



Digital Filters Design for Signal and Image Processing

Edited by
Mohamed Najim

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Introduction

Over the last decade, digital signal processing has matured; thus, digital signal processing techniques have played a key role in the expansion of electronic products for everyday use, especially in the field of audio, image and video processing. Nowadays, digital signal is used in MP3 and DVD players, digital cameras, mobile phones, and also in radar processing, biomedical applications, seismic data processing, etc.

This book aims to be a text book which presents a thorough introduction to digital signal processing featuring the design of digital filters. The purpose of the first part (Chapters 1 to 9) is to initiate the newcomer to digital signal and image processing whereas the second part (Chapters 10 and 11) covers some advanced topics on stability for 2-D filter design. These chapters are written at a level that is suitable for students or for individual study by practicing engineers.

When talking about filtering methods, we refer to techniques to design and synthesize filters with constant filter coefficients. By way of contrast, when dealing with adaptive filters, the filter taps change with time to adjust to the underlying system. These types of filters will not be addressed here, but are presented in various books such as [HAY 96], [SAY 03], [NAJ 06].

Chapter 1 provides an overview of various classes of signals and systems. It discusses the time-domain representations and characterizations of the continuous-time and discrete-time signals.

Chapter 2 details the background for the analysis of discrete-time signals. It mainly deals with the z-transform, its properties and its use for the analysis of linear systems, represented by difference equations.

Chapter 3 is dedicated to the analysis of the frequency properties of signals and systems. The Fourier transform, the discrete Fourier transform (DFT) and the fast Fourier transform (FFT) are introduced along with their properties. In addition, the well-known Shannon sampling theorem is recalled.

As we will see, some of the most popular techniques for digital infinite impulse response (IIR) filter design benefit from results initially developed for analog signals. In order to make the reader's task easy, Chapter 4 is devoted to continuous-time filter design. More particularly, we recall several approximation techniques developed by mathematicians such as Chebyshev or Legendre, who have thus seen their names associated with techniques of filter design.

The following chapters form the core of the book. Chapter 5 deals with the techniques to synthesize finite impulse response (FIR) filters. Unlike IIR filters, these have no equivalent in the continuous-time domain. The so-called windowing method, as a FIR filter design method, is first presented. This also enables us to emphasize the key role played by the windowing in digital signal processing, e.g., for frequency analysis. The Remez algorithm is then detailed.

Chapter 6 concerns IIR filters. The most popular techniques for analog to digital filter conversion, such as the bilinear transform and the impulse invariance method, are presented. As the frequency response of these filters is represented by rational functions, we must tackle the problems of stability induced by the existence of poles of these rational functions.

In Chapter 7, we address the selection of the filter structure and point out its importance for filter implementation. Some problems due to the finite-precision implementation are listed and we provide rules to choose an appropriate structure while implementing filter on fixed point operating devices.

In comparison with many available books dedicated to digital filtering, this title features both 1-D and 2-D systems, and as such covers both signal and image processing. Thus, in Chapters 8 and 9, 2-D filtering is investigated.

Moreover, it is not easy to establish the necessary and sufficient conditions to test the stability of 2-D signals. Therefore, Chapters 10 and 11 are dedicated to the difficult problem of the stability of 2-D digital system, a topic which is still the subject of many works such as [ALA 2003] [SER 06]. Even if these two chapters are not a prerequisite for filter design, they can provide the reader who would like to study the problems of stability in the multi-dimensional case with valuable clarifications. This contribution is another element that makes this book stand out.

The field of digital filtering is often perceived by students as a “patchwork” of formulae and recipes. Indeed, the methods and concepts are based on several specific optimization techniques and mathematical results which are difficult to grasp.

For instance, we have to remember that the so-called Parks-McClellan algorithm proposed in 1972 was first rejected by the reviewers [PAR 72]. This was probably due to the fact that the size of the submitted paper, i.e., 5 pages, did not enable the reviewers to understand every step of the approach [McC 05].

In this book we have tried, at every stage, to justify the necessity of these approaches without recalling all the steps of the derivation of the algorithm. They are described in many articles published during the 1970s in the IEEE periodicals i.e., *Transactions on Acoustics Speech and Signal Processing*, which has since become *Transactions on Signal Processing* and *Transactions on Circuits and Systems*.

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Bordeaux

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Chapter 1

Introduction to Signals and Systems

1.1. Introduction

Throughout a range of fields as varied as multimedia, telecommunications, geophysics, astrophysics, acoustics and biomedicine, signals and systems play a major role. Their frequential and temporal characteristics are used to extract and analyze the information they contain. However, what importance do signals and systems really hold for these disciplines? In this chapter we will look at some of the answers to this question.

First we will discuss different types of continuous and discrete-time signals, which can be termed random or deterministic according to their nature. We will also introduce several mathematical tools to help characterize these signals. In addition, we will describe the acquisition chain and processing of signals.

Later we will define the concept of a system, emphasizing invariant discrete-time linear systems.

1.2. Signals: categories, representations and characterizations

1.2.1. *Definition of continuous-time and discrete-time signals*

The function of a signal is to serve as a medium for information. It is a representation of the variations of a physical variable.

2 Digital Filters Design for Signal and Image Processing

A signal can be measured by a sensor, then analyzed to describe a physical phenomenon. This is the situation of a tension taken to the limits of a resistance in order to verify the correct functioning of an electronic board, as well as, to cite one example, speech signals that describe air pressure fluctuations perceived by the human ear.

Generally, a signal is a function of time. There are two kinds of signals: *continuous* and *discrete-time*.

A *continuous-time* or analog signal can be measured at certain instants. This means physical phenomena create, for the most part, continuous-time signals.

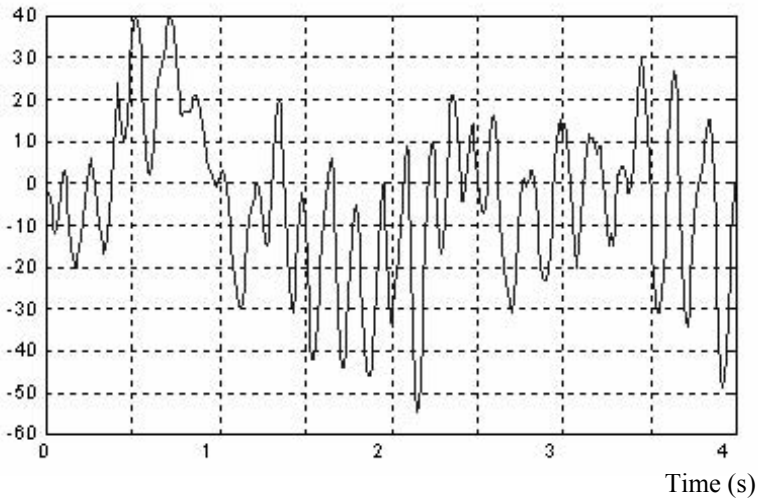


Figure 1.1. Example of the sleep spindles of an electroencephalogram (EEG) signal

The advancement of computer-based techniques at the end of the 20th century led to the development of digital methods for information processing. The capacity to change analog signals to digital signals has meant a continual improvement in processing devices in many application fields. The most significant example of this is in the field of telecommunications, especially in cell phones and digital televisions. The digital representation of signals has led to an explosion of new techniques in other fields as varied as speech processing, audiofrequency signal analysis, biomedical disciplines, seismic measurements, multimedia, radar and measurement instrumentation, among others.

The signal is said to be a *discrete-time* signal when it can be measured at certain instants; it corresponds to a sequence of numerical values. Sampled signals are the result of sampling, uniform or not, of a continuous-time signal. In this work, we are especially interested in signals taken at regular intervals of time, called sampling periods, which we write as $T_s = \frac{1}{f_s}$ where f_s is called the sampling rate or the sampling frequency. This is the situation for a temperature taken during an experiment, or of a speech signal (see Figure 1.2). This discrete signal can be written either as $x(k)$ or $x(kT_s)$. Generally, we will use the first writing for its simplicity. In addition, a digital signal is a discrete-time discrete-valued signal. In that case, each signal sample value belongs to a finite set of possible values.

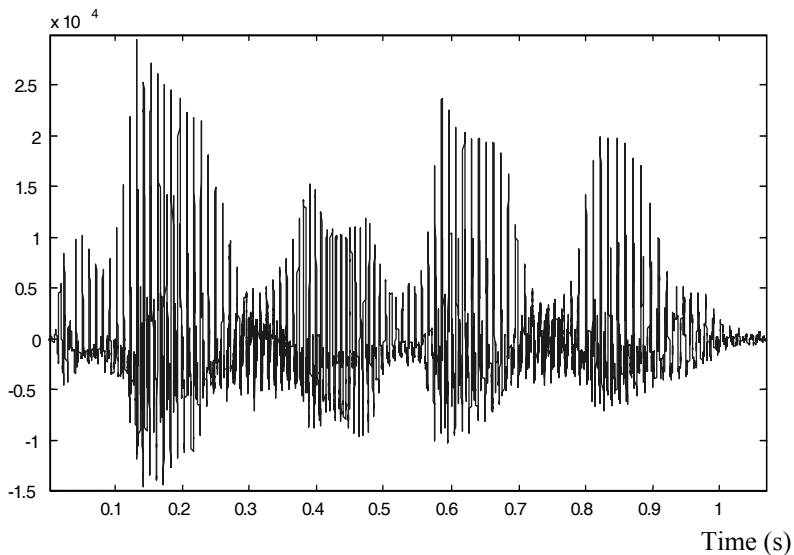


Figure 1.2. Example of a digital voiced speech signal
(the sampling frequency f_s is at 16 KHz)

The choice of a sampling frequency depends on the applications being used and the frequency range of the signal to be sampled. Table 1.1 gives several examples of sampling frequencies, according to different applications.

4 Digital Filters Design for Signal and Image Processing

Signal	f_s	T_s
Speech:		
Telephone band – telephone-	8 KHz	125 μ s
Broadband – audio-visual conferencing-	or 16 KHz	62.5 μ s
Audio: Broadband (Stereo)	32 KHz	31.25 μ s
	44.1 KHz	22.7 μ s
	48 KHz	20.8 μ s
Video	10 MHz	100 ns

Table 1.1. Sampling frequencies according to processed signals

In Figure 1.3, we show an acquisition chain, a processing chain and a signal restitution chain.

The adaptation amplifier makes the input signal compatible with the measurement chain.

A pre-filter which is either pass-band or low-pass, is chosen to limit the width of the input signal spectrum; this avoids the undesirable spectral overlap and hence, the loss of spectral information (aliasing). We will return to this point when we discuss the sampling theorem in section 3.2.2.9. This kind of anti-aliasing filter also makes it possible to reject the out-of-band noise and, when it is a pass-band filter, it helps suppress the continuous component of the signal.

The Analog-to-Digital Converter (A/D) partly carries out sampling, and then quantification, at the sampling frequency f_s , that is, it allocates a coding to each sampling on a certain number of bits.

The digital input signal is then processed in order to give the digital output signal. The reversion into an analog signal is made possible by using a D/A converter and a smoothing filter.

Many parameters influence sampling, notably the quantification step and the response time of the digital system, both during acquisition and restitution. However, by improving the precision of the A/D converter and the speed of the calculators, we can get around these problems. The choice of the sampling frequency also plays an important role.

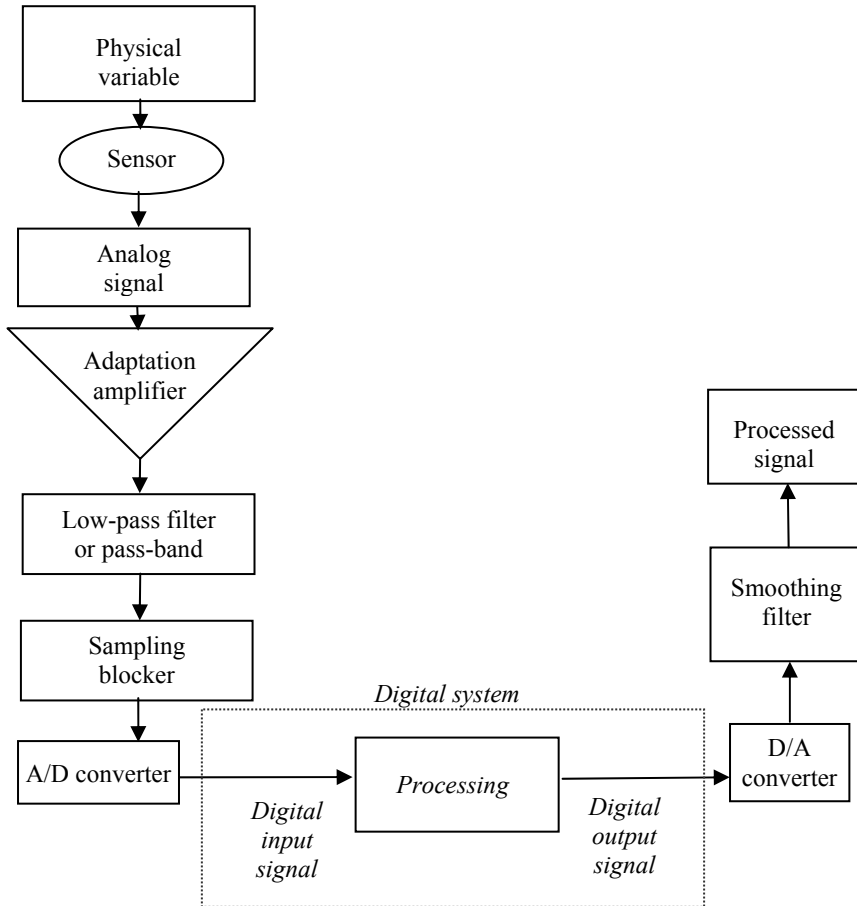


Figure 1.3. Complete acquisition chain and digital processing of a signal

Different types of digital signal representation are possible, such as *functional representations*, *tabulated representations*, *sequential representations*, and *graphic representations* (as in bar diagrams).

Looking at examples of basic digital signals, we return to the unit sample sequence represented by the Kronecker symbol $\delta(k)$, the unit step signal $u(k)$, and the unit ramp signal $r(k)$. This gives us:

$$\text{Unit sample sequence: } \begin{cases} \delta(0) = 1 \\ \delta(k) = 0 \text{ for } k \neq 0 \end{cases}$$

$$\text{Unit step signal: } \begin{cases} u(k) = 1 \text{ for } k \geq 0 \\ u(k) = 0 \text{ for } k < 0 \end{cases}$$

$$\text{Unit ramp signal: } \begin{cases} r(k) = k \text{ for } k \geq 0 \\ r(k) = 0 \text{ for } k < 0. \end{cases}$$

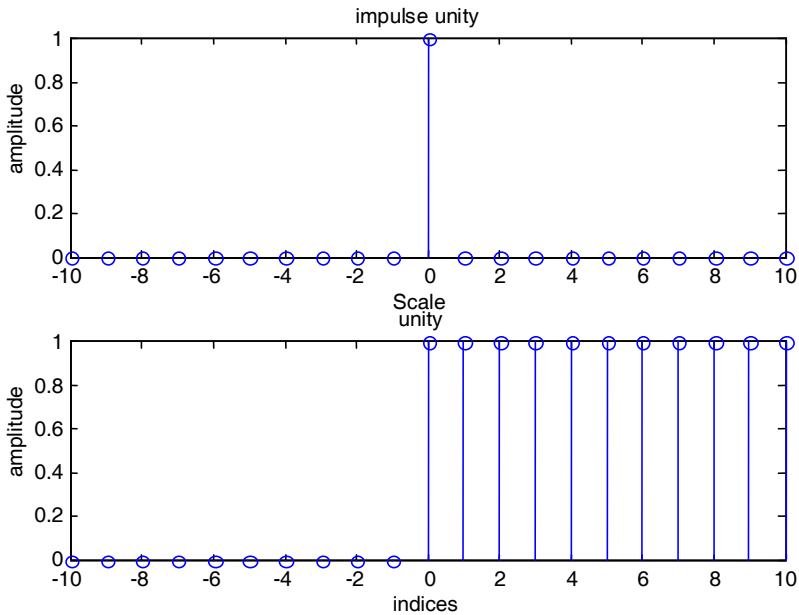


Figure 1.4. Unit sample sequence $\delta(k)$ and unit step signal $u(k)$

1.2.2. Deterministic and random signals

We class signals as being deterministic or random. Random signals can be defined according to the domain in which they are observed. Sometimes, having specified all the experimental conditions of obtaining the physical variable, we see that it fluctuates. Its values are not completely determined, but they can be evaluated in terms of probability. In this case, we are dealing with a random experiment and the signal is called *random*. In the opposite situation, the signal is called *deterministic*.

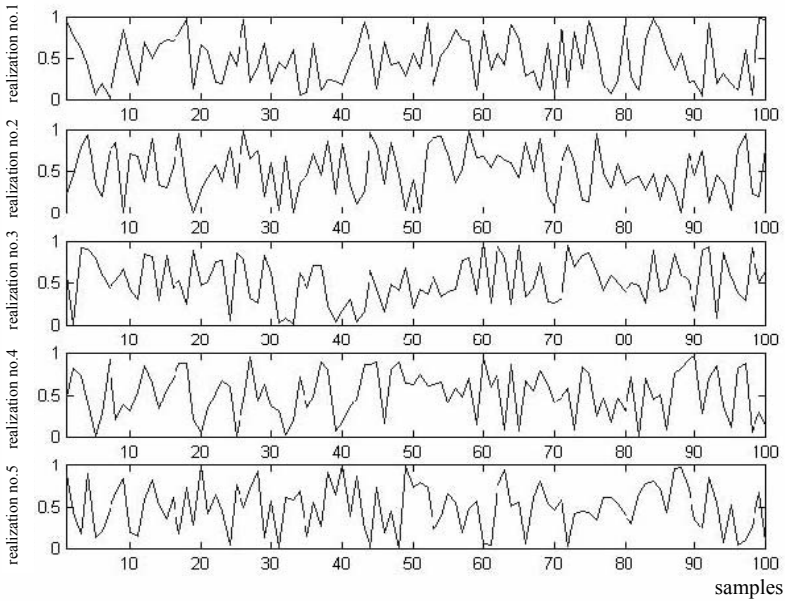


Figure 1.5. Several realizations of a 1-D random signal

EXAMPLE 1.1.– let us look at a continuous signal modeled by a sinusoidal function of the following type.

$$x(t) = a \times \sin(2\pi ft)$$

This kind of model is deterministic. However, in other situations, the signal amplitude and the signal frequency can be subject to variations. Moreover, the signal can be disturbed by an additive noise $b(t)$; then it is written in the following form:

$$x(t) = a(t) \times \sin(2\pi f(t) \times t) + b(t)$$

where $a(t)$, $f(t)$ and $b(t)$ are random variables for each value of t . We say then that $x(t)$ is a random signal. The properties of the received signal $x(t)$ then depends on the statistical properties of these random variables.

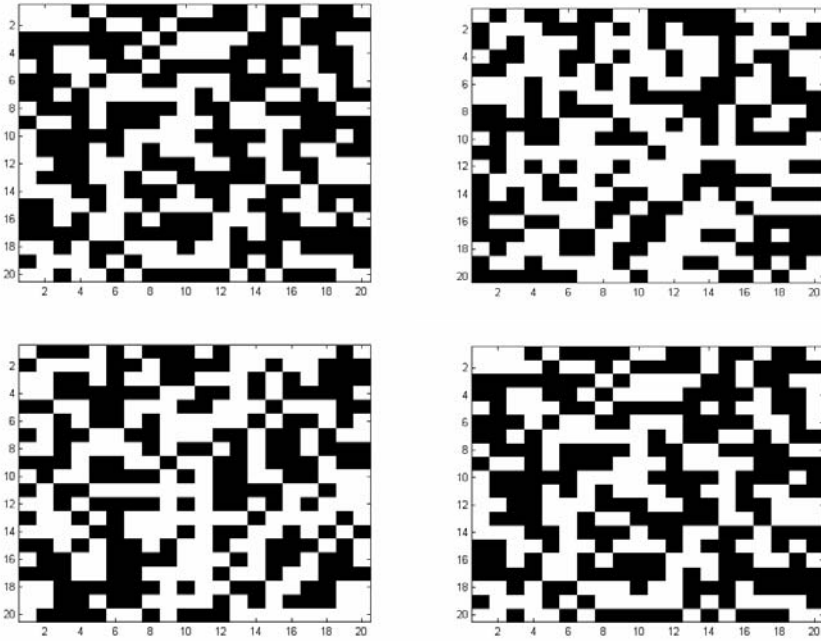


Figure 1.6. Several examples of a discrete random 2-D process

1.2.3. Periodic signals

The class of signals termed periodic plays an important role in signal and image processing. In the case of a continuous-time signal, a signal is called periodic of period T_0 if T_0 is the smallest value verifying the relation:

$$x(t + T_0) = x(t), \quad \forall t .$$

And, for a discrete-time signal, the period of which is N_0 , we have:

$$x(k + N_0) = x(k), \quad \forall k .$$

EXAMPLE 1.2.– examples of periodic signals:

$$x(t) = \sin(2\pi f_0 t), \quad x(k) = (-1)^k, \quad x(k) = \cos\left(\frac{k\pi}{8}\right).$$

1.2.4. Mean, energy and power

We can characterize a signal by its mean value. This value represents the continuous component of the signal.

When the signal is deterministic, it equals:

$$\mu = \lim_{T_1 \rightarrow +\infty} \frac{1}{T_1} \int_{(T_1)} x(t) dt \quad \text{where } T_1 \text{ designates the integration time.} \quad (1.1)$$

When a continuous-time signal is periodic and of period T_0 , the expression of the mean value comes to:

$$\mu = \frac{1}{T_0} \int_{(T_0)} x(t) dt \quad (1.2)$$

PROOF – we can always express the integration time T_1 according to the period of the signal in the following way:

$$T_1 = kT_0 + \xi \quad \text{where } k \text{ is an integer and } \xi \text{ is chosen so that } 0 < \xi \leq T_0.$$

From there, $\mu = \lim_{T_1 \rightarrow +\infty} \frac{1}{T_1} \int_{(T_1)} x(t) dt = \lim_{k \rightarrow +\infty} \frac{1}{kT_0} \int_{(kT_0)} x(t) dt$, since ξ becomes insignificant compared to kT_0 .

By using the periodicity property of the continuous signal $x(t)$, we deduce that

$$\mu = \frac{1}{kT_0} \sum_k \int_{(T_0)} x(t) dt = \frac{1}{T_0} \int_{(T_0)} x(t) dt .$$

When the signal is random, the statistical mean is defined for a fixed value of t , as follows:

$$\mu(t) = E[X(t)] = \int_{-\infty}^{+\infty} x p(x, t) dx , \quad (1.3)$$

where $E[.]$ indicates the mathematical expectation and $p(x, t)$ represents the probability density of the random signal at the instant t . We can obtain the mean value if we know $p(x, t)$; in other situations, we can only obtain an estimated value.

For the class of signals called ergodic in the sense of the mean, we assimilate the statistical mean to the temporal mean, which brings us back to the expression we have seen previously:

$$\mu = \lim_{T_1 \rightarrow +\infty} \frac{1}{T_1} \int_{T_1} x(t) dt .$$

Often, we are interested in the energy ε of the processed signal. For a continuous-time signal $x(t)$, we have:

$$\varepsilon = \int_{-\infty}^{+\infty} |x(t)|^2 dt . \quad (1.4)$$

In the case of a discrete-time signal, the energy is defined as the sum of the magnitude-squared values of the signal $x(k)$:

$$\varepsilon = \sum_k |x(k)|^2 \quad (1.5)$$

For a continuous-time signal $x(t)$, its mean power P is expressed as follows:

$$P = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_T |x(t)|^2 dt . \quad (1.6)$$

For a discrete-time signal $x(k)$, its mean power is represented as:

$$P = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=1}^N |x(k)|^2 \quad (1.7)$$

In signal processing, we often introduce the concept of signal-to-noise ratio (SNR) to characterize the noise that can affect signals. This variable, expressed in decibels (dB), corresponds to the ratio of powers between the signal and the noise. It is represented as:

$$\text{SNR} = 10 \log_{10} \left[\frac{P_{\text{signal}}}{P_{\text{noise}}} \right] \quad (1.8)$$

where P_{signal} and P_{noise} indicate, respectively, the powers of the sequences of the signal and the noise.

EXAMPLE 1.3.– let us consider the example of a periodic signal with a period of 300 Hz signal that is perturbed by a zero-mean Gaussian additive noise with a signal-to-noise ratio varying from 20 to 0 dB at each 10 dB step. Figures 1.7 and 1.8 show these different situations.

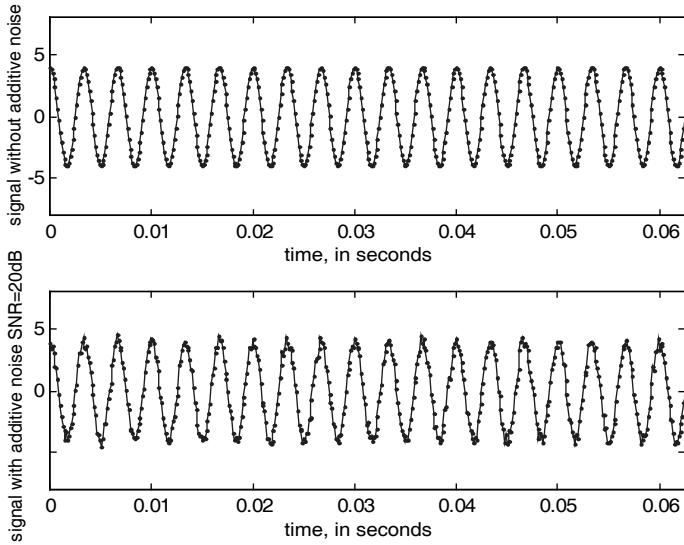


Figure 1.7. Temporal representation of the original signal and of the signal with additive noise, with a signal-to-noise ratio equal to 20 dB

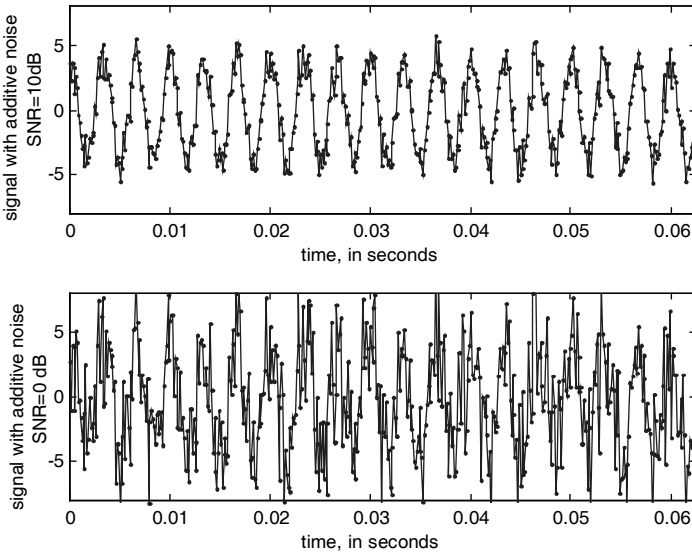


Figure 1.8. Temporal representation of signals with additive noise, with signal-to-noise ratios equal to 10 dB and 0 dB

1.2.5. Autocorrelation function

Let us take the example of a deterministic continuous signal $x(t)$ of finite energy. We can carry out a signal analysis from its autocorrelation function, which is represented as:

$$R_{xx}(\tau) = \int_{-\infty}^{+\infty} x(t)x^*(t-\tau)dt \quad (1.9)$$

The autocorrelation function allows us to measure the degree of resemblance existing between $x(t)$ and $x(t-\tau)$. Some of these properties can then be shown from the results of the scalar products.

From the relations shown in equations (1.4) and (1.9), we see that $R_{xx}(0)$ corresponds to the energy of the signal. We can easily demonstrate the following properties:

$$R_{xx}(\tau) = R_{xx}^*(-\tau) \quad \forall \tau \in \mathbb{R} \quad (1.10)$$

$$|R_{xx}(\tau)| \leq R_{xx}(0) \quad \forall \tau \in \mathbb{R} \quad (1.11)$$

When the signal is periodic and of the period T_0 , the autocorrelation function is periodic and of the period T_0 . It can be obtained as follows:

$$R_{xx}(\tau) = \frac{1}{T_0} \int_0^{T_0} x(t)x^*(t-\tau)dt \quad (1.12)$$

We should remember that the autocorrelation function is a specific instance of the intercorrelation function of two deterministic signals $x(t)$ and $y(t)$, represented as:

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)y^*(t-\tau)dt \quad (1.13)$$

Now, let us look at a discrete-time random process $\{x(k)\}$. We can describe this process from its autocorrelation function, at the instants k_1 and k_2 , written $R_{xx}(k_1, k_2)$ and expressed as

$$R_{xx}(k_1, k_2) = E \left[x(k_1)x^*(k_2) \right] \quad \forall (k_1, k_2) \in \mathbb{Z} \times \mathbb{Z}, \quad (1.14)$$

where $x^*(k_2)$ denotes the conjugate of $x(k_2)$ in the case of complex processes.