} - \frac{1}{6}x^{-\frac{3}{2}}$$

d) $y = 2\sqrt{x}(7x-x^2)$

$$\frac{dy}{dx} = 21x^{\frac{1}{2}} - 5x^{\frac{3}{2}}$$

e) $y = (3+2\sqrt{x})^2$

$$\frac{dy}{dx} = 6x^{-\frac{1}{2}} + 4$$

(a) $y = \frac{4}{x^3} - \frac{4}{3x^2} = 4x^{-3} - \frac{4}{3}x^{-2} \quad \therefore \frac{dy}{dx} = -12x^{-4} + \frac{8}{3}x^{-3}$
 (b) $y = \frac{3}{4x^2} - \frac{12}{x^2\sqrt{x}} = \frac{3}{4}x^{-2} - \frac{12}{x^{\frac{5}{2}}} = \frac{3}{4}x^{-2} - \frac{12}{x^{\frac{5}{2}}} = \frac{3}{4}x^{-2} - 12x^{-\frac{5}{2}}$
 $\therefore \frac{dy}{dx} = -\frac{3}{2}x^{-3} + 30x^{-\frac{7}{2}}$
 (c) $y = \frac{1}{3x} + \frac{2x^3+1}{3\sqrt{x}} = \frac{1}{3}x^{-1} + \frac{2x^3+1}{3x^{\frac{1}{2}}} = \frac{1}{3}x^{-1} + \frac{2x^3}{3x^{\frac{1}{2}}} + \frac{1}{3x^{\frac{1}{2}}}$
 $= \frac{1}{3}x^{-1} + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{3}x^{-\frac{1}{2}}$
 $\therefore \frac{dy}{dx} = -\frac{1}{3}x^{-2} + \frac{5}{3}x^{\frac{3}{2}} - \frac{1}{6}x^{-\frac{3}{2}}$
 (d) $y = 2\sqrt{x}(7x-x^2) = 2x^{\frac{1}{2}}(7x-x^2) = 14x^{\frac{3}{2}} - 2x^{\frac{5}{2}}$
 $\therefore \frac{dy}{dx} = 21x^{\frac{1}{2}} - 5x^{\frac{3}{2}}$
 (e) $y = (3+2\sqrt{x})^2 = 9 + 2 \times 3 \times 2\sqrt{x} + (2\sqrt{x})^2 = 9 + 12\sqrt{x} + 4x$
 $= 9 + 12x^{\frac{1}{2}} + 4x$
 $\therefore \frac{dy}{dx} = 6x^{-\frac{1}{2}} + 4$
 OR MULTIPLY TWO BINOMIALS
 $= (3+2\sqrt{x})(3+2\sqrt{x}) = 9 + 6\sqrt{x} + 6\sqrt{x} + 4x$
 $= 9 + 12\sqrt{x} + 4x = 9 + 12x^{\frac{1}{2}} + 4x$

Question 4

Evaluate the following.

a) $\frac{d}{dx} \left(6x^{\frac{4}{3}} - 2x^{\frac{5}{2}} \right)$

$$8x^{\frac{1}{3}} - 5x^{\frac{3}{2}}$$

b) $\frac{d}{dx} \left(\frac{1}{x} - \frac{1}{\sqrt{x}} \right)$

$$-x^{-2} + \frac{1}{2}x^{-\frac{3}{2}}$$

c) $\frac{d}{dx} \left(\sqrt[3]{x} - \frac{27}{x} \right)$

$$\frac{1}{3}x^{-\frac{2}{3}} + 27x^{-2}$$

d) $\frac{d}{dx} \left(\frac{3\sqrt{x} - 2}{x^{\frac{3}{2}}} \right)$

$$-3x^{-2} + 3x^{-\frac{5}{2}}$$

e) $\frac{d}{dx} \left[\frac{1}{3\sqrt{x}} \left(\frac{2}{x} - 3 \right) \right]$

$$-x^{-\frac{5}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$

$$\begin{aligned} \text{a)} \quad & \frac{d}{dx} [6x^{\frac{4}{3}} - 2x^{\frac{5}{2}}] = 8x^{\frac{1}{3}} - 5x^{\frac{3}{2}} \\ \text{b)} \quad & \frac{d}{dx} \left[\frac{1}{x} - \frac{1}{\sqrt{x}} \right] = \frac{d}{dx} [x^{-1} - x^{-\frac{1}{2}}] = -x^{-2} + \frac{1}{2}x^{-\frac{3}{2}} \\ \text{c)} \quad & \frac{d}{dx} \left[\sqrt[3]{x} - \frac{27}{x} \right] = \frac{d}{dx} [x^{\frac{1}{3}} - 27x^{-1}] = \frac{1}{3}x^{-\frac{2}{3}} + 27x^{-2} \\ \text{d)} \quad & \frac{d}{dx} \left[\frac{3\sqrt{x} - 2}{x^{\frac{3}{2}}} \right] = \frac{d}{dx} \left[\frac{3x^{\frac{1}{2}} - 2}{x^{\frac{3}{2}}} \right] = \frac{d}{dx} \left[\frac{3x^{\frac{1}{2}} - 2}{x^{\frac{3}{2}}} \right] \\ & = \frac{d}{dx} [3x^{-1} - 2x^{-\frac{3}{2}}] = -3x^{-2} + 3x^{-\frac{5}{2}} \\ \text{e)} \quad & \frac{d}{dx} \left[\frac{1}{3\sqrt{x}} \left(\frac{2}{x} - 3 \right) \right] = \frac{d}{dx} \left[\frac{1}{3} x^{-\frac{1}{2}} (2x^{-1} - 3) \right] = \frac{d}{dx} \left[\frac{2}{3} x^{-\frac{3}{2}} - x^{-\frac{1}{2}} \right] \\ & = -x^{-\frac{5}{2}} + \frac{1}{2}x^{-\frac{3}{2}} \end{aligned}$$

Question 5

Evaluate the following.

a) $\frac{d}{dx} \left(\frac{x+x^2}{\sqrt{x}} \right)$

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{\frac{1}{2}}$$

b) $\frac{d}{dx} \left(\frac{4x+\sqrt{x}}{2x^2} \right)$

$$-2x^{-2} - \frac{3}{4}x^{-\frac{5}{2}}$$

c) $\frac{d}{dx} \left(\frac{x^2+2}{x^3} \right)$

$$-x^{-2} - 6x^{-4}$$

d) $\frac{d}{dx} \left(\frac{1-\sqrt{x}}{4x^3} \right)$

$$-\frac{3}{4}x^{-4} + \frac{5}{8}x^{-\frac{7}{2}}$$

e) $\frac{d}{dx} \left[\frac{\sqrt[3]{x^5} - 2x\sqrt{x}}{3x} \right]$

$$\frac{2}{9}x^{-\frac{1}{3}} - \frac{1}{3}x^{-\frac{1}{2}}$$

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx} \left(\frac{x+x^2}{\sqrt{x}} \right) &= \frac{d}{dx} \left(\frac{x}{\sqrt{x}} + \frac{x^2}{\sqrt{x}} \right) = \frac{d}{dx} \left(x^{\frac{1}{2}} + x^{\frac{3}{2}} \right) = \frac{1}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{\frac{1}{2}} \\ \text{(b)} \quad \frac{d}{dx} \left(\frac{4x+\sqrt{x}}{2x^2} \right) &= \frac{d}{dx} \left(\frac{4x}{2x^2} + \frac{\sqrt{x}}{2x^2} \right) = \frac{d}{dx} \left(2x^{-1} + \frac{1}{2}x^{-\frac{3}{2}} \right) = -2x^{-2} - \frac{3}{4}x^{-\frac{5}{2}} \\ \text{(c)} \quad \frac{d}{dx} \left(\frac{x^2+2}{x^3} \right) &= \frac{d}{dx} \left(\frac{x^2}{x^3} + \frac{2}{x^3} \right) = \frac{d}{dx} \left(x^{-1} + 2x^{-3} \right) = -x^{-2} - 6x^{-4} \\ \text{(d)} \quad \frac{d}{dx} \left(\frac{1-\sqrt{x}}{4x^3} \right) &= \frac{d}{dx} \left(\frac{1}{4x^3} - \frac{\sqrt{x}}{4x^3} \right) = \frac{d}{dx} \left(\frac{1}{4}x^{-3} - \frac{1}{4}x^{-\frac{5}{2}} \right) = -\frac{3}{4}x^{-4} + \frac{5}{8}x^{-\frac{7}{2}} \\ \text{(e)} \quad \frac{d}{dx} \left(\frac{\sqrt[3]{x^5} - 2x\sqrt{x}}{3x} \right) &= \frac{d}{dx} \left(\frac{x^{\frac{5}{3}} - 2x^{\frac{3}{2}}}{3x} \right) = \frac{d}{dx} \left(\frac{x^{\frac{5}{3}}}{3x} - \frac{2x^{\frac{3}{2}}}{3x} \right) \\ &= \frac{d}{dx} \left(\frac{1}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{\frac{1}{2}} \right) = \frac{2}{9}x^{-\frac{1}{3}} - \frac{1}{3}x^{-\frac{1}{2}} \end{aligned}$$

Question 6

Differentiate the following expressions with respect to x

a) $y = \frac{4+x}{2x^3}$

$$\frac{dy}{dx} = -6x^{-4} - x^{-3}$$

b) $y = \frac{x^2+3x}{2\sqrt{x}}$

$$\frac{dy}{dx} = \frac{3}{4}x^{-\frac{1}{2}} + \frac{3}{4}x^{\frac{1}{2}}$$

c) $y = \frac{x+4\sqrt{x}}{2x^3}$

$$\frac{dy}{dx} = -5x^{-\frac{7}{2}} - x^{-3}$$

d) $y = \frac{\sqrt{x}(2x-4)}{3x^2}$

$$\frac{dy}{dx} = -\frac{1}{3}x^{-\frac{3}{2}} + 2x^{-\frac{5}{2}}$$

e) $y = \frac{(x+2)(2x-3)}{4x^5}$

$$\frac{dy}{dx} = -\frac{3}{2}x^{-4} - x^{-5} + \frac{15}{2}x^{-6}$$

Handwritten solutions for Question 6:

a) $y = \frac{4+x}{2x^3} = \frac{4}{2x^3} + \frac{x}{2x^3} = 2x^{-3} + \frac{1}{2}x^{-2}$
 $\frac{dy}{dx} = -6x^{-4} - x^{-3}$

b) $y = \frac{x^2+3x}{2\sqrt{x}} = \frac{x^2}{2\sqrt{x}} + \frac{3x}{2\sqrt{x}} = \frac{1}{2}x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{1}{2}}$
 $\frac{dy}{dx} = \frac{3}{4}x^{-\frac{1}{2}} + \frac{3}{4}x^{-\frac{1}{2}}$

c) $y = \frac{x+4\sqrt{x}}{2x^3} = \frac{x}{2x^3} + \frac{4\sqrt{x}}{2x^3} = \frac{1}{2}x^{-2} + 2x^{-\frac{5}{2}}$
 $\frac{dy}{dx} = -x^{-3} - 5x^{-\frac{7}{2}}$

d) $y = \frac{\sqrt{x}(2x-4)}{3x^2} = \frac{2x\sqrt{x} - 4\sqrt{x}}{3x^2} = \frac{2}{3}x^{-\frac{1}{2}} - \frac{4}{3}x^{-\frac{3}{2}}$
 $\frac{dy}{dx} = -\frac{1}{3}x^{-\frac{3}{2}} + \frac{2}{3}x^{-\frac{5}{2}}$

e) $y = \frac{(x+2)(2x-3)}{4x^5} = \frac{2x^2-3x+4x-6}{4x^5} = \frac{2x^2-x-6}{4x^5} = \frac{1}{2}x^{-3} - \frac{1}{4}x^{-4} - \frac{3}{2}x^{-6}$
 $\frac{dy}{dx} = -\frac{3}{2}x^{-4} - x^{-5} + \frac{15}{2}x^{-6}$

Question 7

Find $f'(x)$ for each of the following functions.

a) $f(x) = x(\sqrt{x} + x^{-4})$

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}} - 3x^{-4}$$

b) $f(x) = \frac{1}{\sqrt{x}}\left(\frac{2}{x} - \frac{3}{4x^2}\right)$

$$f'(x) = -3x^{-\frac{5}{2}} + \frac{15}{8}x^{-\frac{7}{2}}$$

c) $f(x) = 4x^{\frac{7}{2}}\left(\frac{6}{x^2} - \frac{5}{\sqrt{x}}\right)$

$$f'(x) = 36x^{\frac{1}{2}} - 60x^2$$

d) $f(x) = 2\sqrt{x}\left(\frac{5}{x} + x^2\right)$

$$f'(x) = -5x^{-\frac{3}{2}} + 5x^{\frac{3}{2}}$$

e) $f(x) = \frac{2}{x^{\frac{3}{2}}}\left(\frac{7x^3 - 5x^2}{4x}\right)$

$$f'(x) = \frac{7}{4}x^{-\frac{1}{2}} + \frac{5}{4}x^{-\frac{3}{2}}$$

$$\begin{aligned} \textcircled{a} \quad f(x) &= x(\sqrt{x} + x^{-4}) = x(x^{\frac{1}{2}} + x^{-4}) = x^{\frac{3}{2}} + x^{-3} \\ \therefore f'(x) &= \frac{3}{2}x^{\frac{1}{2}} - 3x^{-4} \\ \textcircled{b} \quad f(x) &= \frac{1}{\sqrt{x}}\left(\frac{2}{x} - \frac{3}{4x^2}\right) = x^{-\frac{1}{2}}\left(2x^{-1} - \frac{3}{4}x^{-2}\right) = 2x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}} \\ \therefore f'(x) &= -3x^{-\frac{5}{2}} + \frac{15}{8}x^{-\frac{7}{2}} \\ \textcircled{c} \quad f(x) &= 4x^{\frac{7}{2}}\left(\frac{6}{x^2} - \frac{5}{\sqrt{x}}\right) = 4x^{\frac{7}{2}}\left(6x^{-2} - 5x^{-\frac{1}{2}}\right) = 24x^{\frac{3}{2}} - 20x^3 \\ \therefore f'(x) &= 36x^{\frac{1}{2}} - 60x^2 \\ \textcircled{d} \quad f(x) &= 2\sqrt{x}\left(\frac{5}{x} + x^2\right) = 2x^{\frac{1}{2}}(5x^{-1} + x^2) = 10x^{-\frac{1}{2}} + 2x^{\frac{5}{2}} \\ \therefore f'(x) &= -5x^{-\frac{3}{2}} + 5x^{\frac{3}{2}} \\ \textcircled{e} \quad f(x) &= \frac{2}{x^{\frac{3}{2}}}\left(\frac{7x^3 - 5x^2}{4x}\right) = \frac{2}{x^{\frac{3}{2}}}\left(\frac{7x^2 - 5x}{4}\right) = \frac{1}{2}x^{\frac{1}{2}}\left(\frac{7x - 5}{2}\right) \\ &= \frac{7}{4}x^{\frac{3}{2}} - \frac{5}{4}x^{\frac{1}{2}} \\ \therefore f'(x) &= \frac{7}{4}x^{\frac{1}{2}} + \frac{5}{4}x^{-\frac{1}{2}} \end{aligned}$$

Question 8

Differentiate the following expressions with respect to x

a) $y = \frac{(2x-1)(3x-2)}{2x^{\frac{3}{2}}}$

$$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} + \frac{7}{4}x^{-\frac{3}{2}} - \frac{3}{2}x^{-\frac{5}{2}}$$

b) $y = \frac{(3+2\sqrt{x})^2}{4x}$

$$\frac{dy}{dx} = -\frac{3}{2}x^{-\frac{3}{2}} - \frac{9}{4}x^{-2}$$

c) $y = \frac{4x^3 + \sqrt{x^5}}{4\sqrt{x}}$

$$\frac{dy}{dx} = \frac{1}{2}x + \frac{5}{2}x^{\frac{3}{2}}$$

d) $y = \frac{(4x + \sqrt{x})(x^2 - 3)}{3\sqrt{x}}$

$$\frac{dy}{dx} = \frac{2}{3}x + \frac{10}{3}x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}$$

e) $y = \frac{(2x^{\frac{1}{2}} + 6x^{-\frac{1}{2}})(6x^{\frac{3}{2}} - 2x^{-\frac{1}{2}})}{3x}$

$$\frac{dy}{dx} = \frac{4}{3}x^{-2} + 8x^{-3} + 4$$

Handwritten solutions for Question 8:

a) $y = \frac{(2x-1)(3x-2)}{2x^{\frac{3}{2}}} = \frac{6x^2 - 2x + 3x - 2}{2x^{\frac{3}{2}}} = \frac{6x^2 - 2x + 3x - 2}{2x^{\frac{3}{2}}}$
 $\therefore y = 3x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}} - \frac{2}{2x^{\frac{3}{2}}}$
 $\therefore \frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} + \frac{7}{4}x^{-\frac{3}{2}} - \frac{3}{2}x^{-\frac{5}{2}}$

b) $y = \frac{(3+2\sqrt{x})^2}{4x} = \frac{9 + 12\sqrt{x} + 4x}{4x} = \frac{9}{4x} + \frac{12\sqrt{x}}{4x} + \frac{4x}{4x}$
 $\therefore y = \frac{9}{4}x^{-1} + 3x^{\frac{1}{2}} + 1$
 $\therefore \frac{dy}{dx} = -\frac{9}{4}x^{-2} + \frac{3}{2}x^{-\frac{1}{2}}$

c) $y = \frac{4x^3 + \sqrt{x^5}}{4\sqrt{x}} = \frac{4x^3}{4\sqrt{x}} + \frac{\sqrt{x^5}}{4\sqrt{x}} = x^{\frac{5}{2}} + \frac{1}{4}x^{\frac{5}{2}}$
 $\therefore \frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}} + \frac{5}{8}x^{\frac{3}{2}}$

d) $y = \frac{(4x + \sqrt{x})(x^2 - 3)}{3\sqrt{x}} = \frac{4x^3 - 12x + x^{\frac{5}{2}} - 3x^{\frac{1}{2}}}{3\sqrt{x}} = \frac{4x^3}{3\sqrt{x}} - \frac{12x}{3\sqrt{x}} + \frac{x^{\frac{5}{2}}}{3\sqrt{x}} - \frac{3x^{\frac{1}{2}}}{3\sqrt{x}}$
 $= \frac{4}{3}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + \frac{1}{3}x^2 - 1$
 $\therefore \frac{dy}{dx} = \frac{10}{3}x^{\frac{3}{2}} - 2x^{-\frac{1}{2}} + 2x$

e) $y = \frac{(2x^{\frac{1}{2}} + 6x^{-\frac{1}{2}})(6x^{\frac{3}{2}} - 2x^{-\frac{1}{2}})}{3x} = \frac{12x^2 - 2x + 18x - 12x^{-1}}{3x} = \frac{12x^2}{3x} - \frac{2x}{3x} + \frac{18x}{3x} - \frac{12x^{-1}}{3x}$
 $= 4x - \frac{2}{3} + 6 - \frac{4}{3}x^{-2}$
 $\therefore \frac{dy}{dx} = 4 + \frac{8}{3}x^{-3} + 4$

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TANGENTS & NORMALS

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Question 1 (non calculator)

For each of the following curves find an equation of the tangent to the curve at the point whose x coordinate is given.

a) $y = x^2 - 9x + 13$, where $x = 6$

$y = 3x - 23$

b) $y = x^4 + x + 1$, where $x = 1$

$y = 5x - 2$

c) $y = 2x^2 + 6x + 7$, where $x = -1$

$y = 2x + 5$

d) $y = 2x^3 - 4x + 5$, where $x = 1$

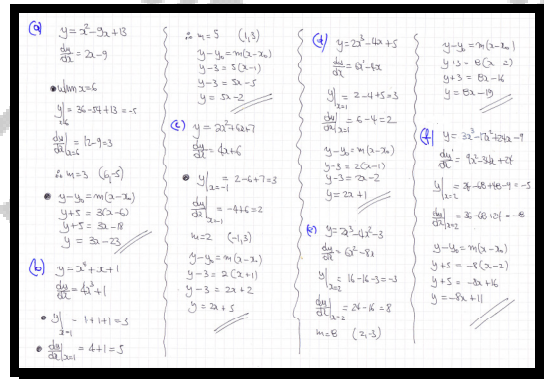
$y = 2x + 1$

e) $y = 2x^3 - 4x^2 - 3$, where $x = 2$

$y = 8x - 19$

f) $y = 3x^3 - 17x^2 + 24x - 9$, where $x = 2$

$y = -8x + 11$



Question 2 (non calculator)

For each of the following curves find an equation of the tangent to the curve at the point whose x coordinate is given.

a) $f(x) = x^3 - 4x^2 + 2x - 1$, where $x = 2$

$y = -2x - 1$

b) $f(x) = 3x^3 + x^2 - 8x - 5$, where $x = 1$

$y = 3x - 12$

c) $f(x) = 2x^3 - 5x^2 + 2x - 1$, where $x = 2$

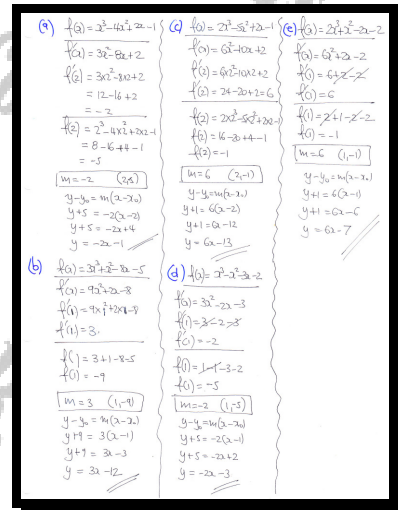
$y = 6x - 13$

d) $f(x) = x^3 - x^2 - 3x - 2$, where $x = 1$

$y = -2x - 3$

e) $f(x) = 2x^3 + x^2 - 2x - 2$, where $x = 1$

$y = 6x - 7$



Question 3 (non calculator)

For each of the following curves find an equation of the tangent to the curve at the point whose x coordinate is given.

a) $y = x^2 - \frac{3}{x} - \frac{1}{2}$, where $x = -2$

$13x + 4y + 6 = 0$

b) $y = x^3 - 6x + \frac{8}{x} + 1$, where $x = 2$

$y = 4x - 7$

c) $y = 4x^2 + \frac{5}{x} - 1$, where $x = 1$

$y = 3x + 5$

d) $y = 2\sqrt{x} - \frac{6}{\sqrt{x}}$, where $x = 4$

$7x - 8y - 20 = 0$

e) $y = 3x^{\frac{3}{2}} - \frac{32}{x}$, where $x = 4$

$y = 11x - 28$

Handwritten solutions for questions a, b, c, and d. The work shows the differentiation of each function, substitution of the given x-value to find the gradient and the point on the curve, and then the use of the point-slope formula to find the equation of the tangent line.

Handwritten solution for question e. It shows the differentiation of the function, substitution of x=4 to find the gradient and the point on the curve, and then the use of the point-slope formula to find the equation of the tangent line.

Question 4 (non calculator)

For each of the following curves find an equation of the normal to the curve at the point whose x coordinate is given.

a) $f(x) = x^3 - 4x^2 + 1$, where $x = 2$

$4y = x - 30$

b) $f(x) = x^3 - 7x^2 + 11x$, where $x = 3$

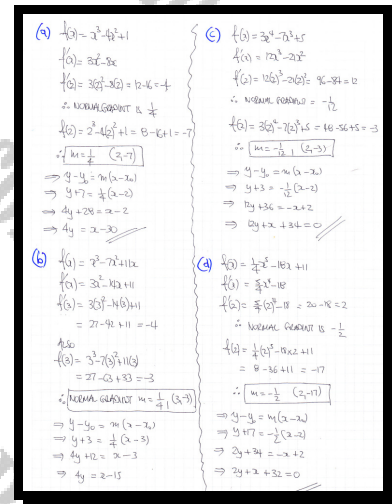
$4y = x - 15$

c) $f(x) = 3x^4 - 7x^3 + 5$ where $x = 2$

$12y + x + 34 = 0$

d) $f(x) = \frac{1}{4}x^5 - 18x + 11$ where $x = 2$

$2y + x + 32 = 0$



Question 5 (non calculator)

For each of the following curves find an equation of the normal to the curve at the point whose x coordinate is given.

a) $f(x) = 2x^3 - 3x^2 - 10x + 18$, where $x = 2$

$x + 2y = 6$

b) $f(x) = x^3 - 4x^2 + 6x + 1$, where $x = 1$

$x + y = 5$

c) $f(x) = 4x^3 + 2x^2 - 18x - 10$ where $x = -2$

$22y + x = 42$

d) $f(x) = -2x^3 + 4x^2 - 1$, where $x = 2$

$8y = x - 10$

Handwritten solutions for Question 5:

a) $f(x) = 2x^3 - 3x^2 - 10x + 18$
 $f'(x) = 6x^2 - 6x - 10$
 $f'(2) = 24 - 12 - 10 = 2$
 $f(2) = 16 - 12 - 20 + 18 = 2$
 \therefore Normal $m = -\frac{1}{2}$ (22)
 $y - y_1 = m(x - x_1)$
 $y - 2 = -\frac{1}{2}(x - 2)$
 $2y - 4 = -x + 2$
 $2y + x = 6$

b) $f(x) = x^3 - 4x^2 + 6x + 1$
 $f'(x) = 3x^2 - 8x + 6$
 $f'(1) = 3 - 8 + 6 = 1$
 $f(1) = 1 - 4 + 6 + 1 = 4$
 \therefore Normal $m = -1$ (14)
 $y - y_1 = m(x - x_1)$
 $y - 4 = -1(x - 1)$
 $y - 4 = -x + 1$
 $y + x = 5$

c) $f(x) = 4x^3 + 2x^2 - 18x - 10$
 $f'(x) = 12x^2 + 4x - 18$
 $f'(-2) = 48 - 8 - 18 = 22$
 $f(-2) = -32 + 8 + 36 - 10 = 2$
 \therefore Normal $m = -\frac{1}{22}$ (22)
 $y - y_1 = m(x - x_1)$
 $y - 2 = -\frac{1}{22}(x + 2)$
 $22y - 44 = -x - 2$
 $22y + x = 42$

d) $f(x) = -2x^3 + 4x^2 - 1$
 $f'(x) = -6x^2 + 8x$
 $f'(2) = -24 + 16 = -8$
 $f(2) = -16 + 16 - 1 = -1$
 \therefore Normal $m = \frac{1}{8}$ (8, 1)
 $y - y_1 = m(x - x_1)$
 $y + 1 = \frac{1}{8}(x - 2)$
 $8y + 8 = x - 2$
 $8y = x - 10$

Question 6 (non calculator)

For each of the following curves find an equation of the normal to the curve at the point whose x coordinate is given.

a) $y = x^2(x-6) + \frac{5}{x} - 1$, where $x=1$

$x - 14y - 15 = 0$

b) $y = 2x^{\frac{3}{2}} - \frac{16}{x}$, where $x=4$

$x + 7y = 88$

c) $y = 4x^2 + x^{-\frac{3}{2}}$, where $x=1$

$2x + 13y = 67$

d) $y = 2x^2 - 4x^{\frac{3}{2}} - \frac{8}{x} - 1$, where $x=4$

$2x + 9y + 19 = 0$

The image shows handwritten solutions for the four parts of Question 6. Each part (a, b, c, d) is solved by first differentiating the given curve to find the gradient of the tangent at the specified x-value. The normal gradient is then found as the negative reciprocal of the tangent gradient. Finally, the equation of the normal line is determined using the point-slope form.

- Part (a):** $y = x^2(x-6) + \frac{5}{x} - 1$. At $x=1$, the normal gradient is $-\frac{1}{14}$. The normal equation is $x - 14y - 15 = 0$.
- Part (b):** $y = 2x^{\frac{3}{2}} - \frac{16}{x}$. At $x=4$, the normal gradient is $-\frac{1}{7}$. The normal equation is $x + 7y = 88$.
- Part (c):** $y = 4x^2 + x^{-\frac{3}{2}}$. At $x=1$, the normal gradient is $-\frac{2}{13}$. The normal equation is $2x + 13y = 67$.
- Part (d):** $y = 2x^2 - 4x^{\frac{3}{2}} - \frac{8}{x} - 1$. At $x=4$, the normal gradient is $-\frac{2}{9}$. The normal equation is $2x + 9y + 19 = 0$.

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STATIONARY POINTS

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Question 1 (non calculator)

For each of the following cubic equations find the coordinates of their stationary points and determine their nature.

a) $y = x^3 - 3x^2 - 9x + 3$

b) $y = x^3 + 12x^2 + 45x + 50$

c) $y = 2x^3 - 6x^2 + 12$

d) $y = 25 - 24x + 9x^2 - x^3$

$\min(3, -24), \max(-1, 8)$, $\min(-3, -4), \max(-5, 0)$, $\min(2, 4), \max(0, 12)$,
 $\min(2, 5), \max(4, 9)$

The image shows handwritten solutions for the four cubic equations. Each solution follows a similar pattern: differentiate the function, set the derivative to zero to find stationary points, and then use the second derivative to determine their nature.

(a) $y = x^3 - 3x^2 - 9x + 3$
 $\frac{dy}{dx} = 3x^2 - 6x - 9$
 For S.T.P., $\frac{dy}{dx} = 0$
 $\Rightarrow 3x^2 - 6x - 9 = 0$
 $\Rightarrow x^2 - 2x - 3 = 0$
 $\Rightarrow (x+1)(x-3) = 0$
 $\Rightarrow x = -1, 3$
 $\frac{d^2y}{dx^2} = 6x - 6$
 $\frac{d^2y}{dx^2} \Big|_{x=-1} = 6(-1) - 6 = -12 < 0$ $\therefore (-1, 8)$ is a Mx
 $\frac{d^2y}{dx^2} \Big|_{x=3} = 6(3) - 6 = 12 > 0$ $\therefore (3, -24)$ is a Mx

(b) $y = x^3 + 12x^2 + 45x + 50$
 $\frac{dy}{dx} = 3x^2 + 24x + 45$
 $\frac{d^2y}{dx^2} = 6x + 24$
 For S.T.P., $\frac{dy}{dx} = 0$
 $\Rightarrow 3x^2 + 24x + 45 = 0$
 $\Rightarrow x^2 + 8x + 15 = 0$
 $\Rightarrow (x+5)(x+3) = 0$
 $\Rightarrow x = -5, -3$
 $\frac{d^2y}{dx^2} \Big|_{x=-5} = 6(-5) + 24 = -6 < 0$ $\therefore (-5, 0)$ is a Mx
 $\frac{d^2y}{dx^2} \Big|_{x=-3} = 6(-3) + 24 = 6 > 0$ $\therefore (-3, -4)$ is a Mx

(c) $y = 2x^3 - 6x^2 + 12$
 $\frac{dy}{dx} = 6x^2 - 12x$
 $\frac{d^2y}{dx^2} = 12x - 12$
 For S.T.P., $\frac{dy}{dx} = 0$
 $\Rightarrow 6x^2 - 12x = 0$
 $\Rightarrow 6x(x-2) = 0$
 $\Rightarrow x = 0, 2$
 $y = 2x^3 - 6x^2 + 12$
 $y = 12$
 $y = 0$
 $\frac{d^2y}{dx^2} \Big|_{x=0} = 12(0) - 12 = -12 < 0$ $\therefore (0, 12)$ is a Mx
 $\frac{d^2y}{dx^2} \Big|_{x=2} = 12(2) - 12 = 12 > 0$ $\therefore (2, 4)$ is a Mx

(d) $y = 25 - 24x + 9x^2 - x^3$
 $\frac{dy}{dx} = -24 + 18x - 3x^2$
 $\frac{d^2y}{dx^2} = 18 - 6x$
 For S.T.P., $\frac{dy}{dx} = 0$
 $\Rightarrow -24 + 18x - 3x^2 = 0$
 $\Rightarrow x^2 - 6x + 8 = 0$
 $\Rightarrow (x-2)(x-4) = 0$
 $\Rightarrow x = 2, 4$
 $y = 25 - 24x + 9x^2 - x^3$
 $y = 5$
 $y = 9$
 $\frac{d^2y}{dx^2} \Big|_{x=2} = 18 - 6(2) = 6 > 0$ $\therefore (2, 5)$ is a Mx
 $\frac{d^2y}{dx^2} \Big|_{x=4} = 18 - 6(4) = -6 < 0$ $\therefore (4, 9)$ is a Mx

Question 2

For each of the following equations find the coordinates of their stationary points and determine their nature.

a) $y = x + \frac{4}{x}, x \neq 0$

b) $y = x^2 + \frac{16}{x}, x \neq 0$

c) $y = x - 4\sqrt{x}, x > 0$

d) $y = 4x^2 + \frac{1}{x}, x \neq 0$

$\min(2, 4), \max(-2, -4), \min(2, 12), \min(4, -4), \min\left(\frac{1}{2}, 3\right)$

Handwritten solutions for questions a, b, c, and d. The work shows the differentiation of each function, setting the first derivative to zero to find stationary points, and using the second derivative test to determine their nature. For (a), the stationary point is at (2, 4). For (b), the stationary point is at (2, 12). For (c), the stationary point is at (4, -4). For (d), the stationary point is at (1/2, 3).

A boxed handwritten solution for question d. It shows the differentiation of $y = 4x^2 + \frac{1}{x}$ to get $\frac{dy}{dx} = 8x - \frac{1}{x^2}$. Setting this to zero gives $8x^3 = 1$, so $x = \frac{1}{2}$. Substituting back into the original equation gives $y = 3$. The second derivative test is also shown, confirming it is a minimum.

Question 3

For each of the following equations find the coordinates of their stationary points and determine their nature.

a) $y = 12\sqrt{x} - x^{\frac{3}{2}}, x > 0$

b) $y = x^{\frac{3}{2}} - 6x^{\frac{1}{2}}, x > 0$

c) $y = 6x^{\frac{1}{2}} - 4x - 2, x > 0$

d) $y = x^{\frac{7}{2}} - 14x^2 + 100, x > 0$

$\boxed{\max(4, 16)}, \boxed{\min(2, -4\sqrt{2})}, \boxed{\max\left(\frac{9}{16}, \frac{1}{4}\right)}, \boxed{\min(4, 4)}$

The image shows handwritten solutions for each part of Question 3. Part (a) uses the chain rule to find $\frac{dy}{dx} = 6x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$ and sets it to zero to find $x=4$, then uses the second derivative to confirm a maximum at $(4, 16)$. Part (b) finds $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$ and finds a minimum at $(2, -4\sqrt{2})$. Part (c) finds $\frac{dy}{dx} = 3x^{-\frac{1}{2}} - 4$ and finds two stationary points, both confirmed as maxima. Part (d) finds $\frac{dy}{dx} = \frac{7}{2}x^{\frac{5}{2}} - 28x$ and finds a minimum at $(4, 4)$.

Question 4

For each of the following equations find the coordinates of their stationary points and determine their nature.

a) $y = x^3 - 16x^{\frac{3}{2}} + 60, x > 0$

b) $y = 5x^2 - 6x^{\frac{5}{3}} + 10, x > 0$

c) $y = 6x^{\frac{4}{3}} - x^2 - 20, x > 0$

d) $y = 5x^2 - 2x^{\frac{5}{2}} - 10, x > 0$

$\min(4, -4), \min(1, 9), \max(8, 12), \max(4, 6)$

(a) $y = x^3 - 16x^{\frac{3}{2}} + 60$

• Find Min/Max $\frac{dy}{dx} = 0$ • CHECK NATURE

$\frac{dy}{dx} = 3x^2 - 24x^{\frac{1}{2}} = 0$
 $\Rightarrow 3x^2 = 24x^{\frac{1}{2}}$
 $\Rightarrow x^{\frac{3}{2}} = 8$
 $\Rightarrow x^{\frac{3}{2}} = 2^3$
 $\Rightarrow x = 4$

$\frac{d^2y}{dx^2} = 6x - 12x^{-\frac{1}{2}}$
 $\frac{d^2y}{dx^2} = 6(4) - 12(4)^{-\frac{1}{2}}$
 $= 24 - 6 = 18 > 0$
 $\therefore (4, -4)$ is a min

$\therefore y = 4^3 - 16(4)^{\frac{3}{2}} + 60$
 $y = 64 - 128 + 60 = -4$
 $y = -4$

(b) $y = 5x^2 - 6x^{\frac{5}{3}} + 10$

• Find Min/Max $\frac{dy}{dx} = 0$ • CHECK NATURE

$\frac{dy}{dx} = 10x - 10x^{\frac{2}{3}} = 0$
 $\Rightarrow 10x = 10x^{\frac{2}{3}}$
 $\Rightarrow x = x^{\frac{2}{3}}$
 $\Rightarrow x^{\frac{1}{3}} = 1$
 $\Rightarrow x = 1$

$\frac{d^2y}{dx^2} = 10 - \frac{20}{3}x^{-\frac{1}{3}}$
 $\frac{d^2y}{dx^2} = 10 - \frac{20}{3}(1)^{-\frac{1}{3}}$
 $= 10 - \frac{20}{3} = \frac{10}{3} > 0$
 $\therefore (1, 9)$ is a min

$\therefore y = 5(1)^2 - 6(1)^{\frac{5}{3}} + 10$
 $y = 5 - 6 + 10 = 9$
 $y = 9$

(c) $y = 6x^{\frac{4}{3}} - x^2 - 20$

• Find Min/Max $\frac{dy}{dx} = 0$ • CHECK NATURE

$\frac{dy}{dx} = 8x^{\frac{1}{3}} - 2x = 0$
 $\Rightarrow 8x^{\frac{1}{3}} = 2x$
 $\Rightarrow 4 = \frac{2x}{x^{\frac{1}{3}}}$
 $\Rightarrow 4 = 2x^{\frac{2}{3}}$
 $\Rightarrow (4)^{\frac{3}{2}} = (2x^{\frac{2}{3}})^{\frac{3}{2}}$
 $\Rightarrow 8 = 2x$
 $\Rightarrow x = 4$

$\frac{d^2y}{dx^2} = \frac{8}{3}x^{-\frac{2}{3}} - 2$
 $\frac{d^2y}{dx^2} = \frac{8}{3}(4)^{-\frac{2}{3}} - 2$
 $= \frac{8}{3} \cdot \frac{1}{2} - 2 = \frac{4}{3} - 2 = -\frac{2}{3} < 0$
 $\therefore (8, 12)$ is max

$\therefore y = 6(4)^{\frac{4}{3}} - (4)^2 - 20$
 $y = 96 - 16 - 20 = 60$
 $y = 60$

(d) $y = 5x^2 - 2x^{\frac{5}{2}} - 10$

• Find Min/Max $\frac{dy}{dx} = 0$ • CHECK NATURE

$\frac{dy}{dx} = 10x - 5x^{\frac{3}{2}} = 0$
 $\Rightarrow 10x = 5x^{\frac{3}{2}}$
 $\Rightarrow 2x = x^{\frac{3}{2}}$
 $\Rightarrow 2 = x^{\frac{1}{2}}$
 $\Rightarrow x = 4$

$\frac{d^2y}{dx^2} = 10 - \frac{15}{2}x^{\frac{1}{2}}$
 $\frac{d^2y}{dx^2} = 10 - \frac{15}{2}(4)^{\frac{1}{2}}$
 $= 10 - 15 = -5 < 0$
 $\therefore (4, 6)$ is max

$\therefore y = 5(4)^2 - 2(4)^{\frac{5}{2}} - 10$
 $y = 80 - 64 - 10 = 6$
 $y = 6$

Question 5

For each of the following equations find the coordinates of their stationary points and determine their nature.

a) $y = \frac{1}{x} - \frac{1}{\sqrt{x}}, x > 0$

b) $y = \frac{3\sqrt{x}-2}{x^{\frac{3}{2}}}, x > 0$

c) $y = \sqrt[3]{x} + \frac{27}{x}, x > 0$

d) $y = \frac{1}{3\sqrt{x}} \left[\frac{2}{x} - 3 \right], x > 0$

$\min\left(4, -\frac{1}{4}\right), \max(1, 1), \min(27, 4), \min\left(2, -\frac{\sqrt{2}}{3}\right)$

The image shows handwritten solutions for each part of Question 5:

- Part (a):** $y = \frac{1}{x} - \frac{1}{\sqrt{x}}$. Derivative: $\frac{dy}{dx} = -\frac{1}{x^2} + \frac{1}{2x^{\frac{3}{2}}}$. Setting to zero: $-\frac{1}{x^2} + \frac{1}{2x^{\frac{3}{2}}} = 0 \Rightarrow \frac{1}{2x^{\frac{3}{2}}} = \frac{1}{x^2} \Rightarrow \frac{1}{2} = \frac{1}{x^{\frac{1}{2}}} \Rightarrow \frac{1}{2} = x^{-\frac{1}{2}} \Rightarrow \frac{1}{2} = \frac{1}{\sqrt{x}} \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$. Then $y = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$. Nature: $\frac{d^2y}{dx^2} = \frac{2}{x^3} - \frac{3}{2x^{\frac{5}{2}}}$. At $x=4$, $\frac{d^2y}{dx^2} = \frac{2}{64} - \frac{3}{2 \cdot 32} = \frac{1}{32} - \frac{3}{64} = -\frac{1}{64} < 0$. So it's a local maximum.
- Part (b):** $y = \frac{3\sqrt{x}-2}{x^{\frac{3}{2}}}$. Derivative: $\frac{dy}{dx} = \frac{\frac{3}{2}\sqrt{x} \cdot x^{-\frac{3}{2}} - (3\sqrt{x}-2) \cdot \frac{3}{2}x^{-\frac{5}{2}}}{x^3}$. Setting to zero: $\frac{3}{2}\sqrt{x} \cdot x^{-\frac{3}{2}} - (3\sqrt{x}-2) \cdot \frac{3}{2}x^{-\frac{5}{2}} = 0 \Rightarrow \frac{3}{2}x^{-1} - \frac{3}{2}(3\sqrt{x}-2)x^{-\frac{5}{2}} = 0 \Rightarrow \frac{3}{2}x^{-1} = \frac{3}{2}(3\sqrt{x}-2)x^{-\frac{5}{2}} \Rightarrow x^{-1} = (3\sqrt{x}-2)x^{-\frac{5}{2}} \Rightarrow x^{\frac{3}{2}} = 3\sqrt{x}-2 \Rightarrow x^{\frac{3}{2}} - 3\sqrt{x} + 2 = 0$. Let $u = \sqrt{x}$, then $u^3 - 3u + 2 = 0 \Rightarrow (u-1)(u^2+u-2) = 0 \Rightarrow (u-1)(u+2)(u-1) = 0 \Rightarrow u=1$ or $u=-2$. Since $u = \sqrt{x} > 0$, $u=1 \Rightarrow x=1$. Then $y = \frac{3\sqrt{1}-2}{1^{\frac{3}{2}}} = \frac{3-2}{1} = 1$. Nature: $\frac{d^2y}{dx^2} = \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{2}(3\sqrt{x}-2)x^{-\frac{5}{2}}$. At $x=1$, $\frac{d^2y}{dx^2} = \frac{3}{2} - \frac{3}{2}(3-2) = \frac{3}{2} - \frac{3}{2} = 0$. Use the first derivative test: for $x < 1$, $\frac{dy}{dx} > 0$; for $x > 1$, $\frac{dy}{dx} < 0$. So it's a local maximum.
- Part (c):** $y = \sqrt[3]{x} + \frac{27}{x}$. Derivative: $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} - \frac{27}{x^2}$. Setting to zero: $\frac{1}{3}x^{-\frac{2}{3}} - \frac{27}{x^2} = 0 \Rightarrow \frac{1}{3}x^{-\frac{2}{3}} = \frac{27}{x^2} \Rightarrow \frac{1}{3}x^{\frac{4}{3}} = 27 \Rightarrow x^{\frac{4}{3}} = 81 \Rightarrow x^{\frac{4}{3}} = 3^4 \Rightarrow x^{\frac{4}{3}} = 3^4 \Rightarrow x = 3^3 = 27$. Then $y = \sqrt[3]{27} + \frac{27}{27} = 3 + 1 = 4$. Nature: $\frac{d^2y}{dx^2} = -\frac{2}{9}x^{-\frac{5}{3}} + \frac{54}{x^3}$. At $x=27$, $\frac{d^2y}{dx^2} = -\frac{2}{9} \cdot \frac{1}{27^{\frac{5}{3}}} + \frac{54}{27^3} = -\frac{2}{9} \cdot \frac{1}{27^2} + \frac{54}{27^3} = -\frac{2}{9 \cdot 729} + \frac{54}{27^3} = -\frac{2}{6561} + \frac{54}{27^3} = -\frac{2}{6561} + \frac{54}{27^3} = \frac{54 \cdot 100 - 2}{6561} = \frac{5400 - 2}{6561} > 0$. So it's a local minimum.
- Part (d):** $y = \frac{1}{3\sqrt{x}} \left[\frac{2}{x} - 3 \right]$. Derivative: $\frac{dy}{dx} = \frac{1}{3} \left[-\frac{1}{2}x^{-\frac{3}{2}} \left(\frac{2}{x} - 3 \right) + \frac{1}{3\sqrt{x}} \left(-\frac{2}{x^2} \right) \right]$. Setting to zero: $-\frac{1}{2}x^{-\frac{3}{2}} \left(\frac{2}{x} - 3 \right) + \frac{1}{3\sqrt{x}} \left(-\frac{2}{x^2} \right) = 0 \Rightarrow -\frac{1}{2}x^{-\frac{3}{2}} \left(\frac{2}{x} - 3 \right) = \frac{2}{3\sqrt{x}x^2} = \frac{2}{3x^{\frac{5}{2}}}$. Multiply by $2x^{\frac{5}{2}}$: $-x \left(\frac{2}{x} - 3 \right) = \frac{4}{3} \Rightarrow -2 + 3x = \frac{4}{3} \Rightarrow 3x = \frac{4}{3} + 2 = \frac{10}{3} \Rightarrow x = \frac{10}{9}$. Then $y = \frac{1}{3\sqrt{\frac{10}{9}}} \left[\frac{2}{\frac{10}{9}} - 3 \right] = \frac{1}{3 \cdot \frac{\sqrt{10}}{3}} \left[\frac{2 \cdot 9}{10} - 3 \right] = \frac{1}{\sqrt{10}} \left[\frac{18}{10} - 3 \right] = \frac{1}{\sqrt{10}} \left[\frac{18 - 30}{10} \right] = \frac{1}{\sqrt{10}} \left[-\frac{12}{10} \right] = -\frac{6}{5\sqrt{10}} = -\frac{6\sqrt{10}}{50} = -\frac{3\sqrt{10}}{25}$. Nature: $\frac{d^2y}{dx^2} = \frac{1}{3} \left[\frac{3}{4}x^{-\frac{5}{2}} \left(\frac{2}{x} - 3 \right) - \frac{1}{2}x^{-\frac{3}{2}} \left(-\frac{4}{x^3} \right) \right]$. At $x = \frac{10}{9}$, $\frac{d^2y}{dx^2} > 0$. So it's a local minimum.

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INCREASING and DECREASING FUNCTIONS

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Question 1

For each of the following equations find the range of the values of x , for which y is increasing or decreasing.

a) $y = 2x^3 - 3x^2 - 12x + 2$, increasing

b) $y = x^3 - 6x^2 + 12$, decreasing

c) $y = x^3 - 3x + 8$, increasing

d) $y = 1 - 3x^2 - x^3$, decreasing

$x < -1$ or $x > 2$, $0 < x < 4$, $x < -1$ or $x > 1$, $x < -2$ or $x > 0$

The image shows handwritten solutions for the four parts of Question 1. Each part (a, b, c, d) is solved by differentiating y with respect to x to find $\frac{dy}{dx}$. The derivative is then set greater than zero (for increasing) or less than zero (for decreasing). The resulting quadratic inequality is solved by finding the roots and testing the intervals. Sign diagrams are drawn to show the regions where the derivative is positive or negative. The final answers are boxed and match the printed solutions above.

(a) $y = 2x^3 - 3x^2 - 12x + 2$
 $\frac{dy}{dx} = 6x^2 - 6x - 12$
 INCREASING $\Rightarrow \frac{dy}{dx} > 0$
 $6x^2 - 6x - 12 > 0$
 $x^2 - x - 2 > 0$
 $(x+1)(x-2) > 0$
 C.V. = $\begin{matrix} -1 \\ 2 \end{matrix}$
 $\therefore x < -1$ or $x > 2$

(b) $y = x^3 - 6x^2 + 12$
 $\frac{dy}{dx} = 3x^2 - 12x$
 DECREASING $\Rightarrow \frac{dy}{dx} < 0$
 $3x^2 - 12x < 0$
 $3x(x-4) < 0$
 C.V. = $\begin{matrix} 0 \\ 4 \end{matrix}$
 $\therefore 0 < x < 4$

(c) $y = x^3 - 3x + 8$
 $\frac{dy}{dx} = 3x^2 - 3$
 INCREASING $\Rightarrow \frac{dy}{dx} > 0$
 $3x^2 - 3 > 0$
 $x^2 - 1 > 0$
 $(x+1)(x-1) > 0$
 C.V. = $\begin{matrix} -1 \\ 1 \end{matrix}$
 $\therefore x < -1$ or $x > 1$

(d) $y = 1 - 3x^2 - x^3$
 $\frac{dy}{dx} = -6x - 3x^2$
 DECREASING $\Rightarrow \frac{dy}{dx} < 0$
 $-6x - 3x^2 < 0$
 $6x + 3x^2 > 0$
 $3x(2+x) > 0$
 C.V. = $\begin{matrix} 0 \\ -2 \end{matrix}$
 $\therefore x < -2$ or $x > 0$

Question 2

Find the range of the values of x , for which $f(x)$ is increasing or decreasing.

a) $f(x) = x^3 - 3x^2 - 9x + 10$, increasing

b) $f(x) = -x^3 + 9x^2 - 15x - 13$, increasing

c) $f(x) = 4x^3 - 3x^2 - 6x$, decreasing

d) $f(x) = 4x^3 - 3x$, decreasing

$$x < -1 \text{ or } x > 3, \quad 1 < x < 5, \quad -\frac{1}{2} < x < 1, \quad -\frac{1}{2} < x < \frac{1}{2}$$

Handwritten solution for Question 2:

(a) $f(x) = x^3 - 3x^2 - 9x + 10$
 $f'(x) = 3x^2 - 6x - 9$
 • INCREASING $\Rightarrow f'(x) > 0$
 $3x^2 - 6x - 9 > 0$
 $x^2 - 2x - 3 > 0$
 $(x+1)(x-3) > 0$
 $x < -1 \text{ or } x > 3$

(b) $f(x) = -x^3 + 9x^2 - 15x - 13$
 $f'(x) = -3x^2 + 18x - 15$
 • INCREASING $\Rightarrow f'(x) > 0$
 $-3x^2 + 18x - 15 > 0$
 $x^2 - 6x + 5 < 0$
 $(x-1)(x-5) < 0$
 $1 < x < 5$

(c) $f(x) = 4x^3 - 3x^2 - 6x$
 $f'(x) = 12x^2 - 6x - 6$
 • DECREASING $\Rightarrow f'(x) < 0$
 $12x^2 - 6x - 6 < 0$
 $2x^2 - x - 1 < 0$
 $(2x+1)(x-1) < 0$
 $-\frac{1}{2} < x < 1$

(d) $f(x) = 4x^3 - 3x$
 $f'(x) = 12x^2 - 3$
 • DECREASING $\Rightarrow f'(x) < 0$
 $12x^2 - 3 < 0$
 $4x^2 - 1 < 0$
 $(2x-1)(2x+1) < 0$
 $-\frac{1}{2} < x < \frac{1}{2}$

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DIFFERENTIATION PRACTICE IN CONTEXT

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Question 1

The curve C has equation

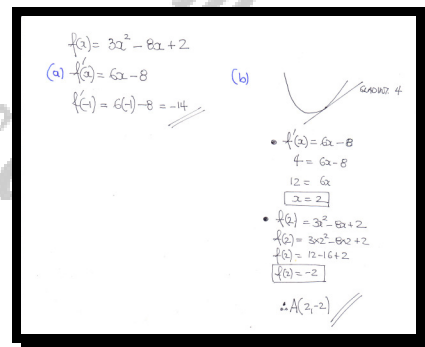
$$f(x) = 3x^2 - 8x + 2.$$

- a) Find the gradient at the point on C , where $x = -1$.

The point A lies on C and the gradient at that point is 4.

- b) Find the coordinates of A .

-14 , $A(2, -2)$



Question 2

The curve C has equation

$$y = x^3 - 11x + 1.$$

- a) Find the gradient at the point on C , where $x = 3$.

The point P lies on C and the gradient at that point is 1.

- b) Find the possible coordinates of P .

16, $P(2, -13)$ or $P(-2, 15)$

The image shows a handwritten solution for Question 2. It is divided into two parts, (a) and (b).
Part (a) starts with the equation $y = x^3 - 11x + 1$. It then differentiates to find $\frac{dy}{dx} = 3x^2 - 11$. At $x = 3$, the gradient is calculated as $3(3)^2 - 11 = 27 - 11 = 16$.
Part (b) starts with the condition that the gradient is 1, so $\frac{dy}{dx} = 1$. This leads to the equation $3x^2 - 11 = 1$, which simplifies to $3x^2 = 12$ and then $x^2 = 4$. Solving for x gives $x = 2$ or $x = -2$. For $x = 2$, $y = 8 - 22 + 1 = -13$. For $x = -2$, $y = -8 + 22 + 1 = 15$. The final answer is $\therefore P(2, -13)$ or $P(-2, 15)$.

Question 3

The curve C has equation

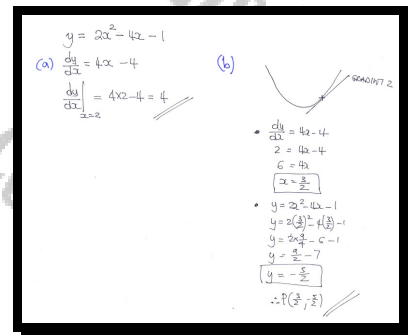
$$y = 2x^2 - 4x - 1.$$

- a) Find the gradient at the point on C , where $x = 2$.

The point P lies on C and the gradient at that point is 2.

- b) Find the coordinates of P .

$$\boxed{4}, \quad \boxed{P\left(\frac{3}{2}, -\frac{5}{2}\right)}$$



Question 4

The curve C has equation

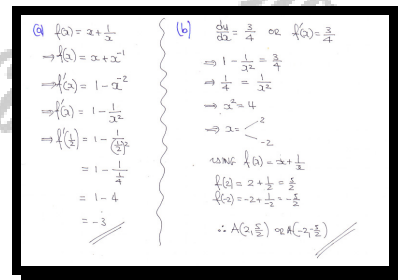
$$f(x) = x + \frac{1}{x}, \quad x \neq 0.$$

- a) Find the gradient at the point on C , where $x = \frac{1}{2}$.

The point A lies on C and the gradient at that point is $\frac{3}{4}$.

- b) Find the possible coordinates of A .

$$\boxed{-3}, \quad \boxed{A\left(2, \frac{5}{2}\right) \text{ or } A\left(-2, -\frac{5}{2}\right)}$$



(a) $f(x) = x + \frac{1}{x}$
 $\rightarrow f'(x) = 1 - \frac{1}{x^2}$
 $\rightarrow f'(x) = 1 - \frac{1}{\left(\frac{1}{2}\right)^2}$
 $\rightarrow f'(x) = 1 - \frac{1}{\frac{1}{4}}$
 $= 1 - 4$
 $= -3$

(b) $\frac{dy}{dx} = \frac{3}{4}$ or $f'(x) = \frac{3}{4}$
 $\rightarrow 1 - \frac{1}{x^2} = \frac{3}{4}$
 $\rightarrow \frac{1}{x^2} = \frac{1}{4}$
 $\rightarrow x^2 = 4$
 $\rightarrow x = \pm 2$
 using $f(x) = x + \frac{1}{x}$
 $f(2) = 2 + \frac{1}{2} = \frac{5}{2}$
 $f(-2) = -2 + \frac{1}{-2} = -\frac{5}{2}$
 $\therefore A\left(2, \frac{5}{2}\right)$ or $A\left(-2, -\frac{5}{2}\right)$

Question 5

The curve C has equation

$$y = x^3 - x^2 - 5x + 2.$$

Find the x coordinates of the points on C with gradient 3.

$$x = -\frac{4}{3}, 2$$

$$\begin{aligned} y &= x^3 - x^2 - 5x + 2 & \Rightarrow 0 &= 3x^2 - 2x - 6 \\ \frac{dy}{dx} &= 3x^2 - 2x - 5 & \Rightarrow 0 &= (3x+4)(x-2) \\ 3 &= 3x^2 - 2x - 5 & \therefore x &= -\frac{4}{3} \end{aligned}$$

Question 6

The curve C has equation

$$y = x^5 - 6x^3 - 3x + 25.$$

Find an equation of the tangent to C at the point where $x = 2$.

$$y = 5x - 7$$

$$\begin{aligned} \bullet y &= x^5 - 6x^3 - 3x + 25 & \frac{dy}{dx} &= 5x^4 - 18x^2 - 3 \\ \text{when } x &= 2 & \frac{dy}{dx} &= 5(2^4) - 18(2^2) - 3 \\ \Rightarrow y &= 2^5 - 6(2^3) - 3(2) + 25 & &= 80 - 72 - 3 \\ \Rightarrow y &= 32 - 48 - 6 + 25 & &= 5 \\ \Rightarrow y &= 57 - 54 & & \\ \Rightarrow y &= 3 & & \\ \therefore (2, 3) & & & \\ & & & y - y_1 = m(x - x_1) \\ & & & y - 3 = 5(x - 2) \\ & & & y - 3 = 5x - 10 \\ & & & y = 5x - 7 \end{aligned}$$

Question 7

The curve C has equation

$$y = -x^2(x+1), \quad x \in \mathbb{R}.$$

The curve meets the coordinate axes at the origin O and at the point A .

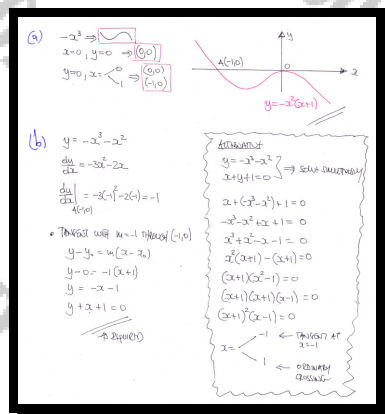
a) Sketch the graph of C , indicating clearly the coordinates of A .

b) Show that the straight line with equation

$$x + y + 1 = 0,$$

is a tangent to C at A .

$$A(-1, 0)$$



Question 8

The curve C has equation

$$y = \frac{6}{x^2} + \frac{5x}{4} - 4, \quad x \neq 0.$$

- a) Find an expression for $\frac{dy}{dx}$.
- b) Determine an equation of the normal to the curve at the point where $x = 2$.

$$\frac{dy}{dx} = \frac{5}{4} - \frac{12}{x^3}, \quad y = 4x - 8$$

(a) $y = \frac{6}{x^2} + \frac{5x}{4} - 4$
 $y = 6x^{-2} + \frac{5x}{4} - 4$
 $\frac{dy}{dx} = -12x^{-3} + \frac{5}{4}$
 $\frac{dy}{dx} = \frac{5}{4} - \frac{12}{x^3}$

(b) when $x=2$
 $y = \frac{6}{2^2} + \frac{5 \cdot 2}{4} - 4$
 $y = \frac{6}{4} + \frac{10}{4} - 4$
 $y = \frac{16}{4} - 4$
 $y = 0$
 $\therefore (2, 0)$
 $\frac{dy}{dx} = \frac{5}{4} - \frac{12}{2^3} = \frac{5}{4} - \frac{12}{8} = \frac{5}{4} - \frac{3}{2} = \frac{5}{4} - \frac{6}{4} = -\frac{1}{4}$
 \therefore Normal gradient is 4, $(2, 0)$
 $y - y_1 = m(x - x_1)$
 $\Rightarrow y - 0 = 4(x - 2)$
 $\Rightarrow y = 4x - 8$

Question 9

The curve C has equation

$$f(x) = 4x\sqrt{x} - \frac{25x^2}{16}, \quad x \geq 0.$$

- a) Find a simplified expression for $f'(x)$.
- b) Determine an equation of the tangent to C at the point where $x = 4$, giving the answer in the form $ax + by = c$, where a , b and c are integers.

$$f'(x) = 6x^{\frac{1}{2}} - \frac{25}{8}x, \quad x + 2y = 18$$

(a) $f(x) = 4x\sqrt{x} - \frac{25x^2}{16}$
 $\Rightarrow f'(x) = 4 \cdot \frac{1}{2}x^{\frac{1}{2}} - \frac{25 \cdot 2x}{16}$
 $\Rightarrow f'(x) = 2x^{\frac{1}{2}} - \frac{25x}{8}$
 $\Rightarrow f'(x) = 2\sqrt{x} - \frac{25x}{8}$

(b) $f(x) = 4x\sqrt{x} - \frac{25x^2}{16}$
 $= 4x^{\frac{3}{2}} - \frac{25x^2}{16}$
 $= 4 \cdot \frac{3}{2}x^{\frac{1}{2}} - \frac{25 \cdot 2x}{16}$
 $= 6x^{\frac{1}{2}} - \frac{25x}{8}$
 $= 6\sqrt{x} - \frac{25x}{8}$
 $= 6\sqrt{4} - \frac{25 \cdot 4}{8}$
 $= 6 \cdot 2 - \frac{100}{8}$
 $= 12 - \frac{25}{2}$
 $= \frac{24}{2} - \frac{25}{2}$
 $= -\frac{1}{2}$

when $x=4$
 $y = f(4) = 4 \cdot 4\sqrt{4} - \frac{25 \cdot 4^2}{16}$
 $= 32 - 25$
 $= 7$
 $\therefore (4, 7)$ is on the curve

$\Rightarrow y - y_1 = m(x - x_1)$
 $\Rightarrow y - 7 = -\frac{1}{2}(x - 4)$
 $\Rightarrow 2y - 14 = -x + 4$
 $\Rightarrow 2y + x = 18$

Question 10

A curve has the following equation

$$f(x) = \frac{(2x-3)(x+2)}{\sqrt{x}}, \quad x > 0.$$

- a) Express $f(x)$ in the form $Ax^{\frac{3}{2}} + Bx^{\frac{1}{2}} + Cx^{-\frac{1}{2}}$, where A , B and C are constants to be found.
- b) Show that the tangent to the curve at the point where $x = 1$ is parallel to the line with equation

$$2y = 13x + 2.$$

$$\boxed{A=2}, \quad \boxed{B=1}, \quad \boxed{C=-6}$$

(a) $f(x) = \frac{(2x-3)(x+2)}{\sqrt{x}} = \frac{2x^2 + 2x - 6}{\sqrt{x}} = \frac{2x^2}{\sqrt{x}} + \frac{2x}{\sqrt{x}} - \frac{6}{\sqrt{x}}$
 $= 2x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}$
 $A=2, B=1, C=-6$

(b) $f'(x) = 3x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} - 3x^{-\frac{3}{2}}$
 $f'(1) = 3 + \frac{1}{2} - 3 = \frac{1}{2}$
 (It gradient of tangent is $\frac{13}{2}$)

Also $2y = 13x + 2$
 $y = \frac{13}{2}x + 1$
 \therefore SAME GRADIENT AS TANGENT
 \therefore PARALLEL

Question 11

A cubic curve has equation

$$f(x) = 2x^3 - 7x^2 + 6x + 1.$$

The point $P(2,1)$ lies on the curve.

- a) Find an equation of the tangent to the curve at P .

The point Q lies on the curve so that the tangent to the curve at Q is parallel to the tangent to the curve at P .

- b) Determine the x coordinate of Q .

$$y = 2x - 3, \quad x_Q = \frac{1}{3}$$

(a) $f(x) = 2x^3 - 7x^2 + 6x + 1$
 $f'(x) = 6x^2 - 14x + 6$
 $f(2) = 1$
 $y - y_1 = m(x - x_1)$
 $y - 1 = 2(x - 2)$
 $y - 1 = 2x - 4$
 $y = 2x - 3$

(b) PARALLEL TANGENTS \Rightarrow SAME GRADIENT = 2
 $2 = 6x^2 - 14x + 6$
 $0 = 6x^2 - 14x + 4$
 $0 = 3x^2 - 7x + 2$
 $0 = (3x - 2)(x - 1)$
 $x = \frac{2}{3}$ ← Point P (ALREADY KNOWN)
 $x = \frac{1}{3}$ ← Point Q

Question 12

The curve C has equation

$$y = 2x^3 - 9x^2 + 12x - 10.$$

- a) Find the coordinates of the two points on the curve where the gradient is zero.

The point P lies on C and its x coordinate is -1 .

- b) Determine the gradient of C at the point P .

The point Q lies on C so that the gradient at Q is the same as the gradient at P .

- c) Find the coordinates of Q .

$(1, -5), (2, -6), 36, Q(4, 22)$

Handwritten solution for Question 12:

$y = 2x^3 - 9x^2 + 12x - 10$
 (a) $\frac{dy}{dx} = 6x^2 - 18x + 12$
 $\frac{dy}{dx} = 0$
 $0 = 6x^2 - 18x + 12$
 $0 = 2x^2 - 3x + 2$
 $0 = (2x-1)(x-2)$
 $x = \frac{1}{2}, y = \frac{2 \cdot 1^3 - 9 \cdot 1^2 + 12 \cdot 1 - 10}{2} = \frac{2 - 9 + 12 - 10}{2} = \frac{-5}{2} = -2.5$ $\therefore P(\frac{1}{2}, -2.5)$
 $x = 2, y = \frac{2 \cdot 2^3 - 9 \cdot 2^2 + 12 \cdot 2 - 10}{2} = \frac{16 - 36 + 24 - 10}{2} = \frac{-6}{2} = -3$ $\therefore Q(2, -3)$

(b) $\frac{dy}{dx} = 6x^2 - 18x + 12$
 $\frac{dy}{dx} \Big|_{x=-1} = 6(-1)^2 - 18(-1) + 12 = 6 + 18 + 12 = 36$

(c) $\frac{dy}{dx} = 36$
 $6x^2 - 18x + 12 = 36$
 $6x^2 - 18x - 24 = 0$
 $x^2 - 3x - 4 = 0$
 $(x+1)(x-4) = 0$
 $x = -1$ (Already known)
 $x = 4$
 $y = 2 \cdot 4^3 - 9 \cdot 4^2 + 12 \cdot 4 - 10$
 $y = 128 - 144 + 48 - 10$
 $y = 176 - 154 = 22$
 $\therefore Q(4, 22)$

Question 13

The curve C has equation

$$y = ax^3 + bx^2 - 10,$$

where a and b are constants.

The point $A(2,2)$ lies on C .

Given that the gradient at A is 4, determine the value of a and the value of b .

$$a = -2, \quad b = 7$$

Handwritten solution for Question 13:

$$y = ax^3 + bx^2 - 10 \Rightarrow \frac{dy}{dx} = 3ax^2 + 2bx$$

• $(2,2)$ lies on the curve

$$2 = 8a + 4b - 10$$

$$12 = 8a + 4b$$

$$3 = 2a + b$$

• With $x=2$, $\frac{dy}{dx} = 4$

$$4 = 12a + 4b$$

$$1 = 3a + b$$

Solving the system:

$$b = 3 - 2a$$

$$b = 1 - 3a$$

$$3 - 2a = 1 - 3a$$

$$a = -2$$

Substituting $a = -2$ into $b = 3 - 2a$:

$$b = 3 - 2(-2) = 3 + 4 = 7$$

Final answer: $a = -2, b = 7$

Question 14

The curve C has equation

$$y = x^3 - 4x^2 + 6x - 3.$$

The point $P(2,1)$ lies on C and the straight line L_1 is the tangent to C at P .

- a) Find an equation of L_1 .

The straight line L_2 is a tangent to C at the point Q .

- b) Given that L_2 is parallel to L_1 , determine ...

- i. ... the exact coordinates of Q .
- ii. ... an equation of L_2 .

$$y = 2x - 3, \quad Q\left(\frac{2}{3}, -\frac{13}{27}\right), \quad 27y = 54x - 49$$

(a) $y = x^3 - 4x^2 + 6x - 3$
 $\frac{dy}{dx} = 3x^2 - 8x + 6$
 $\frac{dy}{dx}\bigg|_{x=2} = 12 - 16 + 6 = 2$
 • $(2, 1)$, $m = 2$, TANGENT AT $(2, 1)$
 $\Rightarrow y - 1 = m(x - 2)$
 $\Rightarrow y - 1 = 2(x - 2)$
 $\Rightarrow y - 1 = 2x - 4$
 $\Rightarrow y = 2x - 3$

(b) (i) $\frac{dy}{dx} = 3x^2 - 8x + 6$
 $2 = 3x^2 - 8x + 6$
 $0 = 3x^2 - 8x + 4$
 $0 = (x - 2)(3x - 2)$
 $x = 2$ or $x = \frac{2}{3}$
 $x = \frac{2}{3}$ is the point Q
 $y = \left(\frac{2}{3}\right)^3 - 4\left(\frac{2}{3}\right)^2 + 6\left(\frac{2}{3}\right) - 3 = \frac{8}{27} - \frac{16}{9} + 4 - 3 = \frac{8}{27} - \frac{48}{27} + \frac{108}{27} - \frac{81}{27} = \frac{8 - 48 + 108 - 81}{27} = \frac{-13}{27}$
 $\therefore Q\left(\frac{2}{3}, -\frac{13}{27}\right)$

(ii) $y - y_1 = m(x - x_1)$
 $y + \frac{13}{27} = 2\left(x - \frac{2}{3}\right)$
 $y + \frac{13}{27} = 2x - \frac{4}{3}$
 $27y + 13 = 54x - 36$
 $27y = 54x - 49$

Question 15

A curve C and a straight line L have respective equations

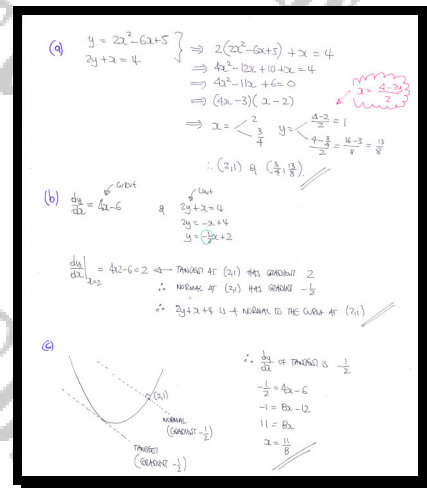
$$y = 2x^2 - 6x + 5 \quad \text{and} \quad 2y + x = 4.$$

- a) Find the coordinates of the points of intersection between C and L .
- b) Show that L is a normal to C .

The tangent to C at the point P is parallel to L .

- c) Determine the x coordinate of P .

$$\boxed{(2, 1), \left(\frac{3}{4}, \frac{13}{8}\right)}, \quad \boxed{x_P = \frac{11}{8}}$$



Question 16

The curve C has equation

$$y = 2x^3 - 6x^2 + 3x + 5.$$

The point $P(2,3)$ lies on C and the straight line L_1 is the tangent to C at P .

- a) Find an equation of L_1 .

The straight lines L_2 and L_3 are parallel to L_1 , and they are the respective normals to C at the points Q and R .

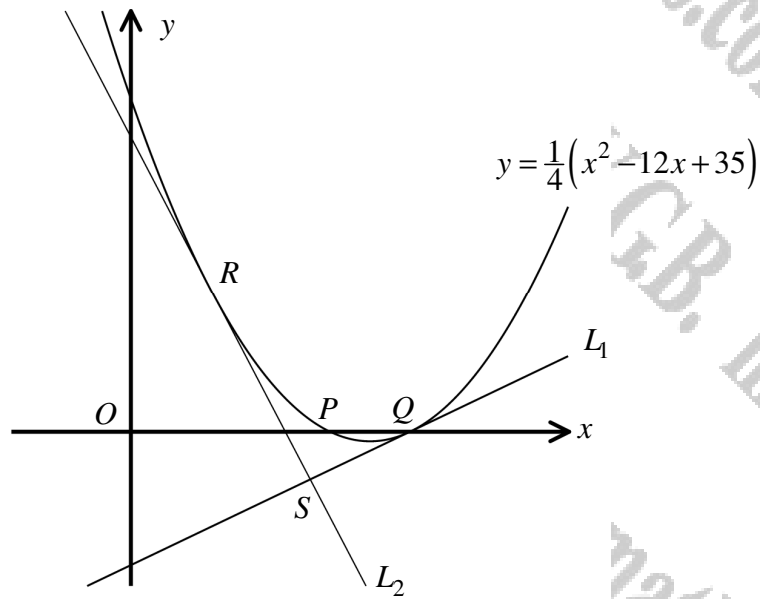
- b) Determine the x coordinate of Q and the x coordinate of R .

$$y = 3x - 3, \quad x = \frac{1}{3}, \frac{5}{3}$$

(a) $y = 2x^3 - 6x^2 + 3x + 5$
 $\frac{dy}{dx} = 6x^2 - 12x + 3$
 $\frac{dy}{dx} \Big|_{x=2} = 6(2)^2 - 12(2) + 3 = 24 - 24 + 3 = 3$
 *TANGENT AT $(2,3)$ GRADIENT 3
 $\Rightarrow y - 3 = 3(x - 2)$
 $\Rightarrow y - 3 = 3x - 6$
 $\Rightarrow y - 3 = 3x - 6$
 $\Rightarrow y = 3x - 3$

(b) $\frac{dy}{dx} = 6x^2 - 12x + 3$
 $\Rightarrow -\frac{1}{3} = 6x^2 - 12x + 3$
 $\Rightarrow 0 = 18x^2 - 36x + 10$
 $\Rightarrow 9x^2 - 18x + 5 = 0$
 $\Rightarrow (3x - 5)(3x - 1)$
 $\Rightarrow x = \frac{1}{3}$
 $\Rightarrow x = \frac{5}{3}$

Question 17



The figure above shows the curve with equation

$$y = \frac{1}{4}(x^2 - 12x + 35).$$

The curve crosses the x axis at the points $P(x_1, 0)$ and $Q(x_2, 0)$, where $x_2 > x_1$.

The tangent to the curve at Q is the straight line L_1 .

- a) Find an equation of L_1 .

The tangent to the curve at the point R is denoted by L_2 . It is further given that L_2 meets L_1 at right angles, at the point S .

- b) Find an equation of L_2 .
- c) Determine the exact coordinates of S .

\square ,
 $y = \frac{1}{2}x - \frac{7}{2}$,
 $4y + 8x = 31$,
 $S\left(\frac{9}{2}, -\frac{5}{4}\right)$

$y = \frac{1}{4}(x^2 - 12x + 35)$
 $y = \frac{1}{4}(x-5)(x-7)$
 $\Rightarrow x = 5, 7$
 $\therefore P(5, 0), Q(7, 0)$
 Gradient of L_1 is $\frac{0 - \frac{1}{4}(7^2 - 12 \cdot 7 + 35)}{7 - 5} = \frac{-\frac{1}{4}(49 - 84 + 35)}{2} = \frac{-\frac{1}{4}(-10)}{2} = \frac{10}{8} = \frac{5}{4}$
 \therefore Equation of L_1 is $y - 0 = \frac{5}{4}(x - 7)$
 $\Rightarrow 4y = 5x - 35$
 $\Rightarrow 4y + 8x = 31$

Gradient of L_2 is $-\frac{5}{4}$ (since $L_1 \perp L_2$)
 \therefore Equation of L_2 is $y - \frac{1}{4}(x^2 - 12x + 35) = -\frac{5}{4}(x - x_1)$
 $\Rightarrow \frac{1}{4}(x^2 - 12x + 35) = -\frac{5}{4}(x - x_1) + \frac{1}{4}(x^2 - 12x + 35)$
 $\Rightarrow -12x + 35 = -5x + 5x_1 + x^2 - 12x + 35$
 $\Rightarrow 0 = x^2 - 7x + 5x_1 - 35$
 $\Rightarrow x^2 - 7x + 5x_1 - 35 = 0$
 $\Rightarrow x = 7$ (since Q is on L_2)
 $\Rightarrow 49 - 49 + 5x_1 - 35 = 0$
 $\Rightarrow 5x_1 = 35$
 $\Rightarrow x_1 = 7$
 $\therefore R(7, 0)$

Gradient of L_2 is $-\frac{5}{4}$
 \therefore Equation of L_2 is $y - 0 = -\frac{5}{4}(x - 7)$
 $\Rightarrow 4y = -5x + 35$
 $\Rightarrow 4y + 5x = 35$

Solving $4y + 8x = 31$ and $4y + 5x = 35$
 $\Rightarrow 3x = -4$
 $\Rightarrow x = -\frac{4}{3}$
 $\Rightarrow 4y + 8(-\frac{4}{3}) = 31$
 $\Rightarrow 4y - \frac{32}{3} = 31$
 $\Rightarrow 4y = 31 + \frac{32}{3} = \frac{93 + 32}{3} = \frac{125}{3}$
 $\Rightarrow y = \frac{125}{12}$
 $\therefore S\left(-\frac{4}{3}, \frac{125}{12}\right)$

Question 18

The point $P(1,0)$ lies on the curve C with equation

$$y = x^3 - x, \quad x \in \mathbb{R}.$$

- a) Find an equation of the tangent to C at P , giving the answer in the form $y = mx + c$, where m and c are constants.

The tangent to C at P meets C again at the point Q .

- b) Determine the coordinates of Q .

$$y = 2x - 2, \quad Q(-2, -6)$$

(a) $y = x^3 - x$
 $\frac{dy}{dx} = 3x^2 - 1$
 $\left. \frac{dy}{dx} \right|_{x=1} = 3(1)^2 - 1 = 2$

Hence $y - y_1 = m(x - x_1)$
 $y - 0 = 2(x - 1)$
 $y = 2x - 2$

(b) $y = x^3 - x$
 $y = 2x - 2$
 $\Rightarrow x^3 - 2x + 2 = 0$

SINCE $P(1,0)$ MUST BE SOLUTION TO THE PROBLEM
 $\Rightarrow x = 1$ IS A ROOT (TRIAL AND ERROR)
 \Rightarrow FURTHER FACTOR $(x - 1)$ MUST BE EXTRACTED (TRUNCATE REMAINDER) I.E. $(x - 1)^2$

$\Rightarrow (x - 1)(x^2 + 2x + 2) = 0$

← CHECK
 $(x + 2)(x - 1)^2$
 $= (x + 2)(x^2 - 2x + 1)$
 $= x^3 - 2x^2 + 2x + 2x^2 - 4x + 2$
 $= x^3 - 2x + 2$

$\therefore x = -2$
 $y = 2(-2) = 2(-2) - 2 = -6$
 So $Q(-2, -6)$

Question 19

A curve C with equation

$$y = 4x^3 + 7x^2 + x + 11, \quad x \in \mathbb{R}.$$

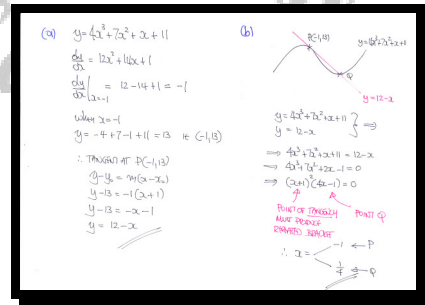
The point P lies on C , where $x = -1$.

- a) Find an equation of the tangent to C at P .

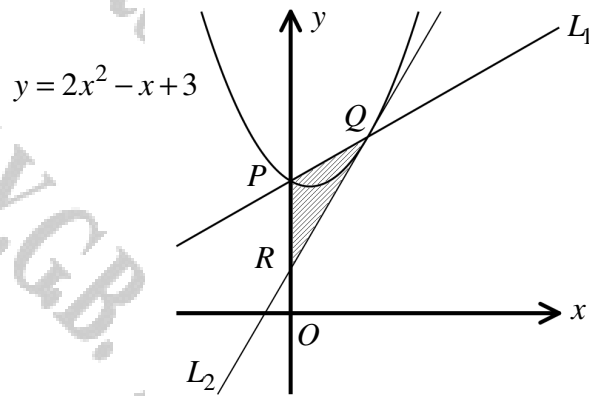
The tangent to C at P meets C again at the point Q .

- b) Determine the x coordinate of Q .

$$y = 12 - x, \quad x_Q = \frac{1}{4}$$



Question 20



The figure above shows the curve C with equation

$$y = 2x^2 - x + 3.$$

C crosses the y axis at the point P . The normal to C at P is the straight line L_1 .

- a) Find an equation of L_1 .

L_1 meets the curve again at the point Q .

- b) Determine the coordinates of Q .

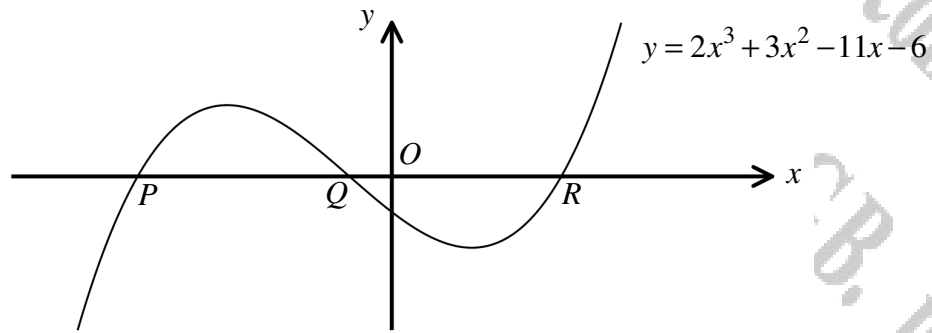
The tangent to C at Q is the straight line L_2 .

L_2 meets the y axis at the point R .

- c) Show that the area of the triangle PQR is one square unit.

$$y = x + 3, \quad Q(1,4)$$

Question 21



The figure above shows the curve C with equation

$$y = 2x^3 + 3x^2 - 11x - 6.$$

The curve crosses the x axis at the points P , Q and $R(2,0)$.

The tangent to C at R is the straight line L_1 .

- a) Find an equation of L_1 .

The normal to C at P is the straight line L_2 .

The straight lines L_1 and L_2 meet at the point S .

- b) Show that $\angle PSR = 90^\circ$.

$$y = 25x - 50$$

(a) $y = 2x^3 + 3x^2 - 11x - 6$
 $\frac{dy}{dx} = 6x^2 + 6x - 11$
 $\frac{dy}{dx} = 6x^2 + 6x - 11$
 $= 24 + 12 - 11 = 25$
 Thinking: $x = 25, (2,0)$
 $y - y_1 = m(x - x_1)$
 $y - 0 = 25(x - 2)$
 $y = 25x - 50$

(b) $y = 2x^3 + 3x^2 - 11x - 6$
 $y = (x-2)(2x^2 + 4x + 3)$
 From part a
 Gradient of normals is 2
 $2x - 2Ax = -11x$
 $3 - 2A = -11$
 $14 = 2A$
 $A = 7$
 $y = (x-2)(2x^2 + 2x + 3)$
 $y = (x-2)(2x+1)(x+3)$
 $x = -\frac{1}{2}$
 $x = -3$
 $x = 2$
 $\frac{dy}{dx} = 2(-\frac{1}{2})^2 + 6(-\frac{1}{2}) - 11 = 54 - 18 - 11 = 25$
 Gradient of the normal at P is $\frac{1}{25}$
 Gradient of the tangent at R is 25
 $\therefore L_1 \perp L_2$ meet at 90°

Question 22

A curve has equation

$$y = 6\sqrt[3]{x^5} - 15\sqrt[3]{x^4} - 80x + 16, \quad x \in \mathbb{R}, \quad x \geq 0.$$

Find the coordinates of the stationary point of the curve and determine whether it is a local maximum, a local minimum or a point of inflexion.

local minimum at (16, -2800)

WRITE THE EQUATION IN INDEXIAL FORM AND DIFFERENTIATE

$$\Rightarrow y = 6\sqrt[3]{x^5} - 15\sqrt[3]{x^4} - 80x + 16$$

$$\Rightarrow y = 6x^{\frac{5}{3}} - 15x^{\frac{4}{3}} - 80x + 16$$

$$\Rightarrow \frac{dy}{dx} = 10x^{\frac{2}{3}} - 20x^{\frac{1}{3}} - 80$$

FIND STATIONARY POINTS, SOLVING $\frac{dy}{dx} = 0$

$$\Rightarrow 0 = 10x^{\frac{2}{3}} - 20x^{\frac{1}{3}} - 80$$

$$\Rightarrow 2x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 8 = 0$$

$$\Rightarrow (2x^{\frac{1}{3}})^2 - 2x^{\frac{1}{3}} - 8 = 0$$

$$\Rightarrow a^2 - 2a - 8 = 0, \text{ where } a = x^{\frac{1}{3}}$$

$$\Rightarrow (a+2)(a-4) = 0$$

$$\Rightarrow a = \begin{cases} -2 \\ 4 \end{cases}$$

$$\Rightarrow x^{\frac{1}{3}} = \begin{cases} -2 \\ 4 \end{cases}$$

$$\Rightarrow x = \begin{cases} -8 \\ 64 \end{cases} \quad (x \geq 0)$$

$$\Rightarrow y = 6 \times 64^{\frac{5}{3}} - 15 \times 64^{\frac{4}{3}} - 80 \times 64 + 16$$

$$= 6 \times 1024 - 15 \times 256 - 80 \times 64 + 16$$

$$= 6144 - 3840 - 5120 + 16$$

$$= -2800$$

DETERMINING THE NATURE BY THE SECOND DERIVATIVE TEST

$$\frac{d^2y}{dx^2} = 10x^{-\frac{1}{3}} - 20x^{-\frac{2}{3}} - 80$$

$$\frac{d^2y}{dx^2} = \frac{10}{\sqrt[3]{x}} - \frac{20}{\sqrt{x}} - 80$$

$$\frac{d^2y}{dx^2} \Big|_{x=64} = \frac{10}{\sqrt[3]{64}} - \frac{20}{\sqrt{64}} - 80 = \frac{5}{4} > 0$$

$\therefore (16, -2800)$ IS A LOCAL MINIMUM

Question 23

A curve has equation

$$y = x^2 - 6x\sqrt[3]{x} + 2, \quad x \in \mathbb{R}, \quad x \geq 0.$$

Find the coordinates of the stationary points of the curve and classify them as local maxima, local minima or a points of inflexion.

local minimum at $(8, -30)$, local maximum at $(0, 2)$

REWRITE THE EQUATION IN INDICES & DIFFERENTIATE

$$\begin{aligned} \rightarrow y &= x^2 - 6x\sqrt[3]{x} + 2 \\ \rightarrow y &= x^2 - 6x^{\frac{4}{3}} + 2 \\ \rightarrow y &= x^2 - 6x^{\frac{4}{3}} + 2 \\ \rightarrow \frac{dy}{dx} &= 2x - 8x^{\frac{1}{3}} \end{aligned}$$

SETTING THE ZERO, SEEKING STATIONARY POINTS

$$\begin{aligned} \Rightarrow 2x - 8x^{\frac{1}{3}} &= 0 \\ \Rightarrow 2x &= 8x^{\frac{1}{3}} \\ \Rightarrow x &= 4x^{\frac{1}{3}} \end{aligned}$$

EITHER $x=0$ (BY INSPECTION) OR IF WE DIVIDE WE OBTAIN

$$\begin{aligned} \Rightarrow x^{\frac{2}{3}} &= 4 \\ \Rightarrow (x^{\frac{1}{3}})^2 &= 4 \\ \Rightarrow \sqrt{x^{\frac{1}{3}}} &= \sqrt{2} \\ \Rightarrow x &= 2^{\frac{3}{2}} \end{aligned}$$

FIND THE CORRESPONDING y COORDINATES

$$\begin{aligned} x=0, \quad y &= 2 \\ x=8, \quad y &= 8^2 - 6 \times 8^{\frac{4}{3}} + 2 = 64 - 6 \times 16 + 2 = 66 - 96 + 2 = -30 \\ \therefore (0, 2) \quad \& \quad (8, -30) \end{aligned}$$

DETERMINING THE NATURE OF THESE POINTS BY USING THE SECOND DERIVATIVE TEST

$$\begin{aligned} \Rightarrow \frac{d^2y}{dx^2} &= 2 - \frac{8}{3}x^{-\frac{2}{3}} \\ \Rightarrow \frac{d^2y}{dx^2} &= 2 - \frac{8}{3 \times 2^{\frac{2}{3}}} \\ \Rightarrow \frac{d^2y}{dx^2} &= 2 - \frac{8}{3 \times 2^{\frac{2}{3}}} \end{aligned}$$

$$\left. \frac{d^2y}{dx^2} \right|_{x=8} = 2 - \frac{8}{3 \times 8^{\frac{2}{3}}} = 2 - \frac{8}{12} = \frac{4}{3} > 0$$

$\therefore (8, -30)$ IS A LOCAL MINIMUM

$$\left. \frac{d^2y}{dx^2} \right|_{x=0} = 2 - \frac{8}{3 \times 0^{\frac{2}{3}}} = 2 - \frac{8}{0}$$

WE CANNOT USE THIS TEST WITHOUT KNOWLEDGE OF LIMITS TECHNIQUE

CHECKING EITHER THE GRADIENT OR THE VALUE OF $\frac{dy}{dx}$ TO THE RIGHT OF $x=0$ (AS $x > 0$)

- $\left. \frac{dy}{dx} \right|_{x=0.1} \approx -3.51 \dots$
- OR
- $\left. \frac{dy}{dx} \right|_{x=0.1} \approx 1.73 \dots$

$\therefore (0, 2)$ IS A LOCAL MAX

Question 24

A curve has equation

$$y = x(x^2 - 128\sqrt{x}), \quad x \in \mathbb{R}, \quad x > 0.$$

The curve has a single stationary point with coordinates $(2^\alpha, -2^\beta)$, where α and β are positive integers.

Find the value of β and justify that the stationary point is a local minimum.

$\beta = 12$

REWRITE THE EQUATION IN INDEXIAL FORM & DIFFERENTIATE

$$\Rightarrow y = x(x^2 - 128\sqrt{x})$$

$$\Rightarrow y = x(x^2 - 128x^{\frac{1}{2}})$$

$$\Rightarrow y = x^3 - 128x^{\frac{3}{2}}$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 192x^{\frac{1}{2}}$$

FOR STATIONARY POINTS SET $\frac{dy}{dx} = 0$

$$\Rightarrow 3x^2 - 192x^{\frac{1}{2}} = 0$$

$$\Rightarrow x^2 - 64x^{\frac{1}{2}} = 0$$

$$\Rightarrow x^2 = 64x^{\frac{1}{2}}$$

$$\Rightarrow \frac{x^2}{x^{\frac{1}{2}}} = 64 \quad (\text{WE ARE NOT CONCERNED WITH } x=0)$$

$$\Rightarrow x^{\frac{3}{2}} = 64$$

$$\Rightarrow (x^{\frac{3}{2}})^{\frac{2}{3}} = (64)^{\frac{2}{3}}$$

$$\Rightarrow x^1 = (\sqrt[3]{64})^2$$

$$\Rightarrow x = 16$$

CHECK THE NATURE OF THE POINT BY THE SECOND DERIVATIVE TEST

$$\Rightarrow \frac{d^2y}{dx^2} = 6x - 96x^{-\frac{1}{2}}$$

$$\Rightarrow \left. \frac{d^2y}{dx^2} \right|_{x=16} = 6 \times 16 - 96 \times 16^{-\frac{1}{2}} = 96 - 96 \times \frac{1}{4} = 96 - 24 = 72 > 0$$

LOCAL MINIMUM

FINALLY TO FIND THE y COORDINATE IN THE STATIONARY POINT

$$y = x(x^2 - 128\sqrt{x})$$

$$y = 16(16^2 - 128\sqrt{16})$$

$$y = -4096$$

$$y = -2^{12} \quad (\text{TOTAL \& POWER OF POWER OF 2})$$

\therefore LOCAL MINIMUM AT $(16, -2^{12})$

Question 25

The point P , whose x coordinate is $\frac{1}{4}$, lies on the curve with equation

$$y = \frac{k + 4x\sqrt{x}}{7x}, \quad x \in \mathbb{R}, \quad x > 0,$$

where k is a non zero constant.

- a) Determine, in terms of k , the gradient of the curve at P .

The tangent to the curve at P is parallel to the straight line with equation

$$44x + 7y - 5 = 0.$$

- b) Find an equation of the tangent to the curve at P .

$$\left. \frac{dy}{dx} \right|_{x=\frac{1}{4}} = \frac{4-16k}{7}, \quad \boxed{44x + 7y = 25}$$

a) TRY THE EQUATION INTO INDIVIDUAL FORM AND DIFFERENTIATE

- $y = \frac{4x\sqrt{x} + k}{7x} = \frac{4x\sqrt{x}}{7x} + \frac{k}{7x} = \frac{4}{7}x^{\frac{1}{2}} + \frac{k}{7}x^{-1}$
- $\frac{dy}{dx} = \frac{4}{7}x^{-\frac{1}{2}} - \frac{k}{7}x^{-2}$
- $\frac{dy}{dx} \Big|_{x=\frac{1}{4}} = \frac{4}{7}\left(\frac{1}{4}\right)^{-\frac{1}{2}} - \frac{k}{7}\left(\frac{1}{4}\right)^{-2} = \frac{4}{7} \times 2 - \frac{k}{7} \times 16 = \frac{8-16k}{7}$

b) REARRANGE THE EQUATION OF THE LINE TO FIND THE GRADIENT

$$\begin{aligned} 44x + 7y - 5 &= 0 \\ 7y &= -44x + 5 \\ y &= -\frac{44}{7}x + \frac{5}{7} \end{aligned}$$

FIND THE GRADIENT AT P WITH 36 - 34 (EQUATE)

$$\begin{aligned} \frac{8-16k}{7} &= -\frac{44}{7} \\ 8-16k &= -44 \\ 48 &= 16k \\ k &= 3 \end{aligned}$$

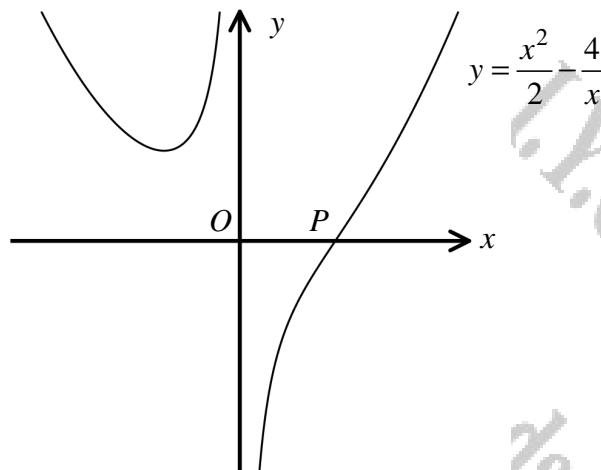
FIND THE y COORDINATE OF P

$$y = \frac{4x\sqrt{x} + 3}{7x} = \frac{4 \times \frac{1}{4} \times \sqrt{\frac{1}{4}} + 3}{7 \times \frac{1}{4}} = \frac{1 + 3}{\frac{7}{4}} = \frac{4}{\frac{7}{4}} = 2$$

EQUATION OF TANGENT AT P(1/4, 2)

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 2 &= -\frac{44}{7}(x - \frac{1}{4}) \\ 7y - 14 &= -44x + 11 \end{aligned} \quad \therefore 44x + 7y = 25$$

Question 26



The figure above shows the curve C with equation

$$y = \frac{x^2}{2} - \frac{4}{x}, \quad x \neq 0.$$

The curve crosses the x axis at the point P .

The straight line L is the normal to C at P .

- a) Find ...
 - i. ... the coordinates of P .
 - ii. ... an equation of L .
- b) Show that L does not meet C again.

$$P(2,0), \quad x + 3y = 2$$

Question 27

The curve C has equation

$$y = (x-1)(x^2 + 4x + 5), \quad x \in \mathbb{R}.$$

- a) Show that C meets the x axis at only one point.

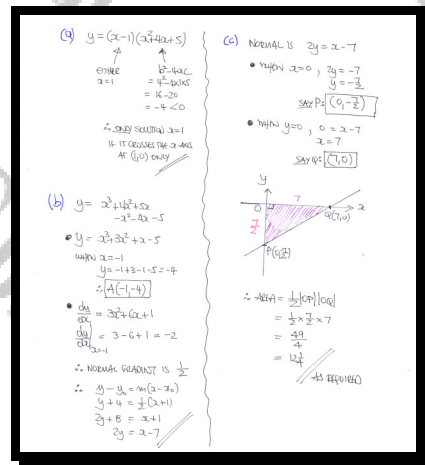
The point A , where $x = -1$, lies on C .

- b) Find an equation of the normal to C at A .

The normal to C at A meets the coordinate axes at the points P and Q .

- c) Show further that the area of the triangle OPQ , where O is the origin, is $12\frac{1}{4}$ square units.

$$2y = x - 7$$



Question 28

A curve has equation

$$y = x - 8\sqrt{x}, \quad x \in \mathbb{R}, \quad x \geq 0.$$

The curve meets the coordinate axes at the origin and at the point P .

- a) Determine the coordinates of P .

The point Q , where $x = 4$, lies on the curve.

- b) Find an equation of the normal to curve at Q .
- c) Show clearly that the normal to the curve at Q does not meet the curve again.

$$P(64,0), \quad y = x - 16$$

Handwritten solution for Question 28:

(a) $y = x - 8\sqrt{x}$ with $y=0$, $0 = x - 8\sqrt{x}$
 $8\sqrt{x} = x$
 $64x = x^2$
 $x = \frac{64}{x}$
 $x^2 = 64$
 $x = 8$
 $\therefore P(64,0)$

(b) When $x=4$, $y = 4 - 8\sqrt{4} = 4 - 16 = -12$ $P(4, -12)$
 $y = x - 8x^{\frac{1}{2}}$
 $\frac{dy}{dx} = 1 - 4x^{-\frac{1}{2}}$
 $\frac{dy}{dx} = 1 - \frac{4}{\sqrt{x}}$
 $\frac{dy}{dx} = 1 - \frac{4}{\sqrt{4}} = 1 - 2 = -1$
 \therefore normal gradient is 1
 \therefore equation of normal
 $y - y_1 = m(x - x_1)$
 $y + 12 = 1(x - 4)$
 $y = x - 16$

(c) $y = x - 16$
 $y = x - 8\sqrt{x}$ $\} \text{ SET EQUAL }$
 $\Rightarrow x - 16 = x - 8\sqrt{x}$
 $\Rightarrow 8\sqrt{x} = 16$
 $\Rightarrow \sqrt{x} = 2$
 $\Rightarrow x = 4 \leftarrow \text{Point } P$
 \therefore NORMAL DOES NOT CROSS THE CURVE AGAIN

Question 29

The curve C has equation

$$y = x^3 - 9x^2 + 24x - 19, \quad x \in \mathbb{R}.$$

- Show that the tangent to C at the point P , where $x=1$, has gradient 9.
- Find the coordinates of another point Q on C at which the tangent also has gradient 9.

The normal to C at Q meets the coordinate axes at the points A and B .

- Show further that the **approximate** area of the triangle OAB , where O is the origin, is 11 square units.

$Q(5,1)$

(a) $y = x^3 - 9x^2 + 24x - 19$
 $\frac{dy}{dx} = 3x^2 - 18x + 24$
 $\left. \frac{dy}{dx} \right|_{x=1} = 3 - 18 + 24 = 9$
 At $x=1$

(b) $\frac{dy}{dx} = 3x^2 - 18x + 24$
 $9 = 3x^2 - 18x + 24$
 $0 = 3x^2 - 18x + 15$
 $0 = x^2 - 6x + 5$
 $(x-1)(x-5) = 0$
 $x = 1$ or $x = 5$
 $\therefore y = 5^3 - 9 \times 5^2 + 24 \times 5 - 19$
 $y = 125 - 225 + 120 - 19$
 $y = 1$
 $\therefore Q(5,1)$

(c) $Q(5,1)$
 • Gradient at Q is 9
 • Normal gradient is $-\frac{1}{9}$
 • Equation of normal
 $y - y_0 = m(x - x_0)$
 $y - 1 = -\frac{1}{9}(x - 5)$
 $9y - 9 = -x + 5$
 $9y + x = 14$
 • When $x=0$, $9y = 14$
 $y = \frac{14}{9}$
 $A(0, \frac{14}{9})$
 • When $y=0$, $x = 14$
 $B(14, 0)$

Area of $\triangle OAB = \frac{1}{2} |OA| |OB|$
 $= \frac{1}{2} \times \frac{14}{9} \times 14$
 $= \frac{98}{9}$
 $\approx 10.9 \approx 11$
 At $x=14$

Question 30

The point $A(2,1)$ lies on the curve with equation

$$y = \frac{(x-1)(x+2)}{2x}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

- a) Find the gradient of the curve at A .
 b) Show that the tangent to the curve at A has equation

$$3x - 4y - 2 = 0.$$

The tangent to the curve at the point B is parallel to the tangent to the curve at A .

- c) Determine the coordinates of B .

gradient at $A = \frac{3}{4}$, $B(-2,0)$

(a) $y = \frac{(x-1)(x+2)}{2x} = \frac{x^2 - x - 2}{2x} = \frac{x^2}{2x} + \frac{-x}{2x} - \frac{2}{2x} = \frac{1}{2}x + \frac{1}{2} - \frac{1}{x}$
 $\therefore y = \frac{1}{2}x + \frac{1}{2} - x^{-1}$
 $\frac{dy}{dx} = \frac{1}{2} + x^{-2} = \frac{1}{2} + \frac{1}{x^2}$
 $\left. \frac{dy}{dx} \right|_{x=2} = \frac{1}{2} + \frac{1}{2^2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

(b) $A(2,1)$ GRADIENT = $\frac{3}{4} \Rightarrow y - y_1 = m(x - x_1)$
 $y - 1 = \frac{3}{4}(x - 2)$
 $4y - 4 = 3x - 6$
 $0 = 3x - 4y - 2$ ✓

(c) PARALLEL TANGENTS \Rightarrow SAME GRADIENT OF $\frac{3}{4}$
 $\frac{dy}{dx} = \frac{1}{2} + \frac{1}{x^2}$
 $\Rightarrow \frac{3}{4} = \frac{1}{2} + \frac{1}{x^2}$
 $\Rightarrow \frac{1}{4} = \frac{1}{x^2}$
 $\Rightarrow 4 = x^2$
 $\Rightarrow x = \begin{cases} 2 & 4 - 4 \\ -2 & 4 - 8 \end{cases}$
 Using $y = \frac{(x-1)(x+2)}{2x}$ with $x = -2$
 $y = \frac{(-2-1)(-2+2)}{2(-2)} = 0$
 $\therefore B(-2,0)$ ✓

Question 31

The curve C has equation $y = f(x)$ given by

$$f(x) = 2(x-2)^3, \quad x \in \mathbb{R}.$$

- a) Sketch the graph of $f(x)$.
- b) Find an expression for $f'(x)$.

The point $P(3,2)$ lies on C and the straight line l_1 is the tangent to C at P .

- c) Find an equation of l_1 .

The straight line l_2 is another tangent at a different point Q on C .

- d) Given that l_1 is parallel to l_2 show that an equation of l_2 is

$$y = 6x - 8.$$

$$f'(x) = 6x^2 - 24x + 24, \quad y = 6x - 16$$

The handwritten solution is divided into several parts:

- (a)** Two coordinate planes showing the graph of $y = 2(x-2)^3$. The first graph shows the curve passing through the origin and the point $(3, 2)$. The second graph shows the curve with a tangent line at $(3, 2)$ and another point $(1, -16)$ marked on the curve.
- (b)** Differentiation steps:

$$f(x) = 2(x-2)^3$$

$$f'(x) = 2(x-2)(2x-2)$$

$$f'(x) = 2(x-2)(2x-2)$$

$$f'(x) = (2x-2)(2x-2)$$

$$f'(x) = 2x^2 - 2x^2 + 4x - 4x$$

$$f'(x) = 2x^2 - 2x^2 + 4x - 4x$$

$$f'(x) = 2x^2 - 2x^2 + 4x - 4x$$

$$f'(x) = 6x^2 - 24x + 24$$
- (c)** Finding the gradient at $P(3, 2)$:

$$f'(3) = 6(3)^2 - 24(3) + 24$$

$$f'(3) = 54 - 72 + 24$$

$$f'(3) = 6$$
 Then, using the point-slope form:

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 6(x - 3)$$

$$y - 2 = 6x - 18$$

$$y = 6x - 16$$
- (d)** Finding the equation of the second tangent line l_2 which is parallel to l_1 (gradient 6):
 - Parallel lines \Rightarrow same gradient $\Rightarrow l_2$ has gradient 6
 - Need another point on C with gradient 6
 - $f'(x) = 6$
 - $6x^2 - 24x + 24 = 6$
 - $6x^2 - 24x + 18 = 0$
 - $2x^2 - 4x + 3 = 0$
 - $(2x-1)(x-3) = 0$
 - $x = \frac{1}{2}$ or $x = 3$
 - $x = 3$ is point P
 - Therefore $x = \frac{1}{2}$, $y = 2\left(\frac{1}{2}-2\right)^3 = -2$
 - $\therefore Q\left(\frac{1}{2}, -2\right)$
 - Using point-slope form:

$$y - y_1 = m(x - x_1)$$

$$y + 2 = 6\left(x - \frac{1}{2}\right)$$

$$y + 2 = 6x - 3$$

$$y = 6x - 5$$

Question 32

The point $P(2,9)$ lies on the curve C with equation

$$y = x^3 - 3x^2 + 2x + 9, \quad x \in \mathbb{R}, \quad x \geq 1.$$

- a) Find an equation of the tangent to C at P , giving the answer in the form $y = mx + c$, where m and c are constants.

The point Q also lies on C so that the tangent to C at Q is perpendicular to the tangent to C at P .

- b) Show that the x coordinate of Q is

$$\frac{6 + \sqrt{6}}{6}.$$

$$y = 2x + 5$$

(a) $y = x^3 - 3x^2 + 2x + 9$
 $\frac{dy}{dx} = 3x^2 - 6x + 2$
 $\left. \frac{dy}{dx} \right|_{x=2} = 3(2)^2 - 6(2) + 2 = 12 - 12 + 2 = 2$

So $y - 9 = m(x - 2)$
 $\Rightarrow y - 9 = 2(x - 2)$
 $\Rightarrow y - 9 = 2x - 4$
 $\Rightarrow y = 2x + 5$

(b) Gradient of l_1 is 2 (From a)
 \therefore Gradient of l_2 must be $-\frac{1}{2}$

$\therefore \frac{dy}{dx} = -\frac{1}{2}$
 $\Rightarrow 3x^2 - 6x + 2 = -\frac{1}{2}$
 $\Rightarrow 6x^2 - 12x + 4 = -1$
 $\Rightarrow 6x^2 - 12x + 5 = 0$

By quadratic formula or
 By completing the square

$x = \frac{12 \pm \sqrt{144 - 120}}{12}$
 $x = \frac{12 \pm \sqrt{24}}{12}$
 $x = \frac{6 \pm \sqrt{6}}{6}$
 $x = \frac{6 + \sqrt{6}}{6} \quad (x > 1)$

Question 33

The volume, $V \text{ cm}^3$, of a soap bubble is modelled by the formula

$$V = (p - qt)^2, \quad t \geq 0,$$

where p and q are positive constants, and t is the time in seconds, measured after a certain instant.

When $t = 1$ the volume of a soap bubble is 9 cm^3 and at that instant its volume is decreasing at the rate of 6 cm^3 per second.

Determine the value of p and the value of q .

$$p = 4, \quad q = 1$$

Handwritten solution for Question 33:

$V = (p - qt)^2 \Rightarrow V = p^2 - 2pqt + q^2t^2$
 $t = 1, V = 9$
 $9 = (p - q)^2$
 $q = p - q$
 $p - q = 3$

$\frac{dV}{dt} = -2pq + 2q^2t$
 $-6 = -2pq + 2q^2 \times 1$
 $2pq - 6 = 2q^2$

• $p - q = 3$
 $p = q + 3$
 $2q(q + 3) - 6 = 2q^2$
 $2q^2 + 6q - 6 = 2q^2$
 $6q = 6$
 $q = 1$
 $p = 4$

• $p - q = -3$
 $p = q - 3$
 $2q(q - 3) - 6 = 2q^2$
 $2q^2 - 6q - 6 = 2q^2$
 $-6q = 6$
 $q = -1$ ($q > 0$)

Question 34

A curve C has equation

$$y = 2x^3 - 5x^2 + a, \quad x \in \mathbb{R},$$

where a is a constant.

The tangent to C at the point where $x = 2$ and the normal to C at the point where $x = 1$, meet at the point Q .

Given that Q lies on the x axis, determine in any order ...

- ... the value of a .
- ... the coordinates of Q .

$$a = \frac{8}{3}, \quad Q\left(\frac{7}{3}, 0\right)$$

Handwritten solution for Question 34:

$y = 2x^3 - 5x^2 + a$
 $\frac{dy}{dx} = 6x^2 - 10x$
 $\left. \frac{dy}{dx} \right|_{x=2} = 6(2)^2 - 10(2) = 4$
 $\left. \frac{dy}{dx} \right|_{x=1} = 6(1)^2 - 10(1) = -4$
 When $x=2$:
 $y = 2(2)^3 - 5(2)^2 + a$
 $y = 16 - 20 + a$
 $y = a - 4$ at $(2, a-4)$
 When $x=1$:
 $y = 2(1)^3 - 5(1)^2 + a$
 $y = 2 - 5 + a$
 $y = a - 3$ at $(1, a-3)$

EQUATION OF TANGENT
 $y - (a-4) = 4(x-2)$
 $y - (a-3) = \frac{1}{4}(x-1)$

NEXT FIND THE 2 INTERSECTING POINTS
 $4x - 4 = \frac{1}{4}x - \frac{1}{4}$
 $16x - 16 = x - 1$
 $15x = 15$
 $x = 1$
 $y = a - 3$

BUT THESE 2 POINTS MUST BE THE SAME (SAME NUMBER)
 $16 - 4a = 15 - 4a$
 $16 - 4a = 15 - 4a$
 $16 - 15 = 4a - 4a$
 $1 = 0$
 $a = \frac{8}{3}$

$Q = \left(\frac{7}{3}, 0\right)$

Question 35

The curve C has equation

$$y = \frac{x^3(5x\sqrt{x} - 128)}{\sqrt{x}}, \quad x \in \mathbb{R}, \quad x > 0.$$

- Determine expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.
- Show that the y coordinate of the stationary point of C is $-k\sqrt[3]{4}$, where k is a positive integer.
- Evaluate $\frac{d^2y}{dx^2}$ at the stationary point of C .
Give the answer in terms of $\sqrt[3]{2}$.
- Find the value of $\frac{d^3y}{dx^3}$ at the point on C , where $\frac{d^2y}{dx^2} = 0$.

$$\boxed{}, \quad \frac{dy}{dx} = 20x^3 - 320x^{\frac{3}{2}}, \quad \frac{d^2y}{dx^2} = 60x^2 - 480x^{\frac{1}{2}}, \quad \frac{d^3y}{dx^3} = 120x - 240x^{-\frac{1}{2}},$$

$$\boxed{k = 3072}, \quad \boxed{960\sqrt[3]{2}}, \quad \boxed{360}$$

a) SIMPLY BE POSSIBLE THE EQUATION IN INDEXAL FORM, THEN DIFFERENTIATE

$$y = \frac{x^3(5x\sqrt{x} - 128)}{\sqrt{x}} = \frac{5x^4\sqrt{x} - 128x^3}{\sqrt{x}} = \frac{5x^{\frac{9}{2}} - 128x^3}{x^{\frac{1}{2}}}$$

- $y = 5x^{\frac{9}{2}} - 128x^{\frac{3}{2}}$
- $\frac{dy}{dx} = 20x^{\frac{7}{2}} - 320x^{\frac{1}{2}}$
- $\frac{d^2y}{dx^2} = 60x^{\frac{5}{2}} - 480x^{-\frac{1}{2}}$
- $\frac{d^3y}{dx^3} = 120x^{\frac{3}{2}} - 240x^{-\frac{3}{2}}$

b) THE STATIONARY POINT $\frac{dy}{dx} = 0$ SUBSTITUTE INTO y & TWO

$$\Rightarrow 20x^{\frac{7}{2}} - 320x^{\frac{1}{2}} = 0$$

$$\Rightarrow x^{\frac{7}{2}} - 16x^{\frac{1}{2}} = 0$$

$$\Rightarrow x^{\frac{7}{2}} = 16x^{\frac{1}{2}} \quad (x \neq 0)$$

$$\Rightarrow x^{\frac{7}{2} - \frac{1}{2}} = 16$$

$$\Rightarrow (x^3)^{\frac{1}{2}} = 16$$

$$\Rightarrow x^3 = (16^2)$$

$$\Rightarrow x = \sqrt[3]{256}$$

SUBSTITUTE INTO y & TWO

$$\Rightarrow y = 5x^{\frac{9}{2}} - 128x^{\frac{3}{2}}$$

$$\Rightarrow y = 5x^{\frac{9}{2}} [x^{-\frac{1}{2}} - 128x^{-\frac{1}{2}}]$$

$$\Rightarrow y = 5x^{\frac{9}{2}} [x^{-\frac{1}{2}}(5 - 128)]$$

$$\Rightarrow y = 5x^{\frac{9}{2}}(-48)$$

$$\Rightarrow y = -240x^{\frac{9}{2}}$$

$$\Rightarrow y = -240 \times (256)^{\frac{9}{2}}$$

$$\Rightarrow y = -3072 \times \sqrt[3]{4}$$

c) $\frac{d^2y}{dx^2} = 60x^2 - 480x^{\frac{1}{2}}$

$$\Rightarrow \frac{d^2y}{dx^2} = 60x^2(1 - 8x^{-\frac{1}{2}})$$

$$\Rightarrow \frac{d^2y}{dx^2} = 60(2^{\frac{3}{2}})^{\frac{1}{2}}(1 - 8)$$

$$= 60 \times 2^{\frac{3}{2}} \times 8 = 60 \times 2^{\frac{3}{2}} \times 2^3 \times 8$$

$$\Rightarrow \frac{d^2y}{dx^2} = 960\sqrt[3]{2}$$

d) $\frac{d^2y}{dx^2} = 0$ FINALLY

$$\Rightarrow 60x^2 - 480x^{\frac{1}{2}} = 0$$

$$\Rightarrow x^2 - 8x^{\frac{1}{2}} = 0$$

$$\Rightarrow x^2 = 8x^{\frac{1}{2}}$$

$$\Rightarrow \frac{x^2}{x^{\frac{1}{2}}} = 8 \quad (x \neq 0)$$

$$\Rightarrow x^{\frac{3}{2}} = 8$$

$$\Rightarrow (x^{\frac{3}{2}})^{\frac{2}{3}} = 8^{\frac{2}{3}}$$

$$\Rightarrow x = \sqrt[3]{64} = 4$$

FINALLY

$$\Rightarrow \frac{d^3y}{dx^3} = 120x - 240x^{-\frac{1}{2}}$$

$$\Rightarrow \frac{d^3y}{dx^3} = 120 \times 4 - 240 \times 4^{-\frac{1}{2}}$$

$$= 480 - 240 \times \frac{1}{2}$$

$$= 480 - 120 = 360$$