DIMENSIONS OF SELF-SIMILAR FRACTALS

BY

MELISSA GLASS

A Thesis Submitted to the Graduate Faculty of

WAKE FOREST UNIVERSITY GRADUATE SCHOOL OF ARTS AND SCIENCES

in Partial Fulfillment of the Requirements

for the Degree of

MASTER OF ARTS

Mathematics

May 2011

Winston-Salem, North Carolina

Approved By:

Sarah Raynor, Ph.D., Advisor

James Kuzmanovich, Ph.D., Chair

Jason Parsley, Ph.D.

Acknowledgments

First and foremost, I would like to thank God for making me who I am, for His guidance, and for giving me the strength and ability to write this thesis.

Second, I would like to thank my husband Andy for all his love, support and patience even on the toughest days. I appreciate the countless dinners he has made me while I worked on this thesis.

Next, I would like to thank Dr. Raynor for all her time, knowledge, and guidance throughout my time at Wake Forest. I would also like to thank her for pushing me to do my best.

A special thank you goes out to Dr. Parsley for putting up with me in class! I also appreciate him serving as one of my thesis committee members.

I also appreciate all the fun conversations with Dr. Kuzmanovich whether it was about mathematics or mushrooms! Also, I would like to thank him for serving as one of my thesis committee members.

A thank you is also in order for all the professors at Wake Forest who have taught me mathematics.

Table of Contents

List of Figures

Abstract

Melissa Glass

We investigate the topological, similarity and Hausdorff dimensions of self-similar fractals that are the invariant sets of iterated function systems. We start with the Contraction Mapping Theorem, which will give us a constructive method in which to find fractals using iterated function systems. We then define the Hausdorff metric in order to use the Contraction Mapping Theorem to prove that each iterated function system has a unique invariant set.

Next, we discuss three different types of dimensions: topological, similarity, and Hausdorff dimension. The main theorem of this thesis tells us that the similarity dimension equals the Hausdorff dimension if the iterated function system satisfies the Open Set Condition. This is important since the similarity dimension is much easier to compute than the Hausdorff dimension.

Finally, we apply the theory that we have developed to some famous examples. For each example we give its construction, discuss the three dimensions, and explain the strange properties each fractal possesses. We begin with the Cantor set. Then move on to the Sierpiński gasket and Koch snowflake. Next, we discuss the Menger sponge. Last, we present an example, Barnsley's wreath, whose associated iterated function system does not satisfy the Open Set Condition.

Chapter 1: Introduction

There are several different ways to define a fractal. Conceptually, one can define a fractal as being self-similar. This means that no matter how much the set is magnified, you get a copy of the original set. There are different types of self-similarity and you may not get an exact copy when the set is magnified. So one may choose to think of a fractal as an object that is more "irregular" than classic geometric objects such as lines, squares, cubes and so on. That is, no matter how much the set is magnified, the set does not become any simpler. Thus, one can think of a fractal as being infinitely complex.

On the other hand, Mandelbrot defines a fractal mathematically as an object whose topological dimension is strictly less than its Hausdorff dimension [18]. But this definition is not perfect. Even Mandelbrot admits that this definition leaves out some objects that may be considered fractals according to the conceptual definition and includes objects of "true geometric chaos" (as quoted in [11]). There is a way to resolve the second problem that involves another fractal dimension, the packing dimension. S. James Taylor defines a fractal to be a set such that the Hausdorff dimension is the same as the packing dimension. But Taylor admits this isn't a good definition of a fractal either [10]. Thus, there still is not a perfect mathematical definition of a fractal.

In this thesis we first develop the necessary theory in order to compute the dimensions of self-similar fractals that are the invariant sets of iterated function systems. We start by proving the Contraction Mapping Theorem, which results in a constructive method to find the fixed point of a contraction mapping. But in order to apply the Contraction Mapping Theorem and the resulting constructive method to iterated function systems, we need to create a metric in which we can work. Thus we define the Hausdorff metric. We then define iterated function systems and apply the Contraction Mapping Theorem to ensure that we get a unique invariant set.

In the next chapter, we discuss three different dimensions: topological, similarity, and Hausdorff dimension. We begin with topological dimension, which is not a fractal dimension. We then define a fractal dimension, the similarity dimension. Similarity dimension is based on the iterated function system and is fairly easy to compute. Finally, we want to define Hausdorff dimension. But before we can define the Hausdorff dimension, we need to define the Hausdorff measure and related concepts. We also prove that the similarity dimension is the same as the Hausdorff dimension of a fractal if the iterated function system associated to the fractal satisfies the Open Set Condition.

Last, we apply the theory we have developed to four examples: the Cantor set, Sierpiński gasket, Koch snowflake, and Menger sponge. We compute the topological, similarity and Hausdorff dimension for each. We include a fifth example, Barnsley's wreath, which does not satisfy the Open Set Condition. We see that the similarity dimension is strictly greater than the Hausdorff dimension of Barnsley's wreath.

1.1 Motivation

Fractals produce beautiful pictures, but why do we care about them other than that? In fact, at first mathematicians were not impressed by the Mandelbrot set. At the time, fractals such as Julia sets and the Mandelbrot set were considered mathematical monsters [1]. Mandelbrot's unique ideas the reaction to the Mandelbrot set inspired him to write his famous book *The Geometry of Nature* [18]. Mandelbrot took note of the fact that objects in nature are often not smooth or regular [18]. On page one of his book he writes, "Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightening travel in a straight line" (as quoted in [11]). Mandelbrot suggests that nature can be better modeled by fractals than by classic geometric figures [18]. Ferns, broccoli, and snowflakes are more examples of natural objects that display fractal-like properties. So, as we can see, fractals are all around us.

Currently, fractals are being used in a variety of applications including ecology, physics, meteorology, and medicine. For example, engineers use fractals to simulate turbulence. Fractals have also been used to create special fonts, such as Chinese calligraphy, that can be resized yet retain the appearance of being written with a brush [25]. But one of the most widely used applications of fractals can be found inside your cell phone. In the 1990s, Nathaniel Cohen needed to create a smaller antenna for his ham radios. He tried bending a wire into the shape of a Koch curve. As a result, Cohen created a smaller antenna, but to his surprise it was also more powerful. Using a fractal shape to create an antenna allows the antenna to receive a wider range of frequencies. As technology has progressed, more features have become available requiring a wider range of frequencies. For example, Bluetooth is made possible through the use of fractal antennas [1].

Fractals also have uses in the entertainment industry. Surprisingly, fractals have been used to create music through the use of iterated function systems [25]. Another surprising application is creating special affects in movies. Fractals can be used to create landscapes such as mountains and lunar surfaces. They can also be used to create scenes such as the one in Star Wars Episode III, where Darth Vader and Obi-Wan Kenobi are fighting on a large mechanical arm and the lava is shooting up around them. The lava was created using an iterated function system on different amounts and colors of lava and then layered to get the desired result [1].

Applications of fractals also include early detection of cancer. Blood networks within our bodies are actually fractals. Blood networks that are created by cancer have a different fractal dimension than healthy blood networks [1]. So fractals are not only all around us, they are also inside us!

1.2 History

Unknown to mathematicians at the time, the study of fractals started to emerge around 1872 when a paper by Karl Weierstrass was read at the Royal Academy of Sciences. In this paper, Weierstrass produced an example of a function that is continuous everywhere but differentiable nowhere [24]. This example, however, was not published until 1875 by P. du Bois-Reymond. Since then, several famous mathematicians have analyzed continuous, non-differentiable functions. One such mathematician's analysis in 1904 resulted in a fractal. That mathematician was Helge Von Koch and the resulting fractal is famously known as the Koch curve [15]. But before the discovery of the Koch curve, Georg Cantor produced the first known fractal in 1884, the Cantor set [7].

The next set of developments in the study of fractals was in the area of measure theory. Constantin Carathéodory gave us a new and very useful definition of a measurable set in 1914. He also introduced a measure that was more general than Lebesgue measure. This new measure defined the p-dimensional measure of a set in q-dimensional space [8]. In 1918 Felix Hausdorff took Carathéodory's measure and generalized it even further so that p does not have to be an integer. This lead to an essential part of the study of fractals, namely the Hausdorff Dimension [13]. In 1934 A. S. Besicovitch was one of the first to actually compute fractional dimensions of subsets of $\mathbb{R}$ using Hausdorff dimension [4] [3]. Besicovitch teamed up with H. D. Ursell in 1937 to explore the Hausdorff dimensions of some continuous curves. They discovered that if a function is differentiable, then the graph has dimension 1 [6]. In

1946 P. A. P. Moran discovered a very useful connection between the similarity ratios of certain self-similar sets and their Hausdorff dimensions. Moran's result gives us an easy way to compute the Hausdorff dimensions of self-similar sets, which we will discuss in detail later in this thesis [20].

Many of the first fractals discovered were exactly self-similar, meaning that when you "zoom in" on the set you see an exact copy of the original. For example, in 1926 Karl Menger created another self-similar fractal now known as the Menger sponge [19]. In 1938 ,thirty-four years after the discovery of the Koch curve in 1904, Paul Lévy showed that there are several other curves and surfaces which possess the same type of self-similarity. We refer to these self-similar connected curves as dragon curves. One such curve is now known as Lévy's dragon [16].

Perhaps the most recognizable name in the history of fractals is Benoit Mandelbrot. In fact he was the first to use the term "fractal" in 1975 [11]. Mandelbrot was also the first to introduce the idea that fractals were not just interesting mathematical objects, but that they existed in the natural world. His first contribution to this way of thinking was in 1967 when he published "How Long is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension" [17]. It wasn't until 1982, when he published his famous book *The Fractal Geometry of Nature* [18], that fractals exploded as a major area of study.

1.3 Basic Definitions and Notation

In order to understand the main concepts of this thesis, one must first be familiar with the basics of topology and measure theory. We provide here a list of basic definitions and results that will be used thought the thesis.

1.3.1 Topology

Definition 1. *A **topology** on a set X is a collection τ of subsets of X such that*

1. *Both X and $\emptyset$ are in τ,*
2. *The union of arbitrarily many elements of τ is again in τ, and*
3. *The finite intersection of elements of τ is again in τ.*

Definition 2. *A **topological space**, (X, τ), is a set X with a topology τ on X.*

Definition 3. *Let X be a topological space with topology τ. Then $O \subseteq X$ is an **open set** if $O \in \tau$.*

Definition 4. *Let X be a topological space. Then $C \subseteq X$ is a **closed set** if $X - C$ is an open set.*

Definition 5. *If X is a set, then a **basis** for a topology on X is a collection, $\mathcal{B}$, of subsets of X such that*

1. *For each $x \in X$, there exists a $B \in \mathcal{B}$ such that $x \in B$*
2. *If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.*

*The elements of the basis $\mathcal{B}$ are called **basis elements**.*

Definition 6. *The topology τ **generated** by $\mathcal{B}$ is the collection of sets $\{O\}$ such that for each $x \in O$ there exists a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq O$.*

Definition 7. *The **standard topology** on $\mathbb{R}^n$ is the topology generated by the collection $\{B_\varepsilon(x) : x \in \mathbb{R}^n \text{ and } \varepsilon > 0\}$ where B_ε represents the n-dimensional open ball of radius ε centered at x.*

Definition 8. *Let X be a topological space with topology τ. Let $Y \subseteq X$. Then the* ***subspace topology*** *is the topology $\tau_Y = \{Y \cap O : O \in \tau\}$.*

Definition 9. *Let X be a topological space and $A \subseteq X$. A collection $\mathcal{A}$ of subsets of X is a* ***cover*** *of A if A is contained in the union of the elements of $\mathcal{A}$. The collection $\mathcal{A}$ is an* ***open cover*** *if all elements of $\mathcal{A}$ are open subsets of X.*

Definition 10. *Let X be a topological space and $A \subseteq X$. The subset A is* ***compact*** *if every open cover $\mathcal{A}$ of A contains a finite subcollection that also covers A.*

Theorem 1.1 (Extreme Value Theorem)**.** *The image of a compact set under a continuous map is compact.*

Proof. See Theorem 26.5 on page 166 of [21]. □

Theorem 1.2 (Heine-Borel Theorem)**.** *Let E be a closed and bounded subset of $\mathbb{R}$. Then every open cover of E has a finite subcover.*

Proof. See [22] page 18. □

1.3.2 Metric Spaces

Definition 11. *A set X together with a function $d : X \times X \to \mathbb{R}$ is a* ***metric space*** *if for all $x, y, z \in X$:*

1. $d(x, y) \geq 0$ *with equality if and only if* $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 12. *A sequence $\{x_n\}$ in a metric space is called a* ***Cauchy sequence*** *if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m, n \geq N$ then $d(x_n, x_m) < \varepsilon$.*

Definition 13. *A metric space X is* ***complete*** *if every Cauchy sequence in X converges in X.*

Definition 14. *Let $x, y \in \mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$. The* ***Euclidean metric*** *d_E on $\mathbb{R}^n$ is defined by*

$$d_E(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}.$$

Theorem 1.3. *The set $\mathbb{R}^n$ is a complete metric space with respect to the Euclidean metric d_E.*

Proof. See [22] page 193 for further explanation. □

Definition 15. *The* ***diameter*** *of a subset A of a metric space X is*

$$diam(A) = \sup\{d(x, y) : x, y \in A\}.$$

Definition 16. *Let $r > 0$. A subset B of a metric space X is an* ***r-net*** *for X if every point of X is within distance at most r of some element of B.*

Definition 17. *A set A is* ***totally bounded*** *if for all $r > 0$ there exists a finite r-net for A.*

1.3.3 Measure Theory

Definition 18. *An* ***outer measure*** *on X is a set-function $\overline{M}$ that assigns to every subset $A \subseteq X$ a value $\overline{M}(A) \in [0, \infty]$ and satisfies*

1. $\overline{M}(\emptyset) = 0$
2. *Countable subadditivity: If $\{A_n\}$ is a countable sequence of subsets of X then*

$$\overline{M}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \overline{M}(A_n).$$

Definition 19. *A set $E \subseteq X$ is called $\overline{M}$**-measurable** if every set $A \subseteq X$ satisfies*

$$\overline{M}(A) = \overline{M}(A \cap E) + \overline{M}(A \cap E^c).$$

Definition 20. *Let*

$$\overline{M}(A) = \inf \left\{ \sum_{k=1}^{\infty} l(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}$$

*where $\{I_k\}_{k=1}^{\infty}$ is a collection of nonempty open, bounded intervals that cover A and $l(I_k)$ is the length of the interval, that is, the difference of its endpoints. The restriction of the outer measure, $\overline{M}$, to the class of $\overline{M}$-measurable sets is called **Lebesgue measure** and is denoted by $|A|$ where A is an $\overline{M}$-measurable set.*

Definition 21. *A collection of subsets of $\mathbb{R}$ is called a σ**-algebra** provided it contains $\mathbb{R}$ and is closed with respect to the formation of complements and countable unions.*

Definition 22. *The intersection of all the σ-algebras of subsets of $\mathbb{R}$ that contain the open sets of $\mathbb{R}$ in the standard topology is a σ-algebra called the Borel σ-algebra. Sets in this collection are called **Borel sets**.*

Chapter 2: Iterated Function Systems

Iterated function systems were introduced in 1981 by John E. Hutchinson in [14]. But they did not become popular in the study of fractals until Michael Barnsley's book *Fractals Everywhere* [2] was published in 1988. Even though the construction of a self-similar fractal is quite intuitive, iterated function systems allow us to analyze the fractal in a more mathematical way.

2.1 Contraction Mapping Theorem

The Contraction Mapping Theorem, also referred to as the Banach Contraction Principle or the Banach Fixed Point Theorem, states that any contraction mapping has a unique fixed point. As we will see, the Contraction Mapping Theorem gives us more than just proof of existence and uniqueness, it also gives us constructive method for finding the fixed point of the contraction mapping starting with any initial set in the associated metric space. This is especially important since we are using iterated function systems consisting of contraction mappings to produce fractals.

Definition 23. *A point $x \in X$ is a* ***fixed point*** *of a function $f : X \to X$ if and only if $f(x) = x$.*

Definition 24. *A function $f : X \to X$ is a* ***contraction mapping*** *if there exists a constant $r < 1$ such that for all $x, y \in X$*

$$d(f(x), f(y)) \leq r \ d(x, y).$$

Theorem 2.1 (Contraction Mapping Theorem)**.** *(Thm 2.1.36 in [11]) Let f be a contraction mapping on a complete nonempty metric space, X. Then f has a unique fixed point.*

Proof. Let X be a complete nonempty metric space and $f : X \to X$ be a contraction mapping.

[Uniqueness] Assume $x, y \in X$ are fixed points. This implies $d(x, y) = d(f(x), f(y))$. But since f is a contraction we have $d(x, y) = d(f(x), f(y)) \leq r\ d(x, y)$ for $0 \leq r < 1$. This is impossible if $d(x, y) > 0$. Thus $d(x, y) = 0$. Therefore $x = y$.

[Existence] Let $x_0 \in X$. Define the sequence $\{x_n\}$ recursively by

$$x_{n+1} = f(x_n)$$

for $n \geq 0$. We claim $\{x_n\}$ is a Cauchy sequence:

Let $a = d(x_0, x_1)$. Now consider $d(x_{n+1}, x_n)$. Since f is a contraction mapping, we have

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq rd(x_n, x_{n-1}).$$

By repeating this n times, we see that $d(x_{n+1}, x_n) \leq r^n d(x_0, x_1) = ar^n$. Now if $m < n$, then

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{j=m}^{n-1} d(x_{j+1}, x_j) \\ &\leq \sum_{j=m}^{n-1} ar^j \\ &= \frac{ar^m - ar^n}{1 - r} \\ &= \frac{ar^m(1 - r^{n-m})}{1 - r} \\ &\leq \frac{ar^m}{1 - r}. \end{aligned}$$

We used the triangle inequality to obtain the first inequality. For the second inequality, we used the previous calculation that $d(x_{n+1}, x_n) \leq ar^n$. The two subsequent equalities are from the geometric series summation formula. The last inequality holds since $m < n$ and $r < 1$, so we have $1 - r^{n-m} < 1$.

Let $\varepsilon > 0$. Choose N large enough such that $\frac{ar^N}{1-r} < \varepsilon$. Now for $n \geq m > N$ we have $d(x_m, x_n) \leq \frac{ar^m}{1-r} < \frac{ar^N}{1-r} < \varepsilon$. Thus $\{x_n\}$ is a Cauchy sequence.

Now since X is complete and $\{x_n\}$ is Cauchy we know that $\{x_n\}$ converges in X. Let $x = \lim_{n\to\infty} x_n$. Since f is continuous, we have that $f(x) = \lim_{n\to\infty} f(x_n)$. But since $f(x_n) = x_{n+1}$ we have $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$. Thus $f(x) = x$. Therefore x is a fixed point. □

This proof actually gives a constructive method for finding the fixed point, via iteration.

Corollary 1. *(Cor 2.1.37 in [11]) Let f be a contraction mapping on a complete nonempty metric space X. If x_0 is any point of X, and $x_{n+1} = f(x_n)$ for $n \geq 0$ then the sequence $\{x_n\}$ converges to the fixed point of f.*

Using Corollary 1 we can start with any initial set in the metric space X and apply the contraction mapping repeatedly in order to obtain the fixed point. We will use this method to find the invariant sets of iterated function systems.

2.2 Hausdorff Metric

In order to use the Contraction Mapping Theorem to find the invariant sets of iterated function systems, we need to define a metric on subsets of $\mathbb{R}^n$. We will use the Hausdorff metric, which will be defined on the set of nonempty compact subsets of $\mathbb{R}^n$.

Definition 25. *Let A be a subset of a metric space X and let $r > 0$. Then the* ***open r-neighborhood*** *of A is*

$$\mathcal{N}_r(A) = \{y \in X : \exists\, x \in A \ s.t.\ d(x,y) < r\}.$$

Let X be a metric space. We will consider the collection, $\mathcal{K}(X)$, of all nonempty compact subsets of X.

Definition 26. *The* ***Hausdorff metric****, D, is defined on $\mathcal{K}(X)$ by*

$$D(A,B) = \inf\{r > 0 : A \subseteq \mathcal{N}_r(B) \ and\ B \subseteq \mathcal{N}_r(A)\}.$$

If we did not restrict D to $\mathcal{K}(X)$, then D would not be a metric. For example let $X = \mathbb{R}$, and consider $\{0\}$ and $[0,\infty)$. We see that $D(\{0\},[0,\infty)) = \infty$ and hence D is not a metric on $\mathbb{R}$. We also exclude $\emptyset$, because $D(\{0\},\emptyset) = \infty$ and thus if $\emptyset$ were included D would not be a metric on $\mathbb{R}$.

Theorem 2.2. *(Thm 2.4.1 in [11]) Let X be a metric space. Then the Hausdorff metric D is a metric on $\mathcal{K}(X)$.*

Proof. Let X be a metric space and let $A, B, C \in \mathcal{K}(X)$.

Since A and B are compact we know that they are also bounded. This means that A and B have finite diameter. That is $\sup\{d(x,y)|x,y \in A\} < \infty$ and $\sup\{d(x,y)|x,y \in B\} < \infty$. This means for all $x \in A$ and $y \in B$ we have $d(x,y) < \infty$. Let $\varepsilon > 0$. Let $r = \sup\{d(x,y) : x \in A \text{ and } y \in B\} + \varepsilon$. Hence there exists an r with $0 < r < \infty$ such that $A \subseteq \mathcal{N}_r(B)$ and $B \subseteq \mathcal{N}_r(A)$. Thus $D(A,B) < \infty$ and is well defined. It is easy to see from the definition that $D(A,B) \geq 0$ and $D(A,B) = D(B,A)$.

Now suppose $A = B$. This means for any $\varepsilon > 0$ we have $\mathcal{N}_\varepsilon(A) = \mathcal{N}_\varepsilon(B)$. So for every $\varepsilon > 0$ we have that $A \subseteq \mathcal{N}_\varepsilon(B)$ and $B \subseteq \mathcal{N}_\varepsilon(A)$. Thus $D(A,B) = 0$.

Now suppose $D(A,B) = 0$. If $x \in A$, then for every $\varepsilon > 0$ we have that $x \in \mathcal{N}_\varepsilon(B)$. Consider any open set U containing x. Since U is open we know there exists $\varepsilon > 0$

such that $B_\varepsilon(x) \subseteq U$. But since $x \in \mathcal{N}_\varepsilon(B)$ we know there exists $y \in B$ such that $d(x,y) < \varepsilon$. Thus $y \in B_\varepsilon(x) \subseteq U$. So we have $y \in U$ and either $y = x$ or $y \neq x$. If $x = y$ then $x \in B$. Otherwise we know x is a limit point of B since U was an arbitrary open set. But since B is compact, we know that it is also closed. Thus $x \in B$. Hence $A \subseteq B$. Similarly, $B \subseteq A$. Therefore $A = B$.

Last, we need to prove the triangle inequality. Let $\varepsilon > 0$. Let $r = D(A,B) + \varepsilon$. We see that $A \subseteq \mathcal{N}_r(B)$. If $x \in A$, then $x \in \mathcal{N}_r(B)$. So there exists $y \in B$ such that $d(x,y) < D(A,B) + \varepsilon$. Similarly, there exists $z \in C$ such that $d(y,z) < D(B,C) + \varepsilon$. Now let $a = D(A,B) + D(B,C) + 2\varepsilon$. Then by the triangle inequality for the metric d, we have that there exists a $z \in C$ such that $d(x,z) \leq d(x,y) + d(y,z) < a$. Since $x \in A$ was arbitrary we can concluded that $A \subseteq \mathcal{N}_a(C)$. Similarly, $C \subseteq \mathcal{N}_a(A)$.

Thus, we have that $D(A,C) \leq a = D(A,B) + D(B,C) + 2\varepsilon$. Since this is true for all $\varepsilon > 0$ we know $D(A,C) \leq D(A,B) + D(B,C)$.

Therefore, we have shown that D is a metric on $\mathcal{K}(X)$. □

So we have that $\mathcal{K}(X)$ is a metric space. But in order to use the Contraction Mapping Theorem, we need that $\mathcal{K}(X)$ is complete.

Lemma 1. *(In proof of Thm 2.4.4 in [11]) Let X be a metric space. If $A \subseteq X$ is totally bounded and closed then it is compact.*

Proof. We need to show that A is limit point compact, which will imply that A is compact [21]. Let F be an infinite subset of A. Since A is totally bounded we know there exists a finite 1/2-net, B_1, for A. This means that each element of F is within distance at most 1/2 of some element of B_1.

Since F is infinite and B_1 is finite, by the Pigeonhole Principle we know there exists $b_1 \in B_1$ that is within distance 1/2 of infinitely many points of F. Let $F_1 \subseteq F$

be the infinite subset of all points of F that are distance at most $1/2$ of $b_1 \in B_1$. Clearly, diam $F_1 \leq 1$.

We will apply the same reasoning to the infinite set F_1. Note that there also exists a finite $1/4$-net, B_2, for A. Again, since $F_1 \subseteq A$ is infinite and B_2 is finite there exists a $b_2 \in B_2$ that is within at most distance $1/4$ of infinitely many points of F_1. Let F_2 be the infinite subset of all points of F_1 that are at most distance $1/4$ of $b_2 \in B_2$. Note that diam $F_2 \leq 1/2$. We can keep repeating this process to obtain infinitely many F_j, each of which are infinite, such that diam $F_j \leq 2^{-j}$ and $F_{j+1} \subseteq F_j$ for all j.

Now choose any element $x_j \in F_j$, creating a sequence $\{x_j\}$. Since for each j we know that diam $(F_j) \leq 2^{-j}$ and $F_{j+1} \subseteq F_j$ we have $d(x_j, x_k) \leq 2^{-j}$ for $j < k$, and thus $\{x_j\}$ is a Cauchy sequence. Since X is complete, we know $\{x_k\}$ converges. Let $x = \lim\limits_{k\to\infty} x_k$. Then since A is closed, we know $x \in A$. But x is also a limit point of F since x is a limit of elements of F. Thus A is limit point compact, which implies A is compact by Theorem 28.2 in [21] since X is a metric space. $\square$

Theorem 2.3. *(Thm 2.4.4 in [11]) Let X be a complete metric space. Then the space $\mathcal{K}(X)$ is complete.*

Proof. Let X be a complete metric space. Let $\{A_n\}$ be a Cauchy sequence in $\mathcal{K}(X)$. We need to show $\{A_n\}$ converges in $\mathcal{K}(X)$.

Let $A = \{x \in X : \exists\ \{x_k\} \text{ s.t. } x_k \to x \text{ and for each } k,\ x_k \in A_k\}$.

We want to show that $A \in \mathcal{K}(X)$ and that A is the limit of the Cauchy sequence $\{A_n\}$ in the Hausdorff metric topology.

First, we must show $\{A_n\} \to A$. To do this, we will show $D(A_n, A) \to 0$.

Let $\varepsilon > 0$. Since $\{A_n\}$ is a Cauchy sequence there exists $N \in \mathbb{N}$ such that if $n, m \geq N$ then $D(A_n, A_m) < \varepsilon/2$. Let $n \geq N$.

We claim that $D(A_n, A) \leq \varepsilon$:

If $x \in A$, then there exists a sequence $\{x_k\}$ such that $x_k \in A_k$ and $x_k \to x$. Thus, for k sufficiently large, we have $d(x_k, x) < \varepsilon/2$.

Now if $k \geq N$ then $D(A_k, A_n) < \varepsilon/2$. This means there exists $y \in A_n$ such that $d(x_k, y) < \varepsilon/2$.

So we have $d(y, x) \leq d(y, x_k) + d(x_k, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus $A \subseteq \mathcal{N}_\varepsilon(A_n)$.

Now let $y \in A_n$. Since $\{A_n\}$ is a Cauchy sequence, we can choose integers $k_1 < k_2 < \ldots$ such that $k_1 = n$ and $D(A_{k_j}, A_m) < 2^{-j}\varepsilon$ for all $m \geq k_j$. We will define a sequence $\{y_k\}$ such that $y_k \in A_k$ by the following recursive process:

Pick $y_k \in A_k$ arbitrarily for $k < n$. Let $y_n = y$. Now if y_{k_j} has been chosen and $k_j < k \leq k_{j+1}$, then choose $y_k \in A_k$ such that $d(y_{k_j}, y_k) < 2^{-j}\varepsilon$, which is possible by the construction of the $k_j s$.

We will show that $\{y_k\}$ is a Cauchy sequence in X. Let $\delta > 0$. We want to show that there exists $N \in \mathbb{N}$ such that for all $k, m \geq N$ we have $d(y_k, y_m) < \delta$. Suppose that k, m have been chosen and choose j such that $k_j < k \leq k_{j+1}$ and choose i such that $k_i < m \leq k_{i+1}$. Using the construction of the sequence $\{y_k\}$ and the construction of the subsequence $\{y_{k_j}\}$, we know there exists l such that if $k > m \geq l$, then

$$\begin{aligned}
d(y_k, y_m) &\leq d(y_k, y_{k_j}) + d(y_{k_j}, y_{k_i}) + d(y_{k_i}, y_m) \\
&\leq d(y_k, y_{k_j}) + d(y_{k_j}, y_{k_{j+1}}) + d(y_{k_{j+1}}, y_{k_{j+2}}) + \ldots + d(y_{k_{i-1}}, y_{k_i}) + d(y_{k_i}, y_m) \\
&< 2^{-j}\varepsilon + 2^{-j}\varepsilon + 2^{-(j+1)}\varepsilon + \ldots + 2^{-(i-1)}\varepsilon + 2^{-i}\varepsilon \\
&= \varepsilon\left[2^{-j} + \sum_{l=j}^{i} 2^{-l}\right] \\
&= \varepsilon\left[2^{-j} + \frac{2^{-j} - 2^{-(i+1)}}{1 - 1/2}\right] \\
&= \varepsilon[2^{-j} + 2^{-j+1} - 2^{i}] \\
&< 3(2^{-j})\varepsilon.
\end{aligned}$$

So we have $d(y_k, y_m) < 3(2^{-j})\varepsilon$ for $m > k \geq k_j$. Choose j sufficiently large such that $3(2^{-j})\varepsilon < \delta$. Now for that j, choose $N > k_j$. Thus, by the calculation above, we have $d(y_k, y_m) < \delta$.

Thus, since X is complete, $\{y_k\}$ converges. Let $x = \lim_{k\to\infty} y_k$. But this means $x \in A$. By an argument similar to the argument above that $\{y_k\}$ is a Cauchy sequence, using $j = 1$, we have

$$d(y, x) = \lim_{k\to\infty} d(y, y_k) < (3/2)\varepsilon,$$

using the continuity of the metric. Thus $y \in \mathcal{N}_\varepsilon(A)$. Hence, $A_n \subseteq \mathcal{N}_\varepsilon(A)$.

Since we already checked that $A \subseteq \mathcal{N}_\varepsilon(A_n)$, we have shown that $D(A, A_n) \leq \varepsilon$. Since for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > N$, $D(A, A_n) \leq \varepsilon$, we know $D(A, A_n) \to 0$. Therefore $\{A_n\} \to A$, because D is a metric by Theorem 2.2.

Note that if we choose any specific $\varepsilon > 0$ in the argument used above to prove that $A_n \subseteq \mathcal{N}_\varepsilon(A)$, we can construct a Cauchy sequence as above that converges to some point $x \in X$, which must then be an element of A. Thus A is nonempty.

Lastly we need to show that A is compact. In order to do this we will show that A is totally bounded and closed. We will then apply Lemma 1.

We claim A is totally bounded:

Choose n such that $D(A_n, A) < \varepsilon/3$. We know A_n is a nonempty compact subset of X. Hence, A_n is sequentially compact. According to [11], Proposition 2.2.5, there exists a finite $(\varepsilon/3)$-net for A_n. Call it $\{y_1, y_2, \ldots, y_m\}$. Since $y_i \in A$ and $D(A_n, A) < \varepsilon/3$ we have that for each y_i, there exists an $x_i \in A$ such that $d(x_i, y_i) < \varepsilon/3$. Since $D(A, A_n) < \varepsilon/3$ we know that for any $a \in A$ there exists $z \in A_n$ such that $d(a, z) < \varepsilon/3$. Also since the collection $\{y_i\}$ is an $\varepsilon/3$-net for A_n we know there exists an i such that $d(z, y_i) < \varepsilon/3$. So for this i we have for any $a \in A$,

$$d(a, x_i) \leq d(a, z) + d(z, y_i) + d(y_i, x_i) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Since $a \in A$ was arbitrary, the set $\{x_1, x_2, \ldots, x_m\}$ is a finite ε-net for A. Thus A is totally bounded.

Next, we claim that A is closed:

Let $x \in \overline{A}$. If $x \notin A$, then this means x is a limit point of A. So we have that for every $\varepsilon > 0$ the ball $B_\varepsilon(x)$ contains a point of A other than x. Now consider the points $y_n \in A$ such that $y_n \neq x$ and $y_n \in B_{2^{-n}}(x)$ for all $n \in \mathbb{N}$. This gives us a sequence $\{y_n\}$ in A such that $d(x, y_n) < 2^{-n}$. For each n, there exists a point $z_n \in A_n$ such that $d(z_n, y_n) < D(A_n, A) + 2^{-n}$. So we have

$$d(z_n, x) \leq d(z_n, y_n) + d(y_n, x) < D(A_n, A) + 2^{-n} + 2^{-n}$$

which converges to 0 as $n \to \infty$. Thus $z_n \to x$. This means that $x \in A$ by construction of A. Therefore A is closed.

We have that A is totally bounded and closed. Thus, by Lemma 1, we know A is compact.

Therefore we have shown that if X is a complete metric space then $\mathcal{K}(X)$ is also complete. $\square$

First note that $\mathcal{K}(X)$ is nonempty since any finite set of points is nonempty, closed, and compact. We have also shown that $\mathcal{K}(X)$ is a complete metric space. Therefore we can apply the Contraction Mapping Theorem and Corollary 1 to $\mathcal{K}(X)$.

2.3 Iterated Function Systems

We have mentioned the usefulness of iterated function systems in producing fractals. But what is an iterated function system? An iterated function system is a finite collection of similarities.

Definition 27. *A function $f : X \to Y$ is a **similarity** if there exists $r > 0$ such that for all $x, y \in X$ we have $d(f(x), f(y)) = r\ d(x, y)$.*

Proposition 1. *A similarity $f : X \to X$ with ratio $r > 0$ is continuous.*

Proof. Let $f : X \to X$ be a similarity with ratio $r > 0$. This means for all $x, y \in X$ we have $d(f(x), f(y)) = r\ d(x, y)$. Let $\varepsilon > 0$. Consider $\delta = \varepsilon/r$. We see that if $d(x, y) < \varepsilon/r$ then

$$d(f(x), f(y)) = r\ d(x, y) < r(\varepsilon/r) = \varepsilon.$$

Therefore f is continuous. □

Definition 28. *A **ratio list** is a finite list of positive numbers, $(r_1, r_2, \ldots, r_n)$.*

Definition 29. *A ratio list is called **contracting** if $r_i < 1$ for all i.*

Definition 30. *An **iterated function system** realizing a ratio list $(r_1, r_2, \ldots, r_n)$ in a metric space X is a list of functions $(f_1, f_2, \ldots, f_n)$ where $f_i : X \to X$ is a similarity with ratio r_i.*

Definition 31. *A nonempty compact set $K \subseteq X$ is an **invariant set** for the iterated function system $(f_1, f_2, \ldots, f_n)$ if $K = f_1[K] \cup f_2[K] \cup \ldots \cup f_n[K]$.*

In order to use the Contraction Mapping Theorem, we require that each function be a contraction mapping. This means that each similarity f_i must have a ratio $r_i < 1$.

Theorem 2.4. *(Thm 4.1.3 in in [11]) Let X be a complete metric space. Let $(r_1, r_2, \ldots, r_n)$ be a contracting ratio list and let $(f_1, f_2, \ldots, f_n)$ be an iterated function system of similarities on X that realizes this ratio list. Then there exists a unique nonempty compact invariant set for the iterated function system.*

Proof. Consider the metric space $\mathcal{K}(X)$ with the Hausdorff metric, D. We have already shown that if X is complete then so is $\mathcal{K}(X)$. Now define a function

$F : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$F(A) = \bigcup_{i=1}^{n} f_i[A].$$

We have already showed that each f_i is continuous in Proposition 1. We know that the continuous image of a compact set is also compact. Also the union of finitely many compact sets is also compact. Thus if A is compact then so is $F(A)$. Also, if A is nonempty, then clearly $F(A)$ is nonempty. Thus f maps $\mathcal{K}(X)$ to itself.

We claim that F is a contraction mapping on $\mathcal{K}(X)$:

Let $r = \max\{r_1, r_2, \ldots, r_n\}$. Clearly $r < 1$. We will show that $D(F(A), F(B)) \leq r\ D(A, B)$.

Let $q > D(A, B)$ be given. If $x \in F(A)$, then $x = f_i(x')$ for some i and some $x' \in A$. Since $q > D(A, B)$ we know there exists $y' \in B$ such that $d(x', y') < q$. But then the point $y = f_i(y') \in F(B)$ satisfies

$$d(x, y) = r_i\ d(x', y') < rq.$$

This is true for all $x \in F(A)$. Thus $F(A) \subseteq \mathcal{N}_{rq}(F(B))$.

Similarly, $F(B) \subseteq \mathcal{N}_{rq}(F(A))$.

Therefore $D(F(A), F(B)) \leq rq$. This is true for all $q > D(A, B)$. Thus $D(F(A), F(B)) \leq r\ D(A, B)$.

So F is a contraction mapping on a complete metric space $\mathcal{K}(X)$. Now by the Contraction Mapping Theorem, we know that F has a unique fixed point in $\mathcal{K}(X)$, which is therefore compact and nonempty. Call this fixed point K. So $F(K) = f_1[K] \cup f_2[K] \cup \ldots \cup f_n[K] = K$. Thus a fixed point of F is equivalent to an invariant set of $(f_1, f_2, \ldots, f_n)$. $\square$

Now we can use Corollary 1 to find the unique invariant set of the iterated function system F. Recall that using Corollary 1 starting with $K_0 \in \mathcal{K}(X)$ we can create a sequence $\{K_m\}$ defined by $K_{n+1} = F(K_n)$ that converges to the invariant set K. Note that if $K \subseteq K_0$ then the sequence of sets $\{K_m\}$ is a nested sequence of nonempty compact sets. For an iterated function system, $(f_1, f_2, \ldots, f_n)$, being applied to an initial set K_0, we call the m^{th} iteration of the function system K_m.

Chapter 3: Dimensions

Our concept of dimension stems from objects in classical geometry. For example, a point is 0-dimensional, a line is 1-dimensional, a square is 2-dimensional and so on. We can think of dimension in several different ways. In fact, there are also several ways to measure dimension. Here we discuss three different types of dimension: topological, similarity, and Hausdorff dimension.

3.1 Topological Dimension

Topological dimension actually refers to a category of dimensions. The type of topological dimension we discuss here is called small inductive dimension. Topological dimension always refers to a nonnegative integer. In fact, topological dimension is how we normally think of dimension. For example, the dimension of a rectangle is 2. Other topological dimensions include large inductive dimension, covering dimension and equation dimension. These four topological dimensions are usually not all equal.

Definition 32. *A subset of a metric space is **clopen** if it is both open and closed.*

Definition 33. *A metric space is **zero-dimensional** if there exists a basis for its topology consisting of clopen sets.*

Topological dimension works the way we expect it should. A finite set of points is 0-dimensional, a curve is 1-dimensional and plane is 2-dimensional and so on. So as we would expect, the real line, $\mathbb{R}$, is not zero-dimensional.

Theorem 3.1. *(Thm 3.1.1 in [11]) The only clopen sets in $\mathbb{R}$ are $\mathbb{R}$ and $\emptyset$. Therefore $\mathbb{R}$ is not zero-dimensional.*

Proof. Let $A \subseteq \mathbb{R}$ such that $A \neq \emptyset$ and $A \neq \mathbb{R}$. We need to show that A is not clopen. In other words, A has a boundary point.

Recursively define two sequences $\{x_n\}$ and $\{y_n\}$. First, since $A \neq \emptyset$, choose $x_0 \in A$. Next, since $A \neq \mathbb{R}$, choose $y_0 \notin A$. Now, after defining x_n and y_n we need to define x_{n+1} and y_{n+1}.

Consider $z_n = \frac{x_n+y_n}{2}$. If $z_n \in A$, then define $x_{n+1} = z_n$ and $y_{n+1} = y_n$. If $z_n \notin A$, then define $x_{n+1} = x_n$ and $y_{n+1} = z_n$.

In either case we have $x_{n+1} \in A$ and $y_{n+1} \notin A$. We also have, if $z_n \in A$,

$$|x_{n+1} - y_{n+1}| = \left|\frac{x_n + y_n}{2} - y_n\right| = \frac{|x_n - y_n|}{2}.$$

Also if $z_n \notin A$, then

$$|x_{n+1} - y_{n+1}| = \left|x_n - \frac{x_n + y_n}{2}\right| = \frac{|y_n - x_n|}{2} = \frac{|x_n - y_n|}{2}.$$

Thus $|x_{n+1} - y_{n+1}| = \dfrac{|x_n - y_n|}{2}$.

Therefore, by induction, $|x_n - y_n| = \dfrac{|x_0 - y_0|}{2^n}$. So as $n \to \infty$ we have $|x_n - y_n| \to 0$. If $x_{n+1} = x_n$ then

$$|x_{n+1} - x_n| = |x_n - x_n| = 0 \leq |x_n - y_n| = \frac{|x_0 - y_0|}{2^n}.$$

Also if $x_{n+1} = \dfrac{x_n + y_n}{2}$ then we have

$$\begin{aligned} |x_{n+1} - x_n| &= \left|\frac{x_n + y_n}{2} - x_n\right| \\ &= \left|\frac{x_n}{2} + \frac{y_n}{2} - \frac{2x_n}{2}\right| \end{aligned}$$

$$\leq \frac{|x_n + y_n - 2x_n|}{2}$$

$$= \frac{|x_n - y_n|}{2}$$

$$\leq \frac{|x_0 - y_0|}{2^n}.$$

This means, by a similar argument as in the proof of the Contraction Mapping Theorem, that $\{x_n\}$ is a Cauchy sequence, which converges since $\mathbb{R}$ is complete. Let $x = \lim_{n\to\infty} x_n$. Hence x is a limit point of A so $x \in \overline{A}$. But then since $|x_n - y_n| \to 0$ we know $y_n \to x$. Hence $x \in \overline{\mathbb{R} - A}$. So we have $x \in \overline{A} \cap \overline{\mathbb{R} - A} = \partial A$. Therefore x is a boundary point of A and A is not clopen. □

Corollary 2. *(Cor 3.1.2 in [11]) Let $a < b$. The only clopen subsets of $[a, b]$ are $[a, b]$ and $\emptyset$. Therefore $[a, b]$ is not zero-dimensional.*

Proof. This corollary is proved using a similar argument to the previous theorem. □

Now that we have defined what it means to be zero-dimensional, we will use it to define what it means to have topological dimension 1.

Definition 34. *A metric space X has **small inductive dimension 1** if X is not zero-dimensional and there exists a basis for the topology consisting of sets with zero-dimensional boundary.*

Definition 35. *We denote the **small inductive dimension** of a metric space X by ind (X). Each metric space X is assigned a small inductive dimension from the set $\{-1, 0, 1, 2, 3, \ldots, \infty\}$. We set ind $\emptyset = -1$. Now if k is a nonnegative integer, then ind $X \leq k$ if there exists a basis of the topology of X consisting of sets U such that ind $\partial U \leq k - 1$. We say ind $X = k$ if ind $X \leq k$ but ind $X \nleq k - 1$. We say ind $X = \infty$ if, for all, k ind $X \nleq k$.*

Theorem 3.2. *Let X be a metric space and $Y \subseteq X$. Then ind $Y \leq$ ind X.*

Proof. See [11] Theorem 3.1.7. $\square$

We will need the previous theorem when evaluating the topological dimension of a given subset of $\mathbb{R}^n$.

3.2 Similarity Dimension

Similarity dimension is a fractal dimension associated with fractals that are produced from iterated function systems. As we will see, the similarity dimension is the easiest dimension to compute out of the three we consider here. Later, we will show that if the iterated function system satisfies certain conditions, then the similarity dimension is the same as the Hausdorff dimension.

Definition 36. *The **dimension** associated with a ratio list $(r_1, \ldots, r_n)$ is the unique positive number s such that $r_1^s + \ldots + r_n^s = 1$.*

Definition 37. *The number s is called the **similarity dimension** of a nonempty compact set K if there exists a finite decomposition of K*

$$K = \bigcup_{i=1}^{n} f_i[K]$$

where $(f_1, \ldots, f_n)$ is an iterated function system of similarities realizing a ratio list with dimension s.

3.3 Hausdorff Dimension

Following the work of Constantin Carathéodory in measure theory, Felix Hausdorff defined a dimension in which non-integer values are possible. At the time Hausdorff did not realize what a huge impact his definition of dimension would make in the study of fractals. Of course, Hausdorff's definition is now known as Hausdorff dimension. Although there are other definitions of fractal dimensions, the Hausdorff dimension is one of the most commonly-used fractal dimensions.

Suppose that the underlying space is $\mathbb{R}^n$ with the Euclidean metric. Let $s \geq 0$ be any proposed dimension.

Definition 38. *Let $\delta > 0$. Define the **Hausdorff s-dimensional δ outer measure** as*

$$H^s_\delta(A) = \inf\left\{\sum_{i=1}^{\infty}(diam\ (B_i))^s : A \subseteq \bigcup_{i=1}^{\infty} B_i, diam\ (B_i) \leq \delta\ \forall i\right\}.$$

Definition 39. *We define the **s-dimensional Hausdorff outer measure** of A to be*

$$H^s(A) = \lim_{\delta\to 0} H^s_\delta(A)$$

.

As $\delta \to 0$, we see that the infimum increases. This means that the limit as $\delta \to 0$ of H^s_δ exists, although it may be $+\infty$.

Theorem 3.3. *For any $s \geq 0$, the s-dimensional Hausdorff measure, H^s, is an outer measure.*

Proof. Let $\delta > 0$. Clearly $H^s_\delta(\emptyset) = 0$ for all s and $H^s_\delta(A)$ is defined for all $A \subseteq \mathbb{R}^n$. Also, if $A \subseteq B$, we have $H^s_\delta(A) \leq H^s_\delta(B)$ since any cover of B also covers A.

Next we need to show that H^s_δ has the countable subadditivity property.

Let $\{A_j\}$ be a sequence of disjoint subsets of $\mathbb{R}^n$. Let $\varepsilon > 0$. There exists $\{E^k_j\}^{\infty}_{k=1}$ with $\text{diam}(E^k_j) \leq \delta$ covering A_j such that

$$H^s_\delta(A_j) + 2^{-j}\varepsilon \geq \sum_{k=1}^{\infty}(\text{diam}\ (E^k_j))^s.$$

We know we can find such a cover of each A_j, since if one didn't exist we would have

$$H^s_\delta(A_j) + 2^{-j}\varepsilon < \sum_{k=1}^{\infty} (\text{diam } (E^j_k))^s$$

for all covers, $\{E^j_k\}$, of A_j such that $\text{diam}(E^k_j) \leq \delta$, which would imply that

$$H^s_\delta(A_j) < \sum_{k=1}^{\infty} (\text{diam } (E^k_j))^s - 2^{-j}\varepsilon.$$

However, by definition $H^s_\delta(A_j)$ is the infimum over all covers of A_j of $\sum_{k=1}^{\infty} (\text{diam } (E^k_j))^s$ such that $\text{diam}(E^k_j) \leq \delta$, and in this case $H^s_\delta(A_j) + 2^{-j}\varepsilon$ becomes a lower bound that is greater than the infimum. This is a contradiction. Thus there must exist such a covering of A_j.

Now we have

$$\begin{aligned} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (\text{diam } (E^k_j))^s &\leq \sum_{j=1}^{\infty} (H^s_\delta(A_j) + 2^{-j}\varepsilon) \\ &= \sum_{j=1}^{\infty} H^s_\delta(A_j) + \varepsilon. \end{aligned}$$

But $A = \bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j,k=1}^{\infty} E^k_j$. This means $\{E^k_j\}$ is a cover for A. Thus

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (\text{diam } (E^k_j))^s \in \left\{ \sum_{i=1}^{\infty} (\text{diam } (B_i))^s : A \subseteq \bigcup_{i=1}^{\infty} B_i, \text{diam } (B_i) \leq \delta \ \forall i \right\}.$$

Hence,

$$\begin{aligned} H^s_\delta(A) &= \inf\left\{\sum_{i=1}^{\infty}(\text{diam } (B_i))^s : A \subseteq \bigcup_{i=1}^{\infty} B_i, \text{diam } (B_i) \leq \delta \ \forall i\right\} \\ &\leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(\text{diam } (E_j^k))^s. \end{aligned}$$

So for all $\varepsilon > 0$ we have

$$H^s_\delta(A) \leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(\text{diam } (E_j^k))^s \leq \sum_{j=1}^{\infty} H^s_\delta(A_j) + \varepsilon.$$

Thus, letting $\varepsilon \to 0$, $H^s_\delta(A) \leq \sum_{j=1}^{\infty} H^s_\delta(A_j)$.

Therefore H^s_δ is an outer measure. By letting $\delta \to 0$ we have that H^s is an outer measure. $\square$

Since all the fractals we will consider are closed, we only need to show that all Borel sets are H^s-measurable. To do this, we will need the following lemma.

Lemma 2. *Let $A, B \in \mathcal{K}(X)$. If $D(A, B) > 0$ then $H^s(A \cup B) = H^s(A) + H^s(B)$.*

Proof. By Theorem 3.3 we know H^s is an outer measure. Thus $H^S(A \cup B) \leq H^s(A) + H^s(B)$.

Choose $\delta < D(A, B)$. Let $\{C_i\}_{i=1}^{\infty}$ be a cover of $A \cup B$ with $\text{diam}(C_i) \leq \delta$ for all i.

Now let $I_1 = \{i : C_i \cap A \neq \emptyset\}$ and $I_2 = \{i : C_i \cap B \neq \emptyset\}$. Let $i \in I$ and let $x \in A \cap C_i$. Since $D(A, B) > \delta$ we have $d(x, B) > \delta$. This means $B \cap C_i = \emptyset$ since $\text{diam}(C_i) \leq \delta$. Thus, we see that I_1 and I_2 are disjoint. So we have $A \subseteq \bigcup_{i \in I_1} C_i$ and $B \subseteq \bigcup_{i \in I_2} C_i$. But this means

$$H^s_\delta(A) \leq \sum_{i \in I_1}(\text{diam}(C_i))^s$$

and

$$H^s_\delta(B) \leq \sum_{i \in I_2} (\mathrm{diam}(C_i))^s.$$

This implies that

$$H^s_\delta(A) + H^s_\delta(B) \leq \sum_{i \in I_1 \cup I_2} (\mathrm{diam}(C_i))^s \leq \sum_{i \in \mathbb{N}} (\mathrm{diam}(C_i))^s.$$

Now take the infimum over all such C_i. Then for δ sufficiently small, we have $H^s_\delta(A)+H^s_\delta(B) \leq H^s_\delta(A \cup B)$. Let $\delta \to 0$. Therefore $H^s(A)+H^s(B) = H^s(A \cup B)$. $\square$

Proposition 2. *All Borel sets are H^s-measurable.*

Proof. Let F be a closed set in $\mathbb{R}^n$. We want to show that for all $A \subseteq \mathbb{R}^n$, $H^s(A) = H^s(A \cap F) + H^s(A \cap F^c)$. We always have that $H^s(A) \leq H^s(A \cap F) + H^s(A \cap F^c)$.

If $H^s(A) = \infty$ then $H^s(A) \geq H^s(A \cap F) + H^s(A \cap F^c)$. Thus $H^s(A) = H^s(A \cap F) + H^s(A \cap F^c)$.

Therefore, we may assume that $H^s(A) < \infty$. Let $B_n = \{x \in A \cap F^c : d(x, F) \geq 1/n\}$. We see that $B_{n-1} \subseteq B_n$ for all n. So we have $\displaystyle\bigcup_{n=1}^{\infty} B_n = \{x \in A \cap F^c : d(x, F) > 0\} = A \cap F^c$ since F is closed.

By Lemma 2 we have that $H^s(A) \geq H^s((A \cap F) \cup B_n) = H^s(A \cap F) + H^s(B_n)$ because $D(B_n, F) \geq 1/n$ for all n.

Now we need to show that $\displaystyle\lim_{n \to \infty} H^s(B_n) = H^s(A \cap F^c)$. Let $C_n = B_{n+1} - B_n$. If $x \in C_{n+1}$ and $d(x, y) < \dfrac{1}{n(n+1)}$ for some y then

$$\begin{aligned} d(y,F) &\leq d(y,x)+d(x,F) \\ &< \frac{1}{n(n+1)}+\frac{1}{n+1} \\ &= \frac{1}{n}. \end{aligned}$$

Below in Figure 3.1 is a general picture of the sets. Keep in mind that the B_i sets are not necessarily disks, but are pictured here as disks for illustration purposes.

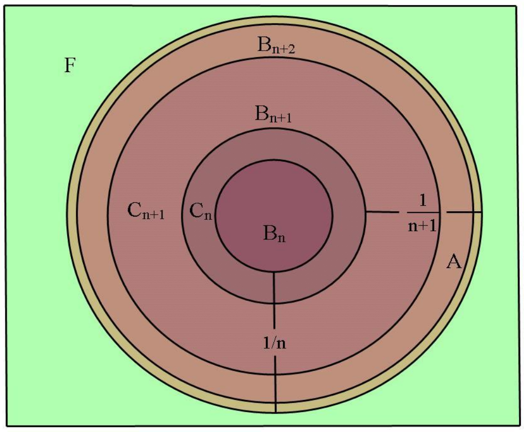

Figure 3.1: F is represented by the green area and F^c is the interior of the first circle.

Since $d(y,F) < 1/n$ we know $y \notin B_n$. Now since $x \in C_{n+1}$, $y \notin B_n$ and $d(x,y) < \frac{1}{n(n+1)} = r$ we know B_n is not contained in $\mathcal{N}_r(C_{n+1})$. Thus $D(C_{n+1}, B_n) \geq \frac{1}{n(n+1)}$.

We claim that $H^s(B_{2k+1}) \geq \sum_{j=1}^{k} H^s(C_{2j})$:

We will prove this by induction. For $k = 1$ we have $H^s(B_3) \geq H^s(C_2)$ since $C_2 = B_3 - B_2 \subseteq B_3$.

Now assume $H^s(B_{2k+1}) \geq \sum_{j=1}^{k} H^s(C_{2j})$. Since $C_{2k+2} = B_{2k+3} - B_{2k+2}$ and $B_{2k+1} \subseteq B_{2k+2}$ we know

$$C_{2k+2} \cup B_{2k+1} \subseteq C_{2k+2} \cup B_{2k+2} = B_{2k+3}.$$

This means that $H^s(B_{2k+3}) \geq H^s(C_{2k+2} \cup B_{2k+1})$. We also have that

$$D(C_{2k+2}, B_{2k+1}) \geq \frac{1}{(2k+1)(2k+2)} > 0.$$

Thus by Lemma 2, we know $H^s(C_{2k+2} \cup B_{2k+1}) = H^s(C_{2k+2}) + H^s(B_{2k+1})$. So now we have

$$\begin{aligned}
H^s(B_{2k+3}) &\geq H^s(C_{2k+2} \cup B_{2k+1}) \\
&= H^s(C_{2k+2}) + H^s(B_{2k+1}) \\
&\geq H^s(C_{2k+2}) + \sum_{j=1}^{k} H^s(C_{2j}) \\
&= \sum_{j=1}^{k+1} H^s(C_{2j}).
\end{aligned}$$

Similarly, we have $H^s(B_{2k}) \geq \sum_{j=1}^{k} H^s(C_{2j-1})$.

Now we see that $H^s(B_n) \leq H^s(A) < \infty$ for all n since each $B_n \subseteq A$ for all n. So we have

$$\begin{aligned}
\sum_{j=1}^{k} H^s(C_{2j}) &\leq H^s(B_{2k+1}) \\
&\leq H^s(A) \\
&< \infty
\end{aligned}$$

and

$$\begin{aligned}\sum_{j=1}^{k} H^s(C_{2j-1}) &\leq H^s(B_{2k}) \\ &\leq H^s(A) \\ &< \infty\end{aligned}$$

Thus both $\sum_{j=1}^{\infty} H^s(C_{2j})$ and $\sum_{j=1}^{\infty} H^s(C_{2j-1})$ are convergent series. This means

$$\sum_{j=1}^{\infty} H^s(C_{2j}) + \sum_{j=1}^{\infty} H^s(C_{2j-1}) = \sum_{j=1}^{\infty} H^s(C_j)$$

is a convergent series.

Now we see that $A \cap F^c = \bigcup_{n=1}^{\infty} B_n = B_n \cup \bigcup_{k=n+1}^{\infty} C_k$. So we have

$$\begin{aligned}H^s(A \cap F^c) &= H^s\left(B_n \cup \bigcup_{k=n+1}^{\infty} C_k\right) \\ &\leq H^s(B_n) + \sum_{k=n+1}^{\infty} H^s(C_k).\end{aligned}$$

by subadditivity since B_n is disjoint from each C_k and the C_k are pairwise disjoint. But since $\sum_{j=1}^{k} H^s(C_j)$ is a convergent series we know $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} H^s(C_k) = 0$.

So now we have

$$H^s(A \cap F^c) \leq \lim_{n\to\infty} H^s(B_n) + \lim_{n\to\infty} \sum_{k=n+1}^{\infty} H^s(C_k).$$

Thus

$$H^s(A \cap F^c) \leq \lim_{n\to\infty} H^s(B_n).$$

Since $B_n \subseteq A \cap F^c$ we have $H^s(A \cap F^c) \geq H^s(B_n)$.

Thus

$$H^s(A \cap F^c) \geq \lim_{n\to\infty} H^s(B_n).$$

Therefore

$$\lim_{n\to\infty} H^s(B_n) = H^s(A \cap F^c).$$

So we have $H^s(A) \geq H^s(A \cap F) + \lim_{n\to\infty} H^s(B_n)$. Thus $H^s(A) \geq H^s(A \cap F) + H^s(A \cap F^c)$. We always have that $H^s(A) \leq H^s(A \cap F) + H^s(A \cap F^c)$. Hence $H^s(A) = H^s(A \cap F) + H^s(A \cap F^c)$.

This means F is H^s-measurable. Since F was an arbitrary closed set we know all closed sets are H^s-measurable. We know that the set of all H^s-measurable sets forms a σ-algebra. The smallest σ-algebra containing all the closed set is the Borel σ-algebra which contains all Borel sets. Therefore all Borel sets are H^s-measurable. □

Proposition 3. *(Thm 6.1.5 in [11]) If $p < q$ and $H^p(A) < \infty$ then $H^q(A) = 0$. If $p > q$ and $H^p(A) > 0$ then $H^q(A) = +\infty$.*

Proof. First note that the second statement is the contrapositive of the first statement. Thus we only need to prove the first statement.

Suppose $H^p(A) < \infty$. This means for all $\delta > 0$ there exists $\{B_j\}$ such that $A \subseteq \bigcup_{j=1}^{\infty} B_j$ with $\operatorname{diam} B_j \leq \delta$ for all j and $\displaystyle\sum_{j=1}^{\infty}(\operatorname{diam}(B_j))^p \leq H^p(A) + 1 = M$. Now if $q > p$ then

$$
\begin{aligned}
\sum_{j=1}^{\infty}(\mathrm{diam}(B_j))^q &= \sum_{j=1}^{\infty}(\mathrm{diam}(B_j))^p(\mathrm{diam}(B_j))^{q-p} \\
&\leq \delta^{q-p}\sum_{j=1}^{\infty}(\mathrm{diam}(B_j))^p \\
&\leq \delta^{q-p}M.
\end{aligned}
$$

But this means $H^q_\delta(A) \leq \delta^{q-p}M$. Now as $\delta \to 0$ we conclude that $H^q(A) = 0$. $\square$

This means that for any Borel set A there is a unique $s > 0$ such that for all $p < s$, $H^p(A) = +\infty$ and for all $p > s$, $H^p(A) = 0$.

Definition 40. *For a given Borel set A, we call the unique $s > 0$ such that for all $p < s$, $H^p(A) = +\infty$ and for all $p > s$, $H^p(A) = 0$ the **Hausdorff dimension** of A.*

Now that we have established the definition of the Hausdorff dimension, we need to relate the Hausdorff dimension to the similarity dimension to create more ease in computation. The similarity dimension will equal the Hausdorff dimension when the iterated function system satisfies the Open Set Condition defined below. Basically, the images of each similarity of the iterated function system cannot "overlap" too much.

Definition 41. *An iterated function system, $(f_1, \ldots, f_n)$, satisfies the **Open Set Condition** if there exists a nonempty open set $O \subseteq \mathbb{R}^m$ such that $\bigcup_{i=1}^{n} f_i(O) \subseteq O$ and $f_i(O) \cap f_j(O) = \emptyset$ for $i \neq j$.*

In order to show that the similarity dimension equals the Hausdorff dimension of an invariant set, K, of an iterated function system satisfying the Open Set Condition, we must first prove a lemma about additive functions on half-open figures. Since H^s is an additive function, this will allow us to show that $H^s(K) > 0$ by only considering covers that consist of finitely many half-open q-dimensional intervals and then bounding $H^s(K)$ below by a positive constant .

Definition 42. *A function f is* ***additive*** *if for a finite collection of sets $\{A_j\}_{j=1}^n$ we have*

$$f\left(\bigcup_{j=1}^{N} A_j\right) = \sum_{j=1}^{N} f(A_j).$$

Definition 43. *A* ***half-open figure*** *in $\mathbb{R}^q$ is a finite union of half-open q-dimensional intervals, that is, open on the right and closed on the left.*

Lemma 3. *(Thm I in [20]) Let $E \subseteq \mathbb{R}^q$ be any closed and bounded set such that $H^s(E) < \infty$. Then $H^s(E) > 0$ if there exists an additive function $\psi(A)$ on half-open figures, A, such that*

1. *$\psi(A) \geq 0$ for all A,*

2. *If $E \subseteq A$ then $0 < b \leq \psi(A)$ for some fixed constant b, and*

3. *There exists $0 < k < \infty$ such that if diam$(A) = d$, then $\psi(A) \leq kd^s$.*

In particular, $H^s(E) \geq b/k$.

Proof. We want to show that if $\{O_i\}$ is a countable collection of open sets covering E with diameters d_i then $\displaystyle\sum_{i=1}^{\infty} d_i^s \geq b/k$. Since E is compact, by the Heine-Borel Theorem we may assume $\{O_i\}_{i=1}^N$ is a finite collection of open sets. If one $d_i = \infty$ then the inequality is trivial. So assume $d_i < \infty$ for all i. Let $\varepsilon_3 > 0$. Cover O_i with half-open q-dimensional intervals with diameters $d_i(\varepsilon_3/2)$. Since the diameter of O_i is finite we can take a finite subcover of half-open q-dimensional intervals. Let the diameter of the union of these q-dimensional half-open intervals be d_i'. Let h_1 and h_2 be the half-open q-dimensional intervals that are the furthest apart in the finite cover. Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be the smallest radii such that $h_1 \subseteq \mathcal{N}_{\varepsilon_1}(O_i)$ and $h_2 \subseteq \mathcal{N}_{\varepsilon_2}(O_i)$. We see that $\varepsilon_1 < d_i(\varepsilon_3/2)$ and $\varepsilon_2 < d_i(\varepsilon_3/2)$. By the triangle inequality we have

$$d_i' \leq d_i + \varepsilon_1 + \varepsilon_2 < d_i + 2d_i(\varepsilon_3/2) = d_i(1 + \varepsilon_3).$$

Thus we can let each O_i be contained in a half-open figure O_i' with diameters d_i' such that $(d_i')^s < (1+\varepsilon)d_i^s$ for an arbitrary $\varepsilon > 0$. By part 3, for each O_i' we have

$$(d_i')^s \geq \frac{1}{k}\psi(O_i').$$

Thus we have

$$\begin{aligned}
\sum_{i=1}^{\infty} d_i^s &\geq \frac{1}{1+\varepsilon}\sum_{i=1}^{N}(d_i')^s \\
&\geq \frac{1}{k(1+\varepsilon)}\sum_{i=1}^{N}\psi(O_i') \\
&\geq \frac{1}{k(1+\varepsilon)}\psi\left(\bigcup_{i=1}^{N} O_i'\right)
\end{aligned}$$

since ψ is an additive function on half-open figures and the half-open figures O_i' may overlap. We also have $E \subseteq \bigcup_{i=1}^{N} O_i \subseteq \bigcup_{i=1}^{N} O_i'$. Thus by part 2 we have

$$\psi\left(\bigcup_{i=1}^{N} O_i'\right) \geq b.$$

Therefore

$$\sum_{i=1}^{\infty} d_i^s \geq \frac{1}{k(1+\varepsilon)}\psi\left(\bigcup_{i=1}^{N} O_i'\right)$$
$$\geq \frac{b}{k(1+\varepsilon)}.$$

Now let $\varepsilon \to 0$. Thus $\sum_{i=1}^{\infty} d_i^s \geq b/k$.

Let $\delta > 0$. Since $\{O_i\}$ was an arbitrary open cover of E we have

$$H_\delta^s(E) = \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam}(O_i))^s : E \subseteq \bigcup_{i=1}^{\infty} O_i,\ \operatorname{diam}(O_i) \leq \delta\right\} \geq b/k.$$

Let $\delta \to 0$. Therefore $H^s(E) \geq b/k$. □

Theorem 3.4. *(Thm III in [20]) Let $(f_1, \ldots, f_n)$ be an iterated function system realizing the contracting ratio list $(r_1, \ldots, r_n)$ and satisfying the Open Set Condition. Let K be the invariant set of the iterated function system with similarity dimension s. Then*

$$0 < H^s(K) < \infty.$$

Proof. First we will show $H^s(K) < \infty$.

Let O_1 be an open set with diam $(O_1) = d$ such that $K \subseteq O_1$. By Corollary 1 we can start with any initial set and obtain the invariant set of the iterated function system. Let K_0 be the initial set such that $K_{m+1} \subseteq K_m$ where K_i is the i^{th} iteration. Thus $K \subseteq K_m$ for all m. Let $O_2^i = f_i(O_1)$ where $i = 1, \ldots, n$ be the n open sets with diameters dr_i. Let $O_3^{ij} = f_j(f_i(O_1))$ where $i, j = 1, \ldots, n$ be the n^2 open sets with diameters $dr_i r_j$. Now let $O_m^{ij\ldots k} = f_k(\ldots(f_j(f_i(O_1)))\ldots)$ where $i, j, \ldots, k = 1, \ldots, n$, be the n^m open sets with diameters in the form $dr_i r_j \cdots r_k$. Fix m and let $O_m =$

$\bigcup_{ij\ldots k} O_m^{ij\ldots k}$. We have $K_m \subseteq O_m$ since $K \subseteq O_1$ and both are being reduced by the same ratios. We see that $\{O_m^{ij\ldots k}\}$ for a fixed m is a cover of K since $K \subseteq K_m \subseteq O_m$ and $O_m = \bigcup_{ij\ldots k} O_m^{ij\ldots k}$.

Let $r = \max\{r_1, \ldots, r_n\} < 1$. Let $\delta > 0$ and choose α so that $\delta \geq r^\alpha$.

Now we see that

$$\begin{aligned} H_\delta^s(K) &= \inf\left\{\sum_{m=1}^{\infty}(\mathrm{diam}(B_m))^s : K \subseteq \bigcup_{m=1}^{\infty} B_m \text{ and } \mathrm{diam}(B_m) \leq \delta\right\} \\ &\leq \sum(\mathrm{diam}(O_\alpha^{ij\ldots k}))^s \\ &= \sum(dr_ir_j\cdots r_k)^s \end{aligned}$$

where the sum is taken over all sets $O_\alpha^{ij\ldots k}$.

Now we see that for $\alpha = 1$ we have

$$\sum(\mathrm{diam}(O_1))^s = d^s.$$

For $\alpha = 2$ we have

$$\begin{aligned} \sum(\mathrm{diam}(O_2^{ij}))^s &= d^sr_1^s + d^sr_2^s + \ldots d^sr_n^s \\ &= d^s(r_1^s + \ldots + r_n^s) \\ &= d^s(1) \\ &= d^s. \end{aligned}$$

For $\alpha = 3$ we have

$$\begin{aligned}
\sum (\mathrm{diam}(O_3^{ij}))^s &= d^s r_1^s r_1^s + d^s r_1^s r_2^s + \ldots + d^s r_1^s r_n^s + d^s r_2^s r_1^s + \ldots + d^s r_n^s r_n^s \\
&= d^s r_1^s (r_1^s + \ldots + r_n^s) + d^s r_2^s (r_1^s + \ldots + r_n^s) + \ldots + d^s r_n^s (r_1^s + \ldots + r_n^s) \\
&= d^s (r_1^s(1) + \ldots + r_n^s(1)) \\
&= d^s (r_1^s + \ldots + r_n^s) \\
&= d^s(1) \\
&= d^s.
\end{aligned}$$

Proceeding inductively, $\sum_{ij\ldots k} (\mathrm{diam}(O_\alpha^{ij\ldots k}))^s = d^s$ for any α.

Thus $H_\delta^s(K) \leq \sum (d r_i r_j \cdots r_k)^s = d^s < \infty$.

Let $\delta \to 0$. Therefore $H^s(K) < \infty$.

Now we need to show that $H^s(K) > 0$.

Let O_1 be the open set from the Open Set Condition such that $K \subseteq \overline{O_1}$ with $\mathrm{diam}(O_1) \leq 1$. If not, we can scale K and O_1 by a ratio less than one so that $K \subseteq \overline{O_1}$ and $\mathrm{diam}(O_1) \leq 1$ and the dimension will be unaffected. We want to define an additive function $\psi(A)$ on half open figures A such that

$$\psi(A) \leq c(\mathrm{diam}(A))^s$$

where c is a constant.

Define

$$\varphi(O_m^{ij\ldots k}) = (r_i r_j \cdots r_k)^s.$$

Now define for any half-open figure A,

$$\Phi(A) = \lim_{m \to \infty} \sum \varphi(O_m^{ij\cdots k})$$

where the sum is taken over all sets $O_m^{ij\cdots k}$ whose closures lie entirely in A.

Now let $A_\eta = \{\vec{x} \in \mathbb{R}^q : \vec{x} + \vec{h} \in A \text{ where } 0 \leq h_\beta \leq \eta \text{ for all } \beta = 1, \ldots, q\}$.

Lastly, define

$$\psi(A) = \lim_{\eta \to 0^+} \Phi(A_\eta).$$

We need to show that $\psi(A)$ is an additive function on half-open figures:

Let A and B be non-intersecting half-open figures.

Since all $O_m^{ij\ldots k} \subseteq A_\eta$ and all $O_m^{ij\ldots k} \subseteq B_\eta$ are also contained in $A_\eta \cup B_\eta$ then by the principle of inclusion-exclusion we have $\Phi(A_\eta) + \Phi(B_\eta) \leq \Phi(A_\eta \cup B_\eta) - \Phi(A_\eta \cap B_\eta)$. Hence $\lim_{\eta \to 0^+}(\Phi(A_\eta) + \Phi(B_\eta)) \leq \lim_{\eta \to 0^+} \Phi(A_\eta \cup B_\eta)$ since $\text{diam}(A_\eta \cap B_\eta) \to 0$ as $\eta \to 0^+$. Thus $\psi(A) + \psi(B) \leq \psi(A \cup B)$.

Now, for a fixed η choose m so that $\eta/2 >\text{diam}(O_m^{ij\cdots k})$ for all $i, j, \ldots, k$. Pick $O_m^{ij\ldots k} \subseteq (A \cup B)_{\eta/2}$. Now choose some $\vec{x} \in (A \cup B)_{\eta/2} = A_{\eta/2} \cup B_{\eta/2}$. This means $\vec{x} + \vec{h} \in A \cup B$ where $0 \leq h_\beta \leq \eta/2$ for all $\beta = 1, \ldots, q$. Without loss of generality we can assume $\vec{x} + \vec{h} \in A$. Thus $\vec{x} \in A_{\eta/2}$. Since $\text{diam}(O_m^{ij\cdots k}) < \eta/2$ and $\vec{x} \in O_m^{ij\ldots k}$ we have $O_m^{ij\ldots k} \subseteq B_{\eta/2}(\vec{x})$. Thus, since $\vec{x} \in A_{\eta/2}$ we have that $O_m^{ij\ldots k} \subseteq A_\eta$.

Thus since each $O_m^{ij\ldots k} \subseteq A_{\eta/2} \cup B_{\eta/2}$ is entirely contained in either A_η or B_η we have $\Phi(A_{\eta/2} \cup B_{\eta/2}) \leq \Phi(A_\eta) + \Phi(B_\eta)$. Let $\eta \to 0^+$. Hence $\psi(A \cup B) \leq \psi(A) + \psi(B)$

Therefore $\psi(A \cup B) = \psi(A) + \psi(B)$ and $\psi(A)$ is an additive function on half-open figures.

Now we need to show that $\psi(A)$ satisfies the conditions of the Lemma.

Suppose $r_1 \geq r_2 \geq \ldots \geq r_n$. Then $O_2^i, O_3^{ij}, \ldots$ are similar to O_1 but reduced by the ratios $r_i, r_i r_j, \ldots$ respectively. Now arrange all possible ratios in decreasing order and denote them by $R_1 \geq R_2 \geq \ldots$ which converge to 0. Let A be any half-open figure with $\text{diam}(A) = d$. Choose t such that $R_t \geq d \geq R_{t+1}$.

Now let C be a q-dimensional ball whose center is some point in A and has a radius of $2d$. Consider all sets $O_m^{ij\ldots k} \subseteq C$ such that the reduction ratios lie between R_{t+1} and $R_{t+1}r_n$. If some are contained in others, then only count the largest set. Call this collection $\{P_i\}_{i=1}^N$. Note that if the iterated function system did not satisfy the Open Set Condition, the $O_m^{ij\ldots k}$ may not be contained in one another but just overlap. Thus we could have infinitely many P_i.

Since the radius of C is twice the diam(A), we have for η sufficiently small and m sufficiently large that every $O_m^{ij\ldots k}$ whose closure is contained in A_η is also contained in some P_i. Figure 3.2 gives an illustration of this situation.

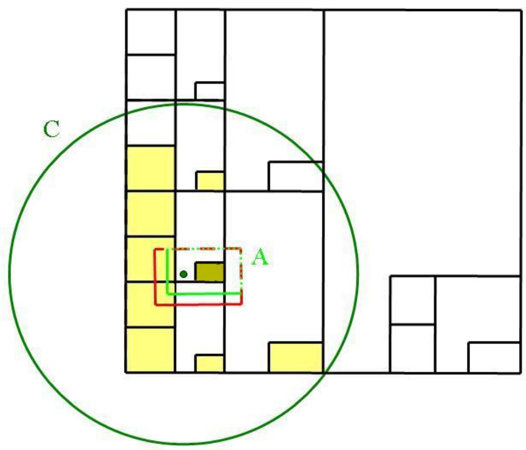

Figure 3.2: The yellow sets represent the P_i. The darker yellow set is contained in A and is also a P_i.

Thus we have

$$\sum \varphi(O_m^{ij\ldots k}) \leq \sum_{i=1}^{N} \varphi(P_i)$$

where the first sum is over all sets $O_m^{ij\ldots k}$ such that the closure of $O_m^{ij\ldots k}$ is contained in A_η. Thus

$$\begin{aligned} \psi(A) &= \lim_{\eta \to 0^+} \lim_{m \to \infty} \sum \varphi(O_m^{ij\ldots k}) \\ &\leq \sum_{i=1}^{N} \varphi(P_i). \end{aligned}$$

We have $|C| = a\pi d^q$ for some $a > 0$. For each P_i we have $|P_i| \geq (R_{t+1} r_n)^q |O_1|$. This implies that

$$N \leq \frac{a\pi d^q}{(R_{t+1} r_n)^q |O_1|}.$$

We see that $\varphi(P_i) \leq R_t^s$ for all i. Also, we have $\dfrac{R_t}{R_{t+1}} \leq \dfrac{1}{r_n}$ and $\dfrac{d^{q-s}}{R_t^{q-s}} \leq (1)^{q-s} = 1$ since $d \leq R_t$ and $s \leq q$.

Thus

$$\begin{aligned} \psi(A) &\leq \sum_{i=1}^{N} \varphi(P_i) \\ &\leq \sum_{i=1}^{N} R_t^s \\ &= N R_t^s \\ &\leq \frac{a\pi d^q R_t^s}{(R_{t+1} r_n)^q |O_1|} \\ &= a\pi d^s \frac{d^{q-s}}{R_t^{q-s}} \frac{R_t^q}{R_{t+1}^q} \frac{1}{r_n^q |O_1|} \\ &\leq \frac{1}{r_n^q} \frac{a\pi d^s}{r_n^q |O_1|} \\ &= \frac{a\pi}{r_n^{2q} |O_1|} d^s. \end{aligned}$$

Therefore we have satisfied the conditions of Lemma 3. Hence $H^s(K) > 0$. $\square$

Corollary 3. *The Hausdorff dimension of an invariant set, K, of an iterated function system satisfying the conditions of Theorem 3.4 is equal to the similarity dimension of K.*

Note that the Open Set Condition is only required to show that $H^s(K) > 0$. This means that for any invariant set, K, of an iterated function system we have $H^s(K) < \infty$ where s is the similarity dimension of the iterated function system.

Chapter 4: Examples

The first fractals discovered were all self-similar fractals. Here we present some of the most famous self-similar fractals. We provide the iterated function systems, their constructions and compute their topological, similarity and Hausdorff dimensions. We will begin with the Cantor set; the first known fractal. We will then proceed to the Sierpinski gasket, followed by the Koch snowflake, Menger sponge, and end with Barnsley's wreath, an example that does not satisfy the Open Set Condition.

4.1 Cantor Set

In 1884 Georg Cantor published a paper on the now-famous Cantor set. Cantor had no idea at the time that he had actually discovered a fractal or how huge the impact of the Cantor set would be in the study of fractals. Many think of the Cantor set as the archetypical fractal. In fact, the Cantor set was explored by Felix Hausdorff in [13] and Besicovitch and Taylor in [5]. Hausdorff computes the Hausdorff dimension of the Cantor set in [13] along with a generalized formula for variations of the Cantor set. Here we will provide an indirect approach to computing the Hausdorff dimension of the Cantor set.

Cantor created the Cantor set, which is a nowhere dense perfect set, in order to prove that all perfect sets have the same cardinality; that is all perfect sets are uncountable. A perfect set is a subset of $\mathbb{R}$ that is closed and has no isolated points. But what is surprising is that even though the Cantor set is uncountable, it is a set of Lebesgue measure zero. In this case the Lebesgue measure can be thought of as the total length of the Cantor set. The total length of the Cantor set is zero, so it can not contain an interval of positive length. But the Cantor set has no isolated points.

This goes against our intuition. As we will see, the dimension of the Cantor set will help us to explain these strange properties.

4.1.1 Construction

We define the Cantor set as the invariant set of an iterated function system. Consider the iterated function system $T = \{T_1, T_2\}$ acting on $\mathcal{K}(\mathbb{R})$ where

$$T_1(x) = x/3 \text{ and } T_2(x) = x/3 + 2/3.$$

We define the iterated function system $T : \mathcal{K}(\mathbb{R}) \to \mathcal{K}(\mathbb{R})$ by

$$T(A) = T_1(A) \cup T_2(A)$$

where $T_1(A) = \{y \in \mathbb{R} : y = x/3 \text{ and } x \in A\}$ and $T_2(A) = \{y \in \mathbb{R} : y = x/3 + 2/3 \text{ and } x \in A\}$. By Theorem 2.4 we know the iterated function system, T, must have a unique invariant set, since it realizes the ratio list $(1/3, 1/3)$ and $1/3 < 1$.

Definition 44. *The* ***Cantor set*** *is the unique fixed point of the iterated function system T.*

By Corollary 1, when applying the iterated function system we can start with any nonempty compact initial set. We choose to start with the closed interval $C_0 = [0, 1]$. We see that $C_1 = [0, 1/3] \cup [2/3, 1]$. Notice that this is the same as removing the middle third of C_0. The next iteration gives us $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. Again, notice that this is the same as removing the middle third of each interval in C_1. We repeat this process to obtain the sequence of approximations $C_0, C_1, C_2, \ldots$ One should also notice that $C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$ We need to show that the Cantor set C is the same as $\bigcap_{k=1}^{\infty} C_k$.

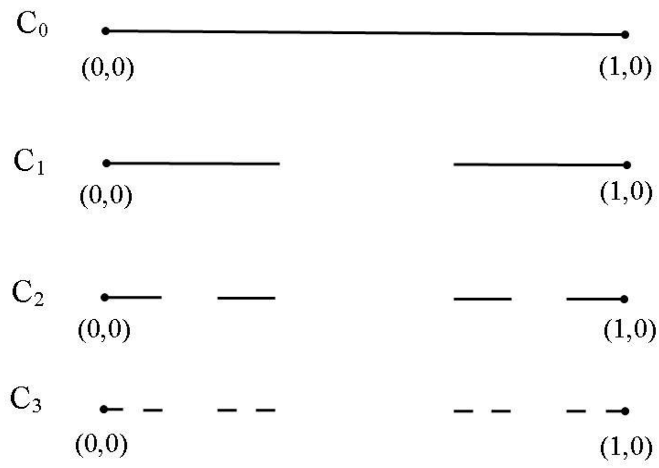

Figure 4.1: C_0 and the first three iterations of T

Proposition 4. *The Cantor set, C, can be written as $C = \bigcap_{k=1}^{\infty} C_k$ where each C_k consists of 2^k disjoint closed intervals of length 3^{-k}. Call these intervals $I_{k,j}$, so that we have $C_k = \bigcup_{j=1}^{2^k} I_{k,j}$.*

Proof. We claim that $C_k \subseteq C_{k-1}$ for all $k \in \mathbb{N}$ where C_k is the disjoint union of 2^k closed intervals of length 3^{-k}:

(By induction) Let $C_0 = [0,1]$ and let $C_k = T(C_{k-1})$.

Let $k = 1$. We have $C_0 = [0,1]$. We see that

$$C_1 = T(C_0) = T_1(C_0) \cup T_2(C_0) = [0,1/3] \cup [2/3,1] \subseteq [0,1] = C_0.$$

Thus $C_1 \subseteq C_0$. We also see that C_1 is the disjoint union of 2 closed intervals of length 1/3.

Now assume for some $k \in \mathbb{N}$ that $C_k \subseteq C_{k-1}$ where C_k is the disjoint union of 2^k closed intervals of length 3^{-k}. So we have that

$$C_{k+1} = T(C_k) = T_1(C_k) \cup T_2(C_k).$$

Now since T_1 and T_2 are linear and one-to-one similarities with ratio $1/3$, we have that $T_1(C_k)$ is the union of 2^k disjoint closed intervals of length $3^{-(k+1)}$ and $T_2(C_k)$ is the disjoint union of 2^k closed intervals of length $3^{-(k+1)}$. Also, $T_1(C_k) \cap T_2(C_k) = \emptyset$. Thus C_{k+1} is the disjoint union of 2^{k+1} closed intervals of length $3^{-(k+1)}$.

Then since $C_k \subseteq C_{k-1}$ we have that

$$T_1(C_k) \subseteq T_1(C_{k-1}) \subseteq C_k$$

and

$$T_2(C_k) \subseteq T_2(C_{k-1}) \subseteq C_k.$$

Thus

$$C_{k+1} = T_1(C_k) \cup T_2(C_k) \subseteq C_k.$$

Therefore for all $k \in \mathbb{N}$ we have $C_k \subseteq C_{k-1}$ where C_k is the disjoint union of 2^k closed intervals of length 3^{-k}.

This gives us that $C = \bigcap_{k=1}^{\infty} C_k$. □

4.1.2 Topological Dimension

The Cantor set has total length 0 which implies the Cantor set contains no interval of positive length. So one would expect the topological dimension to be 0. By using the construction above we can find a basis for the subspace topology that allows us to compute the topological dimension, which is in fact 0.

Proposition 5. *(Prop 3.1.3 in [11]) The sets $M_{k,j} = C \cap I_{k,j}$ where $k = 0, 1, \ldots$ and $j = 1, 2, \ldots, 2^k$ form a basis for the subspace topology on C. They are clopen. Thus C is topologically zero-dimensional.*

Proof. We know that the interval $I_{k,j}$ has length 3^{-k}. Since each interval that is removed at the kth level also has length 3^{-k} we know the distance from $I_{k,j}$ to any other interval, $I_{k,j'}$, is at least 3^{-k}.

Now if $I_{k,j} = [a, b]$, then $M_{k,j} = C \cap [a, b]$, which is closed in C since $[a, b]$ is closed in $\mathbb{R}$. Also since the distance from one interval, $I_{k,j}$, to any other interval, $I_{k,j'}$, is at least 3^{-k} we see that $M_{k,j} = C \cap (a - 3^{-k}, b + 3^{-k})$, which is open in C since $(a - 3^{-k}, b + 3^{-k})$ is open in $\mathbb{R}$. Thus each $M_{k,j}$ is clopen.

Let $\varepsilon > 0$. If $x \in C$, then choose k such that $3^{-k} < \varepsilon$ and choose j such that $x \in I_{k,j}$. This means $x \in M_{k,j} \subseteq C \cap B_\varepsilon(x)$. Thus the collection of all M_{kj} is a basis for the topology of C because $\{B_\varepsilon(x)\}$ is a basis for the standard topology on $\mathbb{R}$. □

4.1.3 Similarity and Hausdorff Dimension

Recall that the Cantor set is the invariant set of an iterated function system with ratio list $(1/3, 1/3)$. So the similarity dimension is the solution s to

$$(1/3)^s + (1/3)^s = 2(1/3)^s = 1.$$

Thus $s = \dfrac{\ln 2}{\ln 3} \approx 0.6309$. Now we can show that the Hausdorff dimension of the Cantor set is equal to s.

Proposition 6. *The Cantor set, C, has Hausdorff dimension $s = \dfrac{\ln 2}{\ln 3}$.*

Proof. We need to satisfy the conditions of Theorem 3.4. The iterated function system T realizes a contracting ratio list. Thus we only need to show that T satisfies the Open Set Condition. Consider $O = (0, 1)$. We see that $T_1(O) = (0, 1/3)$ and $T_2(O) = (2/3, 1)$. Thus $T_1(O) \cup T_2(O) = T(O) \subseteq O$ and $T_1(O) \cap T_2(O) = \emptyset$.

Therefore, by Corollary 3, we have that the Hausdorff dimension of the Cantor set is $\dfrac{\ln 2}{\ln 3}$. □

We see that the dimension of the Cantor set is between 0 and 1. Recall that the Cantor set has total length zero, but has no isolated points. The fractional dimension helps explain why the Cantor set has properties that seem to be somewhere between the properties of 0-dimensional objects and 1-dimensional objects.

4.2 Sierpiński Gasket

In 1915 Waclaw Sierpiński provided an example of a curve that intersects itself at every point. This curve turned out to be the Sierpiński gasket. Although Sierpiński received the credit for this fractal thanks to Mandelbrot, it had been contemplated decades before by several other mathematicians. For example, in 1890 Edouard Lucas discovered that the odd binomial coefficients of Pascal's triangle form a variation of the Sierpiński gasket. Of course he did not realize that his theorem could be interpreted in this way. [23]

The Sierpiński gasket is considered a curve, but the total length is infinite. We could also consider the total area of the Sierpiński gasket. We will see that the total area is 0. So we have a curve of infinite length that appears to take up space but has a total area of 0. Again the dimension will helps us explain these strange properties.

4.2.1 Construction

Consider the iterated function system, $A = \{A_1, A_2, A_3\}$ where the $A_i : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by

$$A_1(\vec{x}) = 1/2\vec{x}$$

$$A_2(\vec{x}) = 1/2\vec{x} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

$$A_3(\vec{x}) = 1/2\vec{x} + \begin{pmatrix} 1/4 \\ \sqrt{3}/4 \end{pmatrix},$$

respectively.

By Theorem 2.4 we know the iterated function system, A, has a unique invariant set, since it realizes the ratio list $(1/2, 1/2, 1/2)$.

Definition 45. *The **Sierpiński gasket**, S, is the unique fixed point of A, i.e. $S = A(S) = A_1(S) \cup A_2(S) \cup A_3(S)$.*

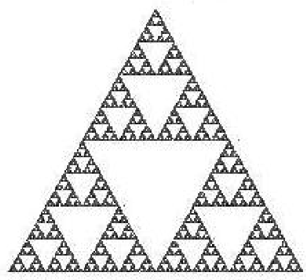

Figure 4.2: Sierpiński gasket

By Corollary 1, we know that by starting with any nonempty compact initial condition, we can obtain the unique fixed point of A, which in this case is the Sierpiński gasket. Define S_0 to be the equilateral triangle with side lengths 1 having corner points $(0,0), (1/2, \sqrt{3}/2)$, and $(1,0)$. The first iteration of A gives us S_1. We can see that this is the same as dividing S_0 into four smaller equilateral triangle and removing the middle triangle. The second iteration, S_2, is the same as dividing each triangle in S_1 into four smaller triangles and again removing the middle triangle from each of the larger triangles.

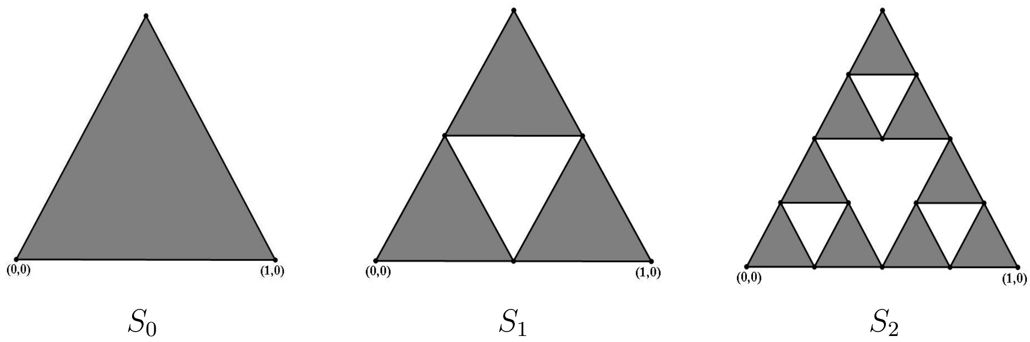

Figure 4.3: S_0 and the first two iterations of A

If we continue with this process we see that the Sierpiński gasket can be written as $S = \bigcap_{k \in \mathbb{N}} S_k$ where each S_k consists of 3^k triangles with side lengths 2^{-k}. We can think of each $S_k = \bigcup_{j=1}^{3^k} S_{k,j}$ where each $S_{k,j}$ is a triangle of side length 2^{-k}. This is called construction by tremas.

4.2.2 Topological Dimension

As we mentioned previously, the Sierpiński gasket is actually a curve. Thus we would expect its topological dimension to be 1. Using the construction by tremas, we can find a basis to compute the topological dimension.

Proposition 7. *The Sierpiński gasket, S, has small inductive dimension 1.*

Proof. First we need to show that S is not zero-dimensional. So consider the equilateral triangle, S_0, in $\mathbb{R}^2$ with side lengths equal to 1 containing the interval $I = [0,1] \times \{0\}$.

We claim $I \subseteq A(I)$:

We see that $A_1(I) = [0, 1/2] \times \{0\}$ and $A_2(I) = [1/2, 1] \times \{0\}$. Thus

$$A_1(I) \cup A_2(I) = ([0, 1/2] \times \{0\}) \cup ([1/2, 1] \times \{0\}) = [0,1] \times \{0\} = I.$$

Recall $A(I) = A_1(I) \cup A_2(I) \cup A_3(I)$. Also by Corollary 1, we know $A(I) \subseteq A(S)$. So we have $I = A_1(I) \cup A_2(I) \subseteq A(I) \subseteq A(S) = S$.

We have shown that $I \subseteq S$. Notice that $\mathbb{R} \cong \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$. Since $[0,1] \times \{0\} = I \subseteq \mathbb{R} \times \{0\}$ we know $I \cong [0,1]$. We proved earlier that an interval in $\mathbb{R}$ is not zero-dimensional. Thus I is not zero-dimensional. Now by Theorem 3.2, we know $0 < \operatorname{ind} I \leq \operatorname{ind} S$. Thus S is not zero-dimensional.

Next we need to show there exists a basis for the topology such that the basis elements have zero-dimensional boundary. We need to consider two different types of basis elements. Let $T_{k,j}$ be $S_{k,j}$ minus any corner points except the corner points of S_0. We have two possibilities of $T_{k,j}$. Examples of each type are shown below in Figure 4.4.

Let the first type of basis element be $M_{k,j} = T_{k,j} \cap S$.

Next, let R_{k,j_1,j_2} be the union of two adjacent triangles, S_{k,j_1} and S_{k,j_2} minus any corner points not shared between the two adjacent triangles except the corner points of S_0. Again, we have two different possibilities of R_{k,j_1,j_2} as shown below in Figure 4.5.

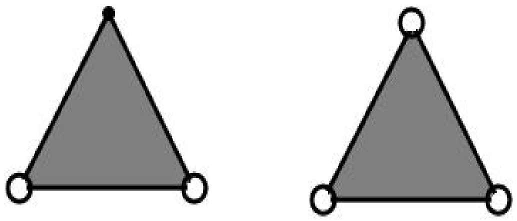

Figure 4.4: Two possible $T_{k,j}$

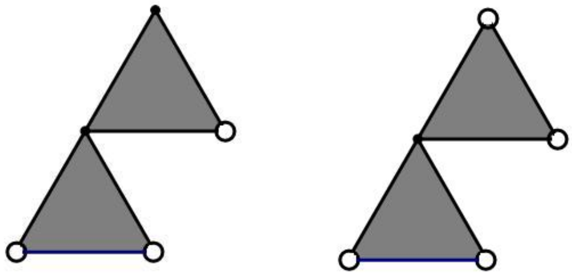

Figure 4.5: Two possible R_{k,j_1,j_2}

The second type of basis element is $N_{k,j_1,j_2} = R_{k,j_1,j_2} \cap S$.

We claim that the collection of all $M_{k,j}$ and N_{k,j_1,j_2} forms a basis for the topology of S:

Let $\varepsilon > 0$. The topology on S is the subspace topology. So the standard basis is $\{B_\varepsilon(x) \cap S\}$. If $x \in S$, then either x is a corner point of $S_{k,j}$ for some k and j or not. First, if it is not a corner point of some triangle $S_{k,j}$, then choose k such that $2^{-k} < \varepsilon$ and j such that $x \in M_{k,j}$. Since the length of the sides of $S_{k,j}$ are 2^{-k} we know $M_{k,j} \subseteq S \cap B_\varepsilon(x)$. Now if x is a corner point of some triangle $S_{k,j}$, then choose k such that $2^{-k+1} < \varepsilon$ and j_1 and j_2 such that $x \in N_{k,j_1,j_2}$. Since the length of the sides of the triangle S_{k,j_1} and S_{k,j_2} are 2^{-k} we know $N_{k,j} \subseteq S \cap B_\varepsilon$. Thus we have shown that the collection of $M_{k,j}$ and $N_{k,j}$ forms a basis of the topology of S.

Lastly, we need to show that the boundary of the basis elements are zero-dimensional. We will show this for both basis element types. First consider some basis element

$M_{k,j}$. We need to find $\partial M_{k,j}$. So we have

$$\begin{aligned}\partial M_{k,j} &= \overline{M_{j,k}} \cap \overline{S - M_{k,j}} \\ &= \overline{T_{k,j} \cap S} \cap \overline{S - M_{k,j}} \\ &= (\overline{T_{k,j}} \cap S) \cap \overline{S - M_{k,j}}.\end{aligned}$$

Since S is closed we know $\overline{T_{k,j} \cap S} = \overline{T_{k,j}} \cap S$. We also know that $\overline{S - M_{k,j}} = S - M_{k,j}$ since $M_{k,j}$ is open and S is closed. Thus we need to find $\overline{T_{k,j}}$.

Recall that $T_{k,j}$ is a triangle missing either two or three corner points. Choose one of these missing points. Call it x. Let $\varepsilon > 0$. Consider $B_\varepsilon(x)$. Since x is an endpoint of a some open interval in $T_{k,j}$, we know there exists a $y \in T_{k,j}$ such that $y \neq x$ and $y \in B_\varepsilon(x)$. Thus x is a limit point of $T_{k,j}$.

Now choose a point other than the missing points and not in $T_{k,j}$. Call it z. We see that there is a positive distance, δ, from z to $T_{k,j}$. This means there exists $B_\delta(z)$ such that $B_\delta(z) \cap T_{k,j} = \emptyset$. Thus z is not a limit point of $T_{k,j}$.

Therefore $\overline{T_{k,j}} = S_{k,j}$.

So we have $\overline{T_{k,j}} \cap S = S_{k,j} \cap S = \overline{M_{j,k}}$.

We see that

$$\partial M_{k,j} = \overline{M_{j,k}} \cap \overline{S - M_{k,j}} = (S_{k,j} \cap S) \cap (S - M_{k,j})$$

is the two or three missing corner points. Therefore, since a finite set of points is zero-dimensional, we know $\partial M_{k,j}$ is zero-dimensional.

Next, we need to consider the other type of basis element, N_{k,j_1,j_2}. Recall that N_{k,j_1,j_2} is the union of two adjacent triangles missing either 2, 3, or 4 corner points. So by a similar argument as above, we have that $\partial N_{k,j_1,j_2}$ consists of the 2, 3, or 4 missing corner points. Thus since a finite number of points is zero-dimensional, $\partial N_{k,j_1,j_2}$ is zero-dimensional.

Therefore we have shown that there exists a basis of S such that the boundary of the basis elements is zero-dimensional. Hence, S has small inductive dimension 1. $\square$

4.2.3 Similarity and Hausdorff dimension

The Sierpiński Gasket is the invariant set of an iterated function system realizing the ratio list $(1/2, 1/2, 1/2)$. This means the similarity dimension is the solution s to

$$(1/2)^s + (1/2)^s + (1/2)^s = 3(1/2)^s = 1.$$

Thus $s = \dfrac{\log 3}{\log 2} \approx 1.585$.

Proposition 8. *The Hausdorff dimension of the Sierpiński Gasket is* $s = \dfrac{\log 3}{\log 2}$.

Proof. We need to show that the iterated function system, A, satisfies the Open Set Condition. Consider O to be the open equilateral triangle with corner points $(0, 0), (1/2, \sqrt{3}/2)$, and $(1, 0)$.

We see that $O_1 = A_1(O)$ is the open triangle with corner points $(0, 0), (1/4, \sqrt{3}/2)$, and $(1/2, 0)$. Next we have $A_2(O) = O_2$ is the open triangle with corner points $(1/2, 0)$, and $(3/4, (\sqrt{3} + 2)/4), (1, 0)$. Last we have $A_3(O) = O_3$ is the open triangle with corner points $(1/4, \sqrt{3}/4), (1/2, \sqrt{3}/2)$, and $(3/4, (\sqrt{3}+2)/4)$. Thus we see that $A_i(O) \subseteq O$ for $i = 1, 2, 3$ and $A_i(O) \cap A_2(O) = \emptyset$ for $i \neq j$ and $i, j = 1, 2, 3$. See Figure 4.6.

Thus A satisfies the Open Set Condition. Therefore by Corollary 3 we know the Hausdorff dimension of the Sierpinski gasket is $s = \dfrac{\log 3}{\log 2}$. $\square$

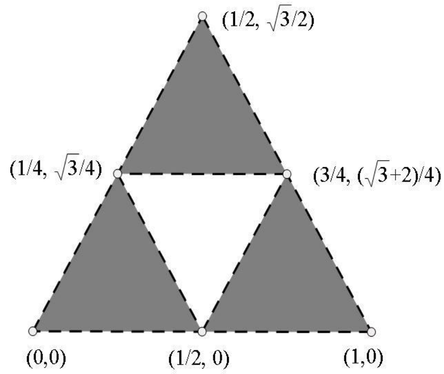

Figure 4.6: Open triangles that satisfy the Open Set Condition

We see that the dimension of the Sierpiński gasket is between 1 and 2. Recall that the Sierpiński gasket is a curve of infinite length but takes up 0 total area. Again, the fractional dimension helps explain how we can have such an object that possesses properties that lie between 1-dimensional and 2-dimensional properties.

4.3 Koch Snowflake

Karl Weierstrass introduced a continuous function that is nondifferentiable in 1872. Weierstrass described this function using only formulas and analysis with no pictures included. For a visual person, it wasn't apparent why the function was not differentiable. It wasn't until 1904 that Helge von Koch described such a curve from a geometrical view. The curve he describes in [15] is now referred to as the Koch curve.

The Koch snowflake in Figure 4.7 is made up of three copies of the Koch curve. Like our previous two examples, the Koch snowflake has some interesting properties. The boundary of the Koch snowflake is a curve of infinite length that encloses a finite area.

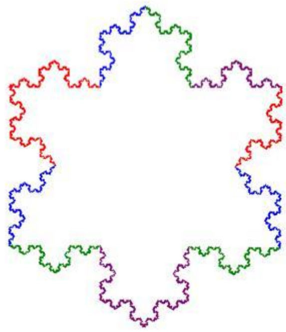

Figure 4.7: Koch snowflake

4.3.1 Construction

First, let's look at the Koch curve. Consider the iterated function system, $B = \{B_1, B_2, B_3, B_4\}$ where the $B_i : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by

$$B_1(\vec{x}) = 1/3\vec{x},$$

$$B_2(\vec{x}) = 1/3 \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix},$$

$$B_3(\vec{x}) = 1/3 \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1/2 \\ \sqrt{3}/6 \end{pmatrix},$$

$$B_4(\vec{x}) = 1/3\vec{x} + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}.$$

By Theorem 2.4 we know the iterated function system, B, has a unique invariant set, because it realizes the ratio list $(1/3, 1/3, 1/3, 1/3)$.

Definition 46. *The **Koch Curve**, K, is the unique fixed point of B, i.e. $K = B(K) = B_1(K) \cup B_2(K) \cup B_3(K) \cup B_4(K)$.*

Using Corollary 1 we can start with any nonempty compact set. Let $K_0 = [0, 1]$. Now applying the iterated function system we get

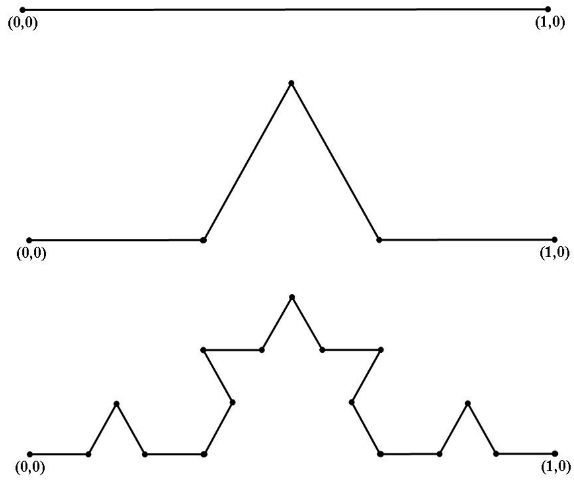

Figure 4.8: K_0, K_1 and K_2

4.3.2 Topological, Similarity and Hausdorff Dimension

The Koch curve is considered a curve since it has topological dimension 1. We do not include a proof here, but note that we cannot use the same method as the Sierpiński gasket since the Koch curve does not contain an interval of positive length.

The Koch curve is the invariant set of an iterated function system realizing the ratio list $(1/3, 1/3, 1/3, 1/3)$. Thus the similarity dimension of the Koch curve is the solution, s, to

$$(1/3)^s + (1/3)^s + (1/3)^s + (1/3)^s = 1.$$

Hence $s = \dfrac{\log 4}{\log 3} \approx 1.26186$.

Proposition 9. *The Hausdorff dimension of the Koch curve, K, is $s = \dfrac{\log 4}{\log 3}$.*

Proof. Let O be an open equilateral triangle with corner points $(0,0), (1/2, \sqrt{3}/2)$ and $(1,0)$. We see that $B_1(O) = O_1$ is the interior of the triangle with corner points $(0,0), (1/6, \sqrt{3}/6)$, and $(1/3, 0)$. Next, $B_2(O) = O_2$ is the interior of the triangle with corner points $(1/3, 0), (1/6, \sqrt{3}/6)$, and $(1/2, \sqrt{3}/6)$ and $B_3(O) = O_3$ is the interior of the triangle with corner points $(1/2, \sqrt{3}/6), (5/6, \sqrt{3}/6)$, and $(2/3, 0)$. Last we have $B_4(O) = O_4$ is the interior of the triangle with corner points $(2/3, 0), (5/6, \sqrt{3}/6)$, and $(1, 0)$. See Figure 4.9.

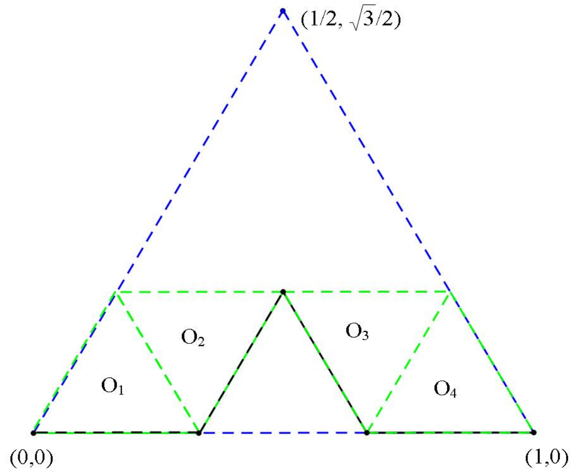

Figure 4.9: The open triangles O_1, O_2, O_3 and O_4.

We see that $B_i(O) \subseteq O$ for $i = 1, 2, 3, 4$. Also $B_i \cap B_j = \emptyset$ for $i, j = 1, 2, 3, 4$ and $i \neq j$. Thus the iterated function system $B = (B_1, B_2, B_3, B_4)$ satisfies the open set condition.

Therefore by Corollary 3 we know the Hausdorff dimension of the Koch curve is $s = \dfrac{\log 4}{\log 3}$. $\square$

We see that $l(K_0) = 1, l(K_1) = 4/3, l(K_2) = 16/9 = (4/3)^2, l(K_3) = 64/27 = (4/3)^3, \ldots, l(K_n) = (4/3)^n, \ldots$ where $l(K_i)$ is the total length of K_i. So the total length of the Koch curve is $\lim_{n \to \infty} (4/3)^n = \infty$. Thus the Koch curve has infinite length.

Now, to obtain the Koch Snowflake we apply the iterated function system, B, separately to the sides of an equilateral triangle with side lengths 1. Each side of the triangle will create a copy of the Koch curve. The area of the equilateral triangle is $\sqrt{3}/4$ and one can calculate that the area contained inside the Koch snowflake is $8/5$

of the area of the original equilateral triangle. So we see that the area of the Koch Snowflake is finite. Thus we have that the Koch Snowflake has an infinite perimeter but a finite area.

4.4 Menger Sponge

Karl Menger discovered the Menger sponge in 1926 in the form of a universal curve. A universal curve, I, has topological dimension 1 and every compact metric space of topological dimension 1 is homeomorphic to a subset of I [19]. As with all fractals, the Menger sponge possesses properties that are not intuitive. The Menger sponge may have the most bizarre properties out of the fractals we have considered so far. Even though the Menger sponge has topological dimension 1, it has infinite surface area but zero volume [12].

4.4.1 Construction

The Menger sponge is the invariant set of an iterated function system, G, consisting of 20 similarities each having a ratio of 1/3. Since G realizes a contracting ratio list we know by Theorem 2.4 that G has a unique invariant set.

The Menger sponge is easy to describe using construction by tremas, that is removing certain pieces. By Corollary 1 we know we can start with any nonempty compact initial set. Let M_0 be the unit cube in $\mathbb{R}^3$ pictured in Figure 4.10. Now divide M_0 into 27 smaller cubes with side length 1/3 and remove the middle cube out of every face and the center most cube. Thus M_1 in Figure 4.11 consists of the 20 cubes left. Now divide each smaller cube into 27 cubes each with a side length of 1/9 and remove the middle cube of each face and the center most cube. So M_2 consists of the 20^2 cubes with side lengths 1/9 shown in Figure 4.12. So M_n will be 20^n cubes with side lengths $(1/3)^n$. This creates a sequence of sets $M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$. Thus the Menger sponge can be written as $\bigcap_{k=1}^{\infty} M_k$ where M_k consists of 20^n cubes with side lengths $(1/3)^n$.

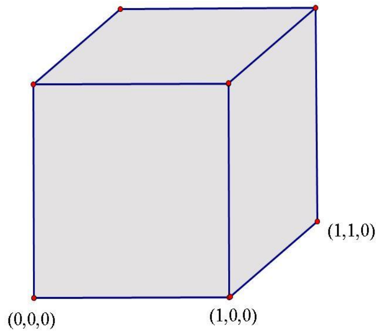

Figure 4.10: M_0

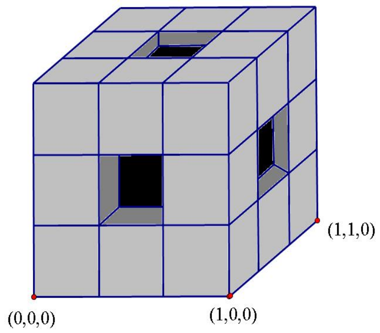

Figure 4.11: M_1

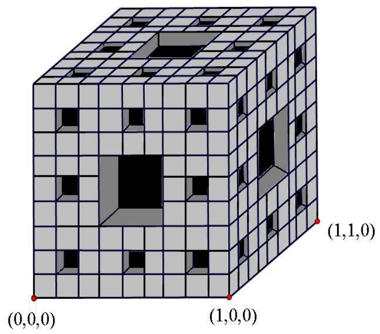

Figure 4.12: M_2

4.4.2 Topological, Similarity and Hausdorff Dimension

As mentioned before, the Menger sponge is considered a curve since it has topological dimension 1. Again, we do not include a proof.

Recall that the Menger sponge is the invariant set of the iterated function system G consisting of 20 similarities each with a ratio of 1/3. So the similarity dimension is the solution s to

$$(1/3)^s + \ldots + (1/3)^s = 20(1/3)^s = 1.$$

Thus $s = \dfrac{\log 20}{\log 3} \approx 2.72683$.

Proposition 10. *The Menger sponge, M, has Hausdorff dimension $s = \dfrac{\log 20}{\log 3}$.*

Proof. Let O be the open unit cube. From the construction above, we can see that the 20 subcubes are each a subset of O. Thus $G(O) \subseteq O$. Also each subcube does not intersect any of the others. So we have $G_i \cap G_j = \emptyset$ for $i, j = 1, \ldots, 20$ and $i \neq j$.

Therefore by Corollary 3 we know the Hausdorff dimension of the Menger sponge is $s = \dfrac{\log 20}{\log 3}$. □

Recall that the Menger sponge has infinite surface area but zero volume. Again, we see that the fractal dimension gives us an explanation for why the Menger sponge has properties lying between 2 and 3-dimensional properties, even though it is topologically 1-dimensional.

4.5 Barnsley Wreath

The Barnsley wreath in Figure 4.14 was created by Michael Barnsley in [2] as an example of a fractal that is the invariant set of an iterated function system that does not satisfy the open set condition. In a sense, the images of the different maps in the iterated function system overlap too much. The iterated function system associated to the Barnsley wreath consists of six similarities, $(f_1, f_2, f_3, f_4, f_5, f_6)$. The first three similarities, f_1, f_2, and f_3, rotate by 180 degrees and contract by a ratio of 1/2. The last three similarities, f_4, f_5, and f_6 also rotate 180 degrees but contract by a ratio of 1/4. In Figure 4.13, the dark blue hexagon is the initial set, the bright blue hexagons have been reduced by a ratio of 1/2 and the green hexagons have been reduced by a ratio of 1/4. So as we can see in Figure 4.13 the six hexagons in the first iteration overlap at more than an edge or points. This is why the Open Set Condition is not satisfied.

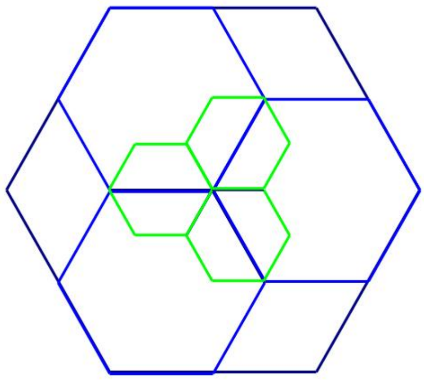

Figure 4.13: The first iteration of the iterated function system.

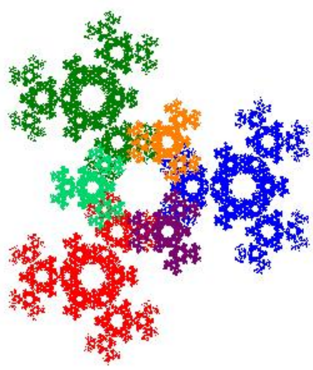

Figure 4.14: Barnsley wreath

From the iterated function system describe above we have that the ratio list of the iterated function system is $(1/2, 1/2, 1/2, 1/4, 1/4, 1/4)$. Thus we can compute the similarity dimension. We have that the similarity dimension is the solution s to

$$(1/2)^s + (1/2)^s + (1/2)^s + (1/4)^s + (1/4)^s + (1/4)^s = 3(1/2)^s + 3(1/4)^s = 1.$$

Thus $s = \dfrac{\log(3+\sqrt{21})}{\log 2} - 1 \approx 1.9227$. Now since the iterated function system does not satisfy the open set condition, we cannot use Corollary 3 to conclude that the similarity dimension is equal to the Hausdorff dimension.

In order to compute the actual Hausdorff dimension of the Barnsley wreath, Gerald Edgar uses a technique called graph self-similarity, which is a generalization of standard self-similarity. The method can be found in [11] and the specific calculation for the Barnsley wreath can be found in [9]. We read in [9] that the Hausdorff dimension of the Barnsley wreath is actually is $s_1 \approx 1.8459$. We see that this is strictly less than the similarity dimension.

Chapter 5: Conclusion

As we discussed earlier, fractals are found all around us, even inside us. Fractals are more than just interesting mathematical objects; they are very applicable to major areas of study such as physics, biology, and medicine. Therefore the study of fractals is not only fun and interesting from a mathematical view, but is also helping in the advancement of technology and medicine.

5.1 Further Study

One property that sets fractals apart is their dimension. Thus fractal dimension is a major concept in the study of fractals and their applications. Further study may include the study of other fractals dimensions. The Hausdorff dimension is the most common fractal dimension used but other fractal dimensions include packing dimension, box dimension, and Bouligand dimension [10], [11]. One could also study the concept of graph self-similarity to compute the Hausdorff dimension of fractals that do not satisfy the Open Set Condition.

Other directions of study include calculating the fractal dimensions of more complicated sets such as the Mandelbrot set and investigating more deeply into the practical and mathematical applications of fractals.

Bibliography

[1] Apsell, Paula (Producer); "Hunting the Hidden Dimension", *NOVA*,PBS (2008), DVD, Accessed April 2, 2011.

[2] Barnsley, Michael; *Fractals Everywhere*, San Diego, CA: Academic Press, Inc. (1988).

[3] Besicovitch, A. S.; "On Rational Approximations to Real Numbers", *Journal of the London Mathematical Society*, **9**: 126-131 (1934).

[4] Besicovitch, A. S.; "On the Sum of Digits of Real Numbers Represented in the Dyadic System", *Mathematische Annalen*, **110**:321-330 (1934).

[5] Besicovitch, A. S. and Taylor, S. J.; " On the Complementary Intervals of a Linear Closed Set", *Journal of the London Mathematical Society*, **29**: 449-459 (1954).

[6] Besicovitch, A. S. and Ursell, H. D.; "On Dimensional Numbers of Some Continuous Curves", *Journal of the London Mathematical Society*, **12**: 18-25 (1937).

[7] Cantor, Georg; "On the Power of Perfect Sets of Points", *Classics on Fractals*, Reading, MA: Addison-Wesley (1993).

[8] Carathéodory, Constantin; "On the Linear Measure of Point Sets-a Generalization of the Concept of Length", *Classics on Fractals*, Reading, MA: Addison-Wesley (1993).

[9] Edgar, G. A. "A Fractal Puzzle", *The Mathematical Intelligencer*, **13** No. 3: 44-50 (1991).

[10] Edgar, Gerald A.; *Integral, Probability, and Fractal Measures*, New York: Springer-Verlag (1998).

[11] Edgar, Gerald A.;*Measure, Topology, and Fractal Geometry*, New York: Springer-Verlag (1990).

[12] Gleick, James; *Chaos: Making a New Science*, New York: Peguine Books (1987).

[13] Hausdorff, Felix; "Dimension and Outer Measure", *Classics on Fractals*, Reading, MA: Addison-Wesley (1993).

[14] Hutchinson, John E.; "Fractals and Self Similarity", *Indiana Mathematics Journal*, **30**: 713-747 (1981).

[15] Koch, Helge von; "On a Continuous Curve without Tangent Constructible from Elementary Geometry", *Classics on Fractals*, Reading, MA: Addison-Wesley (1993).

[16] Lévy, Paul; "Plane or Space Curve and Surfaces Consisting of Parts Similar to the Whole", *Classics on Fractals*, Reading, MA: Addison-Wesley (1993).

[17] Mandelbrot, Benoit; "How Long is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension", *Science*, **156**: 636-639 (1967).

[18] Mandelbrot, Benoit; *The Fractal Geometry of Nature*, San Fransisco: Freeman (1982).

[19] Menger, Karl; "General Spaces and Cartesian Spaces", *Classics on Fractals*, Reading, MA: Addison-Wesley (1993).

[20] Moran, P. A. P.; "Additive Functions of Intervals and Hausdorff Measure", *Proceedings of the Cambridge Philosophical Society*, **42**: 15-23 (1946).

[21] Munkres, James R.; *Topology*, 2nd Ed., Upper Saddle River, NJ: Prentice Hall, Inc. (2000).

[22] Royden, H. L. and Fitzpatrick, P. M.; *Real Analysis*, 4th Ed., Boston, MA: Pearson Education, Inc. (2010).

[23] Stewart, Ian; "Four Encounters with Sierpinski's Gasket", *The Mathematical Intelligencer*, **17** No. 1: 52-64 (1995).

[24] Weierstrass,Karl; "On Continuous Functions of a Real Argument that do not have a Well-Defined Differential Quotient", *Classics on Fractals*, Reading, MA: Addison-Wesley (1993).

[25] Wu, William; *Fractals* (2005), `http://www.ocf.berkeley.edu/~wwu/fractals/fractals.html`, Accessed March 30, 2011.

Vita

Melissa A. Glass was born in Marshall, IL on December 14, 1986. She did her undergraduate study at Berry College in Mount Berry, GA graduating in May 2009 with a B. S. with Honors in mathematics. She then obtained a M. A. in mathematics at Wake Forest University in May 2011. She is a member of the American Mathematical Society, Mathematical Association of American, and Pi Mu Epsilon, Wake Forest chapter. Melissa attended Wake Forest under a Teaching Assistantship for her first three semesters. During her last semester at Wake Forest University, she worked as an adjunct at Forsyth Technical Community College teaching mathematics. She plans to continue teaching mathematics at the college level and earn a Ph.D. in mathematics.