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# Arithmetic Aspects of Noncommutative Geometry: Motives of Noncommutative Tori and Phase Transitions on GL(n) and Shimura Varieties Systems

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# ARITHMETIC ASPECTS OF NONCOMMUTATIVE GEOMETRY: MOTIVES OF NONCOMMUTATIVE TORI AND PHASE TRANSITIONS ON GL(N) AND SHIMURA VARIETIES SYSTEMS

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To my wife, and my son for their unconditional love.
To my advisors and all my friends for their support.
I love you all.

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# ABSTRACT

In this dissertation, we study three important cases in noncommutative geometry. We first observe the standard noncommutative object, noncommutative torus, in noncommutative motives. We work with the category of holomorphic bundles on a noncommutative torus, which is known to be equivalent to the heart of a nonstandard t-structure on coherent sheaves of an elliptic curve. We then introduce a notion of (weak) t-structure in dg categories. By lifting the nonstandard t-structure to the t-structure that we defined, we find a way of seeing a noncommutative torus in noncommutative motives. By applying the t-structure to a noncommutative torus and describing the cyclic homology of the category of holomorphic bundle on the noncommutative torus, we finally show that the periodic cyclic homology functor induces a decomposition of the motivic Galois group of the Tannakian category generated by the associated auxiliary elliptic curve.

In the second case, we generalize the results of Laca, Larsen, and Neshveyev on the $\mathrm{GL}_2$-Connes-Marcolli system to the $\mathrm{GL}_n$-Connes-Marcolli systems. We introduce and define the $\mathrm{GL}_n$-Connes-Marcolli systems and discuss the existence and uniqueness questions of the KMS equilibrium states. Using the ergodicity argument and Hecke pair calculation, we classify the KMS states at different inverse temperatures $\beta$. Specifically, we show that in the range of $n-1 < \beta \leq n$, there exists only one KMS state. We prove that there are no KMS states when $\beta < n-1$ and $\beta \neq 0, 1, \ldots, n-1$, while we actually construct KMS states for integer values of $\beta$ in $1 \leq \beta \leq n-1$. For $\beta > n$, we characterize the extremal KMS states.

In the third case, we push the previous results to more abstract settings. We mainly study the connected Shimura dynamical systems. We give the definition of the essential and superficial KMS states. We further develop a set of arithmetic tools to generalize the results in the previous case. We then prove the uniqueness of the essential KMS states and show the existence of the essential KMS stats for high inverse temperatures.

# CHAPTER 1

# INTRODUCTION

Noncommutative Geometry (NCG) is considered as one of the most important branches in modern mathematics. It is playing an increasingly important role. In the past few decades, it has been developed by leading mathematicians with different viewpoints, such as A. Connes' noncommutative differential geometry, M. Artin's noncommutative algebraic geometry, and M. Kontsevich's noncommutative motives and deformation quantization. NCG has many applications to number theory, differential geometry, algebraic geometry and physics (especially in Quantum Field Theory and Statistical Mechanics).

In this dissertation, we discuss three cases in Noncommutative Geometry, concerning the questions on the connections of different noncommutative geometries and the application in number theory and quantum statistical mechanics.

We first talk about the motives of noncommutative tori. The category of noncommutative motives was first introduced by mathematician Kontsevich at the beginning of this century. To construct the category of noncommutative motives, we shall ask: "What will be the correct objects we should observe in the noncommutative world, which act like the smooth projective varieties in the commutative world and can connect the classic cases?" Kontsevich suggested we should look into dg categories. He first proposed some construction of the category of noncommutative numerical motives in dg categories [27] and also a possible notion and construction of noncommutative Chow motives. Following that, Tabuada constructed the category of noncommutative Chow motives in [48] and showed how they are related to the classic Chow motives in commutative cases. Recently, Marcolli and Tabuada wrote a series of papers on this subject, [35], [36], [37], to further develop the theory of noncommutative motives. Their work extends the classical results on pure motives to the noncommutative settings, such as the semi-simplicity of the category of numerical motives, Grothendieck's standard conjectures on pure motives, Artin motives, etc. Many properties of classic motives are also reformulated in terms of noncommutative motives.

The noncommutative motives successfully extend the notion of pure motives to noncommutative (algebraic) geometry. A natural question would be: can we observe those typical noncommutative objects (like noncommutative tori) in this construction? In the construction of the noncommutative motives, we somehow rely on those "nice" dg categories, called algebraic spaces ([27]). So, one main problem might be how we can see these objects as algebraic spaces, because as algebras they are not "nice", usually not smooth or Noetherian.

However, thanks to the remarkable work of Schwartz and Polishchuk [42], [40], the situation is much better for noncommutative tori. Schwartz and Polishchuk showed that the category of holomorphic bundles on a noncommutative torus is equivalent to the heart of a nonstandard t-structure on some elliptic curve related to the noncommutative torus. By the help of this relation, we can introduce a notion of a weak t-structure over dg categories. We further show that an additive invariant (like the periodic cyclic homology functor) splits into direct sum over the weak t-structure.

As a second case in noncommutative geometry, we study the $\mathrm{GL}_n$-Connes-Marcolli system. Twenty years ago, Bost and Connes introduced a C*-dynamical system from Hecke operators, which we refer to as the original BC-system nowadays (see [3]). This system connects quantum statistical mechanics and number theory with the Riemann zeta function as its partition function. Following the original BC-system, Connes and Marcolli then constructed the $\mathrm{GL}_2$ system, called the Connes-Marcolli system [13]. They observed that the values of the ground equilibrium states of the system on a subalgebra of rational observables span the maximal abelian extension of the rational number field $\mathbb{Q}$, which is related to the primary case of the Hilbert's 21th problem (see [13] and Chapter 3 of [14]). Because of the success of these systems, since then, people attempted to construct various similar (BC-type) systems with interesting properties. Among them, Ha and Paugam gave a most unified construction of systems by using Shimura varieties. Their construction includes most of the existing BC-type systems and produces new systems uniformly. For example, BC-type systems can be built for any number field (see [21]).

As we briefly mentioned in Connes-Marcolli system above, the most interesting properties of a BC type system are reflected by the action of the symmetric group on equilibrium states. These equilibrium states are described by the KMS (Kubo-Martin-Schwinger)-condition and are called the KMS states. Therefore, it is significant to study the phase transitions of a system and how the set of KMS states changes by the effect of these phase transitions. As a successful example, Laca, Larson and Neshveyev thoroughly studied the phase transitions of the Connes-Marcolli system in [30]. At the end of [30], they also suggested that their results might be further generalized to $\mathrm{GL}_n$-systems and asked a list of questions about these systems. In chapter 3 of this dissertation, we follow their work and finish the generalization of the $\mathrm{GL}_n$-systems and also answer the questions asked by them.

As the third case we discuss in noncommutative geometry in this dissertation, we continue the study of the BC-type systems. From the previous case, we see that the study of the KMS states is successful in these concrete constructions. We naturally wish we can obtain similar results in more abstract settings. The construction introduced by Ha and Paugam through Shimura varieties have great significance in unifying the BC-like systems. But, we have to point out, their primary construction is through *Stacks*. In other words, maybe these systems are too abstract to be calculated. We first need to identify some of the cases that can be computed concretely.

As a first step of our plan, we focus on the construction by using connected Shimura varieties. A Shimura variety is defined by a inverse limit of the level structures (see Chapter 4 for details). The connected Shimura varieties have computable structures to be studied in the related systems. The algebraic group in the definition can be set as a simply connected group of noncompact type. Those groups have class number 1 and the limit variety can be determined concretely. We use this as a starting point, and use a modified and simplified definition from [21]. We then introduce two classes of KMS states: the essential and the superficial states. We show, with certain conditions, by using the tools generalized from the previous case, the essential state is unique. We also show the existence of the essential KMS states with high inverse temperature.

**Outline of the Dissertation** The rest of the dissertation is organized as follows. We start Chapter 2 by reviewing the development of the category of Motives. We introduce the concepts of Noncommutative Motives. Then we observe the t-structure related to the noncommutative tori and define a weak t-structure in the dg frame. Finally, we prove there is a decomposition for an additive invariant on the t-structure.

In chapter 3, we study the $\mathrm{GL}_n$-Connes-Marcolli systems. We first introduce the $\mathrm{GL}_n$-systems. Then we show the uniqueness of the KMS state when the inverse temperature $n-1 < \beta \leq n$. We also extensively discuss the existence of the KMS states according to the inverse temperature in different ranges.

In chapter 4, we discuss the connected Shimura systems. We review some basic concepts in Shimura data and define the C*-dynamical systems out of the Shimura Data. We give the definition of the essential and superficial KMS states. We then mainly focus on the connected Shimura setting cases and prove the uniqueness of the essential KMS states. Finally, we briefly discuss the existence of the essential KMS states for higher inverse temperatures.

# CHAPTER 2

# MOTIVES OF NONCOMMUTATIVE TORI

## 2.1 Motives

### 2.1.1 Motives in Algebraic Geometry

The notion of motives in algebraic geometry was first suggested by Grothendieck for a universal cohomology theory for Weil cohomologies and a possible approach to the Weil conjectures. There is, due to the result of Jannsen [23], a good construction of an abelian category of pure movies for smooth projective varieties. But for general varieties or schemes (called mixed motives), so far we only know there are several candidates of triangulated categories having the desired properties of the bounded derived category of the category of mixed motives, due to work of Voevodsky, Hanamura, and Levine [51], [20], [29]. In general, even the existence of mixed motives still remains conjectural [38].

Let us first quickly review the pure motives in algebraic geometry. We know in topology, say for a complex manifold, when we calculate the singular cohomology, Čech cohomology or the de Rham cohomolgy, because they all satisfy the Eilenberg-Steenrod axioms, we finally get the same group.

In algebraic geometry, the situation is more complicated. Let $X$ be a smooth projective variety over a field $k$. A cohomolgy $H(X)$ is viewed as a functor from the category of smooth projective varieties to the category of vector spaces over some field $K$. There are several classical cohomology theories:

i) Betti Cohomology: if $k$ can be embedded into $\mathbb{C}$, we fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$.

$$H^*_B(X) = H^*(X_\sigma(\mathbb{C}), \mathbb{Q}).$$

These are vector spaces over $K = \mathbb{Q}$.

ii) de Rham Cohomology: $H^*_{dR}(X) = \mathbb{H}^*(X, \Omega^*_{X/k})$. These are vector spaces over $K = k$.

iii) Étale Cohomology: if $\ell \neq \mathrm{Char}(k)$,

$$H^*_{\acute{e}t}(X)_\ell = H^*_{\acute{e}t}(X \times_k k^{sep}, \mathbb{Q}_\ell).$$

These are vector spaces over $K = \mathbb{Q}_\ell$.

Skipping the details of how are these cohomologies defined, we already notice a main problem here that these groups are different, because as vector spaces there are not, although somehow they can be characterized as Weil cohomolgies with several axioms. Apparently, we can ask the question: is there a way that we can unify them as we did in topology? A first attempt is naturally to find an algebraic cohomology theory $H^*(X)_{\mathbb{Q}}$ with coefficient in $\mathbb{Q}$, if we notice the base fields of the vector spaces above all contain $\mathbb{Q}$ as a subfield. Then those cohomologies are modified from $H^*(X)_{\mathbb{Q}}$ by tensoring the base fields. An algebraic geometer easily tells you "No! There is not!" (see [34] for details). To solve this problem, Grothendieck suggested: as a functor, a Weil cohomology should factor through a $\mathbb{Q}$-linear category of over $k$ which acts as a universal cohomology theory for smooth projective varieties over $k$, and called this category the category of "motives".

In the following, we briefly list the constructing and some important properties of pure motives. The intention is not to go deep, but to shed some light on how things are concerned when we talk about the noncommutative motives later. One can consult literatures (for example [1] ) for details.

Let $X$ be a smooth projective variety with dimension $d_X$. Let $\mathcal{Z} = \mathcal{Z}^*(X) = \bigoplus_r \mathcal{Z}^r(X)$ be the graded group of algebraic cycles on $X$, where $\mathcal{Z}^r(X)$ is the set of algebraic cycles of codimension $r$. We also use the notation $\mathcal{Z}_k^*$ to denote the group of algebraic cycles with coefficient $k$. For $a$, $b \in \mathcal{Z}^*(X)$ and irreducible, we say that $a$, $b$ *intersect properly* if for any irreducible component $c \subset a \cap b$,

$$\text{codim}_X c = \text{codim}_X a + \text{codim}_X b.$$

Let $X$, $Y$ be smooth projective varieties, and let $p_1$, $p_2$ be the projections from $X \times Y$ to $X$, $Y$. An equivalence relation $\sim$ is said to be *adequate* if i) for any $a, b \in \mathcal{Z}^*$, there exists an $a' \sim a$ such that $a', b$ intersect properly. ii) for $a \in \mathcal{Z}$ and $b \in \mathcal{Z}(X \times Y)$, $a \sim 0$ implies $p_{2*}(p_1^*(a) \cdot b) \sim 0$, where $\cdot$ is the intersection product of cycles. Under an adequate equivalence $\sim$, $\mathcal{Z}_\sim = \mathcal{Z}/\sim$ becomes a ring with the intersection product as its product.

There are several examples of adequate equivalence relations. Let $a \in \mathcal{Z}(X)$.

- $\sim_R$ Rational equivalence: $a \sim_R 0$ iff there is a $b \in \mathcal{Z}(X \times \mathbb{P}^1)$ such that $p_{1*}((X \times 0 - X \times \infty) \cdot b) = a$;
- $\sim_A$ Algebraic equivalence: $a \sim_A 0$ iff there is a smooth projective curve $C$ with two points $q_1, q_2$ on it and there is a $b \in \mathcal{Z}(X \times C)$ such that $p_{1*}((X \times q_1 - X \times q_2) \cdot b) = a$;
- $\sim_H$ Homological equivalence: for a Weil cohomology (one can think of the examples given above), there is a cycle map $\gamma_H$ sending an algebraic cycle to a cohomological cycle (maybe twisted) in the cohomology group. $a \sim_H 0$ iff $\gamma_H(a) = 0$;

$\sim_N$ Numerical equivalence: $a \sim_N 0$ iff $a \in \mathcal{Z}^r$ and for any $b \in \mathcal{Z}^{d_X - r}(X)$, whence $a \cdot b$ exists, $\deg(a \cdot b) = 0$.

$\sim_R$ is the finest adequate equivalence and $\sim_N$ is the coarsest. We denote the ring $\mathcal{Z}/\sim_*$ by $\mathcal{Z}_*$, where $* = R, A, H, N$. $\mathcal{Z}_R$ is also called the *Chow* ring.

Given an adequate equivalence $\sim$, we define the *category of correspondences* $\mathrm{Cor}_\sim$ as following: Its objects are in the form of $(X, n)$, in which $X$ is a smooth projective variety and $n$ is an integer. The morphisms are

$$Hom_{\mathrm{Cor}_\sim}((X,r),(Y,s)) = \mathcal{Z}_\sim^{d_X+s-r}(X \times Y).$$

Traditionally, we denote the object $(X, n)$ by $h(X)(n)$. It is known $\mathrm{Cor}_\sim$ is an additive category.

A *pseudo-abelian* category $\mathcal{A}$ is an additive category such that any idempotent $e : A \to A$ has a kernel, i.e. $A \simeq \ker e \oplus \ker 1 - e$. For any additive category $\mathcal{C}$, there is always a pseudo-abelian category $\mathcal{C}^\natural$ and a universal functor from $\mathcal{C}$ to $\mathcal{C}^\natural$. $\mathcal{C}^\natural$ is called the pseudo-abelian hull of $\mathcal{C}$.

The category of motives $\mathcal{M}_\sim(k)$ is defined as the pseudo-abelian hull of the category $\mathrm{Cor}_\sim$. Moreover, $\mathcal{M}_R(k)$ is called the category of *Chow Motives* and $\mathcal{M}_N(k)$ is called the category of *Grothendieck Motives.*

Here are some important known properties of motives.

- $\mathcal{M}_N(k)$ is abelian and semisimple. $\mathcal{M}_R(k)$ is not abelian except for some special fields.
- Every Weil cohomology factor through the category $\mathcal{M}_H$ universally.
- There is a tensor structure on $\mathcal{M}_\sim(k)$. $\mathcal{M}_N$ is not Tannakian (We will discuss Tannakian categories in noncommutative motives later). But $\mathcal{M}_N$ can be modified into a Tannakian category, if assuming the standard conjectures C and D.

Other important results of pure motives are in the observation of standard conjectures. But these conjectures are beyond the scope of this dissertation. So here we do not state them.

After reviewing the classic pure motives in algebraic geometry, we are going to discuss noncommutative motives.

Throughout this chapter, we fix $k$ as a field.

### 2.1.2 DG Categories and Noncommutative Motives

**DG Categories.** For dg categories, we refer Keller's article [24] to be the main reference.

**Definition 2.1.1.** *A dg (stands for differential graded) category $\mathcal{C}$ is a $k$-category whose morphisms $\mathcal{C}(X,Y)$ are a complex of modules over $k$ and a composition of two morphisms is a morphism of complexes over $k$*

$$\mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z).$$

*Where $X$, $Y$ and $Z$ are objects of $\mathcal{C}$.*

*A dg functor is a functor between two dg categories and which is a morphism of complexes of modules over $k$ between the morphism sets.*

As an example, let $A$ be a $k$-algebra, then we define the dg category $\mathcal{C}_{dg}(A)$ as the following: the objects are complexes of right $A$ modules and for complexes $M = (M^m, d_M)$, $N = (N^m, d_N)$, the $n$-th component $\mathcal{C}_{dg}(A)(M,N)^n$ of the hom sets are morphisms of $A$-modules $f : M^m \to N^{m+n}$, $\forall m \in \mathbb{Z}$ and whose differential operator is given by $d(f) = d_M \circ f - (-1)^n f \circ d_N$. The composition of two morphisms in $\mathcal{C}_{dg}(A)$ is evidently the composition of the graded maps.

All the small dg categories form the category $\mathrm{dgcat}_k$. From now on, we always assume a dg category is in $\mathrm{dgcat}_k$.

Given a dg category $\mathcal{C}$, we can also construct two new categories associated to $\mathcal{C}$, categories $Z^0(\mathcal{C})$ and $H^0(\mathcal{C})$.

**Definition 2.1.2.** *$Z^0(\mathcal{C})$ and $H^0(\mathcal{C})$ are the categories with the same objects as in $\mathcal{C}$ but with morphisms $Z^0(\mathcal{C})(X,Y) = Z^0(\mathcal{C}(X,Y))$ and $H^0(\mathcal{C})(X,Y) = H^0(\mathcal{C}(X,Y))$ for objects $X$ and $Y$.*

**Definition 2.1.3.** *For $\mathcal{C}$, $\mathcal{C}' \in \mathrm{dgcat}_k$, we say the dg functor $Q : \mathcal{C} \to \mathcal{C}'$ is a quasi-equivalence if $Q(X,Y) : \mathcal{C}(X,Y) \to \mathcal{C}'(Q(X),Q(Y))$ is a quasi-isomorphism on complexes over $k$ and the functor $H^0(Q) :\ H^0(\mathcal{C}) \to H^0(\mathcal{C}')$ is an equivalence of categories.*

A left (right) dg $\mathcal{C}$-module is a dg functor $\mathcal{C} \to \mathcal{C}_{dg}(k)$ ($\mathcal{C}^{op} \to \mathcal{C}_{dg}(k)$, where $\mathcal{C}^{op}$ is the opposite category of $\mathcal{C}$). Apparently, for an object $X$ in $\mathcal{C}$, we see $X^\wedge = \mathcal{C}(?,\, X)$ as a right dg module. A morphism between (left or right) dg modules is a natural transformation which is compatible with the dg structures for all objects of $\mathcal{C}$. So all the dg $\mathcal{C}$-modules form a category. We say a morphism of two dg modules is a quasi-isomorphism if it induces an isomorphism in homologies for all objects. If we further localize the category of dg $\mathcal{C}$-modules with respect to the quasi-isomorphisms, we have the derived category of $\mathcal{C}$ which is denoted by $\mathcal{D}(\mathcal{C})$.

The category of $\mathcal{D}(\mathcal{C})$ is known to be a triangulated category. We recall that an object $X$ of $\mathcal{D}(\mathcal{C})$ is compact if and only if $\mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(X,\, ?)$ commutes with arbitrary coproducts in $\mathcal{D}(\mathcal{C})$. All the compact objects in $\mathcal{D}(\mathcal{C})$ form the full subcategory $\mathcal{D}_c(\mathcal{C})$.

**The DG Enhancement.** For any dg category $\mathcal{C}$, there is a dg category embedding $\mathcal{C} \hookrightarrow \text{Pre-Tr}(\mathcal{C})$. $\text{Pre-Tr}(\mathcal{C})$ is called the *pretriangulated hull* of $\mathcal{C}$, which is the smallest dg category containing $\mathcal{C}$ and admitting the shift functor $X[1]$ and the mapping cone $C_f$ for any morphism $f$ in $Z^0(\text{Pre-Tr}(\mathcal{C}))$. If we set $\text{Tr}(\mathcal{C}) = H^0(\text{Pre-Tr}(\mathcal{C}))$, then $\text{Tr}(\mathcal{C})$ is a triangulated category with the shift functor and mapping cones inherited from $\text{Pre-Tr}(\mathcal{C})$. The concrete construction can be found in [5].

A dg category is called *pretriangulated* if the imbedding $\mathcal{C} \hookrightarrow \text{Pre-Tr}(\mathcal{C})$ is also a quasi-equivalence. So the equivalence of categories $H^0(\mathcal{C}) \to \text{Tr}(\mathcal{C})$ makes $H^0(\mathcal{C})$ a triangulated category.

**Definition 2.1.4.** *Let $\mathcal{K}$ be a triangulated category. We say $\mathcal{K}$ is dg enhanced if there is a pretriangulated dg category $\mathcal{C}$ and an equivalence of triangulated categories $H^0(\mathcal{C}) \to \mathcal{K}$. In this case, $\mathcal{C}$ is called a dg enhancement of $\mathcal{K}$.*

For example, let $\mathcal{A}$ be an abelian category with enough injectives (like quasi-coherent sheaves on schemes). It is not hard to show the derived category $\mathscr{D}^b(\mathcal{A})$ is equivalent to the full subcategory in the homotopy category $\mathscr{K}^+(\mathcal{A})$ consisting of the complexes whose terms are all injectives and whose homology groups are almost all zeros. As showed in [5], we know there is a full subcategory in $\mathcal{C}^+_{dg}(\mathcal{A})$ (cf. the example in section 2.1) enhancing $\mathscr{D}^b(\mathcal{A})$ and we will denote this enhancement by $\mathcal{D}^b_{dg}(\mathcal{A})$.

**Noncommutative Motives.**

**Definition 2.1.5.** *Two dg categories $\mathcal{B}$, $\mathcal{C}$ are said to be Morita equivalent if there is a dg functor $\mathcal{B} \to \mathcal{C}$ that gives an equivalence on the derived categories $\mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{C})$.*

We can make $\text{dgcat}_k$ a Quillen model category with the Morita equivalences as its weak equivalences and its homotopy category is denoted by $\text{Hmo}_k$. Moreover, a tensor product $\otimes^{\mathbf{L}}$ can be defined on $\text{Hmo}_k$, with which we can make $\text{Hmo}_k$ a symmetric monoidal category. [17]

We also define the category $\text{Hmo}_0$ as the category which has the same objects as $\text{Hmo}_k$ and whose morphisms $\mathcal{B} \to \mathcal{C}$ are given by the Grothendieck group of $\text{rep}(\mathcal{B}, \mathcal{C})$, which is denoted by $\text{K}_0\text{rep}(\mathcal{B}, \mathcal{C})$. $\text{rep}(\mathcal{B}, \mathcal{C})$ is subcategory of bimodules $X$ in $\mathcal{D}(\mathcal{B}^{op} \otimes^{\mathbf{L}} \mathcal{C})$ such that $X(?, B)$ is in $\mathcal{D}_c(\mathcal{C})$, for any object $B$ of $\mathcal{B}$. There is a canonical functor $\text{Hmo}_k \to \text{Hmo}_0$. Let $\mathcal{A}$ be an additive category, then the functor $F : \text{Hmo}_k \to \mathcal{A}$ is said an *additive invariant* if $F$ factors through $\text{Hmo}_k \to \text{Hmo}_0$ [24].

According to Kontsevich [27], we have the following important definitions.

**Definition 2.1.6.** *A dg category $\mathcal{C}$ is called smooth if the bimodule $\mathcal{C}(?,?)$ is a compact $\mathcal{C}^{op} \otimes^{\mathbf{L}} \mathcal{C}$ module. It is called proper if for any objects $X, Y$ of $\mathcal{C}$, the complex $\mathcal{C}(X,Y)$ is perfect.*

**Definition 2.1.7.** *[48]The category of noncommutative Chow motives,* $\mathrm{NChow}(k)_F$*, with coefficients in the field $F$ is defined as the pseudo-abelian hull of the category whose objects are smooth and proper dg categories, whose morphisms from dg category $\mathcal{B}$ to $\mathcal{C}$ are the $F$-coefficient Grothendieck group $\mathbf{K}_0(\mathcal{B}^{op} \otimes^{\mathbf{L}} \mathcal{C})_F$, and where the composition of two morphisms is induced by $\otimes^{\mathbf{L}}$ on bimodules.*

Similarly to what happens in the theory of motives in classical algebraic geometry, we still call a morphism in the category of noncommutative Chow motives a correspondence. Moreover, one can define the intersection number on these correspondences, and similarly to the case of algebraic cycles, one can define a numerical equivalence relation on them. If we mod out these relations, we get a new category. It is called the category of noncommutative numerical motives and is denoted by $\mathrm{NNum}(k)_F$. The main result in the paper [35] shows that $\mathrm{NNum}(k)_F$ is abelian and semisimple.

Let $X$ be a projective variety over $k$, $D^b(\mathrm{Coh}(X))$ be the bounded derived category of coherent sheaves on $X$, and $per(X)$ the subcategory of perfect complexes on $X$. When $X$ is smooth in addition, it is well-known that $D^b(\mathrm{Coh}(X))$ is equivalent to $per(X)$. In this case, we also know $D^b(\mathrm{Coh}(X))$ ($=per(X)$) has a dg enhancement, denoted by $D^b_{per}(X)$. According to the result of Toën [50], the dg category $D^b_{per}(X)$ is smooth and proper. So for any smooth projective variety $X$, $D^b_{per}(X)$ gives an object of the noncommutative Chow motives $\mathrm{NChow}(k)_F$. In this way, the category of classical pure Chow motives are "embedded" (mod the Tate twists) the into $\mathrm{NChow}(k)_F$ and so are the numerical motives [49].

### 2.1.3 Tannakian Categories

Let $F$ be a field, and $L/F$ a field extension. Let $\mathcal{A}$ be a rigid symmetric monoidal $F$-linear abelian category ([9]).

**Definition 2.1.8.** *([9])*

- *i), An $L$-valued* fibre functor *is a faithful exact tensor functor $\omega : \mathcal{A} \to Vect_L$, where $Vect_L$ means the category of finite generated $L$-modules;*
- *ii), $\mathcal{A}$ is called a* Tannakian *category if there is an $L$-valued fibre functor on $\mathcal{A}$;*
- *iii), $\mathcal{A}$ is called a* neutral Tannakian *category if there is an $F$-valued fibre functor on $\mathcal{A}$.*

*iv), When $\mathcal{A}$ is neutral Tannakian, we can define the $F$-algebraic group, which is called the Galois group of $\mathcal{A}$, by*

$$Gal(\mathcal{A})(R) = \{\text{isomorphisms of tensor functors } \omega \otimes R \to \omega \otimes R\},$$

*for any $F$-algebras $R$.*

If we let $G_{\mathcal{A}} = Gal(\mathcal{A})$ and $Rep_F(G_{\mathcal{A}})$ be the category of finite dimensional $F$-representations of $G_{\mathcal{A}}$, then there is an equivalence of categories $\mathcal{A} \tilde{\to} Rep_F(G_{\mathcal{A}})$.

In Grothendieck's theory of pure motives, there are several standard conjectures. They are named as standard conjecture B, C, D and I. When the standard conjectures C and D hold, one can make the category of pure numerical motives (the so-called Grothendieck motives) a neutral Tannakian category by changing the signs of the symmetric isomorphisms and using a Weil cohomology as the fiber functor [34].

**Definition 2.1.9.** *With the same notations above, we call a tensor functor $\omega : \mathcal{A} \to sVect_L$ a $L$-valued* super-fibre functor *if it is exact and faithful, where $sVect_L$ stands for the finite dimensional super ($\mathbb{Z}_2$-graded) vector spaces over $L$.*

Similarly, one can give the definitions of *super-Tannakian* and *neutral super-Tannakian* categories (please consult [18], also Appendix A of [36]).

It has been shown in [36], the category $\mathrm{NNum}(k)_F$ is super-Tannakian. Even more, in the same paper, Marcolli and Tabuada stated the standard conjecture C and D in noncommutative motives. Moreover, they proved if the noncommutative standard conjecture C and D hold, then $\mathrm{NNum}(k)_F$ can be modified to a neutral Tannakian category.

## 2.2 Holomorphic Bundles on the Noncommutative Tori

### 2.2.1 Noncommutative Tori and Holomorphic Bundles

If we now fix an irrational real number $\theta$, then the algebra $\mathcal{A}_\theta$ of smooth functions on the noncommutative torus $\mathbb{T}_\theta$ is

$$\mathcal{A}_\theta = \left\{ \sum_{(m,n)\in\mathbb{Z}} a_{m,n} U_1^m U_2^n \mid a_{m,n} \in \mathcal{S}(\mathbb{Z}^2) \right\}$$

where $\mathcal{S}(\mathbb{Z}^2)$ denotes the Schwartz space on $\mathbb{Z}^2$ and $U_1$, $U_2$ satisfy the relation $U_1U_2 = e^{2\pi i\theta}U_2U_1$.

A vector bundle on $\mathbb{T}_\theta$ is a right finitely generated projective $\mathcal{A}_\theta$-module. We denote by $\mathbf{proj}\mathcal{A}_\theta$ the category of vector bundles on $\mathbb{T}_\theta$.

If $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$ with $\tau_1, \tau_2 \in \mathbb{R}$ and $\tau_2 \neq 0$, then there is a complex structure on $\mathbb{T}_\theta$ given by the derivation $\delta_\tau$ on $\mathcal{A}_\theta$ such that

$$\delta_\tau(U_1) = 2\pi\tau i, \quad \delta_\tau(U_2) = 2\pi i.$$

From now on, we use the notation $\mathbb{T}_{\theta,\tau}$ to indicate that there is a complex structure $\delta_\tau$ on $\mathbb{T}_\theta$.

Let $E$ be an object in $\mathbf{proj}\mathcal{A}_\theta$. A *holomorphic structure* on $E$ is a map $\nabla : E \to E$ such that $\nabla(ea) = \nabla(e)a + e\delta_\tau(a)$ for $e \in E, a \in \mathcal{A}_\theta$. We may say that the pair $(E, \nabla)$ is a *holomorphic bundle* on $\mathbb{T}_\theta$. A morphism between two holomorphic bundles is a morphism of $\mathcal{A}_\theta$-modules which is also compatible with the holomorphic structures. We also denote the category of holomorphic bundles by the notation $\mathbf{Vect}(\mathbb{T}_\theta)$. Given a pair of integers $(m, n)$ such that $n + \theta m \neq 0$, in addition to satisfying $m \neq 0$, there is a holomorphic bundle $(E_{n,m}, \nabla_z)$ in $\mathbf{Vect}(\mathbb{T}_\theta)$, such that $E_{n,m}$ is the Schwartz space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}/m\mathbb{Z})$ and $\nabla_z(f) = \frac{\partial f}{\partial x} + 2\pi i(\tau\mu x + z)f$, where $\mu = \frac{m}{n+\theta m}$ and $z$ is a complex number. When $m = 0$, we let $E_{n,0} = \mathcal{A}_\theta^{|n|}$ and define the holomorphic structure as $\nabla_z(a) = \delta_\tau(a) + 2\pi iza$. For any vector bundle $E$ on $\mathbb{T}_\theta$, $E$ is isomorphic to $E_{n,m}^{\oplus k}$ for some $E_{n,m}$ [44]. So, we can naturally give a holomorphic structure on $E$. However, the converse is not always true. Not every holomorphic bundle is given by a direct sum of these $(E_{n,m}, \nabla_z)$'s.

The holomorphic bundles are closely related to elliptic curves. In fact, according to a remarkable result of Schwarz and Polishchuck [42], [40], the category of holomorphic vector bundles on a noncommutative torus is equivalent to the heart of some t-structure on $D^b(X)$ for some $\theta$.

More precisely, let $X = X_\tau = \frac{\mathbb{C}}{\mathbb{Z}+\mathbb{Z}\tau}$ be an elliptic curve and $\mathrm{Coh}(X)$ be the category of coherent sheaves on $X$. For any real number $\theta$, we denote by $\mathrm{Coh}_{>\theta}$ the subcategory of all the coherent sheaves whose semistable factors all have slopes $> \theta$ and $\mathrm{Coh}_{\leq\theta}$ as the subcategory of all the coherent sheaves whose semistable factors all have slopes $\leq \theta$. Then, $(\mathrm{Coh}_{>\theta}, \mathrm{Coh}_{\leq\theta})$ is a torsion pair of $\mathrm{Coh}(X)$. If we let $D^b(X) = D^b(\mathrm{Coh}X)$ and continue to define

$$D^{\theta,\leq 0} = \{F \in D^b(X) \mid H^{>0}(F) = 0,\ H^0(F) \in \mathrm{Coh}_{>\theta}\},$$

$$D^{\theta,\geq 1} = \{F \in D^b(X) \mid H^{<0}(F) = 0,\ H^0(F) \in \mathrm{Coh}_{\leq\theta}\},$$

then $(D^{\theta,\leq 0},\ D^{\theta,\geq 0})$ is a t-structure on $D^b(X)$. We denote the heart of this t-structure by $\mathcal{C}^\theta$. So the result of Schwarz and Polishchuck states that the categories $\mathbf{Vect}(\mathbb{T}_\theta)$ and $\mathcal{C}^\theta$ are equivalent.

A heart of a t-structure is always an abelian category. So immediately, the category of holomorphic vector bundles on a noncommutative torus is abelian.

### 2.2.2 Cyclic Homology of Holomorphic Bundles

It is well-known that for any $k$-algebra $A$, the category of finitely generated projective modules over $A$ is an exact category. By this comment we know $\mathbf{proj}\mathcal{A}_\theta$ is an exact category. According to last paragraph we also know that $\mathbf{Vect}(\mathbb{T}_\theta)$ is an abelian category. The following proposition gives a relation between exact sequences in these two categories.

**Proposition 2.2.1.** *Let* $F : \mathrm{Vect}(\mathbb{T}_\theta) \to \mathrm{proj}\mathcal{A}_\theta$ *be the functor that forgets the holomorphic structure on the vector bundle. Thus* $F$ *is faithful and exact. Moreover every exact sequence in* $\mathrm{proj}\mathcal{A}_\theta$ *comes from an exact sequence in* $\mathrm{Vect}(\mathbb{T}_\theta)$*.*

*Proof.* It is easy to see that $F$ is faithful and exact.

We first show that if $f : E_1 \to E_2$ is surjective in $\mathbf{proj}\mathcal{A}_\theta$, then if there is a holomorphic structure $\nabla_2$ on $E_2$, then we can find a holomorphic structure $\nabla_1$ on $E_1$, which makes $f$ a morphism in $\mathbf{Vect}(\mathbb{T}_\theta)$.

We consider the case where $E_1$ is free at first and then we generalize it to the finitely generated projective case.

If $E_1$ is free and has basis $\{e_1, \dots, e_n\}$, then $\{f(e_1) \dots f(e_n)\}$ spans $E_2$. Moreover, if $\nabla_2(f(e_i)) = \sum_j f(e_j)b_i^j$, we simply define $\nabla_1(e_i a^i) = e_i\delta_\tau(a^i) + \sum_j e_j b_i^j a^i$. One can easily check that this is a holomorphic structure on $E_1$, which makes $f$ a morphism in $\mathbf{Vect}(\mathbb{T}_\theta)$.

Now we consider the case where $E_1$ is a finitely generated projective $\mathcal{A}_\theta$-module. Therefore, $E_1$ is a direct summand of a finitely generated free module $E_0$, with the projection $p : E_0 \to E_1$. Moreover, the composition map $f \circ p : E_0 \to E_2$ is a surjection. Thus, as it is showed in the previous case, there is a holomorphic structure $\nabla_0$ on $E_0$ making $f \circ p$ a morphism in $\mathbf{Vect}(\mathbb{T}_\theta)$. We define $\nabla_1 = p\nabla_0$. We will see that $\nabla_1$ is a holomorphic structure on $E_1$, which makes $f : (E_1, \nabla_1) \to (E_2, \nabla_2)$ a morphism in $\mathbf{Vect}(\mathbb{T}_\theta)$.

We have $p\nabla_0(ea) = p(\nabla_0(e)a + e\delta_\tau(a)) = p\nabla_0(e)a + e\delta_\tau(a)$. This determines a holomorphic structure on $E_1$. We also have $f(p\nabla_0(e)) = f \circ p(\nabla_0(e)) = \nabla_2(f \circ p(e)) = \nabla_2(f(e))$. Hence $f$ is compatible with these holomorphic structures.

If $0 \to E_0 \to E_1 \to E_2 \to 0$ is a short exact sequence in $\mathbf{proj}\mathcal{A}_\theta$ and $(E_2, \nabla_2)$ is an object in $\mathbf{Vect}(\mathbb{T}_\theta)$, then there is a $\nabla_1$ on $E_1$ making $E_1 \to E_2$ a morphism in $\mathbf{Vect}(\mathbb{T}_\theta)$. Moreover if we see $E_0$ as a kernel, then $E_0$ naturally has a holomorphic structure $\nabla_0$ inherited from $(E_1, \nabla_1)$. Thus, $0 \to (E_0, \nabla_0) \to (E_1, \nabla_1) \to (E_2, \nabla_2) \to 0$ is an short exact sequence in $\mathbf{Vect}(\mathbb{T}_\theta)$ and with the same morphisms if we forget the holomorphic structures.

For a long exact sequence $0 \to E_0 \to E_1 \to \cdots \to E_n \to 0$, if we break it into short exact sequences and apply the argument above, then we have a long exact sequence in $\mathbf{Vect}(\mathbb{T}_\theta)$. The morphisms remain the same, when we forget the holomorphic structures.

□

Now let us turn to the cyclic homology of these categories. In the spirit of Mitchell, for a dg category $\mathcal{B}$, its mixed complex $C(\mathcal{B})$ is given by the following constructions:

i) the n-th term of precyclic complex:

$$\coprod_{B_0,\ldots,B_n\in\mathrm{Ob}(\mathcal{B})} \mathcal{B}(B_n, B_0) \otimes \mathcal{B}(B_{n-1}, B_n) \otimes \cdots \otimes \mathcal{B}(B_0, B_1);$$

ii) the degeneracy maps and the cyclic operator:

$$\partial_i(f_n, \ldots, f_i, f_{i-1}, \ldots, f_0) = \begin{cases} (f_n, \ldots, f_i f_{i-1}, \ldots, f_0) & i \neq 0 \\ (-1)^\alpha (f_0 f_n, \ldots, f_1) & i = 0 \end{cases},$$

$$t(f_{n-1}, \ldots, f_0) = (-1)^\alpha (f_0, f_{n-1}, f_{n-2}, \ldots, f_1),$$

where $\alpha = n + \deg f_0 \cdot (\deg f_1 + \cdots + \deg f_{n-1})$.

Given an exact category $\mathcal{A}$, we denote the dg category of bounded complexes over $\mathcal{A}$ by $\mathscr{C}^b(\mathcal{A})$ and its subcategory of acyclic complexes by $\mathscr{A}\mathscr{C}^b(\mathcal{A})$. The mixed complex of the exact category $\mathcal{A}$ is defined as the mapping cone of the embedding of complexes $\mathscr{A}\mathscr{C}^b(\mathcal{A}) \hookrightarrow \mathscr{C}^b(\mathcal{A})$. The mixed complexes can be also be viewed as dg modules over the dg algebra $k[\epsilon]/(\epsilon^2)$ with $\deg\epsilon = 1$ and trivial differentials. In the derived category of $\Lambda$, we then define the cyclic homologies $HC$, $HN$, $HP$ of $\mathcal{A}$ by applying functors $- \otimes^{\mathbf{L}}_\Lambda k$, $\mathbf{R}\mathrm{Hom}_\Lambda(k, -)$ and $\mathbf{R}\varprojlim P_k[-2n] \otimes_\Lambda -$ to its mixed complex and taking the homology. So in the sense of Keller, all the cyclic homologies are "unified" by the mixed complex [25]. In the following, when used without further specification, the word "cyclic homology" of an exact category will really stand for the mixed complex of it.

Recall that for a smooth projective variety $X$, the mixed complex can be defined on the category of coherent sheaves over $X$ [26].

**Proposition 2.2.2.** *The cyclic homology of* $\mathbf{Vect}(\mathbb{T}_\theta)$ *is the same as the cyclic homology of* $X$.

*Proof.* By the equivalence of categories in the previous section, we only consider the heart $\mathcal{C}^\theta$ of some t-structure of $D^b(X)$. The proof is similar to the proof of the statement $D^b(\mathcal{C}^\theta) = D^b(X)$.

It is known that, for elliptic curves, every bounded t-structure is given by a cotilting torsion pair. In our case, it is $(\mathrm{Coh}_{>\theta}, \mathrm{Coh}_{\leq\theta})$ [19]. So $(\mathrm{Coh}_{\leq\theta}[1], \mathrm{Coh}_{>\theta})$ is a tilting torsion pair in $\mathcal{C}^\theta$.

$\mathrm{Coh}_{\leq\theta}$ is an exact category. Moreover, by the lemma and proposition and their duals of [4], we have $D^b(X) = D^b(\mathrm{Coh}_{\leq\theta})$ and $D^b(\mathcal{C}^\theta) = D^b(\mathrm{Coh}_{\leq\theta}[1])$.

By the main theorem of [25], we have the quasi-isomorphisms of cyclic complexes induced by the inclusions: $C(\mathrm{Coh}_{\leq\theta})\tilde{\to}C(X)$ and $C(\mathrm{Coh}_{\leq\theta}[1])\tilde{\to}C(\mathcal{C}^\theta)$. So there is a quasi-isomorphism $C(\mathcal{C}^\theta) \simeq C(X)$. □

**Proposition 2.2.3.** *The functor $F$ in proposition 2.2.1 induces an injection*

$$C(\mathbf{Vect}(\mathbb{T}_\theta)) \rightarrowtail C(\mathbf{proj}\mathcal{A}_\theta)\tilde{\to}C(\mathcal{A}_\theta).$$

*Proof.* The last quasi-isomorphism is due to the theorem of McCarthy on the special homotopy [32].

To show $F$ induces the injection $C(\mathbf{Vect}(\mathbb{T}_\theta)) \rightarrowtail C(\mathbf{proj}\mathcal{A}_\theta)$, we need to introduce a third category $\mathbf{proj}\mathcal{A}_\theta^\nabla$. $\mathbf{proj}\mathcal{A}_\theta^\nabla$ has the same objects as $\mathbf{Vect}(\mathbb{T}_\theta)$. For any $(E_1, \nabla_1)$, $(E_2, \nabla_2)$ in $\mathbf{proj}\mathcal{A}_\theta^\nabla$, we define $\mathrm{Hom}_{\mathbf{proj}\mathcal{A}_\theta^\nabla}((E_1, \nabla_1), (E_2, \nabla_2)) = \mathrm{Hom}_{\mathcal{A}_\theta}(E_1, E_2)$.

We can define a natural functor $\mathbf{proj}\mathcal{A}_\theta^\nabla \to \mathbf{proj}\mathcal{A}_\theta$ simply as the forgetful functor that drops the $\nabla$'s. It is obvious that this functor is fully faithful. Moreover, it is surjective on objets (hence essentially surjective). This is an equivalence of categories. This gives the quasi-isomorphism $C(\mathbf{proj}\mathcal{A}_\theta^\nabla)\tilde{\to}C(\mathbf{proj}\mathcal{A}_\theta)$ by Keller's theorem again.

On the other hand, $\mathbf{Vect}(\mathbb{T}_\theta)$ is a subcategory of $\mathbf{proj}\mathcal{A}_\theta^\nabla$ (but not full). So the functor $F : \mathbf{Vect}(\mathbb{T}_\theta) \to \mathbf{proj}\mathcal{A}_\theta$ is just the composition of $\mathbf{Vect}(\mathbb{T}_\theta) \hookrightarrow \mathbf{proj}\mathcal{A}_\theta^\nabla \to \mathbf{proj}\mathcal{A}_\theta$. According to the definition of mixed complex of exact categories, we see $\mathscr{C}^b(\mathbf{Vect}(\mathbb{T}_\theta)) \subset \mathscr{C}^b(\mathbf{proj}\mathcal{A}_\theta^\nabla)$ and $\mathscr{A}\mathscr{C}^b(\mathbf{Vect}(\mathbb{T}_\theta)) \subset \mathscr{A}\mathscr{C}^b(\mathbf{proj}\mathcal{A}_\theta^\nabla)$. It follows that $C(\mathbf{Vect}(\mathbb{T}_\theta)) \hookrightarrow C(\mathbf{proj}\mathcal{A}_\theta^\nabla)$. Thus $F$ induces the morphism of complexes:

$$C(\mathbf{Vect}(\mathbb{T}_\theta)) \hookrightarrow C(\mathbf{proj}\mathcal{A}_\theta^\nabla)\tilde{\to}C(\mathbf{proj}\mathcal{A}_\theta).$$

□

## 2.3 Motives of Noncommutative Tori

### 2.3.1 The (Weak) t-Structures

In the previous section, we recall the known fact that the category of holomorphic bundles on the noncommutative torus is equivalent to a heart of a nonstandard t-structure on the derived category of some elliptic curve. We know that an elliptic curve, as a smooth projective variety, can be used to define an objet in the category of pure motives and so an object in the category of noncommutative pure motives. In this section we propose a way to extend the concept of t-structure to the frame of noncommutative motives. In this way, we can obtain objects with the properties of noncommutative tori in the category of noncommutative motives.

Assume $\mathcal{C}$ is a dg category which has a shift functor $E[i]$ and such that for any closed and degree 0 morphism $f : E \to F$ in $\mathcal{C}$ there is a mapping cone $C_f)$ in $\mathcal{C}$.

**Definition 2.3.1.** *Let $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$ be full subcategories of $\mathcal{C}$, $\mathcal{C}^{\leq n} = \mathcal{C}^{\leq 0}[-n]$ and $\mathcal{C}^{\geq n} = \mathcal{C}^{\geq 0}[-n]$. One says that $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is a* ***(weak)-t-structure*** *if:*

*(i)* $\mathcal{C}^{\leq -1} \subset \mathcal{C}^{\leq 0}$ *and* $\mathcal{C}^{\geq 1} \subset \mathcal{C}^{\geq 0}$.

*(ii) For* $X \in \mathrm{Ob}(\mathcal{C}^{\leq 0})$ *and* $Y \in \mathrm{Ob}(\mathcal{C}^{\geq 1})$, $\mathrm{Hom}(X, Y) = 0$ *in* $H^0(\mathcal{C})$.

*(iii) For any* $X \in \mathrm{Ob}(\mathcal{C})$*, there is a triangle* $X_0 \to X \to X_1 \to X_0[1]$ *in* $\mathcal{C}$ *with* $X_0 \in \mathrm{Ob}(\mathcal{C}^{\leq 0})$ *and* $X_1 \in \mathrm{Ob}(\mathcal{C}^{\geq 1})$*, which is quasi-equivalent to a triangle having the form* $E \to F \to C_f \to E[1]$ *for some closed degree* 0 *morphism* $f : E \to F$.

**Remark 2.3.2.** *Here, we may use "weak" in the sense of quasi-equivalence. As usual, the word "strong" stands for the dg-equivalence.*

Recall that a triangulated category $\mathcal{K}$ is dg enhanced, if there is a pretriangulated category $\mathcal{C}$ with an equivalence $H^0(\mathcal{C}) \to \mathcal{K}$ of triangulated categories. We know that the category of $H^0(\mathcal{C})$ has the same objects as $\mathcal{C}$. So, if $(\mathcal{K}^{\leq 0}, \mathcal{K}^{\geq 0})$ is a t-structure of triangulated categories on $\mathcal{K} = H^0(\mathcal{C})$, then we get two full subcategories of $\mathcal{C}$ generated by the objects of $\mathcal{K}^{\leq 0}$ and $\mathcal{K}^{\geq 0}$, and we use the notation $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ to denote them. We will see that $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is a t-structure on the dg category $\mathcal{C}$. Actually, condition (i) and (ii) in our definition are naturally satisfied. For condition (iii), because $\mathcal{K}$ is pretriangulated, every distinguished triangle is induced in that way. So (iii) holds. Hence, if $\mathcal{K}$ is dg enhanced, the t-structure on $\mathcal{K}$ always induces a t-structure on its dg enhancement.

The following proposition is a standard result for t-structures.

**Proposition 2.3.3.** *For any $X \in \mathcal{C}^{\leq n}$ (resp. $X \in \mathcal{C}^{\geq n}$), $Y \in \mathcal{C}$ and $k \in \mathbb{Z}$, there is a $Y_0 \in \mathcal{C}^{\leq n+k}$ (resp. $Y_1 \in \mathcal{C}^{\geq n+k}$), such that*

$$\mathcal{H}^k(X, Y_0) \xrightarrow{\sim} \mathcal{H}^k(X, Y),$$

$$\big(\textit{resp.} \quad \mathcal{H}^k(Y_1, X) \xrightarrow{\sim} \mathcal{H}^k(Y, X).\big)$$

*Proof.* First, we notice that if $(\mathcal{K}^{\leq 0}, \mathcal{K}^{\geq 0})$ is a t-structure of a triangulated category $\mathcal{K}$, then for any $X \in \mathcal{K}$, there is an exact triangle

$$X_0 \to X[n] \to X_1 \to X_0[1]$$

with $X_0 \in \mathcal{K}^{\leq 0}$ and $X_1 \in \mathcal{K}^{\geq 1}$. Then, we have

$$X_0[-n] \to X \to X_1[-n] \to X_0[-n+1]$$

with $X_0[-n] \in \mathcal{K}^{\leq 0}[-n] = \mathcal{K}^{\leq n}$ and $X_1[-n] \in \mathcal{K}^{\geq 1}[-n] = \mathcal{K}^{\geq n+1}$. Therefore, $(\mathcal{K}^{\leq n}, \mathcal{K}^{\geq n})$ is also a t-structure for $\mathcal{K}$. Hence the standard argument gives the adjoint functors $\tau^{\leq n}$, $\tau^{\geq n}$:

$$\mathrm{Hom}_{\mathcal{K}^{\leq n}}(X,\, \tau^{\leq n}Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}}(X, Y), \text{ with } X \in \mathrm{Ob}(\mathcal{K}^{\leq n}),\, Y \in \mathrm{Ob}(\mathcal{K})$$

and

$$\mathrm{Hom}_{\mathcal{K}^{\geq n}}(\tau^{\geq n}X,\, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}}(X, Y) \text{ with } X \in \mathrm{Ob}(\mathcal{K}),\, Y \in \mathrm{Ob}(\mathcal{K}^{\geq n}).$$

Now let $X$ be an object of $\mathcal{C}^{\leq n}$. Then $\mathcal{H}^k(X, Y) = \mathcal{H}^0(\mathrm{Hom}(X[-k], Y))$, with $X[-k]$ is in $\mathcal{C}^{\leq n+k}$. Thus, if we set $Y_0 = \tau^{\leq n+k}Y$, we have

$$\mathrm{Hom}(X[-k],\, Y_0) \xrightarrow{\sim} \mathrm{Hom}(X[-k], Y).$$

Moreover, we have

$$\mathcal{H}^k(X, Y_0) \xrightarrow{\sim} \mathcal{H}^k(X, Y).$$

The other statement is proven similarly. □

### 2.3.2 Decomposition of Motivic Galois Groups

Let $\mathcal{D}$ be a triangulated category which is dg enhanced and idempotent complete. Thus, there is a dg pre-triangulated category $\mathcal{A}$ such that $H^0(\mathcal{A}) = \mathcal{D}$. As Toën and Vaquié showed in [50], there is an equivalence $\mathcal{D} \to \mathcal{D}_c(\mathcal{A})$.

**Theorem 2.3.4.** *If $(\mathcal{D}^{\leq 0},\, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$, then it also gives a (weak-)t-structure on $\mathcal{A}$, which may be denoted by $(\mathcal{D}_{dg}^{\leq 0}, \mathcal{D}_{dg}^{\geq 0})$. Then for any additive invariant $F : \mathcal{A} \to \mathcal{B}$, we have $F(\mathcal{D}_{dg}^{\leq 0}) \oplus F(\mathcal{D}_{dg}^{\geq 1}) = F(\mathcal{A})$.*

*Proof.* It is sufficient to show that, for any $\mathcal{U} \in \mathrm{dgcat}_k$,

$$\mathrm{K}_0\mathrm{rep}(\mathcal{U}, \mathcal{D}_{dg}^{\leq 0}) \oplus \mathrm{K}_0\mathrm{rep}(\mathcal{U}, \mathcal{D}_{dg}^{\geq 1}) = \mathrm{K}_0\mathrm{rep}(\mathcal{U}, \mathcal{A}).$$

Recall $\mathrm{rep}(\mathcal{U},\mathcal{A}) \subset \mathcal{D}(\mathcal{U}^{op} \otimes \mathcal{A})$ is the sub-triangulated category of bimodules $X$ such that for any $x \in \mathcal{U}$, $X(\ ,x) \in \mathcal{D}_c(\mathcal{A})$.

This theorem is proven if we notice the fact that $H^0(\mathcal{A}) \to \mathcal{D}_c(\mathcal{A})$ is an equivalence. For any $X \in \mathrm{Ob}(H^0(\mathcal{A}))$, there is an exact triangle

$$X_0 \to X \to X_1 \to X_0[1]$$

in $H^0(\mathcal{A})$ with $X_0$ in $\mathcal{D}^{\leq 0} = H^0(\mathcal{D}^{\leq 0}_{dg}) \hookrightarrow H^0(\mathcal{A})$ and $X_1$ in $\mathcal{D}^{\geq 1} = H^0(\mathcal{D}^{\geq 1}_{dg}) \hookrightarrow H^0(\mathcal{A})$. By the equivalence, this gives an exact triangle in $\mathcal{D}_c(\mathcal{A})$, hence in $\mathrm{rep}(\mathcal{U},\mathcal{A})$. After passing to the Grothendieck group, the exact triangle becomes the desired direct sum decomposition. □

It is known [24], [17], the periodic cyclic homology $HP : \mathrm{NChow} \to \mathrm{sVect}(k)$ is an additive invariant. Also, Marcolli and Tabuada showed in [36], the category NChow is super-Tannakinan with the super-fibre functor $HP$.

Let $X$ be the elliptic curve above and $D^b_{dg}(X)$ the object associated to $X$ in NChow. Actually, by the Hodge conjecture, the Tannakian subcategory $M_X$ generated by $D^b_{dg}(X)$ in NChow is Tannakian with the fibre functor $HP$ [36] and the motivic Galois group $Gal(D^b_{dg}(X))$. The t-structure $(\mathbf{D}^{\leq 0}_{dg}, \mathbf{D}^{\geq 0}_{dg})$ on $D^b_{dg}(X)$ gives the direct sum

$$HP(\mathbf{D}^{\leq 0}_{dg}) \oplus HP(\mathbf{D}^{\geq 1}_{dg}) = HP(D^b_{dg}(X)).$$

We define $HP^{\leq 0}$ as the composition $D^b_{dg}(X) \to HP(D^b_{dg}(X)) \to HP(\mathbf{D}^{\leq 0}_{dg})$ and $HP^{\geq 0}$ as the composition $D^b_{dg}(X) \to HP(D^b_{dg}(X)) \to HP(\mathbf{D}^{\geq 0}_{dg})$. Moreover, $HP^{\leq 0}$ and $HP^{\geq 0}$ induce two subgroups $G^{\leq 0}$, $G^{\geq 0}$ of $Gal(D^b_{dg}(X))$.

**Corollary 2.3.5.** *There are two subgroups $G^{\leq 0}$, $G^{\geq 0}$ of $Gal(D^b_{dg}(X))$ associated to the t-structure on $D^b_{dg}(X)$ and $G^{\leq 0} \oplus G^{\geq 0} \subset Gal(D^b_{dg}(X))$*

# CHAPTER 3

# THE $\mathrm{GL}_N$-CONNES-MARCOLLI SYSTEMS

## 3.1 Background and Questions

The original BC-system introduced by Bost and Connes ([3]) is based on the Hecke Algebras in number theory. Later, it is discovered that the BC system could be constructed in different settings. In the common language of noncommutative geometry, the BC-system is interpreted through the groupoid construction:

$$Z_{BC} = \{(g, \rho) \in \mathbb{Q}_{\times} \times \hat{\mathbb{Z}} \mid \ g > 0, \ g\rho \in \hat{\mathbb{Z}}\}.$$

As mentioned before, this system has the Riemann zeta function as its partition function. Remarkably, the Galois action on the equilibrium states of the system shows a deep relation between quantum statistical mechanics and the class field theory. The dual system of the BC system is also used to study the Riemann Hypothesis (see [12] and Chapter 4 of [14]).

Because of the arithmetic significance of the BC system, several generalizations are constructed later. Particularly, Connes and Marcolli introduce the $\mathrm{GL}_2$ system (CM system) in [13]. Similarly, the CM system is constructed by using the CM-algebra:

$$C_r^*\big(\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{GL}_2^+(\mathbb{Q}) \boxtimes_{\mathrm{SL}_2(\mathbb{Z})} (\mathrm{Mat}_2(\hat{\mathbb{Z}}) \times \mathbb{H})\big),$$

After a careful discussion of the properties of this system, Connes, Marcolli and Ramachandran construct another quantum statistical mechanical system related to the CM system, which has the same properties of the BC system for imaginary quadratic fields, and connects to a known case of Hilbert's 12th problem: the construction of the maximal abelian extension of imaginary quadratic fields in terms of values of modular functions at CM points (see [15], [16]). The remaining known cases of the Hilbert's 12th problem, namely abelian extensions of CM fields, have also been interpreted from the quantum statistical mechanical point of view in [52].

As we mentioned before, the most significant properties of the BC type systems are reflected by the action of symmetric group of the system on the KMS states. In [30], Laca, Larsen and Neshveyev developed a method based on Hecke operators and ergodic theory and gave a thorough study of the KMS states over the ($\mathrm{GL}_2$-) Connes-Marcolli system. At the end of their paper (see

[30], Remark 4.10), they suggested that their results could be generalized to a $\mathrm{GL}_n$-Connes-Marcolli system (Definition 3.2.1) and predicted that

(i) There is no $\mathrm{KMS}_\beta$ state on the system for each $\beta < n-1$ and $\beta \neq 0, 1, \cdots, n-1$.

(ii) There exists only one $\mathrm{KMS}_\beta$ state when $\beta \in (n-1,\, n]$

(iii) The $\mathrm{KMS}_\beta$ states are determined by the probability measures on $\Gamma\backslash Y$ when $\beta > n$.

(iv) At the dividing points $\beta = 1, \cdots, n-1$, the Haar measure on $\mathbb{A}_f^{\beta n}$ determines a $\mathrm{KMS}_\beta$ state once we see $\mathbb{A}_f^{\beta n}$ as the set of those matrices in $\mathrm{Mat}_n(\mathbb{A}_f)$ with the first $n-\beta$ columns all occupied by zeros.

In this chapter, we develop a similar set of tools, with Hecke operators and ergodic theory, adapted to the $\mathrm{GL}_n$-setup and we prove the corresponding results of the $\mathrm{GL}_n$-Connes-Marcolli systems (assertion (i)$\sim$(iv) above).

## 3.2 The $\mathrm{GL}_n$-Connes-Marcolli Systems

We first fix some notations for this chapter. Let $\mathbb{P} = \mathrm{PGL}_n^+(\mathbb{R})$ be the connected component of the real Lie group $\mathrm{PGL}_n(\mathbb{R})$ containing the identity and let $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. We also use $\mathrm{Mat}_n^+$ and $\mathrm{GL}_n^+$ to denote those corresponding matrices with positive determinants. Let $Y = \mathbb{P} \times \mathrm{Mat}_n(\hat{\mathbb{Z}})$ and $X = \mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$ where $\mathbb{A}_f$ is the finite Adele ring over $\mathbb{Q}$, and $\hat{\mathbb{Z}} = \varinjlim_n \mathbb{Z}/n\mathbb{Z}$ is the ring of profinite integers.

Let $\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \times_\Gamma X$ be the quotient space of $\mathrm{GL}_n^+(\mathbb{Q}) \times X$ by the $\Gamma \times \Gamma$-action:

$$(\gamma_1, \gamma_2)(g, x) = (\gamma_1 g \gamma_2^{-1}, \gamma_2 x) \quad \text{with}\,(g,x) \in \mathrm{GL}_n^+(\mathbb{Q}) \times X\,\text{and}\,\gamma_{1,2} \in \Gamma.$$

The subspace $\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_\Gamma Y$ of $\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \times_\Gamma X$ is the space whose elements can be represented by pairs $(g, y) \in \mathrm{GL}_n^+(\mathbb{Q}) \times Y$ such that $gy \in Y$.

We construct an involutive associative algebra and a Hilbert space representation associated to the space $\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_\Gamma Y$, which is analogous to the algebra of the $\mathrm{GL}_2$ system constructed in [13], see also §5.2 of [14].

For $f_1, f_2 \in C_c(\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_\Gamma Y)$, there is naturally a convolution defined as

$$f_1 * f_2(g, y) = \sum_{h \in \Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}), hy \in Y} f_1(gh^{-1}, hy) f_2(h, y), \tag{3.1}$$

and there is also an involution

$$f_1^*(g, y) = \overline{f_1(g^{-1}, gy)}.$$

If we set $G_y = \{g \in \mathrm{GL}_n^+(\mathbb{Q}) \mid gy \in Y\}$, then the action

$$\pi_y(f)\delta(g) = \sum_{h \in \Gamma\backslash G_y} f(gh^{-1}, hy)\delta(h)$$

gives a representation of $C_c(\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_\Gamma Y)$ on the Hilbert space $\mathfrak{H}_y = \ell^2(\Gamma\backslash G_y)$. Then one defines the norm as in Definition 3.43 of [14] as

$$\|f\| = \sup_{y \in Y}\{\|\pi_y(f)\|\}$$

on $C_c(\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_\Gamma Y)$. The completion of $C_c(\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_\Gamma Y)$ in this norm is a C*-algebra, which is denoted by $\mathrm{C}_r^*(\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_\Gamma Y)$. Then we have the following definition (referring to the Remark 4.10 of [30])

**Definition 3.2.1.** *The* $\mathrm{GL}_n$*-Connes-Marcolli algebra is the* $C^*$*-algebra*

$$\mathcal{A} = \mathrm{C}_r^*(\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_\Gamma Y).$$

Recall that a time evolution on a $C^*$-algebra $\mathcal{A}$ is a continuous one-parameter family of automorphisms $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$.

**Lemma 3.2.2.** *Setting* $\sigma_t(f)(g, x) = \det(g)^{it} f(g, x)$*, for* $f \in C_c(\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_\Gamma Y)$ *determines a time evolution* $\sigma_t$ *on the* $C^*$*-algebra* $\mathcal{A}$ *of Definition 3.2.1.*

*Proof.* In general, for an algebra of the form $C_c(\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_\Gamma Y)$, with the convolution product (4.1), any group homomorphism $N : \mathrm{GL}_n^+(\mathbb{Q}) \to \mathbb{R}_+^*$ with the property that $\Gamma \subset \mathrm{Ker}(N)$ determines a time evolution by setting $\sigma_t(f)(g, x) = N(g)^{it} f(g, x)$, see Section 2 of [30]. Indeed, this property suffices to ensure that, with respect to the convolution product (4.1), one has $\sigma_t(f_1 * f_2) = \sigma_t(f_1) * \sigma_t(f_2)$. Clearly setting $N(g) = \det(g)$ has the desired properties. □

Recall that an element $a \in \mathcal{A}$ is said to be *entire* if the function $t \mapsto \sigma_t(a)$ can be extended to an entire function on the complex number field $\mathbb{C}$, see Definition 2.5.20 of [7]. We also recall that a $\mathrm{KMS}_\beta$ state is a $\sigma_t$ invariant state $\varphi$ over $\mathcal{A}$ such that, for all entire elements $a, b$ of $\mathcal{A}$, the relation $\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$ holds, see Definition 5.3.1 of [8].

If we identify the space $\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \times_\Gamma \mathbb{P} \times \{0\}$ with the space $\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \times_\Gamma \mathbb{P}$, then the specialization of the *-algebra structure and representation of

$$C_c(\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_\Gamma Y)$$

to

$$C_c(\Gamma\backslash \mathrm{GL}_n^+(\mathbb{Q}) \times_\Gamma \mathbb{P} \times \{0\})$$

makes

$$C_c(\Gamma\backslash \mathrm{GL}_n^+(\mathbb{Q}) \times_\Gamma \mathbb{P})$$

a normed *-algebra. Then the completion of

$$C_c(\Gamma\backslash \mathrm{GL}_n^+(\mathbb{Q}) \times_\Gamma \mathbb{P})$$

in the induced norm, which is denoted by

$$\mathcal{B} = \mathrm{C}_r^*(\Gamma\backslash \mathrm{GL}_n^+(\mathbb{Q}) \times_\Gamma \mathbb{P}),$$

is also a $C^*$ algebra. In a similar way, the specialization of $\sigma_t$ to

$$C_c(\Gamma\backslash \mathrm{GL}_n^+(\mathbb{Q}) \times_\Gamma \mathbb{P} \times \{0\})$$

defines a time evolution on $\mathcal{B}$. So $(\mathcal{B}, \sigma_t)$ also forms a $C^*$-dynamical system.

Given a matrix of the form

$$g = \begin{pmatrix} k & & \\ & \ddots & \\ & & k \end{pmatrix}$$

with $k \in \mathbb{N}$, consider the function $u_g$ that takes values $u_g(h, x) = 1$ if $h \in \Gamma.g$ and $u_g(h, x) = 0$ otherwise. This defines a unitary multiplier $u_g$ of $\mathcal{B}$ which is an eigenfunction of $\sigma_t$, namely such that

$$\sigma_t(u_g) = k^{nit} u_g.$$

The definition of KMS states and a direct calculation then show the following lemma.

**Lemma 3.2.3.** *There is no KMS$_\beta$ state over $\mathcal{B}$ if $\beta \neq 0$.*

*Proof.* If $\phi$ is a KMS$_\beta$ state over $\mathcal{B}$, then by the property of the KMS state we have

$$1 = \phi(u_g * u_g^*) = \phi(u_g^* * \sigma_{i\beta}(u_g)) = k^{-n\beta}.$$

This is absurd if $\phi \neq 0$. So all the KMS$_\beta$ states vanish on $\mathcal{B}$. □

The next lemma characterizes the fixed points of $Y$ under the action of $\mathrm{GL}_n^+(\mathbb{Q})$.

**Lemma 3.2.4.** *Let $g \in \mathrm{GL}_n^+(\mathbb{Q})$ and $g \neq I$. If $gy = y$ with $y \in Y = \mathbb{P} \times \mathrm{Mat}_n(\hat{\mathbb{Z}})$, then $y = (x, 0)$, $x \in \mathbb{P}$.*

*Proof.* Since $\mathbb{P} = \mathrm{PGL}_n^+(\mathbb{R}) \subset \mathrm{PGL}_n(\mathbb{R}) = \mathrm{GL}_n(\mathbb{R})/\mathbb{R}^*$, for $g \in \mathrm{GL}_n^+(\mathbb{Q})$, $x \in \mathbb{P}$, $gx = x$ only if $g = rI$ where $r \in \mathbb{R}^*$ (actually $r \in \mathbb{Q}^*$ here). However, for $h \in \mathrm{Mat}_n(\mathbb{A}_f)$, $rh = h$ only if $r = 1$ or $h = 0$. By assumption $g \neq I$, so $r \neq 1$. This means we must have $h = 0$. So when $gy = y$ where $y = (x, h) \in Y$, we obtain $y = (x, 0)$. □

Let $\mu$ be a Radon measure $\mu$ on $Y$ satisfying the scaling condition

$$\mu(gB) = \det(g)^{-\beta}\mu(B) \tag{3.2}$$

for $g \in \mathrm{GL}_n^+(\mathbb{Q})$ and Borel set $B \subset Y$ with $gB \subset Y$.

The scaling condition (3.2) also implies that $\mu$ is a $\Gamma$-invariant measure on $Y$. It determines the measure $\nu$ on $\Gamma\backslash Y$ such that

$$\int_Y f d\mu = \int_{\Gamma\backslash Y} \sum_{x \in \Gamma y} f(x) d\nu([y]).$$

Let $E : \mathcal{A} \to C_0(\Gamma\backslash Y)$ be the canonical conditional expectation (see [44], Proposition 2.3.22.) defined by

$$E(f)(y) = f(I, y) \ \text{ for } f \in C_c(\Gamma\backslash \mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_\Gamma Y).$$

Then the map

$$\mu \mapsto (\varphi : f \mapsto \varphi(f) = \int_{\Gamma\backslash Y} E(f)(y)\ d\nu(y))$$

defines a $\mathrm{KMS}_\beta$ weights over $\mathcal{A}$. Moreover, for a measure $\mu$ satisfying the scaling condition (3.2), the corresponding $\mathrm{KMS}_\beta$ weight is a state iff the induced measure $\nu$ on $\Gamma\backslash Y$ is a probability measure, i.e. $\nu(\Gamma\backslash Y) = 1$. In the chapter, sometimes to save notation, we still use $\mu$ for the induced measure on $\Gamma\backslash Y$ if there is no otherwise meaning in context.

The Poposition 2.1 in [30] implies that, if the action of $\mathrm{GL}_n^+(\mathbb{Q})$ is free, then any $\mathrm{KMS}_\beta$ state $\varphi$ over $\mathcal{A}$ is in this form for some Radon measure $\mu$ satisfying the scaling condition (3.2). Unfortunately, as we have seen in Lemma 3.2.4 above, the action of $\mathrm{GL}_n^+(\mathbb{Q})$ in the $\mathrm{GL}_n$-system is not totally free. But the fixed points are all contained in $\mathbb{P} \times \{0\}$. So combining the previous two lemmata we still have the following theorem.

**Theorem 3.2.5.** *For the* $\mathrm{GL}_n$*-Connes-Marcolli system, when* $\beta \neq 0$*, there is a one-to-one correspondence between* $KMS_\beta$*-states over* $\mathcal{A}$ *and Radon measures* $\mu$ *on* $Y$ *such that* $\mu$ *satisfies the scaling condition* (3.2) *and induces a probability measure* $\nu$ *on* $\Gamma\backslash Y$*.*

*Proof.* Let $\mathcal{J}$ be the ideal $C^*_r(\Gamma\backslash G \boxtimes_\Gamma (Y - (\mathbb{P}\times\{0\})))$. Notice that if a $\mathrm{KMS}_\beta$ state $\phi$ on $\mathcal{A}$ vanishes on $\mathcal{J}$ then it is a KMS state over $\mathcal{A}/\mathcal{J} = \mathcal{B}$. Then by Lemma 3, we see that $\phi$ vanishes on $\mathcal{A}$. Thus, given a KMS state $\varphi$ on $\mathcal{A}$, if we extend the KMS state $\varphi|_\mathcal{J}$ to a KMS state $\phi$ on $\mathcal{A}$, then $\phi - \varphi$ vanishes over $\mathcal{A}$. Thus, the $\mathrm{KMS}_\beta$ states on $\mathcal{A}$ are totally determined by the $\mathrm{KMS}_\beta$ states on $\mathcal{J}$.

On the other hand, if we have a measure $\mu$ on $Y$ satisfying the scale condition (3.2), since all scalar matrices act on $\mathbb{P}\times\{0\}$ trivially, the scale condition implies that $\mathbb{P}\times\{0\}$ is of measure 0.

By Lemma 4 we see that the action of $\mathrm{GL}_n^+(\mathbb{Q})$ on $Y - (\mathbb{P}\times\{0\})$ is free. So by The Poposition 2.1 in [30], there is a one-to-one correspondence between $\mathrm{KMS}_\beta$-states and Radon measures on $Y$ that satisfy the scaling condition (3.2). □

**Remark 3.2.6.** *Most general formulas and lemmas in [30] are established under the condition of the freeness of the G-action. But for the concrete systems, the G-action is not totally free. This kind of issue is addressed in the same manner of [30] as demonstrated in the proof of Theorem 3.2.5: If the statement involves a measure $\mu$ with scaling condition* (3.2) *and the action is free outside a G-invariant measure zero set, the results may still be applied. We may point out similar arguments in the following context if necessary.*

**Remark 3.2.7.** *In the theorem above, the only restriction on $\beta$ is nonzero. However, for the* $\mathrm{GL}_n$*-system, the KMS states do not exist for all $\beta$'s. Namely, due to the 1-1 correspondence between the KMS states and the Radon measures above, Radon measures satisfying the scaling condition* (3.2) *only exist for a certain range of values of the parameter $\beta$. This will be explained in the next section.*

## 3.3 The Phase Transition

### 3.3.1 Hecke Pairs

To prove our main theorem, we extend the approach and the results of [30] to the case of the $\mathrm{GL}_n$ groups. First we need to recall some concepts of Hecke Pairs and some related formulas.

We recall the following notions from [28], I. A Hecke pair is a pair of groups $(G, H)$ such that $H \subset G$ and for any $g \in H$ the set $H/(g^{-1}Hg \cap H)$ is finite. From [28], I. lemma 3.1, we also see $\#(H/(g^{-1}Hg\cap H)) = \#(H\backslash HgH)$. If $f$ is an $H$-invariant function over a space $X$ with a $G$ action, then for the Hecke pair we define the Hecke operator $T_g$ for $g \in G$ to be

$$T_g f(x) = \frac{1}{\#(H/(g^{-1}Hg \cap H))} \sum_{h \in H\backslash HgH} f(hx).$$

In this chapter, we mainly focus on the Hecke pair $(G, \Gamma) = (\mathrm{GL}_n^+(\mathbb{Q}), \mathrm{SL}_n(\mathbb{Z}))$ ([2], Lemma 3.3.1. or [28], V. Corollary 5.3).

If the measure $\mu$ satisfies the scaling conditions (3.2) on $X$, one has

$$\int_{\Gamma\backslash X} f d\nu = \det(g)^{-\beta} \int_{\Gamma\backslash X} T_g f d\nu,$$

where $\nu$ is the induced measure by $\mu$. (see [30], Lemma 2.6 and the comment after the lemma. Again, this is a formula about integrals. Once the group action is free outside a invariant measure zero set, the same result holds).

Also, if we let $Y \subset X$ for some $X$ with a free $G$-action (outside a $G$-invariant measure zero set), and let $Z \subset Y$ such that if $h \in G$ and $hz \in Z$ for some $z \in Z$ then $h \in \Gamma$, then once we have some $g \in G$ with $gZ \subset Y$ the following formula holds

$$\nu(\Gamma\backslash\Gamma g\Gamma Z) = \det(g)^{-\beta} \#(\Gamma/(g^{-1}\Gamma g \cap \Gamma))\nu(\Gamma\backslash\Gamma Z). \tag{3.3}$$

This formula can be seen in a direct calculation or by [30], Lemma 2.7.

### 3.3.2 Phase Transition

Because of the one-to-one correspondence between the KMS states on $\mathcal{A}$ and the Borel measures $\mu$ with scaling condition (3.2), instead of studying the structure of the KMS, we rather study the properties of the corresponding Borel measures on $Y$. From the definition of the algebra $\mathcal{A}$, we see it is not very convenient to work directly with the measure on $Y$. For instance, the group action on $Y$ is only partially defined: for $g \in G$ and $y \in Y$, we cannot always guarantee that $gy \in Y$. So we want to extend the measure to a larger space to make the discussion more convenient. For this purpose, we need a lemma from [30].

**Lemma 3.3.1.** *([30], Lemma 2.2) Let $X$ be a space with a free $G$-action and $Y \subset X$ a clopen subset. Given a measure $\mu$ on $Y$ satisfying condition* (3.2)*, then we can uniquely extend $\mu$ to a Radon measure on $GY = \{gy \mid g \in G, y \in Y\} \subset X$ which also satisfies the scaling condition* (3.2) *for any Borel set $B \subset GY$.*

Back to our specific case of the $\mathrm{GL}_n$-Connes-Marcolli system, where $X = \mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$ and $Y = \mathbb{P} \times \mathrm{Mat}_n(\hat{\mathbb{Z}})$, the group $G = \mathrm{GL}_n^+(\mathbb{Q})$ does not act on $X$ freely, but it acts on the space $X - (\mathbb{P} \times \{0\})$ freely. Once we have a Radon measure $\mu$ on $X$ satisfying the scaling condition (3.2) for any Borel set $B \subset X$, the same diagonal matrix argument (see the proof of Lemma 3.2.3) shows $\mu(\mathbb{P} \times \{0\}) = 0$. Then if we note that $GY = X$ and $G\big(Y - (\mathbb{P} \times \{0\})\big) = X - (\mathbb{P} \times \{0\})$, by applying the Lemma 3.3.1 above, every Radon measure on $Y$ satisfying the scaling condition (3.2) can be uniquely extended to a Radon measure on $X$, which is also satisfies the scaling condition (3.2).

With the discussion above, we can restate Theorem 3.2.5 in the following way.

**Theorem 3.2.5′.** *For the* $\mathrm{GL}_n$*-Connes-Marcolli system, when* $\beta \neq 0$*, there is a one-to-one correspondence between KMS*$_\beta$*-states over* $\mathcal{A}$ *and Radon measures* $\mu$ *on* $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$ *such that* $\mu$ *satisfies the scaling condition* (3.2) *and induces a probability measure* $\nu$ *on* $\Gamma\backslash Y$*.*

So from now on, we can just study the measures on $X = \mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$. This is more convenient, because $X$ admits a well defined $G$-action, instead of having only a partially defined action.

**Lemma 3.3.2.** *(Key lemma) Let* $p$ *be a fixed prime number and* $\mu_p$ *be a* $\Gamma$*-invariant measure on the space* $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{Q}_p)$*, which induces a probability measure on* $\Gamma\backslash\mathbb{P} \times \mathrm{Mat}_n(\mathbb{Z}_p)$*, such that* $\mu_p(\mathbb{P} \times \{0\}) = 0$ *and* $\mu_p(gB) = \det(g)^{-\beta}\mu_p(B)$*, where* $g \in \mathrm{Mat}_n(\mathbb{Z})$ *and in addition* $\det(g) = p^l$*, and* $B$ *is a Borel set in* $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{Q}_p)$*.*

*Then, for* $\beta \neq 0, 1, 2, \ldots, n-1$*, we have that* $\mathbb{P} \times \mathrm{GL}_n(\mathbb{Q}_p)$ *is a subset of full measure.*

*Proof.* First, we need to show that, for any nonzero singular $A \in \mathrm{Mat}_n(\mathbb{Q}_p)$, there are matrices $B \in \mathrm{SL}_n(\mathbb{Z}_p)$ and $C \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that

$$BAC = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & p^{k_1} & & \\ & & & & \ddots & \\ & & & & & p^{k_l} \end{pmatrix}$$

where $0 < l < n$ and $k_{i-1} \leq k_i$. Moreover, the expression on the right hand side is unique.

For any $A = (a_{ij})_{n\times n}$, we see that each entry is the form of $a_{ij} = ap^k$ where $a$ is invertible in $\mathbb{Z}_p$ or 0. So there is a smallest $p^k$ among those nonzero $a_{ij}$'s. Then by using the elementary matrices of exchanging rows and columns and of changing signs, we can make the smallest nonzero $ap^k$ to be the entry at $(1,1)$. For simplicity purposes, we still call the result matrix $A$.

If there is another entry element in the first column, for example, we can assume the entry $a_{21} = bp^l$ in $A = (a_{ij})$. Remember that now $a_{11} = ap^k$, if we multiply the elementary matrix $E_{21} = (e_{ij})$, in which $e_{ii} = 1$, $e_{21} = -a^{-1}bp^{l-k}$ and $e_{ij} = 0$ for any other entries, to the left of $A$, it makes the entry $a_{21}$ 0. Notice that $\det(E_{21}) = 1$ and because of the minimality of $ap^k$, $l - k \geq 0$, so $-a^{-1}bp^{l-k} \in \mathbb{Z}_p$. Hence $E_{21} \in \mathrm{SL}_n(\mathbb{Z}_p)$. By doing this, we make all the remaining entries in the first column 0. Similarly, by multiplying elementary matrices on the right, we use $ap^k$ to kill other nonzero entries in the first row. By multiplying by the diagonal matrix $\mathrm{diag}(a^{-1}, 1, \cdots, 1)$ on the right, we cancel out $a$. That is to say, we can find $B_1 \in \mathrm{SL}_n(\mathbb{Z}_p)$ and $C_1 \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that

$$B_1 A C_1 = \begin{pmatrix} p^k & 0 & \dots & 0 \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

By iterating this procedure, finally we get a diagonal matrix whose entries are all either powers of $p$ or zero. Then, by changing rows, columns and signs again, we obtain a matrix in the desired form. The uniqueness is a direct application of the Cauchy-Binet formula (*Cf.* the proof of the Theorem of Elementary Divisors in [2]).

By the previous argument, we have actually shown that the set of (nonzero) singular matrices in $\mathrm{Mat}_n(\mathbb{Q}_p)$ is a disjoint union of the sets $Z_{k_1 \dots k_l}$, where

$$Z_{k_1 \dots k_l} = \mathrm{SL}_n(\mathbb{Z}_p) \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & p^{k_1} & & \\ & & & & \ddots & \\ & & & & & p^{k_l} \end{pmatrix} \mathrm{GL}_n(\mathbb{Z}_p), \quad 1 \leq l \leq n-1. \tag{3.4}$$

Let $\mathscr{B}_\Gamma$ be the $\sigma$-field of $\Gamma$-invariant Borel sets in $\mathrm{Mat}_n(\mathbb{Q}_p)$. We now define a measure on $\mathrm{Mat}_n(\mathbb{Q}_p)$ by $v(B) = \mu_p(\Gamma \backslash \mathbb{P} \times B)$ for $B \in \mathscr{B}_\Gamma$. To show $\mathbb{P} \times \mathrm{GL}_n(\mathbb{Q}_p)$ is of full measure is the same as showing that its complement has measure 0. We have shown that the complement set is covered by $\mathbb{P} \times Z_{k_1 \dots k_l}$'s and that the set of $Z_{k_1 \dots k_l}$'s is countable, so we only need to show that each $\mu_p(\mathbb{P} \times Z_{k_1 \dots k_l}) = 0$. Namely, it is enough to show that these $Z_{k_1 \dots k_l}$ have $v$-measure zero.

To do this we need to give a description of the right cosets of

$$\mathrm{SL}_n(\mathbb{Z}) \backslash \mathrm{SL}_n(\mathbb{Z}) T_l \mathrm{SL}_n(\mathbb{Z}),$$

where $T_l$ stands for the diagonal matrix

$$T_l = \mathrm{diag}(\overbrace{1, \dots, 1}^{n-l}, \underbrace{p, \dots, p}_{l}).$$

This is actually done in [28], V. 7. In [28], V. Proposition 7.2, it is shown that $\mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{GL}_n(\mathbb{Z}) T_l \mathrm{GL}_n(\mathbb{Z})$ has a set of representatives given by lower triangular matrices

$$\begin{pmatrix} p^{k_1} & & \\ a_{ij} & \ddots & \\ & & p^{k_n} \end{pmatrix}$$

in which, $k_i = 0$ or $1$ and $\sum k_i = l$, and $0 \le a_{ij} < p^{k_j(1-k_i)}$.

We show that these are also representatives in $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{Z})T_l\mathrm{SL}_n(\mathbb{Z})$. First notice that, for any $g \in \mathrm{GL}_n(\mathbb{Z})$, we have $\det(g) = \pm 1$. If there are $g_1, g_2 \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$g_1 T_l g_2 = \begin{pmatrix} p^{k_1} & & \\ a_{ij} & \ddots & \\ & & p^{k_n} \end{pmatrix},$$

since $\det(T_1)$ and the determinant of the matrix on the right side are both positive, then we have $\det(g_1) = \det(g_2) = \pm 1$. If $\det(g_1) = \det(g_2) = -1$, we replace $g_1, g_2$ by $g_1 F, F g_2$ where $F = \mathrm{diag}(-1, 1, \ldots, 1)$. We see that, since $T_l$ is diagonal, $F T_l F = T_l$. Hence

$$g_1 F T_l F g_2 = g_1 T_l g_2.$$

However, this time we have $\det(g_1 F) = \det(F g_2) = 1$. Namely $g_1 F, F g_2 \in \mathrm{SL}_n(\mathbb{Z})$. Thus, the matrices

$$\begin{pmatrix} p^{k_1} & & \\ a_{ij} & \ddots & \\ & & p^{k_n} \end{pmatrix}$$

are representatives of $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{Z})T_l\mathrm{SL}_n(\mathbb{Z})$ and they are not equivalent. Indeed, since $\mathrm{SL}_n(\mathbb{Z})$ is a subgroup of $\mathrm{GL}_n(\mathbb{Z})$, if two matrices above are equivalent under $\mathrm{SL}_n(\mathbb{Z})$, then they will also be equivalent under $\mathrm{GL}_n(\mathbb{Z})$, but from [28], V. Proposition 7.2 we already know those matrices are inequivalent representatives in $\mathrm{GL}_n(\mathbb{Z})\backslash\mathrm{GL}_n(\mathbb{Z})T_l\mathrm{GL}_n(\mathbb{Z})$.

To prove that all the $v(Z_{k_1 \ldots k_l}) = 0$, we only need to show that $v(Z_{k_1,\ldots,k_{n-1}}) = 0$, then view the diagonal elements $0 = p^{-\infty}$ in $Z_{k_1 \ldots k_l}$ when $l < n-1$.

Let $n_l = \#(\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{Z})T_l\mathrm{SL}_n(\mathbb{Z}))$ and let

$$E_l(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_l \le n} x_{i_1} x_{i_2} \cdots x_{i_l}$$

be the $l$-th elementary symmetric polynomial. Then we see that

$$n_l = E_l(p, p^2, \ldots, p^n)/p^{l(l+1)/2},$$

by counting the representatives in the form above.

Let $f_{k_1,\ldots,k_{n-1}} = \mathbb{1}_{Z_{k_1,\ldots,k_{n-1}}}$ be the characteristic function of the set $Z_{k_1,\ldots,k_{n-1}}$. We want to use the Hecke operater $T_{T_l}$ (see 3.1) to act on $f_{k_1,\ldots,k_{n-1}}$. From the decomposition of $Z_{k_1,\ldots,k_{n-1}}$ in (3.4), we see since $\mathrm{GL}_n(\mathbb{Z}_p)$ is open, $Z_{k_1,\ldots,k_{n-1}}$ is also an open set. So $f_{k_1,\ldots,k_{n-1}}$ is a continuous function. So is $T_{T_l} f_{k_1,\ldots,k_{n-1}}$. $T_{T_l} f_{k_1,\ldots,k_{n-1}}$ is left $\mathrm{SL}_n(\mathbb{Z})$-invariant by definition. Notice that, as an algebraic group, $\mathrm{SL}_n/\mathbb{Q}$ has strong approximation property ([43]). So $\mathrm{SL}_n(\mathbb{Z})$ is dense in $\mathrm{SL}_n(\mathbb{Z}_p)$. Hence, $T_{T_l} f_{k_1,\dots,k_{n-1}}$ is also left $\mathrm{SL}_n(\mathbb{Z}_p)$-invariant. Again, by the standard decomposition (3.4) of $Z_{k_1,\dots,k_{n-1}}$, to calculate $T_{T_l} f_{k_1,\dots,k_{n-1}}$, we need to check the representatives in $\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{Z}) T_l \mathrm{SL}_n(\mathbb{Z})$ act on

$$\begin{pmatrix} 0 & & & \\ & p^{k_1} & & \\ & & \ddots & \\ & & & p^{k_{n-1}} \end{pmatrix}.$$

For example, the representatives in $\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{Z}) T_1 \mathrm{SL}_n(\mathbb{Z})$ are in the form that they are lower-triangular and the diagonal contains one $p$ and $(n-1)$ 1's. The action only depends on the position of that $p$. There are $p^{n-1}$ representatives when $p$ is at the entry-(1,1). These representatives do not change

$$\begin{pmatrix} 0 & & & \\ & p^{k_1} & & \\ & & \ddots & \\ & & & p^{k_{n-1}} \end{pmatrix}.$$

if they are multiplied to the left. We see that

$$T_{T_1} f_{k_1,\dots,k_{n-1}} = \frac{1}{n_1}(p^{n-1} f_{k_1,\dots,k_{n-1}} + \triangle_1),$$

where $\triangle_1$ is some linear combination of other $f_{k_1,\dots,k_l}$'s. By iterating this procedure, we have

$$\begin{aligned} T_{T_1} f_{k_1,\dots,k_{n-1}} &= \frac{1}{n_1}(p^{n-1} f_{k_1,\dots,k_{n-1}} + \triangle_1) \\ T_{T_2} f_{k_1,\dots,k_{n-1}} &= \frac{1}{n_2}(p^{n-2} \triangle_1 + \triangle_2) \\ T_{T_3} f_{k_1,\dots,k_{n-1}} &= \frac{1}{n_3}(p^{n-3} \triangle_2 + \triangle_3) \\ &\cdots \\ T_{T_{n-1}} f_{k_1,\dots,k_{n-1}} &= \frac{1}{n_{n-1}}(p \triangle_{n-2} + f_{k_1-1,k_2-1,\dots,k_{n-1}-1}). \end{aligned}$$

Then by using the integral formula $\int T_g f dv = \det(g)^\beta \int f dv$, we get

$$\begin{aligned} n_1 p^\beta v(Z_{k_1,\dots,k_{n-1}}) &= p^{n-1} v(Z_{k_1,\dots,k_{n-1}}) + v(\triangle_1) \\ n_2 p^{2\beta} v(Z_{k_1,\dots,k_{n-1}}) &= p^{n-2} v(\triangle_1) + v(\triangle_2) \\ n_3 p^{3\beta} v(Z_{k_1,\dots,k_{n-1}}) &= p^{n-3} v(\triangle_2) + v(\triangle_3) \\ &\cdots \\ n_{n-1} p^{(n-1)\beta} v(Z_{k_1,\dots,k_{n-1}}) &= p v(\triangle_{n-2}) + p^{n\beta} v(Z_{k_1,\dots,k_{n-1}}), \end{aligned}$$

where $v(\triangle_i)$ means the value of the integral.

We set $z = v(Z_{k_1,\ldots,k_{n-1}})$ and we cancel out all the $v(\triangle_i)$'s recursively. We then have an equation

$$\begin{aligned} n_{n-1}p^{(n-1)\beta}z &- p^{n\beta}z - p\Bigg(n_{n-2}p^{(n-2)\beta}z \\ &- p^2\big(n_{n-3}p^{(n-3)\beta}z - \ldots - p^{n-2}(n_1p^\beta z - p^{n-1}z)\big)\Bigg) = 0. \end{aligned}$$

If $z \neq 0$, we can divide out $z$ and let $x = p^\beta$. Then we get an equation of $x$:

$$\begin{aligned} -x^n + n_{n-1}x^{n-1} &- pn_{n-2}x^{n-2} + p \cdot p^2 n_{n-3}x^{n-3} \\ &- \cdots + (\pm 1)p \cdot p^2 \cdots p^{n-2}n_1 x \pm p \cdot p^2 \cdots p^{n-1} = 0. \end{aligned}$$

Recall that the classical Vieta's formulas for polynomials states,

$$\prod_{i=1}^{n}(x - \lambda_i) = \Sigma_{i=0}^{n} \pm E_{n-i}(\lambda_1, \lambda_2, \ldots, \lambda_n)x^i.$$

In our polynomial above, the coefficient in front of $x^l$ is $pp^2 \ldots p^{n-1-l}n_l$ with a possible positive or negative sign as in the alternative sum. By the definition of $n_l$, we have

$$\begin{aligned} pp^2 \ldots p^{n-1-l}n_l &= p^{(n-1-l)(n-l)/2}E_l(p, p^2, \cdots, p^n)/p^{l(l+1)/2} \\ &= p^{(n-1-l)(n-l)/2}\Big(\sum_{1 \le i_1 < \cdots < i_{n-l} \le n} p^{n(n+1)/2}/p^{i_1+i_2+\ldots+i_{n-l}}\Big)/p^{l(l+1)/2} \\ &= \sum_{1 \le i_1 < \cdots < i_{n-l} \le n} p^{n(n+1)/2+(n-1-l)(n-l)/2-l(l+1)/2-(i_1+i_2+\ldots+i_{n-l})} \\ &= \sum_{1 \le i_1 < \cdots < i_{n-l} \le n} p^{n(n-l)-(i_1+i_2+\ldots+i_{n-l})} \\ &= \sum_{1 \le i_1 < \cdots < i_{n-l} \le n} p^{(n-i_1)+(n-i_2)+\ldots+(n-i_{n-i})} \\ &= E_{n-l}(1, p, \ldots, p^{n-1}). \end{aligned}$$

So according to the Vieta's formula, we have $p^\beta = x = 1, p, \ldots, \text{or}\, p^{n-1}$. However, this is ruled out by the assumption that $\beta \neq 0, 1, 2, \ldots, n-1$ in the lemma. Thus we must have $z = 0$. Whence $v(Z_{k_1,\ldots,k_{n-1}}) = 0$, we have all $v(Z_{k_1,\ldots,k_l}) = 0$. Then $\mu_p(\mathbb{P} \times \{g \in \mathrm{Mat}_n(\mathbb{Q}_p) | \det(g) = 0\}) = v(\{g \in \mathrm{Mat}_n(\mathbb{Q}_p) | \det(g) = 0\}) = 0$. Thus, the set $\mathbb{P} \times \mathrm{GL}_n(\mathbb{Q}_p)$ is a subset of full measure with respect to $\mu_p$.

□

The following lemma shows we have similar result on $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$.

**Lemma 3.3.3.** *Let $\mu$ be a measure on $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$, which induces a probability measure on $\Gamma\backslash(\mathbb{P} \times \mathrm{Mat}_n(\hat{\mathbb{Z}}))$, such that $\mu(gB) = \det(g)^{-\beta}\mu(B)$ for $g \in \mathrm{GL}_n^+(\mathbb{Q})$ and any Borel set $B$ of $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$. Then, when $\beta \neq 0, 1, 2, \ldots, n-1$, the set*

$$\mathbb{P} \times \mathrm{Mat}'_n(\mathbb{A}_f) = \{(x, (m_p)) \in \mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f) \mid m_p \in \mathrm{GL}_n(\mathbb{Q}_p)\}$$

*has full measure.*

*Proof.* For any $\Gamma$-invariant Borel set $B \subset \mathrm{Mat}_n(\mathbb{Q}_p)$, the measure $\mu$ restricts to $\Gamma\backslash\mathbb{P} \times B \times \prod_{q\neq p} \mathrm{Mat}_n(\mathbb{Z}_q)$ and gives a measure $v_p$ on $\mathrm{Mat}_n(\mathbb{Q}_p)$. Notice that the matrix $g = \mathrm{diag}(p, \ldots, p)$ is invertible in $\mathrm{Mat}_n(\mathbb{Z}_q)$, for $q \neq p$. Thus,

$$g(\mathbb{P} \times \{0\} \times \prod_{q\neq p} \mathrm{Mat}_n(\mathbb{Z}_q)) = \mathbb{P} \times \{0\} \times \prod_{q\neq p} \mathrm{Mat}_n(\mathbb{Z}_q).$$

Since $\det(g) \neq 1$ and the set $\mathbb{P} \times \{0\} \times \prod_{q\neq p} \mathrm{Mat}_n(\mathbb{Z}_q)$ is $\Gamma$-invariant, the scaling condition of the measure implies that

$$\mu(\Gamma\backslash\mathbb{P} \times \{0\} \times \prod_{q\neq p} \mathrm{Mat}_n(\mathbb{Z}_q)) = 0.$$

One then has $v_p(\mathrm{Mat}_n(\mathbb{Z}_p)) = \mu(\Gamma\backslash\mathbb{P} \times \mathrm{Mat}_n(\hat{\mathbb{Z}})) = 1$. Thus, $v_p$ is a measure that satisfies all the conditions of the auxiliary measure $v$ constructed in the previous lemma. Thus, the set

$$\{m \in \mathrm{Mat}_n(\mathbb{Q}_p) \mid m \notin \mathrm{GL}_n(\mathbb{Q}_p)\}$$

has zero $v_p$-measure and by the definition of $v_p$ and the scaling condition on $\mu$ together with the fact $\mathrm{GL}_n^+(\mathbb{Q})(\mathrm{Mat}_n(\hat{\mathbb{Z}})) = \mathrm{Mat}_n(\mathbb{A}_f)$ (see in the proof of proposition 3.4.7)

$$\mu(\{(x, (m_q)) \in \mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f) | m_p \notin \mathrm{GL}_n(\mathbb{Q}_p)\}) = 0.$$

If we set

$$Z_p = \{(x, (m_q)) \in \mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f) | m_p \notin \mathrm{GL}_n(\mathbb{Q}_p)\},$$

then $\mathbb{P} \times \mathrm{Mat}'_n(\mathbb{A}_f)$ is the complement of the union of all the $Z_p$'s in $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$. □

Now let us discuss the nonexistence of the KMS states. In order to do this, we need the following definition (also see Definition 2.8 of [30]).

**Definition 3.3.4.** *Let $\beta \in \mathbb{R}$ and let $S$ be a semigroup such that $\Gamma \subset S \subset G$. We define*

$$\zeta(\Gamma, S, \beta) = \sum_{g\in\Gamma\backslash S} \det(g)^{-\beta}.$$

*The datum $(\Gamma, S, \beta)$ is* summable *if $\zeta(\Gamma, S, \beta) < \infty$.*

Recall also that a lower triangular integral matrix $R = (r_{ij})_{n\times n}$ is called *reduced* if $0 \le r_{ij} \le r_{jj}$, $i \ge j$.

**Proposition 3.3.5.** *([2] Lemma 3.2.7 and Exercise 3.2.10) The following identity of sets holds:*

$$\mathrm{SL}_n(\mathbb{Z})\backslash\{M \in \mathrm{Mat}_n(\mathbb{Z}) \mid \det(M) = l\} = \{R \in \mathrm{Mat}_n(\mathbb{Z}) \mid R \ \text{ reduced}, \det(R) = l\}$$

*for any positive integer $l$.*

With the help of the proposition above, we can calculate $\zeta(\Gamma, S, \beta)$ for some special choices of the semigroup $S$. Let

$$\mathrm{Mat}_n(p) = \{M \in \mathrm{Mat}_n(\mathbb{Z}) \mid \det(M) = p^l\}$$

for some prime $p$.

**Lemma 3.3.6.** *The datum $(\Gamma, \mathrm{Mat}_n(p), \beta)$ is summable only if $\beta > n-1$. In this range, the sum is given by*

$$\zeta(\Gamma, \mathrm{Mat}_n(p), \beta) = \frac{1}{(1-p^{-\beta+n-1})(1-p^{-\beta+n-2})\cdots(1-p^{-\beta})}. \tag{3.5}$$

*For $\beta > n$ we have*

$$\zeta(\Gamma, \mathrm{Mat}_n^+(\mathbb{Z}), \beta) = \zeta(\beta-n+1)\zeta(\beta-n+2)\cdots\zeta(\beta), \tag{3.6}$$

*where $\zeta(s)$ is the Riemann zeta-function.*

*Proof.* If the entry-$(j,j)$ of a reduced matrix is $p^l$, then the element in the $j$-th column under $p^l$ only has $p^l$ choices. So the $j$-th column totally has $p^{l(n-j)}$ different cases if the entry-$(j,j)$ is $p^l$. This gives us a way of computing

$$\begin{aligned}\zeta(\Gamma, \mathrm{Mat}_n(p), \beta) &= \sum_{k_1,k_2,\ldots,k_n=0}^{\infty} p^{-\beta(k_1+k_2+\cdots+k_n)} p^{k_1(n-1)} p^{k_2(n-2)} \cdots p^{k_{n-1}} \\ &= \sum_{k_1,k_2,\ldots,k_n=0}^{\infty} p^{k_1\left((n-1)-\beta\right)} p^{k_2\left((n-2)-\beta\right)} \cdots p^{k_{n-1}(1-\beta)} p^{-\beta k_n}.\end{aligned}$$

Then sum converges only if $\beta > n-1$. When $\beta > n-1$, we can sum it up and we obtain (3.5).

The Euler product formula of the Riemann zeta function states that $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$, for $\mathfrak{Re}(s) > 1$. Noticing that every integer can be represented by a finite product of powers of prime numbers, so by the same counting method on each factor, when $\beta > n$, the product of each (3.5) and the Euler product formula give us (3.6). $\square$

**Proposition 3.3.7.** (Polarization formula: [30], Lemma 2.9 and also [31], Proposition 1.2). *Let $(G, \Gamma)$ be a Hecke pair. Also let $\mu$ be the measure on $Y$ satisfying the scaling condition (3.2) for some $\beta$. Let $\nu$ be the induced measure on $\Gamma\backslash Y$. If there is a semigroup $S$ such that $\Gamma \subset S \subset G$ and $(\Gamma, S, \beta)$ is summable, and there is a left $\Gamma$ invariant subset $Y_0 \subset Y$ such that $gY_0 \cap Y_0 = \emptyset$ if $g \in G - \Gamma$, $\Gamma$ acts on $Y_0$ freely, and the set $SY_0$ is conull with respect to $\mu$, then for any left $S$-invariant function $f \in L^2(\Gamma\backslash Y, d\nu)$, we have*

$$\int_{\Gamma\backslash Y} f d\nu = \zeta(\Gamma, S, \beta) \int_{\Gamma\backslash Y_0} f d\nu. \tag{3.7}$$

*As a consequence, if $P$ is the projection operator from $L^2(\Gamma\backslash Y)$ to its subspaces of $S$-invariant functions, then the following projection formula holds,*

$$Pf = T_S(f), \tag{3.8}$$

*where $T_S$ is the Hecke operator given by the formula*

$$T_S(f)(x) = \frac{1}{\zeta(\Gamma, S, \beta)} \sum_{g \in \Gamma\backslash S/\Gamma} \det(s)^{-\beta} \#(\Gamma/(g^{-1}\Gamma g \cap \Gamma)) T_g f(x).$$

When $\beta \neq 0, 1, \cdots, n-1$, for some prime $p$ we take

$$Y_p = \mathbb{P} \times \mathrm{GL}_n(\mathbb{Z}_p) \times \prod_{q \neq p} \mathrm{Mat}_n(\mathbb{Z}_q),$$

and

$$\mathrm{GL}_n(p) = \mathrm{GL}_n^+(\mathbb{Z}[1/p]).$$

Lemma 3.3.2 implies this set $Y_p$ has the correct properties of the set $Y_0$ described in Proposition 3.3.7 with respect to the group $G = \mathrm{GL}_n(p)$ and the semigroup $S = \mathrm{Mat}_n(p)$.

If $J$ is a finite set of primes, say $J = \{p_1, p_2, \ldots, p_l\}$, $l < \infty$, we let $\mathrm{GL}_n(J) = \mathrm{GL}_n^+(\mathbb{Z}[p_1^{-1}, p_2^{-1}, \ldots, p_l^{-1}])$ and $\mathrm{Mat}_n(J) = \{M \in \mathrm{Mat}_n(\mathbb{Z}) \mid \det(M) = p_1^{a_1} p_2^{a_2} \ldots p_l^{a_l}\}$. Whence we set

$$Y_J = \mathbb{P} \times \prod_{p \in J} \mathrm{GL}_n(\mathbb{Z}_p) \times \prod_{q \notin J} \mathrm{Mat}_n(\mathbb{Z}_q).$$

Corollary 3.3.3 shows the set $Y_J$ has the correct properties of the set $Y_0$ described in Proposition 3.3.7 for the group $G = \mathrm{GL}_n(J)$ and the semigroup $S = \mathrm{Mat}_n(J)$.

Let us take another look at

$$\zeta(\Gamma, \mathrm{Mat}_n(p), \beta) = \sum_{g \in \Gamma\backslash \mathrm{Mat}_n(p)} \det(g)^{-\beta}.$$

Let $\Gamma$ act on the right coset $\Gamma g$ from the right. So, the stabilizers are $g^{-1}\Gamma g \cap \Gamma$. If we see $\Gamma g \Gamma$ as a set of right coset in $\Gamma \backslash \mathrm{Mat}_n(p)$, then there are $\#(\Gamma/(g^{-1}\Gamma g \cap \Gamma))$ right cosets there. So we can also calculate $\zeta(\Gamma, \mathrm{Mat}_n(p), \beta)$ in terms of double cosets,

$$\zeta(\Gamma, \mathrm{Mat}_n(p), \beta) = \sum_{g \in \Gamma \backslash \mathrm{Mat}_n(p)/\Gamma} \det(g)^{-\beta} \#(\Gamma/(g^{-1}\Gamma g \cap \Gamma)).$$

Now, if $g_1,\ g_2 \in \mathrm{Mat}_n(p)$ and $\Gamma g_1 \Gamma \neq \Gamma g_2 \Gamma$, then $\Gamma g_1 \mathrm{GL}_n(\mathbb{Z}_p) \neq \Gamma g_2 \mathrm{GL}_n(\mathbb{Z}_p)$. Actually, if $\Gamma g_1 \mathrm{GL}_n(\mathbb{Z}_p) = \Gamma g_2 \mathrm{GL}_n(\mathbb{Z}_p)$, we have $\gamma g_1 = g_2 g'$ for some $\gamma \in \Gamma$ and $g' \in \mathrm{GL}_n(\mathbb{Z}_p)$. So $g' \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}_p)$ and has determinant 1. This implies that $g' \in \Gamma$. It contradicts that $\Gamma g_1 \Gamma \neq \Gamma g_2 \Gamma$. On the other hand, if we let $\mathrm{Mat}'_n(\mathbb{Z}_p) = \mathrm{Mat}_n(\mathbb{Z}_p) \cap \mathrm{GL}_n(\mathbb{Q}_p)$, then we see $\mathrm{Mat}'_n(\mathbb{Z}_p) = \mathrm{Mat}_n(p)\mathrm{GL}_n(\mathbb{Z}_p)$.

From the discussion above, we have a disjoint union

$$\mathbb{P} \times \mathrm{Mat}'_n(\mathbb{Z}_p) \times \prod_{q \neq p} \mathrm{Mat}_n(\mathbb{Z}_q) = \biguplus_{g \in \Gamma \backslash \mathrm{Mat}_n(p)/\Gamma} g Y_p.$$

From Lemma 3.3.3, we see $\mathbb{P} \times \mathrm{Mat}'_n(\mathbb{Z}_p) \times \prod_{q \neq p} \mathrm{Mat}_n(\mathbb{Z}_q)$ is a subset of full measure of $Y$. So by the formula (3.3) we have,

$$1 = \nu(\Gamma \backslash Y) = \zeta(\Gamma, \mathrm{Mat}_n(p), \beta)\nu(\Gamma \backslash Y_p).$$

Where $\nu$ is the induced probability measure on $\Gamma \backslash Y$ and we see $\zeta(\Gamma, \mathrm{Mat}_n(p), \beta)$ as a sum. However, this makes sense only if $(\Gamma, \mathrm{Mat}_n(p), \beta)$ is summable.

By the correspondence of the $\mathrm{KMS}_\beta$ states and the Radon measures on $Y$ with the scaling condition (3.2), thus, we can summarize the result as the following statement.

**Proposition 3.3.8.** *For the $\mathrm{GL}_n$-Connes-Marcolli system, when $\beta \neq 0, 1, \ldots, n-1$, the $KMS_\beta$ states exist only if $\beta > n-1$.*

To prove our main theorem we also need the following theorem.

**Theorem 3.3.9.** (Real approximation theorem, [33]). *Let $G$ be a connected algebraic group over $\mathbb{Q}$. Then $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$, where $G(\mathbb{R})$ is the real Lie group. In particular, $\mathrm{GL}_n(\mathbb{Q})$ is dense in $\mathrm{GL}_n(\mathbb{R})$.*

Now we want to show the uniqueness of the KMS state when $n-1 < \beta \leq n$. Because of the one-to-one correspondence between the KMS states and the Borel measures satisfying the conditions in Theorem 3.2.5, we only need to show the uniqueness of the measure. To show the uniqueness of the measure, a standard method is to use an argument based on ergodicity (for a full discussion, we refer the readers to [30], Proof of Theorem 4.2). This main idea in this type of argument is that the measures with the desired properties in Theorem 3.2.5 form a (Choquet) simplex and the ones with ergodic action are the vertices. If the group action with respect to every measure is ergodic, then the simplex must be made of just one point. We also need to mention a common technique used in proving ergodicity: a group $G$ acts on a probability space $(X, \mu)$ ergodically if and only if the subspace of all $G$-invariant functions in $L^2(X)$ consists of constant functions.

**Theorem 3.3.10.** *Let $n-1 < \beta \leq n$. Let $\mu$ be a Borel measure on $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$ and $\nu$ be the corresponding measure on $\Gamma\backslash\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$ induced by $\mu$, such that $\mu$ satisfies the scaling condition* (3.2) *with respect to $\beta$ for any Borel set $B$ and $\nu$ is a probability measure. The action of $\mathrm{GL}_n^+(\mathbb{Q})$ on $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$ is ergodic with respect to the measure $\mu$.*

*Proof.* We use the same strategy as in [30]. The proof will be done in two steps. These two steps are based on the [30], Proposition 4.6. But for the sake of completeness, it is worth reviewing the details of the [30], Proposition 4.6. We present the two steps as a specified argument of the Proposition 4.6 in [30].

First, we show that $\mathrm{GL}_n^+(\mathbb{Q})$ acts on $(\mathbb{P} \times (\mathrm{Mat}_n(\mathbb{A}_f)/\mathrm{GL}_n(\hat{\mathbb{Z}})), \mu)$ ergodically. Here and in the following, the quotients always should be interpreted from the measure-theoretic point of view. The action of $\mathrm{GL}_n(\hat{\mathbb{Z}})$ on $\mathbb{P}\times(\mathrm{Mat}_n(\mathbb{A}_f)$ from the right by multiplying to the factor $\mathrm{Mat}_n(\mathbb{A}_f)$ is compatible with the left action of $\mathrm{GL}_n^+(\mathbb{Q})$. So to say the $\mathrm{GL}_n^+(\mathbb{Q})$-action on $(\mathbb{P} \times (\mathrm{Mat}_n(\mathbb{A}_f)/\mathrm{GL}_n(\hat{\mathbb{Z}})), \mu)$ is ergodic is the same as to say that the $\mathrm{GL}_n(\hat{\mathbb{Z}})$-action on the measure theoretic quotient space $(\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)/\mathrm{GL}_n^+(\mathbb{Q}), \mu)$ is ergodic.

As a second step, since $\mathrm{GL}_n(\hat{\mathbb{Z}})$ is compact, the measure $\mu$ is supported only in one orbit (see [53], corollary 2.1.13), so we can think of $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)/\mathrm{GL}_n^+(\mathbb{Q})$ as a measure theoretic quotient space $\mathrm{GL}_n(\hat{\mathbb{Z}})/H$, for some subgroup $H$ of $\mathrm{GL}_n(\hat{\mathbb{Z}})$. Noticing that $\mathbb{P}$ is actually a group, if we define the $\mathbb{P}$ action on $\mathbb{P}\times\mathrm{Mat}_n(\mathbb{A}_f)$ as a multiplier on $\mathbb{P}$ from the right, then this action is compatible with the diagonal action of $\mathrm{GL}_n^+(\mathbb{Q})$ on $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$ from the left. The group $\mathbb{P}$ acts on the quotient space $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)/\mathrm{GL}_n^+(\mathbb{Q})$ hence on $\mathrm{GL}_n(\hat{\mathbb{Z}})/H$. The $\mathbb{P}$-action on $\mathrm{GL}_n(\hat{\mathbb{Z}})/H$ is continuous and $\mathrm{GL}_n(\hat{\mathbb{Z}})$ is totally disconnected, so it is a trivial action. If we can show that the action of $\mathbb{P}$ is ergodic, then we get that all the $\mathrm{GL}_n^+(\mathbb{Q})$-invariant functions on $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$ are constant. Thus the $\mathrm{GL}_n^+(\mathbb{Q})$ action is ergodic.

*(1) $\mathrm{GL}_n^+(\mathbb{Q})$ acts on $(\mathbb{P} \times (\mathrm{Mat}_n(\mathbb{A}_f)/\mathrm{GL}_n(\hat{\mathbb{Z}})), \mu)$ ergodically.*

Before showing this, we observe the fact that any continuous $\mathrm{Mat}_n^+(\mathbb{Z})$-invariant function $f$ on $\Gamma\backslash\mathbb{P}$ is constant. Thus, we can view $f$ as a $\Gamma$-invariant function on $\mathbb{P}$. This is true, because for any $\Gamma$-invariant function $f$ on $\mathbb{P}$ that is also $\mathrm{Mat}_n^+(\mathbb{Z})$ invariant, we have the following facts. First, for any $g \in \mathrm{GL}_n^+(\mathbb{Q})$, there is a $k \in \mathrm{Mat}_n^+(\mathbb{Z})$ making $kg \in \mathrm{Mat}_n^+(\mathbb{Z})$, hence $f(gx) = f(kgx) = f(x)$. Namely, $f$ is $\mathrm{GL}_n^+(\mathbb{Q})$-invariant. Second, $\mathrm{GL}_n(\mathbb{Q})$ is dense in $\mathrm{GL}_n(\mathbb{R})$ by the real approximation theorem. It follows that $f$ is also $\mathrm{GL}_n^+(\mathbb{R})$-invariant, i.e., $f$ is constant on $\Gamma\backslash\mathbb{P}$.

Let $J$ be some finite set of primes. For any bounded Borel function $f$ over $\Gamma\backslash\mathbb{P}$, we define $f_J = f\mathbb{1}_{Y_J}$ on $\mathbb{P} \times \mathrm{Mat}_n(\hat{\mathbb{Z}})$, so that $f_J(x,\rho) = f(x)\mathbb{1}_{Y_J}(x,\rho)$, for $(x,\,\rho) \in \mathbb{P} \times \mathrm{Mat}_n(\hat{\mathbb{Z}})$. We also need two operators. Consider the Hilbert space $L^2(\Gamma\backslash(\mathbb{P} \times \mathrm{Mat}_n(\hat{\mathbb{Z}})),\nu)$, where $\nu$ is the probability measure on $\Gamma\backslash(\mathbb{P} \times \mathrm{Mat}_n(\hat{\mathbb{Z}}))$ induced by $\mu$. Let $P_J$ be the projection to the subspace of $\mathrm{Mat}_n(J)$-invariant functions, while $P$ is the projection to the subspace of $\mathrm{Mat}_n^+(\mathbb{Z})$-invariant functions.

For another finite set $F$ of primes disjoint from $J$, by the projection formula, we have

$$P_F f_J = (T_F f)_J, \tag{3.9}$$

where $T_F = T_{\mathrm{Mat}_n(F)}$ with respect to the action of $\mathrm{GL}_n^+(\mathbb{Q})$ on $\Gamma\backslash\mathbb{P}$. By definition, $T_F f$ is $\mathrm{Mat}_n(F)$-invariant. From the observation at the beginning of this step, we see an $\mathrm{Mat}_n^+(\mathbb{Z})$-invariant function on $\Gamma\backslash\mathbb{P}$ is also $\mathrm{GL}_n^+(\mathbb{Q})$-invariant. Similarly, we can see an $\mathrm{Mat}_n(J)$-invariant function on $\Gamma\backslash\mathbb{P}$ is also $\mathrm{Mat}_n^{-1}(J)\mathrm{Mat}_n(J)$-invariant, where $\mathrm{Mat}_n^{-1}(J) = \{g^{-1} \mid g \in \mathrm{Mat}_n(J)\}$ and $\mathrm{Mat}_n^{-1}(J)\mathrm{Mat}_n(J)$ is the group generated by $\mathrm{Mat}_n^{-1}(J)$ and $\mathrm{Mat}_n(J)$. Notice that $PP_F = P$ and $P = \lim_I P_I$, where $I$ runs through all the finite set of primes. Then from equation (3.9) we see that $Pf_J$ is in the form of $Ph_J$ in which $h$ is some $G_J$-invariant function. Where

$$G_J = \{g \in \mathrm{GL}_n^+(\mathbb{Q}) \mid \det(g) = p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k},\, p_i \notin J\}$$

is a dense subgroup of $\mathrm{GL}_n^+(\mathbb{Q})$. Hence $G_J$ is also dense in $\mathrm{GL}_n^+(\mathbb{R})$. Thus, $h = a$ is a constant. We then have

$$Pf_J = aP\mathbb{1}_{\Gamma\backslash Y_J}$$

by the definition of the restriction operator $f \mapsto f_J$. Since $PP_J = P$ and $P_J\mathbb{1}_{\Gamma\backslash Y_J}$ is a constant by the projection formula (3.7), we obtain that

$$Pf_J = aP\mathbb{1}_{\Gamma\backslash Y_J} = aPP_J\mathbb{1}_{\Gamma\backslash Y_J}$$

is a constant.

We always regard a $\prod_{p\in J}\mathrm{GL}_n(\mathbb{Z}_p)$-right-invariant function $f$ on the quotient

$$\Gamma\backslash(\mathbb{P}\times\prod_{p\in J}\mathrm{Mat}_n(\mathbb{Z}_p))$$

as a $\mathrm{GL}_n(\hat{\mathbb{Z}})$-right-invariant function on the quotient $\Gamma\backslash(\mathbb{P}\times\mathrm{Mat}_n(\hat{\mathbb{Z}}))$. Moreover, all such functions for all $J$'s are dense in the space of $\mathrm{GL}_n(\hat{\mathbb{Z}})$-right-invariant functions. So we can simply restrict our observation to $Pf$. To calculate $Pf$, we need the operator $P_J$ again and the fact that $PP_J = P$. From the formula (3.8), we see $P_J$ is compatible with the right $\prod_{p\in J}\mathrm{GL}_n(\mathbb{Z}_p)$ action (seen as $\mathrm{GL}_n(\hat{\mathbb{Z}})$ action). So $P_Jf$ is still $\mathrm{GL}_n(\hat{\mathbb{Z}})$ invariant. Since $\mathrm{Mat}_n(J)Y_J$ is of full measure, we can think $P_Jf$ is supported in $\mathrm{Mat}_n(J)Y_J$. Recall the construction of $Y_J$. The middle part of $Y_J$ is exactly the group $\prod_{p\in J}\mathrm{GL}_n(\mathbb{Z}_p)$. The property of $\mathrm{GL}_n(\hat{\mathbb{Z}})$ invariance implies that $P_Jf$ is only determined by the factor in $\mathbb{P}$. Namely $P_Jf$ is actually in the form of $g_J$ for some $g$ on $\Gamma\backslash\mathbb{P}$. So $Pf = PP_Jf = Pg_J$. By the previous paragraph, $Pg_J$ is a constant, so is $Pf$. This shows all the $\mathrm{GL}_n^+(\mathbb{Q})$ invariant functions on

$$(\mathbb{P}\times(\mathrm{Mat}_n(\mathbb{A}_f)/\mathrm{GL}_n(\hat{\mathbb{Z}})),\mu)$$

are constants. So the ergodicity follows.

*(2) The group $\mathbb{P}$ acts on the quotient space $\mathbb{P}\times\mathrm{Mat}_n(\mathbb{A}_f)/\mathrm{GL}_n^+(\mathbb{Q})$ ergodically.*

Recall the group $\mathbb{P}$ acts on $\mathbb{P}\times\mathrm{Mat}_n(\mathbb{A}_f)$ at the first factor from the right. So the $\mathbb{P}$ action and the $\mathrm{GL}_n^+(\mathbb{Q})$ action are compatible. We show that the action of $\mathrm{GL}_n^+(\mathbb{Q})$ on $\mathbb{P}\times\mathrm{Mat}_n(\mathbb{A}_f)/\mathbb{P} = \mathrm{Mat}_n(\mathbb{A}_f)$ is ergodic.

Let $P$ be the projection from the Hilbert space $L^2(\mathrm{Mat}_n(\hat{\mathbb{Z}}))$ to the subspace of $\mathrm{Mat}_n^+(\mathbb{Z})$ invariant functions. It is enough to show that $Pf$ is a constant function for $f\in L^2(\mathrm{Mat}_n(\hat{\mathbb{Z}}))$.

For any $J$, we need to show that all the $\mathrm{GL}_n^+(\mathbb{Q})$ invariant functions on the product $\prod_J\mathrm{Mat}_n(\mathbb{Q}_p)$ are constant. Since $\prod_J\mathrm{GL}_n(\mathbb{Q}_p)$ is of full measure in $\prod_J\mathrm{Mat}_n(\mathbb{Q}_p)$, we show that $\mathrm{GL}_n^+(\mathbb{Q})$ invariant functions on $\prod_J\mathrm{GL}_n(\mathbb{Q}_p)$ are constant. This is equivalent to the fact that $\mathrm{Mat}_n^+(\mathbb{Z})$ invariant functions on $\prod_J\mathrm{GL}_n(\mathbb{Z}_p)$ are constant. Once this is true for a $J$, we can vary all the possible $J$'s. Because that all the such functions for all $J$'s span a dense subspace of the space of $\mathrm{Mat}_n^+(\mathbb{Z})$ invariant functions, it follows that all the $\mathrm{Mat}_n^+(\mathbb{Z})$ invariant functions are constant.

To find $Pf$, we notice that $\Gamma\subset\mathrm{Mat}_n^+(\mathbb{Z})$, so we always can map $f$ to the subspace of $\Gamma$ invariant functions first. We therefore assume that $f$ is $\Gamma$ invariant. We define

$$f_J = f\mathbb{1}_{\prod_J\mathrm{GL}_n(\mathbb{Z}_p)},\quad\text{such that}\quad f_J((m_p)_p) = f((m_p)_p)\mathbb{1}_{\prod_J\mathrm{GL}_n(\mathbb{Z}_p)}((m_p)_p),$$

on $\mathrm{Mat}_n(\hat{\mathbb{Z}})$, for any $\Gamma$ invariant function $f$ over $\prod_J \mathrm{GL}_n(\mathbb{Z}_p)$. The group $\Gamma$ is dense in $\mathrm{SL}_n(\hat{\mathbb{Z}})$, hence $f$ only depends on the value of $\det(m) \in \prod_J \mathbb{Z}_p^*$. Thus, it is of the form $f(m) = \chi(\det(m))$, where $\chi$ is a character in the dual group of $\prod_J \mathbb{Z}_p^*$. Thus, if $\chi$ is trivial, then $f_J = \mathbb{1}_{Y_J}$. So $Pf_J$ is a constant as we showed in step (1). When $\chi$ is not trivial, we use the projection formula (3.8), and we find

$$P_F f_J(m) = \chi(\det(m_p)_{p\in J}) \times \prod_{q\in F} \frac{(1-p^{-\beta})(1-p^{-(\beta-1)})\cdots(1-p^{-(\beta-n+1)})}{(1-\chi(p)p^{-\beta})(1-\chi(p)p^{-(\beta-1)})\cdots(1-\chi(p)p^{-(\beta-n+1)})}, \tag{3.10}$$

where $F$ is a set of finite primes that is disjoint from $J$.

We see that $P = \lim_F P_F$ and $PP_F = P$. If we keep enlarging $F$, by the properties of Dirichlet series ([39], Lemma VII.13.3), whence $n-1 < \beta \leq n$, the numerator of the right hand side of (3.10) is approaching to 0, while the denominator is approaching a nonzero number (or infinity). So $Pf_J = 0$. In both cases, we find that $Pf_J$ is constant. Then, by varying all the possible $J$'s, we can see that all the $Pf$'s are constant functions.

□

## 3.4 The Existence of the KMS States

In the previous section, we have shown there is no KMS states when $\beta \in (-\infty, 0) \cup (0,1) \cup (1,2) \cup \cdots \cup (n-1, n)$ and the uniqueness of the KMS state when $n-1 < \beta \leq n$. In the section, we discuss how to construct the KMS states in the possible intervals and the existence at the dividing points of $\beta = 0, 1, 2, \ldots, n-1$.

### 3.4.1 The Case $\beta = 0$

When $\beta = 0$, the Lemma 3.2.3 fails. Actually, the Haar measure on $\mathbb{P}$ does define a $\mathrm{KMS}_0$ state on the subalgebra $\mathcal{B} = \mathrm{C}_r^*(\Gamma\backslash\mathrm{GL}_n^+(\mathbb{Q}) \times_\Gamma \mathbb{P})$. Moreover, recall $\mathcal{J} = \mathrm{C}_r^*(\Gamma\backslash G \boxtimes_\Gamma (Y - (\mathbb{P}\times\{0\})))$. So $\mathcal{B} = \mathcal{A}/\mathcal{J}$. The following proposition shows the $\mathrm{KMS}_0$ states only come from $\mathcal{B}$.

**Proposition 3.4.1.** *There are no KMS states on the $\mathcal{J}$ for $\beta = 0$.*

*Proof.* In this case, the KMS states are tracial. If there is a trace, say $\varphi$, on $\mathcal{J}$, $\varphi$ gives a nonzero $\mathrm{GL}_n^+(\mathbb{Q})$ invariant measure on $\mathbb{P} \times (\mathrm{Mat}_n(\mathbb{A}_f) - \{0\})$. We can similarly define

$$v(B) = \mu(\Gamma\backslash\mathbb{P} \times B \times \prod_{q\neq p}(\mathrm{Mat}_n(\mathbb{Z}_q) - \{0\}))$$

for $B \subset (\mathrm{Mat}_n(\mathbb{Q}_p) - \{0\})$ and $\Gamma$-invariant. So if there is a $Z_{k_1 \dots k_l}$ with $v(Z_{k_1 \dots k_l} \neq 0)$, then

$$v(Z_{k_1+1 \dots k_l+1}) = v(pZ_{k_1 \dots k_l}) \neq 0.$$

So, $0 \neq v(Z_{k_1 \dots k_l}) = v(Z_{k_1+1 \dots k_l+1}) = v(Z_{k_1+2 \dots k_l+2}) = \cdots$. Notice all these $Z_{k_1 \dots k_l}$'s are disjoint in $\mathrm{Mat}_n(\mathbb{Z}_p)$ and

$$v(\bigcup_{n \geq 0} Z_{k_1+n \dots k_l+n}) = \sum_{n \geq 0} v(Z_{k_1+n \dots k_l+n}) = \infty.$$

So this contradicts the fact that $v(\mathrm{Mat}_n(\mathbb{Z}_p)) < \infty$, since $\mathrm{Mat}_n(\mathbb{Z}_p)$ is compact. Then we have $v(Z_{k_1 \dots k_l}) = 0$ for all $Z_{k_1 \dots k_l}$. Again, under this condition, following the argument in Lemma 3.3.3, we have the similar result that $\mathbb{P} \times \mathrm{Mat}'_n(\mathbb{A}_f)$ is of full measure. We again let $\nu$ be the measure induced by $\mu$ on $\Gamma \backslash Y$. By formulas (3.3), we have

$$\nu(\Gamma \backslash Y) = \zeta(\Gamma, \mathrm{Mat}_n(p), 0) \nu(\Gamma \backslash Y_p).$$

This is a contradiction, because $\zeta(\Gamma, \mathrm{Mat}_n(p), 0) = \infty$ but $\nu(\Gamma \backslash Y)$ and $\nu(\Gamma \backslash Y_p)$ are both positive finite numbers. So $\mu \equiv 0$ and there is no KMS state on $\mathcal{J}$ for $\beta = 0$. □

### 3.4.2 The Cases $\beta = 1, 2, \dots, n-1$

First let $\mu$ be a Borel measure corresponding to some $\mathrm{KMS}_\beta$ state as in Theorem 3.2.5′ and let $\nu$ be the induced probability measure on $\Gamma \backslash Y$. Note that by formula (3.3), we have

$$\nu(\Gamma \backslash \mathrm{Mat}_n(p) Y_p) = \zeta(\Gamma, \mathrm{Mat}_n(p), \beta) \nu(\Gamma \backslash Y_p).$$

When $\beta = 1, 2, \dots, n-1$, the series defining $\zeta(\Gamma, \mathrm{Mat}_n(p), \beta)$ is divergent. So $\nu(\Gamma \backslash Y_p) = 0$. Let $p$ run over all the primes, we conclude that the set

$$\mathbb{P} \times \{M = (m_p)_p \in \mathrm{Mat}_n(\mathbb{A}_f) \mid \det(m_p) = 0\}$$

is a set of full measure. Moreover, as shown in Corollary 3.3.3, the set

$$\mathbb{P} \times \{0\} \times \prod_{q \neq p} \mathrm{Mat}_n(\mathbb{Z}_q)$$

is always of $\mu$-measure 0. So the set $\mathbb{P} \times \mathrm{Mat}_n^{0,\beta}(\mathbb{A}_f)$, where

$$\mathrm{Mat}_n^{0,\beta}(\mathbb{A}_f) = \{M = (m_p)_p \in \mathrm{Mat}_n(\mathbb{A}_f) \mid m_p \neq 0, \, \det(m_p) = 0\},$$

is of full measure with respect to $\mu$.

Moreover, actually there are a lot of such measures.

**Proposition 3.4.2.** *Let $\beta = k \in \{1, 2, \ldots, n-1\}$ and let $\mu_k = \mu_{\mathbb{P}} \times \mu'$, where $\mu_{\mathbb{P}}$ is a $\mathrm{GL}_n^+(\mathbb{Q})$ invariant measure on $\mathbb{P}$ and $\mu'$ is the Haar measure on $\mathbb{A}_f^{kn} \simeq \mathrm{Mat}_n^k(\mathbb{A}_f)$. Here*

$$\mathrm{Mat}_n^k(\mathbb{A}_f) = \{M \in \mathrm{Mat}_n(A) \mid M = (\mathbf{0}_{n-k}|N_k)\},$$

*and $\mathbf{0}_{n-k}$ is the zero matrix of size $n \times (n-k)$ and $N_k \in \mathrm{Mat}_{n\times k}(\mathbb{A}_f)$. For $g \in \mathrm{GL}_n(\hat{\mathbb{Z}})$, the measure $\mu_{kg} = \mu_k(\,\cdot\, g^{-1})$ with support in $\mathbb{P} \times \mathrm{Mat}_n^k(\mathbb{A}_f)g$ satisfies the scaling condition (3.2), hence it defines a KMS$_\beta$ state.*

*Proof.* As an additive group, $\mathrm{Mat}_n^k(\mathbb{A}_f)$ is the same as $\mathbb{A}_f^{kn}$. So the Haar measure on $\mathbb{A}_f^{kn}$ gives a measure $\mu'$ on $\mathrm{Mat}_n(\mathbb{A}_f)$ supported in $\mathrm{Mat}_n^k(\mathbb{A}_f)$. Let $\mu_k = \mu_{\mathbb{P}} \times \mu'$. The measure $\mu_k$ also naturally satisfies the scaling condition (3.2) for $\beta = k$ (One can think of the case of the Lebesgue measure over $\mathbb{R}^n$ which is a Haar measure if we see $\mathbb{R}^n$ is an additive group). We also can normalize $\mu_k$ so that the corresponding measure $\nu_k$ is a probability measure on $\Gamma\backslash Y$. So a KMS$_\beta$ state has been constructed. Let $g \in \mathrm{GL}_n(\hat{\mathbb{Z}})$. So $g$ acts on $\mathbb{P} \times \mathrm{Mat}_n(\mathbb{A}_f)$ from the right by multiplying to the second factor on the right. Then we define $\mu_{kg} = \mu_k(\,\cdot\, g^{-1})$. The measure $\mu_{kg}$ is also a measure satisfying the scaling condition (3.2) with support in $\mathbb{P} \times \mathrm{Mat}_n^k(\mathbb{A}_f)g$. □

Thus, we have obtained a construction of a non-empty set of KMS$_\beta$ states for each $\beta = k \in \{1, 2, \ldots, n-1\}$.

### 3.4.3 The $\mathrm{GL}_2$-Case for $\beta = 1$

Now, let us specifically turn to the case $n = 2$, $\beta = 1$. In this case, $\mathbb{P} = \mathrm{PGL}_2^+(\mathbb{R}) = \mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^*$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We let

$$\mathrm{Mat}_2^1(\mathbb{A}_f)' = \{B = (b_p)_p \in \mathrm{Mat}_2^1(\mathbb{A}_f) \mid b_p \neq 0\}.$$

Let $M = (m_p)_p \in \mathrm{Mat}_2(\mathbb{A}_f)$ such that $m_p \neq 0$ and $\det(m_p) = 0$ for all $p$. We have

$$m_p = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Q}_p).$$

Since $m_p \neq 0$ and $\det(m_p) = 0$, the matrix $m_p$ has rank 1. So

$$\begin{pmatrix} n_{11} \\ n_{21} \end{pmatrix} = ap^\alpha \begin{pmatrix} n_{12} \\ n_{22} \end{pmatrix},$$

where $a$ is invertible in $\mathbb{Z}_p$. We can assume $\alpha > 0$, if not we multiply the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p)$$

to the right of $m_p$ to swap the columns. Let

$$g_p = \begin{pmatrix} 1 & 0 \\ -ap^{\alpha} & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p)$$

and

$$m_p g_p = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -ap^{\alpha} & 1 \end{pmatrix} = \begin{pmatrix} 0 & n_{12} \\ 0 & n_{22} \end{pmatrix} \in \mathrm{Mat}_2^1(\mathbb{Q}_p)'.$$

Considering all $p$'s, we actually show that for any $M = (m_p)_p \in \mathrm{Mat}_2(\mathbb{A}_f)$ such that $m_p \neq 0$ and $\det(m_p) = 0$, there is a $g = (g_p)_p \in \mathrm{GL}_2(\hat{\mathbb{Z}}) = \prod_p \mathrm{GL}_2(\mathbb{Z}_p)$ such that $Mg \in \mathrm{Mat}_2^1(\mathbb{A}_f)'$.

Let $B = (b_p)_p \in \mathrm{Mat}_2^1(\mathbb{A}_f)'$ and $h = (h_p)_p \in \mathrm{GL}_2(\hat{\mathbb{Z}})$. We want to see what happens when $Bh \in \mathrm{Mat}_2^1(\mathbb{A}_f)'$. For some $p$, we see that

$$b_p = \begin{pmatrix} 0 & c_1 \\ 0 & c_2 \end{pmatrix}$$

and $c_1$, $c_2$ can not be both 0. If

$$h_p = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then

$$b_p h_p = \begin{pmatrix} a_{21}c_1 & a_{22}c_1 \\ a_{21}c_2 & a_{22}c_2 \end{pmatrix}.$$

If $Bh \in \mathrm{Mat}_2^1(\mathbb{A}_f)'$, this only can happen if $a_{21} = 0$. Since $p$ is taken arbitrarily, we actually have shown that $h$ is an upper triangular matrix in $\mathrm{GL}_2(\hat{\mathbb{Z}})$, i.e. the stabilizer of $\mathrm{Mat}_2^1(\mathbb{A}_f)'$ is the subgroup of upper triangular matrices in $\mathrm{GL}_2(\hat{\mathbb{Z}})$, which is denoted by $U$. Immediately it follows that, if

$$\mathrm{Mat}_2^1(\mathbb{A}_f)' M_1 \bigcap \mathrm{Mat}_2^1(\mathbb{A}_f)' M_2 \neq \varnothing,$$

then $M_1 = uM_2$ with $u \in U$. Combining with the previous paragraph, we have shown that

$$\{\mathrm{Mat}_2^1(\mathbb{A}_f)' g\}_{g \in U \backslash \mathrm{GL}_2(\hat{\mathbb{Z}})}$$

forms a partition of the set $\mathrm{Mat}_2^{0,1}(\mathbb{A}_f)$ and there is a map

$$\pi : \mathrm{Mat}_2^{0,1}(\mathbb{A}_f) \to U \backslash \mathrm{GL}_2(\hat{\mathbb{Z}})$$

such that $\pi(M) = g$ if $M \in \mathrm{Mat}_2^1(\mathbb{A}_f)' g$. We also consider this map as

$$\pi : \mathbb{P} \times \mathrm{Mat}_2^{0,1}(\mathbb{A}_f) \to U \backslash \mathrm{GL}_2(\hat{\mathbb{Z}})$$

so that $\pi(x, M) = g$ for

$$(x, M) \in \mathbb{P} \times \mathrm{Mat}_2^{0,1}(\mathbb{A}_f),$$

if $M \in \mathrm{Mat}_2^1(\mathbb{A}_f)'g$.

For $\mathrm{Mat}_2^1(\mathbb{A}_f)'g$, there is a measure $\mu_g$ supported in $\mathbb{P} \times \mathrm{Mat}_2^1(\mathbb{A}_f)'g$ and defining a $\mathrm{KMS}_\beta$ state, as we shown in Proposition 3.4.2 above. We are going to show that the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $(\mathbb{P} \times \mathrm{Mat}_2^1(\mathbb{A}_f)'g,\, \mu_g)$, with $g \in U\backslash \mathrm{GL}_2(\hat{\mathbb{Z}})$ is ergodic. To accomplish this, we need the help of the following important theorem in algebraic number theory (see [39], Chapter III, §1, Exercise 1 or [11], Chapter II, Theorem 15).

**Theorem 3.4.3.** *(Strong Approximation Theorem) Let $\mathfrak{p}_0$ be an arbitrary place of $\mathbb{Q}$ and $S$ be a finite set of places such that $\mathfrak{p}_0 \notin S$. Given any $(\alpha_\mathfrak{p})_{\mathfrak{p} \in S} \in \prod_{\mathfrak{p} \in S} \mathbb{Q}_\mathfrak{p}$ and $\epsilon > 0$, there exists some $c \in \mathbb{Q}$ such that $|\, c - \alpha_\mathfrak{p}|_\mathfrak{p} < \epsilon$ for $\mathfrak{p} \in S$ and $|\, c\,|_\mathfrak{p} \leq 1$ for $\mathfrak{p} \notin S, \mathfrak{p} \neq \mathfrak{p}_0$.*

*In particular, if we take $\mathfrak{p}_0 = \infty$, this says that $\mathbb{Q}$ is dense in $\mathbb{A}_f$ via the diagonal embedding.*

We then have the following result.

**Proposition 3.4.4.** *Let $\lambda$ be a quasi-invariant measure on $\mathbb{P} \times \mathrm{Mat}_2^1(\mathbb{A}_f)'$ with respect to the action of $\mathrm{GL}_2^+(\mathbb{Q})$. Then the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $(\mathbb{P} \times \mathrm{Mat}_2^1(\mathbb{A}_f)',\, \lambda)$ is ergodic. Moreover, if $\lambda$ is normalized, then such a measure is unique.*

*Proof.* Let $\mathrm{Mat}_2^1(\mathbb{A}_f)'_J$ be the image of the canonical map

$$\mathrm{Mat}_2^1(\mathbb{A}_f)' \to \prod_{p \in J} \mathrm{Mat}_2^1(\mathbb{Q}_p)$$

for some finite set $J$ of primes over $\mathbb{Q}$. Note that

$$\mathbb{P} = \mathrm{PGL}_2^+(\mathbb{R}) = \mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^*.$$

For any

$$r = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}) \quad \text{and} \quad (m_p)_{p \in J} = \left( \begin{pmatrix} m_{11}^{(p)} & m_{12}^{(p)} \\ m_{21}^{(p)} & m_{22}^{(p)} \end{pmatrix} \right)_{p \in J} \in \prod_{p \in J} \mathrm{GL}_2(\mathbb{Q}_p),$$

by the Strong Approximation Theorem above, there is a

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$$

such that

$q_{11}$ is close enough to $(r_{11}, (m_{11}^{(p)})_{p \in J})$,

$q_{22}$ is close enough to $(r_{22}, (m_{22}^{(p)})_{p \in J})$,

$q_{12}$ is close enough to $(r_{12}, (m_{12}^{(p)})_{p\in J})$,

$q_{21}$ is close enough to $(r_{21}, (m_{21}^{(p)})_{p\in J})$,

where $(r_{11}, (m_1^{(p)})_{p\in J})$ means an element in $\mathbb{R}\times\prod_{p\in J}\mathbb{Q}_p$, and we see the rational number $q_{11}$ as an element in $\mathbb{R}\times\prod_{p\in J}\mathbb{Q}_p$ via the diagonal embedding.

We see $\mathrm{GL}_n^+(\mathbb{Q})$ is a dense subgroup of $\mathrm{GL}_n^+(\mathbb{R})\times\prod_{p\in J}\mathrm{GL}_2(\mathbb{Q}_p)$. $\mathrm{GL}_n^+(\mathbb{R})\times\prod_{p\in J}\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\mathbb{P}\times\mathrm{Mat}_2^1(\mathbb{A}_f)'_J$ transitively, so the $\mathrm{GL}_2^+(\mathbb{Q})$-invariant functions on $\mathbb{P}\times\mathrm{Mat}_2^1(\mathbb{A}_f)'_J$ are almost constant. Those functions also can be seen as functions on $\mathbb{P}\times\mathrm{Mat}_2^1(\mathbb{A}_f)'$. By taking all the possible $J$'s, we find a dense subspace of $\mathrm{GL}_2^+(\mathbb{Q})$ invariant functions over $\mathbb{P}\times\mathrm{Mat}_2^1(\mathbb{A}_f)'$ consisting of constants. Thus we obtain that the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $(\mathbb{P}\times\mathrm{Mat}_2^1(\mathbb{A}_f)',\ \lambda)$ is ergodic. Moreover, we also see that, if $\lambda$ is normalized, for example $\lambda(\Gamma\backslash\mathbb{P}\times\mathrm{Mat}_2^1(\hat{\mathbb{Z}})')=1$, then such a measure is unique (*Cf.* the paragraph before Theorem 3.3.10). $\square$

Let $\rho$ be a measure on $\mathbb{P}\times\mathrm{Mat}_2^{0,1}(\mathbb{A}_f)$ that satisfies conditions in Theorem 3.2.5$'$. So $\mathrm{GL}_2^+(\mathbb{Q})$ acts on

$$(\mathbb{P}\times\mathrm{Mat}_2^1(\mathbb{A}_f)'g, \rho|_{\mathrm{Mat}_2^1(\mathbb{A}_f)'g}),$$

with $g\in U\backslash\mathrm{GL}_2(\hat{\mathbb{Z}})$ ergodically. If we set $\kappa=\rho\circ\pi^{-1}$, then

$$\rho=\int_{U\backslash\mathrm{GL}_2(\hat{\mathbb{Z}})}\mu_g d\kappa(g).$$

So we have obtained the following result, which confirms the conjecture in the Remark 4.8(ii) of [30].

**Proposition 3.4.5.** *For $n=2$, $\beta=1$, the set of extremal KMS$_1$ states on the $\mathrm{GL}_2$-Connes-Marcolli system is identified with the set of $U\backslash\mathrm{GL}_2(\hat{\mathbb{Z}})$.*

### 3.4.4 The Case $\beta\in(n-1,n]$: Existence

We have proved in Theorem 3.3.10 that, in the range $\beta\in(n-1,n]$, if the set of KMS states is non-empty, then it consists of a single point. Here we show the existence.

**Proposition 3.4.6.** *For $\beta\in(n-1,n]$, the set of KMS$_\beta$ states is non-empty, hence by Theorem 3.3.10 it consists of a unique element.*

*Proof.* A KMS state can be constructed in the following way (*cf.* [30], the first part of section 4). Let $n-1 < \beta \le n$. At place $p$, there is the normalized Haar measure $\mu_p$ on the compact group $\mathrm{GL}_n(\mathbb{Z}_p)$ such that

$$\mu_p(\mathrm{GL}_n(\mathbb{Z}_p)) = (1-p^{-\beta+n-1})(1-p^{-\beta+n-2})\cdots(1-p^{-\beta}).$$

This measure can be uniquely extended (we still call it $\mu_p$) to the group $\mathrm{GL}_n(\mathbb{Q}_p)$ by

$$\mu_p(K) = \sum_{g\in \mathrm{GL}_n(\mathbb{Z}_p)\backslash \mathrm{GL}_n(\mathbb{Q}_p)} |\det(g)|_p^{-\beta}\mu(gK\cap \mathrm{GL}_n(\mathbb{Z}_p)),$$

for compact $K \subset \mathrm{GL}_n(\mathbb{Q}_p)$. Here $|\cdot|_p$ is the standard $p$-adic norm of $\mathbb{Q}_p$. Set $\mu_{\mathbb{P}}$ to be the Haar measure of $\mathbb{P}$. Since $\Gamma$ is a lattice of $\mathrm{SL}_n(\mathbb{R})$, we normalize $\mu_{\mathbb{P}}$ so that $\mu_{\mathbb{P}}(\Gamma\backslash\mathbb{P}) = 1$. Then we let

$$\mu_\beta = \mu_{\mathbb{P}} \times \prod_p \mu_p.$$

The measure $\mu_\beta$ satisfies the conditions in Theorem 3.2.5($'$), hence there is a KMS at inverse temperature $\beta$. □

### 3.4.5 The Case $\beta > n$

As before, we use the fact that there is a one-to-one correspondence between the $\mathrm{KMS}_\beta$ states and the Borel measures satisfying the conditions in Theorem 3.2.5$'$. There may be many of these, like the one constructed in the previous subsection. Using the same argument in [30], Remark 4.8 or [14], Theorem 3.97, we focus on classifying the extremal states.

**Proposition 3.4.7.** *Let $\mu$ be a Borel measure satisfying the conditions in Theorem 3.2.5$'$, with $\beta > n$, and $\nu$ be the induced probability measure on $\Gamma\backslash Y$. Then there is a one-to-one correspondence between the set $\Gamma\backslash\mathbb{P}\times \mathrm{GL}_n(\hat{\mathbb{Z}})$ and the set of extremal $KMS_\beta$ states on $\mathcal{A}$*

*Proof.* On one hand, from the formula (3.3), we have

$$\nu(\Gamma\backslash \mathrm{Mat}_n^+(\mathbb{Z})(\mathbb{P}\times \mathrm{GL}_n(\hat{\mathbb{Z}}))) = \zeta(\Gamma, \mathrm{Mat}_n^+(\mathbb{Z}), \beta)\nu(\Gamma\backslash\mathbb{P}\times \mathrm{GL}_n(\hat{\mathbb{Z}})).$$

On the other hand, let $J$ be a finite set of primes. By formula 3.7, we have

$$1 = \nu(\Gamma\backslash Y) = \zeta(\Gamma, \mathrm{Mat}_n(J), \beta)\nu(\Gamma\backslash Y_J).$$

So $\nu(\Gamma\backslash Y_J) = \zeta(\Gamma, \mathrm{Mat}_n(J), \beta)^{-1}$. Enlarging $J$, we finally have

$$\nu(\Gamma\backslash\mathbb{P}\times \mathrm{GL}_n(\hat{\mathbb{Z}})) = \zeta(\Gamma, \mathrm{Mat}_n^+(\mathbb{Z}), \beta)^{-1}.$$

This shows that

$$\nu(\Gamma\backslash \mathrm{GL}_n^+(\mathbb{Z})(\mathbb{P}\times \mathrm{GL}_n(\hat{\mathbb{Z}}))) = \zeta(\Gamma, \mathrm{Mat}_n^+(\mathbb{Z}), \beta)\nu(\Gamma\backslash\mathbb{P}\times \mathrm{GL}_n(\hat{\mathbb{Z}})) = 1.$$

So $\mathrm{Mat}_n^+(\mathbb{Z})(\mathbb{P}\times \mathrm{GL}_n(\hat{\mathbb{Z}}))$ is a subset of full measure of $\mathbb{P}\times \mathrm{Mat}_n(\hat{\mathbb{Z}})$. By multiplying $\mathrm{GL}_n^+(\mathbb{Q})$, we have that

$$\mathrm{GL}_n^+(\mathbb{Q})(\mathbb{P}\times \mathrm{GL}_n(\hat{\mathbb{Z}})) = \mathbb{P}\times \mathrm{GL}_n(\mathbb{A}_f)$$

is a subset of full measure of $\mathbb{P}\times \mathrm{Mat}_n(\mathbb{A}_f)$. Here we use two facts:

1. $\mathrm{GL}_n^+(\mathbb{Q})\mathrm{GL}_n(\hat{\mathbb{Z}}) = \mathrm{GL}_n(\mathbb{A}_f)$,
2. $\mathrm{GL}_n^+(\mathbb{Q})\mathrm{Mat}_n(\hat{\mathbb{Z}}) = \mathrm{Mat}_n(\mathbb{A}_f)$.

(1) holds because $\mathrm{GL}_n$ (as an algebraic group over $\mathbb{Q}$) has class number 1.
(2) is more straightforward. Let $m = (m_p)_p \in \mathrm{Mat}_n(\mathbb{A}_f)$. By the definition of the Adele ring, there are finitely many $m_p$'s lie in $\mathrm{Mat}_n(\mathbb{Q}_p)$. Each $m_p$ is an $n\times n$ matrix with entries in $\mathbb{Q}_p$. So each entry has the form $p^{-k}l_p$ with $l_p \in \mathbb{Z}_p$. So we can find a integer $n_p$ (we also think $n_p$ as an element in $\mathrm{GL}_n^+(\mathbb{Q})$) such that $n_p m_p \in \mathrm{Mat}_n(\mathbb{Z}_p)$. Since only finitely many $m_p$'s lie in $\mathrm{Mat}_n(\mathbb{Q}_p)$, we can take the finite product of all these $n_p$'s, say $\tilde{n}$. Thus, $\tilde{n}m \in \mathrm{Mat}_n(\hat{\mathbb{Z}})$ with $\tilde{n} \in \mathrm{GL}_n^+(\mathbb{Q})$. So $m \in \mathrm{GL}_n^+(\mathbb{Q})\mathrm{Mat}_n(\hat{\mathbb{Z}})$. Thus, $\mathrm{Mat}_n(\mathbb{A}_f) \subset \mathrm{GL}_n^+(\mathbb{Q})\mathrm{Mat}_n(\hat{\mathbb{Z}})$. Since $\mathrm{GL}_n^+(\mathbb{Q}) \subset \mathrm{Mat}_n(\mathbb{A}_f)$ and $\mathrm{Mat}_n(\hat{\mathbb{Z}}) \subset \mathrm{Mat}_n(\mathbb{A}_f)$, we also have $\mathrm{Mat}_n(\mathbb{A}_f) = \mathrm{GL}_n^+(\mathbb{Q})\mathrm{Mat}_n(\hat{\mathbb{Z}})$.

Given a $y \in \Gamma\backslash\mathbb{P}\times \mathrm{GL}_n(\hat{\mathbb{Z}})$, the point mass measure $\nu_y$ at $y$ with mass

$$\zeta(\Gamma, \mathrm{Mat}_n^+(\mathbb{Z}), \beta)^{-1}$$

determines an ergodic measure $\mu_y$ on $\mathbb{P}\times \mathrm{GL}_n(\mathbb{A}_f)$ satisfying the scaling condition (3.2) such that

$$\mu_y(g\Gamma y) = \det(g)^{-\beta}\mu_y(\Gamma y),$$

with $g \in \mathrm{GL}_n^+(\mathbb{Q})$. This map $y \mapsto \mu_y$ gives a one-to-one correspondence between the set $\Gamma\backslash\mathbb{P}\times \mathrm{GL}_n(\hat{\mathbb{Z}})$ and the set of extremal $\mathrm{KMS}_\beta$ states on $\mathcal{A}$ (*cf.* [14], Theorem 3.97). □

# CHAPTER 4

# QUANTUM STATISTICAL MECHANICS OVER CONNECTED SHIMURA VARIETIES

## 4.1 Bost-Connes-Marcolli Systems in Shimura Data

### 4.1.1 Shimura Data

The Shimura varieties are originally seen as the analogue of the higher-dimensional modular curves. A Shimura variety is defined by an inverse limit system of varieties via Shimura data. With the Shimura data, in [21], Ha and Paugam introduce a construction of BC-type systems unifying most the known systems. But the structures of the KMS states on these systems have not been deeply observed due to some difficulties. In this chapter, we will develop the methods used in the previous chapter and analyze some of the Shimura cases with some arithmetic tools.

We want to first establish some notational conventions about this chapter. For a variety we will use the notation $X_k$ to indicate it is a variety over the ground filed $k$. For a $k$-algebra $R$, we use $X(R)$ to indicate the $R$ points on $X_k$. Even more, when $R$ is a topological ring (field), $X(R)$ is also a topological space that is topologized by $R$, for example $\mathrm{GL}_n(\mathbb{Q}_p)$.

In this chapter, an algebraic group always means a linear algebraic group. Given an algebraic group $G$, $Z$ denotes the center of $G$. We use the notation $G^a$ for the adjoint group of $G$, which is defined to be the quotient group $G/Z$. The Lie algebra $\mathbf{L}(G_k)$ of an algebraic group $G$ over $k$ is the kernel of the morphism $G(k[\epsilon]) \to G(k)$, where $k[\epsilon]$ is the ring generated by $\epsilon$ with the relation $\epsilon^2 = 0$. As usual, we use the notation $\mathbb{S}$ for real algebraic group of the Weil torus $\mathbb{S} = \mathbf{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. Also recall that a Cartan involution of a real algebraic group $G$ is an involution

$$\cdot^o : G \to G$$

such that the following group is compact,

$$G^o(\mathbb{R}) = \{g \in G(\mathbb{C}) \mid g = \bar{g}^o\}.$$

One of the basic results about Cartan involution is the following theorem ([47] I 4.3).

**Theorem 4.1.1.** *Cartan involution exists iff $G$ is reductive and the set of Cartan involutions (if nonempty) is a $G(\mathbb{R})$-conjugate class.*

Our definition of the Shimura data is based on the expository survey paper [33].

**Definition 4.1.2.** *A Shimura datum is a pair* $(G, X)$ *in which* $G$ *is a reductive algebraic group over* $\mathbb{Q}$*,* $X$ *is a set of conjugates of homomorphisms* $x : \mathbb{S} \to G_{\mathbb{R}}$ *by* $G(\mathbb{R})$ *such that the following axioms hold for* $x$

*S1. in the representation defined by* $x$ *over* $\mathbf{L}(G^a_{\mathbb{C}})$*,* $z/\bar{z}$*,* $1$*,* $\bar{z}/z$ *are the only possible characters;*

*S2.* $\mathrm{ad}(i)$ *is a Cartan involution of* $G^a$*;*

*S3. there is no* $\mathbb{Q}$*-factor of* $G^a$ *on which the projection of* $x$ *is trivial.*

The setting of the Shimura data can be interpreted in a slightly more concrete way. There is always a complex structure on $X$ such that $X$ is a finite disjoint union of hermitian symmetric domains (for hermitian symmetric domains, we refer the reader to Helgason's masterpiece [22] and also [47] for algebraic theory). Let $X^+$ be a connected component of $X$ and let $G^+(\mathbb{R})$ be the connected component of the identity of the real Lie group $G(\mathbb{R})$. Then $\forall g \in G(\mathbb{R})$, $gX^+ = X^+$ iff $g \in G^+(\mathbb{R})$.

**Example 4.1.3.** *let* $G = \mathrm{GL}_2$ *(over* $\mathbb{Q}$*) and* $X = \mathbb{C} - \mathbb{R}$*. Then* $(\mathrm{GL}_2, X)$ *is a Shimura datum.*

*Let* $X^+ = \mathbb{H} = \{z \in \mathbb{C} \mid \mathfrak{Im}(z) > 0\}$*. It is well known that* $\mathrm{GL}_2^+(\mathbb{R})$ *acts on* $\mathbb{H}$ *transitively by the action*

$$z \mapsto \frac{az+b}{cz+d}, \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}).$$

**Example 4.1.4.** Siegel modular variety. *Let*

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

*where* $I$ *is the* $n \times n$ *identity matrix.*

*The* general symplectic group *over* $k$ *is the group of* $2n \times 2n$ *matrices such that*

$$\mathrm{Gsp}(k) = \left\{M \mid M^T J M = sJ \textit{ for some } s \in k - \{0\}\right\}.$$

*The subgroup* $\mathrm{Sp}(k) = \left\{M \mid M^T J M = J\right\}$ *is called the* symplectic group*. Also, we call the monoid* $\mathrm{Msp}(k) = \left\{M \mid M^T J M = sJ \textit{ for some } s \in k\right\}$ *the* symplectic monoid*.*

*Let* $X^+$ $(X^-)$ $= \left\{Z = X + iY \mid Z = Z^T, Y > 0\,(Y < 0)\right\}$*, where* $X, Y$ *are* $n \times n$ *real matrices, and* $Y > 0$ $(Y < 0)$ *means* $Y$ *is positive (negative) definite. Here,* $X^+$ *is called the* Siegel upper space*.*

*If we set $X = X^+ \cup X^-$, $(\mathrm{Gsp}_{\mathbb{Q}}, X)$ is a Shimura datum. We see that*

$$\mathrm{Gsp}^+(\mathbb{R}) = \left\{ M \mid M^T J M = sJ,\, s \in \mathbb{R}_{>0} \right\}$$

*and for any $g \in \mathrm{Gsp}^+(\mathbb{R})$ if we write*

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D\ n \times n \text{ matrices},$$

*then there is a transitive action of $\mathrm{Gsp}^+(\mathbb{R})$ on $X^+$ defined by*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}, \quad Z \in X^+.$$

*Specifically in the case $n = 1$, we have $\mathrm{Gsp} = \mathrm{GL}_2$, $\mathrm{Sp} = \mathrm{SL}_2$ and $X^+ = \mathbb{H}$. This is the usual half complex plane case in the previous example.*

**Remark 4.1.5.** Shimura varieties. *The Shimura variety of a Shimura datum $(G, X)$ is defined by an inverse system as follows ([33]).*

*Recall $\mathbb{A}_f$ is the ring of finite adèles of $\mathbb{Q}$. For any compact open subgroup $K$ of $G(\mathbb{A}_f)$, let $\mathrm{Sh}_K(G, X)$ be the double coset space*

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

*where $G(\mathbb{Q})$ acts diagonally on $X \times G(\mathbb{A}_f)$ from the left, and $K$ acts on $G(\mathbb{A}_f)$ from the right. It is easy to see any inclusion of $K' \subset K$ induces a map $\mathrm{Sh}_{K'}(G, X) \to \mathrm{Sh}_K(G, X)$. So all the $\mathrm{Sh}_K(G, X)$'s form an inverse system.*

*The* Shimura variety *attached to $(G, X)$ is defined as the inverse system with respect to $\mathrm{Sh}_K(G, D)$. It is interesting that the limit object*

$$\mathrm{Sh}(G, X) := \varprojlim_K \mathrm{Sh}_K(G, X)$$

*is a scheme but not necessary a variety. Moreover, one can actually show that the limit is of the form*

$$\mathrm{Sh}(G, X) := \varprojlim_K \mathrm{Sh}_K(G, X) = G(\mathbb{Q})/Z(\mathbb{Q})\backslash X \times (G(\mathbb{A}_f)/\overline{Z(\mathbb{Q})}),$$

*where $Z$ is the center of $G$ and $\overline{Z(\mathbb{Q})}$ is the closure of $Z(\mathbb{Q})$ in $G(\mathbb{A}_f)$. If $G$ satisfies additional axioms, like the requirement that $Z(\mathbb{Q})$ is discrete in $G(\mathbb{A}_f)$, then the limit can be simplified as*

$$\varprojlim_K \mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f).$$

### 4.1.2 Bost-Connes-Marcolli Systems and KMS States

In order to construct the Bost-Connes-Marcolli (BCM) type systems from the Shimura data, we first need to recall some basic concepts about the linear algebraic monoids ([10], [45]).

**Definition 4.1.6.** *A linear algebraic monoid $M$ is an affine variety over a ground field $k$ along with an associative product $M \times M \to M$, which is a morphism of varieties and a neutral element $e$ for the product. We denote by $G(M)$ the unit group of $M$ and by $E(M)$ the set of all idempotents in $M$.*

*We say $M$ is connected if the underlying variety of $M$ is irreducible.*

Here we list some basic results of linear algebraic monoids and and linear algebraic groups.

1. $G(M)$ is a linear algebraic group and $G(M)$ is open in $M$.
2. $M - G(M)$ is an ideal of $M$, i.e. $M(M - G(M)) = (M - G(M))M = M - G(M)$.
3. If the character group of a connected linear algebraic $G$ group is not trivial, then $G$ is the unit group of some connected linear algebraic monoid $M$ and $G \neq M$. If we see $M$ as a closed submonoid of $\mathrm{Mat}_n$ (see below), then $0 \in M$.
4. Any linear algebraic monoid is isomorphic to some closed submonoid of $\mathrm{Mat}_n$, where $\mathrm{Mat}_n$ is the monoid of $n \times n$ matrices over the ground field. Also, $G(M)$ is a closed subgroup of $\mathrm{GL}_n$, where $\mathrm{GL}_n$ is the unit group of $\mathrm{Mat}_n$, namely the group of invertible $n \times n$ matrices.
5. A connected linear algebraic monoid is said to be reductive if $G(M)$ is a reductive algebraic group. Also $M$ is regular if $M = E(M)G(M) = G(M)E(M)$. When $M$ contains the zero element 0, $M$ is reductive if and only if $M$ is regular.
6. $E(M)$ is a smooth subvariety of a connected linear algebraic monoid $M$.

To begin with the construction of the BCM system, we need to fix an embedding $\iota : M \to \mathrm{Mat}_n$ (there is always a such embedding due to 4. in the list above). So when we talk about elements in $M$ ( and its unit group $G$ ) we always see them as contained in the image $\iota(M)$, i.e. as matrices.

Let $(G, X)$ be a Shimura datum and let $M$ be a connected algebraic monoid with $G(M) = G$. We set

$$G^+(\mathbb{Q}) = G^+(\mathbb{R}) \cap G(\mathbb{Q}) \quad \text{and} \quad \Gamma = G^+(\mathbb{Q}) \cap G(\hat{\mathbb{Z}}).$$

We also let $X^+$ be a connected component of $X$, which is a hermitian symmetric domain and on which $G^+(\mathbb{R})$ acts transitively and properly.

Let $Y = X^+ \times M(\hat{\mathbb{Z}})$. We define

$$\Gamma \backslash G^+(\mathbb{Q}) \boxtimes_\Gamma Y$$

to be the quotient space of

$$\{(g,y) \in G^+(\mathbb{Q}) \times Y \mid y=(x,\rho),\ gy=(gx,g\rho) \in Y = X^+ \times M(\hat{\mathbb{Z}})\}$$

by the $\Gamma^2$ action defined by

$$(\gamma_1,\gamma_2)(g,y) = (\gamma_1 g \gamma_2^{-1},\ \gamma_2 y).$$

On $C_c(\Gamma\backslash G^+(\mathbb{Q}) \boxtimes_\Gamma Y)$ we define a convolution:

$$f_1 * f_2(g,y) = \sum_{h\in\Gamma\backslash G^+(\mathbb{Q}), hy\in Y} f_1(gh^{-1},hy) f_2(h,y), \tag{4.1}$$

for $f_1, f_2 \in C_c(\Gamma\backslash G^+(\mathbb{Q}) \boxtimes_\Gamma Y)$, and also an involution:

$$f_1^*(g,y) = \overline{f_1(g^{-1},gy)}.$$

Let $G_y = \{g \in G^+(\mathbb{Q}) \mid gy \in Y\}$. There is a representation of $C_c(\Gamma\backslash G^+(\mathbb{Q}) \boxtimes_\Gamma Y)$ on the Hilbert space $\mathfrak{H}_y = \ell^2(\Gamma\backslash G_y)$ given by

$$\varpi_y(f)\delta_g = \sum_{h\in\Gamma\backslash G_y} f(gh^{-1},hy)\delta_h.$$

**Lemma 4.1.7.**

$$\|f\| = \sup_{y\in Y}\{\|\varpi_y(f)\|\}$$

*defines a norm on* $C_c(\Gamma\backslash G^+(\mathbb{Q}) \boxtimes_\Gamma Y)$.

*Proof.* For $\chi, \psi \in \mathfrak{H}_y$, from the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |(\varpi(f)\chi,\psi)| &\le \sum_{g,h\in\Gamma\backslash G_y} f(gh^{-1},hy)\|\chi(h)\|\|\psi(g)\| \\ &\le \Big(\sum_{g,h\in\Gamma\backslash G_y} f(gh^{-1},hy)\|\chi(h)\|^2\Big)^{\frac{1}{2}}\Big(\sum_{g,h\in\Gamma\backslash G_y} f(gh^{-1},hy)\|\psi(g)\|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

The only thing we need to show here is that $|\sum_{h\in\Gamma\backslash G_y} f(h,y)|$ is uniformly bounded for all $y$.

Since $f$ is compactly supported and the action of $\Gamma$ is proper, there are only finitely many nonzero terms in the sum $\sum_{h\in\Gamma\backslash G_y} f(h,y)$ for all $y$. So there is an upper bound and $\|f\| = \sup_{y\in Y}\{\|\varpi_y(f)\|\}$ defines a norm. □

We define the C*-algebra $\mathcal{A}_{(G,X)} = \mathrm{C}_r^*(\Gamma\backslash G^+(\mathbb{Q}) \boxtimes_\Gamma Y)$ to be the completion of

$$C_c(\Gamma\backslash G^+(\mathbb{Q}) \boxtimes_\Gamma Y)$$

with respect to the norm in the lemma above. Further, if we set

$$\sigma_t(f)(g,x) = \det(g)^{it} f(g,x) \quad \forall t \in \mathbb{R}, f \in C_c(\Gamma\backslash G^+(\mathbb{Q}) \boxtimes_\Gamma Y),$$

this $\sigma_t$ then defines a time evolution $\mathbb{R} \to \mathrm{Aut}(\mathcal{A}_{(G,X)})$.

**Definition 4.1.8.** *We call the $C^*$-algebra $\mathcal{A}_{(G,X)}$ the $(G,X)$-Bost-Connes-Marcolli (BCM) algebra. The $C^*$-dynamical system $(\mathcal{A}_{(G,X)}, \sigma_t)$ is called the $(G,X)$-Bost-Connes-Marcolli (BCM) system.*

**Remark 4.1.9.** *In [21], Ha and Paugam use stacks to construct these dynamical systems with Shimura data. We take a more concrete construction to define these systems. But as they have shown ([21]), under certain conditions, these constructions are the same.*

**Example 4.1.10.** The BC system. *The original Bost-Connes (BC) system can be formulated from the pair $(\mathbb{G}_m, \{\pm 1\})$. Let*

$$Z_{BC} = \{(g,\rho) \in \mathbb{Q}_\times \times \hat{\mathbb{Z}} \mid \; g > 0, \;\; g\rho \in \hat{\mathbb{Z}}\}.$$

*The BC system is the Hecke $C^*$-algebra $\mathrm{C}^*_r(Z_{BC})$ with the time evolution $\sigma_t(f)(g,\rho) = g^{it} f(g,\rho)$.*

**Example 4.1.11.** The CM system. *The Connes-Marcolli system can be defined from the Shimura datum $(\mathrm{GL}_2, \mathbb{C} - \mathbb{R})$. So the CM system is the the $C^*$-algebra*

$$C^*_r(\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{GL}_2^+(\mathbb{Q}) \boxtimes_{\mathrm{SL}_2(\mathbb{Z})} (\mathbb{H} \times \mathrm{Mat}_2(\hat{\mathbb{Z}})),$$

*with the time evolution $\sigma_t(f)(g,\rho) = \det(g)^{it} f(g,\rho)$.*

**Example 4.1.12.** The Siegel BCM systems. *It is natural to generalize the previous example to the the Siegel modular varieties setting. Let $(\mathrm{Gsp}_{\mathbb{Q}}, X)$ be the Siegel Shimura datum as in Example 4.1.4. The Siegel BCM algebra is to be*

$$\mathrm{C}^*_r(\mathrm{Sp}(\mathbb{Z})\backslash (\mathrm{Gsp}^+(\mathbb{R}) \boxtimes_{\mathrm{Sp}(\mathbb{Z})} (X^+ \times \mathrm{Msp}(\hat{\mathbb{Z}})))).$$

**Example 4.1.13.** The connected Shimura systems. *We will discuss this in detail in the next section.*

**Remark 4.1.14.** *In the previous chapter, we discussed the $\mathrm{GL}_n$-CM systems. These systems are not constructed from Shimura data, since $\mathrm{PGL}_n(\mathbb{R})$ is not a hermitian symmetric domain. But they still have some properties in common. For example, $\Gamma$ acts on $\mathrm{PGL}_n^+(\mathbb{R})$ properly and $\mathrm{GL}_n^+(\mathbb{R})$ acts on it transitively.*

Recall that, for an element $a \in \mathcal{A}_{(G,X)}$, we say that $a$ is entire if the function $t \mapsto \sigma_t(a)$ can be extended to an entire function on the whole complex plane $\mathbb{C}$. Given a real number $\beta$ (called the inverse temperature) and a weight $\phi$ on $\mathcal{A}_{(G,X)}$, we say $\phi$ is a $\mathrm{KMS}_\beta$ weight if for any entire elements $a$, $b$, $\phi(ab) = \phi(b\sigma_{i\beta}(a))$. A $\mathrm{KMS}_\beta$ state is a $\mathrm{KMS}_\beta$ weight that is also a state.

We know that, for an algebraic monoid $M$, $M - G(M)$ is an ideal. If for some prime $p$ we let

$$M'_p(\hat{\mathbb{Z}}) = M(\hat{\mathbb{Z}}) \cap \big(G(\mathbb{Q}_p) \times \prod_{q \neq p} M(\mathbb{Q}_q)\big)$$

and

$$M^0_p(\hat{\mathbb{Z}}) = M(\hat{\mathbb{Z}}) - M'_p(\hat{\mathbb{Z}}),$$

we have an ideal in $\mathcal{A}_{(G,X)}$

$$\mathcal{I}_p = \mathrm{C}^*_r(\Gamma\backslash G^+(\mathbb{Q}) \boxtimes_\Gamma (X^+ \times M'_p)).$$

We also take $M'(\hat{\mathbb{Z}}) = \cap_p M'_p(\hat{\mathbb{Z}})$ and $\mathcal{I}_e = \cap_p \mathcal{I}_p$.

**Definition 4.1.15.** *Let $\phi$ be a $KMS_\beta$ state over the $(G, X)$-BCM system. We say $\phi$ is*

*ı)* essential, *if the restriction $\phi|_{\mathcal{A}_{(G,X)} - \mathcal{I}_p} \equiv 0$ for all $p$;*

*ıı)* superficial, *if the restriction $\phi|_{\mathcal{I}_e} \equiv 0$;*

*ııı)* mixed, *if $\phi$ is neither essential nor superficial.*

**Example 4.1.16.** *The Connes-Marcolli system. A KMS state on the CM system is either to be essential or to be superficial.*

**Remark 4.1.17.** *Similar results hold for the $\mathrm{GL}_n$-Connes-Marcolli systems. But those systems are slightly modified from the Shimura constructions.*

**Theorem 4.1.18.** *Let $\phi$ be an essential $KMS_\beta$ state for the $(G, X)$-BCM system $(\mathcal{A}_{(G,X)}, \sigma_t)$, then $\phi$ is of the form*

$$\phi(f) = \int_{\Gamma\backslash Y} f(1, y)d\nu(y), \quad \text{for any } f \in \mathcal{A}_{(G,X)},$$

*where $\nu$ is a probability measure on $\Gamma\backslash(X^+ \times M(\hat{\mathbb{Z}}))$ supported in $\Gamma\backslash(X^+ \times M'(\hat{\mathbb{Z}}))$.*

*Proof.* Let $h \in G^+(\mathbb{Q})$ and $f \in C_c(\Gamma\backslash G^+(\mathbb{Q}) \boxtimes_\Gamma Y)$ such that $f(g, (x, \rho)) = 0$ for any $g \notin \Gamma h \Gamma$. Since every function in $\mathrm{C}_c(\Gamma\backslash G^+(\mathbb{Q}) \boxtimes_\Gamma Y)$ is a finite sum of such functions, we need to show $\phi(f) = 0$ for such a function when $h \notin \Gamma$.

Let $\nu$ be the probability measure given by $\phi$. We are assuming that $\phi$ is essential, so $\nu$ is supported in $\Gamma\backslash(X^+ \times M'(\hat{\mathbb{Z}}))$. Let $Y^h$ be the set of points in $Y$ that are fixed by the action of

$h \notin \Gamma$. If $y = (x, \rho) \in Y$ such that $hy = y$, then $h\rho = \rho, \rho = (m_p)_p \in M(\hat{\mathbb{Z}}) = \prod_p M(\mathbb{Z}_p)$. For some $m_p$ if $m_p \in M(\mathbb{Z}_p) \cap G(\mathbb{Q}_p)$, then we must have $h = 1$, which is not the case here, since $1 \in \Gamma$ and $h \notin \Gamma$. So if $hy = y$, then $y \in X^+ \times M_p^0$ for all $p$.

We can further require that $f(g, y) = 0$ unless $y \in K$, where $K$ is some compact set in $\Gamma\backslash Y$ with $K \cap Y^h = \emptyset$. Otherwise, we can find a sequence $f_n \to f$ in which each $f_n$ has compact support disjoint from $Y^h$, since $Y^h$ is not in the support of $\nu$ and it is of measure 0.

Let $\mathrm{pr} : Y \to \Gamma\backslash Y$ be the quotient map. By the properness of the action of $\Gamma$, we find a finite over $\{U_j\}$ of $K$ such that $\mathrm{pr}(h\mathrm{pr}^{-1}(U_j)) \cap U_j = \emptyset$. We also denote the corresponding partition of unity by $\{\psi_j\}$. Let $\alpha_j$ be the function such that $\alpha_j(g, y) = \psi_j(y)^{\frac{1}{2}}$, if $g \in \Gamma$ and $\alpha_j(g, y) = 0$, otherwise. Then $\alpha_j * f(g, y) = \psi(gy)^{\frac{1}{2}} f(g, y)$ by calculating the convolution. It follows that

$$f = \sum_j \alpha_j * \alpha_j * f.$$

The KMS condition implies that $\phi(\alpha_j * \alpha_j * f) = \phi((\alpha_j * f) * \alpha_j)$. But

$$(\alpha_j * f) * \alpha_j(h, y) = (\alpha_j * f)(h, y)\alpha_j(1, y) = \psi_j(hy)^{\frac{1}{2}}\psi_j(y)^{\frac{1}{2}} f(h, y).$$

By the choice of $U_j$ we see $\psi_j(hy)^{\frac{1}{2}}\psi_j(y)^{\frac{1}{2}} = 0$. So $\phi(f) = 0$. □

## 4.2 Connected Shimura Variety Systems

### 4.2.1 Connected Shimura Systems

Now, let us turn to the main part of this chapter. To extend the methods used in the previous chapter to the abstract settings, we will focus on the data with more computable properties. We are going to discuss the case of connected Shimura varieties. But before doing this, we need to review more concepts from algebraic groups (also see [41], chapter 7).

**Definition 4.2.1.** *A semisimple group $G$ is said to be simply connected if any isogeny $G' \to G$ with $G'$ connected is an isomorphism.*

The following theorem tells us that, theoretically, there are many simply connected algebraic groups.

**Theorem 4.2.2.** *For any semisimple linear algebraic group $G$, there is an isogeny $\tilde{G} \to G$ such that $\tilde{G}$ is a simply connected algebraic group. $\tilde{G}$ is called the universal covering of $G$.*

**Definition 4.2.3.** *A linear algebraic group $G_{\mathbb{Q}}$ is of compact type if $G(\mathbb{R})$ is compact. If $G_{\mathbb{Q}}$ contains no nonzero normal subgroup of compact type, then we say $G$ is of noncompact type.*

Let $G_{\mathbb{Q}}$ be a simply connected linear algebraic group over $\mathbb{Q}$. We also collect some basic facts about $G$.

1. Weak approximation. Let $S$ be a finite set of places over $\mathbb{Q}$ and let $G_S = \prod_{v \in S} G(\mathbb{Q}_v)$. The diagonal embedding $G(\mathbb{Q}) \hookrightarrow G_S$ is dense in $G_S$.

2. Strong approximation. Let $\mathbb{A}_f$ be the finite adele ring of $\mathbb{Q}$. In addition if $G$ is of noncompact type, then the diagonal embedding $G(\mathbb{Q}) \hookrightarrow G(\mathbb{A}_f)$ is dense in $G(\mathbb{A}_f)$.

3. Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ be the total adele ring of $\mathbb{Q}$. The diagonal embedding $G(\mathbb{Q}) \hookrightarrow G(\mathbb{A})$ is discrete.

4. $G(\mathbb{R})$ is connected. $G(\mathbb{C})$ is simply connected.

5. Real approximation. $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$.

The strong approximation property passes to the algebraic monoid, as shown in the following the following lemma.

**Lemma 4.2.4.** *Let $M$ be a connected linear algebraic monoid over $\mathbb{Q}$. If $G(M)$ is simply connected of noncompact type, then the strong approximation property holds for $M$, i.e. the image of the diagonal embedding $M(\mathbb{Q}) \hookrightarrow M(\mathbb{A}_f)$ is dense.*

*Proof.* Since $G(M)$ is semisimple, $M$ is regular, i.e. $M = G(M)E(M)$. $E(M)$ is smooth, so is $M$. $M$ is connected, so $G(M)$ is an open dense subvariety of $M$. The smoothness of $M$ implies that $G(M)(\mathbb{Q}_p)$ is dense in $M(\mathbb{Q}_p)$ for any prime $p$. $G(M)$ is integral, then $G(M)(\mathcal{O}_p) \neq \emptyset$ for all but finitely many $p$'s. Here $\mathcal{O}_p$ is the ring of integers in $\mathbb{Q}_p$. Hence, $G(M)(\mathbb{A}_f)$ is dense in $M(\mathbb{A}_f)$. The strong approximation property follows. □

Recall that we say two subgroups $H_1$, $H_2$ of a group $G$ are commensurable if $H_1 \cap H_2$ is a subgroup of finite index in both $H_1$ and $H_2$.

**Definition 4.2.5.** *A subgroup $\Gamma$ of $G(\mathbb{Q})$ is arithmetic if $\Gamma$ is commensurable with $G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$ for some embedding $G \hookrightarrow \mathrm{GL}_n$.*

A theorem of Borel ([6], 7.13) shows that if $\Gamma$ is commensurable with $G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$ for one embedding, then the same is true for any embedding.

Particularly, if we let $\Gamma = G(\mathbb{Q}) \cap G(\hat{\mathbb{Z}})$, where the intersection takes place in $G(\mathbb{A}_f)$, then $\Gamma$ is a so-called congruence subgroup of $G(\mathbb{Q})$. A congruence subgroup is always arithmetic.

Let $S$ be a subgroup of some group $H$. We recall the pair $(H, S)$ is a Hecke pair if for any $h \in H$, $S$ is commensurable with $hSh^{-1}$.

Back to our previous settings, let $\Gamma$ be the congruence subgroup of $G(\mathbb{Q})$ above. Since $\Gamma$ is an arithmetic subgroup as well, one can show that $g\Gamma g^{-1}$ is also an arithmetic subgroup for any $g \in G(\mathbb{Q})$. As commensurability is an equivalence relation of groups, $\Gamma$ is commensurable with $g\Gamma g^{-1}$. So we conclude that

**Proposition 4.2.6.** $(G(\mathbb{Q}), \Gamma)$ *is a Hecke pair.*

**Definition 4.2.7.** *A connected Shimura datum is a pair* $(G, D)$ *such that* $G$ *is a semisimple algebraic group over* $\mathbb{Q}$ *of noncompact type,* $D$ *is a hermitian symmetric domain, and there is an action of* $G(\mathbb{R})^+$ *on* $D$ *defined by a surjective homomorphism* $G(R)^+ \to \mathrm{Hol}(D)^+$ *with compact kernel.*

**Remark 4.2.8.** *From now on, we use the letter* $D$ *instead of the letter* $X$ *to denote the hermitian symmetric domain. We use the letter* $X$ *in the definition of the* $C^*$*-dynamical systems*

From now on, let us focus on $G_{\mathbb{Q}}$ which is simply connected of noncompact type. We then see, $G(\mathbb{R})$ is connected. So $G(\mathbb{R})$ acts on $D$ directed through the surjection $G(\mathbb{R}) \to \mathrm{Hol}(D)^+$ with a compact kernel. We also know, actually, $D = \mathrm{Hol}^+/K$ for some compact subgroup of $\mathrm{Hol}^+$.

**Definition 4.2.9.** *The connected Shimura variety is the inverse limit*

$$\mathrm{Sh}(G, D) := \varprojlim_K \mathrm{Sh}_K(G, D).$$

**Proposition 4.2.10.** *([33], 4.19)* $\varprojlim_K \mathrm{Sh}_K(G, D) = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)$.

Let $G$ be a simply connected linear algebraic group of noncompact type over $\mathbb{Q}$. We also assume the character group of $G$ is nontrivial. So G is a unit group of some connected linear algebraic monoid such that $G \neq M$. Again let us fix an embedding of $M \hookrightarrow \mathrm{Mat}_n$. So $0 \in M$ and we can always think the elements of $M$ or $G$ are matrices and we can talk about the determinant det of their elements. We still use $\Gamma$ for the congruence subgroup of $G(\mathbb{Q}) \cap G(\hat{\mathbb{Z}})$.

For a simply connected linear algebraic group of noncompact type $G$ over $\mathbb{Q}$, because of the strong approximation property, it has class number 1, i.e. $G(\mathbb{A}_f) = G(\mathbb{Q})G(\hat{\mathbb{Z}})$.

**Definition 4.2.11.** *With the settings above, the Bost-Connes-Marcolli (BCM) algebra for the connected Shimura datum* $(G, D)$ *is the* $\mathrm{C}^*$*-algebra*

$$\mathcal{A}_{(G,D)} = \mathrm{C}^*_r(\Gamma\backslash G(\mathbb{Q}) \boxtimes_\Gamma Y),$$

*Where* $Y = D \times M(\hat{\mathbb{Z}}) \subset X = D \times M(\mathbb{A}_f)$ *and* $G(\mathbb{Q})$ *acts diagonally on* $X$*. The dynamics* $\sigma_t$ *on* $A$ *is given by the action* $\sigma_t(f)(g, x) = \det(g)^{it} f(g, x)$ *for* $f \in C_c(\Gamma\backslash G(\mathbb{Q}) \times_\Gamma X)$*.*

### 4.2.2 KMS States on the Connected Shimura Systems

As shown in Theorem 4.1.18, every essential KMS state is determined by a probability measure $\nu$ on the space $\Gamma\backslash Y$. We can now do a more detailed analysis of this measure. The relation

$$\int_Y f(y)d\mu = \int_{\Gamma\backslash Y} \Big( \sum_{y\in \mathrm{pr}^{-1}([x])} f(y)\Big) d\nu([x])$$

defines a $\Gamma$-invariant measure $\mu$ on $Y$. Moreover, by the KMS condition, $\mu$ also satisfies

$$\mu(gB) = \det(g)^{-\beta}\mu(B), \tag{4.2}$$

here $B$ is a Borel set in $Y$ with $gB \subset Y$.

Since $G$ is simply connected of noncompact type, so it has the strong approximation property, i.e. $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_f)$. This gives that $G_{\mathbb{Q}}$ is of class number 1, i.e. $G(\mathbb{Q})G(\hat{\mathbb{Z}}) = G(\mathbb{A}_f)$. Thus, we have

$$G(\mathbb{Q})M'(\hat{\mathbb{Z}}) = M'(\mathbb{A}_f) = M(\mathbb{A}_f) \cap \prod_p G(\mathbb{Q}_p).$$

It follows that $G(\mathbb{Q})(D\times M'(\hat{\mathbb{Z}})) = D\times M'(\mathbb{A}_f)$. So, by the lemma 3.3.1 in the previous chapter, the measure $\mu$ can be uniquely extended to a measure on $D\times M(\mathbb{A}_f)$ which is supported in $D\times M'(\mathbb{A}_f)$ and satisfies the scaling condition (4.2). From this discussion, we have shown, there is a one-to-one correspondence between the set of essential KMS states and the measures $\mu$ on $D \times M(\mathbb{A}_f)$ supported in $D \times M'(\mathbb{A}_f)$ and satisfying the scaling condition (4.2).

Now we state the main result of this chapter.

**Theorem 4.2.12.** *The essential KMS$_\beta$ state on $\mathcal{A}_{(G,D)}$ is unique if it exists.*

The proof of Theorem 4.2.12 will follow from the previous observation, together with the results of Theorem 4.2.13 and Theorem 4.2.14. Because of the one-to-one correspondence between the essential KMS states and the measures $\mu$ described above, to prove the uniqueness we only need to show the ergodicity of the measure as we do in the $\mathrm{GL}_n$ case in the previous chapter.

We need a more general version of strong approximation for algebraic groups. Let $V$ be a finite set of primes on $\mathbb{Q}$. We denote the image of $G(\mathbb{A})$ under the projection to $\prod_{v\notin V} G(\mathbb{Q}_v)$ by $G(\mathbb{A}_V)$.

**Theorem 4.2.13.** Strong Approximation.*([41], Theorem 7.12) Let $G_{\mathbb{Q}}$ be an reductive algebraic group and let $V$ be a finite set of primes (infinite or finite) on $\mathbb{Q}$. If $G$ is simply connected of noncompact type, then $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_V)$ via the diagonal embedding.*

Strong approximation always implies the weak approximation, i.e. the embedding

$$G(\mathbb{Q}) \hookrightarrow \prod_{v \notin V} G(\mathbb{Q}_v)$$

has dense image.

With the discussion after the statement of theorem 4.2.12, to prove the theorem, we only need to prove the following theorem.

**Theorem 4.2.14.** *With the measure $\mu$ on $D \times M(\mathbb{A}_f)$ which is supported in $D \times M'(\mathbb{A}_f)$ and satisfies the scaling condition* (4.2)*, the action of $G(\mathbb{Q})$ on $D \times M(\mathbb{A}_f)$ is ergodic.*

*Proof.* Recall $G^a = G/Z$ is the adjoint group of $G$. So $G^a(\mathbb{R}) = G(\mathbb{R})/Z(\mathbb{R})$. Then $G^a \tilde{\to} \mathrm{Hol}^+(D)$ is an isomorphism. So $D \simeq G^a/K$ for some compact subgroup $K$. Again, by using the formula,

$$\int_{G^a(\mathbb{R}) \times M(\mathbb{A}_f)} f d\mu = \int_{D \times M(\mathbb{A}_f)} (\int_K f(\cdot g) dg) d\mu,$$

$\mu$ is determined by the measure on $G^a(\mathbb{R}) \times M(\mathbb{A}_f)$ supported in $G^a(\mathbb{R}) \times M'(\mathbb{A}_f)$, which we still call $\mu$. We want to show the action of $G(\mathbb{Q})$ on $G^a(\mathbb{R}) \times M(\mathbb{A}_f)$ is ergodic with respect this measure $\mu$.

Let $J$ be a finite set of finite primes of $\mathbb{Q}$. Let $f$ be a $G(\mathbb{Q})$-invariant function on $G^a(\mathbb{R}) \times \prod_{p \in J} G(\mathbb{Q}_p)$. We see $f$ as a function on $G^a \times M'(\mathbb{A}_f)$. By the strong approximation, we see $G(\mathbb{Q})$ is a dense subgroup of $G(\mathbb{R}) \times \prod_{p \in J} G(\mathbb{Q}_p)$. So if $f$ is $G(\mathbb{Q})$-invariant, then $f$ is almost constant. By taking all possible $J$'s, we see a dense subspace of functions over $G^a \times M'(\mathbb{A}_f)$ consists of almost constant functions. So the action of $G(\mathbb{Q})$ is ergodic. □

Now we turn to discuss the existence of the KMS states.

**Definition 4.2.15.** *Let $S$ be a semigroup in $G(\mathbb{Q})$ containing $\Gamma$. We define*

$$\zeta_{S,\Gamma}(\beta) = \sum_{s \in \Gamma \backslash S} \det(s)^{-\beta} = \sum_{s \in \Gamma \backslash S / \Gamma} \#(\Gamma \backslash \Gamma s \Gamma) \det(s)^{-\beta}. \tag{4.3}$$

Lemma 4.2.4 says in our case that the strong approximation property holds for the linear algebraic monoid $M$. This means that, if we fix a realization $M \hookrightarrow \mathrm{Mat}_n$, then $M(\mathbb{Z})$ is dense in $M(\hat{\mathbb{Z}})$. For any prime number $p$, this shows that $M(\mathbb{Z})$ is dense in $M(\mathbb{Z}_p)$. For all $g \in M'(\mathbb{Z}_p) = M(\mathbb{Z}_p) \cap G(\mathbb{Q}_p)$, $gG(\mathbb{Z}_p)$ is a neighborhood of $g$. So there is some $h \in M(\mathbb{Z})$ such that $h \in gG(\mathbb{Z}_p)$. It implies that $h = gk$ with $k \in G(\mathbb{Z}_p)$. So $g = hk^{-1} \in hG(\mathbb{Z}_p)$. Also since $gG(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$,

$h \in M'(\mathbb{Z}) = M(\mathbb{Z}) \cap G(\mathbb{Q})$. So $M'(\mathbb{Z})G(\mathbb{Z}_p) = M'(Z_p)$. If we set $S_p = M(\mathbb{Z}) \cap G(\mathbb{Z}[p^{-1}])$, then $S_p G(\mathbb{Z}_p) = M'(\mathbb{Z}_p)$.

In the previous chapter, we have establish the formula (3.3) , which is repeated here.

$$\nu(\Gamma\backslash\Gamma g \Gamma Z) = \det(g)^{-\beta} \#(\Gamma/(g^{-1}\Gamma g \cap \Gamma))\nu(\Gamma\backslash\Gamma Z). \tag{4.4}$$

Let $Y_p = G^a(\mathbb{R}) \times G(\mathbb{Z}_p) \times \prod_{q \neq p} M(\mathbb{Z}_q)$. So, $S_p Y_p = G^a(\mathbb{R}) \times M'(\mathbb{Z}_p) \times \prod_{q \neq p} M(\mathbb{Z}_q)$. By the formula (4.4), we have

$$1 = \nu(\Gamma\backslash Y) = \nu(\Gamma\backslash S_p Y_p) = \zeta_{S_p,\Gamma}(\beta)\nu(Y_p).$$

It makes sense only if $\zeta_{S_p,\Gamma}(\beta) < \infty$ and it depends on the inverse temperature $\beta$.

First of all, for small $\beta$ there is no essential $\mathrm{KMS}_\beta$ state. For example, if $\beta \leq 0$, the right hand side of (4.3) is always infinite.

Next, $\Gamma$ is a congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$, so the index of $\Gamma$ in $\mathrm{SL}_n(\mathbb{Z})$ is finite, say $l$. We know that if $\beta$ is big enough (for example $\beta > n$ and as shown in the previous chapter), the sum

$$\sum \#(\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{Z}) s \mathrm{SL}_n(\mathbb{Z})) \det(s)^{-\beta}$$

is convergent.

We see that

$$\#(\Gamma\backslash\Gamma s\Gamma) \leq \#(\Gamma\backslash \mathrm{SL}_n(\mathbb{Z}) s \mathrm{SL}_n(\mathbb{Z})) = l\#(\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{Z}) s \mathrm{SL}_n(\mathbb{Z})).$$

So for $\beta$ large enough, $\zeta_{S_p,\Gamma}(\beta) < \infty$. For such a $\beta$, the $\mathrm{KMS}_\beta$ state can be determined by the following measure.

For a prime number $p$, there is the normalized Haar measure $\mu_p$ on group $G(\mathbb{Z}_p)$ such that

$$\mu_p(G(\mathbb{Z}_p)) = \frac{1}{\zeta_{S_p,\Gamma}(\beta)}.$$

This measure can be uniquely extended (we still call it $\mu_p$) to the group $G(\mathbb{Q}_p)$ by

$$\mu_p(K) = \sum_{g \in G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)} |\det(g)|_p^{-\beta} \mu(gK \cap G(\mathbb{Z}_p)),$$

for compact $K \subset G(\mathbb{Q}_p)$. Here $|\cdot|_p$ is the standard $p$-adic norm of $\mathbb{Q}_p$.

Let $\mu_a$ be the Haar measure of $G^a(\mathbb{R})$. Since $\Gamma$ is a lattice (because it is arithmetic), we normalize $\mu_a$ so that $\mu_a(\Gamma\backslash G^a(\mathbb{R})) = 1$.

Then we let

$$\mu_\beta = \mu_a \times \prod_p \mu_p.$$

The measure $\mu_\beta$ satisfies the conditions in (4.2), hence there is an essential KMS at inverse temperature $\beta$ defined by this $\mu_\beta$.

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