
Dimensional Analysis, Scale Analysis, and Similarity Theories

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Dimensions and units

So far as we know, nature can be described using the four “dimensions” or “primary quantities” of length, time, mass, and electric charge. Here the word “dimension” is used to refer to aspects of nature that are independent in the sense that they are not inter-convertible. Length cannot be re-scaled as a mass. Time cannot be re-scaled as an electric charge.

All physical quantities can be measured as combinations of these four dimensions. For example, velocity is length divided by time, and energy can be expressed as mass times length squared divided by time squared.

Temperature is an anomaly; there has been a debate about whether or not it should be included as a primary quantity independent of mass, length, and time (Huntley, 1967). It is a statistic of the molecular motions, which can be defined in terms of energy per unit mass. In atmospheric science, temperature is usually treated as a fifth primary quantity.

Units are different from dimensions. The various primary quantities are measured using units, which can be defined in very arbitrary ways. For example, length can be measured using meters, feet, furlongs, or the size of Henry VIII’s foot. Today, scientists almost always use the metric or “International” system of units.

Dimensions are physical; units are not. It is possible to define natural or fundamental units (e.g., Barrow, 2002; Wilczek, 2005, 2006 a, b), but these are not convenient for use in atmospheric science.

The starting point for the discussion below is that physical principles must be independent of the choice of units. For example, Newton’s law $F = ma$, i.e., force equals mass times acceleration, must predict the same physical phenomena whether we use English units or SI units.

We need some definitions:

1. *Physical quantity*: A quantitative conceptual property of a physical system, which can be expressed numerically in terms of one or more standards. Example: The radius of the Earth.

2. *Primary quantities*: A set of quantities (hereafter called q s) chosen arbitrarily for the description of a problem, subject to the constraint that the units of measurement chosen for the quantities can be assigned independently. Primary quantities are sometimes called “fundamental quantities.” Examples: Length, time, mass.
3. *Dimension*: The relationship of a derived physical quantity to whatever primary quantities have been selected. Example: Velocity = Length / Time.
4. *Standard*: An arbitrary reference measure adopted for purposes of communication. Example: meter.
5. *Unit*: An arbitrary fraction or multiple of a standard, used to avoid inconveniently large (or small) numbers. Example: Kilometer. All standards are units, but not all units are standards.
6. *Extraneous standard*: A standard that is irrelevant for a particular problem. Example: The length of Henry VIII’s foot, which is irrelevant except when Henry VIII visits a shoe store.
7. *Extraneous unit*: a unit based on an extraneous standard. Example: 1 mile = 5280 ft.
8. *Dimensionless quantity*: A quantity that is expressed in units *derivable from the problem* (that is, *not* in extraneous units). There are no *intrinsically* or universally dimensionless quantities. A quantity is dimensionless or not only with respect to a particular problem. Example: Rossby number.
9. *Dimensional analysis*: The process of removing extraneous information from a problem by forming dimensionless groups.
10. *Nondimensionalization*: Conversion of a system of dimensional equations to a system that contains only nondimensional quantities.
11. *Scale analysis, sometimes called “scaling:”* Using *chosen* numerical values for the dimensional parameters to compare the orders of magnitude of various terms of a system of non-dimensional equations. A scale analysis can only be performed when the governing equations are known.
12. *Similar systems*: Those for which the dimensionless quantities have identical values, even though the dimensional quantities may be very different. Example: A model airplane in a wind tunnel can be “similar” to a full-size airplane in the atmosphere.
13. *Similarity theory*: A theory based on the hypothesis that functional relationships exist among the nondimensional parameters describing a physical system. The functions themselves must be determined empirically. Similarity theories can be

useful when the desired functions cannot be derived from the governing equations, or even when the governing equations are not known.

Consistent use of dimensional and nondimensional quantities in equations

It makes no sense to add or subtract quantities that have different dimensions. You cannot add a length to a mass. One consequence of this is that all terms of an equation must have the same dimensions. This is called the requirement of *dimensional homogeneity*.

An exponent must be nondimensional, because it is the number (pure number) of times that something is multiplied by itself.

Similarly, the arguments of (inputs to) mathematical functions must be nondimensional. Examples of such functions include exponentials, logarithms and trigonometric functions. You can take the sine of 2 but you cannot take the sine of 2 meters.

We often see dimensional quantities as the arguments of logarithms, even though it makes no sense. For example, you might see something like this:

$$\frac{1}{\theta} \frac{D\theta}{Dt} = \frac{D}{Dt}(\ln\theta),$$

where θ is the potential temperature. A more correct (but slightly longer and slightly more complicated) statement is this:

$$\frac{1}{\theta} \frac{D\theta}{Dt} = \frac{D}{Dt} \left[\ln \left(\frac{\theta}{\theta_{ref}} \right) \right],$$

where θ_{ref} is a constant reference value. It seems that some people just don't want to be bothered with including θ_{ref} and then saying "where θ_{ref} is a constant reference value." Too much trouble.

The Buckingham Pi Theorem

The fundamental theorem of dimensional analysis is due to Buckingham, and is stated here without proof:

The Buckingham Pi Theorem

If the equation

$$\phi(q_1, q_2, \dots, q_n) = 0 \quad (1)$$

is the only relationship among the q_i , and if it holds for any arbitrary choice of the units in which q_1, q_2, \dots, q_n are measured, then (1) can be written in the form

$$\phi(\pi_1, \pi_2, \dots, \pi_m) = 0, \quad (2)$$

where $\pi_1, \pi_2, \dots, \pi_m$ are independent dimensionless products of the q 's.

Further, if k is the minimum number of primary quantities necessary to express the dimensions of the q s, then

$$\boxed{m = n - k}. \quad (3)$$

Since $k \geq 0$, (3) implies that $m \leq n$. According to (3), the number of dimensionless products is the number of dimensional parameters minus the number of primary quantities.

Another way of writing (2) is

$$\phi'(\pi_1, \pi_2, \dots, \pi_m; 1, \dots, 1) = 0 \quad (4)$$

where the number of “1s” appearing in the argument list is k . *Clearly the 1s carry no information about the functional relationship among the π s, so that we can just omit them, as was done in (2).* In (4), the 1s clearly represent “extraneous” information, which entered the problem through the extraneous units used to measure the q s. By dropping the extraneous information, we simplify the problem. We don't actually change the problem. We just boil it down to its essentials.

The choice of the q s can be made by inspection of the governing equations (if known), or by thinking about the physical system of interest.

The dimensions of the q s can be determined in terms of chosen primary quantities. The primary quantities can be chosen arbitrarily, provided that their units can be assigned independently. It is necessary to choose enough primary quantities to ensure that nondimensional combinations can be formed in all cases.

Here is an example of a dimensional analysis, based on the theory of Rayleigh-Bénard convection. Consider thermal convection of a shallow fluid that fills a laboratory tank, with no mean flow or rotation. The depth of the tank is h . The linearized governing equations are:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) u' &= - \frac{\partial}{\partial x} \left(\frac{p'}{\rho_0} \right) \\ \left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) v' &= - \frac{\partial}{\partial y} \left(\frac{p'}{\rho_0} \right) \\ \left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) w' &= - \frac{\partial}{\partial z} \left(\frac{p'}{\rho_0} \right) + g \alpha T' , \\ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} &= 0 , \\ \left(\frac{\partial}{\partial t} - \kappa \nabla^2 \right) T' &= -w' \Gamma . \end{aligned}$$

Here x , y , and z are the spatial coordinates, u , v , and w are the corresponding components of the velocity vector, T is temperature, p is pressure, ρ_0 is a constant reference-state density, g is the acceleration of gravity, α is a thermal expansion coefficient, ν is the molecular viscosity, κ is the molecular conductivity, and $\Gamma \equiv \overline{\partial T} / \partial z$ is the rate at which temperature increases upward in the mean state. An overbar denotes the mean state, and a prime denotes the departure from the mean state.

By inspection of the equations, we see that the six dimensional *parameters* of the problem are

$$g, \Gamma, h, \nu, \kappa, \alpha .$$

Here g is gravity, Γ is the lapse rate, h is the depth of the fluid (which enters through the boundary conditions), ν is the molecular viscosity, κ is the molecular thermal conductivity, and α is a parameter that measures the amount of thermal expansion per unit temperature change. These six parameters are determined by the design of the laboratory experiment, e.g., the depth of the tank and the choice of convecting fluid (water, air, oil, etc.).

We don't include the dependent variables u , v , w , p , or T in the list of parameters, because they are part of the solution, and so depend on the parameters listed above. We don't include ρ_0 in the list because it appears only in the combination p' / ρ_0 , and so we just think of p' / ρ_0 as one of the dependent variables.

As primary quantities, we choose length (L), time (T), and temperature (Θ). There are three primary quantities, and six dimensional parameters, so the π -theorem tells us that we should be able to eliminate $6 - 3 = 3$ pieces of extraneous information.

We can tabulate the dimensions of the q s as follows:

Quantity	Dimensions		
primary quantities	L	T	Θ
g	1	-2	0
α	0	0	-1
Γ	-1	0	1
h	1	0	0
ν	2	-1	0
κ	2	-1	0

As our pertinent (non-extraneous) unit of length, we choose h . Forming products, we systematically eliminate the "lengths" from our set of quantities:

Quantity	Dimensions		
primary quantities	L	T	Θ
gh^{-1}	0	-2	0
α	0	0	-1
Γh	0	0	1
1	0	0	0
νh^{-2}	0	-1	0
κh^{-2}	0	-1	0

As our unit of time, we use $h^2\nu^{-1}$. (We could just as well take $h^2\kappa^{-1}$.) Again forming products, we obtain

Quantity	Dimensions		
primary quantities	L	T	Θ
$gh^3\nu^{-2}$	0	0	0
α	0	0	-1
Γh	0	0	1
1	0	0	0
1	0	0	0
$\kappa\nu^{-1}$	0	0	0

Obviously, to complete the procedure, we simply form the product $\Gamma\alpha h$, and the final version of the table is shown below.

Quantity	Dimensions		
primary quantities	L	T	Θ
$gh^3\nu^{-2}$	0	0	0
$\alpha\Gamma h$	0	0	0
1	0	0	0
1	0	0	0
1	0	0	0
$\kappa\nu^{-1}$	0	0	0

All together, we then have three 1s in the header column of the final version of the table. This means that three pieces of extraneous information have been eliminated, as promised by the π -theorem. But we have the *three* nondimensional combinations

$$gh^3\nu^{-2} \equiv x_1, \quad (5)$$

$$\kappa\nu^{-1} \equiv Pr, \quad (6)$$

and

$$\Gamma\alpha h \equiv x_2. \quad (7)$$

The Rayleigh number is defined by

$$Ra \equiv Pr x_1 x_2 \quad (8)$$

so that we can, alternatively, regard Pr , Ra , and x_1 (say) as our three combinations.

Chandrasekhar (1961) shows that only *two* dimensionless combinations matter for the convection problem, namely Pr and Ra . Why then have we found three? The reason is that, in the governing equations, g and α appear only in the combination $g\alpha$, so they don't have to be *separately* included in our list of q s. Because this reduces the number of q s by one, without reducing the number primary quantities, the π -theorem tells us that the number of dimensionless combinations will also be reduced by one. The lesson is that we should not enter dimensional quantities separately if they enter the equations only in some unique combination.

Scale analysis

In atmospheric science and oceanography, scale analysis is sometimes used to justify various approximations that can aid in the solution of the governing equations. Scale analysis involves the use of nondimensional parameters, but it is quite different from nondimensionalization itself. Scale analysis always starts from a set of governing equations, and it always involves the choice of the numerical values of dimensional scales, based on the intended application of the equations. A given set of equations may be applicable to a wide variety of physical problems, in which case it is possible to scale-analyze them in more than one way.

Consider an important example: The quasi-geostrophic theory of large-scale motions was derived by Charney (1948) using scale analysis. We now perform a simple scale analysis of the equation of horizontal motion, in the spirit of Charney's work. The equation of motion can be written in simplified form as

$$\frac{D\mathbf{V}}{Dt} + f\mathbf{k} \times \mathbf{V} = -\nabla_p \phi . \quad (9)$$

Here \mathbf{V} is the horizontal wind vector. We have omitted friction, for simplicity. The three terms included in (9) suffice to describe the evolution of the large-scale horizontal wind throughout most of the atmosphere. Note that (9) involves a single dimensional parameter, namely the Coriolis parameter, f , which we will treat as a constant here. Also note that the dimensions of mass and temperature do not enter into (9).

We now introduce U as a (dimensional) velocity scale, and write

$$\mathbf{V} = U\hat{\mathbf{V}} ,$$

where the “carat” notation denotes a nondimensional variable. Similarly, let L be a (dimensional) length scale. *Although U and L do not appear in (9), we introduce here them as a way of specifying what type of motion we want to analyze.* We now have a total of three dimensional parameters, namely U , L , and the Coriolis parameter.

We can combine U and L to construct a time scale, $T = L/U$, and write

$$t = T\hat{t} = (L/U)\hat{t} , \text{ and}$$

$$\frac{D}{Dt} = \frac{1}{T} \frac{D}{D\hat{t}} = \frac{U}{L} \frac{D}{D\hat{t}} .$$

Making the various substitutions into (9), we can rewrite it as

$$\frac{U^2}{L} \frac{D\hat{\mathbf{V}}}{D\hat{t}} + f\mathbf{k} \times U\hat{\mathbf{V}} = -\nabla_p \phi.$$

Dividing through by fU , we find that

$$\text{Ro} \frac{D\hat{\mathbf{V}}}{D\hat{t}} + \mathbf{k} \times \hat{\mathbf{V}} = -\frac{\nabla_p \phi}{f_0 U}, \quad (10)$$

where $\text{Ro} \equiv \frac{U}{Lf}$ is the (nondimensional) Rossby number. Equation (10) is the nondimensional form of (9).

In converting the dimensional equation (9) to the nondimensional form (10), we have not really changed anything. The problem to be solved remains the same. We have reduced the number of parameters from three to one. This is consistent with the Buckingham Pi Theorem. Our nondimensionalization has revealed a single nondimensional parameter of physical importance, namely the Rossby number.

As mentioned earlier, two systems are said to be “similar” if their nondimensional parameters have the same values, even though their dimensional parameters are quite different. The most familiar example is a wind-tunnel, in which small models are used to investigate the aerodynamic properties of much larger, full-scale aircraft. We can also construct laboratory analogues of the atmosphere. The simplified and incomplete scale analysis presented above suggests that, in order to be a useful analogue of the atmosphere, the laboratory system must be designed to have the same Rossby number as the atmosphere. A more complete scale analysis would show the importance of several additional nondimensional parameters. Similarity theories are discussed in the next section.

Nondimensionalization is the first step of a scale analysis. The second step is to choose the numerical values of the scales. *They are chosen so that the nondimensional dependent variables are of order one, for the problem of interest.* For example, suppose that we want to investigate large-scale midlatitude motions in the Earth’s atmosphere. We choose U to be 10 m s^{-1} , L to be 10^6 m , and f to be 10^{-4} s^{-1} , because these are about the right size for midlatitude large-scale motions.

Using these scales, we find that $\text{Ro} = 0.1$. This means that the (leading) acceleration term of (10) is an order of magnitude smaller than the Coriolis term. In order for the equation to be satisfied, the Coriolis term has to be balanced by the only remaining term, which represents the pressure-gradient force. Then (10) reduces to

$$\hat{f}\mathbf{k} \times \hat{\mathbf{V}} \cong -\frac{\nabla_p \phi}{f_0 U},$$

which is an expression of geostrophic balance. Since $\nabla_p \sim 1/L$, we can conclude that the horizontal variations of ϕ on pressure surfaces, denoted by $\delta\phi$, satisfy

$$\delta\phi \sim fUL.$$

The scale analysis has thus allowed us to deduce the order of magnitude of $\delta\phi$, and to conclude that for large-scale motions in middle latitudes the acceleration term of (10) is smaller (in absolute value) than the other two terms.

The conclusions drawn above depend on our *choices* for the numerical values of the dimensional parameters U , L , and f . If we were interested in a different meteorological phenomenon, such as turbulent eddies in the boundary layer, we would make different choices for the numerical values of U and L , and arrive at different conclusions.

Similarity theories

Scale analysis makes use of the equations that describe relationships among variables of interest in a physical system. In principle, such relationships are determined by the solutions of the governing differential equations, but unfortunately there are many cases in which we don't know how to solve the differential equations in order to derive the desired relationships.

For example: The wind experiences a drag force as it moves near the Earth's surface. We can hypothesize that there exists (i.e., it is possible to find) a "formula" that tells us how to compute the drag given the wind speed, the lapse rate of temperature, the roughness of the ground (or ocean), and perhaps several other dimensional quantities. Unfortunately, we don't know how to derive the formula from the equation of motion and the other basic physical principles of our science, but we believe that the formula exists.

Similarity theories aim to find such formulas by means of hypotheses, which could be described as inspired guesses. The first step is to write down the list of dimensional parameters that appear in the governing equations, including the boundary conditions. In general, this would include such things as the spatial coordinates and time, as well as parameters like the Earth's rotation rate and the acceleration of gravity. The list of dimensional parameters can be very long, especially when you consider that it could in principle include detailed information about such things as the Earth's topography.

We want to make the list shorter. As a first step, we can hypothesize, often with good reason, that some of the dimensional quantities are irrelevant to the problem at hand. For example, I'm pretty sure that the height of Mt. Everest is irrelevant to the drag that the air experiences as it flows over Kansas City. As a less ludicrous example, we might omit the density

of water vapor from the list of dimensional parameters to be used in our search for the drag formula.

Having settled on a reasonably short list of dimensional parameters, we then nondimensionalize. The Buckingham Pi Theorem tells us that nondimensionalization will yield an even shorter list of nondimensional parameters.

We then assert that there are functional relationships among the nondimensional combinations of interest. If the number of parameters is small enough, we have a reasonable hope of finding the functional relationships empirically. This is exactly what has been done in developing the famous Monin-Obukhov similarity theory, which provides very useful empirical formulas for determining the surface fluxes of momentum and sensible heat in terms of the mean wind and temperature profiles near the Earth's surface.

As a second example, which connects with the discussion of the preceding section, we can imagine two laboratory convection experiments with different numerical values for h , ν , κ , $g\alpha$, and Γ . We would like to find a formula that relates the upward heat flux in the convection tank to the nondimensional parameters of the problem, namely Pr and Ra . We don't know how to derive such a formula from first principles. A similarity hypothesis might be that if the nondimensional parameters Pr and Ra are the same in the two experiments, then the (nondimensional) heat flux will also be the same.

This turns out to be true. If the suitable nondimensionalized heat fluxes from many "similar" experiments are plotted in nondimensional form, all of the data fall neatly onto families of curves. For example, for a given value of Pr , we can plot the experimentally determined nondimensional heat flux against Ra . For each value of Pr , the data fall onto an orderly curve. *The similarity theory does not tell us the shapes of the curves, which have to be determined empirically, but at least it tells us what to look for.* If the same data were plotted in dimensional form, a "scattered" set of points would result, and no order would be apparent.

A famous example of similarity analysis was provided by G. I. Taylor (1950 a, b), who analyzed nuclear explosions. Using his (formidable) intuition, he chose the following q s for the problem:

Symbol	Definition	Representative value or first guess
R	radius of wavefront	10^2 m
t	time	10^{-2} s
p_0	ambient pressure	10^5 Pa
ρ_0	ambient density	1 kg m ⁻³
E	energy released	10^{14} J

Starting from the five dimensional parameters in the table, which can be described using the three primary quantities length, time, and mass, Taylor used dimensional analysis to identify the following two nondimensional parameters:

$$\frac{\rho_0 R^5}{Et^2}, \text{ and } \frac{p_0 R^3}{E}. \quad (11)$$

He could then assert that

$$\frac{\rho_0 R^5}{Et^2} = f\left(\frac{p_0 R^3}{E}\right), \quad (12)$$

where f is a function to be determined empirically. But actually, he did better than that.

With the numerical values given in the third column of the table above, including a “first guess” at the value of E , Taylor estimated that

$$\frac{\rho_0 R^5}{Et^2} = 1,$$

$$\frac{p_0 R^3}{E} = 10^{-3}.$$

Because the second parameter is much less than one, Taylor concluded that it was physically irrelevant. The second parameter involves the ambient pressure, so this amounts to the (plausible) assumption that the nuclear fireball, with its huge internal pressure, does not give a damn about the relatively puny ambient pressure. This similarity hypothesis led Taylor to conclude that (12) can be replaced by

$$\frac{\rho_0 R^5}{Et^2} = A = \text{constant},$$

where A is expected to be close to one. It follows that

$$R^5 = \left(\frac{AE}{\rho_0} \right) t^2.$$

(13)

The factor AE / ρ_0 is expected to be independent of time, because A is a constant and both ρ_0 and E are independent of time. Taylor was able to confirm that R^5 increased in proportion to t^2 using published (unclassified) magazine photos of the explosion. He then estimated E by assuming that $A=1$. His accurate estimate embarrassed the government, which had not declassified the amount of energy released in the explosion.

Summary

Dimensional reasoning is very common in atmospheric science and engineering. It is used in several different ways.

Mass, length, time, and temperature suffice to describe the physical quantities used in most atmospheric science work. The Buckingham Pi Theorem tells us that a physical problem can be described most concisely if it is expressed using only nondimensional combinations. Dimensional analysis can be used to identify such combinations.

Scale analysis goes further, by using chosen dimensional scales, i.e., numerical values expressed in units, to determine the dominant terms of the nondimensionalized equations that describe a physical system. The scales are chosen so that the nondimensional unknowns are of order one for the problem of interest. Scale analysis always starts from the known governing equations.

Finally, similarity theories are used to find empirical relationships among the nondimensional parameters that characterize a physical system. Similarity theories are used when the formulas that are sought cannot be derived from first principles.

Similar systems are those for which the relevant nondimensional parameters take the same values. The concept of similar systems is relevant to both scale analysis and similarity theories.

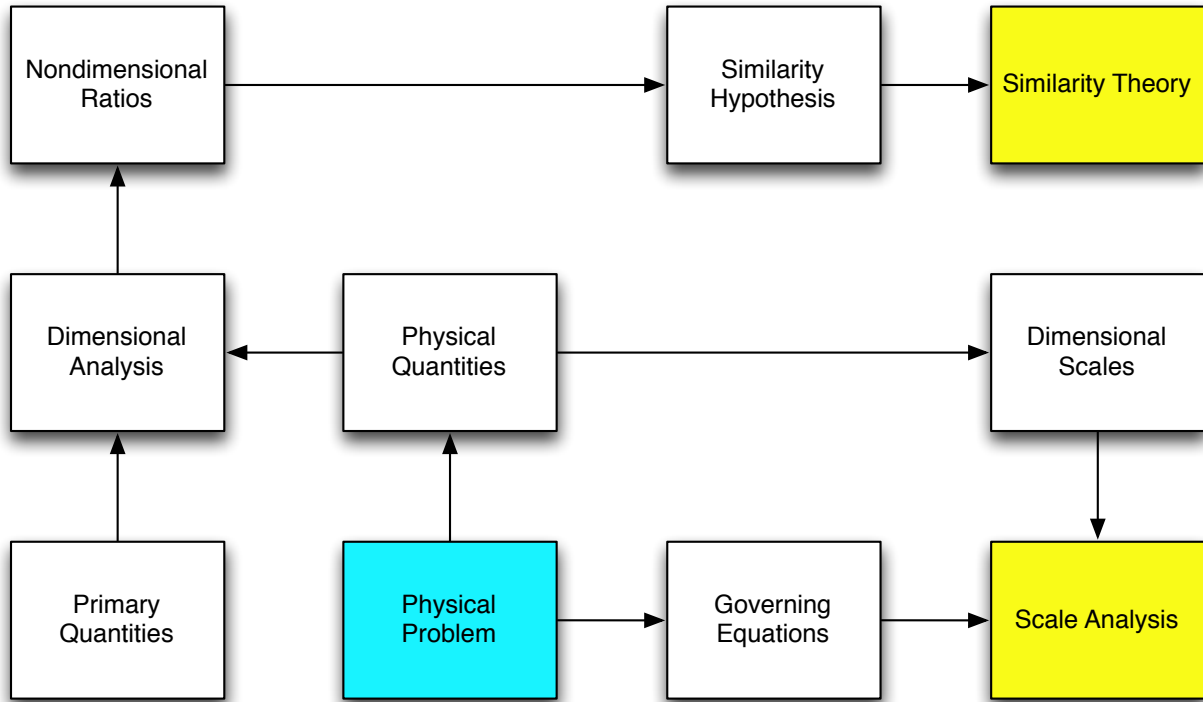


Fig. 1: An attempt to summarize the logical connections among the various topics discussed in this essay. Everything flows from the physical problem (the blue box), and the end-points are similarity theory and scale analysis (the yellow boxes).

Figure 1 summarizes the logical relationships between dimensional analysis, scale analysis, and similarity theories.

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