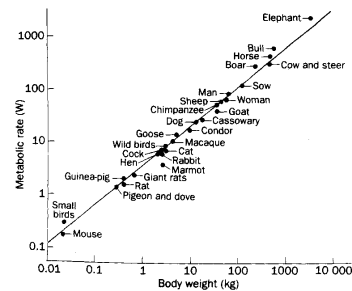
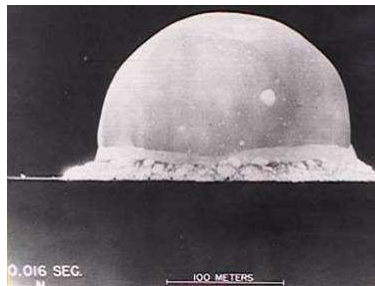


Dimensional analysis

- Scaling - a powerful idea
- Similitude
- Buckingham Pi theorem
- Examples of the power of dimensional analysis
- Useful dimensionless quantities and their interpretation



Scaling and similitude

- *Scaling* is a notion from physics and engineering that should really be second nature to you as you solve problems:
- How does dependent quantity y scale with variations of independent quantity x ?
- *Similitude* is more of an engineering idea: suppose we scaled parameter x by a factor C . If we wanted to maintain the same value of quantity y , how would we need to rescale the other independent parameters?
- Put another way: if we measure an **invariant quantity**, like the speed of light, we get 3×10^8 m/s. We know that **if we want to change our length and time units** so that the speed of light is “1” in our new units, **we have to rescale both space and time**.
- The invariance of actual physics and the requirement of a consistent system of units is information at our disposal that we usually don’t think about consciously. **Often we can use this information to solve some pretty amazing problems without resorting to gory computations!**

Buckingham Pi theorem: statement

Buckingham, Phys. Rev. 4, 345 (1914).

If a physical problem involves n variables $v_1 \dots v_n$ that depend on r independent dimensional units, then the solution of the problem can always be written in the form:

$$0 = f(\Pi_1, \Pi_2, \dots, \Pi_{n-r})$$

where the Π_k are $(n - r)$ dimensionless variables of the form

$$\Pi_k = v_1^{\mu_{k1}} v_2^{\mu_{k2}} \dots v_n^{\mu_{kn}}$$

and the μ_{kj} are rational numbers chosen to make the product dimensionless.

Units

We have to pick m fundamental units F_i - examples of these are length L, time T, and mass M.

We can write one of our variables v_j as: $v_j = \rho_j [v_j]$

Then write $[v_j] = \prod_{i=1}^m F_i^{a_{ij}}$ Example: $[g] = L^1 T^{-2}$

Note that the fundamental units have to be independent:

$$\prod_{i=1}^m F_i^{x_i} = 1 \rightarrow x_1 = x_2 = \dots = x_m = 0.$$

Can define a dimension matrix for our problem:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \quad \text{Tells us how all the variables } v_1, \dots, v_n \text{ depend on the fundamental units.}$$

Dimensionless quantities

What are the dimensionless combinations of our variables?

$$[\Pi_k] = [v_1^{\mu_{k1}} v_2^{\mu_{k2}} \dots v_n^{\mu_{kn}}] = 1$$

This is equivalent to writing

$$\sum_i a_{i1} \mu_{k1} + a_{i2} \mu_{k2} + \dots + a_{in} \mu_{kn} = 0$$
$$\rightarrow A \mu_k = 0$$

That is, the *null space* of the dimension matrix A is spanned by a basis of $n-m$ vectors μ_k , each of which corresponds to an independent dimensionless product of our original variables.

Buckingham Pi theorem

Suppose we can write a physical relation as $\Phi(v_1, \dots, v_n) = 0$

By the restrictions that:

- (a) This should be true independent of any units system we use, provided the fundamental units make sense, and
- (b) The units of Φ have to be some combination of the units of the v_p

we can show that it has to be possible to write the physical relation as: $0 = f(\Pi_1, \Pi_2, \dots, \Pi_{n-r})$

I'll provide a link to a rigorous proof on the course website.

Note that the above means we can also write: $\Pi_1 = g(\Pi_2, \dots, \Pi_{n-r})$

That is, we can always write one dimensionless product as a function of the other dimensionless products.

Example of the Pi theorem in action: a simple pendulum

Consider a pendulum of mass m at the end of a rope of length l , and worry about describing the displacement of the pendulum θ as a function of its initial position, θ_0 .

What are our variables to work with? $\theta, \theta_0, m, l, g, t$.

That's 6 variables.

How many dimensions do we need to worry about?

$[\theta]$	dimensionless	} three dimensions.
$[\theta_0]$	dimensionless	
$[m]$	mass	
$[l]$	length	
$[g]$	length/time ²	
$[t]$	time	

Simple pendulum

So, we should be able to find 3 independent dimensionless parameters.

- Note that m is the only variable to contain the mass unit. Therefore it *cannot* be part of any dimensionless parameters - there's no way to "cancel" out the mass dependence.

Common sense and inspection suggest the following dimensionless parameters:

$$\Pi_1 = \theta$$

$$\Pi_2 = \theta_0$$

$$\Pi_3 = \frac{g}{l} t^2$$

We know we can write $\theta = f\left(\theta_0, \frac{g}{l} t^2\right)$ where f is dimensionless.

Simple pendulum

$$\theta = f\left(\theta_0, \frac{g}{l} t^2\right)$$

If we know that the motion is periodic, we can write

$$f(\theta_0, 0) - f\left(\theta_0, \frac{g}{l} P^2\right) = 0$$

We can solve this equation in principle for P , the period, and find, in terms of another dimensionless function $\Psi(\theta_0)$:

$$P = \Psi(\theta_0) \sqrt{\frac{g}{l}}$$

So, knowing only the units involved and the fact that pendulum motion is periodic, we've found:

- The period is independent of pendulum mass, and scales as the square root of g/l .
- The period in general depends on initial position.

Caveats:

- In exactly the same way that a basis set of vectors is not unique (a new basis may be formed by an appropriate linear combination of an old basis), our choice of dimensionless parameters is not unique!
- The dimensionless parameters must be independent.
- Start with your initial variable list. If you try to use too few of them, you'll find that you can't make the right number of dimensionless parameters.
- Can go through and find dimensionless quantities by brute force, but intuition and inspection are generally used.

Fluid flow in a pipe

Suppose what we want to know is the pressure drop along a length of pipe with fluid flowing through it.

Relevant variables:

$ML^{-2}T^{-2}$	∇P	average pressure gradient
L	D	pipe diameter
L	e	average size of pipe wall roughness
ML^{-3}	ρ	density of flowing fluid
$ML^{-1}T^{-1}$	μ	viscosity of flowing fluid
LT^{-1}	v	velocity of flowing fluid

Fluid flow in a pipe

Six variables, three units implies three independent dimensionless products describe this phenomenon.

Again, an infinite number of ways to choose dimensionless parameters, but we go with:

$$\frac{\rho v D}{\mu} \quad \text{Reynolds number}$$

$$\varepsilon \equiv \frac{e}{D} \quad \text{relative roughness}$$

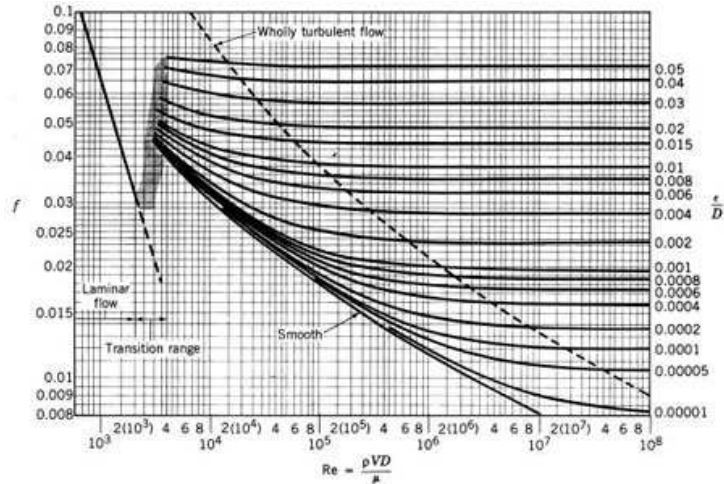
$$\frac{D \nabla P}{\rho v^2} \quad (\text{no name})$$

Using the Pi theorem, we can write this as $\frac{D \nabla P}{\rho v^2} = f(\text{Re}, \varepsilon)$

or, playing with numerical factors, $\nabla P = \frac{2 \rho v^2}{D} f(\text{Re}, \varepsilon)$

Moody plot

$$\nabla P = \frac{2\rho v^2}{D} f(\text{Re}, \varepsilon)$$



Here f is called Fanning's friction factor.

Note that this allows the scaling of numerous experiments onto a single plot, and gives us important similitude information....

Stokes' drag law

How fast does a spherical particle fall in a viscous fluid, as a function of radius?

Relevant variables:

L	R	particle radius
ML^{-3}	ρ_p	particle density
LT^{-2}	g	gravitational acceleration
ML^{-3}	ρ_f	density of fluid
$ML^{-1}T^{-1}$	μ	viscosity of fluid
LT^{-1}	v	terminal velocity of particle

Stokes' drag law

Again, six variables, three dimensions implies three dimensionless products.

$$\Pi_1 = \frac{\rho_p}{\rho_f} \quad \Pi_2 = \frac{v^2}{Rg} \quad \Pi_3 = \frac{\rho_f v R}{\mu}$$

\uparrow \uparrow
 “Froude number” Reynolds number

$$\frac{v^2}{Rg} = f\left(\frac{\rho_f v R}{\mu}, \frac{\rho_p}{\rho_f}\right)$$

Being clever, and knowing the physics at work, we can rewrite:

$$v = \frac{R^2 \rho_f g}{\mu} \psi\left(\frac{\rho_p}{\rho_f}\right)$$

Stokes' drag law

$$v = \frac{R^2 \rho_f g}{\mu} \psi\left(\frac{\rho_p}{\rho_f}\right)$$

Better even than that, we know that if the two densities are equal the particle should remain motionless!

So, knowing that what we're really doing is buoyancy, we know that g only shows up as:

$$g(\rho_p - \rho_f)$$

The only way for this to be true is if we can write:

$$v = C \frac{R^2 \rho_f g}{\mu} \left(\frac{\rho_p}{\rho_f} - 1\right)$$

where C is just some number. For a spherical particle, it turns out from detailed calculations that C is $2/9$.

Again, with a bit of intuition and dimensional analysis, we got quite far!

Atomic bomb blast

A very famous example: when the atomic bomb was tested at Alamogordo, NM in 1945, a number of high speed photographs were taken.

After the war, these pictures appeared in a 1947 issue of Life Magazine. At this time, the yield of the bomb was still classified.

However, based on these pictures, British (and Soviet) scientists were able to come up with a very good estimate for the bomb yield!

Let's give this a try....

Atomic bomb blast

Assume a total amount of energy E dumped into an infinitesimal volume very rapidly.

Assume a resulting spherical shockwave of radius $r(t)$ expanding into the surrounding undisturbed air of density ρ .

We have four parameters:

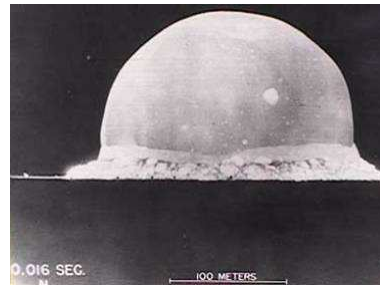
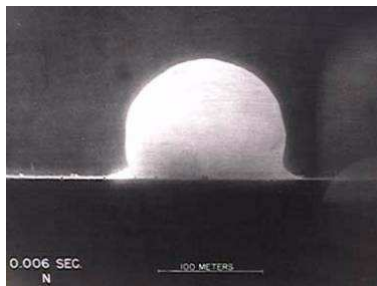
ML^2/T^2	E	bomb yield
L	r	shock radius
T	t	time
M/L^3	ρ	air density

and three units, which means there should just be *one* dimensionless parameter(!) that remains constant during this process!

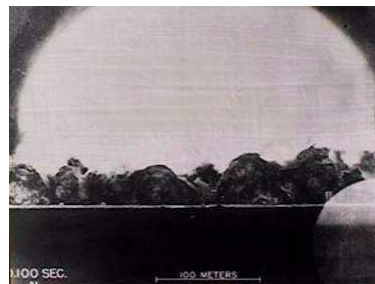
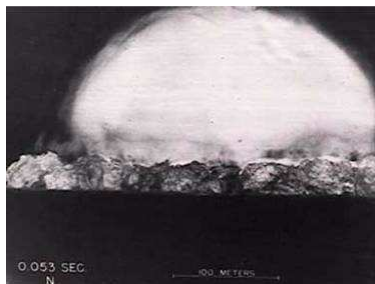
Atomic bomb blast

Plugging in, we can quickly find that:

$\frac{Et^2}{\rho r^5}$ should be constant during the expansion of the shockwave.



Atomic bomb blast



Assuming the constant of proportionality is just 1, and using the scale bar so thoughtfully provided by the Army, one can calculate that the yield of the Trinity test was around 25 kT.

Because of the beauty of scaling, one can actually do tests with, say, dynamite to determine the actual value of that numerical coefficient.

Dimensionless numbers and their interpretation

We've shown that dimensional analysis can be very useful for getting information out of complicated systems without necessarily performing a huge calculation.

An added ingredient, however, to avoid picking a really awkward set of dimensionless products, is intuition.

What follows are some common dimensionless numbers and their interpretations....

Reynolds number:

$$\frac{\rho v d}{\mu} \sim \frac{(\rho v^2 d^2)}{(\mu v / d)(d^2)}$$

inertial forces
viscous forces

Dimensionless numbers and their interpretation

Froude number

$$\frac{v^2}{Rg} \sim \frac{\rho v^2}{\rho g R}$$

kinetic energy
gravitational potential energy

Summary

- Dimensional consistency places significant (if subtle) constraints on the relationships between quantities described by dimensioned variables.
- Dimensional analysis can give us an impressive amount of information when supplemented with a little physical intuition and insight.
- Understanding dimensionless quantities and their relationships is essential for similitude tests....

Fluid mechanics: an overview

Several different regimes of fluid mechanics and dynamics:

- Viscous flow vs. inviscid flow
- Steady vs. unsteady flow
- Laminar vs. turbulent flow
- Incompressible vs. compressible flow
- Open vs. confined flow

Most relevant to microfluidics:

Viscous, steady, laminar, incompressible flow.

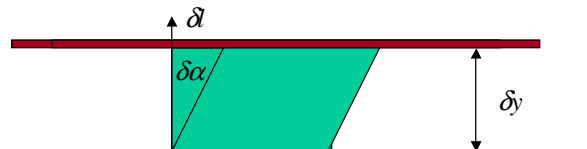
In open flow, must worry about surface tension, surface chemistry effects.

Two main ways of approaching fluids problems:

- Control volume analysis
- Eulerian approach (following fluid elements)

Viscosity

Defined by considering the stresses involved in shearing a volume of fluid:



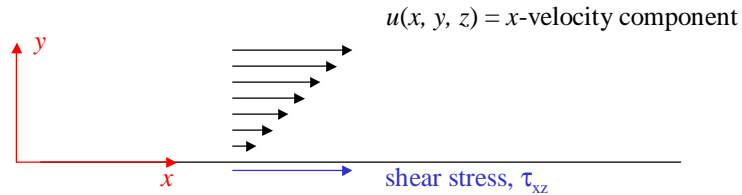
$$\delta\alpha = \frac{\delta l}{\delta y} \rightarrow \frac{\partial\alpha}{\partial t} = \frac{\partial(\partial l / \partial t)}{\partial y} = \frac{\partial u}{\partial y}$$

For Newtonian viscosity, shear stress is linearly proportional to the strain rate:

$$\tau_{xz} = \mu \frac{\partial u}{\partial y}$$

Viscosity

Thinking in terms of velocity fields,



Newtonian viscosity defined as: $\tau_{xz} = \mu \frac{\partial u}{\partial y}$

The *kinematic viscosity* is another useful quantity: $\nu \equiv \frac{\mu}{\rho}$

Viscosity is sometimes referred to as a diffusion coefficient for momentum.

Because a force is actually a rate of change of momentum, a stress is a *momentum flux*. In that sense the shear stress is a flux of x momentum in the $-y$ direction, driven by a gradient in the concentration of x momentum.

Newtonian vs. nonNewtonian

Newtonian fluids are those for which the viscosity is independent of strain rate.

NonNewtonian fluids are certainly possible, for which

$$\mu = \mu\left(\frac{\partial \alpha}{\partial t}, t\right) = \mu\left(\frac{\partial u}{\partial y}, t\right)$$

Several types:

- Pseudoplastic - viscosity drops with increasing shear rate (shear-thinning; e.g. mayonnaise, toothpaste)
- Dilatant - viscosity increases with increasing shear rate (shear-thickening; e.g. corn starch in water, silly putty)
- Thixotropic - viscosity drops with time under constant shear
- Rheopexic - viscosity increases with time under constant shear (rare)

NonNewtonian fluids can be important for microfluidics! Colloids and other suspensions (e.g. blood) are often pseudoplastic / thixotropic.

Incompressible vs. compressible

Much of what we care about is flow of liquids.

Under our conditions of interest, most liquids are practically incompressible: ρ is constant in space and time.

- Note that for gas-based microfluidic devices (microscale turbines, refrigerators, etc.) this may not be a reasonable approximation.
- Also note that if we assume incompressibility, we are ignoring potential physics in longitudinal sound waves.

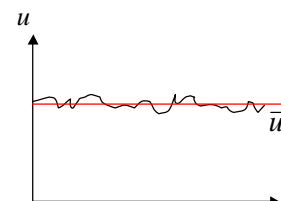
Turbulence

You know from experience that the details of fluid flow can be incredibly complicated - just look at the evolution of a puff of smoke in air, or a drop of cream in coffee.

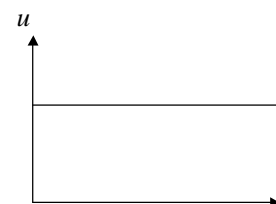
The cascade of detailed fluid motion over a broad range of length scales superposed on top of background average motion is called turbulence.

We are fortunate in two respects:

- Control volume analysis lets us treat average behavior of steady turbulent flow readily.
- In micro- and nanofluidic systems, we almost never need to worry about turbulence!



steady turbulent flow



laminar flow

Laminar flow

Most micro- and nano-fluidic devices are in the laminar regime.

This regime is typified by *low Reynolds number*.

Reynolds number:

$$\frac{\rho v d}{\mu} \sim \frac{(\rho v^2 d^2)}{(\mu v / d)(d^2)}$$

inertial forces
viscous forces

- **Viscous forces dominate**, resulting in a smooth velocity field that can sometimes even be solved analytically, given correct boundary conditions.

- No mixing of adjacent fluid layers (except by particle diffusion, which still happens).



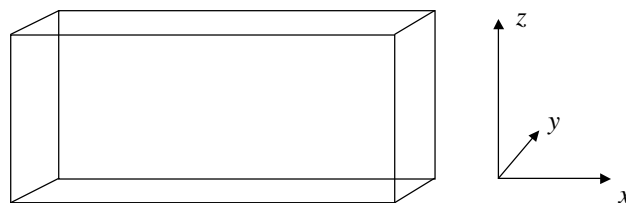
Kenis *et al.*, Science **285**, 83 (1999)

Control volume analysis - steady flow

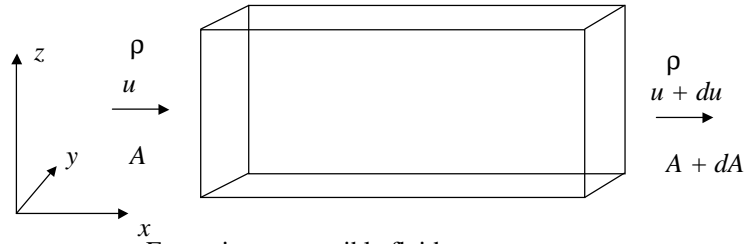
Draw a fixed box around a volume of interest and consider the conservation laws that must be obeyed:

- Total mass of fluid must be conserved.
- Total momentum must be conserved: rate of change of momentum of the fluid flowing through the volume must be made up by the forces acting on the fluid.
- Total energy must be conserved (we'll get to this one later).

Start off with 1d case:



Continuity



For an incompressible fluid,

$$\rho A u = \rho (A + dA)(u + du)$$

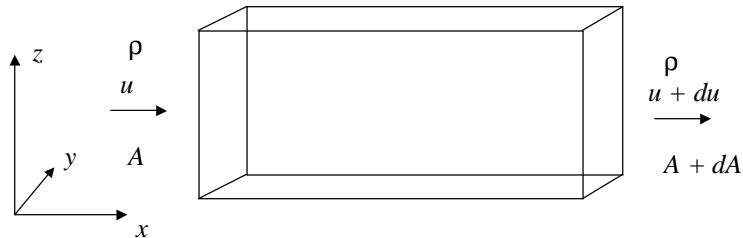
In integral form,
$$0 = \frac{\partial}{\partial t} \int_{cv} \rho dV + \int_{cs} \rho \mathbf{u} \cdot d\mathbf{A}$$

This should be old-hat for you to convert to differential form:

$$\nabla \cdot (\rho \mathbf{u}) = 0$$

For a compressible fluid,
$$\nabla \cdot (\rho \mathbf{u}) = -\frac{\partial \rho}{\partial t}$$

Momentum equation



Now consider the *momentum flux* in and out of the control volume.

x -momentum in left side: $A(\rho u)u$

x -momentum out right side: $(A + dA)(\rho(u + du))(u + du)$

Difference has to be equal to total forces (body + surface) acting on the fluid in the volume.

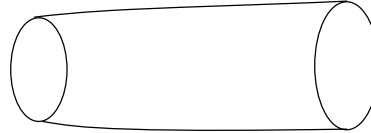
In integral form,
$$\mathbf{F}_s + \mathbf{F}_b = \frac{\partial}{\partial t} \int_{cv} (\rho \mathbf{u}) dV + \int_{cs} (\rho \mathbf{u}) \mathbf{u} \cdot d\mathbf{A}$$

Momentum equation along stream lines, no viscosity

In laminar flow, can think about stream lines - trajectories traced out by, e.g., tag particles in flow.

Stream lines do not cross each other, so one may define a control volume that follows stream lines and only have to worry about fluxes of matter out the ends.

Say s is coordinate along direction of stream line.



$$\rho A u_s = \rho (A + dA)(u_s + du_s)$$

An example of a force acting on the fluid: hydrostatic pressure.

Assume for now that the pressure is uniform across each end.

Pressure force term: $F_s = pA - (p + dp)(A + dA) + (p + \frac{dp}{2})(dA)$

Momentum equation along stream lines, no viscosity

Simplifying the pressure term,

$$F_s = -Adp - \frac{1}{2}dpdA$$



Body force? How about gravity:

$$F_b = -\rho g \sin \theta dV = -\rho g \sin \theta (A + \frac{dA}{2})ds = -\rho g (A + \frac{dA}{2})dz$$

Momentum flux will be just out the ends:

$$F_s + F_b = u_s (-\rho u_s A) + (u_s + du_s)(\rho(u_s + du_s)(A + dA))$$

Recall, $\rho A u_s = \rho (A + dA)(u_s + du_s)$

So, we're left with:

$$-Adp - \frac{1}{2}dpdA - \rho g Adz - \cancel{\frac{1}{2}\rho g dAdz} = \rho u_s A du_s$$

Bernoulli equation

Regrouping terms, we get $-\frac{dp}{\rho} - g dz = u_s du_s$

or $\frac{dp}{\rho} + g dz + d\left(\frac{1}{2}u_s^2\right) = 0$ where these differentials are taken along a stream line.

Dropping the subscript and integrating, we see that for steady, inviscid, incompressible flow along a streamline,

$$\frac{p}{\rho} + g dz + \left(\frac{1}{2}u_s^2\right) = \text{const}$$

This is the Bernoulli equation that you've probably seen somewhere before.

Remember, this comes from solving simultaneously the continuity and momentum equations for a differential control volume along a streamline, under quite restrictive conditions.

Control volume summary

- Define some volume of interest.
- Add up mass fluxes in and out of volume. Continuity will provide a condition on the velocities (for incompressible flow) and flow areas to assure conservation of mass.
- Add up momentum fluxes in and out of volume. The net rate of change of momentum of the fluid going through the volume must equal the total force on the fluid from surfaces and "body" forces.
- For more complicated situations, must keep track of all different components of momentum!
- For specific case of steady, incompressible, inviscid flow along streamlines, can use Bernoulli equation.

Eulerian analysis

Instead of a fixed volume, consider following an individual fluid particle as it passes through the velocity field assumed to exist everywhere.

That particle has to obey Newton's laws, and it can be deformed (rotated, sheared, stretched).

So, at time t have particle at position x, y, z , with a velocity $\mathbf{u}(x, y, z, t)$.

An instant later, it's velocity should be given by

$$\mathbf{u}(x + \delta x, y + \delta y, z + \delta z, t + \delta t)$$

Use the chain rule:

$$d\mathbf{u}_p = \frac{\partial \mathbf{u}}{\partial x} \delta x + \frac{\partial \mathbf{u}}{\partial y} \delta y + \frac{\partial \mathbf{u}}{\partial z} \delta z + \frac{\partial \mathbf{u}}{\partial t} \delta t$$

Acceleration of the particle is then

$$\mathbf{a}_p = \frac{d\mathbf{u}_p}{dt} = \frac{\partial \mathbf{u}}{\partial x} \frac{\delta x}{dt} + \frac{\partial \mathbf{u}}{\partial y} \frac{\delta y}{dt} + \frac{\partial \mathbf{u}}{\partial z} \frac{\delta z}{dt} + \frac{\partial \mathbf{u}}{\partial t}$$

Eulerian analysis

If we call the components of the velocity field u, v, w , then

$$\frac{D\mathbf{u}}{Dt} \equiv \mathbf{a}_p = \frac{d\mathbf{u}_p}{dt} = \frac{\partial \mathbf{u}}{\partial x} u + \frac{\partial \mathbf{u}}{\partial y} v + \frac{\partial \mathbf{u}}{\partial z} w + \frac{\partial \mathbf{u}}{\partial t}$$

This kind of derivative is called a *total* or *convective* or *advective* derivative.

In differential form,
$$\frac{D\mathbf{u}}{Dt} = \mathbf{u} \cdot (\nabla \mathbf{u}) + \frac{\partial \mathbf{u}}{\partial t} \mathbf{u}$$

local acceleration

“Convective” or “advective” acceleration of particle

Eulerian analysis: momentum equation

Now that we know what the actual acceleration of a particular blob of fluid is, we can figure out what the forces on that particular blob are, and set them equal. This should give us a microscopic form of the momentum equation.

Consider a little volume of fluid $dx dy dz$. The forces acting on the little blob can include a body force (gravity, in our example), as well as forces from stresses acting at the surfaces of the blob.

Look at the x direction for example, and allow the stresses to vary in space!

$$\begin{aligned} F_{sx} = & \left(\sigma_{xx} + \frac{1}{2} \frac{\partial \sigma_{xx}}{\partial x} dx \right) dy dz - \left(\sigma_{xx} - \frac{1}{2} \frac{\partial \sigma_{xx}}{\partial x} dx \right) dy dz \\ & + \left(\tau_{yx} + \frac{1}{2} \frac{\partial \tau_{yx}}{\partial y} dy \right) dx dz - \left(\tau_{yx} - \frac{1}{2} \frac{\partial \tau_{yx}}{\partial y} dy \right) dx dz \\ & + \left(\tau_{zx} + \frac{1}{2} \frac{\partial \tau_{zx}}{\partial z} dz \right) dx dy - \left(\tau_{zx} - \frac{1}{2} \frac{\partial \tau_{zx}}{\partial z} dz \right) dx dy \end{aligned}$$

Eulerian analysis: momentum equation

Simplifying, and writing for all 3 components of the surface forces,

$$\begin{aligned} F_{sx} &= \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) dx dy dz \\ F_{sy} &= \left(\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} \right) dx dy dz \\ F_{sz} &= \left(\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} \right) dx dy dz \end{aligned}$$

To go further, we need to write the stresses in terms of the pressure and the viscosity (which you know shows up because of velocity gradients).

Eulerian analysis: momentum equation

$$\begin{aligned}\tau_{xy} = \tau_{yx} &= \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & \sigma_{xx} &= -p - \frac{2}{3} \mu \nabla \cdot \mathbf{u} + 2\mu \frac{\partial u}{\partial x} \\ \tau_{yz} = \tau_{zy} &= \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) & \sigma_{yy} &= -p - \frac{2}{3} \mu \nabla \cdot \mathbf{u} + 2\mu \frac{\partial v}{\partial y} \\ \tau_{zx} = \tau_{xz} &= \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) & \sigma_{zz} &= -p - \frac{2}{3} \mu \nabla \cdot \mathbf{u} + 2\mu \frac{\partial w}{\partial z}\end{aligned}$$

Recall, also, that for incompressible flow,

$$\nabla \cdot (\rho \mathbf{u}) = -\frac{\partial \rho}{\partial t} \rightarrow \nabla \cdot \mathbf{u} = 0$$

Navier-Stokes equations - incompressible case

Substituting, and adding in gravity, we get:

$$\begin{aligned}F_x &= \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\ F_y &= \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\ F_z &= \rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)\end{aligned}$$

Rewriting the whole momentum equation in a coordinate-free way, assuming constant density and viscosity:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{u}$$

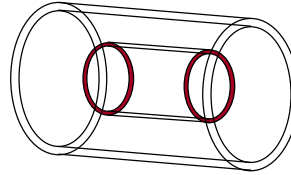
This is the Navier-Stokes equation for viscous, incompressible flow.

Laminar, viscous flow in a circular pipe

We want to find the velocity distribution for “fully-developed” viscous flow down a circular cross-section pipe:

Can solve this either with a control volume analysis, or using Navier-Stokes. First, the former.

Annular control volume, centered on point x . Flow velocity $u(r)$.



$$\text{Force on left end: } \left(p - \frac{1}{2} \frac{\partial p}{\partial x} dx \right) 2\pi r dr$$

$$\text{Force on right end: } - \left(p + \frac{1}{2} \frac{\partial p}{\partial x} dx \right) 2\pi r dr$$

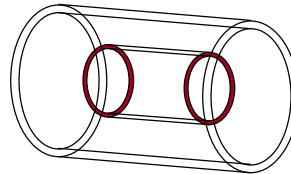
Laminar, viscous flow in a circular pipe

Shear force on inner surface:

$$- \left(\tau_{rx} - \frac{1}{2} \frac{\partial \tau_{rx}}{\partial r} dr \right) 2\pi \left(r - \frac{1}{2} dr \right) dx$$

Shear force on outer surface:

$$\left(\tau_{rx} + \frac{1}{2} \frac{\partial \tau_{rx}}{\partial r} dr \right) 2\pi \left(r + \frac{1}{2} dr \right) dx$$



For fully developed flow, total force on fluid must be zero:

$$\frac{\partial p}{\partial x} = \frac{\tau_{rx}}{r} + \frac{\partial \tau_{rx}}{\partial r} = \frac{1}{r} \frac{\partial (r \tau_{rx})}{\partial r}$$

Integrating w.r.t. r :

$$\begin{aligned} r \tau_{rx} &= \frac{1}{2} r^2 \frac{\partial p}{\partial x} + c_1 & \rightarrow & \frac{\partial u}{\partial r} = \frac{1}{2\mu} r \frac{\partial p}{\partial x} + \frac{c_1}{r\mu} \\ \tau_{rx} &= \mu \frac{\partial u}{\partial r} & \rightarrow & u = \frac{1}{4\mu} r^2 \frac{\partial p}{\partial x} + \frac{c_1}{\mu} \ln r + c_2 \end{aligned}$$

Laminar, viscous flow in a circular pipe

Apply boundary condition that $u = 0$ at $r = R$: (“no slip”)

$$u = \frac{R^2}{4\mu} \left(\frac{\partial p}{\partial x} \right) \left[\left(\frac{r}{R} \right)^2 - 1 \right]$$

Parabolic velocity profile.

Can compute volume flow rate for this case:

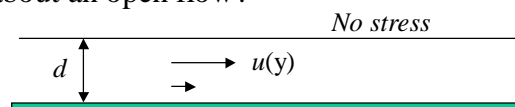
$$Q = \int_0^R u(r) \cdot 2\pi r dr = -\frac{\pi R^4}{8\mu} \left(\frac{\partial p}{\partial x} \right)$$

This kind of fully developed laminar flow is also called [Poiseuille flow](#).

This is the flow regime most commonly encountered in long, closed microfluidic channels.

Note: Could have written Navier-Stokes eqn. in cylindrical coords., and solved with appropriate boundary condition, getting same answer.

What about an open flow?



Again, ignore gravity and assume a driving pressure gradient.

Also assume that the flow is much wider than it is deep.

Do same kind of control volume analysis for a rectangular control volume, and find:

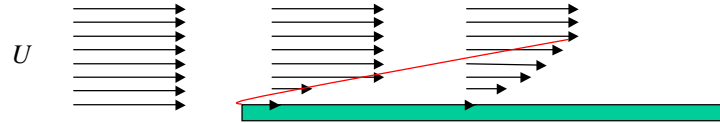
$$\frac{\partial \tau_{yx}}{\partial y} = \frac{\partial p}{\partial x} \longrightarrow \mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}$$

What are correct boundary conditions for this case?

$$u(y=0) = 0, \quad \left. \frac{\partial u}{\partial y} \right|_{y=d} = 0$$

Again, could integrate and use b.c. to find $u(y)$.

Boundary layer



With no pressure gradient, steady viscous incompressible flow over a plate develops a *boundary layer* of fluid moving at reduced velocity compared to the bulk.

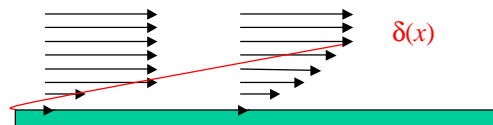
The plate also experiences a shear force.

Exact solution requires solving a boundary value problem:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2}$$

$$u(y=0) = 0, \quad u(y=\infty) = U$$

Boundary layer



Requires numerical solution. Still, casting problem in correct nondimensional variables leads to the expressions

$$\delta(x) \approx \frac{5x}{\sqrt{\rho U x / \mu}} \quad \text{where } u(y=\delta) = 0.99 U.$$

$$\tau_w \approx \frac{0.332 \rho U^2}{\sqrt{\rho U x / \mu}}$$

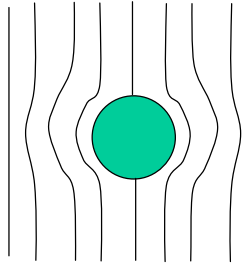
These expressions hold in the *laminar* regime. Far enough downstream, the effective Re in the denominator becomes large ($> \sim 2300$), implying a transition to turbulence.

For water, $\mu \sim 0.001$ Pa-s; for a bulk flow speed of $10 \mu\text{m/s}$, this occurs at a point ~ 230 m downstream = never, in microfluidic case.

Stokes' drag law

What is the drag force on a sphere in a medium moving with uniform flow speed U far from the sphere?

Again, boundary value problem for Navier-Stokes.



Stokes introduced a stream function ψ , defined so that

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \quad u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}$$

Can then rewrite NS equation as:

$$E^2(E^2\psi) = 0, \text{ where } E^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right)$$

Stokes' drag law

$$E^2(E^2\psi) = 0, \text{ where } E^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right)$$

This is a 4th order PDE, requiring four boundary conditions:

$$u_r(r=a) = 0 \quad \text{No flow normal to sphere surface at surface.}$$

Shear stress prop. to velocity gradient at sphere surface.

$$u_r(r \rightarrow \infty) = U \cos \theta$$

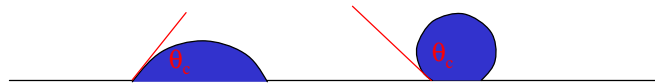
$$u_\theta(r \rightarrow \infty) = -U \sin \theta$$

Final end result: $F_D = 6\pi\mu Ua$

- Very useful for describing forces on small particles (colloid, cells, etc.) in microfluidic systems.
- Quite easy to have drag forces be substantial at low Re!

Surface interactions

- Details of the interactions between fluids and surfaces can be of crucial importance in micro- and nanofluidic environments.
- Obviously, for small volumes of material, the surface to volume ratio can become large, leading to surface forces swamping out body forces.
- Surface interactions often determined by surface chemistry – energetic competition between liquid cohering to itself and wetting the surface.



Contact angle clearly varies depending on affinity of liquid for solid surface.

What is surface tension?

- Surface tension has units of energy per area, or force per length.
- Accounts for the energetic differences per molecule between molecules in the bulk of a liquid and those at the interface between the liquid and another material.
- Can think of it as the energy cost for creating a unit area of that interface.
- Analogous, thermodynamically, to pressure (though defined with a minus sign).

That is, in 3d: $W = -pdV$

For a 2d interface, $W = \sigma dA$

Basic surface tension intuition

Surface energy per unit volume for a spherical drop:

$$\frac{4\pi r^2 \sigma}{(4/3)\pi r^3} = \frac{3\sigma}{r}$$

Clearly surfaces become very expensive at small size scales.

All other things being equal, two small drops will merge to minimize the surface area.

Similarly, small bubbles are energetically very expensive, and will tend to collapse if pressure forces from the vapor inside can't balance the surface tension force.

This bubble collapse can be very intense in terms of energy density - cavitation, sonoluminescence.

Water: $\sigma_{LV} = 0.072 \text{ J/m}^2$

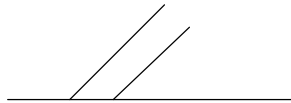
Hg: $\sigma_{LV} = 0.436 \text{ J/m}^2$

Octane: $\sigma_{LV} = 0.022 \text{ J/m}^2$

Surface interactions

Contact angle determined by energetics. Consider moving the interface slightly.

Defining solid-liquid, liquid-air, and solid-air surface tensions as σ_{sl} , σ_{lv} , and σ_{sv} , respectively,



$$\sigma_{sv} - \sigma_{sl} = \sigma_{lv} \cos \theta_c$$

Surface interactions (surface tensions) can be tailored chemically.

Under certain circumstances, they can also be tuned on the fly!

Wetability

$$\sigma_{sv} - \sigma_{sl} = \sigma_{lv} \cos \theta_c$$

Spreading coefficient is defined as $S \equiv \sigma_{sv} - \sigma_{sl} - \sigma_{lv}$

Large positive S implies that the liquid likes to spread.

“High energy” surfaces have large values of σ_{sv} and are energetically expensive.

Surface energies between the liquid and solid come about because of microscopic interactions at the interface.

The extreme version of this would be covalent bonding, but for most liquid/solid interactions, it will be some variant of Van der Waals.

Wetability

So, if the interaction potential is what determines surface affinities, we can write:

$$\sigma_{sl} = \sigma_{sv} + \sigma_{lv} - V_{SL}$$

where we’ve defined an attractive interaction to give a positive V_{SL} .

Similarly, two identical liquid regions that then touch have *no* surface energy associated with that interface:

$$0 = \sigma_{lv} + \sigma_{lv} - V_{LL}$$

So, combining these we can see that the wetting coefficient is then

$$S = V_{SL} - V_{LL}$$

This is also pretty much why teflon is effectively water-proof: the hydrogen bonding interaction between water molecules is much more strong than the VdW interaction between the water and the fluoropolymer.

Wetability

If we're really talking about Van der Waals, the potential between two different materials is related to their polarizabilities:

$$V_{LL} \sim \alpha_L^2 \quad V_{SL} \sim \alpha_S \alpha_L$$

Since the polarizability of most hard solids is larger than those of liquids, wetting ends up being favored.

Complications:

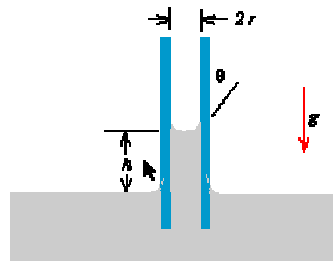
- Polarizabilities are frequency-dependent, so there are corrections for this to the above.
- As we said, other stronger interactions can be relevant (hydrogen bonding, chemistry)
- Liquid microstructure near the surface can be quite different than in the bulk (extremely important for nanoconfined fluids).

Capillary action

The interaction between the liquid and the solid surface can be powerful enough to draw liquid up against gravity: this is capillary action.

$$\rho g (\pi r^2 h) = 2\pi r \sigma_{sl} \cos \theta_c$$

$$h = \frac{2\sigma_{sl} \cos \theta_c}{\rho g r}$$



Note that for nonwetting interactions, the contact angle is greater than 90 degrees, so h is actually negative!

This kind of capillarity is one way to measure surface tensions.

Strength of capillary action: assuming zero contact angle, water in a 1 micron tube would get drawn up 15 m (!).

Can surface interactions be tuned?

Yes, in several ways, some irreversible, some not:

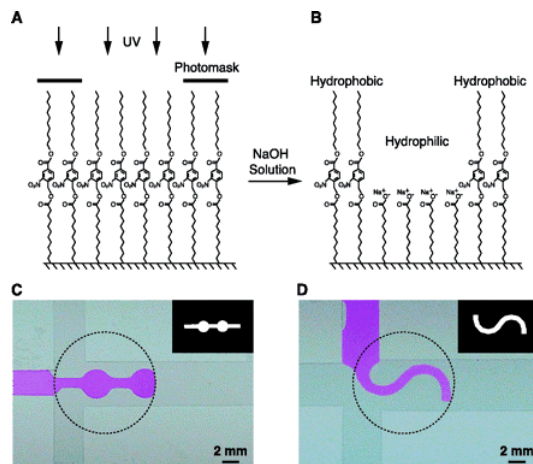
- Chemical modification of liquid surface - surfactants.
- Chemical modification of surface termination
 - Self-assembled monolayer
 - Temperature-dependent monolayer
 - Optically active monolayer
 - Electrochemically modified interactions
- Electrokinetic effects
- Electrostatics and solutions

Self-assembled monolayers

Using soft lithography / microcontact printing with PDMS stamps, can pattern surfaces into hydrophilic and hydrophobic regions.

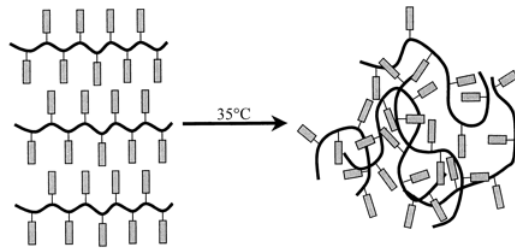
Also, optical patterning can be done to modify the SAM once it's in place.

At right, dye in aqueous solution only goes where hydrophobic surface termination has been removed.



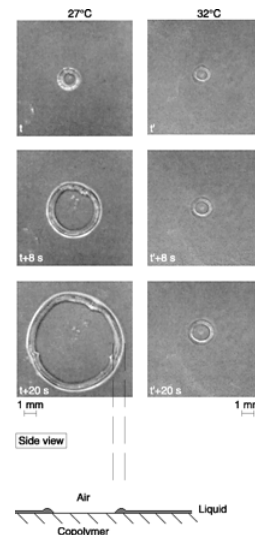
Zhao *et al.*, Science **291**, 1023 (2001).

Temperature-dependent layer



Coat the surface with a liquid crystal polymer film. Below transition temperature, film is ordered and presents a hydrophobic sidegroup.

Above transition, film is disordered: more hydrophilic interaction.

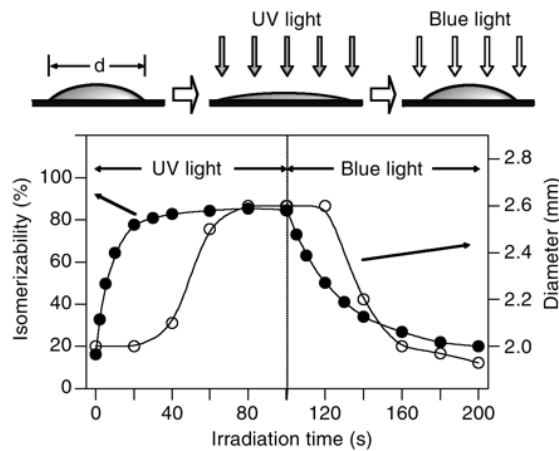


Crevoisier *et al.*, Science **285**, 1246 (1999).

Optically active monolayer

In this case, SAM has two isomeric forms between which it may be reversibly switched optically.

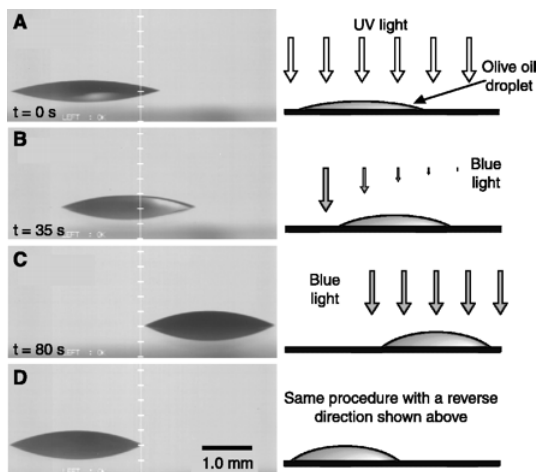
UV light conformation favors wetting of the surface by oil, while blue light conformation favors dewetting.



Ichimura *et al.*, Science **288**, 1624 (2000).

Optically active monolayer

Because these surface modifications are reversible, it is possible to move droplets of liquid around the surface purely by optical modification of the surface energies.



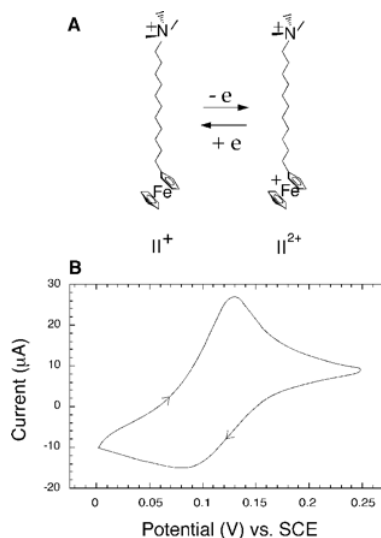
Ichimura *et al.*, *Science* **288**, 1624 (2000).

Electrochemically modified interactions

Can also use electronic modification of interfaces.

At right, a surfactant molecule that can be switched between a relatively low and high surface tension state depending on its ionization state.

Using this on an electrolyte / electrode interface, it is possible to modify the surface energy on-the-fly by electrochemistry.



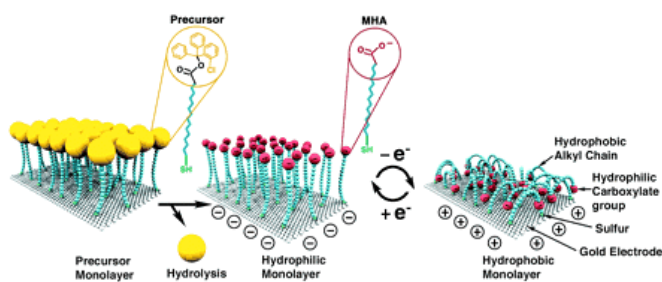
Gallardo *et al.*, *Science* **283**, 57 (1998).

Electrostatically modified interactions

Dense alkanethiol SAMs are hydrophobic.

Furthermore, because of their close packing, applying an electric field to the SAM is ineffective at moving the SAM around if the monolayer is dense and well ordered.

Other SAMs can be designed, however:

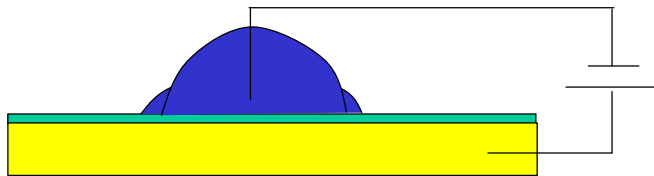


Lahann *et al.*, Science **299**, 371 (2003).

Electrostatically modified interactions

Under the right circumstances, similar to those discussed on your problem set, it is possible to use basic electrostatics to manipulate wetting. Assume a conductive drop.

The idea is to use electrostatic forces to alter the balance between body and surface effects:



$$\cos \theta(V) \approx \cos \theta(V = 0) + \frac{1}{2} \frac{\kappa \epsilon_0}{\sigma_{lv}} \frac{V^2}{d}$$

This is called *electrowetting*.

Solution electrostatics

Now that we've introduced electrically conductive liquids as a possibility, we have to contend with the actual electrostatic interactions between the liquid and the surrounding materials.