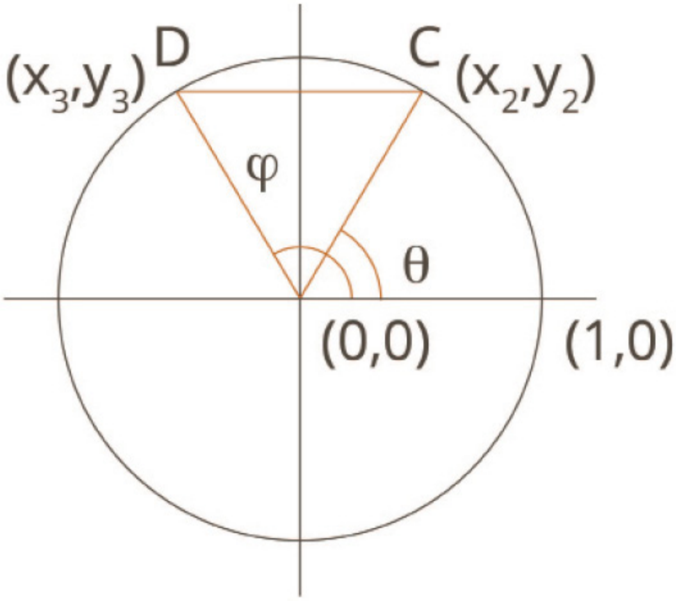
The University of Texas at Austin
OnRamps

Discovery Precalculus:

A Creative and Connected Approach



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University of Texas at Austin

In memory of Ted Odell, Professor of Mathematics at the University of Texas at Austin, whose efforts helped launch this project. Also, in memory of Professor of Mathematics Efraim Armedariz, whose ideas and creativity inspired some of the explorations in this text.

Contents

PREFACE

Discovery Precalculus: A Creative and Connected Approach

To the Student

It is my genuine hope that the student/learner experiences this text in an exploratory and collaborative manner. The presentation of topics in this text is done in an inquiry-based format so as to invite the learner to actively participate in the Lessons that develop the material of this innovative Precalculus course. The topics covered in this text encourage students to recall previous knowledge, analyze the current Exploration, and then synthesize prior knowledge and new concepts. The Precalculus concepts in this course are introduced in such a way as to emphasize the connections between these topics at a university level of thinking. I believe that the material and mathematical connections made in this text are essential to preparing students to be creative and independent thinkers who are prepared for success in various future paths.

Based on future degree and career choices, some students/learners may need to understand the mathematical concepts in this text in more depth than others. The reason for the exploratory and collaborative nature of this mathematics text is to engage all students in the synthesis of content to deepen critical thinking skills; this is why you may notice less of a focus on a particular topic and more emphasis placed on the merging of prior and new knowledge.

This course is meant to deepen students' mathematical content knowledge, but also meant to foster creativity, collaboration, communication, and persistence. As students work through the Explorations in this text, they will interact with their peers and their instructor at a level they may not have previously experienced in a mathematics course. If the student completes this course with better critical thinking skills, communication skills, and an appreciation for productive inquiry, then the student is empowered for success in any future path chosen.

Sincerely,

The Author

INTRODUCTION

Discovery Precalculus: A Creative and Connected Approach

The layout of this text is presented in sections labeled by “Lesson” of instruction. These Lessons are of varied length, some of which can be covered in less than one class period and others may take multiple class periods to complete. It may be the case that you don’t use all of the Explorations within, or that you don’t finish all of the Explorations within each Lesson. Always keep in mind that the implementation of this course should not be done with the mindset of “racing to the end of the book.”

The key to experiencing this course appropriately is to approach the associated teaching and learning with a flexible mindset. Let the Explorations take you in many directions based upon the presentation and discussion of the material under investigation. Consider and be open to the fact that there are often multiple ways to approach or obtain the desired result for a given Exploration. Much of the learning in this course will come from listening to others’ justifications and explanations of how a result was obtained.

Let us begin!

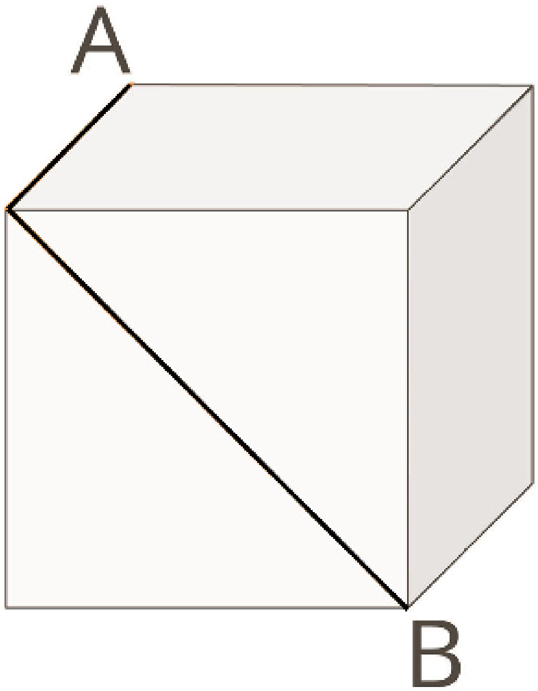
UNIT 0

Essential Pre-Requisite Unit

Lesson 0.1: Getting Started – Thinking Like a Mathematician

The three Explorations of this section are meant to "get you started" on the right track regarding the expectations of this text. These three Explorations of Lesson 0.1 are designed to be accessible to you no matter what your mathematics background is at this point. This text is meant to accompany a mathematics course that, among other things, is about "thinking." Starting with Lesson 1.1, it is our intention that the activities presented within will entice you to think deeply about the mathematics of precalculus, about new ideas presented, and about the connections between the two. With this in mind, let's get started.

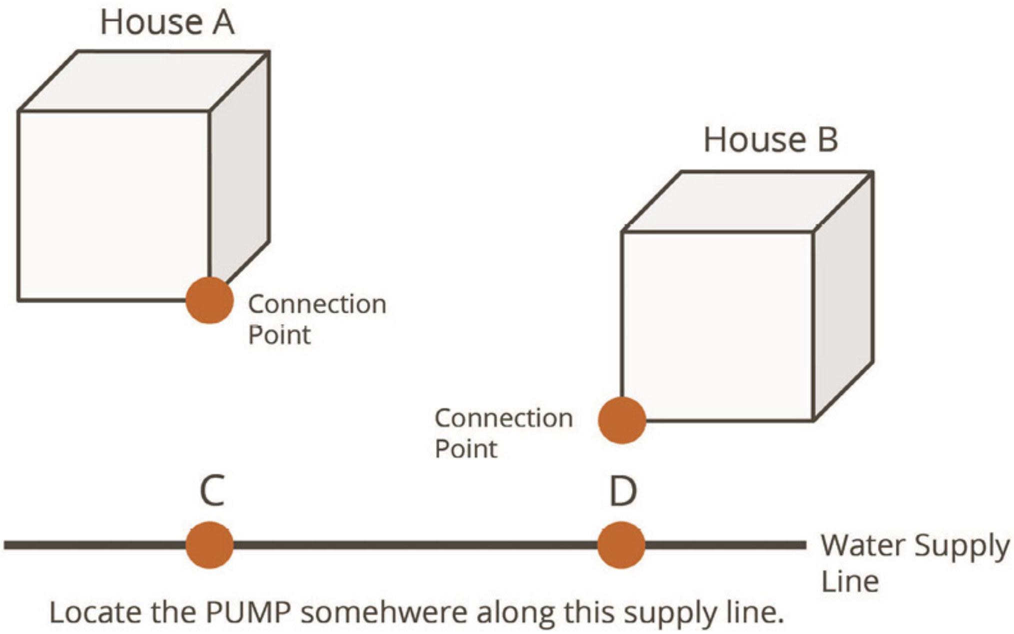
Exploration 0.1.1: A Common Cube Conundrum



Suppose one desired to move along the surface of the cube from the top far corner (point A) to the near bottom corner (point B). Would the dark line path marked in the picture be the shortest route from point A to point B? If you agree, justify your answer. If you do not agree, describe the shortest path and justify your answer. Is the shortest path solution that you have chosen unique?

Exploration 0.1.2: The Efficient Waterline

The city wishes to connect two houses to an existing water supply line using a single pump at the supply line and a minimal amount of pipe extended to each house. The houses are located at different distances from the street (as pictured). In the interval from C to D, where should the pump connection be located on the supply line? Of course, you must justify your answer. Assume *C* and *D* are located directly beneath the connection point on houses *A* and *B* respectively.



As mentioned previously, it is important that you also make connections between mathematics concepts you are asked to remember and that you newly learn in this course. The next Exploration is a good example of making connections between topics and concepts in mathematics.

Exploration 0.1.3: Making Connections

Given the three topics listed below, devise a visual, verbal, and algebraic way of connecting the concepts:

1. The distance formula
2. The standard equation of a circle (not centered at the origin)
3. The Pythagorean Theorem

This Exploration is a good precursor for the topic of trigonometry based on triangle relationships on the Unit Circle. You will learn about this in a later Unit of the text.

UNIT 1

Functions, Rates, and Patterns

Lesson 1.1: What is a Function?

In this section, we will take a deeper look at the concept of *function* in terms of a function's definition, type, and general properties. You have encountered and worked with functions in your mathematical experiences so far, but have you really thought much about what constitutes a function?

Exploration 1.1.1: Function?

Work in groups to answer the question, "What is your definition of a *function*?" Each group should agree upon and present one definition.

Each group should also display their agreed-upon definition for the class and all definitions should be recorded by each student. At this point, however, a formal definition of function should not yet be formulated.

Exploration 1.1.2: Function Identification

Decide, in groups, which of the relations listed below are examples of functions; justify your answers.

	Function?	Justification
1	$y = -x^2$	

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<tr><th></th><th>Function?</th><th>Justification</th></tr>
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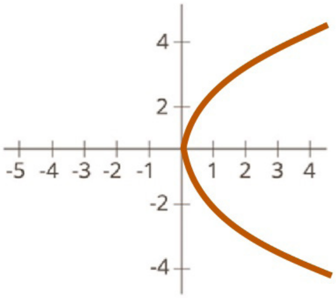
<tr><td>4</td><td>
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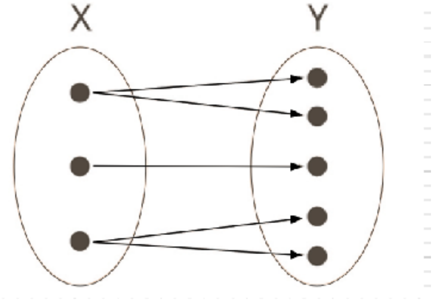
	Function?	Justification
5	$y^4 = 8x^2$	Also identify: What does the graph of this relation look like?
6		
7	Tom: Blue Jill: Brown Bill: Green Harry: Green	

Exploration 1.1.3: Function Identification

As you discuss these problems, consider the meaning of the symbols in the set-builder notation. These symbols should be discussed in class as part of the next section.

1. Let $A = \{a, b, c\}$, $B = \{4, 5, 6\}$, and $f = \{(a, 6), (b, 4), (c, 6)\}$. Is f a function from A to B?

2. Let $A = \{1, 2, 3\}$, $B = \{c, d, e\}$, and $g = \{(1, d), (2, c), (1, e)\}$. Is g a function from A to B?

3. Let M be the set of all museums and N the set of all countries.
   Consider the set of all ordered pairs (m, n) such that museum m is in country n.
   This can be written symbolically as

   $L = \{(m, n) \in M \text{ x } N \mid \text{the museum } m \text{ is in the country } n\}$.

   Is L a function from M to N?

4. Let D be the set of all dogs, and let

   $C = \{(d, o) \in D \text{ x } D \mid \text{the dog } d \text{ is a parent of the offspring } o\}$.

   Is C a function from D to D?

Take-home exercise

Reflect on your informal work in trying to describe what a function is and consider the various group definitions of *function* presented. Now revise the definition you originally created for describing a function in order to develop a more refined definition. Explain your reasons for refining (or not refining) your definition.

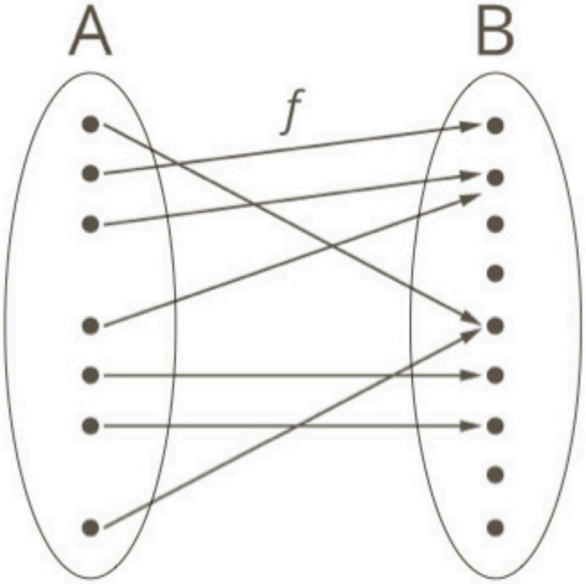
Lesson 1.2: Functions and Types of Functions

We will now attempt to formalize our definition of function by providing three textbook definitions of the concept of function. Note the use of the symbols in each definition. As you read over the three provided definitions of a function, you are asked to consider the commonality and differences between these definitions and the one that you have previously written. Three examples of how a function might be defined are:

DEFINITION Given two sets A and B, the set $A \times B$, the *Cartesian product* of A and B, denoted A x B, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. A subset of $A \times B$ is called a *relation*. Thus:

1. A function from A to B is a pairing of elements in A with elements in B in such a way that each element in A is paired with exactly one element in B.
2. A function f from A to B is a rule or relation between A and B that assigns each element $a \in A$ to a unique element $b \in B$.
3. A function f from A to B is a subset of the Cartesian product

   $A \times B = \{(a, b) \mid a \in A, b \in B\}$ such that b is unique for each $a \in A$.



EXERCISE: Why is it or is it not important to have a precise definition of the term function? Also, which of the supplied function definitions do you like best, and why?

DEFINITION: If we have a function and it is viewed as a subset of the Cartesian cross product $A \times B$, then the set of all elements of A that appear will be called the *domain* of the function. The set of all elements of B that appear will be called the *range* of the function.

DEFINITION: If we have a function and it is viewed as a subset of the Cartesian cross product $A \times B$, then the set B is called the codomain of the function.

DEFINITION: Given a function rule $y = f(x)$, in the event that a domain is not specified, the domain is taken to be the *natural domain* or *implied domain*, which is the set of all real numbers to which it is possible to apply f.

Exploration 1.2.1: Function Foundations

Domain and Range

1. Can the size of the domain be smaller than the size of the range?
2. Can the size of the range be smaller than the size of the domain?
3. What happens if the domain and range have the same number of elements? What if both the domain and range have infinitely many elements?

Are These Two Functions Equal?

For each of these pairs of functions, decide if and why the functions are identical. If they are not identical, what do they have in common?

4. $\{(1, 3), (2, 3), (4, 4)\}$ and $\{(1, 3), (2, 4), (4, 4)\}$.
5. The functions $f(x) = x^3$ with domain $[1, 3]$ and $f(x) = x^3$ with domain $[2, 3]$.
6. The functions $y = x^2$ with domain all real numbers and $y = x^2$ with domain all real numbers except the number 1. (You can do this one using graphs to support your reasoning, if you would like.)
7. The functions $y = x^3 - 6x^2 + 8.5x$ and $y = 0.5x$ with domain $\{0, 2, 4\}$.

Natural/Implied Domain

8. What is the natural domain of $f(x)=\sqrt{x}$? What is the natural domain of $g(x)=\sqrt{x-2}$?

9. What is the natural domain of $h(x) = \frac{1}{x}$? What is the natural domain of $k(x) = \frac{1}{(x-4)\ (x-2)}$?

10. Can you make a generalization or set of rules to help compute the natural domain of a function?

DEFINITION: If a function $y=f(x)$ is defined on an interval $[a, b]$, then $f(x)$ is *increasing on* $[a, b]$, if, whenever $x_1 < x_2$, then $f(x_1) < f(x_2)$. The function $f(x)$ is *decreasing on* $[a, b]$, if, whenever $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Exploration 1.2.2: Function Increase and Decrease

Consider the definition above and the graph of $y = (x-1)(x-2)(x-3)$.

1. Is this graph increasing on $[3, 4]$?

2. Is this graph increasing on $[2, 4]$?

3. Do you think it should make sense to say "This graph is increasing at $x = 3$?" If no, why not? If so, how can we create a definition of what it means for a function to be increasing at a single number instead of an interval?

4. Draw a function $y = f(x)$ with the following properties:

- The domain is $(-\infty, 3]$
- The range is $[-1, \infty)$
- $f(0) = 1$
- The x–intercepts are at -1 and 1

Historical Notes

1. Although the notion of a *function* dates back to the seventeenth century, a relation-based definition, as we use today, was not formulated until the beginning of the twentieth century. The concept of mathematical relations first appears in a text by René Descartes in 1637 named *Geometry*, and the term "function" was introduced about fifty years later by Gottfried Wilhelm Leibniz. It was Leonhard Euler, in the eighteenth century, who first used today's notation $y = f(x)$. Finally, it was Hardy who defined, in 1908, a function as a relation between two variables x and y such that "to some values of x at any rate correspond values of y."

2. French author Nicolas Bourbáki, is not a single author but a group of authors who came together in the late 1950's in an effort to standardize the language of modern mathematics. That language is used in the definitions that follow. Among the group was mathematician John Tate, who later became a faculty member at the University of Texas at Austin.

Function Types

In addition to knowing whether one is working with a function or not, it is often useful to know the type of function under investigation based upon the mapping properties of the function.

Given two sets A and B,

DEFINITION A function $f: A \to B$ is called **surjective** (or is said to map A **onto** B) if $B = \text{range } f$.

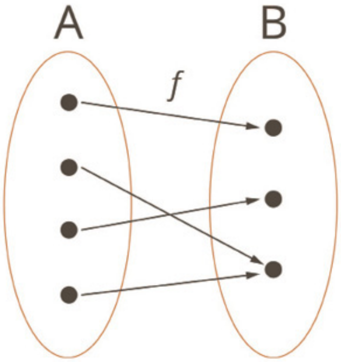
DEFINITION A function $f: A \to B$ is called **injective** (or **one-to-one)** if, for all a and a' in A, $f(a) = f(a')$ implies that $a = a'$.

DEFINITION A function $f: A \to B$ is called **bijective** if it is both surjective and injective.

Exploration 1.2.3: Types of Functions

1. Devise and explain two examples each of a surjective function that is not injective, an injective function that is not surjective, and a bijective function.

2. Characterize the function f pictured here:



3. An *odd function* is defined as a function that meets this criteria:

$$f(-x) = -f(x),$$

and an *even function* is defined as one that meets the criteria:

$$f(-x) = f(x).$$

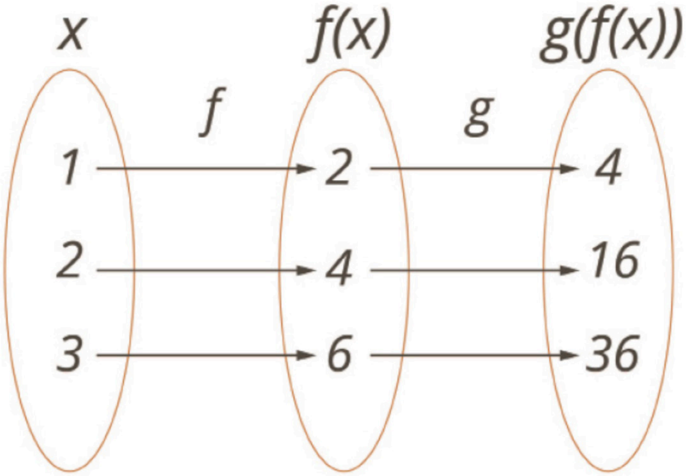
Of course functions can be neither even nor odd. List at least 10 different functions you have studied in the past, and then state and justify whether each function is even, odd, or neither.

4. Sketch graphs of the parent functions for the list you just created. What conjectures can you make about the symmetries of each function and that function's classification as even, odd, or neither? Consider symmetry about the x-axis, y-axis, and the origin.

5. Is it possible for a function to be both even and odd? Justify your reasoning.

Composition of Functions

The *composition* of functions involves taking the output values from one function and using those as the input values for a second function. Visually, it looks like this:



Using a more general approach, the composition of functions can be thought of as a situation where one or more functions are *nested* inside another. An example of this would be

$$h(x) = \sqrt{x^3 + 2}. \quad (1)$$

We think of this function as being composed of the function $f(x) = \sqrt{x}$ with the function $g(x) = x^3 + 2$ nested inside of it. Thus, (1) can be thought of as $h(x) = f(g(x))$ which can also be written as $h(x) = (f \circ g)(x)$ and stated "h of x is equal to f in composition with g."

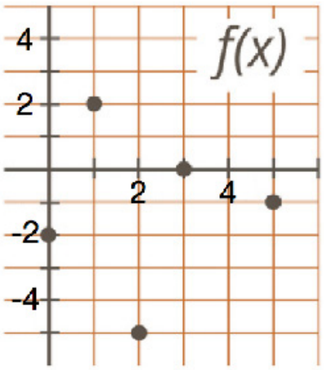
Exploration 1.2.4: Composite Functions

1. Using the same f and g functions described in the previous paragraph, write the function that is "g in composition with f." This new function could be symbolized by $g(f(x))$ or $(g \circ f)(x)$.
2. In general, is it true that $f(g(x)) = g(f(x))$? State your conclusion as a property of the composition of functions.

3. Find the composite function $f(g(x))$ for each pair of functions given below.

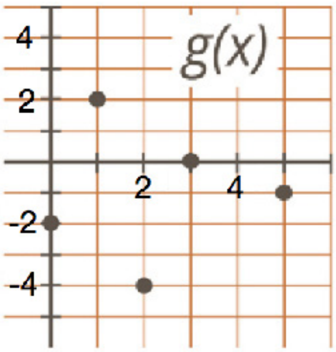
a.

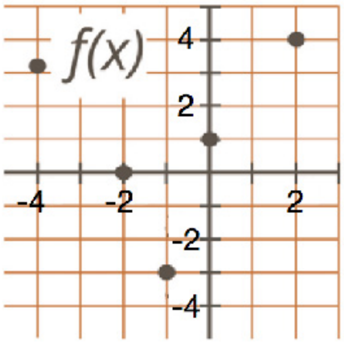
x	$g(x)$
-3	2
0	5
2	0
5	1
9	3

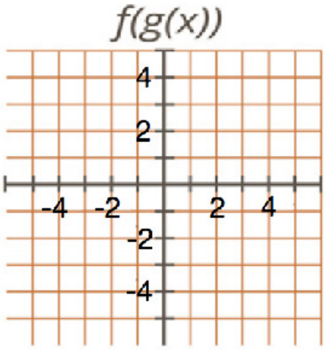


x	$f(g(x))$

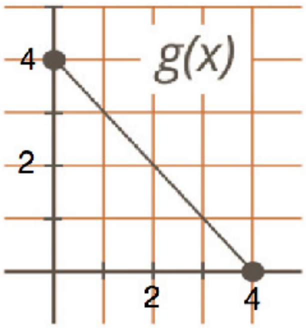
b.

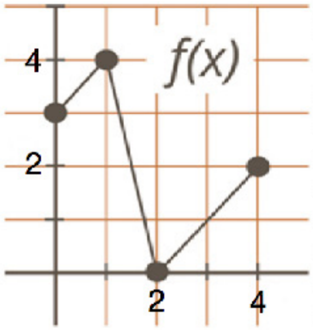


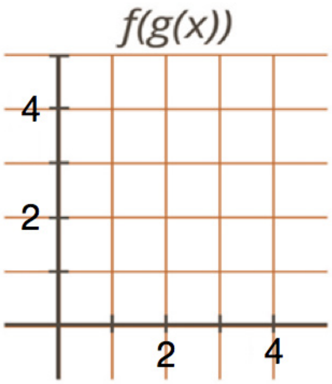




c.







d. $g(x) = 2x - 1$	$f(x) = x^2 + 5$	Find a function rule for $f(g(x))$.

e.

x	g(x)
1	-4
3	2
7	-1
10	9
12	5

x	f(x)
-4	1
-1	7
2	3
5	12
9	10

x	f(g(x))

4. You probably noticed that there is something unusual about problem #3e. Describe what happened when you performed the composition, and see if you can explain why you got the result that you did. State your conclusion as an "If… then…" statement.

Extension — Decomposition of Functions

In calculus, it is often useful to recognize the functions $f(x)$ and $g(x)$ that combine to make the composite function $f(g(x))$. For example, if $f(g(x))=\sqrt{x-8}$, then one possibility is that $g(x)=x-8$ and $f(x)=\sqrt{x}$. For the following composite functions, see if you can identify the component functions $f(x)$ and $g(x)$. Remember, it does make a difference which function you designate as $f(x)$ and which you designate as $g(x)$!

1. $f[g(x)]=(x+3)^2$

$f(x)=$ ____________________

$g(x)=$ ____________________

2. $f[g(x)]=2^{x-1}$

$f(x)=$ ____________________

$g(x)=$ ____________________

3. $f[g(x)]=\log(2x-5)$

$f(x)=$ ____________________

$g(x)=$ ____________________

4. $f[g(x)]=(x+1)!$

$f(x)=$ ____________________

$g(x)=$ ____________________

5. $f[g(x)]=\dfrac{5}{x}$

$f(x)=$ ____________________

$g(x)=$ ____________________

Exploration 1.2.5: Composite Functions — A Cautionary Tale

1. On the same paper, graph the two functions

   $f(x) = -x^2 + 8x - 4$ on the interval $[1, 5]$
   $g(x) = -x + 4$ on the interval $[0, 6]$

2. On the same paper, plot the graph of $h(x) = f\left(g(x)\right)$ and identify the domain and range of $h(x)$.

3. Verify that $f\left(g(4)\right)$ and $g\left(f(4)\right)$ are both undefined. What is the lesson to be learned when dealing with composite functions?

4. Find the explicit equation for $h(x) = f\left(g(x)\right)$.

The Inverse of a Function

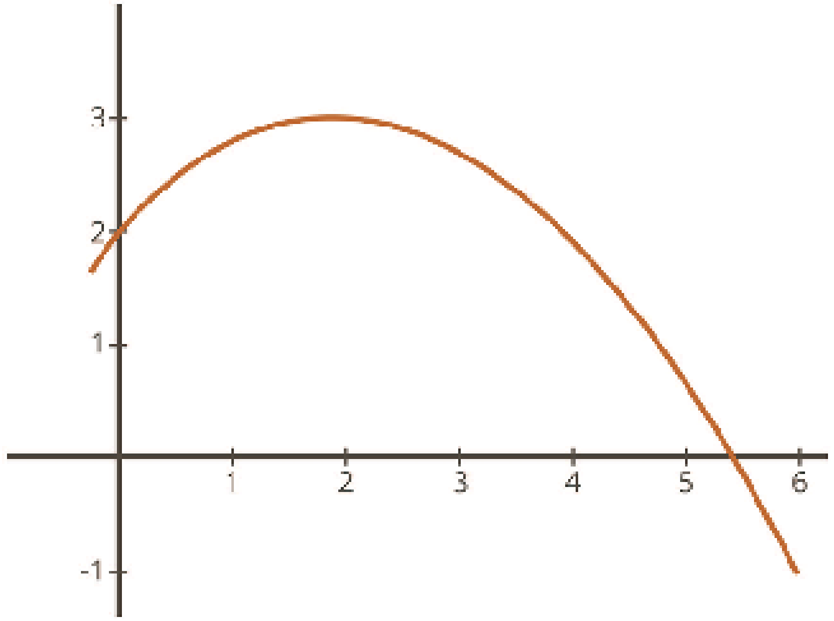
DEFINITION: If a function is injective, or one-to-one, then it is said to be *invertible.* If $y = f(x)$ is an invertible function, then the inverse function f^{-1} is defined by the equation $x = f(y)$. This can be rewritten using the following notation: if $x = f(y)$, then we write $y = f^{-1}\left(f(y)\right) = f^{-1}(x)$. The graphs of f and f^{-1} display symmetry about the line $y = x$.

[**EXERCISE**: Can you verify this property?]

Informally speaking, the inverse function switches the domain and range of the parent function. A typical example would be to suppose that one had a function that takes temperature in degrees Fahrenheit as inputs and returns corresponding temperatures in degrees Celsius as outputs. This parent function's inverse function would take input values in degrees Celsius and output corresponding temperatures in degrees Fahrenheit.

Exploration 1.2.6: Inverse of a Function

Consider the function f corresponding to the graph:

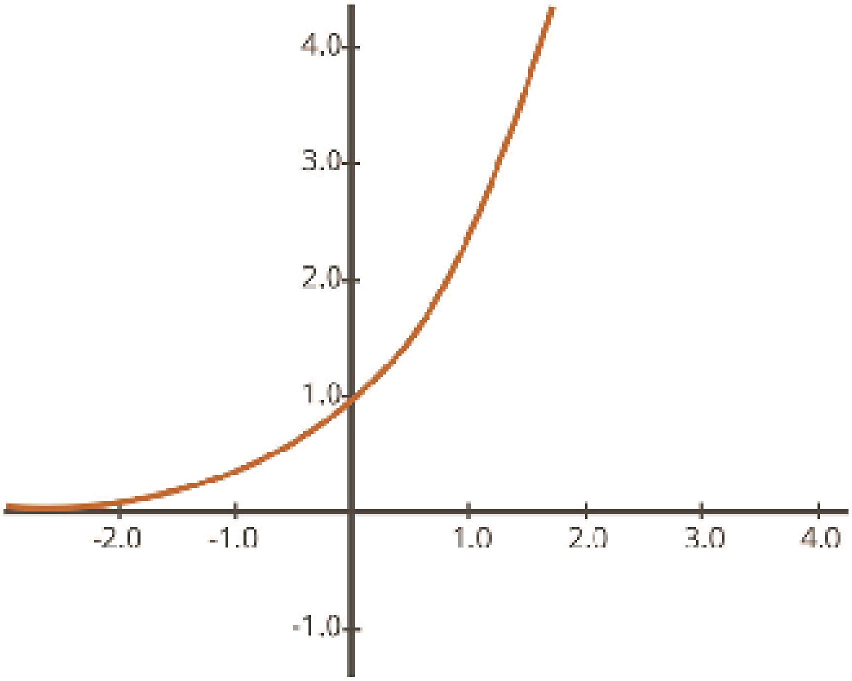


1. Explain why f does not have an inverse function.

2. How might you modify or restrict f such that it does have an inverse function?

3. Graph your “modified” f function and the *inverse* of your modified f function.

Now consider f as a function with this graph:



4. Explain why f has an inverse that is a function and graph it.

5. Can you identify and explain the graphical symmetry between a function and its inverse?

Extension - Invertibility One property of invertible functions that if f is invertible, then $f^{-1}\left(f(x)\right)=x$ and $f\left(f^{-1}(x)\right)=x$. Can you verify this property?

Exploration 1.2.7: Inverse of a Function & Domain and Range

Think of a function that squares any number and, if possible, takes the reciprocal of that number.

1. What is a formula for this function?
2. What is the domain of this function? Is this function invertible? Why or why not?
3. Define a subset of the domain where the *restriction* of this function is actually going to be invertible. Write that smaller domain down, and say what the range of your new graph is.
4. If this function squares an input and then takes the reciprocal of that, what will the inverse function do?
5. Write the equation of the inverse function, and record the domain and range as well.
6. Were there other domain restrictions that you could have tried? How many different domain restrictions can you think of?
7. Can you think of a function that won't be invertible, no matter how much you restrict the domain?

Lesson 1.3: A Qualitative Look at Rates

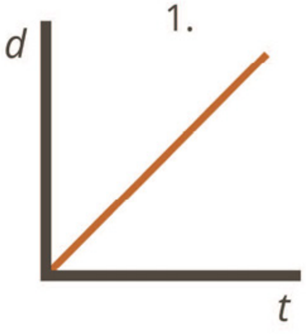
In this section we will concentrate specifically on *rate of change* of functions. That is, we will investigate how incremental changes in the input values of a function lead to changes in the output values of the function. This investigation will largely be accomplished in a qualitative manner.

Exploration 1.3.1: Graphs of Distance and Time

For each graph below, try to describe what type of motion (imagine yourself walking along a path in front of a motion detector) would produce such a graph. Consider the horizontal axis to represent increments of time and the vertical axis to represent increments of distance from a frame of reference. Then write a detailed description of how you produced the graph. The description should be detailed enough that another group could follow your directions to reproduce the graph and should include such vocabulary as *faster* or *slower rate, increasing function values, decreasing function values*, and the *function is constant* on defined intervals.

Warning! Some of these graphs may not be functions, and thus do not make sense in a physical way related to motion.

1.



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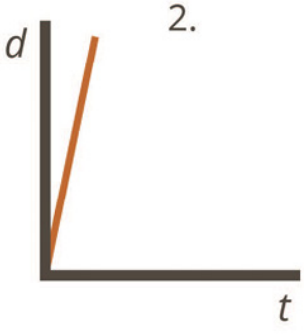

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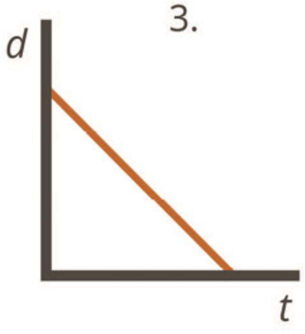
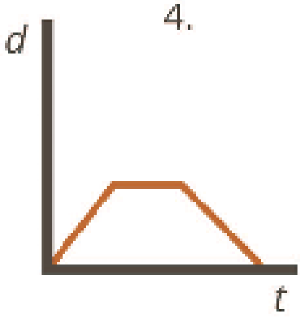


4.



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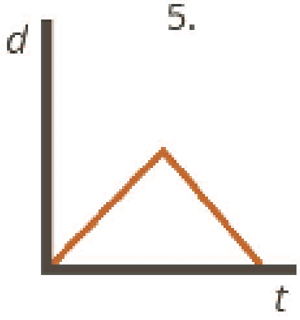

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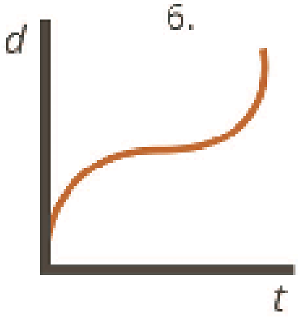

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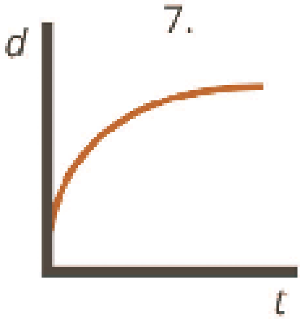

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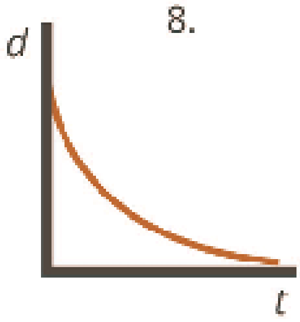

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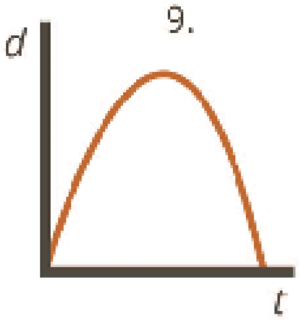

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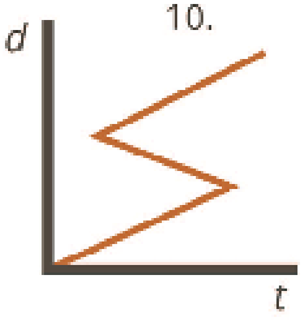

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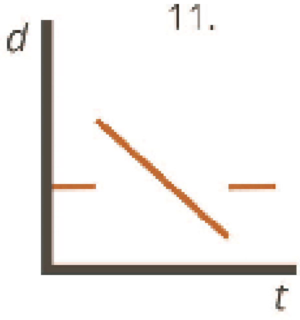

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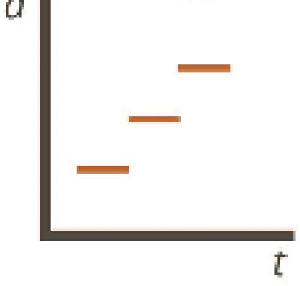
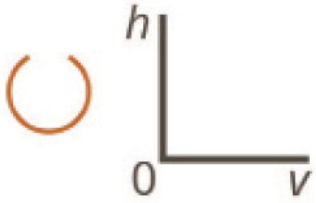


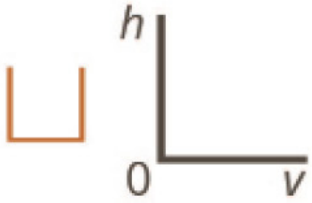
Exploration 1.3.2: Bottles, Rates, and Graphs

1. For each bottle shown below, sketch what its height vs. volume graph would look like if one were to slowly and consistently fill the bottle with a liquid. For our purposes, what matters is the shape of the given bottles and how this affects the shape of the graph. You can assume that the bottles have circular horizontal cross sections.

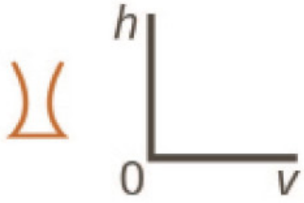
a.



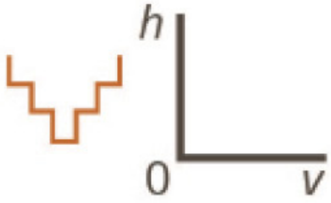
b.



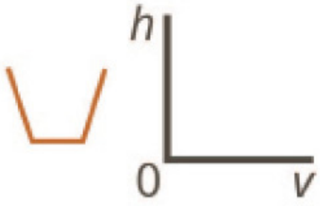
c.



d.

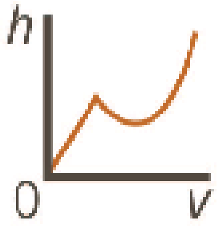


e.

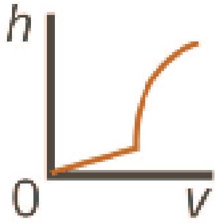


2. For each height vs. volume graph shown, draw a bottle that could produce such a graph when slowly and consistently filled with a liquid. If no such bottle is possible for a given graph, explain why.

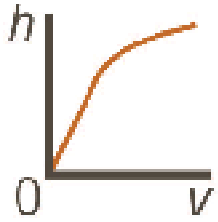
a.



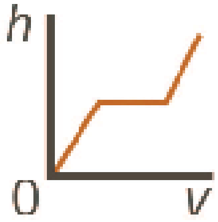
b.



c.



d.



Lesson 1.4: Sequences and Triangular Differences

Our goal in the next few sections will be to devise a way to investigate and categorize certain data sets or ordered pairs to decide what kind of function models the data. To start, we will work with supplied data that can be modeled by the common functions taught in high school. These functions are linear, quadratic, power, exponential, and logarithmic functions. Patterns will be identified in the domain set of a given function that lead to an identifiable pattern in the range of the function.

In order to use the function-identifying technique alluded to in the previous paragraph, we must first focus our attention on exploring some aspects of mathematical sequences generated by

$$f(n) = an^k,$$

where n is an element of the natural numbers, a is a real number, and k is an element of the positive integers.

Your first task, however, is to define what is meant by the term *sequence*.

Exploration 1.4.1: What is a Sequence?

1. Work in groups to settle upon and present a precise definition for the term *sequence*.
2. Also, be prepared to report on where, in your mathematical careers so far, you have encountered and worked specifically with sequences, and in what capacity.

You are now ready for the next Exploration in which you will investigate differences between consecutive terms of a defined sequence.

Exploration 1.4.2: Triangular Differences Involving Sequences of the Form $f(n) = an^k$

Take a look at the sequence of square numbers listed in the first row of numbers below. The first several terms of the sequence are listed and, below that, the first differences of the two numbers immediately above, and below that are the differences of the differences (i.e., the second differences).

1 4 9 16 25 36 49 64 81 ...
3 5 7 9 11 13 15 17 ...
2 2 2 2 2 2 2 ...

Notice that for the sequence n^2 the second differences are all 2.

1. Now explore the differences for the sequence of cubes n^3. Do you notice any patterns?

2. Make a conjecture as to what will happen with the fourth powers and the fifth powers. Make a conjecture about the differences and the resulting constant for the sequence of k^{th} powers of n^k where k is a positive integer. [This Exploration can be done by hand, or with a calculator, or in a spreadsheet program such as Excel.]

3. If you are given a sequence of k^{th} powers an^k, is there a way to figure out the value of the coefficient a?

4. Explain how the triangular differences process is related to the rate of change of your sequence function.

As a challenge activity, you might consider sequences generated by explicit equations containing multiple terms of the form an^k. For example, try applying triangular differences for the sequences generated by n^2, n, and $n^2 - 3n$. Do you notice any patterns? What about $2n^2 + 4n$ or $5n^2 + 2n - 5$? This exercise is most efficiently investigated using a computer spreadsheet program.

Lesson 1.5: Functions Defined by Patterns

Building upon what was learned in Exploration 1.4.2, we will now explore data sets consisting of ordered pairs. The purpose of this exercise is to detect patterns in the domain of a given data set that result in patterns in the range, which we can use to identify the type of function that models the data. The type of patterns that we will look for will be arithmetic, geometric, or triangular difference patterns.

Exploration 1.5.1: Finding Function Patterns

The goal of this exercise is to try to use what you have learned from the sequence Exploration that you previously completed to find a pattern in each domain and related range of a given function in order to identify what kind of model may be present (i.e., exponential, linear, quadratic, etc.). At this point, we are not interested in the actual equation that models the data, only the kind of model present. Keep in mind that using triangular differences is not the only way to discern patterns. Look for ratios or multiplication patterns as well.

For each table of data:

1. Plot the data points and make a conjecture as to what kind of function is present.
2. Find a pattern in the domain and the range for each function provided and try to understand how this pattern affects the shape of the graph that you have plotted (assuming that the graph is continuous).
3. Make a conjecture as to what kind of function can be used as a model for the given data.

Example 1

x	$f(x)$
2	4
4	9
6	14
8	19
10	24

Example 2

x	$f(x)$
1	15
3	5
5	19
7	57
9	119

Example 3

x	$f(x)$
1	15
3	135
5	1215
7	10935

It is a more complicated endeavor to find the pattern relationship in Example 4. Be aware that sometimes a pattern is revealed not by examining consecutive terms in a domain or range sequence; rather, consecutive terms can be considered as those terms, in order, that reveal a pattern. Consequently, sometimes ordered pairs can be "skipped" in order to discern a pattern relationship in the data. All ordered pairs will be points on the graph of the function. It is simply that the proper function pattern may not be revealed by examining each consecutive ordered pair.

Example 4

x	$f(x)$
3	135
6	1080
9	3645
12	8640

Example 5

x	$f(x)$
6	1
18	2
54	3
162	4

Functions Defined by Patterns - Verification

The domain - range patterns that you should have identified in Exploration 1.5.1 that are related to some of the more common functions studied in mathematics courses are:

Sum - Sum

Constant Second Difference

Product - Product

Sum - Product

Product - Sum

Of course, at this point your pattern identifications and associated functions are simply conjectures. In the next Exploration, you are asked to verify these conjectures by working with general forms of the functions identified by pattern.

!! For this Exploration, you will have to recall some basic properties of exponents and logarithmic functions. Your instructor might want to perform a brief review of these topics.

Exploration 1.5.2: Pattern Verification

In this Exploration, you are going to work with general forms of functions identified by patterns in order to discover why certain patterns in the domain of a given function lead to predictable behavior for the values produced in the range.

1. Sum – Sum: A Linear Function

Statement: For a linear function f, adding a constant c to a given domain value results in adding a constant to the corresponding range value:

To verify the statement above, find $f(x_2)$ in terms of $f(x_1)$ and fill in the statement below

If $f(x) = mx + b$ and $x_2 = c + x_1$, then $f(x_2) = \ldots$

2. Sum – Constant Second Difference: A Quadratic Function

Statement: For a function f of the form $ax^2 + bx + c$ with domain values k units apart, then the second differences between consecutive $f(x)$ values are constant and equal to $2ak^2$.

To verify the statement above, find the second differences involving $f(x)$ evaluated at

$$x_1,\ x_2 = (x_1 + k),\ x_3 = (x_1 + 2k).$$

3. Sum – Product: An Exponential Function

Statement: For an exponential function f, adding a constant *c* to a given domain value results in multiplying the corresponding range value by a constant:

To verify the statement above, find $f(x_2)$ in terms of $f(x_1)$ and fill in the statement below

If $f(x) = ab^x$ and $x_2 = x_1 + c$, then $f(x_2) = \ldots$

4. Product - Product: A Power Function

Statement: For a power function *f*, multiplying a given domain value by a constant *c* results in multiplying the corresponding range value by a constant:

To verify the statement above, find $f(x_2)$ in terms of $f(x_1)$ and fill in the statement below

If $f(x) = ax^k$ and $x_2 = cx_1$, then $f(x_2) = \ldots$

5. Product – Sum: A Logarithmic Function

Statement: For a logarithmic function *f*, multiplying a given domain value by a constant *c* results in adding a constant to the corresponding range value:

To verify the statement above, find $f(x_2)$ in terms of $f(x_1)$ and fill in the statement below

Given $f(x) = a + b\log_n x$, if $x_2 = cx_1$, then $f(x_2)$ = ...

6. **EXERCISE:** The "functions defined by patterns" process can also be used to create data sets specific to a desired type of function.

If a function f has values $f(5) = 12$ and $f(10) = 18$, use what you have learned about function patterns to find $f(20)$ if f is:

a. an exponential function

b. a linear function

c. a power function

Piecewise Functions

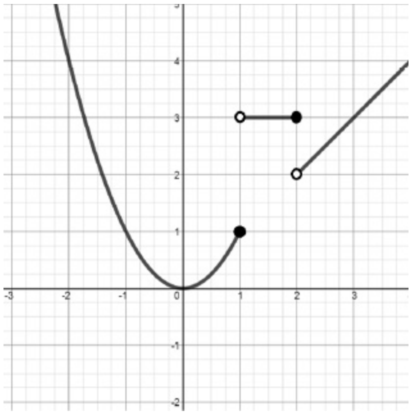
A *piecewise function f* is a function in which different rules for *f* are applied in different "pieces" (or intervals) of the domain. We notate this idea as:

$$f(x)=\begin{cases}\text{formula 1} & \text{if domain to use formula 1}\\ \text{formula 2} & \text{if domain to use formula 2}\\ \text{formula 3} & \text{if domain to use formula 3}\end{cases}$$

You have seen examples of the graphs of piecewise functions in Exploration 1.3.1, graphs 11 and 12. The following exploration gives you the chance to work further with piecewise functions.

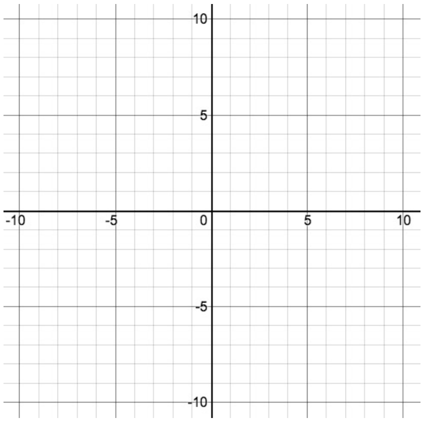
Exploration 1.5.3: Piecewise Functions

1. Write a formula for the piecewise function that will generate the graph below:



2. Sketch the graph of the relation

$$y=\begin{cases}x+3 & if \quad x\le -1\\ x^2-1 & if \quad -1<x\le 2\\ 1 & if \quad x\ge 2\end{cases}$$



3. For each piecewise function evaluate *f(-1), f(0), f(2), f(4)* and *f(6)*. Then graph each function.

$$f(x)=\begin{cases} x^2-2 & \text{if } x<2 \\ 4+x & \text{if } x\geq 2 \end{cases}$$

$$f(x)=\begin{cases} 4-x^2 & \text{if } x<1 \\ \sqrt{x+1} & \text{if } x\geq 1 \end{cases}$$

4. At Gunterville College, tuition rates for in-state residents were $92 per credit for the first 10 credits, $35 per credit for credits 11-18, and for over 18 credits the rate is $74 per credit. Write a piecewise defined function for the total tuition, T, at Guntersville College as a function of the number of credits, c. Be sure to consider reasonable domain and range.

5. A wireless phone provider uses the function below to determine the cost, *C*, in dollars for, *g*, gigabytes of data transfer. Graph this piecewise function and find the cost of using 0.5 gigabytes of data, and the cost of using 6 gigabytes of data.

$$C(g)=\begin{cases} 25 & \text{if } 0<g<2 \\ 25+10(g-2) & \text{if } g\geq 2 \end{cases}$$