CHAPTER 7

DIMENSIONAL ANALYSIS AND MODELING

In this chapter, we first review the concepts of *dimensions* and *units*. We then review the fundamental principle of *dimensional homogeneity*, and show how it is applied to equations in order to *nondimensionalize* them and to identify *dimensionless groups*. We discuss the concept of *similarity* between a *model* and a *prototype*. We also describe a powerful tool for engineers and scientists called *dimensional analysis*, in which the combination of dimensional variables, nondimensional variables, and dimensional constants into *nondimensional parameters* reduces the number of necessary independent parameters in a problem. We present a step-by-step method for obtaining these nondimensional parameters, called the *method of repeating variables*, which is based solely on the dimensions of the variables and constants. Finally, we apply this technique to several practical problems to illustrate both its utility and its limitations.

OBJECTIVES

When you finish reading this chapter, you should be able to

- Develop a better understanding of dimensions, units, and dimensional homogeneity of equations
- Understand the numerous benefits of dimensional analysis
- Know how to use the method of repeating variables to identify nondimensional parameters
- Understand the concept of dynamic similarity and how to apply it to experimental modeling

7–1 ▪ DIMENSIONS AND UNITS

A **dimension** is a measure of a physical quantity (without numerical values), while a **unit** is a way to assign a *number* to that dimension. For example, length is a dimension that is measured in units such as microns (μm), feet (ft), centimeters (cm), meters (m), kilometers (km), etc. (Fig. 7–1). There are seven **primary dimensions** (also called **fundamental** or **basic dimensions**)—mass, length, time, temperature, electric current, amount of light, and amount of matter.

> All nonprimary dimensions can be formed by some combination of the seven primary dimensions.

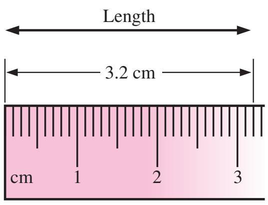

FIGURE 7–1
A *dimension* is a measure of a physical quantity without numerical values, while a *unit* is a way to assign a number to the dimension. For example, length is a dimension, but centimeter is a unit.

For example, force has the same dimensions as mass times acceleration (by Newton's second law). Thus, in terms of primary dimensions,

$$\textit{Dimensions of force:}\quad \{\text{Force}\} = \left\{\text{Mass}\,\frac{\text{Length}}{\text{Time}^2}\right\} = \{\text{mL/t}^2\} \tag{7–1}$$

where the brackets indicate "the dimensions of" and the abbreviations are taken from Table 7–1. You should be aware that some authors prefer force instead of mass as a primary dimension—we do not follow that practice.

TABLE 7–1

Primary dimensions and their associated primary SI and English units

Dimension	Symbol*	SI Unit	English Unit
Mass	m	kg (kilogram)	lbm (pound-mass)
Length	L	m (meter)	ft (foot)
Time[†]	t	s (second)	s (second)
Temperature	T	K (kelvin)	R (rankine)
Electric current	I	A (ampere)	A (ampere)
Amount of light	C	cd (candela)	cd (candela)
Amount of matter	N	mol (mole)	mol (mole)

* We italicize symbols for variables, but not symbols for dimensions.

[†] Note that some authors use the symbol T for the time dimension and the symbol θ for the temperature dimension. We do not follow this convention to avoid confusion between time and temperature.

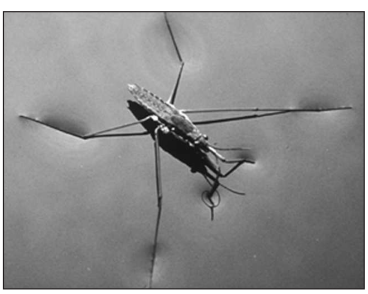

FIGURE 7–2
The water strider is an insect that can walk on water due to surface tension.
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EXAMPLE 7–1 Primary Dimensions of Surface Tension

An engineer is studying how some insects are able to walk on water (Fig. 7–2). A fluid property of importance in this problem is surface tension (σ_s), which has dimensions of force per unit length. Write the dimensions of surface tension in terms of primary dimensions.

SOLUTION The primary dimensions of surface tension are to be determined.

Analysis From Eq. 7–1, force has dimensions of mass times acceleration, or {mL/t²}. Thus,

$$\textit{Dimensions of surface tension:}\quad \{\sigma_s\} = \left\{\frac{\text{Force}}{\text{Length}}\right\} = \left\{\frac{\text{m}\cdot\text{L/t}^2}{\text{L}}\right\} = \mathbf{\{m/t^2\}} \tag{1}$$

Discussion The usefulness of expressing the dimensions of a variable or constant in terms of primary dimensions will become clearer in the discussion of the method of repeating variables in Section 7–4.

7–2 ▪ DIMENSIONAL HOMOGENEITY

We've all heard the old saying, You can't add apples and oranges (Fig. 7–3). This is actually a simplified expression of a far more global and fundamental mathematical law for equations, the **law of dimensional homogeneity,** stated as

> Every additive term in an equation must have the same dimensions.

Consider, for example, the change in total energy of a simple compressible closed system from one state and/or time (1) to another (2), as illustrated in Fig. 7–4. The change in total energy of the system (ΔE) is given by

Change of total energy of a system:
$$\Delta E = \Delta U + \Delta\text{KE} + \Delta\text{PE} \tag{7–2}$$

where E has three components: internal energy (U), kinetic energy (KE), and potential energy (PE). These components can be written in terms of the system mass (m); measurable quantities and thermodynamic properties at each of the two states, such as speed (V), elevation (z), and specific internal energy (u); and the known gravitational acceleration constant (g),

$$\Delta U = m(u_2 - u_1) \qquad \Delta\text{KE} = \frac{1}{2}m(V_2^2 - V_1^2) \qquad \Delta\text{PE} = mg(z_2 - z_1) \tag{7–3}$$

It is straightforward to verify that the left side of Eq. 7–2 and all three additive terms on the right side of Eq. 7–2 have the same dimensions—energy. Using the definitions of Eq. 7–3, we write the primary dimensions of each term,

$$\{\Delta E\} = \{\text{Energy}\} = \{\text{Force} \cdot \text{Length}\} \quad \rightarrow \quad \{\Delta E\} = \{\text{mL}^2/\text{t}^2\}$$

$$\{\Delta U\} = \left\{\text{Mass}\,\frac{\text{Energy}}{\text{Mass}}\right\} = \{\text{Energy}\} \quad \rightarrow \quad \{\Delta U\} = \{\text{mL}^2/\text{t}^2\}$$

$$\{\Delta\text{KE}\} = \left\{\text{Mass}\,\frac{\text{Length}^2}{\text{Time}^2}\right\} \quad \rightarrow \quad \{\Delta\text{KE}\} = \{\text{mL}^2/\text{t}^2\}$$

$$\{\Delta\text{PE}\} = \left\{\text{Mass}\,\frac{\text{Length}}{\text{Time}^2}\,\text{Length}\right\} \quad \rightarrow \quad \{\Delta\text{PE}\} = \{\text{mL}^2/\text{t}^2\}$$

If at some stage of an analysis we find ourselves in a position in which two additive terms in an equation have *different* dimensions, this would be a clear indication that we have made an error at some earlier stage in the analysis (Fig. 7–5). In addition to dimensional homogeneity, calculations are valid only when the *units* are also homogeneous in each additive term. For example, units of energy in the above terms may be J, N · m, or kg · m^2/s^2, all of which are equivalent. Suppose, however, that kJ were used in place of J for one of the terms. This term would be off by a factor of 1000 compared to the other terms. It is wise to write out *all* units when performing mathematical calculations in order to avoid such errors.

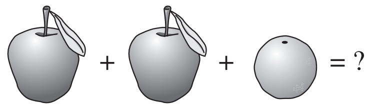


FIGURE 7–3
You can't add apples and oranges!

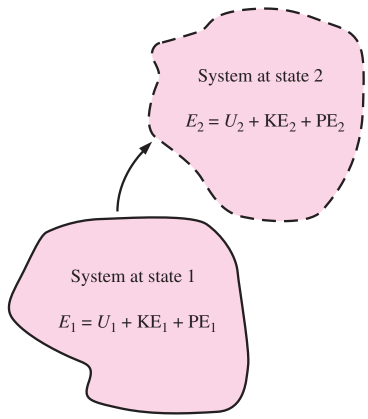


FIGURE 7–4
Total energy of a system at state 1 and at state 2.

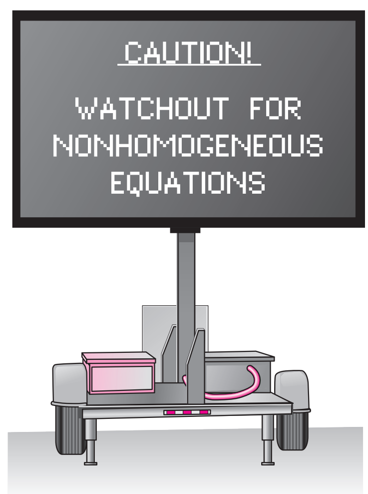


FIGURE 7–5
An equation that is not dimensionally homogeneous is a sure sign of an error.

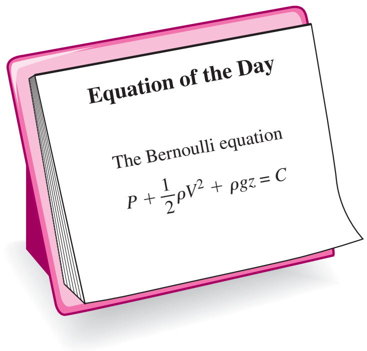


FIGURE 7–6
The Bernoulli equation is a good example of a *dimensionally homogeneous* equation. All additive terms, including the constant, have the *same* dimensions, namely that of pressure. In terms of primary dimensions, each term has dimensions $\{m/(t^2L)\}$.

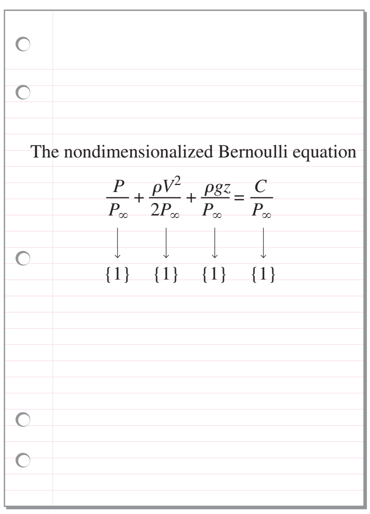


FIGURE 7–7
A *nondimensionalized* form of the Bernoulli equation is formed by dividing each additive term by a pressure (here we use P_∞). Each resulting term is *dimensionless* (dimensions of $\{1\}$).

EXAMPLE 7–2 Dimensional Homogeneity of the Bernoulli Equation

Probably the most well-known (and most misused) equation in fluid mechanics is the Bernoulli equation (Fig. 7–6), discussed in Chap. 5. The standard form of the Bernoulli equation for incompressible irrotational fluid flow is

Bernoulli equation:
$$P + \frac{1}{2}\rho V^2 + \rho gz = C \qquad (1)$$

(*a*) Verify that each additive term in the Bernoulli equation has the same dimensions. (*b*) What are the dimensions of the constant *C*?

SOLUTION We are to verify that the primary dimensions of each additive term in Eq. 1 are the same, and we are to determine the dimensions of constant *C*.

Analysis (*a*) Each term is written in terms of primary dimensions,

$$\{P\} = \{\text{Pressure}\} = \left\{\frac{\text{Force}}{\text{Area}}\right\} = \left\{\text{Mass}\frac{\text{Length}}{\text{Time}^2}\frac{1}{\text{Length}^2}\right\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

$$\left\{\frac{1}{2}\rho V^2\right\} = \left\{\frac{\text{Mass}}{\text{Volume}}\left(\frac{\text{Length}}{\text{Time}}\right)^2\right\} = \left\{\frac{\text{Mass}\times\text{Length}^2}{\text{Length}^3\times\text{Time}^2}\right\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

$$\{\rho gz\} = \left\{\frac{\text{Mass}}{\text{Volume}}\frac{\text{Length}}{\text{Time}^2}\text{Length}\right\} = \left\{\frac{\text{Mass}\times\text{Length}^2}{\text{Length}^3\times\text{Time}^2}\right\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

Indeed, **all three additive terms have the same dimensions.**

(*b*) From the law of dimensional homogeneity, the constant must have the same dimensions as the other additive terms in the equation. Thus,

Primary dimensions of the Bernoulli constant:
$$\{C\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

Discussion If the dimensions of any of the terms were different from the others, it would indicate that an error was made somewhere in the analysis.

Nondimensionalization of Equations

The law of dimensional homogeneity guarantees that every additive term in an equation has the same dimensions. It follows that if we divide each term in the equation by a collection of variables and constants whose product has those same dimensions, the equation is rendered **nondimensional** (Fig. 7–7). If, in addition, the nondimensional terms in the equation are of order unity, the equation is called **normalized.** Normalization is thus more restrictive than nondimensionalization, even though the two terms are sometimes (incorrectly) used interchangeably.

Each term in a nondimensional equation is dimensionless.

In the process of nondimensionalizing an equation of motion, **nondimensional parameters** often appear—most of which are named after a notable scientist or engineer (e.g., the Reynolds number and the Froude number). This process is referred to by some authors as **inspectional analysis.**

As a simple example, consider the equation of motion describing the elevation z of an object falling by gravity through a vacuum (no air drag), as in Fig. 7–8. The initial location of the object is z_0 and its initial velocity is w_0 in the z-direction. From high school physics,

Equation of motion:
$$\frac{d^2z}{dt^2} = -g \tag{7–4}$$

Dimensional variables are defined as dimensional quantities that change or vary in the problem. For the simple differential equation given in Eq. 7–4, there are two dimensional variables: z (dimension of length) and t (dimension of time). **Nondimensional** (or **dimensionless**) **variables** are defined as quantities that change or vary in the problem, but have no dimensions; an example is angle of rotation, measured in degrees or radians which are dimensionless units. Gravitational constant g, while dimensional, remains constant and is called a **dimensional constant.** Two additional dimensional constants are relevant to this particular problem, initial location z_0 and initial vertical speed w_0. While dimensional constants may change from problem to problem, they are fixed for a particular problem and are thus distinguished from dimensional variables. We use the term **parameters** for the combined set of dimensional variables, nondimensional variables, and dimensional constants in the problem.

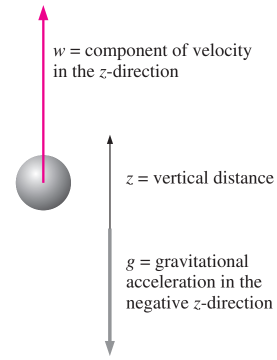


FIGURE 7–8
Object falling in a vacuum. Vertical velocity is drawn positively, so $w < 0$ for a falling object.

Equation 7–4 is easily solved by integrating twice and applying the initial conditions. The result is an expression for elevation z at any time t:

Dimensional result:
$$z = z_0 + w_0t - \frac{1}{2}gt^2 \tag{7–5}$$

The constant $\frac{1}{2}$ and the exponent 2 in Eq. 7–5 are dimensionless results of the integration. Such constants are called **pure constants.** Other common examples of pure constants are π and e.

To nondimensionalize Eq. 7–4, we need to select **scaling parameters,** based on the primary dimensions contained in the original equation. In fluid flow problems there are typically at least *three* scaling parameters, e.g., L, V, and $P_0 - P_\infty$ (Fig. 7–9), since there are at least three primary dimensions in the general problem (e.g., mass, length, and time). In the case of the falling object being discussed here, there are only two primary dimensions, length and time, and thus we are limited to selecting only *two* scaling parameters. We have some options in the selection of the scaling parameters since we have three available dimensional constants g, z_0, and w_0. We choose z_0 and w_0. You are invited to repeat the analysis with g and z_0 and/or with g and w_0. With these two chosen scaling parameters we nondimensionalize the dimensional variables z and t. The first step is to list the primary dimensions of *all* dimensional variables and dimensional constants in the problem,

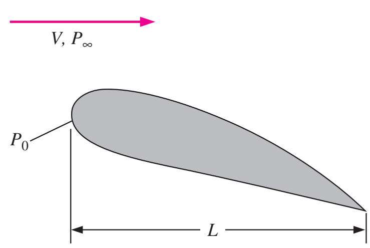


FIGURE 7–9
In a typical fluid flow problem, the scaling parameters usually include a characteristic length L, a characteristic velocity V, and a reference pressure difference $P_0 - P_\infty$. Other parameters and fluid properties such as density, viscosity, and gravitational acceleration enter the problem as well.

Primary dimensions of all parameters:

$$\{z\} = \{\text{L}\} \qquad \{t\} = \{\text{t}\} \qquad \{z_0\} = \{\text{L}\} \qquad \{w_0\} = \{\text{L/t}\} \qquad \{g\} = \{\text{L/t}^2\}$$

The second step is to use our two scaling parameters to nondimensionalize z and t (by inspection) into nondimensional variables z^* and t^*,

Nondimensionalized variables:
$$z^* = \frac{z}{z_0} \qquad t^* = \frac{w_0t}{z_0} \tag{7–6}$$

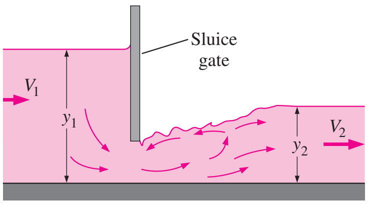


FIGURE 7–10
The *Froude number* is important in free-surface flows such as flow in open channels. Shown here is flow through a sluice gate. The Froude number upstream of the sluice gate is $\mathrm{Fr}_1 = V_1/\sqrt{gy_1}$, and it is $\mathrm{Fr}_2 = V_2/\sqrt{gy_2}$ downstream of the sluice gate.

Substitution of Eq. 7–6 into Eq. 7–4 gives

$$\frac{d^2z}{dt^2} = \frac{d^2(z_0z^*)}{d(z_0t^*/w_0)^2} = \frac{w_0^2}{z_0}\frac{d^2z^*}{dt^{*2}} = -g \quad\rightarrow\quad \frac{w_0^2}{gz_0}\frac{d^2z^*}{dt^{*2}} = -1 \qquad \textbf{(7–7)}$$

which is the desired nondimensional equation. The grouping of dimensional constants in Eq. 7–7 is the square of a well-known **nondimensional parameter** or **dimensionless group** called the **Froude number,**

Froude number: $$\mathrm{Fr} = \frac{w_0}{\sqrt{gz_0}} \qquad \textbf{(7–8)}$$

The Froude number also appears as a nondimensional parameter in free-surface flows (Chap. 13), and can be thought of as the ratio of inertial force to gravitational force (Fig. 7–10). You should note that in some older textbooks, Fr is defined as the *square* of the parameter shown in Eq. 7–8. Substitution of Eq. 7–8 into Eq. 7–7 yields

Nondimensionalized equation of motion: $$\frac{d^2z^*}{dt^{*2}} = -\frac{1}{\mathrm{Fr}^2} \qquad \textbf{(7–9)}$$

In dimensionless form, only one parameter remains, namely the Froude number. Equation 7–9 is easily solved by integrating twice and applying the initial conditions. The result is an expression for dimensionless elevation z^* at any dimensionless time t^*:

Nondimensional result: $$z^* = 1 + t^* - \frac{1}{2\mathrm{Fr}^2}t^{*2} \qquad \textbf{(7–10)}$$

Comparison of Eqs. 7–5 and 7–10 reveals that they are equivalent. In fact, for practice, substitute Eqs. 7–6 and 7–8 into Eq. 7–5 to verify Eq. 7–10.

It seems that we went through a lot of extra algebra to generate the same final result. *What then is the advantage of nondimensionalizing the equation?* Before answering this question, we note that the advantages are not so clear in this simple example because we were able to analytically integrate the differential equation of motion. In more complicated problems, the differential equation (or more generally the coupled *set* of differential equations) *cannot* be integrated analytically, and engineers must either integrate the equations numerically, or design and conduct physical experiments to obtain the needed results, both of which can incur considerable time and expense. In such cases, the nondimensional parameters generated by nondimensionalizing the equations are extremely useful and can save much effort and expense in the long run.

There are two key advantages of nondimensionalization (Fig. 7–11). First, *it increases our insight about the relationships between key parameters*. Equation 7–8 reveals, for example, that doubling w_0 has the same effect as decreasing z_0 by a factor of 4. Second, *it reduces the number of parameters in the problem*. For example, the original problem contains one dependent variable, z; one independent variable, t; and *three* additional dimensional constants, g, w_0, and z_0. The nondimensionalized problem contains one dependent parameter, z^*; one independent parameter, t^*; and only *one* additional parameter, namely the dimensionless Froude number, Fr. The number of additional parameters has been reduced from three to one! Example 7–3 further illustrates the advantages of nondimensionalization.

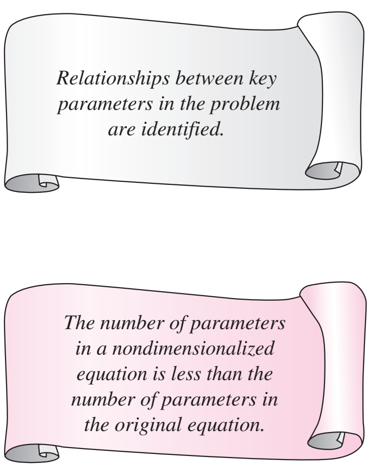


FIGURE 7–11
The two key advantages of nondimensionalization of an equation.

EXAMPLE 7–3 Illustration of the Advantages of Nondimensionalization

Your little brother's high school physics class conducts experiments in a large vertical pipe whose inside is kept under vacuum conditions. The students are able to remotely release a steel ball at initial height z_0 between 0 and 15 m (measured from the bottom of the pipe), and with initial vertical speed w_0 between 0 and 10 m/s. A computer coupled to a network of photosensors along the pipe enables students to plot the trajectory of the steel ball (height z plotted as a function of time t) for each test. The students are unfamiliar with dimensional analysis or nondimensionalization techniques, and therefore conduct several "brute force" experiments to determine how the trajectory is affected by initial conditions z_0 and w_0. First they hold w_0 fixed at 4 m/s and conduct experiments at five different values of z_0: 3, 6, 9, 12, and 15 m. The experimental results are shown in Fig. 7–12*a*. Next, they hold z_0 fixed at 10 m and conduct experiments at five different values of w_0: 2, 4, 6, 8, and 10 m/s. These results are shown in Fig. 7–12*b*. Later that evening, your brother shows you the data and the trajectory plots and tells you that they plan to conduct more experiments at different values of z_0 and w_0. You explain to him that by first nondimensionalizing the data, the problem can be reduced to just *one* parameter, and no further experiments are required. Prepare a nondimensional plot to prove your point and discuss.

SOLUTION A nondimensional plot is to be generated from all the available trajectory data. Specifically, we are to plot z^* as a function of t^*.

Assumptions The inside of the pipe is subjected to strong enough vacuum pressure that aerodynamic drag on the ball is negligible.

Properties The gravitational constant is 9.81 m/s^2.

Analysis Equation 7–4 is valid for this problem, as is the nondimensionalization that resulted in Eq. 7–9. As previously discussed, this problem combines three of the original dimensional parameters (g, z_0, and w_0) into *one* nondimensional parameter, the Froude number. After converting to the dimensionless variables of Eq. 7–6, the 10 trajectories of Fig. 7–12*a* and *b* are replotted in dimensionless format in Fig. 7–13. It is clear that all the trajectories are of the same family, with the Froude number as the only remaining parameter. Fr^2 varies from about 0.041 to about 1.0 in these experiments. If any more experiments are to be conducted, they should include combinations of z_0 and w_0 that produce Froude numbers outside of this range. A large number of additional experiments would be unnecessary, since all the trajectories would be of the same family as those plotted in Fig. 7–13.

Discussion At low Froude numbers, gravitational forces are much larger than inertial forces, and the ball falls to the floor in a relatively short time. At large values of Fr on the other hand, inertial forces dominate initially, and the ball rises a significant distance before falling; it takes much longer for the ball to hit the ground. The students are obviously not able to adjust the gravitational constant, but if they could, the brute force method would require many more experiments to document the effect of g. If they nondimensionalize first, however, the dimensionless trajectory plots already obtained and shown in Fig. 7–13 would be valid for *any* value of g; no further experiments would be required unless Fr were outside the range of tested values.

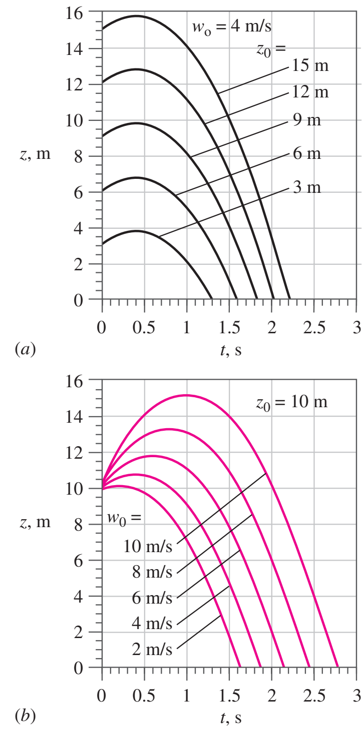


FIGURE 7–12
Trajectories of a steel ball falling in a vacuum: (*a*) w_0 fixed at 4 m/s, and (*b*) z_0 fixed at 10 m (Example 7–3).

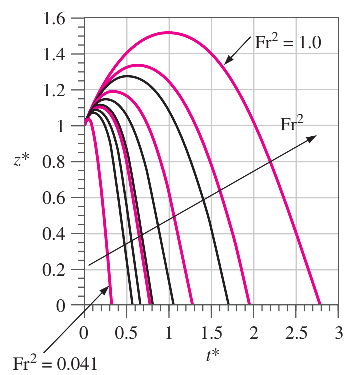


FIGURE 7–13
Trajectories of a steel ball falling in a vacuum. Data of Fig. 7–12*a* and *b* are nondimensionalized and combined onto one plot.

If you are still not convinced that nondimensionalizing the equations and the parameters has many advantages, consider this: In order to reasonably document the trajectories of Example 7–3 for a range of all three of the dimensional parameters g, z_0, and w_0, the brute force method would require several (say a minimum of four) additional plots like Fig. 7–12*a* at various values (levels) of w_0, plus several additional *sets* of such plots for a range of g. A complete data set for three parameters with five levels of each parameter would require $5^3 = 125$ experiments! Nondimensionalization reduces the number of parameters from three to one—a total of only $5^1 = 5$ experiments are required for the same resolution. (For five levels, only five dimensionless trajectories like those of Fig. 7–13 are required, at carefully chosen values of Fr.)

Another advantage of nondimensionalization is that extrapolation to untested values of one or more of the dimensional parameters is possible. For example, the data of Example 7–3 were taken at only one value of gravitational acceleration. Suppose you wanted to extrapolate these data to a different value of g. Example 7–4 shows how this is easily accomplished via the dimensionless data.

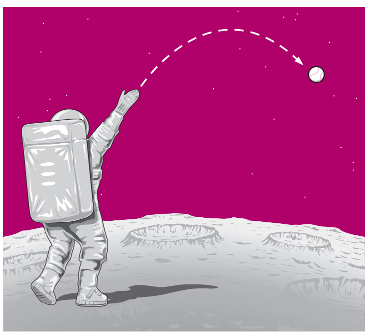


FIGURE 7–14
Throwing a baseball on the moon (Example 7–4).

EXAMPLE 7–4 Extrapolation of Nondimensionalized Data

The gravitational constant at the surface of the moon is only about one-sixth of that on earth. An astronaut on the moon throws a baseball at an initial speed of 21.0 m/s at a 5° angle above the horizon and at 2.0 m above the moon's surface (Fig. 7–14). (*a*) Using the dimensionless data of Example 7–3 shown in Fig. 7–13, predict how long it takes for the baseball to fall to the ground. (*b*) Do an *exact* calculation and compare the result to that of part (*a*).

SOLUTION Experimental data obtained on earth are to be used to predict the time required for a baseball to fall to the ground on the moon.

Assumptions **1** The horizontal velocity of the baseball is irrelevant. **2** The surface of the moon is perfectly flat near the astronaut. **3** There is no aerodynamic drag on the ball since there is no atmosphere on the moon. **4** Moon gravity is one-sixth that of earth.

Properties The gravitational constant on the moon is $g_{\text{moon}} \cong 9.81/6 = 1.63\ \text{m/s}^2$.

Analysis (*a*) The Froude number is calculated based on the value of g_{moon} and the vertical component of initial speed,

$$w_0 = (21.0\ \text{m/s})\sin(5°) = 1.830\ \text{m/s}$$

from which

$$\text{Fr}^2 = \frac{w_0^2}{g_{\text{moon}} z_0} = \frac{(1.830\ \text{m/s})^2}{(1.63\ \text{m/s}^2)(2.0\ \text{m})} = 1.03$$

This value of Fr^2 is nearly the same as the largest value plotted in Fig. 7–13. Thus, in terms of dimensionless variables, the baseball strikes the ground at $t^* \cong 2.75$, as determined from Fig. 7–13. Converting back to dimensional variables using Eq. 7–6,

Estimated time to strike the ground: $$t = \frac{t^* z_0}{w_0} = \frac{2.75(2.0\ \text{m})}{1.830\ \text{m/s}} = \mathbf{3.01\ s}$$

(*b*) An exact calculation is obtained by setting z equal to zero in Eq. 7–5 and solving for time t (using the quadratic formula), *Exact time to strike the ground*:

$$t = \frac{w_0 + \sqrt{w_0^2 + 2z_0 g}}{g}$$

$$= \frac{1.830 \text{ m/s} + \sqrt{(1.830 \text{ m/s})^2 + 2(2.0 \text{ m})(1.63 \text{ m/s}^2)}}{1.63 \text{ m/s}^2} = \mathbf{3.05\ s}$$

Discussion If the Froude number had landed between two of the trajectories of Fig. 7–13, interpolation would have been required. Since some of the numbers are precise to only two significant digits, the small difference between the results of part (*a*) and part (*b*) is of no concern. The final result is $t = 3.0$ s to two significant digits.

The differential equations of motion for fluid flow are derived and discussed in Chap. 9. In Chap. 10 you will find an analysis similar to that presented here, but applied to the differential equations for fluid flow. It turns out that the Froude number also appears in that analysis, as do three other important dimensionless parameters—the Reynolds number, Euler number, and Strouhal number (Fig. 7–15).

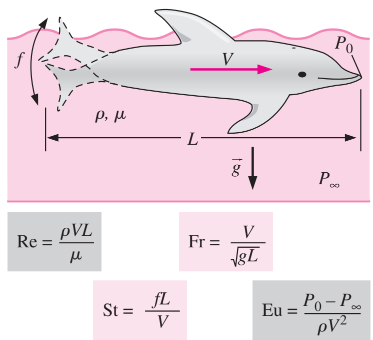


FIGURE 7–15
In a general unsteady fluid flow problem with a free surface, the scaling parameters include a characteristic length L, a characteristic velocity V, a characteristic frequency f, and a reference pressure difference $P_0 - P_\infty$. Nondimensionalization of the differential equations of fluid flow produces four dimensionless parameters: the Reynolds number, Froude number, Strouhal number, and Euler number (see Chap. 10).

7–3 ▪ DIMENSIONAL ANALYSIS AND SIMILARITY

Nondimensionalization of an equation by inspectional analysis is useful only when one knows the equation to begin with. However, in many cases in real-life engineering, the equations are either not known or too difficult to solve; oftentimes *experimentation* is the only method of obtaining reliable information. In most experiments, to save time and money, tests are performed on a geometrically scaled **model,** rather than on the full-scale **prototype.** In such cases, care must be taken to properly scale the results. We introduce here a powerful technique called **dimensional analysis.** While typically taught in fluid mechanics, dimensional analysis is useful in *all* disciplines, especially when it is necessary to design and conduct experiments. You are encouraged to use this powerful tool in other subjects as well, not just in fluid mechanics. The three primary purposes of dimensional analysis are

- To generate nondimensional parameters that help in the design of experiments (physical and/or numerical) and in the reporting of experimental results
- To obtain scaling laws so that prototype performance can be predicted from model performance
- To (sometimes) predict trends in the relationship between parameters

Before discussing the *technique* of dimensional analysis, we first explain the underlying *concept* of dimensional analysis—the principle of **similarity.** There are three necessary conditions for complete similarity between a model and a prototype. The first condition is **geometric similarity**—the model must be the same shape as the prototype, but may be scaled by some constant scale factor. The second condition is **kinematic similarity,** which means that the velocity at any point in the model flow must be proportional (by a constant scale factor) to the velocity at the corresponding point in the prototype flow (Fig. 7–16). Specifically, for kinematic similarity the velocity at corresponding points must scale in magnitude and must point in the same relative direction. You may think of geometric similarity as *length-scale* equivalence and kinematic similarity as *time-scale* equivalence. *Geometric similarity is a prerequisite for kinematic similarity*. Just as the geometric scale factor can be less than, equal to, or greater than one, so can the velocity scale factor. In Fig. 7–16, for example, the geometric scale factor is less than one (model smaller than prototype), but the velocity scale is greater than one (velocities around the model are greater than those around the prototype). You may recall from Chap. 4 that streamlines are kinematic phenomena; hence, the streamline pattern in the model flow is a geometrically scaled copy of that in the prototype flow when kinematic similarity is achieved.

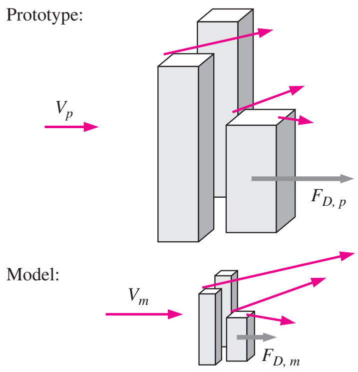


FIGURE 7–16
Kinematic similarity is achieved when, at all locations, the velocity in the model flow is proportional to that at corresponding locations in the prototype flow, and points in the same direction.

The third and most restrictive similarity condition is that of **dynamic similarity.** Dynamic similarity is achieved when all *forces* in the model flow scale by a constant factor to corresponding forces in the prototype flow (*force-scale* equivalence). As with geometric and kinematic similarity, the scale factor for forces can be less than, equal to, or greater than one. In Fig. 7–16 for example, the force-scale factor is less than one since the force on the model building is less than that on the prototype. *Kinematic similarity is a necessary but insufficient condition for dynamic similarity*. It is thus possible for a model flow and a prototype flow to achieve both geometric and kinematic similarity, yet not dynamic similarity. All three similarity conditions must exist for complete similarity to be ensured.

> In a general flow field, complete similarity between a model and prototype is achieved only when there is geometric, kinematic, and dynamic similarity.

We let uppercase Greek letter Pi (Π) denote a nondimensional parameter. You are already familiar with one Π, namely the Froude number, Fr. In a general dimensional analysis problem, there is one Π that we call the **dependent** Π, giving it the notation Π_1. The parameter Π_1 is in general a function of several other Π's, which we call **independent** Π's. The functional relationship is

Functional relationship between Π*'s:*
$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots, \Pi_k) \tag{7–11}$$

where k is the total number of Π's.

Consider an experiment in which a scale model is tested to simulate a prototype flow. To ensure complete similarity between the model and the prototype, each independent Π of the model (subscript m) must be identical to the corresponding independent Π of the prototype (subscript p), i.e., $\Pi_{2,\,m} = \Pi_{2,\,p}$, $\Pi_{3,\,m} = \Pi_{3,\,p}$, . . . , $\Pi_{k,\,m} = \Pi_{k,\,p}$.

> To ensure complete similarity, the model and prototype must be geometrically similar, and all independent Π groups must match between model and prototype.

Under these conditions the *dependent* Π of the model ($\Pi_{1,\,m}$) is guaranteed to also equal the dependent Π of the prototype ($\Pi_{1,\,p}$). Mathematically, we write a conditional statement for achieving similarity,

$$\text{If} \quad \Pi_{2,\,m} = \Pi_{2,\,p} \quad \text{and} \quad \Pi_{3,\,m} = \Pi_{3,\,p} \ldots \quad \text{and} \quad \Pi_{k,\,m} = \Pi_{k,\,p},$$
$$\text{then} \quad \Pi_{1,\,m} = \Pi_{1,\,p} \tag{7–12}$$

Consider, for example, the design of a new sports car, the aerodynamics of which is to be tested in a wind tunnel. To save money, it is desirable to test a small, geometrically scaled model of the car rather than a full-scale prototype of the car (Fig. 7–17). In the case of aerodynamic drag on an automobile, it turns out that if the flow is approximated as incompressible, there are only two Π's in the problem,

$$\Pi_1 = f(\Pi_2) \qquad \text{where} \qquad \Pi_1 = \frac{F_D}{\rho V^2 L^2} \qquad \text{and} \qquad \Pi_2 = \frac{\rho V L}{\mu} \tag{7–13}$$

The procedure used to generate these Π's is discussed in Section 7–4. In Eq. 7–13, F_D is the magnitude of the aerodynamic drag on the car, ρ is the air density, V is the car's speed (or the speed of the air in the wind tunnel), L is the length of the car, and μ is the viscosity of the air. Π_1 is a nonstandard form of the drag coefficient, and Π_2 is the **Reynolds number,** Re. You will find that many problems in fluid mechanics involve a Reynolds number (Fig. 7–18).

The Reynolds number is the most well known and useful dimensionless parameter in all of fluid mechanics.

In the problem at hand there is only one independent Π, and Eq. 7–12 ensures that if the independent Π's match (the Reynolds numbers match: $\Pi_{2,\,m} = \Pi_{2,\,p}$), then the dependent Π's also match ($\Pi_{1,\,m} = \Pi_{1,\,p}$). This enables engineers to measure the aerodynamic drag on the model car and then use this value to predict the aerodynamic drag on the prototype car.

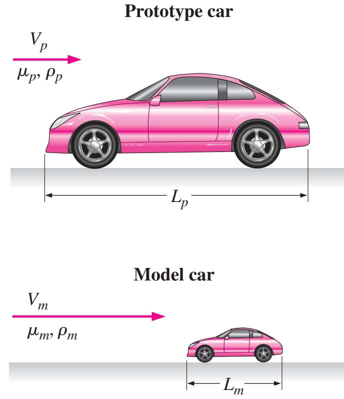


FIGURE 7–17
Geometric similarity between a prototype car of length L_p and a model car of length L_m.

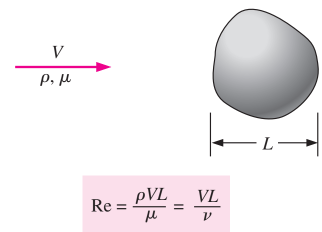


FIGURE 7–18
The *Reynolds number* Re is formed by the ratio of density, characteristic speed, and characteristic length to viscosity. Alternatively, it is the ratio of characteristic speed and length to *kinematic viscosity*, defined as $\nu = \mu/\rho$.

EXAMPLE 7–5 Similarity between Model and Prototype Cars

The aerodynamic drag of a new sports car is to be predicted at a speed of 50.0 mi/h at an air temperature of 25°C. Automotive engineers build a one-fifth scale model of the car to test in a wind tunnel. It is winter and the wind tunnel is located in an unheated building; the temperature of the wind tunnel air is only about 5°C. Determine how fast the engineers should run the wind tunnel in order to achieve similarity between the model and the prototype.

SOLUTION We are to utilize the concept of similarity to determine the speed of the wind tunnel.

Assumptions **1** Compressibility of the air is negligible (the validity of this approximation is discussed later). **2** The wind tunnel walls are far enough away so as to not interfere with the aerodynamic drag on the model car. **3** The model is geometrically similar to the prototype. **4** The wind tunnel has a moving belt to simulate the ground under the car, as in Fig. 7–19. (The moving belt is necessary in order to achieve kinematic similarity everywhere in the flow, in particular underneath the car.)

Properties For air at atmospheric pressure and at $T = 25°\text{C}$, $\rho = 1.184\ \text{kg/m}^3$ and $\mu = 1.849 \times 10^{-5}\ \text{kg/m}\cdot\text{s}$. Similarly, at $T = 5°\text{C}$, $\rho = 1.269\ \text{kg/m}^3$ and $\mu = 1.754 \times 10^{-5}\ \text{kg/m}\cdot\text{s}$.

Analysis Since there is only one independent Π in this problem, the similarity equation (Eq. 7–12) holds if $\Pi_{2,\,m} = \Pi_{2,\,p}$, where Π_2 is given by Eq. 7–13, and we call it the Reynolds number. Thus, we write

$$\Pi_{2,\,m} = \mathrm{Re}_m = \frac{\rho_m V_m L_m}{\mu_m} = \Pi_{2,\,p} = \mathrm{Re}_p = \frac{\rho_p V_p L_p}{\mu_p}$$

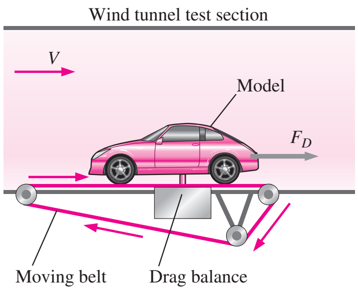


FIGURE 7–19
A *drag balance* is a device used in a wind tunnel to measure the aerodynamic drag of a body. When testing automobile models, a *moving belt* is often added to the floor of the wind tunnel to simulate the moving ground (from the car's frame of reference).

which can be solved for the unknown wind tunnel speed for the model tests, V_m,

$$V_m = V_p\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{L_p}{L_m}\right)$$

$$= (50.0\text{ mi/h})\left(\frac{1.754 \times 10^{-5}\text{ kg/m}\cdot\text{s}}{1.849 \times 10^{-5}\text{ kg/m}\cdot\text{s}}\right)\left(\frac{1.184\text{ kg/m}^3}{1.269\text{ kg/m}^3}\right)(5) = \mathbf{221\text{ mi/h}}$$

Thus, to ensure similarity, the wind tunnel should be run at 221 mi/h (to three significant digits). Note that we were never given the actual length of either car, but the ratio of L_p to L_m is known because the prototype is five times larger than the scale model. When the dimensional parameters are rearranged as nondimensional ratios (as done here), the unit system is irrelevant. Since the units in each numerator cancel those in each denominator, no unit conversions are necessary.

Discussion This speed is quite high (about 100 m/s), and the wind tunnel may not be able to run at that speed. Furthermore, the incompressible approximation may come into question at this high speed (we discuss this in more detail in Example 7–8).

Once we are convinced that complete similarity has been achieved between the model tests and the prototype flow, Eq. 7–12 can be used again to predict the performance of the prototype based on measurements of the performance of the model. This is illustrated in Example 7–6.

EXAMPLE 7–6 Prediction of Aerodynamic Drag Force on the Prototype Car

This example is a follow-up to Example 7–5. Suppose the engineers run the wind tunnel at 221 mi/h to achieve similarity between the model and the prototype. The aerodynamic drag force on the model car is measured with a **drag balance** (Fig. 7–19). Several drag readings are recorded, and the average drag force on the model is 21.2 lbf. Predict the aerodynamic drag force on the prototype (at 50 mi/h and 25°C).

SOLUTION Because of similarity, the model results can be scaled up to predict the aerodynamic drag force on the prototype.

Analysis The similarity equation (Eq. 7–12) shows that since $\Pi_{2,\,m} = \Pi_{2,\,p}$, $\Pi_{1,\,m} = \Pi_{1,\,p}$, where Π_1 is given for this problem by Eq. 7–13. Thus, we write

$$\Pi_{1,\,m} = \frac{F_{D,\,m}}{\rho_m V_m^2 L_m^2} = \Pi_{1,\,p} = \frac{F_{D,\,p}}{\rho_p V_p^2 L_p^2}$$

which can be solved for the unknown aerodynamic drag force on the prototype car, $F_{D,\,p}$,

$$F_{D,\,p} = F_{D,\,m}\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2$$

$$= (21.2\text{ lbf})\left(\frac{1.184\text{ kg/m}^3}{1.269\text{ kg/m}^3}\right)\left(\frac{50.0\text{ mi/h}}{221\text{ mi/h}}\right)^2(5)^2 = \mathbf{25.3\text{ lbf}}$$

Discussion By arranging the dimensional parameters as nondimensional ratios, the units cancel nicely even though they are a mixture of SI and English units. Because both velocity and length are squared in the equation for Π_1, the higher speed in the wind tunnel nearly compensates for the model's smaller size, and the drag force on the model is nearly the same as that on the prototype. In fact, if the density and viscosity of the air in the wind tunnel were *identical* to those of the air flowing over the prototype, the two drag forces would be identical as well (Fig. 7–20).

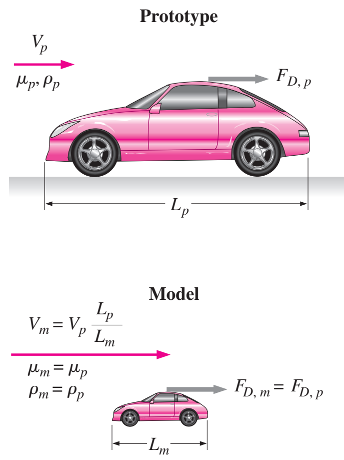


FIGURE 7–20
For the special case in which the wind tunnel air and the air flowing over the prototype have the same properties ($\rho_m = \rho_p$, $\mu_m = \mu_p$), and under similarity conditions ($V_m = V_p L_p / L_m$), the aerodynamic drag force on the prototype is equal to that on the scale model. If the two fluids do *not* have the same properties, the two drag forces are *not* necessarily the same, even under dynamically similar conditions.

The power of using dimensional analysis and similarity to supplement experimental analysis is further illustrated by the fact that the actual values of the dimensional parameters (density, velocity, etc.) are irrelevant. As long as the corresponding independent Π's are set equal to each other, similarity is achieved—*even if different fluids are used*. This explains why automobile or aircraft performance can be simulated in a water tunnel, and the performance of a submarine can be simulated in a wind tunnel (Fig. 7–21). Suppose, for example, that the engineers in Examples 7–5 and 7–6 use a water tunnel instead of a wind tunnel to test their one-fifth scale model. Using the properties of water at room temperature (20°C is assumed), the water tunnel speed required to achieve similarity is easily calculated as

$$V_m = V_p\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{L_p}{L_m}\right)$$

$$= (50.0 \text{ mi/h})\left(\frac{1.002 \times 10^{-3} \text{ kg/m}\cdot\text{s}}{1.849 \times 10^{-5} \text{ kg/m}\cdot\text{s}}\right)\left(\frac{1.184 \text{ kg/m}^3}{998.0 \text{ kg/m}^3}\right)(5) = 16.1 \text{ mi/h}$$

As can be seen, one advantage of a water tunnel is that the required water tunnel speed is much lower than that required for a wind tunnel using the same size model.

7–4 ▪ THE METHOD OF REPEATING VARIABLES AND THE BUCKINGHAM PI THEOREM

We have seen several examples of the usefulness and power of dimensional analysis. Now we are ready to learn how to *generate* the nondimensional parameters, i.e., the Π's. There are several methods that have been developed for this purpose, but the most popular (and simplest) method is the **method of repeating variables,** popularized by Edgar Buckingham (1867–1940). The method was first published by the Russian scientist Dimitri Riabouchinsky (1882–1962) in 1911. We can think of this method as a step-by-step procedure or "recipe" for obtaining nondimensional parameters. There are six steps, listed concisely in Fig. 7–22, and in more detail in Table 7–2. These steps are explained in further detail as we work through a number of example problems.

As with most new procedures, the best way to learn is by example and practice. As a simple first example, consider a ball falling in a vacuum as discussed in Section 7–2. Let us pretend that we do not know that Eq. 7–4 is appropriate for this problem, nor do we know much physics concerning falling objects. In fact, suppose that all we know is that the instantaneous

FIGURE 7–21
Similarity can be achieved even when the model fluid is different than the prototype fluid. Here a submarine model is tested in a wind tunnel.
Courtesy NASA Langley Research Center.

The Method of Repeating Variables

Step 1: List the parameters in the problem and count their total number n.

Step 2: List the primary dimensions of each of the n parameters.

Step 3: Set the *reduction* j as the number of primary dimensions. Calculate k, the expected number of Π's,
$k = n - j$

Step 4: Choose j *repeating parameters*.

Step 5: Construct the k Π's, and manipulate as necessary.

Step 6: Write the final functional relationship and check your algebra.

FIGURE 7–22
A concise summary of the six steps that comprise the *method of repeating variables*.

TABLE 7–2

Detailed description of the six steps that comprise the *method of repeating variables**

Step 1	List the parameters (dimensional variables, nondimensional variables, and dimensional constants) and count them. Let n be the total number of parameters in the problem, including the dependent variable. Make sure that any listed independent parameter is indeed independent of the others, i.e., it cannot be expressed in terms of them. (E.g., don't include radius r and area $A = \pi r^2$, since r and A are *not* independent.)
Step 2	List the primary dimensions for each of the n parameters.
Step 3	Guess the **reduction** j. As a first guess, set j equal to the number of primary dimensions represented in the problem. The expected number of Π's (k) is equal to n minus j, according to the **Buckingham Pi theorem**, *The Buckingham Pi theorem*: $k = n - j$ **(7–14)** If at this step or during any subsequent step, the analysis does not work out, verify that you have included enough parameters in step 1. Otherwise, go back and *reduce j by one* and try again.
Step 4	Choose j **repeating parameters** that will be used to construct each Π. Since the repeating parameters have the potential to appear in each Π, be sure to choose them *wisely* (Table 7–3).
Step 5	Generate the Π's one at a time by grouping the j repeating parameters with one of the remaining parameters, forcing the product to be dimensionless. In this way, construct all k Π's. By convention the first Π, designated as Π_1, is the *dependent* Π (the one on the left side of the list). Manipulate the Π's as necessary to achieve established dimensionless groups (Table 7–5).
Step 6	Check that all the Π's are indeed dimensionless. Write the final functional relationship in the form of Eq. 7–11.

* This is a step-by-step method for finding the dimensionless Π groups when performing a dimensional analysis.

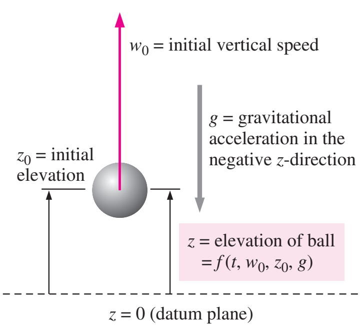


FIGURE 7–23
Setup for dimensional analysis of a ball falling in a vacuum. Elevation z is a function of time t, initial vertical speed w_0, initial elevation z_0, and gravitational constant g.

elevation z of the ball must be a function of time t, initial vertical speed w_0, initial elevation z_0, and gravitational constant g (Fig. 7–23). The beauty of dimensional analysis is that the only other thing we need to know is the primary dimensions of each of these quantities. As we go through each step of the method of repeating variables, we explain some of the subtleties of the technique in more detail using the falling ball as an example.

Step 1

There are five parameters (dimensional variables, nondimensional variables, and dimensional constants) in this problem; $n = 5$. They are listed in functional form, with the dependent variable listed as a function of the independent variables and constants:

List of relevant parameters: $\quad z = f(t, w_0, z_0, g) \quad n = 5$

Step 2

The primary dimensions of each parameter are listed here. We recommend writing each dimension with exponents since this helps with later algebra.

$$\underset{\{L^1\}}{z} \qquad \underset{\{t^1\}}{t} \qquad \underset{\{L^1t^{-1}\}}{w_0} \qquad \underset{\{L^1\}}{z_0} \qquad \underset{\{L^1t^{-2}\}}{g}$$

Step 3

As a first guess, j is set equal to 2, the number of primary dimensions represented in the problem (L and t).

Reduction: $$j = 2$$

If this value of j is correct, the number of Π's predicted by the Buckingham Pi theorem is

Number of expected Π*'s*: $$k = n - j = 5 - 2 = 3$$

Step 4

We need to choose two repeating parameters since $j = 2$. Since this is often the hardest (or at least the most mysterious) part of the method of repeating variables, several guidelines about choosing repeating parameters are listed in Table 7–3.

Following the guidelines of Table 7–3 on the next page, the wisest choice of two repeating parameters is w_0 and z_0.

Repeating parameters: $$w_0 \text{ and } z_0$$

Step 5

Now we combine these repeating parameters into products with each of the remaining parameters, one at a time, to create the Π's. The first Π is always the *dependent* Π and is formed with the dependent variable z.

Dependent Π: $$\Pi_1 = z w_0^{a_1} z_0^{b_1} \tag{7–15}$$

where a_1 and b_1 are constant exponents that need to be determined. We apply the primary dimensions of step 2 into Eq. 7–15 and *force* the Π to be dimensionless by setting the exponent of each primary dimension to zero:

Dimensions of Π_1: $$\{\Pi_1\} = \{L^0t^0\} = \{z w_0^{a_1} z_0^{b_1}\} = \{L^1(L^1t^{-1})^{a_1}L^{b_1}\}$$

Since primary dimensions are by definition independent of each other, we equate the exponents of each primary dimension independently to solve for exponents a_1 and b_1 (Fig. 7–24).

Time: $$\{t^0\} = \{t^{-a_1}\} \qquad 0 = -a_1 \qquad a_1 = 0$$

Length: $$\{L^0\} = \{L^1L^{a_1}L^{b_1}\} \qquad 0 = 1 + a_1 + b_1 \qquad b_1 = -1 - a_1 \qquad b_1 = -1$$

Equation 7–15 thus becomes

$$\Pi_1 = \frac{z}{z_0} \tag{7–16}$$

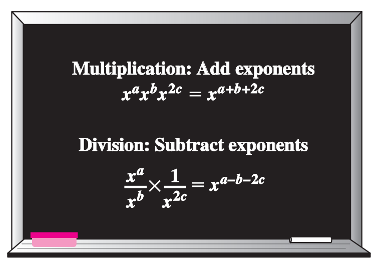


FIGURE 7–24
The mathematical rules for adding and subtracting exponents during multiplication and division, respectively.

TABLE 7–3

Guidelines for choosing *repeating parameters* in step 4 of the method of repeating variables*

Guideline	Comments and Application to Present Problem
1. Never pick the *dependent* variable. Otherwise, it may appear in all the Π's, which is undesirable.	In the present problem we cannot choose z, but we must choose from among the remaining four parameters. Therefore, we must choose two of the following parameters: t, w_0, z_0, and g.
2. The chosen repeating parameters must not *by themselves* be able to form a dimensionless group. Otherwise, it would be impossible to generate the rest of the Π's.	In the present problem, any two of the independent parameters would be valid according to this guideline. For illustrative purposes, however, suppose we have to pick three instead of two repeating parameters. We could not, for example, choose t, w_0, and z_0, because these can form a Π all by themselves (tw_0/z_0).
3. The chosen repeating parameters must represent *all* the primary dimensions in the problem.	Suppose for example that there were *three* primary dimensions (m, L, and t) and *two* repeating parameters were to be chosen. You could not choose, say, a length and a time, since primary dimension mass would not be represented in the dimensions of the repeating parameters. An appropriate choice would be a density and a time, which together represent all three primary dimensions in the problem.
4. Never pick parameters that are already dimensionless. These are Π's already, all by themselves.	Suppose an angle θ were one of the independent parameters. We could not choose θ as a repeating parameter since angles have no dimensions (radian and degree are dimensionless units). In such a case, one of the Π's is already known, namely θ.
5. Never pick two parameters with the *same* dimensions or with dimensions that differ by only an exponent.	In the present problem, two of the parameters, z and z_0, have the same dimensions (length). We cannot choose both of these parameters. (Note that dependent variable z has already been eliminated by guideline 1.) Suppose one parameter has dimensions of length and another parameter has dimensions of volume. In dimensional analysis, volume contains only one primary dimension (length) and *is not dimensionally distinct from length*—we cannot choose both of these parameters.
6. Whenever possible, choose dimensional constants over dimensional variables so that only *one* Π contains the dimensional variable.	If we choose time t as a repeating parameter in the present problem, it would appear in all three Π's. While this would not be *wrong,* it would not be *wise* since we know that ultimately we want some nondimensional height as a function of some nondimensional time and other nondimensional parameter(s). From the original four independent parameters, this restricts us to w_0, z_0, and g.
7. Pick common parameters since they may appear in each of the Π's.	In fluid flow problems we generally pick a length, a velocity, and a mass or density (Fig. 7–25). It is unwise to pick less common parameters like viscosity μ or surface tension σ_s, since we would in general not want μ or σ_s to appear in each of the Π's. In the present problem, w_0 and z_0 are wiser choices than g.
8. Pick simple parameters over complex parameters whenever possible.	It is better to pick parameters with only one or two basic dimensions (e.g., a length, a time, a mass, or a velocity) instead of parameters that are composed of several basic dimensions (e.g., an energy or a pressure).

* These guidelines, while not infallible, help you to pick repeating parameters that usually lead to established nondimensional Π groups with minimal effort.

In similar fashion we create the first independent Π (Π_2) by combining the repeating parameters with independent variable t.

First independent Π: $$\Pi_2 = tw_0^{a_2}z_0^{b_2}$$

Dimensions of Π_2: $$\{\Pi_2\} = \{\mathrm{L}^0\mathrm{t}^0\} = \{tw_0^{a_2}z_0^{b_2}\} = \{\mathrm{t}(\mathrm{L}^1\mathrm{t}^{-1})^{a_2}\mathrm{L}^{b_2}\}$$

Equating exponents,

Time: $\{t^0\} = \{t^1 t^{-a_2}\} \qquad 0 = 1 - a_2 \qquad a_2 = 1$

Length: $\{L^0\} = \{L^{a_2} L^{b_2}\} \qquad 0 = a_2 + b_2 \qquad b_2 = -a_2 \qquad b_2 = -1$

Π_2 is thus

$$\Pi_2 = \frac{w_0 t}{z_0} \tag{7–17}$$

Finally we create the second independent Π (Π_3) by combining the repeating parameters with g and *forcing* the Π to be dimensionless (Fig. 7–26).

Second independent Π: $\Pi_3 = g w_0^{a_3} z_0^{b_3}$

Dimensions of Π_3: $\{\Pi_3\} = \{L^0 t^0\} = \{g w_0^{a_3} z_0^{b_3}\} = \{L^1 t^{-2} (L^1 t^{-1})^{a_3} L^{b_3}\}$

Equating exponents,

Time: $\{t^0\} = \{t^{-2} t^{-a_3}\} \qquad 0 = -2 - a_3 \qquad a_3 = -2$

Length: $\{L^0\} = \{L^1 L^{a_3} L^{b_3}\} \qquad 0 = 1 + a_3 + b_3 \qquad b_3 = -1 - a_3 \qquad b_3 = 1$

Π_3 is thus

$$\Pi_3 = \frac{g z_0}{w_0^2} \tag{7–18}$$

All three Π's have been found, but at this point it is prudent to examine them to see if any manipulation is required. We see immediately that Π_1 and Π_2 are the same as the nondimensionalized variables z^* and t^* defined by Eq. 7–6—no manipulation is necessary for these. However, we recognize that the third Π must be raised to the power of $-\frac{1}{2}$ to be of the same form as an established dimensionless parameter, namely the Froude number of Eq. 7–8:

Modified Π_3:
$$\Pi_{3,\text{ modified}} = \left(\frac{g z_0}{w_0^2}\right)^{-1/2} = \frac{w_0}{\sqrt{g z_0}} = \text{Fr} \tag{7–19}$$

Such manipulation is often necessary to put the Π's into proper established form. The Π of Eq. 7–18 is not *wrong*, and there is certainly no mathematical advantage of Eq. 7–19 over Eq. 7–18. Instead, we like to say that Eq. 7–19 is more "socially acceptable" than Eq. 7–18, since it is a named, established nondimensional parameter that is commonly used in the literature. In Table 7–4 are listed some guidelines for manipulation of your nondimensional Π groups into established nondimensional parameters.

Table 7–5 lists some established nondimensional parameters, most of which are named after a notable scientist or engineer (see Fig. 7–27 and the Historical Spotlight on p. 289). This list is by no means exhaustive. Whenever possible, you should manipulate your Π's as necessary in order to convert them into established nondimensional parameters.

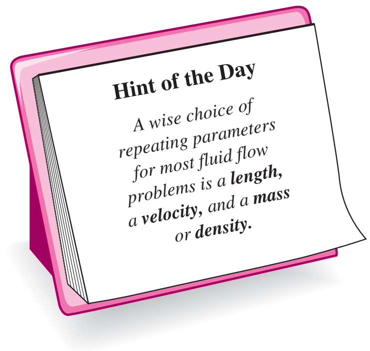


FIGURE 7–25
It is wise to choose *common* parameters as repeating parameters since they may appear in each of your dimensionless Π groups.

$$\{\Pi_1\} = \{m^0 L^0 t^0 T^0 I^0 C^0 N^0\} = \{1\}$$
$$\{\Pi_2\} = \{m^0 L^0 t^0 T^0 I^0 C^0 N^0\} = \{1\}$$
$$\vdots$$
$$\{\Pi_k\} = \{m^0 L^0 t^0 T^0 I^0 C^0 N^0\} = \{1\}$$

FIGURE 7–26
The Π groups that result from the method of repeating variables are guaranteed to be dimensionless because we *force* the overall exponent of all seven primary dimensions to be zero.

TABLE 7–4

Guidelines for manipulation of the Π's resulting from the method of repeating variables.*

Guideline	Comments and Application to Present Problem
1. We may impose a constant (dimensionless) exponent on a Π or perform a functional operation on a Π.	We can raise a Π to any exponent n (changing it to Π^n) without changing the dimensionless stature of the Π. For example, in the present problem, we imposed an exponent of $-\frac{1}{2}$ on Π_3. Similarly we can perform the functional operation sin(Π), exp(Π), etc., without influencing the dimensions of the Π.
2. We may multiply a Π by a pure (dimensionless) constant.	Sometimes dimensionless factors of $\frac{1}{2}$, 2, 4, etc., are included in a Π for convenience. This is perfectly okay since such factors do not influence the dimensions of the Π.
3. We may form a product (or quotient) of any Π with any other Π in the problem to replace one of the Π's.	We could replace Π_3 by $\Pi_3\Pi_1$, Π_3/Π_2, etc. Sometimes such manipulation is necessary to convert our Π into an established Π. In many cases, the established Π would have been produced if we would have chosen different repeating parameters.
4. We may use any of guidelines 1 to 3 in combination.	In general, we can replace any Π with some new Π such as $A\Pi_3^B \sin(\Pi_1^C)$, where A, B, and C are pure constants.
5. We may substitute a dimensional parameter in the Π with other parameter(s) of the same dimensions.	For example, the Π may contain the square of a length or the cube of a length, for which we may substitute a known area or volume, respectively, in order to make the Π agree with established conventions.

*These guidelines are useful in step 5 of the method of repeating variables and are listed to help you convert your nondimensional Π groups into standard, *established* nondimensional parameters, many of which are listed in Table 7–5.

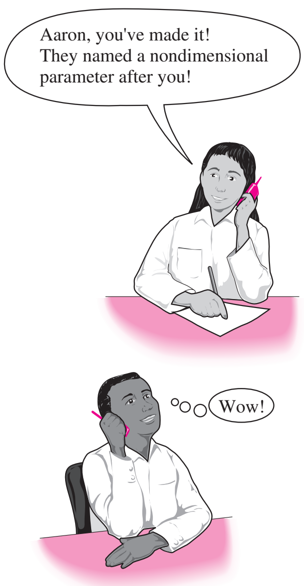


FIGURE 7–27
Established nondimensional parameters are usually named after a notable scientist or engineer.

Step 6

We should double-check that the Π's are indeed dimensionless (Fig. 7–28). You can verify this on your own for the present example. We are finally ready to write the functional relationship between the nondimensional parameters. Combining Eqs. 7–16, 7–17, and 7–19 into the form of Eq. 7–11,

$$\textit{Relationship between } \Pi\textit{'s}: \quad \Pi_1 = f(\Pi_2, \Pi_3) \quad \rightarrow \quad \frac{z}{z_0} = f\left(\frac{w_0 t}{z_0}, \frac{w_0}{\sqrt{g z_0}}\right)$$

Or, in terms of the nondimensional variables z^* and t^* defined previously by Eq. 7–6 and the definition of the Froude number,

$$\textit{Final result of dimensional analysis}: \qquad z^* = f(t^*, \mathrm{Fr}) \qquad \textbf{(7–20)}$$

It is useful to compare the result of dimensional analysis, Eq. 7–20, to the exact analytical result, Eq. 7–10. The method of repeating variables properly predicts the functional relationship between dimensionless groups. However,

> The method of repeating variables cannot predict the exact mathematical form of the equation.

This is a fundamental limitation of dimensional analysis and the method of repeating variables. For some simple problems, however, the form of the equation *can* be predicted to within an unknown constant, as is illustrated in Example 7–7.

TABLE 7–5

Some common established nondimensional parameters or Π's encountered in fluid mechanics and heat transfer*

Name	Definition	Ratio of Significance
Archimedes number	$\mathrm{Ar} = \dfrac{\rho_s g L^3}{\mu^2}(\rho_s - \rho)$	$\dfrac{\text{Gravitational force}}{\text{Viscous force}}$
Aspect ratio	$\mathrm{AR} = \dfrac{L}{W}$ or $\dfrac{L}{D}$	$\dfrac{\text{Length}}{\text{Width}}$ or $\dfrac{\text{Length}}{\text{Diameter}}$
Biot number	$\mathrm{Bi} = \dfrac{hL}{k}$	$\dfrac{\text{Surface thermal resistance}}{\text{Internal thermal resistance}}$
Bond number	$\mathrm{Bo} = \dfrac{g(\rho_f - \rho_v)L^2}{\sigma_s}$	$\dfrac{\text{Gravitational force}}{\text{Surface tension force}}$
Cavitation number	$\mathrm{Ca}\ (\text{sometimes } \sigma_c) = \dfrac{P - P_v}{\rho V^2}$ $\left(\text{sometimes } \dfrac{2(P - P_v)}{\rho V^2}\right)$	$\dfrac{\text{Pressure} - \text{Vapor pressure}}{\text{Inertial pressure}}$
Darcy friction factor	$f = \dfrac{8\tau_w}{\rho V^2}$	$\dfrac{\text{Wall friction force}}{\text{Inertial force}}$
Drag coefficient	$C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}$	$\dfrac{\text{Drag force}}{\text{Dynamic force}}$
Eckert number	$\mathrm{Ec} = \dfrac{V^2}{c_P T}$	$\dfrac{\text{Kinetic energy}}{\text{Enthalpy}}$
Euler number	$\mathrm{Eu} = \dfrac{\Delta P}{\rho V^2}\left(\text{sometimes } \dfrac{\Delta P}{\frac{1}{2}\rho V^2}\right)$	$\dfrac{\text{Pressure difference}}{\text{Dynamic pressure}}$
Fanning friction factor	$C_f = \dfrac{2\tau_w}{\rho V^2}$	$\dfrac{\text{Wall friction force}}{\text{Inertial force}}$
Fourier number	$\mathrm{Fo}\ (\text{sometimes } \tau) = \dfrac{\alpha t}{L^2}$	$\dfrac{\text{Physical time}}{\text{Thermal diffusion time}}$
Froude number	$\mathrm{Fr} = \dfrac{V}{\sqrt{gL}}\left(\text{sometimes } \dfrac{V^2}{gL}\right)$	$\dfrac{\text{Inertial force}}{\text{Gravitational force}}$
Grashof number	$\mathrm{Gr} = \dfrac{g\beta\lvert\Delta T\rvert L^3 \rho^2}{\mu^2}$	$\dfrac{\text{Buoyancy force}}{\text{Viscous force}}$
Jakob number	$\mathrm{Ja} = \dfrac{c_p(T - T_{\mathrm{sat}})}{h_{fg}}$	$\dfrac{\text{Sensible energy}}{\text{Latent energy}}$
Knudsen number	$\mathrm{Kn} = \dfrac{\lambda}{L}$	$\dfrac{\text{Mean free path length}}{\text{Characteristic length}}$
Lewis number	$\mathrm{Le} = \dfrac{k}{\rho c_p D_{AB}} = \dfrac{\alpha}{D_{AB}}$	$\dfrac{\text{Thermal diffusion}}{\text{Species diffusion}}$
Lift coefficient	$C_L = \dfrac{F_L}{\frac{1}{2}\rho V^2 A}$	$\dfrac{\text{Lift force}}{\text{Dynamic force}}$

(Continued)

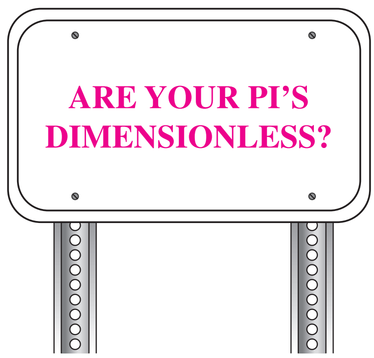


FIGURE 7–28
A quick check of your algebra is always wise.

TABLE 7–5 (Continued)

Name	Definition	Ratio of Significance
Mach number	$\text{Ma (sometimes } M) = \dfrac{V}{c}$	$\dfrac{\text{Flow speed}}{\text{Speed of sound}}$
Nusselt number	$\text{Nu} = \dfrac{Lh}{k}$	$\dfrac{\text{Convection heat transfer}}{\text{Conduction heat transfer}}$
Peclet number	$\text{Pe} = \dfrac{\rho L V c_p}{k} = \dfrac{LV}{\alpha}$	$\dfrac{\text{Bulk heat transfer}}{\text{Conduction heat transfer}}$
Power number	$N_P = \dfrac{\dot{W}}{\rho D^5 \omega^3}$	$\dfrac{\text{Power}}{\text{Rotational inertia}}$
Prandtl number	$\text{Pr} = \dfrac{\nu}{\alpha} = \dfrac{\mu c_p}{k}$	$\dfrac{\text{Viscous diffusion}}{\text{Thermal diffusion}}$
Pressure coefficient	$C_p = \dfrac{P - P_\infty}{\frac{1}{2}\rho V^2}$	$\dfrac{\text{Static pressure difference}}{\text{Dynamic pressure}}$
Rayleigh number	$\text{Ra} = \dfrac{g\beta \lvert \Delta T \rvert L^3 \rho^2 c_p}{k\mu}$	$\dfrac{\text{Buoyancy force}}{\text{Viscous force}}$
Reynolds number	$\text{Re} = \dfrac{\rho V L}{\mu} = \dfrac{VL}{\nu}$	$\dfrac{\text{Inertial force}}{\text{Viscous force}}$
Richardson number	$\text{Ri} = \dfrac{L^5 g\, \Delta\rho}{\rho \dot{V}^2}$	$\dfrac{\text{Buoyancy force}}{\text{Inertial force}}$
Schmidt number	$\text{Sc} = \dfrac{\mu}{\rho D_{AB}} = \dfrac{\nu}{D_{AB}}$	$\dfrac{\text{Viscous diffusion}}{\text{Species diffusion}}$
Sherwood number	$\text{Sh} = \dfrac{VL}{D_{AB}}$	$\dfrac{\text{Overall mass diffusion}}{\text{Species diffusion}}$
Specific heat ratio	$k \text{ (sometimes } \gamma) = \dfrac{c_p}{c_v}$	$\dfrac{\text{Enthalpy}}{\text{Internal energy}}$
Stanton number	$\text{St} = \dfrac{h}{\rho c_p V}$	$\dfrac{\text{Heat transfer}}{\text{Thermal capacity}}$
Stokes number	$\text{Stk (sometimes St)} = \dfrac{\rho_p D_p^2 V}{18\mu L}$	$\dfrac{\text{Particle relaxation time}}{\text{Characteristic flow time}}$
Strouhal number	$\text{St (sometimes S or Sr)} = \dfrac{fL}{V}$	$\dfrac{\text{Characteristic flow time}}{\text{Period of oscillation}}$
Weber number	$\text{We} = \dfrac{\rho V^2 L}{\sigma_s}$	$\dfrac{\text{Inertial force}}{\text{Surface tension force}}$

* A is a characteristic area, D is a characteristic diameter, f is a characteristic frequency (Hz), L is a characteristic length, t is a characteristic time, T is a characteristic (absolute) temperature, V is a characteristic velocity, W is a characteristic width, $\dot{W}$ is a characteristic power, ω is a characteristic angular velocity (rad/s). Other parameters and fluid properties in these Π's include: c = speed of sound, c_p, c_v = specific heats, D_p = particle diameter, D_{AB} = species diffusion coefficient, h = convective heat transfer coefficient, h_{fg} = latent heat of evaporation, k = thermal conductivity, P = pressure, T_{sat} = saturation temperature, $\dot{V}$ = volume flow rate, α = thermal diffusivity, β = coefficient of thermal expansion, λ = mean free path length, μ = viscosity, ν = kinematic viscosity, ρ = fluid density, ρ_f = liquid density, ρ_p = particle density, ρ_s = solid density, ρ_v = vapor density, σ_s = surface tension, and τ_w = shear stress along a wall.

HISTORICAL SPOTLIGHT ■ Persons Honored by Nondimensional Parameters

Guest Author: Glenn Brown, Oklahoma State University

Commonly used, established dimensionless numbers have been given names for convenience, and to honor persons who have contributed in the development of science and engineering. In many cases, the namesake was not the first to define the number, but usually he/she used it or a similar parameter in his/her work. The following is a list of some, but not all, such persons. Also keep in mind that some numbers may have more than one name.

Archimedes (287–212 BC) Greek mathematician who defined buoyant forces.

Biot, Jean-Baptiste (1774–1862) French mathematician who did pioneering work in heat, electricity, and elasticity. He also helped measure the arc of the meridian as part of the metric system development.

Darcy, Henry P. G. (1803–1858) French engineer who performed extensive experiments on pipe flow and the first quantifiable filtration tests.

Eckert, Ernst R. G. (1904–2004) German–American engineer and student of Schmidt who did early work in boundary layer heat transfer.

Euler, Leonhard (1797–1783) Swiss mathematician and associate of Daniel Bernoulli who formulated equations of fluid motion and introduced the concept of centrifugal machinery.

Fanning, John T. (1837–1911) American engineer and textbook author who published in 1877 a modified form of Weisbach's equation with a table of resistance values computed from Darcy's data.

Fourier, Jean B. J. (1768–1830) French mathematician who did pioneering work in heat transfer and several other topics.

Froude, William (1810–1879) English engineer who developed naval modeling methods and the transfer of wave and boundary resistance from model to prototype.

Grashof, Franz (1826–1893) German engineer and educator known as a prolific author, editor, corrector, and dispatcher of publications.

Jakob, Max (1879–1955) German–American physicist, engineer, and textbook author who did pioneering work in heat transfer.

Knudsen, Martin (1871–1949) Danish physicist who helped developed the kinetic theory of gases.

Lewis, Warren K. (1882–1975) American engineer who researched distillation, extraction, and fluidized bed reactions.

Mach, Ernst (1838–1916) Austrian physicist who was first to realize that bodies traveling faster than the speed of sound would drastically alter the properties of the fluid. His ideas had great influence on twentieth-century thought, both in physics and in philosophy, and influenced Einstein's development of the theory of relativity.

Nusselt, Wilhelm (1882–1957) German engineer who was the first to apply similarity theory to heat transfer.

Peclet, Jean C. E. (1793–1857) French educator, physicist, and industrial researcher.

Prandtl, Ludwig (1875–1953) German engineer and developer of boundary layer theory who is considered the founder of modern fluid mechanics.

Lord Raleigh, John W. Strutt (1842–1919) English scientist who investigated dynamic similarity, cavitation, and bubble collapse.

Reynolds, Osborne (1842–1912) English engineer who investigated flow in pipes and developed viscous flow equations based on mean velocities.

Richardson, Lewis F. (1881–1953) English mathematician, physicist, and psychologist who was a pioneer in the application of fluid mechanics to the modeling of atmospheric turbulence.

Schmidt, Ernst (1892–1975) German scientist and pioneer in the field of heat and mass transfer. He was the first to measure the velocity and temperature field in a free convection boundary layer.

Sherwood, Thomas K. (1903–1976) American engineer and educator. He researched mass transfer and its interaction with flow, chemical reactions, and industrial process operations.

Stanton, Thomas E. (1865–1931) English engineer and student of Reynolds who contributed to a number of areas of fluid flow.

Stokes, George G. (1819–1903) Irish scientist who developed equations of viscous motion and diffusion.

Strouhal, Vincenz (1850–1922) Czech physicist who showed that the period of oscillations shed by a wire are related to the velocity of the air passing over it.

Weber, Moritz (1871–1951) German professor who applied similarity analysis to capillary flows.

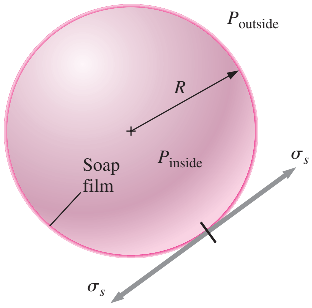


FIGURE 7–29
The pressure inside a soap bubble is greater than that surrounding the soap bubble due to surface tension in the soap film.

EXAMPLE 7–7 Pressure in a Soap Bubble

Some children are playing with soap bubbles, and you become curious as to the relationship between soap bubble radius and the pressure inside the soap bubble (Fig. 7–29). You reason that the pressure inside the soap bubble must be greater than atmospheric pressure, and that the shell of the soap bubble is under tension, much like the skin of a balloon. You also know that the property surface tension must be important in this problem. Not knowing any other physics, you decide to approach the problem using dimensional analysis. Establish a relationship between pressure difference $\Delta P = P_{\text{inside}} - P_{\text{outside}}$, soap bubble radius R, and the surface tension σ_s of the soap film.

SOLUTION The pressure difference between the inside of a soap bubble and the outside air is to be analyzed by the method of repeating variables.

Assumptions **1** The soap bubble is neutrally buoyant in the air, and gravity is not relevant. **2** No other variables or constants are important in this problem.

Analysis The step-by-step method of repeating variables is employed.

Step 1 There are three variables and constants in this problem; $n = 3$. They are listed in functional form, with the dependent variable listed as a function of the independent variables and constants:

List of relevant parameters: $$\Delta P = f(R, \sigma_s) \qquad n = 3$$

Step 2 The primary dimensions of each parameter are listed. The dimensions of surface tension are obtained from Example 7–1, and those of pressure from Example 7–2.

$$\begin{array}{ccc} \Delta P & R & \sigma_s \\ \{m^1L^{-1}t^{-2}\} & \{L^1\} & \{m^1t^{-2}\} \end{array}$$

Step 3 As a first guess, j is set equal to 3, the number of primary dimensions represented in the problem (m, L, and t).

Reduction (first guess): $$j = 3$$

If this value of j is correct, the expected number of Π's is $k = n - j = 3 - 3 = 0$. But how can we have zero Π's? Something is obviously not right (Fig. 7–30). At times like this, we need to first go back and make sure that we are not neglecting some important variable or constant in the problem. Since we are confident that the pressure difference should depend only on soap bubble radius and surface tension, we reduce the value of j by one,

Reduction (second guess): $$j = 2$$

If this value of j is correct, $k = n - j = 3 - 2 = 1$. Thus we expect *one* Π, which is more physically realistic than zero Π's.

Step 4 We need to choose two repeating parameters since $j = 2$. Following the guidelines of Table 7–3, our only choices are R and σ_s, since ΔP is the dependent variable.

Step 5 We combine these repeating parameters into a product with the dependent variable ΔP to create the dependent Π,

Dependent Π: $$\Pi_1 = \Delta P R^{a_1} \sigma_s^{b_1} \tag{1}$$

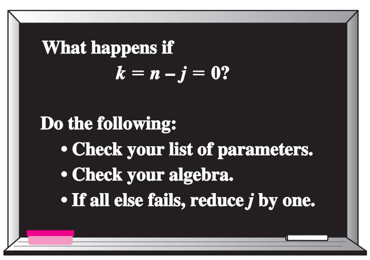


FIGURE 7–30
If the method of repeating variables indicates zero Π's, we have either made an error, or we need to reduce j by one and start over.

We apply the primary dimensions of step 2 into Eq. 1 and force the Π to be dimensionless.

Dimensions of Π_1:

$$\{\Pi_1\} = \{m^0L^0t^0\} = \{\Delta P R^{a_1}\sigma_s^{b_1}\} = \{(m^1L^{-1}t^{-2})L^{a_1}(m^1t^{-2})^{b_1}\}$$

We equate the exponents of each primary dimension to solve for a_1 and b_1:

Time:	$\{t^0\} = \{t^{-2}t^{-2b_1}\}$	$0 = -2 - 2b_1$	$b_1 = -1$
Mass:	$\{m^0\} = \{m^1m^{b_1}\}$	$0 = 1 + b_1$	$b_1 = -1$
Length:	$\{L^0\} = \{L^{-1}L^{a_1}\}$	$0 = -1 + a_1$	$a_1 = 1$

Fortunately, the first two results agree with each other, and Eq. 1 thus becomes

$$\Pi_1 = \frac{\Delta P R}{\sigma_s} \tag{2}$$

From Table 7–5, the established nondimensional parameter most similar to Eq. 2 is the **Weber number,** defined as a pressure (ρV^2) times a length divided by surface tension. There is no need to further manipulate this Π.

Step 6 We write the final functional relationship. In the case at hand, there is only one Π, which is a function of *nothing*. This is possible only if the Π is constant. Putting Eq. 2 into the functional form of Eq. 7–11,

Relationship between Π*'s*:

$$\Pi_1 = \frac{\Delta P R}{\sigma_s} = \mathbf{f}(\text{nothing}) = \text{constant} \quad \rightarrow \quad \boldsymbol{\Delta P = \text{constant}\,\frac{\sigma_s}{R}} \tag{3}$$

Discussion This is an example of how we can sometimes predict *trends* with dimensional analysis, even without knowing much of the physics of the problem. For example, we know from our result that if the radius of the soap bubble doubles, the pressure difference decreases by a factor of 2. Similarly, if the value of surface tension doubles, ΔP increases by a factor of 2. Dimensional analysis cannot predict the value of the constant in Eq. 3; further analysis (or *one* experiment) reveals that the constant is equal to 4 (Chap. 2).

EXAMPLE 7–8 Lift on a Wing

Some aeronautical engineers are designing an airplane and wish to predict the lift produced by their new wing design (Fig. 7–31). The chord length L_c of the wing is 1.12 m, and its **planform area** A (area viewed from the top when the wing is at zero angle of attack) is 10.7 m^2. The prototype is to fly at $V = 52.0$ m/s close to the ground where $T = 25°C$. They build a one-tenth scale model of the wing to test in a pressurized wind tunnel. The wind tunnel can be pressurized to a maximum of 5 atm. At what speed and pressure should they run the wind tunnel in order to achieve dynamic similarity?

SOLUTION We are to determine the speed and pressure at which to run the wind tunnel in order to achieve dynamic similarity.

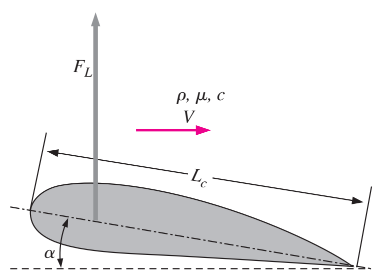


FIGURE 7–31
Lift on a wing of chord length L_c at angle of attack α in a flow of free-stream speed V with density ρ, viscosity μ, and speed of sound c. The angle of attack α is measured relative to the free-stream flow direction.

Assumptions **1** The prototype wing flies through the air at standard atmospheric pressure. **2** The model is geometrically similar to the prototype.

Analysis First, the step-by-step method of repeating variables is employed to obtain the nondimensional parameters. Then, the dependent Π's are matched between prototype and model.

Step 1 There are seven parameters (variables and constants) in this problem; $n = 7$. They are listed in functional form, with the dependent variable listed as a function of the independent parameters:

$$\textit{List of relevant parameters:} \qquad F_L = f(V, L_c, \rho, \mu, c, \alpha) \qquad n = 7$$

where F_L is the lift force on the wing, V is the fluid speed, L_c is the chord length, ρ is the fluid density, μ is the fluid viscosity, c is the speed of sound in the fluid, and α is the angle of attack of the wing.

Step 2 The primary dimensions of each parameter are listed; angle α is dimensionless:

F_L	V	L_c	ρ	μ	c	α
$\{m^1L^1t^{-2}\}$	$\{L^1t^{-1}\}$	$\{L^1\}$	$\{m^1L^{-3}\}$	$\{m^1L^{-1}t^{-1}\}$	$\{L^1t^{-1}\}$	$\{1\}$

Step 3 As a first guess, j is set equal to 3, the number of primary dimensions represented in the problem (m, L, and t).

$$\textit{Reduction:} \qquad j = 3$$

If this value of j is correct, the expected number of Π's is $k = n - j = 7 - 3 = 4$.

Step 4 We need to choose three repeating parameters since $j = 3$. Following the guidelines listed in Table 7–3, we cannot pick the dependent variable F_L. Nor can we pick α since it is already dimensionless. We cannot choose both V and c since their dimensions are identical. It would not be desirable to have μ appear in all the Π's. The best choice of repeating parameters is thus either V, L_c, and ρ or c, L_c, and ρ. Of these, the former is the better choice since the speed of sound appears in only one of the established nondimensional parameters of Table 7–5, whereas the velocity scale is more "common" and appears in several of the parameters (Fig. 7–32).

$$\textit{Repeating parameters:} \qquad V,\ L_c,\ \text{and}\ \rho$$

Step 5 The dependent Π is generated:

$$\Pi_1 = F_L V^{a_1} L_c^{b_1} \rho^{c_1} \quad \rightarrow \quad \{\Pi_1\} = \{(m^1L^1t^{-2})(L^1t^{-1})^{a_1}(L^1)^{b_1}(m^1L^{-3})^{c_1}\}$$

The exponents are calculated by forcing the Π to be dimensionless (algebra not shown). We get $\boldsymbol{a_1 = -2}$, $\boldsymbol{b_1 = -2}$, and $\boldsymbol{c_1 = -1}$. The dependent Π is thus

$$\Pi_1 = \frac{F_L}{\rho V^2 L_c^2}$$

From Table 7–5, the established nondimensional parameter most similar to our Π_1 is the **lift coefficient,** defined in terms of planform area A rather than the square of chord length, and with a factor of $\frac{1}{2}$ in the denominator. Thus, we may manipulate this Π according to the guidelines listed in Table 7–4 as follows:

$$\textit{Modified } \Pi_1\textit{:} \qquad \Pi_{1,\,\text{modified}} = \frac{F_L}{\frac{1}{2}\rho V^2 A} = \text{Lift coefficient} = C_L$$

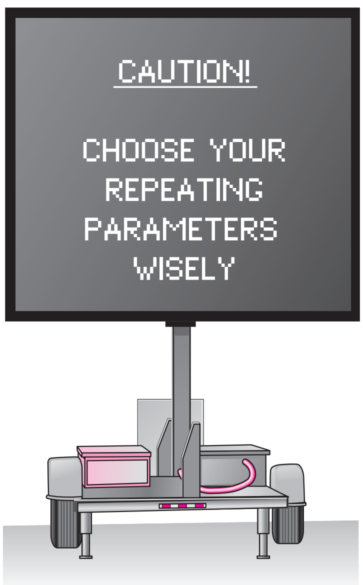


FIGURE 7–32
Oftentimes when performing the method of repeating variables, the most difficult part of the procedure is choosing the repeating parameters. With practice, however, you will learn to choose these parameters wisely.

Similarly, the first independent Π is generated:

$$\Pi_2 = \mu V^{a_2} L_c^{b_2} \rho^{c_2} \quad \rightarrow \quad \{\Pi_2\} = \{(\mathrm{m^1L^{-1}t^{-1}})(\mathrm{L^1t^{-1}})^{a_2}(\mathrm{L^1})^{b_2}(\mathrm{m^1L^{-3}})^{c_2}\}$$

from which $a_2 = -1$, $b_2 = -1$, and $c_2 = -1$, and thus

$$\Pi_2 = \frac{\mu}{\rho V L_c}$$

We recognize this Π as the inverse of the Reynolds number. So, after inverting,

Modified Π_2: $$\Pi_{2,\ \text{modified}} = \frac{\rho V L_c}{\mu} = \text{Reynolds number} = \text{Re}$$

The third Π is formed with the speed of sound, the details of which are left for you to generate on your own. The result is

$$\Pi_3 = \frac{V}{c} = \text{Mach number} = \text{Ma}$$

Finally, since the angle of attack α is already dimensionless, it is a dimensionless Π group all by itself (Fig. 7–33). You are invited to go through the algebra; you will find that all the exponents turn out to be zero, and thus

$$\Pi_4 = \alpha = \text{Angle of attack}$$

Step 6 We write the final functional relationship as

$$C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A} = f(\text{Re, Ma}, \alpha) \qquad (1)$$

To achieve dynamic similarity, Eq. 7–12 requires that all three of the dependent nondimensional parameters in Eq. 1 match between the model and the prototype. While it is trivial to match the angle of attack, it is not so simple to simultaneously match the Reynolds number and the Mach number. For example, if the wind tunnel were run at the same temperature and pressure as those of the prototype, such that ρ, μ, and c of the air flowing over the model were the same as ρ, μ, and c of the air flowing over the prototype, Reynolds number similarity would be achieved by setting the wind tunnel air speed to 10 times that of the prototype (since the model is one-tenth scale). But then the Mach numbers would differ by a factor of 10. At 25°C, c is approximately 346 m/s, and the Mach number of the prototype airplane wing is $\text{Ma}_p = 52.0/346 = 0.150$—subsonic. At the required wind tunnel speed, Ma_m would be 1.50—supersonic! This is clearly unacceptable since the physics of the flow changes dramatically from subsonic to supersonic conditions. At the other extreme, if we were to match Mach numbers, the Reynolds number of the model would be 10 times too small.

What should we do? A common rule of thumb is that for Mach numbers less than about 0.3, as is the fortunate case here, compressibility effects are practically negligible. Thus, it is not necessary to exactly match the Mach number; rather, as long as Ma_m is kept below about 0.3, approximate dynamic similarity can be achieved by matching the Reynolds number. Now the problem shifts to one of how to match Re while maintaining a low Mach number. This is where the pressurization feature of the wind tunnel comes in. At constant temperature, density is proportional to pressure, while viscosity and speed of sound are very weak functions of pressure. If the wind tunnel pressure could be pumped to 10 atm, we could run the model test at the

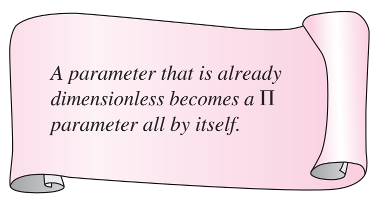


FIGURE 7–33
A parameter that is dimensionless (like an angle) is already a nondimensional Π all by itself—we know this Π without doing any further algebra.

same speed as the prototype and achieve a nearly perfect match in both Re and Ma. However, at the maximum wind tunnel pressure of 5 atm, the required wind tunnel speed would be twice that of the prototype, or 104 m/s. The Mach number of the wind tunnel model would thus be Ma_m = 104/346 = 0.301—approximately at the limit of incompressibility according to our rule of thumb. In summary, the wind tunnel should be run at approximately **100 m/s, 5 atm,** and **25°C.**

Discussion This example illustrates one of the (frustrating) limitations of dimensional analysis; namely, *You may not always be able to match all the dependent* Π*'s simultaneously in a model test.* Compromises must be made in which only the most important Π's are matched. In many practical situations in fluid mechanics, the Reynolds number is not critical for dynamic similarity, provided that Re is high enough. If the Mach number of the prototype were significantly larger than about 0.3, we would be wise to precisely match the Mach number rather than the Reynolds number in order to ensure reasonable results. Furthermore, if a different gas were used to test the model, we would also need to match the specific heat ratio (k), since compressible flow behavior is strongly dependent on k (Chap. 12). We discuss such model testing problems in more detail in Section 7–5.

We return to Examples 7–5 and 7–6. Recall that the air speed of the prototype car is 50.0 mi/h, and that of the wind tunnel is 224 mi/h. At 25°C, this corresponds to a prototype Mach number of Ma_p = 0.065, and at 5°C, the Mach number of the wind tunnel is 0.29—on the borderline of the incompressible limit. In hindsight, we should have included the speed of sound in our dimensional analysis, which would have generated the Mach number as an additional Π. Another way to match the Reynolds number while keeping the Mach number low is to use a *liquid* such as water, since liquids are nearly incompressible, even at fairly high speeds.

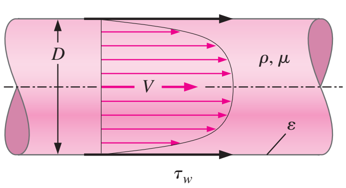


FIGURE 7–34
Friction on the inside wall of a pipe. The shear stress τ_w on the pipe walls is a function of average fluid speed V, average wall roughness height ε, fluid density ρ, fluid viscosity μ, and inside pipe diameter D.

EXAMPLE 7–9 **Friction in a Pipe**

Consider flow of an incompressible fluid of density ρ and viscosity μ through a long, horizontal section of round pipe of diameter D. The velocity profile is sketched in Fig. 7–34; V is the average speed across the pipe cross section, which by conservation of mass remains constant down the pipe. For a very long pipe, the flow eventually becomes **fully developed,** which means that the velocity profile also remains uniform down the pipe. Because of frictional forces between the fluid and the pipe wall, there exists a shear stress τ_w on the inside pipe wall as sketched. The shear stress is also constant down the pipe in the fully developed region. We assume some constant average roughness height ε along the inside wall of the pipe. In fact, the only parameter that is *not* constant down the length of pipe is the pressure, which must decrease (linearly) down the pipe in order to "push" the fluid through the pipe to overcome friction. Develop a nondimensional relationship between shear stress τ_w and the other parameters in the problem.

SOLUTION We are to generate a nondimensional relationship between shear stress and other parameters.

Assumptions **1** The flow is fully developed. **2** The fluid is incompressible. **3** No other parameters are significant in the problem.

Analysis The step-by-step method of repeating variables is employed to obtain the nondimensional parameters.

Step 1 There are six variables and constants in this problem; $n = 6$. They are listed in functional form, with the dependent variable listed as a function of the independent variables and constants:

$$\textit{List of relevant parameters:} \qquad \tau_w = f(V, \varepsilon, \rho, \mu, D) \qquad n = 6$$

Step 2 The primary dimensions of each parameter are listed. Note that shear stress is a force per unit area, and thus has the same dimensions as pressure.

$$\underset{\{m^1L^{-1}t^{-2}\}}{\tau_w} \quad \underset{\{L^1t^{-1}\}}{V} \quad \underset{\{L^1\}}{\varepsilon} \quad \underset{\{m^1L^{-3}\}}{\rho} \quad \underset{\{m^1L^{-1}t^{-1}\}}{\mu} \quad \underset{\{L^1\}}{D}$$

Step 3 As a first guess, j is set equal to 3, the number of primary dimensions represented in the problem (m, L, and t).

$$\textit{Reduction:} \qquad j = 3$$

If this value of j is correct, the expected number of Π's is $k = n - j = 6 - 3 = 3$.

Step 4 We choose three repeating parameters since $j = 3$. Following the guidelines of Table 7–3, we cannot pick the dependent variable τ_w. We cannot choose both ε and D since their dimensions are identical, and it would not be desirable to have μ or ε appear in all the Π's. The best choice of repeating parameters is thus V, D, and ρ.

$$\textit{Repeating parameters:} \qquad V, D, \text{ and } \rho$$

Step 5 The dependent Π is generated:

$$\Pi_1 = \tau_w V^{a_1} D^{b_1} \rho^{c_1} \quad \rightarrow \quad \{\Pi_1\} = \{(m^1L^{-1}t^{-2})(L^1t^{-1})^{a_1}(L^1)^{b_1}(m^1L^{-3})^{c_1}\}$$

from which $a_1 = -2$, $b_1 = 0$, and $c_1 = -1$, and thus the dependent Π is

$$\Pi_1 = \frac{\tau_w}{\rho V^2}$$

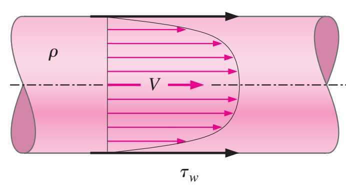
From Table 7–5, the established nondimensional parameter most similar to this Π_1 is the **Darcy friction factor,** defined with a factor of 8 in the numerator (Fig. 7–35). Thus, we may manipulate this Π according to the guidelines listed in Table 7–4 as follows:

$$\textit{Modified } \Pi_1\text{:} \qquad \Pi_{1,\text{ modified}} = \frac{8\tau_w}{\rho V^2} = \text{Darcy friction factor} = f$$

Similarly, the two independent Π's are generated, the details of which are left for the reader:

$$\Pi_2 = \mu V^{a_2} D^{b_2} \rho^{c_2} \quad \rightarrow \quad \Pi_2 = \frac{\rho V D}{\mu} = \text{Reynolds number} = \text{Re}$$

$$\Pi_3 = \varepsilon V^{a_3} D^{b_3} \rho^{c_3} \quad \rightarrow \quad \Pi_3 = \frac{\varepsilon}{D} = \text{Roughness ratio}$$



Darcy friction factor: $\boxed{f = \dfrac{8\tau_w}{\rho V^2}}$

Fanning friction factor: $\boxed{C_f = \dfrac{2\tau_w}{\rho V^2}}$

FIGURE 7–35
Although the *Darcy friction factor* for pipe flows is most common, you should be aware of an alternative, less common friction factor called the *Fanning friction factor*. The relationship between the two is $f = 4C_f$.

Step 6 We write the final functional relationship as

$$f = \frac{8\tau_w}{\rho V^2} = f\left(\text{Re}, \frac{\varepsilon}{D}\right) \tag{1}$$

Discussion The result applies to both laminar and turbulent fully developed pipe flow; it turns out, however, that the second independent Π (roughness ratio ε/D) is not nearly as important in laminar pipe flow as in turbulent pipe flow. This problem presents an interesting connection between geometric similarity and dimensional analysis. Namely, it is necessary to match ε/D since it is an independent Π in the problem. From a different perspective, thinking of roughness as a geometric property, it is necessary to match ε/D to ensure *geometric similarity* between two pipes.

To verify the validity of Eq. 1 of Example 7–9, we use **computational fluid dynamics (CFD)** to predict the velocity profiles and the values of wall shear stress for two physically different but dynamically similar pipe flows:

- ***Air*** at 300 K flowing at an average speed of 14.5 ft/s through a pipe of inner diameter 1.00 ft and average roughness height 0.0010 ft.
- ***Water*** at 300 K flowing at an average speed of 3.09 m/s through a pipe of inner diameter 0.0300 m and average roughness height 0.030 mm.

The two pipes are clearly geometrically similar since they are both round pipes. They have the same average roughness ratio (ε/D = 0.0010 in both cases). We have carefully chosen the values of average speed and diameter such that the two flows are also *dynamically* similar. Specifically, the other independent Π (the Reynolds number) also matches between the two flows.

$$\text{Re}_{\text{air}} = \frac{\rho_{\text{air}} V_{\text{air}} D_{\text{air}}}{\mu_{\text{air}}} = \frac{(1.225\ \text{kg/m}^3)(14.5\ \text{ft/s})(1.00\ \text{ft})}{1.789 \times 10^{-5}\ \text{kg/m} \cdot \text{s}} \left(\frac{0.3048\ \text{m}}{\text{ft}}\right)^2 = 9.22 \times 10^4$$

where the fluid properties are those built into the CFD code, and

$$\text{Re}_{\text{water}} = \frac{\rho_{\text{water}} V_{\text{water}} D_{\text{water}}}{\mu_{\text{water}}} = \frac{(998.2\ \text{kg/m}^3)(3.09\ \text{m/s})(0.0300\ \text{m})}{0.001003\ \text{kg/m} \cdot \text{s}} = 9.22 \times 10^4$$

Hence by Eq. 7–12, we expect that the *dependent* Π's should match between the two flows as well. We generate a computational mesh for each of the two flows, and use a commercial CFD code to generate the velocity profile, from which the shear stress is calculated. Fully developed, time-averaged, turbulent velocity profiles near the far end of both pipes are compared. Although the pipes are of different diameters and the fluids are vastly different, the velocity profile shapes look quite similar. In fact, when we plot *normalized* axial velocity (u/V) as a function of *normalized* radius (r/R), we find that the two profiles fall on top of each other (Fig. 7–36).

Wall shear stress is also calculated from the CFD results for each flow, a comparison of which is shown in Table 7–6. There are several reasons why the wall shear stress in the water pipe is orders of magnitude larger than that in the air pipe. Namely, water is over 800 times as dense as air and over 50 times as viscous. Furthermore, shear stress is proportional to the *gradient* of velocity, and the water pipe diameter is less than one-tenth that of the air

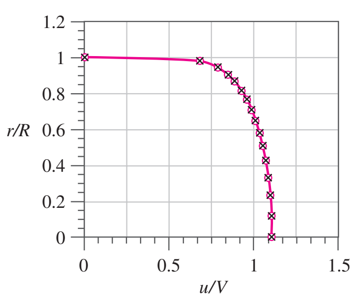


FIGURE 7–36
Normalized axial velocity profiles for fully developed flow through a pipe as predicted by CFD; profiles of air (circles) and water (crosses) are shown on the same plot.

TABLE 7–6

Comparison of wall shear stress and nondimensionalized wall shear stress for fully developed flow through an air pipe and a water pipe as predicted by CFD*

Parameter	Air Flow	Water Flow
Wall shear stress	$\tau_{w,\,\text{air}} = 0.0557\ \text{N/m}^2$	$\tau_{w,\,\text{water}} = 22.2\ \text{N/m}^2$
Dimensionless wall shear stress (Darcy friction factor)	$f_{\text{air}} = \dfrac{8\tau_{w,\,\text{air}}}{\rho_{\text{air}}V^2_{\text{air}}} = 0.0186$	$f_{\text{water}} = \dfrac{8\tau_{w,\,\text{water}}}{\rho_{\text{water}}V^2_{\text{water}}} = 0.0186$

* Data obtained with FLUENT using the standard k-ε turbulence model with wall functions.

pipe, leading to steeper velocity gradients. In terms of the *nondimensionalized* wall shear stress, f, however, Table 7–6 shows that the results are identical due to dynamic similarity between the two flows. Note that although the values are reported to three significant digits, the reliability of turbulence models in CFD is accurate to at most two significant digits (Chap. 15).

7–5 ▪ EXPERIMENTAL TESTING AND INCOMPLETE SIMILARITY

One of the most useful applications of dimensional analysis is in designing physical and/or numerical experiments, and in reporting the results of such experiments. In this section we discuss both of these applications, and point out situations in which complete dynamic similarity is not achievable.

Setup of an Experiment and Correlation of Experimental Data

As a generic example, consider a problem in which there are five original parameters (one of which is the *dependent* parameter). A complete set of experiments (called a **full factorial** test matrix) is conducted by testing every possible combination of several levels of each of the four independent parameters. A full factorial test with five levels of each of the four independent parameters would require $5^4 = 625$ experiments. While experimental design techniques (**fractional factorial** test matrices; see Montgomery, 1996) can significantly reduce the size of the test matrix, the number of required experiments would still be large. However, assuming that three primary dimensions are represented in the problem, we can reduce the number of parameters from five to two ($k = 5 - 3 = 2$ nondimensional Π groups), and the number of *independent* parameters from four to one. Thus, for the same resolution (five tested levels of each independent parameter) we would then need to conduct a total of only $5^1 = 5$ experiments. You don't have to be a genius to realize that replacing 625 experiments by 5 experiments is cost effective. You can see why it is wise to perform a dimensional analysis *before* conducting an experiment.

Continuing our discussion of this generic example (a two-Π problem), once the experiments are complete, we plot the dependent dimensionless parameter (Π_1) as a function of the independent dimensionless parameter (Π_2), as in Fig. 7–37. We then determine the functional form of the relationship by

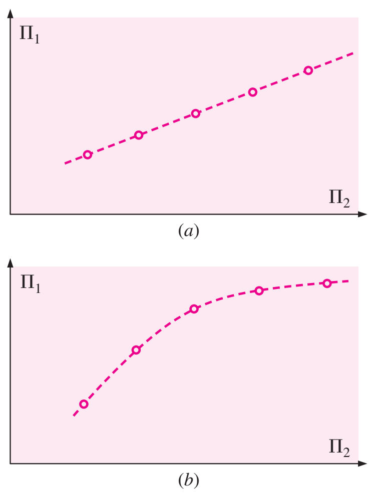


FIGURE 7–37
For a two-Π problem, we plot dependent dimensionless parameter (Π_1) as a function of independent dimensionless parameter (Π_2). The resulting plot can be (*a*) linear or (*b*) nonlinear. In either case, regression and curve-fitting techniques are available to determine the relationship between the Π's.

performing a **regression analysis** on the data. If we are lucky, the data may correlate linearly. If not, we can try linear regression on log–linear or log–log coordinates, polynomial curve fitting, etc., to establish an approximate relationship between the two Π's. See Holman (2001) for details about these curve-fitting techniques.

If there are more than two Π's in the problem (e.g., a three-Π problem or a four-Π problem), we need to set up a test matrix to determine the relationship between the dependent Π and the independent Π's. In many cases we discover that one or more of the dependent Π's has negligible effect and can be removed from the list of necessary dimensionless parameters.

As we have seen (Example 7–7), dimensional analysis sometimes yields only *one* Π. In a one-Π problem, we know the form of the relationship between the original parameters to within some unknown constant. In such a case, only *one* experiment is needed to determine that constant.

Incomplete Similarity

We have shown several examples in which the nondimensional Π groups are easily obtained with paper and pencil through straightforward use of the method of repeating variables. In fact, after sufficient practice, you should be able to obtain the Π's with ease—sometimes in your head or on the "back of an envelope." Unfortunately, it is often a much different story when we go to apply the results of our dimensional analysis to experimental data. The problem is that it is not always possible to match *all* the Π's of a model to the corresponding Π's of the prototype, even if we are careful to achieve geometric similarity. This situation is called **incomplete similarity.** Fortunately, in some cases of incomplete similarity, we are still able to extrapolate model tests to obtain reasonable full-scale predictions.

Wind Tunnel Testing

We illustrate incomplete similarity with the problem of measuring the aerodynamic drag force on a model truck in a wind tunnel (Fig. 7–38). Suppose we purchase a one-sixteenth scale die-cast model of a tractor-trailer rig (18-wheeler). The model is geometrically similar to the prototype—even in the details such as side mirrors, mud flaps, etc. The model truck is 0.991 m long, corresponding to a full-scale prototype length of 15.9 m. The model truck is to be tested in a wind tunnel that has a maximum speed of 70 m/s. The wind tunnel test section is 1.0 m tall and 1.2 m wide—big enough to accommodate the model without needing to worry about wall interference or blockage effects. The air in the wind tunnel is at the same temperature and pressure as the air flowing around the prototype. We want to simulate flow at V_p = 60 mi/h (26.8 m/s) over the full-scale prototype truck.

The first thing we do is match the Reynolds numbers,

$$\text{Re}_m = \frac{\rho_m V_m L_m}{\mu_m} = \text{Re}_p = \frac{\rho_p V_p L_p}{\mu_p}$$

which can be solved for the required wind tunnel speed for the model tests V_m,

$$V_m = V_p\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{L_p}{L_m}\right) = (26.8\text{ m/s})(1)(1)\left(\frac{16}{1}\right) = 429\text{ m/s}$$

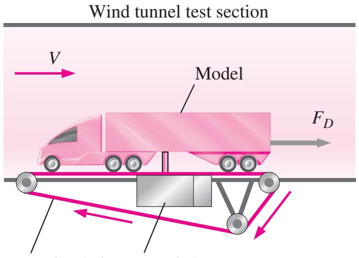


FIGURE 7–38
Measurement of aerodynamic drag on a model truck in a wind tunnel equipped with a *drag balance* and a *moving belt* ground plane.

Thus, to match the Reynolds number between model and prototype, the wind tunnel should be run at 429 m/s (to three significant digits). We obviously have a problem here, since this speed is more than six times greater than the maximum achievable wind tunnel speed. Moreover, even if we *could* run the wind tunnel that fast, the flow would be *supersonic*, since the speed of sound in air at room temperature is about 346 m/s. While the Mach number of the prototype truck moving through the air is 26.8/335 = 0.080, that of the wind tunnel air moving over the model would be 429/335 = 1.28 (if the wind tunnel could go that fast).

It is clearly not possible to match the model Reynolds number to that of the prototype with this model and wind tunnel facility. What do we do? There are several options:

- If we had a bigger wind tunnel, we could test with a larger model. Automobile manufacturers typically test with three-eighths scale model cars and with one-eighth scale model trucks and buses in very large wind tunnels. Some wind tunnels are even large enough for full-scale automobile tests (Fig. 7–39). As you can imagine, however, the bigger the wind tunnel and the model the more expensive the tests. We must also be careful that the model is not too big for the wind tunnel. A useful rule of thumb is that the **blockage** (ratio of the model frontal area to the cross-sectional area of the test section) should be less than 7.5 percent. Otherwise, the wind tunnel walls adversely affect both geometric and kinematic similarity.
- We could use a different fluid for the model tests. For example, water tunnels can achieve higher Reynolds numbers than can wind tunnels of the same size, but they are much more expensive to build and operate.
- We could pressurize the wind tunnel and/or adjust the air temperature to increase the maximum Reynolds number capability. While these techniques can help, the increase in the Reynolds number is limited.
- If all else fails, we could run the wind tunnel at several speeds near the maximum speed, and then extrapolate our results to the full-scale Reynolds number.

Fortunately, it turns out that for many wind tunnel tests the last option is quite viable. While drag coefficient C_D is a strong function of the Reynolds number at low values of Re, C_D often levels off for Re above some value. In other words, for flow over many objects, especially "bluff" objects like trucks, buildings, etc., the flow is **Reynolds number independent** above some threshold value of Re (Fig. 7–40), typically when the boundary layer and the wake are both fully turbulent.

FIGURE 7–39
The Langley full-scale wind tunnel (LFST) is large enough that full-scale vehicles can be tested.
Courtesy NASA Langley Research Center.

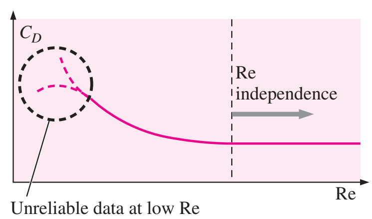


FIGURE 7–40
For many objects, the drag coefficient levels off at Reynolds numbers above some threshold value. This fortunate situation is called *Reynolds number independence*. It enables us to extrapolate to prototype Reynolds numbers that are outside of the range of our experimental facility.

EXAMPLE 7–10 **Model Truck Wind Tunnel Measurements**

A one-sixteenth scale model tractor-trailer truck (18-wheeler) is tested in a wind tunnel as sketched in Fig. 7–38. The model truck is 0.991 m long, 0.257 m tall, and 0.159 m wide. During the tests, the moving ground belt speed is adjusted so as to always match the speed of the air moving through the test section. Aerodynamic drag force F_D is measured as a function of

TABLE 7–7

Wind tunnel data: aerodynamic drag force on a model truck as a function of wind tunnel speed

V, m/s	F_D, N
20	12.4
25	19.0
30	22.1
35	29.0
40	34.3
45	39.9
50	47.2
55	55.5
60	66.0
65	77.6
70	89.9

wind tunnel speed; the experimental results are listed in Table 7–7. Plot the drag coefficient C_D as a function of the Reynolds number Re, where the area used for the calculation of C_D is the frontal area of the model truck (the area you see when you look at the model from upstream), and the length scale used for calculation of Re is truck width W. Have we achieved dynamic similarity? Have we achieved Reynolds number independence in our wind tunnel test? Estimate the aerodynamic drag force on the prototype truck traveling on the highway at 26.8 m/s. Assume that both the wind tunnel air and the air flowing over the prototype car are at 25°C and standard atmospheric pressure.

SOLUTION We are to calculate and plot C_D as a function of Re for a given set of wind tunnel measurements and determine if dynamic similarity and/or Reynolds number independence have been achieved. Finally, we are to estimate the aerodynamic drag force acting on the prototype truck.

Assumptions **1** The model truck is geometrically similar to the prototype truck. **2** The aerodynamic drag on the strut(s) holding the model truck is negligible.

Properties For air at atmospheric pressure and at T = 25°C, ρ = 1.184 kg/m^3 and $\mu = 1.849 \times 10^{-5}$ kg/m · s.

Analysis We calculate C_D and Re for the last data point listed in Table 7–7 (at the fastest wind tunnel speed),

$$C_{D,m} = \frac{F_{D,m}}{\frac{1}{2}\rho_m V_m^2 A_m} = \frac{89.9\text{ N}}{\frac{1}{2}(1.184\text{ kg/m}^3)(70\text{ m/s})^2(0.159\text{ m})(0.257\text{ m})}\left(\frac{1\text{ kg}\cdot\text{m/s}^2}{1\text{ N}}\right)$$
$$= 0.758$$

and

$$\text{Re}_m = \frac{\rho_m V_m W_m}{\mu_m} = \frac{(1.184\text{ kg/m}^3)(70\text{ m/s})(0.159\text{ m})}{1.849 \times 10^{-5}\text{ kg/m}\cdot\text{s}} = 7.13 \times 10^5 \quad (1)$$

We repeat these calculations for all the data points in Table 7–7, and we plot C_D versus Re in Fig. 7–41.

Have we achieved dynamic similarity? Well, we have *geometric* similarity between model and prototype, but the Reynolds number of the prototype truck is

$$\text{Re}_p = \frac{\rho_p V_p W_p}{\mu_p} = \frac{(1.184\text{ kg/m}^3)(26.8\text{ m/s})[16(0.159\text{ m})]}{1.849 \times 10^{-5}\text{ kg/m}\cdot\text{s}} = 4.37 \times 10^6 \quad (2)$$

where the width of the prototype is specified as 16 times that of the model. Comparison of Eqs. 1 and 2 reveals that the prototype Reynolds number is more than six times larger than that of the model. Since we cannot match the independent Π's in the problem, **dynamic similarity has not been achieved.**

Have we achieved Reynolds number independence? From Fig. 7–41 we see that **Reynolds number independence has indeed been achieved**—at Re greater than about 5×10^5, C_D has leveled off to a value of about 0.76 (to two significant digits).

Since we have achieved Reynolds number independence, we can extrapolate to the full-scale prototype, assuming that C_D remains constant as Re is increased to that of the full-scale prototype.

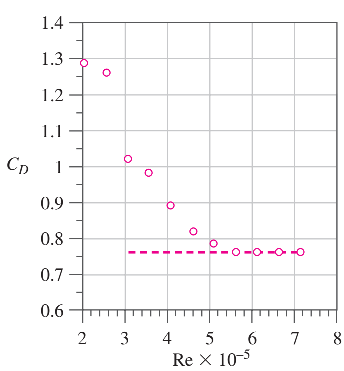


FIGURE 7–41
Aerodynamic drag coefficient as a function of the Reynolds number. The values are calculated from wind tunnel test data on a model truck (Table 7–7).

Predicted aerodynamic drag on the prototype:

$$F_{D,p} = \tfrac{1}{2}\rho_p V_p^2 A_p C_{D,p}$$

$$= \tfrac{1}{2}(1.184 \text{ kg/m}^3)(26.8 \text{ m/s})^2[16^2(0.159 \text{ m})(0.257 \text{ m})](0.76)\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right)$$

$$= \mathbf{3400 \text{ N}}$$

Discussion We give our final result to two significant digits. More than that cannot be justified. As always, we must exercise caution when performing an extrapolation, since we have no guarantee that the extrapolated results are correct.

Flows with Free Surfaces

For the case of model testing of flows with free surfaces (boats and ships, floods, river flows, aqueducts, hydroelectric dam spillways, interaction of waves with piers, soil erosion, etc.), complications arise that preclude complete similarity between model and prototype. For example, if a model river is built to study flooding, the model is often several hundred times smaller than the prototype due to limited lab space. If the vertical dimensions of the model were scaled proportionately, the depth of the model river would be so small that surface tension effects (and the Weber number) would become important, and would perhaps even dominate the model flow, even though surface tension effects are negligible in the prototype flow. In addition, although the flow in the actual river may be turbulent, the flow in the model river may be laminar, especially if the slope of the riverbed is geometrically similar to that of the prototype. To avoid these problems, researchers often use a **distorted model** in which the vertical scale of the model (e.g., river depth) is exaggerated in comparison to the horizontal scale of the model (e.g., river width). In addition, the model riverbed slope is often made proportionally steeper than that of the prototype. These modifications result in incomplete similarity due to lack of geometric similarity. Model tests are still useful under these circumstances, but other tricks (like deliberately roughening the model surfaces) and empirical corrections and correlations are required to properly scale up the model data.

In many practical problems involving free surfaces, both the Reynolds number and Froude number appear as relevant independent Π groups in the dimensional analysis (Fig. 7–42). It is difficult (often impossible) to match both of these dimensionless parameters simultaneously. For a free-surface flow with length scale L, velocity scale V, and kinematic viscosity ν, the Reynolds number is matched between model and prototype when

$$\text{Re}_p = \frac{V_p L_p}{\nu_p} = \text{Re}_m = \frac{V_m L_m}{\nu_m} \tag{7–21}$$

The Froude number is matched between model and prototype when

$$\text{Fr}_p = \frac{V_p}{\sqrt{gL_p}} = \text{Fr}_m = \frac{V_m}{\sqrt{gL_m}} \tag{7–22}$$

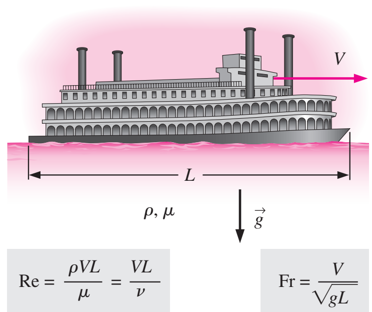


FIGURE 7–42
In many flows involving a liquid with a free surface, both the Reynolds number and Froude number are relevant nondimensional parameters. Since it is not always possible to match both Re and Fr between model and prototype, we are sometimes forced to settle for incomplete similarity.

To match both Re and Fr, we solve Eqs. 7–21 and 7–22 simultaneously for the required length scale factor L_m/L_p,

$$\frac{L_m}{L_p} = \frac{\nu_m}{\nu_p}\frac{V_p}{V_m} = \left(\frac{V_m}{V_p}\right)^2 \tag{7–23}$$

Eliminating the ratio V_m/V_p from Eq. 7–23, we see that

Required ratio of kinematic viscosities to match both Re and Fr:

$$\frac{\nu_m}{\nu_p} = \left(\frac{L_m}{L_p}\right)^{3/2} \tag{7–24}$$

Thus, to ensure complete similarity (assuming geometric similarity is achievable without unwanted surface tension effects as discussed previously), we would need to use a liquid whose kinematic viscosity satisfies Eq. 7–24. Although it is sometimes possible to find an appropriate liquid for use with the model, in most cases it is either impractical or impossible, as Example 7–11 illustrates.

EXAMPLE 7–11 **Model Lock and River**

In the late 1990s the U.S. Army Corps of Engineers designed an experiment to model the flow of the Tennessee River downstream of the Kentucky Lock and Dam (Fig. 7–43). Because of laboratory space restrictions, they built a scale model with a length scale factor of L_m/L_p = 1/100. Suggest a liquid that would be appropriate for the experiment.

SOLUTION We are to suggest a liquid to use in an experiment involving a one-hundredth scale model of a lock, dam, and river.
Assumptions **1** The model is geometrically similar to the prototype. **2** The model river is deep enough that surface tension effects are not significant.
Properties For water at atmospheric pressure and at T = 20°C, the prototype kinematic viscosity is $\nu_p = 1.002 \times 10^{-6}$ m²/s.
Analysis From Eq. 7–24,

Required kinematic viscosity of model liquid:

$$\nu_m = \nu_p\left(\frac{L_m}{L_p}\right)^{3/2} = (1.002 \times 10^{-6}\ \mathrm{m^2/s})\left(\frac{1}{100}\right)^{3/2} = \mathbf{1.00 \times 10^{-9}\ m^2/s} \tag{1}$$

Thus, we need to find a liquid that has a viscosity of 1.00×10^{-9} m²/s. A quick glance through the appendices yields no such liquid. Hot water has a lower kinematic viscosity than cold water, but only by about a factor of 3. Liquid mercury has a very small kinematic viscosity, but it is of order 10^{-7} m²/s—still two orders of magnitude too large to satisfy Eq. 1. Even if liquid mercury would work, it would be too expensive and too hazardous to use in such a test. What do we do? The bottom line is that *we cannot match both the Froude number and the Reynolds number in this model test*. In other words, it is impossible to achieve complete similarity between model and prototype in this case. Instead, we do the best job we can under conditions of incomplete similarity. Water is typically used in such tests for convenience.

FIGURE 7–43
A 1:100 scale model constructed to investigate navigation conditions in the lower lock approach for a distance of 2 mi downstream of the dam. The model includes a scaled version of the spillway, powerhouse, and existing lock. In addition to navigation, the model was used to evaluate environmental issues associated with the new lock and required railroad and highway bridge relocations. The view here is looking upstream toward the lock and dam. At this scale, 52.8 ft on the model represents 1 mi on the prototype. A pickup truck in the background gives you a feel for the model scale.

Photo courtesy of the U.S. Army Corps of Engineers, Nashville.

Discussion It turns out that for this kind of experiment, Froude number matching is more critical than Reynolds number matching. As discussed previously for wind tunnel testing, Reynolds number independence is achieved at high enough values of Re. Even if we are unable to achieve Reynolds number independence, we can often extrapolate our low Reynolds number model data to predict full-scale Reynolds number behavior (Fig. 7–44). A high level of confidence in using this kind of extrapolation comes only after much laboratory experience with similar problems.

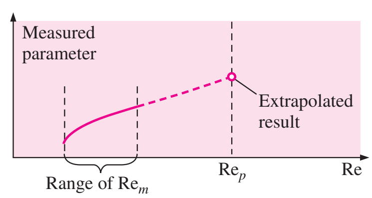


FIGURE 7–44
In many experiments involving free surfaces, we cannot match both the Froude number and the Reynolds number. However, we can often *extrapolate* low Re model test data to predict high Re prototype behavior.

In closing this section on experiments and incomplete similarity, we mention the importance of similarity in the production of Hollywood movies in which model boats, trains, airplanes, buildings, monsters, etc., are blown up or burned. Movie producers must pay attention to dynamic similarity in order to make the small-scale fires and explosions appear as realistic as possible. You may recall some low-budget movies where the special effects are unconvincing. In most cases this is due to lack of dynamic similarity between the small model and the full-scale prototype. If the model's Froude number and/or Reynolds number differ too much from those of the prototype, the special effects don't look right, even to the untrained eye. The next time you watch a movie, be on the alert for incomplete similarity!

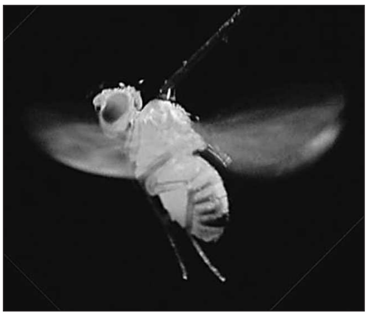
APPLICATION SPOTLIGHT ■ How a Fly Flies

Guest Author: Michael Dickinson,
California Institute of Technology

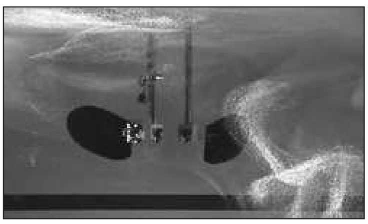
An interesting application of dimensional analysis is in the study of how insects fly. The small size and fast wing speed of an insect, such as a tiny fruit fly, make it difficult to measure the forces or visualize the air motion created by the fly's wings directly. However, using principles of dimensional analysis, it is possible to study insect aerodynamics on a larger-scale, slowly moving model—a mechanical robot. The forces created by a hovering fly and flapping robot are dynamically similar if the Reynolds number is the same for each case. For a flapping wing, Re is calculated as $2\Phi R L_c \omega/\nu$, where Φ is the angular amplitude of the wing stroke, R is the wing length, L_c is the average wing width (chord length), ω is the angular frequency of the stroke, and ν is the kinematic viscosity of the surrounding fluid. A fruit fly flaps its 2.5-mm-long, 0.7-mm-wide wings 200 times per second over a 2.8-rad stroke in air with a kinematic viscosity of 1.5×10^{-5} m^2/s. The resulting Reynolds number is approximately 130. By choosing mineral oil with a kinematic viscosity of 1.15×10^{-4} m^2/s, it is possible to match this Reynolds number on a robotic fly that is 100 times larger, flapping its wings over 1000 times more slowly! If the fly is not stationary, but rather moving through the air, it is necessary to match another dimensionless parameter to ensure dynamic similarity, the reduced frequency, $\sigma = 2\Phi R\omega/V$, which measures the ratio of the flapping velocity of the wing tip ($2\Phi R\omega$) to the forward velocity of the body (V). To simulate forward flight, a set of motors tows *Robofly* through its oil tank at an appropriately scaled speed.

Dynamically scaled robots have helped show that insects use a variety of different mechanisms to produce forces as they fly. During each back-and-forth stroke, insect wings travel at high angles of attack, generating a prominent leading-edge vortex. The low pressure of this large vortex pulls the wings upward. Insects can further augment the strength of the leading-edge vortex by rotating their wings at the end of each stroke. After the wing changes direction, it can also generate forces by quickly running through the wake of the previous stroke.

Figure 7–45*a* shows a real fly flapping its wings, and Fig. 7–45*b* shows *Robofly* flapping its wings. Because of the larger length scale and shorter time scale of the model, measurements and flow visualizations are possible. Experiments with dynamically scaled model insects continue to teach researchers how insects manipulate wing motion to steer and maneuver.



(*a*)



(*b*)

FIGURE 7–45
(*a*) The fruit fly, *Drosophila melanogaster*, flaps its tiny wings back and forth 200 times a second, creating a blurred image of the stroke plane. (*b*) The dynamically scaled model, *Robofly*, flaps its wings once every 5 s in 2 tons of mineral oil. Sensors at the base of the wings record aerodynamic forces, while fine bubbles are used to visualize flow. The size and speed of the robot, as well as the properties of the oil, were carefully chosen to match the Reynolds number of a real fly.

References

Dickinson, M. H., Lehmann, F.-O., and Sane, S., "Wing rotation and the aerodynamic basis of insect flight," *Science*, 284, p. 1954, 1999.
Dickinson, M. H., "Solving the mystery of insect flight," *Scientific American*, 284, No. 6, pp. 35–41, June 2001.
Fry, S. N., Sayaman, R., and Dickinson, M. H., "The aerodynamics of free-flight maneuvers in *Drosophila*," *Science*, 300, pp. 495–498, 2003.

SUMMARY

There is a difference between *dimensions* and *units*; a *dimension* is a measure of a physical quantity (without numerical values), while a *unit* is a way to assign a number to that dimension. There are seven *primary dimensions*—not just in fluid mechanics, but in all fields of science and engineering. They are mass, length, time, temperature, electric current, amount of light, and amount of matter. *All other dimensions can be formed by combination of these seven primary dimensions.*

All mathematical equations must be *dimensionally homogeneous*; this fundamental principle can be applied to equations in order to nondimensionalize them and to identify *dimensionless groups*, also called *nondimensional parameters*. A powerful tool to reduce the number of necessary independent parameters in a problem is called *dimensional analysis*. The *method of repeating variables* is a step-by-step procedure for finding the nondimensional parameters, or Π's, based simply on the dimensions of the variables and constants in the problem. The six steps in the method of repeating variables are summarized here.

Step 1 List the n parameters (variables and constants) in the problem.

Step 2 List the primary dimensions of each parameter.

Step 3 Guess the *reduction* j, usually equal to the number of primary dimensions in the problem. If the analysis does not work out, reduce j by one and try again. The expected number of Π's (k) is equal to n minus j.

Step 4 Wisely choose j *repeating parameters* for construction of the Π's.

Step 5 Generate the k Π's one at a time by grouping the j repeating parameters with each of the remaining variables or constants, forcing the product to be dimensionless, and manipulating the Π's as necessary to achieve established nondimensional parameters.

Step 6 Check your work and write the final functional relationship.

When all the dimensionless groups match between a model and a prototype, *dynamic similarity* is achieved, and we are able to directly predict prototype performance based on model experiments. However, it is not always possible to match *all* the Π groups when trying to achieve similarity between a model and a prototype. In such cases, we run the model tests under conditions of *incomplete similarity*, matching the most important Π groups as best we can, and then extrapolating the model test results to prototype conditions.

We use the concepts presented in this chapter throughout the remainder of the book. For example, dimensional analysis is applied to fully developed pipe flows in Chap. 8 (friction factors, loss coefficients, etc.). In Chap. 10, we normalize the differential equations of fluid flow derived in Chap. 9, producing several dimensionless parameters. Drag and lift coefficients are used extensively in Chap. 11, and dimensionless parameters also appear in the chapters on compressible flow and open-channel flow (Chaps. 12 and 13). We learn in Chap. 14 that dynamic similarity is often the basis for design and testing of pumps and turbines. Finally, dimensionless parameters are also used in computations of fluid flows (Chap. 15).

REFERENCES AND SUGGESTED READING

1. D. C. Montgomery. *Design and Analysis of Experiments*, 4th ed. New York: Wiley, 1996.

2. J. P. Holman. *Experimental Methods for Engineers*, 7th ed. New York: McGraw-Hill, 2001.

PROBLEMS*

Dimensions and Units, Primary Dimensions

7–1C What is the difference between a *dimension* and a *unit*? Give three examples of each.

7–2C When performing a dimensional analysis, one of the first steps is to list the primary dimensions of each relevant parameter. It is handy to have a table of parameters and their primary dimensions. We have started such a table for you (Table P7–2C), in which we have included some of the basic parameters commonly encountered in fluid mechanics. As you work through homework problems in this chapter, add to this table. You should be able to build up a table with dozens of parameters.

* Problems designated by a "C" are concept questions, and students are encouraged to answer them all. Problems designated by an "E" are in English units, and the SI users can ignore them. Problems with the icon are solved using EES, and complete solutions together with parametric studies are included on the enclosed DVD. Problems with the icon are comprehensive in nature and are intended to be solved with a computer, preferably using the EES software that accompanies this text.

TABLE P7–2C

Parameter Name	Parameter Symbol	Primary Dimensions
Acceleration	a	$\mathrm{L^1t^{-2}}$
Angle	θ, ϕ, etc.	1 (none)
Density	ρ	$\mathrm{m^1L^{-3}}$
Force	F	$\mathrm{m^1L^1t^{-2}}$
Frequency	f	$\mathrm{t^{-1}}$
Pressure	P	$\mathrm{m^1L^{-1}t^{-2}}$
Surface tension	σ_s	$\mathrm{m^1t^{-2}}$
Velocity	V	$\mathrm{L^1t^{-1}}$
Viscosity	μ	$\mathrm{m^1L^{-1}t^{-1}}$
Volume flow rate	$\dot{V}$	$\mathrm{L^3t^{-1}}$

7–3C List the seven *primary dimensions*. What is significant about these seven?

7–4 Write the primary dimensions of the *universal ideal gas constant* R_u. (Hint: Use the *ideal gas law*, $PV = nR_uT$ where P is pressure, V is volume, T is absolute temperature, and n is the number of moles of the gas.) *Answer:* $\{\mathrm{m^1L^2t^{-2}T^{-1}N^{-1}}\}$

7–5 On a periodic chart of the elements, molar mass (M), also called *atomic weight,* is often listed as though it were a dimensionless quantity (Fig. P7–5). In reality, atomic weight is the mass of 1 mol of the element. For example, the atomic weight of nitrogen M_{nitrogen} = 14.0067. We interpret this as 14.0067 g/mol of elemental nitrogen, or in the English system, 14.0067 lbm/lbmol of elemental nitrogen. What are the primary dimensions of atomic weight?

6 C 12.011	7 N 14.0067	8 O 15.9994
14 Si 28.086	15 P 30.9738	16 S 32.060

FIGURE P7–5

7–6 Some authors prefer to use *force* as a primary dimension in place of mass. In a typical fluid mechanics problem, then, the four represented primary dimensions m, L, t, and T are replaced by F, L, t, and T. The primary dimension of force in this system is {force} = {F}. Using the results of Prob. 7–4, rewrite the primary dimensions of the universal gas constant in this alternate system of primary dimensions.

7–7 We define the *specific ideal gas constant* R_{gas} for a particular gas as the ratio of the universal gas constant and the molar mass (also called *molecular weight*) of the gas, $R_{\text{gas}} = R_u/M$. For a particular gas, then, the ideal gas law can be written as follows:

$$PV = mR_{\text{gas}}T \qquad \text{or} \qquad P = \rho R_{\text{gas}}T$$

where P is pressure, V is volume, m is mass, T is absolute temperature, and ρ is the density of the particular gas. What are the primary dimensions of R_{gas}? For air, R_{air} = 287.0 J/kg · K in standard SI units. Verify that these units agree with your result.

7–8 The *moment of force* ($\vec{M}$) is formed by the cross product of a moment arm ($\vec{r}$) and an applied force ($\vec{F}$), as sketched in Fig. P7–8. What are the primary dimensions of moment of force? List its units in primary SI units and in primary English units.

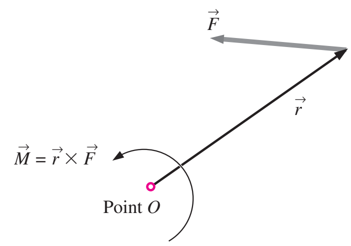


FIGURE P7–8

7–9 Write the primary dimensions of each of the following variables from the field of thermodynamics, showing all your work: (*a*) energy E; (*b*) specific energy $e = E/m$; (*c*) power $\dot{W}$. *Answers:* (*a*) $\{\mathrm{m^1L^2t^{-2}}\}$; (*b*) $\{\mathrm{L^2t^{-2}}\}$; (*c*) $\{\mathrm{m^1L^2t^{-3}}\}$

7–10 What are the primary dimensions of electric voltage (E)? (Hint: Make use of the fact that electric power is equal to voltage times current.)

7–11 You are probably familiar with *Ohm's law* for electric circuits (Fig. P7–11), where ΔE is the voltage difference or *potential* across the resistor, I is the electric current passing through the resistor, and R is the electrical resistance. What are the primary dimensions of electrical resistance? *Answer:* $\{\mathrm{m^1L^2t^{-3}I^{-2}}\}$

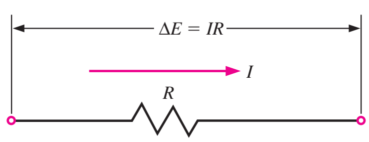


FIGURE P7–11

7–12 Write the primary dimensions of each of the following variables, showing all your work: (*a*) acceleration a; (*b*) angular velocity ω; (*c*) angular acceleration α.

7–13 *Angular momentum*, also called *moment of momentum* ($\vec{H}$), is formed by the cross product of a moment arm ($\vec{r}$) and the linear momentum ($m\vec{V}$) of a fluid particle, as sketched in Fig. P7–13. What are the primary dimensions of angular momentum? List the units of angular momentum in primary SI units and in primary English units. *Answers:* $\{m^1L^2t^{-1}\}$, kg · m²/s, lbm · m²/s, lbm · ft²/s

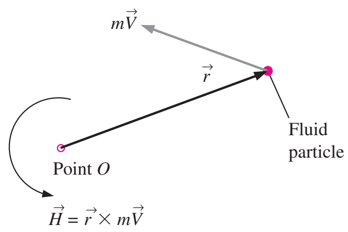


FIGURE P7–13

7–14 Write the primary dimensions of each of the following variables, showing all your work: (*a*) specific heat at constant pressure c_p; (*b*) specific weight ρg; (*c*) specific enthalpy h.

7–15 **Thermal conductivity** k is a measure of the ability of a material to conduct heat (Fig. P7–15). For conduction heat transfer in the x-direction through a surface normal to the x-direction, **Fourier's law of heat conduction** is expressed as

$$\dot{Q}_{\text{conduction}} = -kA\frac{dT}{dx}$$

where $\dot{Q}_{\text{conduction}}$ is the rate of heat transfer and A is the area normal to the direction of heat transfer. Determine the primary dimensions of thermal conductivity (k). Look up a value of k in the appendices and verify that its SI units are consistent with your result. In particular, write the primary SI units of k.

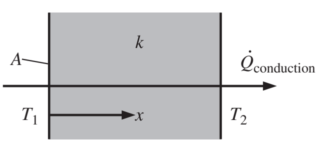


FIGURE P7–15

7–16 Write the primary dimensions of each of the following variables from the study of convection heat transfer (Fig. P7–16), showing all your work: (*a*) heat generation rate $\dot{g}$ (Hint: rate of conversion of thermal energy per unit volume); (*b*) heat flux $\dot{q}$ (Hint: rate of heat transfer per unit area); (*c*) heat transfer coefficient h (Hint: heat flux per unit temperature difference).

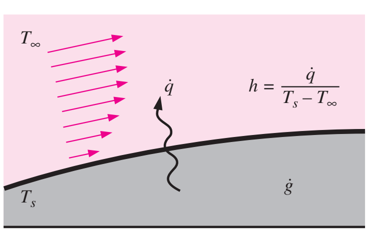


FIGURE P7–16

7–17 Thumb through the appendices of your thermodynamics book, and find three properties or constants not mentioned in Probs. 7–1 to 7–16. List the name of each property or constant and its SI units. Then write out the primary dimensions of each property or constant.

7–18E Thumb through the appendices of this book and/or your thermodynamics book, and find three properties or constants not mentioned in Probs. 7–1 to 7–17. List the name of each property or constant and its English units. Then write out the primary dimensions of each property or constant.

Dimensional Homogeneity

7–19C Explain the *law of dimensional homogeneity* in simple terms.

7–20 In Chap. 4 we defined the *material acceleration*, which is the acceleration following a fluid particle (Fig. P7–20),

$$\vec{a}(x, y, z, t) = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}$$

(*a*) What are the primary dimensions of the gradient operator $\vec{\nabla}$? (*b*) Verify that each additive term in the equation has the same dimensions. *Answers:* (*a*) $\{L^{-1}\}$; (*b*) $\{L^1t^{-2}\}$

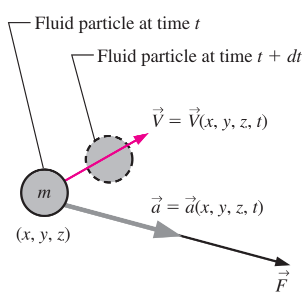


FIGURE P7–20

7–21 Newton's second law is the foundation for the differential equation of conservation of linear momentum (to be discussed in Chap. 9). In terms of the material acceleration following a fluid particle (Fig. P7–20), we write Newton's second law as follows:

$$\vec{F} = m\vec{a} = m\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}\right]$$

Or, dividing both sides by the mass m of the fluid particle,

$$\frac{\vec{F}}{m} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}$$

Write the primary dimensions of each additive term in the equation, and verify that the equation is dimensionally homogeneous. Show all your work.

7–22 In Chap. 4 we defined *volumetric strain rate* as the rate of increase of volume of a fluid element per unit volume (Fig. P7–22). In Cartesian coordinates we write the volumetric strain rate as

$$\frac{1}{\mathcal{V}}\frac{D\mathcal{V}}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Write the primary dimensions of each additive term, and verify that the equation is dimensionally homogeneous. Show all your work.

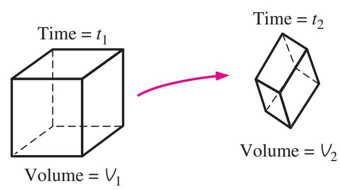


FIGURE P7–22

7–23 In Chap. 9 we discuss the differential equation for conservation of mass, the *continuity equation*. In cylindrical coordinates, and for steady flow,

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

Write the primary dimensions of each additive term in the equation, and verify that the equation is dimensionally homogeneous. Show all your work.

7–24 Cold water enters a pipe, where it is heated by an external heat source (Fig. P7–24). The inlet and outlet water temperatures are T_{in} and T_{out}, respectively. The total rate of heat transfer $\dot{Q}$ from the surroundings into the water in the pipe is

$$\dot{Q} = \dot{m}c_p(T_{\text{out}} - T_{\text{in}})$$

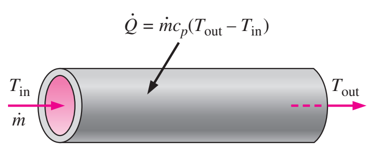


FIGURE P7–24

where $\dot{m}$ is the mass flow rate of water through the pipe, and c_p is the specific heat of the water. Write the primary dimensions of each additive term in the equation, and verify that the equation is dimensionally homogeneous. Show all your work.

7–25 The *Reynolds transport theorem* (RTT) is discussed in Chap. 4. For the general case of a moving and/or deforming control volume, we write the RTT as follows:

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b \, d\mathcal{V} + \int_{\text{CS}} \rho b \vec{V}_r \cdot \vec{n} \, dA$$

where $\vec{V}_r$ is the *relative velocity*, i.e., the velocity of the fluid relative to the control surface. Write the primary dimensions of each additive term in the equation, and verify that the equation is dimensionally homogeneous. Show all your work. (Hint: Since B can be any property of the flow—scalar, vector, or even tensor—it can have a variety of dimensions. So, just let the dimensions of B be those of B itself, $\{B\}$. Also, b is defined as B per unit mass.)

7–26 An important application of fluid mechanics is the study of room ventilation. In particular, suppose there is a **source** S (mass per unit time) of air pollution in a room of volume $\mathcal{V}$ (Fig. P7–26). Examples include carbon monoxide from cigarette smoke or an unvented kerosene heater, gases like ammonia from household cleaning products, and vapors given off by evaporation of **volatile organic compounds** (VOCs) from an open container. We let c represent the **mass concentration** (mass of contaminant per unit volume of air). $\dot{\mathcal{V}}$ is the volume flow rate of fresh air entering the room. If the room air is well mixed so that the mass concentration c is uniform throughout the room, but varies with time, the differential equation for mass concentration in the room as a function of time is

$$\mathcal{V}\frac{dc}{dt} = S - \dot{\mathcal{V}}c - cA_s k_w$$

where k_w is an **adsorption coefficient** and A_s is the surface area of walls, floors, furniture, etc., that adsorb some of the contaminant. Write the primary dimensions of the first three additive terms in the equation, and verify that those terms are dimensionally homogeneous. Then determine the dimensions of k_w. Show all your work.

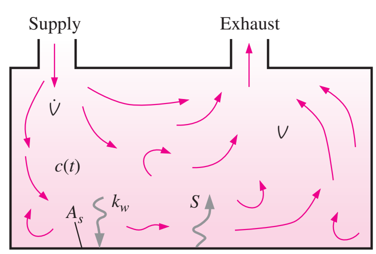


FIGURE P7–26

Nondimensionalization of Equations

7–27C What is the primary reason for *nondimensionalizing* an equation?

7–28 Consider ventilation of a well-mixed room as in Fig. P7–26. The differential equation for mass concentration in the room as a function of time is given in Prob. 7–26 and is repeated here for convenience,

$$V\frac{dc}{dt} = S - \dot{V}c - cA_s k_w$$

There are three characteristic parameters in such a situation: L, a characteristic length scale of the room (assume $L = V^{1/3}$); $\dot{V}$, the volume flow rate of fresh air into the room, and c_{limit}, the maximum mass concentration that is not harmful. (*a*) Using these three characteristic parameters, define dimensionless forms of all the variables in the equation. (Hint: For example, define $c^* = c/c_{\text{limit}}$.) (*b*) Rewrite the equation in dimensionless form, and identify any established dimensionless groups that may appear.

7–29 Recall from Chap. 4 that the volumetric strain rate is zero for a steady incompressible flow. In Cartesian coordinates we express this as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Suppose the characteristic speed and characteristic length for a given flow field are V and L, respectively (Fig. P7–29). Define the following dimensionless variables,

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad z^* = \frac{z}{L},$$

$$u^* = \frac{u}{V}, \quad v^* = \frac{v}{V}, \quad \text{and} \quad w^* = \frac{w}{V}$$

Nondimensionalize the equation, and identify any established (named) dimensionless parameters that may appear. Discuss.

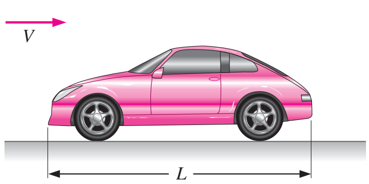


FIGURE P7–29

7–30 In an oscillating compressible flow field the volumetric strain rate is *not* zero, but varies with time following a fluid particle. In Cartesian coordinates we express this as

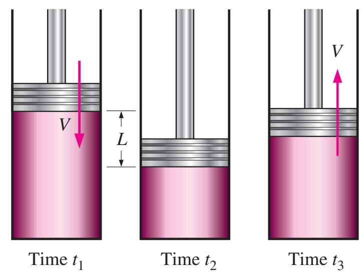
$$\frac{1}{V}\frac{DV}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Suppose the characteristic speed and characteristic length for a given flow field are V and L, respectively. Also suppose that f is a characteristic frequency of the oscillation (Fig. P7–30). Define the following dimensionless variables,

$$t^* = ft, \quad V^* = \frac{V}{L^3}, \quad x^* = \frac{x}{L}, \quad y^* = \frac{y}{L},$$

$$z^* = \frac{z}{L}, \quad u^* = \frac{u}{V}, \quad v^* = \frac{v}{V}, \quad \text{and} \quad w^* = \frac{w}{V}$$

Nondimensionalize the equation and identify any established (named) dimensionless parameters that may appear.



f = frequency of oscillation

FIGURE P7–30

7–31 In Chap. 9 we define the **stream function** ψ for two-dimensional incompressible flow in the xy-plane,

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

where u and v are the velocity components in the x- and y-directions, respectively. (*a*) What are the primary dimensions of ψ? (*b*) Suppose a certain two-dimensional flow has a characteristic length scale L and a characteristic time scale t. Define dimensionless forms of variables x, y, u, v, and ψ. (*c*) Rewrite the equations in nondimensional form, and identify any established dimensionless parameters that may appear.

7–32 In an oscillating incompressible flow field the force per unit mass acting on a fluid particle is obtained from Newton's second law in intensive form (see Prob. 7–21),

$$\frac{\vec{F}}{m} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}$$

Suppose the characteristic speed and characteristic length for a given flow field are V_∞ and L, respectively. Also suppose that ω is a characteristic angular frequency (rad/s) of the oscillation (Fig. P7–32). Define the following nondimensionalized variables,

$$t^* = \omega t, \quad \vec{x}^* = \frac{\vec{x}}{L}, \quad \vec{\nabla}^* = L\vec{\nabla}, \quad \text{and} \quad \vec{V}^* = \frac{\vec{V}}{V_\infty}$$

Since there is no given characteristic scale for the force per unit mass acting on a fluid particle, we assign one, noting that $\{\vec{F}/m\} = \{L/t^2\}$. Namely, we let

$$(\vec{F}/m)^* = \frac{1}{\omega^2 L}\vec{F}/m$$

Nondimensionalize the equation of motion and identify any established (named) dimensionless parameters that may appear.

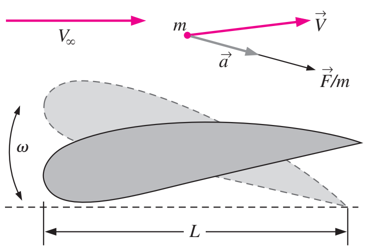


FIGURE P7–32

7–33 A wind tunnel is used to measure the pressure distribution in the airflow over an airplane model (Fig. P7–33). The air speed in the wind tunnel is low enough that compressible effects are negligible. As discussed in Chap. 5, the Bernoulli equation approximation is valid in such a flow situation everywhere except very close to the body surface or wind tunnel wall surfaces and in the wake region behind the model. Far away from the model, the air flows at speed V_∞ and pressure P_∞, and the air density ρ is approximately constant. Gravitational effects are generally negligible in airflows, so we write the Bernoulli equation as

$$P + \frac{1}{2}\rho V^2 = P_\infty + \frac{1}{2}\rho V_\infty^2$$

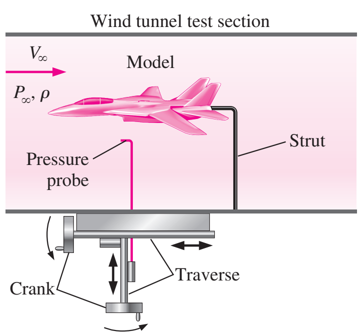


FIGURE P7–33

Nondimensionalize the equation, and generate an expression for the **pressure coefficient** C_p at any point in the flow where the Bernoulli equation is valid. C_p is defined as

$$C_p = \frac{P - P_\infty}{\frac{1}{2}\rho V_\infty^2}$$

Answer: $C_p = 1 - V/V_\infty^2$

Dimensional Analysis and Similarity

7–34C List the three primary purposes of dimensional analysis.

7–35C List and describe the three necessary conditions for complete similarity between a model and a prototype.

7–36 A student team is to design a human-powered submarine for a design competition. The overall length of the prototype submarine is 2.24 m, and its student designers hope that it can travel fully submerged through water at 0.560 m/s. The water is freshwater (a lake) at $T = 15°C$. The design team builds a one-eighth scale model to test in their university's wind tunnel (Fig. P7–36). A shield surrounds the drag balance strut so that the aerodynamic drag of the strut itself does not influence the measured drag. The air in the wind tunnel is at 25°C and at one standard atmosphere pressure. At what air speed do they need to run the wind tunnel in order to achieve similarity? *Answer:* 61.4 m/s

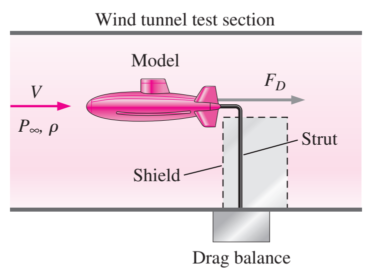


FIGURE P7–36

7–37 Repeat Prob. 7–36 with all the same conditions except that the only facility available to the students is a much smaller wind tunnel. Their model submarine is a one-twenty-fourth scale model instead of a one-eighth scale model. At what air speed do they need to run the wind tunnel in order to achieve similarity? Do you notice anything disturbing or suspicious about your result? Discuss.

7–38E A lightweight parachute is being designed for military use (Fig. P7–38E). Its diameter D is 24 ft and the total weight W of the falling payload, parachute, and equipment is 230 lbf. The design *terminal settling speed* V_t of the parachute at this weight is 20 ft/s. A one-twelfth scale model of the parachute is tested in a wind tunnel. The wind tunnel temperature and pressure are the same as those of the prototype, namely 60°F and standard atmospheric pressure. (*a*) Calculate the drag coefficient of the prototype. (Hint: At terminal settling speed, weight is balanced by aerodynamic drag.) (*b*) At what wind tunnel speed should the wind tunnel be run in order to achieve dynamic similarity? (*c*) Estimate the aerodynamic drag of the model parachute in the wind tunnel (in lbf).

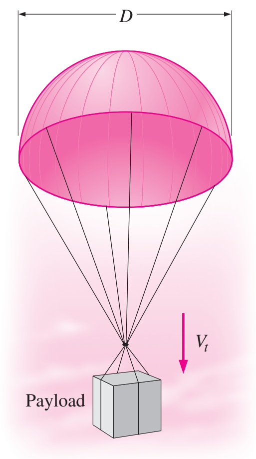


FIGURE P7–38E

7–39 Some wind tunnels are *pressurized*. Discuss why a research facility would go through all the extra trouble and expense to pressurize a wind tunnel. If the air pressure in the tunnel increases by a factor of 1.5, all else being equal (same wind speed, same model, etc.), by what factor will the Reynolds number increase?

7–40 This is a follow-up to Prob. 7–36. The students measure the aerodynamic drag on their model submarine in the wind tunnel (Fig. P7–36). They are careful to run the wind tunnel at conditions that ensure similarity with the prototype submarine. Their measured drag force is 2.3 N. Estimate the drag force on the prototype submarine at the conditions given in Prob. 7–36. *Answer:* 10.3 N

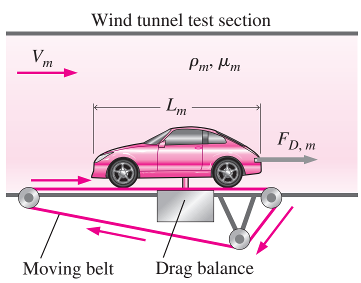


FIGURE P7–41E

7–41E The aerodynamic drag of a new sports car is to be predicted at a speed of 60.0 mi/h at an air temperature of 25°C. Automotive engineers build a one-fourth scale model of the car (Fig. P7–41E) to test in a wind tunnel. The temperature of the wind tunnel air is also 25°C. The drag force is measured with a drag balance, and the moving belt is used to simulate the moving ground (from the car's frame of reference). Determine how fast the engineers should run the wind tunnel to achieve similarity between the model and the prototype.

7–42E This is a follow-up to Prob. 7–41E. The aerodynamic drag on the model in the wind tunnel (Fig. P7–41E) is measured to be 36.5 lbf when the wind tunnel is operated at the speed that ensures similarity with the prototype car. Estimate the drag force (in lbf) on the prototype car at the conditions given in Prob. 7–41E.

7–43 Consider the common situation in which a researcher is trying to match the Reynolds number of a large prototype vehicle with that of a small-scale model in a wind tunnel. Is it better for the air in the wind tunnel to be cold or hot? Why? Support your argument by comparing wind tunnel air at 10°C and at 50°C, all else being equal.

7–44 Some students want to visualize flow over a spinning baseball. Their fluids laboratory has a nice water tunnel into which they can inject multicolored dye streaklines, so they decide to test a spinning baseball in the water tunnel (Fig. P7–44). Similarity requires that they match both the Reynolds number and the Strouhal number between their model test and the actual baseball that moves through the air at 80 mi/h and spins at 300 rpm. Both the air and the water are at 20°C. At what speed should they run the water in the water tunnel, and at what rpm should they spin their baseball? *Answers:* 5.30 mi/h, 20.0 rpm

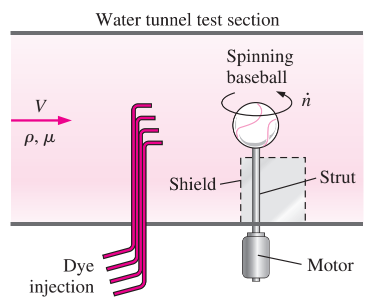


FIGURE P7–44

Dimensionless Parameters and the Method of Repeating Variables

7–45 Using primary dimensions, verify that the Archimedes number (Table 7–5) is indeed dimensionless.

7–46 Using primary dimensions, verify that the Grashof number (Table 7–5) is indeed dimensionless.

7–47 Using primary dimensions, verify that the Rayleigh number (Table 7–5) is indeed dimensionless. What other established nondimensional parameter is formed by the ratio of Ra and Gr? *Answer:* the Prandtl number

7–48 Consider a liquid in a cylindrical container in which both the container and the liquid are rotating as a rigid body (solid-body rotation). The elevation difference h between the center of the liquid surface and the rim of the liquid surface is a function of angular velocity ω, fluid density ρ, gravitational acceleration g, and radius R (Fig. P7–48). Use the method of repeating variables to find a dimensionless relationship between the parameters. Show all your work. *Answer:* $h/R = f(\text{Fr})$

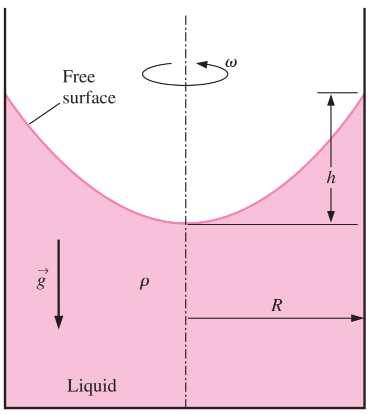


FIGURE P7–48

7–49 Consider the case in which the container and liquid of Prob. 7–48 are initially at rest. At $t = 0$ the container begins to rotate. It takes some time for the liquid to rotate as a rigid body, and we expect that the liquid's viscosity is an additional relevant parameter in the unsteady problem. Repeat Prob. 7–48, but with two additional independent parameters included, namely, fluid viscosity μ and time t. (We are interested in the development of height h as a function of time and the other parameters.)

7–50 A periodic *Kármán vortex street* is formed when a uniform stream flows over a circular cylinder (Fig. P7–50). Use the method of repeating variables to generate a dimensionless relationship for Kármán vortex shedding frequency f_k as a function of free-stream speed V, fluid density ρ, fluid viscosity μ, and cylinder diameter D. Show all your work. *Answer:* $\text{St} = f(\text{Re})$

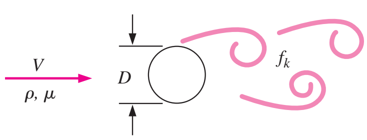


FIGURE P7–50

7–51 Repeat Prob. 7–50, but with an additional independent parameter included, namely, the speed of sound c in the fluid. Use the method of repeating variables to generate a dimensionless relationship for Kármán vortex shedding frequency f_k as a function of free-stream speed V, fluid density ρ, fluid viscosity μ, cylinder diameter D, and speed of sound c. Show all your work.

7–52 A stirrer is used to mix chemicals in a large tank (Fig. P7–52). The shaft power $\dot{W}$ supplied to the stirrer blades is a function of stirrer diameter D, liquid density ρ, liquid viscosity μ, and the angular velocity ω of the spinning blades. Use the method of repeating variables to generate a dimensionless relationship between these parameters. Show all your work and be sure to identify your Π groups, modifying them as necessary. *Answer:* $N_p = f(\text{Re})$

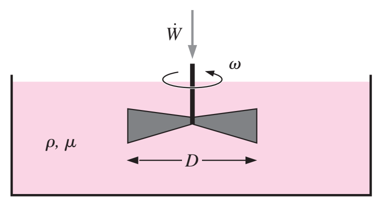


FIGURE P7–52

7–53 Repeat Prob. 7–52 except do not assume that the tank is large. Instead, let tank diameter D_{tank} and average liquid depth h_{tank} be additional relevant parameters.

7–54 A boundary layer is a thin region (usually along a wall) in which viscous forces are significant and within which the flow is rotational. Consider a boundary layer growing along a thin flat plate (Fig. P7–54). The flow is steady. The boundary layer thickness δ at any downstream distance x is a function of x, free-stream velocity V_∞, and fluid properties ρ (density) and μ (viscosity). Use the method of repeating variables to generate a dimensionless relationship for δ as a function of the other parameters. Show all your work.

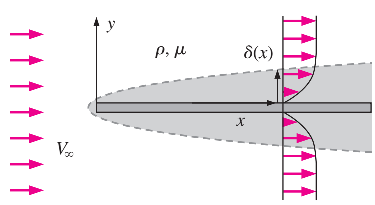


FIGURE P7–54

7–55 Miguel is working on a problem that has a characteristic length scale L, a characteristic velocity V, a characteristic density difference $\Delta\rho$, a characteristic (average) density ρ, and of course the gravitational constant g, which is always available. He wants to define a Richardson number, but does not have a characteristic volume flow rate. Help Miguel define a characteristic volume flow rate based on the parameters available to him, and then define an appropriate Richardson number in terms of the given parameters.

7–56 Consider fully developed **Couette flow**—flow between two infinite parallel plates separated by distance h, with the top plate moving and the bottom plate stationary as illustrated in Fig. P7–56. The flow is steady, incompressible, and two-dimensional in the xy-plane. Use the method of repeating variables to generate a dimensionless relationship for the x-component of fluid velocity u as a function of fluid viscosity μ, top plate speed V, distance h, fluid density ρ, and distance y. Show all your work. *Answer:* $u/V = f(\text{Re}, y/h)$

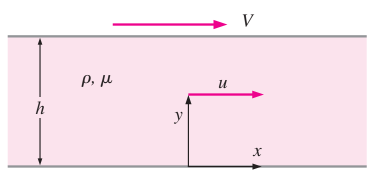


FIGURE P7–56

7–57 Consider *developing* Couette flow—the same flow as Prob. 7–56 except that the flow is not yet steady-state, but is developing with time. In other words, time t is an additional parameter in the problem. Generate a dimensionless relationship between all the variables.

7–58 The speed of sound c in an ideal gas is known to be a function of the ratio of specific heats k, absolute temperature T, and specific ideal gas constant R_{gas} (Fig. P7–58). Showing all your work, use dimensional analysis to find the functional relationship between these parameters.

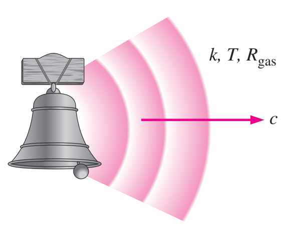


FIGURE P7–58

7–59 Repeat Prob. 7–58, except let the speed of sound c in an ideal gas be a function of absolute temperature T, universal ideal gas constant R_u, molar mass (molecular weight) M of the gas, and ratio of specific heats k. Showing all your work, use dimensional analysis to find the functional relationship between these parameters.

7–60 Repeat Prob. 7–58, except let the speed of sound c in an ideal gas be a function only of absolute temperature T and specific ideal gas constant R_{gas}. Showing all your work, use dimensional analysis to find the functional relationship between these parameters. *Answer:* $c/\sqrt{R_{gas}T}$ = constant

7–61 Repeat Prob. 7–58, except let speed of sound c in an ideal gas be a function only of pressure P and gas density ρ. Showing all your work, use dimensional analysis to find the functional relationship between these parameters. Verify that your results are consistent with the equation for speed of sound in an ideal gas, $c = \sqrt{kR_{gas}T}$.

7–62 When small aerosol particles or microorganisms move through air or water, the Reynolds number is very small (Re $\ll$ 1). Such flows are called **creeping flows.** The aerodynamic drag on an object in creeping flow is a function only of its speed V, some characteristic length scale L of the object, and fluid viscosity μ (Fig. P7–62). Use dimensional analysis to generate a relationship for F_D as a function of the independent variables.

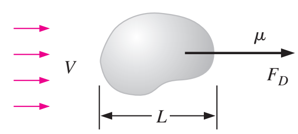


FIGURE P7–62

7–63 A tiny aerosol particle of density ρ_p and characteristic diameter D_p falls in air of density ρ and viscosity μ (Fig. P7–63). If the particle is small enough, the creeping flow approximation is valid, and the terminal settling speed of the particle V depends only on D_p, μ, gravitational constant g, and the density difference $(\rho_p - \rho)$. Use dimensional analysis to generate a relationship for V as a function of the independent variables. Name any established dimensionless parameters that appear in your analysis.

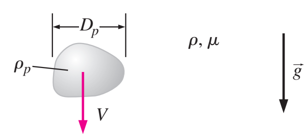


FIGURE P7–63

7–64 Combine the results of Probs. 7–62 and 7–63 to generate an equation for the settling speed V of an aerosol particle falling in air (Fig. P7–63). Verify that your result is consistent with the functional relationship obtained in Prob. 7–63. For consistency, use the notation of Prob. 7–63. (Hint: For a particle falling at constant settling speed, the particle's net weight must equal its aerodynamic drag. Your final result should be an equation for V that is valid to within some unknown constant.)

7–65 You will need the results of Prob. 7–64 to do this problem. A tiny aerosol particle falls at steady settling speed V. The Reynolds number is small enough that the creeping flow approximation is valid. If the particle size is doubled, all else being equal, by what factor will the settling speed go up? If the density difference $(\rho_p - \rho)$ is doubled, all else being equal, by what factor will the settling speed go up?

7–66 An incompressible fluid of density ρ and viscosity μ flows at average speed V through a long, horizontal section of round pipe of length L, inner diameter D, and inner wall roughness height ε (Fig. P7–66). The pipe is long enough that the flow is fully developed, meaning that the velocity profile does not change down the pipe. Pressure decreases (linearly) down the pipe in order to "push" the fluid through the pipe to overcome friction. Using the method of repeating variables, develop a nondimensional relationship between pressure drop $\Delta P = P_1 - P_2$ and the other parameters in the problem. Be sure to modify your Π groups as necessary to achieve established nondimensional parameters, and name them. (Hint: For consistency, choose D rather than L or ε as one of your repeating parameters.) *Answer:* Eu = f (Re, ε/D, L/D)

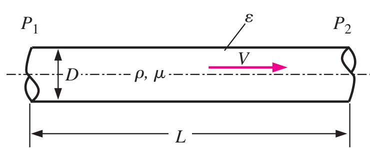


FIGURE P7–66

7–67 Consider *laminar* flow through a long section of pipe, as in Fig. P7–66. For laminar flow it turns out that wall roughness is not a relevant parameter unless ε is very large. The volume flow rate $\dot{V}$ through the pipe is in fact a function of pipe diameter D, fluid viscosity μ, and axial pressure gradient dP/dx. If pipe diameter is doubled, all else being equal, by what factor will volume flow rate increase? Use dimensional analysis.

7–68 The rate of heat transfer to water flowing in a pipe was analyzed in Prob. 7–24. Let us approach that same problem, but now with dimensional analysis. Cold water enters a pipe, where it is heated by an external heat source (Fig. P7–68). The inlet and outlet water temperatures are T_{in} and T_{out}, respectively. The total rate of heat transfer $\dot{Q}$ from the surroundings into the water in the pipe is known to be a function of mass flow rate $\dot{m}$, the specific heat c_p of the water, and the temperature difference between the incoming and outgoing water. Showing all your work, use dimensional analysis to find the functional relationship between these parameters, and compare to the analytical equation given in Prob. 7–24. (Note: We are pretending that we do not know the analytical equation.)

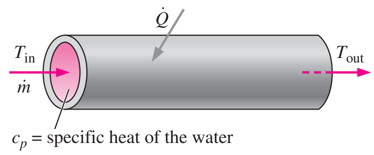


FIGURE P7–68

Experimental Testing and Incomplete Similarity

7–69C Define *wind tunnel blockage*. What is the rule of thumb about the maximum acceptable blockage for a wind tunnel test? Explain why there would be measurement errors if the blockage were significantly higher than this value.

7–70C What is the rule of thumb about the Mach number limit in order that the incompressible flow approximation is reasonable? Explain why wind tunnel results would be incorrect if this rule of thumb were violated.

7–71C Although we usually think of a model as being smaller than the prototype, describe at least three situations in which it is better for the model to be *larger* than the prototype.

7–72C Discuss the purpose of a moving ground belt in wind tunnel tests of flow over model automobiles. Can you think of an alternative if a moving ground belt is unavailable?

7–73 Use dimensional analysis to show that in a problem involving shallow water waves (Fig. P7–73), both the Froude number and the Reynolds number are relevant dimensionless parameters. The wave speed c of waves on the surface of a liquid is a function of depth h, gravitational acceleration g, fluid density ρ, and fluid viscosity μ. Manipulate your Π's to get the parameters into the following form:

$$\text{Fr} = \frac{c}{\sqrt{gh}} = f(\text{Re}) \qquad \text{where } \text{Re} = \frac{\rho c h}{\mu}$$

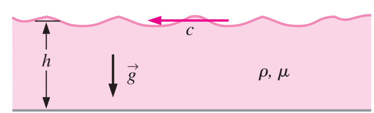


FIGURE P7–73

7–74 Water at 20°C flows through a long, straight pipe. The pressure drop is measured along a section of the pipe of length L = 1.3 m as a function of average velocity V through the pipe (Table P7–74). The inner diameter of the pipe is D = 10.4 cm. (*a*) Nondimensionalize the data and plot the Euler number as a function of the Reynolds number. Has the experiment been run at high enough speeds to achieve Reynolds number independence? (*b*) Extrapolate the experimental data to predict the pressure drop at an average speed of 80 m/s. *Answer:* 1,940,000 N/m^2

TABLE P7–74

V, m/s	ΔP, N/m^2
0.5	77.0
1	306
2	1218
4	4865
6	10,920
8	19,440
10	30,340
15	68,330
20	121,400
25	189,800
30	273,200
35	372,100
40	485,300
45	614,900
50	758,700

7–75 In the model truck example discussed in Section 7–5, the wind tunnel test section is 2.6 m long, 1.0 m tall, and 1.2 m wide. The one-sixteenth scale model truck is 0.991 m long, 0.257 m tall, and 0.159 m wide. What is the wind tunnel blockage of this model truck? Is it within acceptable limits according to the standard rule of thumb?

7–76C Consider again the model truck example discussed in Section 7–5, except that the maximum speed of the wind tunnel is only 50 m/s. Aerodynamic force data are taken for wind tunnel speeds between $V = 20$ and 50 m/s—assume the same data for these speeds as those listed in Table 7–7. Based on these data *alone,* can the researchers be confident that they have reached Reynolds number independence?

7–77E A small wind tunnel in a university's undergraduate fluid flow laboratory has a test section that is 20 by 20 in in cross section and is 4.0 ft long. Its maximum speed is 160 ft/s. Some students wish to build a model 18-wheeler to study how aerodynamic drag is affected by rounding off the back of the trailer. A full-size (prototype) tractor-trailer rig is 52 ft long, 8.33 ft wide, and 12 ft high. Both the air in the wind tunnel and the air flowing over the prototype are at 80°F and atmospheric pressure. (*a*) What is the largest scale model they can build to stay within the rule-of-thumb guidelines for blockage? What are the dimensions of the model truck in inches? (*b*) What is the maximum model truck Reynolds number achievable by the students? (*c*) Are the students able to achieve Reynolds number independence? Discuss.

7–78 A one-sixteenth scale model of a new sports car is tested in a wind tunnel. The prototype car is 4.37 m long, 1.30 m tall, and 1.69 m wide. During the tests, the moving ground belt speed is adjusted so as to always match the speed of the air moving through the test section. Aerodynamic drag force F_D is measured as a function of wind tunnel speed; the experimental results are listed in Table P7–78. Plot drag coefficient C_D as a function of the Reynolds number Re, where the area used for calculation of C_D is the frontal area of the model car (assume A = width × height), and the length scale used for calculation of Re is car width W. Have we achieved dynamic similarity? Have we achieved Reynolds number independence in our wind tunnel test? Estimate the aerodynamic drag force on the prototype car traveling on the highway at 29 m/s (65 mi/h). Assume that both the wind tunnel air and the air flowing over the prototype car are at 25°C and atmospheric pressure. *Answers:* no, yes, 350 N

TABLE P7–78

V, m/s	F_D, N
10	0.29
15	0.64
20	0.96
25	1.41
30	1.55
35	2.10
40	2.65
45	3.28
50	4.07
55	4.91

Review Problems

7–79C For each statement, choose whether the statement is true or false and discuss your answer briefly.

(*a*) Kinematic similarity is a necessary and sufficient condition for dynamic similarity.
(*b*) Geometric similarity is a necessary condition for dynamic similarity.
(*c*) Geometric similarity is a necessary condition for kinematic similarity.
(*d*) Dynamic similarity is a necessary condition for kinematic similarity.

7–80C Think about and describe a prototype flow and a corresponding model flow that have geometric similarity, but not kinematic similarity, even though the Reynolds numbers match. Explain.

7–81C There are many established nondimensional parameters besides those listed in Table 7–5. Do a literature search or an Internet search and find at least three established, named nondimensional parameters that are *not* listed in Table 7–5. For each one, provide its definition and its ratio of significance, following the format of Table 7–5. If your equation contains any variables not identified in Table 7–5, be sure to identify those variables.

7–82 Write the primary dimensions of each of the following variables from the field of solid mechanics, showing all your work: (*a*) moment of inertia I; (*b*) modulus of elasticity E, also called Young's modulus; (*c*) strain ε; (*d*) stress σ.

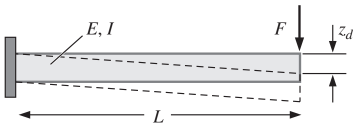


FIGURE P7–83

(*e*) Finally, show that the relationship between stress and strain (Hooke's law) is a dimensionally homogeneous equation.

7–83 Force F is applied at the tip of a cantilever beam of length L and moment of inertia I (Fig. P7–83). The modulus of elasticity of the beam material is E. When the force is applied, the tip deflection of the beam is z_d. Use dimensional analysis to generate a relationship for z_d as a function of the independent variables. Name any established dimensionless parameters that appear in your analysis.

7–84 An explosion occurs in the atmosphere when an anti-aircraft missile meets its target (Fig. P7–84). A **shock wave** (also called a **blast wave**) spreads out radially from the explosion. The pressure difference across the blast wave ΔP and its radial distance r from the center are functions of time t, speed of sound c, and the total amount of energy E released by the explosion. (*a*) Generate dimensionless relationships between ΔP and the other parameters and between r and the other parameters. (*b*) For a given explosion, if the time t since the explosion doubles, all else being equal, by what factor will ΔP decrease?

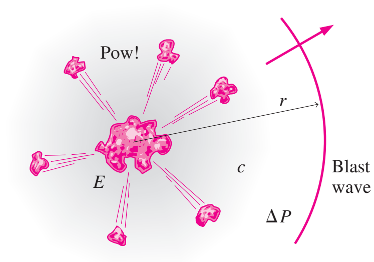


FIGURE P7–84

7–85 The Archimedes number listed in Table 7–5 is appropriate for buoyant *particles* in a fluid. Do a literature search or an Internet search and find an alternative definition of the Archimedes number that is appropriate for buoyant *fluids* (e.g., buoyant jets and buoyant plumes, heating and air-conditioning applications). Provide its definition and its ratio of significance, following the format of Table 7–5. If your equation contains any variables not identified in Table 7–5, be sure to identify those variables. Finally, look through the established dimensionless parameters listed in Table 7–5 and find one that is similar to this alternate form of the Archimedes number.

7–86 Consider steady, laminar, fully developed, two-dimensional **Poiseuille flow**—flow between two infinite parallel plates separated by distance h, with both the top plate and bottom plate stationary, and a forced pressure gradient dP/dx driving the flow as illustrated in Fig. P7–86. (dP/dx is constant and negative.) The flow is steady, incompressible, and two-dimensional in the xy-plane. The flow is also *fully developed*, meaning that the velocity profile does not change with downstream distance x. Because of the fully developed nature of the flow, there are no inertial effects and density does not enter the problem. It turns out that u, the velocity component in the x-direction, is a function of distance h, pressure gradient dP/dx, fluid viscosity μ, and vertical coordinate y. Perform a dimensional analysis (showing all your work), and generate a dimensionless relationship between the given variables.

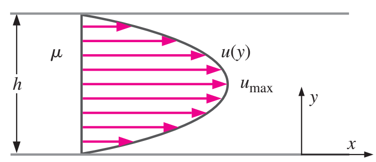


FIGURE P7–86

7–87 Consider the steady, laminar, fully developed, two-dimensional Poiseuille flow of Prob. 7–86. The maximum velocity u_{max} occurs at the center of the channel. (*a*) Generate a dimensionless relationship for u_{max} as a function of distance between plates h, pressure gradient dP/dx, and fluid viscosity μ. (*b*) If the plate separation distance h is doubled, all else being equal, by what factor will u_{max} change? (*c*) If the pressure gradient dP/dx is doubled, all else being equal, by what factor will u_{max} change? (*d*) How many experiments are required to describe the complete relationship between u_{max} and the other parameters in the problem?

7–88 The pressure drop $\Delta P = P_1 - P_2$ through a long section of round pipe can be written in terms of the shear stress τ_w along the wall. Shown in Fig. P7–88 is the shear stress acting by the wall on the fluid. The shaded blue region is a control volume composed of the fluid in the pipe between axial locations 1 and 2. There are two dimensionless parameters related to the pressure drop: the Euler number Eu and the Darcy friction factor f. (*a*) Using the control volume sketched in Fig. P7–88, generate a relationship for f in terms of Eu (and any other properties or parameters in the problem as needed). (*b*) Using the experimental data

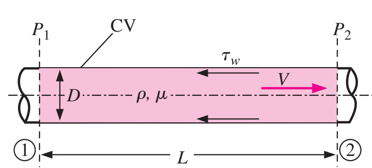


FIGURE P7–88

and conditions of Prob. 7–74 (Table P7–74), plot the Darcy friction factor as a function of Re. Does f show Reynolds number independence at large values of Re? If so, what is the value of f at very high Re? *Answers:* (*a*) $f = 2\frac{D}{L}Eu$; (*b*) yes, 0.0487

7–89 Oftentimes it is desirable to work with an established dimensionless parameter, but the characteristic scales available do not match those used to define the parameter. In such cases, we *create* the needed characteristic scales based on dimensional reasoning (usually by inspection). Suppose for example that we have a characteristic velocity scale V, characteristic area A, fluid density ρ, and fluid viscosity μ, and we wish to define a Reynolds number. We create a length scale $L = \sqrt{A}$, and define

$$\text{Re} = \frac{\rho V \sqrt{A}}{\mu}$$

In similar fashion, define the desired established dimensionless parameter for each case: (*a*) Define a Froude number, given $\dot{V}'$ = volume flow rate per unit depth, length scale L, and gravitational constant g. (*b*) Define a Reynolds number, given $\dot{V}'$ = volume flow rate per unit depth and kinematic viscosity ν. (*c*) Define a Richardson number, given $\dot{V}'$ = volume flow rate per unit depth, length scale L, characteristic density difference $\Delta\rho$, characteristic density ρ, and gravitational constant g.

7–90 A liquid of density ρ and viscosity μ flows by gravity through a hole of diameter d in the bottom of a tank of diameter D (Fig. P7–90). At the start of the experiment, the liquid surface is at height h above the bottom of the tank, as sketched. The liquid exits the tank as a jet with average velocity V straight down as also sketched. Using dimensional analysis, generate a dimensionless relationship for V as a function of the other parameters in the problem. Identify any established nondimensional parameters that appear in your result. (Hint: There are three length scales in this problem. For consistency, choose h as your length scale.)

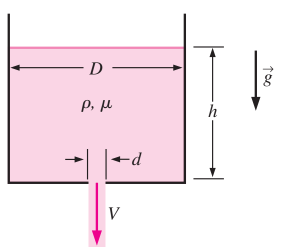


FIGURE P7–90

7–91 Repeat Prob. 7–90 except for a different dependent parameter, namely, the time required to empty the tank t_{empty}. Generate a dimensionless relationship for t_{empty} as a function of the following independent parameters: hole diameter d, tank diameter D, density ρ, viscosity μ, initial liquid surface height h, and gravitational acceleration g.

7–92 A liquid delivery system is being designed such that ethylene glycol flows out of a hole in the bottom of a large tank, as in Fig. P7–90. The designers need to predict how long it will take for the ethylene glycol to completely drain. Since it would be very expensive to run tests with a full-scale prototype using ethylene glycol, they decide to build a one-quarter scale model for experimental testing, and they plan to use *water* as their test liquid. The model is geometrically similar to the prototype (Fig. P7–92). (*a*) The temperature of the ethylene glycol in the prototype tank is 60°C, at which $\nu = 4.75 \times 10^{-6}$ m²/s. At what temperature should the water in the model experiment be set in order to ensure complete similarity between model and prototype? (*b*) The experiment is run with water at the proper temperature as calculated in part (*a*). It takes 4.53 min to drain the model tank. Predict how long it will take to drain the ethylene glycol from the prototype tank. *Answers:* (*a*) 45.8°C, (*b*) 9.06 min

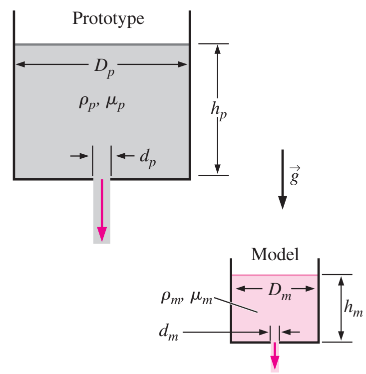


FIGURE P7–92

7–93 Liquid flows out of a hole in the bottom of a tank as in Fig. P7–90. Consider the case in which the hole is very small compared to the tank ($d \ll D$). Experiments reveal that average jet velocity V is nearly independent of d, D, ρ, or μ. In fact, for a wide range of these parameters, it turns out that V depends only on liquid surface height h and gravitational acceleration g. If the liquid surface height is doubled, all else being equal, by what factor will the average jet velocity increase? Answer: $\sqrt{2}$

7–94 An aerosol particle of characteristic size D_p moves in an airflow of characteristic length L and characteristic velocity V. The characteristic time required for the particle to adjust to a sudden change in air speed is called the **particle relaxation time** τ_p,

$$\tau_p = \frac{\rho_p D_p^2}{18\mu}$$

Verify that the primary dimensions of τ_p are time. Then create a dimensionless form of τ_p, based on some characteristic velocity V and some characteristic length L of the airflow (Fig. P7–94). What established dimensionless parameter do you create?

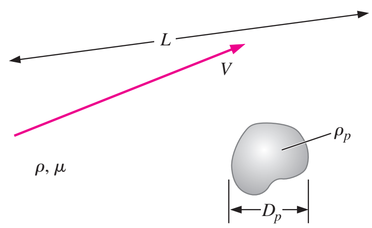


FIGURE P7–94

7–95 Compare the primary dimensions of each of the following properties in the *mass-based* primary dimension system (m, L, t, T, I, C, N) to those in the *force-based* primary dimension system (F, L, t, T, I, C, N): (*a*) pressure or stress; (*b*) moment or torque; (*c*) work or energy. Based on your results, explain when and why some authors prefer to use force as a primary dimension in place of mass.

7–96 In Example 7–7, the mass-based system of primary dimensions was used to establish a relationship for the pressure difference $\Delta P = P_{\text{inside}} - P_{\text{outside}}$ between the inside and outside of a soap bubble as a function of soap bubble radius R and surface tension σ_s of the soap film (Fig. P7–96). Repeat the dimensional analysis using the method of repeating variables, but use the *force-based* system of primary dimensions instead. Show all your work. Do you get the same result?

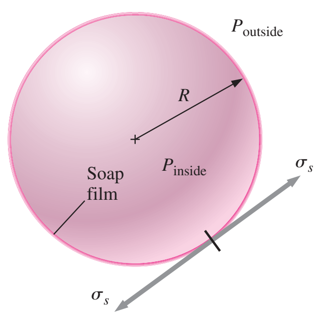


FIGURE P7–96

7–97 Many of the established nondimensional parameters listed in Table 7–5 can be formed by the product or ratio of two other established nondimensional parameters. For each pair of nondimensional parameters listed, find a *third* established nondimensional parameter that is formed by some manipulation of the two given parameters: (*a*) Reynolds number and Prandtl number; (*b*) Schmidt number and Prandtl number; (*c*) Reynolds number and Schmidt number.

7–98 The Stanton number is listed as a named, established nondimensional parameter in Table 7–5. However, careful analysis reveals that it can actually be formed by a combination of the Reynolds number, Nusselt number, and Prandtl number. Find the relationship between these four dimensionless groups, showing all your work. Can you also form the Stanton number by some combination of only *two* other established dimensionless parameters?

7–99 Consider a variation of the fully developed Couette flow problem of Prob. 7–56—flow between two infinite parallel plates separated by distance h, with the top plate moving at speed V_{top} and the bottom plate moving at speed V_{bottom} as illustrated in Fig. P7–99. The flow is steady, incompressible, and two-dimensional in the xy-plane. Generate a dimensionless relationship for the x-component of fluid velocity u as a function of fluid viscosity μ, plate speeds V_{top} and V_{bottom}, distance h, fluid density ρ, and distance y. (Hint: Think carefully about the list of parameters before rushing into the algebra.)

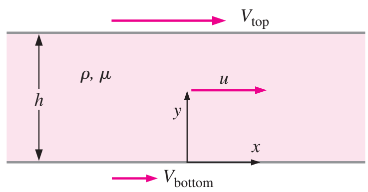


FIGURE P7–99

7–100 What are the primary dimensions of electric charge q, the units of which are **coulombs** (C)? (Hint: Look up the fundamental definition of electric current.)

7–101 What are the primary dimensions of electrical capacitance C, the units of which are **farads**? (Hint: Look up the fundamental definition of electrical capacitance.)

7–102 In many electronic circuits in which some kind of time scale is involved, such as filters and time-delay circuits (Fig. P7–102—*a low-pass filter*), you often see a resistor (R) and a capacitor (C) in series. In fact, the product of R and C is called the *electrical time constant*, RC. Showing all your work, what are the primary dimensions of RC? Using dimen-

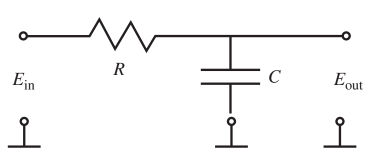


FIGURE P7–102

sional reasoning alone, explain why a resistor and capacitor are often found together in timing circuits.

7–103 From fundamental electronics, the current flowing through a capacitor at any instant of time is equal to the capacitance times the rate of change of voltage across the capacitor,

$$I = C\frac{dE}{dt}$$

Write the primary dimensions of both sides of this equation, and verify that the equation is dimensionally homogeneous. Show all your work.

7–104 A common device used in various applications to clean particle-laden air is the **reverse-flow cyclone** (Fig. P7–104). Dusty air (volume flow rate $\dot{V}$ and density ρ) enters tangentially through an opening in the side of the cyclone and swirls around in the tank. Dust particles are flung outward and fall out the bottom, while clean air is drawn out the top. The reverse-flow cyclones being studied are all geometrically similar; hence, diameter D represents the only length scale required to fully specify the entire cyclone geometry. Engineers are concerned about the pressure drop δP through the cyclone. (*a*) Generate a dimensionless relationship between the pressure drop through the cyclone and the given parameters. Show all your work. (*b*) If the cyclone size is doubled, all else being equal, by what factor will the pressure drop change? (*c*) If the volume flow rate is doubled, all else being equal, by what factor will the pressure drop change? *Answers:* (*a*) $D^4\delta P/\rho\dot{V}^2$ = constant; (*b*) 1/16; (*c*) 4

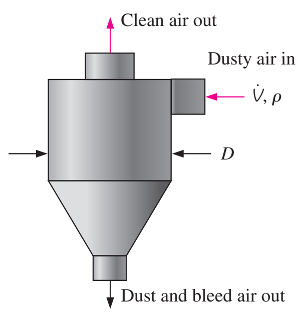


FIGURE P7–104

7–105 An **electrostatic precipitator** (ESP) is a device used in various applications to clean particle-laden air. First, the dusty air passes through the *charging stage* of the ESP, where dust particles are given a positive charge q_p (coulombs) by charged ionizer wires (Fig. P7–105). The dusty air then enters the *collector stage* of the device, where it flows between two oppositely charged plates. The applied *electric field strength* between the plates is E_f (voltage difference per unit distance). Shown in Fig. P7–105 is a charged dust particle of diameter D_p. It is attracted to the negatively charged plate and moves toward that plate at a speed called the **drift velocity** w. If the plates are long enough, the dust particle impacts the negatively charged plate and adheres to it. Clean air exits the device. It turns out that for very small particles the drift velocity depends only on q_p, E_f, D_p, and air viscosity μ. (*a*) Generate a dimensionless relationship between the drift velocity through the collector stage of the ESP and the given parameters. Show all your work. (*b*) If the electric field strength is doubled, all else being equal, by what factor will the drift velocity change? (*c*) For a given ESP, if the particle diameter is doubled, all else being equal, by what factor will the drift velocity change?

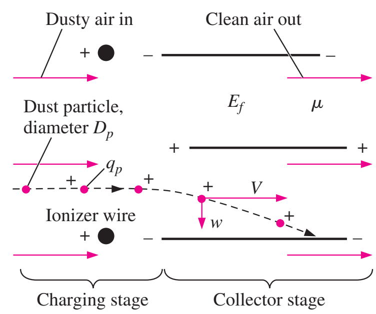


FIGURE P7–105

7–106 When a capillary tube of small diameter D is inserted into a container of liquid, the liquid rises to height h inside the tube (Fig. P7–106). h is a function of liquid density ρ, tube diameter D, gravitational constant g, contact angle ϕ, and the surface tension σ_s of the liquid. (*a*) Generate a dimensionless relationship for h as a function of the given parameters. (*b*) Compare your result to the exact analytical equation for h given in Chap. 2. Are your dimensional analysis results consistent with the exact equation? Discuss.

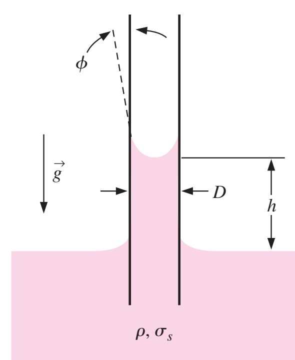


FIGURE P7–106

7–107 Repeat part (*a*) of Prob. 7–106, except instead of height h, find a functional relationship for the time scale t_{rise} needed for the liquid to climb up to its final height in the capillary tube. (Hint: Check the list of independent parameters in Prob. 7–106. Are there any additional relevant parameters?)

7–108 **Sound intensity** I is defined as the acoustic power per unit area emanating from a sound source. We know that I is a function of sound pressure level P (dimensions of pressure) and fluid properties ρ (density) and speed of sound c. (*a*) Use the method of repeating variables in mass-based primary dimensions to generate a dimensionless relationship for I as a function of the other parameters. Show all your work. What happens if you choose three repeating variables? Discuss. (*b*) Repeat part (*a*), but use the force-based primary dimension system. Discuss.

7–109 Repeat Prob. 7–108, but with the distance r from the sound source as an additional independent parameter.