Dimensional Analysis and Black Holes

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Abstract

This is a brief introduction to dimensional analysis for high school students. We start with simple algebraic rules for finding dimensions, then proceed to applications: checking answers and systematically determining scales for a system. We move on to the role of fundamental constants, and finish by "deriving" various properties of black holes. Based on a talk for the UBC Metro Vancouver Physics Circle, February 2019.

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Please email me with any corrections. Check out my website for more maths and physics!

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Acknowledgements

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1 Introduction

1.1 Numbers and measurements

Let's start by making a clear distinction between maths and physics. Maths deals with numbers, and generalisations of numbers, e.g. matrices and functions. Mathematical laws give us relationships between these objects. Pythagoras' Theorem, for instance, tells us about relationships between the sides of a triangle. Physics, on the other hand, deals with *measurements*, things we can detect in experiments. Physical laws give us relationships between measurements, and measurements are more than just numbers.

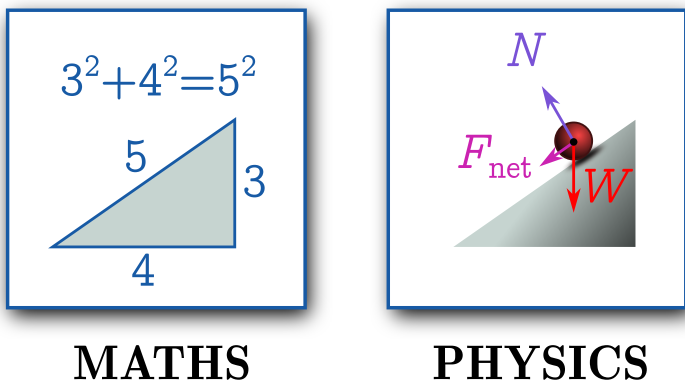

Figure 1: Maths vs physics. Maths gives relationships between numbers, while physics gives relationships between measurements.

To illustrate, suppose I go away and do an experiment on a piece of chalk, then come back and tell you: "The answer is 10!" What's missing? The first problem is that I haven't told you what *attribute* of the chalk I'm measuring. I need to choose some physical property of the chalk to measure, such as its mass or its length. This is a *dimension*: a physical property we can measure. Other examples are mass, length, time, force, and energy.

Box 1: Dimensions

Dimensions are physical properties we can measure in experiments.

Suppose I go away and measure the *length* of the chalk, then come back and say "The length is 10!" What's the problem now? I haven't specified the *units*. It could be 10 centimetres or 10 furlongs; you don't know which.

The problem is that there is no intrinsic measure of length. You have no way to convert that number into a length unless we agree beforehand on what "1 unit" of length means. That's all a "unit" is: a measurement we agree on beforehand, which lets us convert a number into a measurement and vice versa. Put a different way, the numbers appearing before units when we quote measurements are just *ratios*. Saying that something weighs 10 kilograms means that it's 10 times heavier than the kilogram mass kept in a glass jar in Paris.

> **Box 2: Units**
>
> *Units* are special measurements, typically chosen by convention. Working in a particular set of units means that you report you answer as a *ratio* between your measurement and the special measurement.

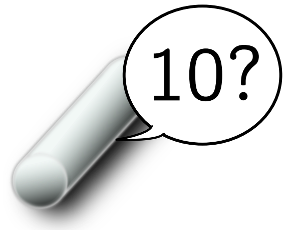

Figure 2: A measurement is more than just a number.

1.2 Dimensional algebra

You're probably used to doing algebra with units, for instance converting between metres and centimetres. We're going to do something similar, but even easier: algebra with dimensions. This is just like regular high school algebra, where we have symbols like x, y and z standing for unknowns. Even if we don't know what the numbers are, we can manipulate them using index laws:

$$(xy)^n = x^n y^n, \quad x^{-n} = \frac{1}{x^n}, \quad (x^m)^n = x^{mn}.$$

Exactly the same algebra works for dimensions, where we replace the unknowns with symbols standing for different dimensions, that is, physical measurements we can make. We're going to choose a set of *basic dimensions* in terms of which we can write everything else.

> **Box 3: Basic dimensions**
>
> We will use the set of *basic dimensions*:
>
> - mass M
> - length L
> - time T.
>
> We can reduce more complicated dimensions to some product of these.

If someone hands me a measurement, I can throw away the number and just ask: "What is the physical property the unit measures?" For instance, if I measure the length of the chalk to be $\ell = 10\,\mathrm{cm}$, then throwing away the number leaves cm, which measures length. We say that the *dimension* of ℓ is length, L, and write

$$[\ell] = [10\,\mathrm{cm}] = [\mathrm{cm}] = L.$$

Let's define this *dimension-taking function* a little more formally, and record some of its algebraic properties.

Box 4: Dimension-taking function

If A is a measurement, $[A]$ is the *dimension of* A. We call $[\cdot]$ the *dimension-taking function*, or *bracket* for short. The bracket has the following properties:

$$[AB] = [A][B] \tag{1}$$
$$[A^n] = [A]^n, \quad \text{for all } n \in \mathbb{R}. \tag{2}$$

The second identity applies to *any* real number n, not just positive integers (which would follow from the first identity). This is all a bit dry, so let's do some examples.

Calculation 1: Area (units)

Suppose the area of this blackboard is $A = 2\,\mathrm{m}^2$. Then the dimension is

$$[A] = \left[2\,\mathrm{m}^2\right] = \left[\mathrm{m}^2\right] = [\mathrm{m}]^2 = L^2.$$

We throw away the number, and take the square outside the bracket using either (1) or (2). We learn that the dimension of this measurement is L^2. In fact, the dimension of area generally is L^2.

Let's do another example but for different units.

Calculation 2: Velocity (units)

Suppose I throw the chalk, and it travels at some velocity $v = 5\,\mathrm{m/s}$. Then the dimension is

$$[v] = \left[5\mathrm{m}\,\mathrm{s}^{-1}\right] = \left[\mathrm{m}\,\mathrm{s}^{-1}\right] = [\mathrm{m}][\mathrm{s}^{-1}] = [\mathrm{m}][\mathrm{s}]^{-1} = \frac{L}{T}.$$

We used (1) for the third equality, and (2) for the fourth. We learn that the dimension of velocity is L/T.

Exercise 1: Odd units

Find the dimension of a measurement

$$H_0 = 70\,\frac{\mathrm{km}}{\mathrm{s}\cdot\mathrm{Mpc}},$$

where $1\,\mathrm{Mpc} \approx 3 \times 10^{19}\,\mathrm{km}$. (This is the conventional set of units for measuring the expansion of the universe.)

In the two examples above, we took the dimension of a specific measurement which included units. From our knowledge of units, we could read off the dimension. You might wonder if it's possible to calculate the dimension in a more general way, which doesn't require units and applies to all measurements of that type. We can solve both problems by using *equations*. This approach is easiest to understand with an example.

Calculation 3: Area and velocity (equations)

To find the units of area in general, we can use the formula for the area of a rectangle,[a] $\text{area} = \text{height} \times \text{width}$. Thus,

$$[\text{area}] = [\text{width} \times \text{height}] = [\text{width}][\text{height}] = L^2.$$

Width and height both have the dimension of length; they're things we can measure with a ruler. Similarly, velocity is defined as distance over time, so

$$[\text{velocity}] = \frac{[\text{distance}]}{[\text{time}]} = \frac{L}{T}.$$

[a] You might reasonably object that not everything is a rectangle. It turns out you can take an arbitrary shape and split it into tiny rectangles!

Exercise 2: Volume and acceleration

Find the dimensions of (a) volume, and (b) acceleration.

The laws of physics provide an even more powerful way of learning about dimensions. Let's take Newton's second law, $F = ma$. We look at the dimension of both sides, and from the dimensions for acceleration we just worked out, we can find the dimension of force.

Calculation 4: Newton's second law

From $F = ma$, and the dimensions of acceleration (see §A.2), we have

$$[F] = [m][a] = \frac{ML}{T^2}.$$

Force appears all over the shop in physics, so you can use its dimensions to quickly find the dimension of other things.

Exercise 3: Energy

Find the dimensions of energy. *Hint.* The quickest way to do this is to use a formula that relates energy and force.

This concludes the boring part of the notes. We can now use dimensional analysis to learn interesting things about physics.

2 Applications

2.1 Checking answers

The first application is a simple one: checking answers. Suppose you solve an elaborate kinematics problem, and end up with a formula which is equally elaborate. You have no idea if it's correct. But if the dimensions on the left hand side don't match the dimensions on the right hand side, then you know the answer is wrong!

Suppose I drill a hole through the middle of the earth, and drop a package into the hole to deliver to the other side of the world. How long does it take to get to the other side? We will

Figure 3: Presents for the antipodes.

call this the *gravitational postal service* (GPS) problem. Suppose that after some calculations, I write down

$$t = \frac{R}{g},$$

where R is the radius of the earth and g is the acceleration due to gravity at the surface. This makes sense: it takes more time if the earth is bigger, and less if gravity is stronger. But let's check the dimensions.

Calculation 5: Gravitational postal service

The LHS clearly has T, so the RHS needs to match:

$$\left[\frac{R}{g}\right] = \frac{L}{[g]} = \frac{L}{L/T^2} = T^2.$$

Our answer is clearly wrong because the dimensions don't match!

Exercise 4: Fall time

The time it takes for an object to fall to the ground from height h is given by the kinematics formula $t = \sqrt{2h/g}$. Check this makes dimensional sense.

2.2 Finding answers

This strategy for checking answers can be upgraded into a strategy for *finding* answers. You may have already guessed a simple way to improve the answer to the GPS problem: take the square root, and we get something with the right dimensions! Indeed, the correct answer is

$$t = \pi\sqrt{\frac{R}{g}}. \tag{3}$$

We miss the factor of π, since taking dimensions doesn't tell us about numbers, but we've somewhat magically produced the right answer from an easily correctable guess. Let's make this less magical by giving a general procedure for finding answers from scratch. (If you are interested in seeing where the π comes from, see Problem 4.1.)

We'll illustrate with GPS once more. We start by writing all the things that could be relevant to the physics of the problem:

- the radius of the earth R;
- the width of the hole w;
- the mass of package m;
- the mass of the earth M;
- the acceleration at the surface g; and
- the gravitational constant G, which governs the strength of gravity.

This list is fairly large (and could be even longer), but let's winnow it down using physical reasoning. We assume the hole is small compared to the size of the earth, and the package light compared to the mass of the earth, so we can neglect h and m. See Fig. 2.2 for an illustration of the scenarios we are ignoring. Finally, we can trade the mass of the earth and Newton's constant for the surface acceleration g, since $F = mg$ and $F = GMm/R^2$ imply

$$\frac{GM}{R^2} = g.$$

Ultimately, we're left with two things, g and R. We proceed systematically, guessing that our target quantity t is *a product of powers of g and R.*

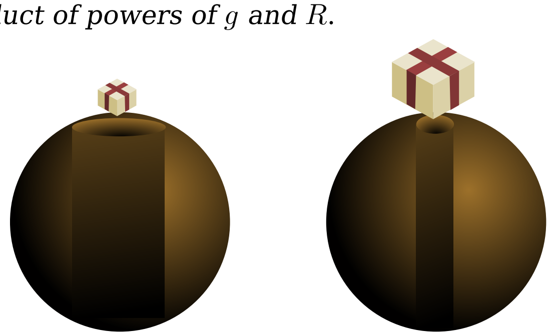

Figure 4: On the left, a large hole. On the right, a heavy package.

Calculation 6: GPS (dimensional analysis)

We guess a relation of the form

$$t \sim g^a R^b$$

where a and b are unknown powers, and the $\sim$ indicates that there may be some numbers in front, like the π we saw earlier. Now we just take dimensions of the RHS and use the algebraic relations (1) and (2):

$$[g^a R^b] = [g]^a [R]^b = \left(\frac{L}{T^2}\right)^a L^b = L^{a+b} T^{-2a}.$$

This needs to match the LHS, which has the dimensions of time. The factor of L must go away, and the power of T should be 1, which we can encode as two linear equations:

$$-2a = 1, \quad a + b = 0.$$

From the first equation we find $a = -1/2$, and the second tells us $b = 1/2$. This means

$$t \sim g^{-1/2} R^{1/2} = \sqrt{\frac{R}{g}}.$$

This is a much less magical way of getting the same answer!

Exercise 5: Fall time revisited

Use dimensional analysis to "derive" the kinematics formula $t \sim \sqrt{h/g}$, assuming that the height h, mass m, and acceleration g are the only relevant factors.

Let's summarise the steps we used above, but now for a general dimensional analysis problem. This will give us a powerful tool for determining how dimensionful quantities in a system are related, subject to some caveats we describe in the next section.

Box 5: Dimensional analysis

1. Write down all the dimensionful parameters that could be relevant, $\{A_1, A_2, \ldots\}$.
2. Use physical reasoning to make the list as small as possible. This produces a shortlist $\{B_1, B_2, \ldots\}$.
3. Find the dimensions of terms on the list, $\{[B_1], [B_2], \ldots\}$.
4. Write the target quantity C as a product of powers $C \sim B_1^a B_2^b \cdots$.
5. Take dimensions of both sides using the rules (1), (2) and Step 3, and solve the resulting system of equations for the unknown powers $\{a, b, \ldots\}$.

2.3 Scales and ratios

Although we have been dealing with dimensions, numbers are obviously important too. For instance, in the GPS example the answer (3) includes a factor of π we can't get from dimensional analysis. In fact, if I ask *any* question with the dimensions of time (e.g. how long does the package take to reach the centre of the earth), our dimensional analysis would give us the same answer: some multiple of $\sqrt{R/g}$.

We could think of this as a weakness. But it's better to think of it as an answer to a different question: how long does it take for something interesting to happen? This is called a *time scale*, and loosely speaking, it's a unit for measuring time adapted to the system. Interesting stuff happens at the *order of magnitude* given by the scale. In general, dimensional analysis gives scales (length scales, time scales, etc.) rather than answers to specific questions.

Box 6: Scales

Dimensional analysis produces *scales* rather than answers to specific questions. These can be viewed as units adapted to the system, giving order-of-magnitude estimates for where interesting behaviour occurs.

But there's a catch. Dimensional analysis only produces these scales when the powers in $C \sim B_1^a B_2^b \cdots$ can be uniquely determined. This can fail when some combination of parameters forms a *dimensionless ratio*.

Calculation 7: GPS (dimensionless ratios)

Suppose the first situation in Fig. 2.2 is realised. If the hole width h becomes comparable to the size of the earth, we need to include it in our analysis. Our guess for the time scale will become

$$t \sim g^a R^b w^c = g^a R^{b+c} \left(\frac{w}{R}\right)^c .$$

Since w/R is dimensionless, there is no way of fixing c by comparing to the LHS.

When it works, dimensional analysis forces things to be related by simple powers. But when a dimensionless ratio arrives on the scene, we can put it into functions like $\cos(x)$, $\sin(x)$ and e^x, which only take dimensionless numbers as arguments.[1] These functions can change our answers by much more than an order of magnitude, and we need to go beyond dimensional analysis to understand things. For a simple example of how a dimensionless ratio controls the physics of a system, see Problem 4.3.

Exercise 6: GPS with a large hole

The presence of dimensionless ratios does not make dimensional analysis useless. Show that $t \propto g^{-1/2}$ in the GPS problem, even if the hole is comparable to the size of the earth.

[1] Why? Because these functions can be expressed as *sums of powers* of x. To add the powers together, x must be dimensionless. For instance, for small x, $e^x \approx 1 + x$. This only makes sense if x, like 1, is dimensionless.

Notice that mass plays no role in our calculation, and we can restrict to the base dimensions L, T. We have three 3 relevant parameters (R, w, g) and 2 base dimensions (L, T). The difference $3 - 2 = 1$ is precisely the number of ratios. This isn't a coincidence, but an example of a more general result called the *Buckingham π theorem*.

Box 7: Buckingham π theorem

Consider a dimensional equation involving n dimensionful parameters, expressed in terms of k base dimensions. If $n > k$, there are $n - k$ independent dimensionless ratios. If $n \leq k$, you can uniquely determine the relation between quantities.

The justification is straightforward. An equation can determine at most one unknown power. There are n unknown powers (one for each term in our list) and k equations (one for each base dimension). If $n > k$, we will have $n - k$ leftover powers, which manifest as dimensionless ratios. There are a few subtleties I'm sweeping under the rug. For our purposes, the main thing to note is that k is the *number of distinct dimensions* needed to express all the quantities involved, rather than the total number of base dimensions.

Exercise 7: GPS and Buckingham π

Consider the GPS problem with a large hole h and heavy package m. How many dimensionless ratios do you get? Is this consistent with the Buckingham π theorem?

2.4 The Trinity Test

We will devote this section to a classic example of dimensional analysis. In 1945, the US tested the first atomic bomb somewhere in the deserts of New Mexico. This was called the *Trinity Test*. They took some photos of the explosion, and a couple of years later, published them in a popular science magazine. The strength of the bomb was strictly classified, but a British physicist called Geoffrey Taylor was able to determine how much energy was released in the explosion just by looking at the photograph!

We're going to skip the first three steps of the systematic dimensional analysis method above, and write down a shortlist of parameters and their dimensions. They are:

- the time after detonation t, with dimension $[t] = T$;
- the radius of the explosion R, with dimension $[R] = L$;
- and the density of air ρ, with dimension $[\rho] = M/L^3$.

Exercise 8: Density and gravity in the Trinity Test

Why do you think the density of air is involved, but not the gravitational acceleration g?

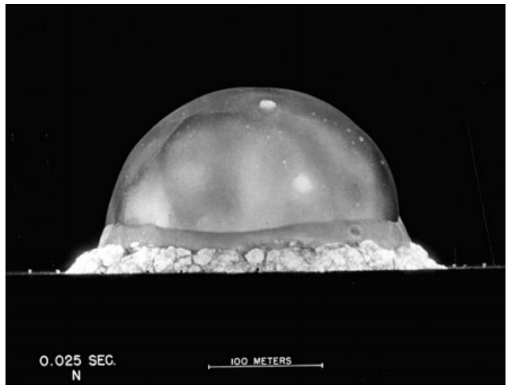

Figure 5: The Trinity explosion, $25\,\mathrm{ms}$ after detonation.

Calculation 8: Trinity Test

We now write down an equation of the form

$$E \sim t^a R^b \rho^c.$$

As we discovered earlier, the dimensions of energy are $[E] = ML^2T^{-2}$. We could solve the simultaneous equations, but here, the dimensions are simple enough to read off the answer. First of all, only ρ has dimensions of mass, so to match energy we must set $c = 1$. Similarly, only t has dimensions of time, so we must set $a = -2$. This gives

$$\frac{ML^2}{T^2} = [E] = [t^{-2}R^b\rho] = \frac{1}{T^2} \cdot L^b \cdot \frac{M}{L^3} = \frac{ML^{b-3}}{T^2} \quad \Longrightarrow \quad b = 5.$$

We learn that

$$E \sim \frac{\rho R^5}{t^2}.$$

Now we can just plug in numbers to make an order-of-magnitude estimate of the energy released by the bomb. From the label on the photo, we see that $t = 25\,\mathrm{ms}$. We guess that $R(t) \approx 140\,\mathrm{m}$. Finally, a cubic metre of air weighs about a kilogram, so $\rho \approx 1\,\mathrm{kg/m}^3$. We plug these into our expression for energy to get

$$E \approx \frac{\rho R^5}{t^2} \approx \frac{1 \cdot 140^5}{0.025^2}\,\mathrm{J} \approx 8.6 \times 10^{12}\,\mathrm{J}.$$

It's conventional to estimate the strength of explosives in terms of equivalent tons on dynamite (or TNT), where

$$1\,\mathrm{ton\,of\,TNT} = 4.2 \times 10^9\,\mathrm{J}.$$

We find

$$E \approx \frac{8.6 \times 10^{12}}{4.2 \times 10^9}\,\mathrm{tons\,of\,TNT} \approx 20\,\mathrm{kilotons\,of\,TNT}.$$

This is not just correct to an order of magnitude. It's correct to the nearest kiloton![a]

[a] Incidentally, this was a rather small bomb. A "typical" nuclear detonation releases energy in the range of *megatons* of TNT.

3 Black holes and dimensional analysis

3.1 Fundamental constants

For our last topic, we will venture into the mysterious realm of quantum gravity, and use dimensional analysis to learn about *black holes*. Before we do that, I need to tell you a little about fundamental constants.[2]

Box 8: Fundamental constants

A *fundamental constant* is a constant with dimensions which appears in physical laws.

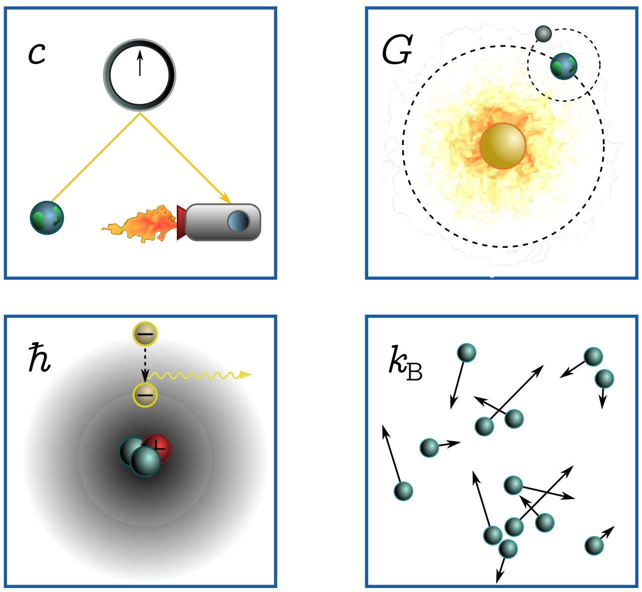

Figure 6: Fundamental constants: the speed of light c; the gravitational constant G; Planck's constant $\hbar$; Boltzmann's constant k_{B}.

A nice unifying theme for fundamental constants is that they tell us about different forms of energy. Our first and most famous example is

$$E_{\mathrm{mass}} = mc^2. \tag{4}$$

This captures Einstein's famous insight that mass and energy are equivalent. Here, c is the speed of light, which has dimensions $[c] = L/T$, and value in SI units

$$c = 3.00 \times 10^8 \, \mathrm{m} \cdot \mathrm{s}^{-1}. \tag{5}$$

[2]We will only discuss fundamental constants *with dimension*; there are also *dimensionless* fundamental constants, but as we might guess from §2.3, these cannot be handled using the tools of dimensional analysis. Indeed, these dimensionless constants are connected to some of the deepest unsolved problems in physics.

The speed of light appears when things travel fast or light is involved; this is the domain of *special relativity*. You might wonder what's actually moving quickly in (4). In fact, (4) is a special case of a less well-known equation which applies to moving objects with momentum p:

$$E = \sqrt{p^2c^2 + m^2c^4}. \tag{6}$$

This isn't quite as punchy! But it reduces to (4) when $p = 0$.

Energy can also be stored in gravitational fields. If two point masses M and m are a distance r apart, they have potential energy[3]

$$E_{\text{grav}} = -\frac{GMm}{r}, \tag{7}$$

where G is the gravitational constant:

$$G = 6.67 \times 10^{-11}\,\text{N} \cdot \text{m}^2 \cdot \text{kg}^{-2}. \tag{8}$$

Exercise 9: Dimensions of G

Check using Newton's law of gravitation that G has the dimensions $[G] = L^3/MT^2$.

There are two other fundamental constants we will need. The first is associated with the energy of *quantum particles*, and in particular, a particle like a photon which oscillates with some angular frequency ω,[4]

$$E_{\text{quant}} = \hbar\omega$$

where $\hbar$ is *Planck's (reduced) constant*:

$$\hbar = 1.05 \times 10^{-34}\,\text{J} \cdot \text{s}. \tag{9}$$

Since frequency has dimensions of inverse time T^{-1}, the dimension of $\hbar$ is

$$[\hbar] = \frac{[E]}{[\omega]} = \frac{ML^2/T^2}{1/T} = \frac{ML^2}{T}.$$

The last constant we need is associated with *thermodynamics*, the physics of big, hot blobs of matter. For a system of N particles at temperature $\mathcal{T}$, such as an ideal gas, the thermal energy is related to the temperature by

$$E_{\text{therm}} \propto Nk_{\text{B}}\mathcal{T}, \tag{10}$$

where k_{B} is *Boltzmann's constant*:

$$k_{\text{B}} = 1.38 \times 10^{-23}\,\text{J} \cdot \text{K}^{-1}. \tag{11}$$

The constant of proportionality is just a number, determined by the properties of the system.We need to introduce another basic dimension to keep track of temperature.

[3]The minus sign is because they are attracted. We need to give them energy to separate them!

[4]Recall that angular frequency $\omega = 2\pi f$, where f is just the regular frequency.

Box 9: Temperature dimension

Temperature is associated with a basic dimension Θ.

In terms of this new base dimension, (10) implies that

$$[k_\mathrm{B}] = \frac{[E]}{[\mathcal{T}]} = \frac{ML^2}{\Theta T^2}.$$

Box 10: Dimensional analysis with fundamental constants

In a dimensional analysis, we should include:

- the gravitational constant G, $[G] = L^3/MT^2$, when things are heavy;
- the speed of light c, $[c] = L/T$, when things are fast;
- Planck's constant $\hbar$, $[\hbar] = ML^2/T$, when things are quantum;
- and Boltzmann's constant k_B, $[k_\mathrm{B}] = ML^2/\Theta T^2$, when things are hot.

Exercise 10: Stefan-Boltzmann law

Max Planck showed that hot objects *glow* due to the quantum excitation of particles on the surface. When these particles de-excite they emit light. We will determine the total radiant power P using dimensional analysis and the rules of thumb in Box 10.

(a) Which fundamental constants do you expect to be involved?

(b) The total amount of energy radiated will depend on the surface area of the object A. Introduce a new dimension, N, for the amount of matter the surface. Find the dimensions of A and P, including this new dimension, and explain why the fundamental constants do not depend on N.

(c) Using dimensional analysis and the result of (a) and (b), show that

$$P \sim \left(\frac{k_\mathrm{B}^4}{\hbar^3 c^2}\right) A\mathcal{T}^4. \qquad (12)$$

The constants in (12) are condensed into the *Stefan-Boltzmann constant*

$$\sigma = \frac{\pi^2 k_\mathrm{B}^4}{60\hbar^3 c^2} \approx 5.67 \times 10^{-8} \frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{K}^4}. \qquad (13)$$

Perfect emitters of radiation are called *blackbodies*, with radiant power

$$P_\mathrm{bb} = \sigma A \mathcal{T}^4. \qquad (14)$$

An imperfect emitter has some *emissitivity* $\epsilon < 1$, and $P = \epsilon P_\mathrm{bb}$.

3.2 Black holes

In black holes, *all* of the fundamental constants will get involved somehow. As you probably know, a black hole is an object so heavy that not even light can escape from it. We can define it a little more formally.

Box 11: Black holes (classical)

A *black hole* is a region of spacetime which traps light. The boundary of this region is called the *event horizon*.

Let's see if we can guess the size of the black hole, i.e. light-trapping region, based on its mass and some dimensional analysis. Since black holes are heavy, we must include G, and since light is involved, we should also include c.

Calculation 9: Black hole size

We guess the size R of the black hole is related to its mass m by

$$R \sim m^a G^b c^d.$$

Taking dimensions, we get L on the LHS, and on the RHS

$$M^a \cdot \left(\frac{L^3}{MT^2}\right)^b \cdot \left(\frac{L}{T}\right)^d = M^{a-b} L^{3b+d} T^{-(2b+d)}.$$

Matching the LHS and RHS, we need to solve the system of linear equations

$$a - b = 0, \quad 3b + d = 1, \quad 2b + d = 0.$$

Subtracting the third from the second gives $b = 1$, hence $a = 1$ and $d = -2$. The size of a black hole is then related to its mass by

$$R \sim \frac{Gm}{c^2}. \tag{15}$$

As usual, asking a specific question will produce a specific number out the front. For instance, the radius of a spherical black hole of mass m is

$$R_{\mathrm{s}} = \frac{2Gm}{c^2}. \tag{16}$$

This is called the *Schwarzschild radius*. Since light travels faster than anything else, *anything* which falls inside the Schwarzschild radius is trapped in the black hole forever!

Exercise 11: Schwarzschild radius

The *escape velocity* v_{esc} for a particle of mass m_{particle} in the gravity well of a mass M is defined as the velocity needed to escape the gravity well (i.e. to get infinitely far away) with no kinetic energy left over. This implies that

$$E_{\text{kin}} + E_{\text{grav}} = \frac{1}{2} m_{\text{particle}} v_{\text{esc}}^2 - \frac{GMm_{\text{particle}}}{R} = 0.$$

Show that the escape velocity equals the speed of light $v_{\text{esc}} = c$ precisely at the Schwarzschild radius (16).

Classically, this is where the story ends. But there are reasons to think that there is more to a black hole than meets (or doesn't meet) the eye. Consider a hot box of gas. From the perspective of a macroscopic experimentalist, it has only a few coarse features, such as volume and energy. But with a microscope, we can zoom in and see the much more detailed features of atomic motion, with $\sim 10^{23}$ particles zipping about and banging into each other.

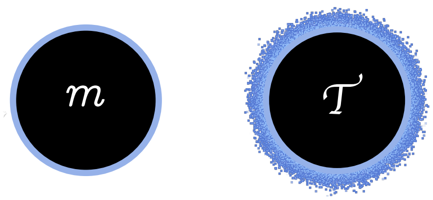

Figure 7: *Left.* A classical black hole of mass m. *Right.* A black hole as a thermodynamic system with temperature $\mathcal{T}$, subject to quantum effects.

The black hole is perhaps similar. From the perspective of an experimentalist stationed outside, the black hole just has a volume $V \sim R^3$ and energy $E = mc^2$. But they could make the black hole by collapsing a box of gas inside its own Schwarzschild radius; with the right sort of microscope, perhaps they could peek inside! Since nothing can escape the black hole, it's not clear how such a microscope could work, though we will get some hints below. For the moment, we will simply hypothesise that, like the hot box of gas, the black hole is a thermodynamic system. This means k_{B} will be involved.

If a black hole is a thermodynamic system, we can contemplate the possibility it has a temperature. But what would this mean? In everyday life, hot objects *radiate*. If you heat up coals in a fire, they glow red; a tungsten filament is incandescent because the current makes it hot. Since black holes form in the vacuum of space, radiation would be the *only* way for us to tell if they have a temperature! As discussed in Exercise 10, hot objects glow because of quantum oscillators on the surface. We therefore throw $\hbar$ into the mix.

Box 12: Black holes (thermodynamics)

Black holes are thermodynamic systems consisting of many particles. Classically, the details of the system are hidden behind the event horizon. Quantum mechanically, they have a temperature and can radiate.

With k_{B} and $\hbar$ on our list of relevant quantities, we can determine the temperature of a black hole. We will treat R in (15) as a *fixed* system size and forget about gravity, so G is not needed for our analysis. We include c since temperature manifests as radiation.

Calculation 10: Black hole temperature

We are led to a guess of the form

$$\mathcal{T} \sim R^a k_{\mathrm{B}}^b \hbar^d c^e.$$

The dimension on the LHS is temperature, while the right is

$$L^a \cdot \left(\frac{ML^2}{\Theta T^2}\right)^b \cdot \left(\frac{ML^2}{T}\right)^d \cdot \left(\frac{L}{T}\right)^e = L^{a+2b+2d+e} M^{b+d} T^{-(2b+d+e)} \Theta^{-b}.$$

We get a system of equations

$$a + 2b + 2d + e = 0 \tag{17}$$

$$b + d = 0 \tag{18}$$

$$2b + d + e = 0 \tag{19}$$

$$-b = 1. \tag{20}$$

We find b from (20), then d from (18), then subtract (19) from (17) to find a. Finally, we plug our numbers back into (17) or (19) to find e. The end result is $a = b = -1$ and $d = e = 1$. This leads to the estimate of black hole temperature

$$\mathcal{T} \sim \frac{\hbar c}{k_{\mathrm{B}} R} \sim \frac{\hbar c^3}{k_{\mathrm{B}} G m}, \tag{21}$$

simplifying with (15).

This agrees with Stephen Hawking's famous calculation of the *Hawking temperature*, up to a number out the front. After some much harder work, he discovered that

$$\mathcal{T} = \frac{\hbar c^3}{8\pi k_{\mathrm{B}} G m}. \tag{22}$$

The factor 8π is the difficult part! But it was a complete shock to Hawking, and the rest of the physics community, to discover that "black" holes could glow.

Exercise 12: Black hole evaporation

The Stefan-Boltzmann law (14) tells us that a black hole loses energy at a rate

$$P = \sigma A \mathcal{T}^4.$$

The total energy is $E = mc^2$. Use these to estimate the lifetime t_{evap} of the black hole in terms of the black hole mass m.

Let's return to the question of whether a black hole is like a hot box of gas. Trying to answer this will give us some clues about the "microscope" needed to look inside, and at the same, raise a profound puzzle. I said before that the energy of a system of N hot particles is

$$E \sim N k_{\mathrm{B}} \mathcal{T}.$$

The energy of the black hole is $E = mc^2$. If we take the thermodynamic nature of black holes seriously, we should be able to equate these two forms of energy.

Calculation 11: Black hole particle number

Since we have an expression for the temperature (21), we can also estimate N, the number of particles in the black hole, using (15) to tidy up:

$$N \sim \frac{E}{k_{\mathrm{B}}T} \sim \frac{Gm \cdot mc^2}{\hbar c^3} \sim \left(\frac{Gm}{c^2}\right)^2 \frac{c^3}{G\hbar} \sim \frac{R^2}{G\hbar/c^3}. \tag{23}$$

Though perhaps not immediately clear, formula (23) is remarkable. Most importantly, it tells us that the effective number of particles N in the black hole scales like the *area* $A \sim R^2$, rather than the *volume* $V \sim R^3$. This suggests that the thermodynamic system underlying the black hole lives on its surface! This is not just a curiousity, but a fundamental limitation of gravity. If you try to pack more than N particles into a region of volume V, it will collapse into a black hole![5] In other words, the maximum number of particles you can fit into a region V, and still describe using gravity, scales as the surface area of the region.

This suggests that gravity in a blob of space is somehow encoded on the *surface* of that region; if our black hole story is to be believed, that encoding should involve thermodynamics and quantum physics. Since this is similar to the way a three-dimensional image emerges from a two-dimensional grating in optical holograms, this property of gravity is called the *holographic principle*.

Box 13: The Holographic Principle

Gravity in a three-dimensional region of space V can be described in terms of a system on the boundary of V exhibiting thermodynamic and quantum properties.

The second interesting thing about (23) is that, in order to get something dimensionless on the RHS, we must divide R^2 by something with the dimensions of area, $\mathcal{A} = G\hbar/c^3$. If we take the square root, we get a special length called the *Planck length*.

[5] This follows from the intepretation of N as the *entropy* of both the box of gas and the black hole. The Second Law of Thermodynamics states that entropy always increases, a fact which can be used to prove that black holes have *maximum entropy density*. Try to violate the Second Law of Thermodynamics, and it will protect itself by making black holes!

Box 14: The Planck length

The *Planck length* is the unique length built out of fundamental constants:

$$\ell_{\mathrm{P}} = \sqrt{\frac{G\hbar}{c^3}}. \tag{24}$$

The Planck length is the smallest distance that makes sense in a theory of quantum gravity. See Appendix A.1 for more on units formed from fundamental constants.

Exercise 13: Planck units

Find expressions for (a) the *Planck time* t_{P}, and (b) the *Planck energy* E_{P}. These are just expressions built out of fundamental constants with the dimensions of time and energy. *Hint.* You can either use dimensional analysis, or the expression for the Planck length and some shortcuts.

Roughly speaking, the Planck length is the smallest sensible length. If you try to zoom in more, quantum fluctuations become so violent that the notion of spacetime breaks down! The result (23) tells us this. In a sphere with diameter ℓ_{P}, we can fit ~ 1 particle before it collapses to form a black hole. This also suggests a cute analogy for the black hole surface: it is like a *screen*, with each Planck length-squared $\mathcal{A}$ a pixel supporting a single particle. Either way, ℓ_{P} is telling us the basic resolution of spacetime.

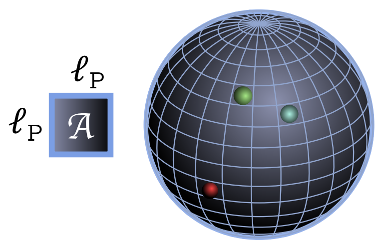

Figure 8: The surface of a black hole is like a screen with resolution ℓ_{P}. Each pixel has area $\mathcal{A} = \ell_{\mathrm{P}}^2$, and corresponds to a single particle; only three are pictured.

⁂

Hopefully, you now have some feeling for the power and limitations of dimensional analysis. You can use it to check your homework, extract classified information from photographs, and even get insights into the deepest mysteries of the universe! Thanks for reading.

4 Problems

This section contains a few longer problems. The first outlines a solution of the GPS problem *without* dimensional analysis, including the factor of π. The next two apply dimensional analysis to tsunamis and turbulence. These questions will also appear on Physics Circle problem sheets, and solutions will be placed on the website in due course.

4.1 Gravitational postal service

We can solve the GPS problem using a couple of helpful results from elsewhere. The first is the marvellous *Sphere Theorem*, first proved by Isaac Newton:

- an object outside a spherical body (of constant density) is gravitationally attracted to it *as if all the mass were concentrated at the centre*;[6]
- an object inside a spherical shell *feels no gravitational attraction to the shell*.

The second result we need is on simple harmonic motion. Unlike the Sphere Theorem, we will derive this from scratch.

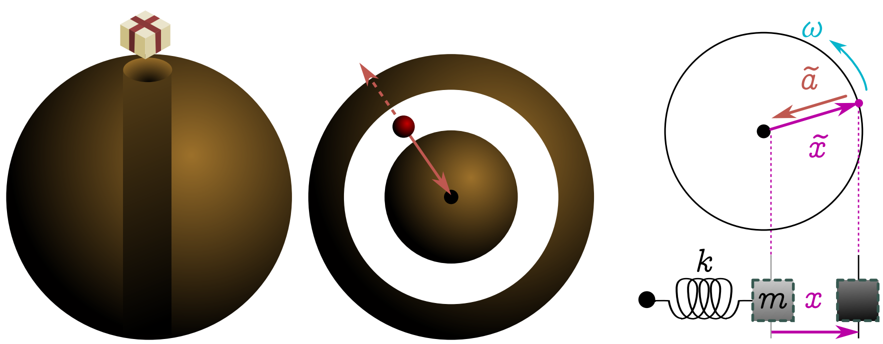

Figure 9: *Left*. A package travelling through the hole in the middle of the earth. *Middle*. The sphere theorem; the red mass feels no attraction to the shell, and attraction to the inner sphere as if all the mass were concentrated at the centre. *Right*. Phasor approach to solving the spring-mass problem.

1. Let r denote the radial distance from the centre of the earth. From the Sphere Theorem, show that a package at position r is subject to a gravitational force

$$F = \left(\frac{mg}{R}\right) r$$

   directed towards the centre, where R is the radius of the earth and g the gravitational acceleration at the surface.

2. The force on the package is proportional to the distance from the centre. This is just like a spring! Let's understand springs first, then return to the delivery problem. If we

[6]This explains why we always just treat planets as point masses in gravity problems.

attach a mass m to a spring of stiffness k, and pull the mass a distance x away from the equilibrium position, there is a restoring force

$$F = -kx.$$

If we displace the mass and let it go, the result is *simple harmonic motion*, where the mass just oscillates back and forth. To understand this motion, we can use the *phasor trick*. The basic idea is to upgrade x to a complex variable $\tilde{x} = re^{i\omega t}$ in uniform circular motion on the complex plane. Treating the acceleration $\tilde{a}$ and position $\tilde{x}$ as phasors, show that the phasor satisfies

$$\tilde{a} = -\omega^2 \tilde{x}.$$

Just so you know, you don't need any calculus!

3. Conclude that the phasor satisfies a spring equation for

$$k = \omega^2 m.$$

4. We must return to the harsh realities of the real line. To pluck out a real component of the phasor, we can use *Euler's formula*:

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t).$$

By taking the real part of the phasor solution, show that a mass m oscillates on a spring of stiffness k according to

$$x(t) = x(0)\cos(\omega t), \quad \omega = \sqrt{\frac{k}{m}},$$

where it is released from rest at $x(0)$.

5. Using questions (1) and (4), argue that the package reaches the other side of the world in time

$$t_{\text{delivery}} = \pi\sqrt{\frac{R}{g}}.$$

4.2 Tsunamis

The full dynamics of ocean waves are complicated. But in the limit that the water is very shallow or very deep, the dynamics simplify enough to be understood using dimensional analysis. This will allow us to understand *tsunami formation*, which occurs in the shallow water regime.

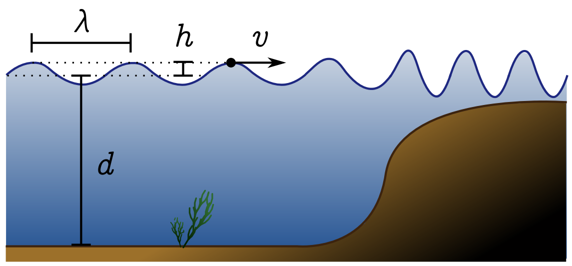

Figure 10: Ocean waves. As the water gets shallower, the waves increase in height.

1. Let λ denote the wavelength of an ocean wave and d the depth of the water. Typically, both are much larger than the height h of the wave, so we can ignore it for the time being. Argue from dimensional analysis that in the *deep water limit* $\lambda \ll d$, the velocity of the wave is proportional to the square root of the wavelength:

$$v \sim \sqrt{g\lambda}.$$

   In the *shallow water limit* $\lambda \gg d$, explain why you expect $v \sim \sqrt{gd}$.

2. Ocean waves can be generated by oscillations beneath the ocean floor. For a source of frequency f, what is the wavelength of the corresponding wave in shallow water? Estimate the wavelength if the source is an earthquake of period $T = 20\,\text{min}$ at depth $d = 4\,\text{km}$, and check your answer is consistent with the shallow water limit.

3. Consider an ocean wave of height h and width w. The energy E carried by a single "cycle" of the wave equals the volume V of water above the mean water level d, multiplied by the gravitational energy density ϵ. By performing a dimensional analysis on each term separately, argue that the total energy in a cycle is approximately

$$E \approx V\epsilon \approx \rho g \lambda w h^2,$$

   where $\rho \approx 10^3\,\text{kg}\,\text{m}^{-3}$ is the density of water and g the gravitational acceleration.

4. Energy in waves is generally *conserved*: the factor E is constant, even as the wavelength λ and height h of the wave change. (We will ignore spreading of the wave.) By applying energy conservation to shallow waves, deduce *Green's law*:

$$h \propto d^{-1/4}.$$

   The increase in height is called *shoaling*. The relation breaks down near shore when the depth d becomes comparable to the height h.

5. The earthquake from earlier creates a tsunami of height $h_0 = 0.5\,\text{m}$. What is the height, speed, and power per unit width of the tsunami close to the shore? (By "close to the shore", we mean $h \approx d$.) You may assume the shallow water equation holds.

4.3 Turbulence

Stir a cup of coffee vigorously enough, and the fluid will begin to mix in a chaotic or *turbulent* way. Unlike the steady flow of water through a pipe, the behaviour of turbulent fluids is unpredictable and poorly understood. However, for many purposes, we can do suprisingly well by modelling a turbulent fluid as a collection of (three-dimensional) eddies of different sizes, with larger eddies feeding into smaller ones and losing energy in the process. This model will also provide an example of a system where a dimensionless ratio appears and plays a physically clear role.

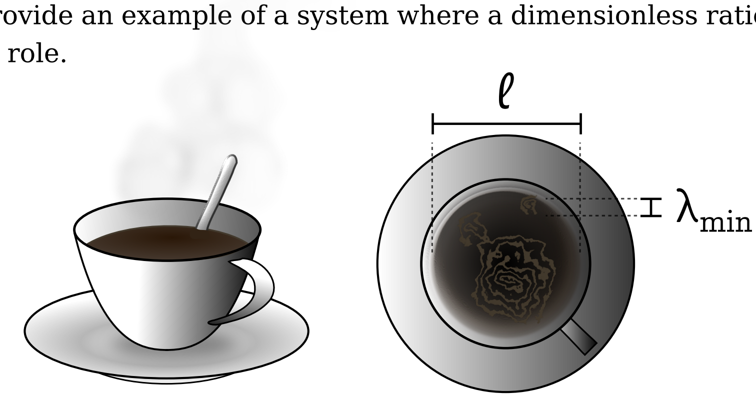

Figure 11: A well-stirred cup of coffee. On the right, a large eddy (size $\sim \ell$) and the smallest eddy (size λ_{min}) are depicted.

Suppose our cup of coffee has characteristic length ℓ, and the coffee has density ρ. When it is turbulently mixed, the largest eddies will be a similar size to the cup, order ℓ, and experience fluctuations in velocity of size Δv due to interaction with other eddies. The fluid also has internal drag[7] or *viscosity* μ, with units $\text{N} \cdot \text{s}/\text{m}^2$.

1. Let ϵ be the rate at which kinetic energy dissipates per unit mass due to eddies. Observation shows that this energy loss is independent of the fluid's viscosity. Argue on dimensional grounds that

$$\epsilon \approx \frac{(\Delta v)^3}{\ell}.$$

   Why doesn't the density ρ appear?

2. Kinetic energy can also be lost due to internal friction. Argue that the time scale for this dissipation due to viscosity is

$$\tau_{\text{drag}} \approx \frac{\ell^2 \rho}{\mu}.$$

3. Using the previous two questions, show that eddy losses[8] dominate viscosity losses pro-

[7] More precisely, viscosity is the resistance to *shear flows*. A simple way to create shear flow is by moving a large plate along the surface of a stationary fluid. Experiments show that the friction per unit area of plate is proportional to the speed we move it, and inversely proportional to the height; the proportionality constant at unit height is the viscosity. Since layers of fluid also generate shear flows, viscosity creates internal friction.

[8] Since ϵ depends on $\ell, \Delta v$, you need not consider it when finding the time scale for eddy losses.

vided

$$\frac{\ell\rho\Delta v}{\mu} \gg 1.$$

The quantity on the left is called the *Reynolds number*, $\mathrm{Re} = \ell\rho\Delta v/mu$. In fact, one *definition* of turbulence is fluid flow where the Reynolds number is high.

4. So far, we have focused on the largest eddies. These feed energy into smaller eddies of size λ and velocity uncertainty Δv_λ, which have an associated *eddy Reynolds number*,

$$\mathrm{Re}_\lambda = \frac{\lambda\rho\Delta v_\lambda}{\mu}.$$

When the eddy Reynolds number is less than 1, eddies of the corresponding size are prevented from forming by viscosity.[9] Surprisingly, the rate of energy dissipation per unit mass in these smaller eddies is ϵ, the same as the larger eddies.[10] Show from dimensional analysis that the minimum eddy size is roughly

$$\lambda_{\min} \approx \left(\frac{\mu^3}{\epsilon\rho^3}\right)^{1/4}.$$

5. If a cup of coffee is stirred violently to Reynolds number $\mathrm{Re} \approx 10^4$, estimate the size of the smallest eddies in the cup.

[9]Lewis Fry Richardson not only invented the eddy model, but this brilliant mnemonic couplet: "Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity."

[10]This is not at all obvious, but roughly, follows because we can fit more small eddies in the container. Intriguingly, this makes the turbulent fluid like a *fractal*: the structure of eddies repeats itself as we zoom in, until viscosity begins to play a role. At infinite Reynolds number, it really is a fractal!

A Appendices

A.1 SI and natural units

The *International System of Units* (in French, *Système international* or SI for short) is the official metric system of units. We summarise the basic units and dimensions below.

Box A1: SI base dimensions and units

There are seven SI base dimensions and corresponding units:

Dimension	**SI unit**
mass (M)	kilogram (kg)
length (L)	metre (m)
time (T)	second (s)
temperature (Θ)	Kelvin (K)
amount of substance (N)	mole (mol)
current (I)	ampere (A)
luminosity (J)	candela (cd)

We have already seen that the amount of substance N can play an important role in the physics of a system; it also plays an important role in chemistry, where it is measured in mol (multiples of Avogadro's constant $N_{\mathrm{A}} \approx 6 \times 10^{23}$). The new dimensions in this list are current I and luminosity J. Current is just the amount of charge passing a point per unit time, while luminosity is a measure of visual brightness (intensity) of a luminous source.[11]

We saw above that we could combine fundamental constants to obtain the Planck length $\ell_{\mathrm{P}} = \sqrt{G\hbar/c^3}$. In fact, from the four fundamental constants $c, G, \hbar, k_{\mathrm{B}}$, we can make four independent dimensionsful quantities called *Planck units*.

Box A2: Planck units

The four fundamental constants let us define four *Planck units*:

Unit	**Expression**	**SI value**
Planck length	$\ell_{\mathrm{P}} = \sqrt{G\hbar/c^3}$	$1.6 \times 10^{-35}\,\mathrm{m}$
Planck mass	$m_{\mathrm{P}} = c^2 \ell_{\mathrm{P}} G^{-1}$	$2.2 \times 10^{-8}\,\mathrm{kg}$
Planck time	$t_{\mathrm{P}} = \ell_{\mathrm{P}} c^{-1}$	$5.4 \times 10^{-44}\,\mathrm{s}$
Planck temperature	$\mathcal{T}_{\mathrm{P}} = m_{\mathrm{P}} c^2 k_{\mathrm{B}}^{-1}$	$1.4 \times 10^{32}\,\mathrm{K}$

Planck units are very different from SI units: they are built out of measurements that nature gives to us, rather than measurements which are chosen for human convenience. For this reason, theoretical physicists usually prefer Planck units to SI units. To go with these very special units is a very special, and initially confusing, convention: we don't bother to label

[11] Here, "visual" means that luminosity is tied to the visual spectrum. This makes it distinct from *radiant intensity*, which measures intensity across the whole spectrum of electromagnetic radiation without reference to the eye.

them at all! This convention is called *natural units*. Since we can invert the expressions for Planck units to recover the fundamental constants, this is equivalent to choosing units where all fundamental constants equal 1.

Why is this confusing? Setting $c = 1$, for instance, seems to mystifyingly suggest that time and length are the same. Although relativity does teach us something like this, it is better to think of $c = 1$ as *trivially* true. A unit is just a choice of measurement to compare to. Here, we have chosen to compare to c itself, so of course the answer is 1! Practically speaking, it may seem hard to convert from natural units back to SI units (or any other units for that matter) without a trailing unit symbol. But all we need to know is the dimension! For instance, we know that an energy $E = 5$ in natural units really means $E = 5E_{\mathrm{P}}$. To convert into another system of units, we just need to convert $E_{\mathrm{P}} = m_{\mathrm{P}}c^2$ into those units first. You can find and keep track of dimensions using the tools in §1.2.

Box A3: Natural units

Natural units are simply Planck units without a trailing unit symbol. In natural units,

$$\ell_{\mathrm{P}} = m_{\mathrm{P}} = t_{\mathrm{P}} = \mathcal{T}_{\mathrm{P}} = 1. \tag{25}$$

Equivalently, we can invert these and write

$$c = G = \hbar = k_{\mathrm{B}} = 1. \tag{26}$$

We can convert a measurement in natural units to any other system of units using the dimension of the measurement, and converting the appropriate Planck unit.

In particle physics and string theory, natural units means something slightly different. Instead of setting all fundamental constants to 1, we just set

$$c = \hbar = k_{\mathrm{B}} = 1,$$

and leave G dimensionful. Choosing units where some fundamental constants equal 1 allows us to collapse the distinction between certain dimensions; for instance, setting $c = 1$ suggests we take $L = T$. Although I argued against this interpretation above, it can be very handy to think this way. You can explore some of the consequences below.

Exercise A1: Mass dimension

We will explore the *mass dimension function*, $[\cdot]_M$, which keeps only the mass part of the dimension. It is also defined as the *power* of mass appearing in the measurement:

$$[m^n]_M = n,$$

as opposed to $[m^n] = M^n$, since we no longer need to keep track of other dimensions. This means that our earlier rules (1) and (2) become:

$$[AB]_M = [A]_M + [B]_M, \quad [A^n]_M = n[A]_M.$$

(a) Since $c = 1$, we will set

$$[c]_M = [v]_M = 0.$$

Thus, for the purposes of mass dimension, $L = T$. Show using similar reasoning that

$$[E]_M = 1.$$

(b) Using $\hbar = k_\mathrm{B} = 1$, show that

$$[t]_M = [d]_M = -1, \quad [\mathcal{T}]_M = 1.$$

In other words, temperature has units of mass, while time and length have units of *inverse* mass.

(c) Restoring fundamental constants, show that the length λ_C corresponding to a mass m is

$$\lambda_\mathrm{C} = \frac{\hbar}{mc}.$$

This is the *Compton wavelength*.

(d) Quantum mechanics teaches us that particles have wave-like properties and waves have particle-like properties. While the energy of a classical particle is perfectly localised, the energy in a wave is spread out over a region the order of the wavelength λ. Thus, λ represents an *uncertainty* in the location of the corresponding particle. According to quantum mechanics, a particle with momentum $p = mv$ forms a matter wave with associated *de Broglie wavelength*

$$\lambda_\mathrm{dB} = \frac{2\pi\hbar}{mv}.$$

In light of this, give a physical interpretation of the Compton wavelength.

A.2 Solutions to exercises

1. We have two different units for length, and some very large number, but don't let that confuse you! We just grind through using our rules (1) and (2):

$$[H_0] = \left[70\,\frac{\text{km}}{\text{s}\cdot\text{MPc}}\right] = \frac{[\text{km}]}{[\text{s}][\text{MPc}]} = \frac{L}{LT} = \frac{1}{T}.$$

   This has units of inverse time. In fact, it's just the reciprocal of the age of the universe!

2. (a) We can use the formula for the volume of a box, for instance, $V = whd$. Then

$$[V] = [w][h][d] = L^3.$$

   (b) Recall that acceleration is the change in velocity over time, $a = \Delta v/t$. So

$$[a] = \frac{[\Delta v]}{[t]} = \frac{L/T}{T} = \frac{L}{T^2}.$$

3. Work, defined by $W = Fd$ for a force F applied over a distance d is a form of energy. Using our result for $[F]$,

$$[E] = [Fd] = [F]L = \frac{ML^2}{T^2}.$$

4. The LHS clearly has dimensions of time. The RHS has dimensions

$$\left[\sqrt{\frac{2h}{g}}\right] = \sqrt{\frac{[h]}{[g]}} = \sqrt{\frac{L}{L/T^2}} = \sqrt{T^2} = T,$$

   as required.

5. We guess $t \sim h^a g^b m^c$, and therefore need to solve

$$[h^a g^b m^c] = L^a \left(\frac{L}{T^2}\right)^b M^c = L^{a+b}T^{-2a}M^c = T.$$

   There is no power of M on the RHS, so $c = 0$. We solve to get $b = -a = 1/2$ as in the GPS problem, or $t \sim \sqrt{h/g}$.

6. We just take dimensions:

$$[g^a R^b w^c] = [g]^a[R]^b[w]^c = \left(\frac{L}{T^2}\right)^a L^{b+c} = L^{a+b+c}T^{-2a} = T.$$

   We can immediately read off that $a = -1/2$, even though we can't solve for b and c.

7. In this case, we make the guess $t \sim g^a R^b w^c m^d$. Taking dimensions, we have to solve

$$[g^a R^b w^c] = [g]^a[R]^b[w]^c[m]^d = L^{a+b+c}T^{-2a}M^d = T.$$

   We can see that $d = 0$, so the mass of the package is not relevant, at least assuming that we can replace G, M with g. (For a very large package, we will need to introduce additional parameters which mean this is no longer true.) We still only have one dimensionless ratio. But since mass was involved in the problem, $k = 3$. Since $n = 4$, we have $n - k = 1$ ratio, consistent with the Buckingham π theorem.

8. The force due to an atomic explosion is many orders of magnitude greater than the force due to gravity. At least initially, the fireball expands as if there is no gravity at all! We can therefore neglect g. However, the density ρ is important because the *inertia* of the air, its resistance to acceleration, is what determines the shape of the explosion early on.

9. We can rearrange Newton's law of gravitation:

$$[G] = \frac{[F][R]^2}{[M][m]} = \frac{(ML/T^2)L^2}{M^2} = \frac{L^3}{MT^2}.$$

10. (a) Since radiation is due to hot quantum oscillators emitting light at the surface, we expect $\hbar$ (quantum), k_B (hot) and c (light) to be involved.

(b) Define the dimension N to the *amount of stuff* for a system consisting of many particles, such as a lump of matter. In our case, we will specifically use N for the measurement of the many particles on the *surface* of the lump. This means that

$$[A] = L^2 N,$$

where as for an atomic-size patch of the surface a, $[a] = L^2$. The power emitted by the object is due to the sum of power emitted from each atomic patch of surface. Thus, we expect the power to also pick up a factor of N, so

$$[P] = \frac{[E]}{[t]} = \frac{MNL^2}{T^3}.$$

Put a different way, A and P scale with the system size. The constants do not, since they are just constants. Hence, they do not contain dimensions of N.

(c) We guess

$$P \sim \hbar^a k_\mathrm{B}^b c^d A^f \mathcal{T}^g.$$

We have five constants and five base dimensions (M, L, T, Θ, N) so by the Buckingham π theorem, we should expect to be able to determine the powers uniquely. On the LHS, we have

$$[P] = ML^2T^{-3}N.$$

On the RHS, we have

$$\begin{aligned}[\hbar]^a[k_\mathrm{B}]^b[c]^d[A]^f[\mathcal{T}]^g &= \left(\frac{ML^2}{T}\right)^a \left(\frac{ML^2}{\Theta T^2}\right)^b \left(\frac{L}{T}\right)^d \left(L^2N\right)^f \Theta^g \\ &= M^{a+b}L^{2a+2b+d+2f}T^{-(a+2b+d)}\Theta^{g-b}N^f.\end{aligned}$$

This leads to the system of equation

$$\begin{aligned} a+b &= 1 \\ 2a+2b+d+2f &= 2 \\ a+2b+d &= 3 \\ g-b &= 0 \\ f &= 1. \end{aligned}$$

We can plug $f = 1$ into the second equation, and subtract it from the third, to find $a = -3$. The first equation implies $b = 4$, and the fourth then gives $g = 4$. Finally, plugging our values for a, b into the second or third equation tells us that $d = -2$. So we discover as required that

$$P \sim \left(\frac{k_{\mathrm{B}}^4}{\hbar^3 c^2}\right) A\mathcal{T}^4.$$

11. Let's set $v_{\mathrm{esc}} = c$ and solve for r:

$$\frac{1}{2} m_{\mathrm{particle}} c^2 = \frac{G m_{\mathrm{BH}} m_{\mathrm{particle}}}{R} \implies c^2 R = 2 G m_{\mathrm{BH}} \implies R = \frac{2 G m_{\mathrm{BH}}}{c^2}.$$

12. The rate of power loss is $P = \sigma A \mathcal{T}^4$, and the total energy is $E = mc^2$. We guess an evaporation time scale $t_{\mathrm{evap}} \sim E/P$, since P is the rate the black hole is radiating away its energy stores. Of course, the rate will change, but this only results in a different dimensionless number sitting out the front! Thus, this should be the evaporation time scale. We can make the dependence on m explicit by expanding out our definitions using $A \sim R^2$, (13) , (15) and (21):

$$\begin{aligned} t_{\mathrm{evap}} \sim \frac{E}{P} &\sim \frac{mc^2}{\sigma A \mathcal{T}^4} \\ &\sim \frac{mc^2 k_{\mathrm{B}}^4 R^4}{\sigma R^2 \hbar^4 c^4} \\ &\sim \frac{m k_{\mathrm{B}}^4}{\hbar^4 c^2} \cdot \frac{\hbar^3 c^2}{k_{\mathrm{B}}^4} \cdot \frac{G^2 m^2}{c^4} = \frac{G^2 m^3}{\hbar c^4}. \end{aligned}$$

13. (a) The speed of light c lets us convert the Planck length into the *Planck time* t_{P}:

$$t_{\mathrm{P}} = \frac{c}{\ell_{\mathrm{P}}} = \sqrt{\frac{c^5}{G\hbar}}.$$

(b) Planck's constant $\hbar$ has units

$$[\hbar] = \frac{ML^2}{T} = \frac{ML^2}{T^2} \cdot T = [E][t].$$

Thus, it lets us convert the Planck time into the *Planck energy* E_{P}:

$$E_{\mathrm{P}} = \frac{\hbar}{t_{\mathrm{P}}} = \hbar c \ell_{\mathrm{P}} = \sqrt{\frac{G\hbar^3}{c}}.$$

A1. (a) Since $E = mc^2$, we have

$$[E]_M = [m]_M + 2[c]_M = 1 + 2 \cdot 0 = 1.$$

(b) Since $[\hbar] = [Et]$, it follows that in mass dimension,

$$0 = [1]_M = [\hbar]_M = [Et]_M = [E]_M + [t]_M \implies [t]_M = -1.$$

From part (a), it follows that $[d]_M = [t_M] = -1$. Similarly, since $[k_\mathrm{B}] = [E\mathcal{T}^{-1}]$, it follows that

$$0 = [1]_M = [k_\mathrm{B}]_M = [E\mathcal{T}^{-1}]_M = [E]_M - [\mathcal{T}]_M \implies [\mathcal{T}]_M = 1.$$

(c) Since $[d]_M = -1$, in these units there is a length corresponding to a mass m:

$$\lambda_\mathrm{C} = \frac{1}{m}.$$

To restore fundamental constants, we must find a combination α with dimension $[\alpha] = LM$, so that α/m has units of length. It's easy to check that $\alpha = \hbar/c$ works:

$$\left[\frac{\hbar}{c}\right] = \frac{[\hbar]}{[c]} = \frac{ML^2/T}{L/T} = ML.$$

So the *Compton wavelength* is

$$\lambda_\mathrm{C} = \frac{\hbar}{mc}.$$

(d) The Compton wavelength looks similar to the de Broglie wavelength, except for the factor of 2π and the replacement $v \to c$. As we have learned, numbers like 2π are irrelevant for determining scales, so we will ignore it. The replacement $v \to c$, however, is important.

Since nothing can travel faster than the speed of light, and the de Broglie wavelength decreases with speed, the Compton wavelength represents the *minimum* de Broglie wavelength. Put a different way, the Compton wavelength is the *minimum uncertainty* in the position of a particle of mass m.