Chapter

5

Discrete Dependent Variable Models

CHAPTER 5; SECTION A: LOGIT, NESTED LOGIT, & PROBIT

Purpose of Logit, Nested Logit, and Probit Models:

Logit, Nested Logit, and Probit models are used to model a relationship between a dependent variable Y and one or more independent variables X. The dependent variable, Y, is a discrete variable that represents a choice, or category, from a set of mutually exclusive choices or categories. For instance, an analyst may wish to model the choice of automobile purchase (from a set of vehicle classes), the choice of travel mode (walk, transit, rail, auto, etc.), the manner of an automobile collision (rollover, rear-end, sideswipe, etc.), or residential location choice (high-density, suburban, exurban, etc.). The independent variables are presumed to affect the choice or category or the choice maker, and represent *a priori* beliefs about the causal or associative elements important in the choice or classification process. In the case of ordinal scale variables, an ordered logit or probit model can be applied to take advantage of the additional information provided by the ordinal over the nominal scale (not discussed here).

Examples: An analyst wants to model:
1. *The effect of household member characteristics, transportation network characteristics, and alternative mode characteristics on choice of transportation mode; bus, walk, auto, carpool, single occupant auto, rail, or bicycle.*
2. *The effect of consumer characteristics on choice of vehicle purchase: sport utility vehicle, van, auto, light pickup truck, or motorcycle.*
3. *The effect of traveler characteristics and employment characteristics on airline carrier choice; Delta, United Airlines, Southwest, etc.*
4. *The effect of involved vehicle types, pre-crash conditions, and environmental factors on vehicle crash outcome: property damage only, mild injury, severe injury, fatality.*

Basic Assumptions/Requirements of Logit, Nested Logit, and Probit Models:

1) The observations on dependent variable Y are assumed to have been randomly sampled from the population of interest (even for stratified samples or choice-based samples).

2) Y is caused by or associated with the X's, and the X's are determined by influences (variables) 'outside' of the model.

3) There is uncertainty in the relation between Y and the X's, as reflected by a scattering of observations around the functional relationship.

4) The distribution of error terms must be assessed to determine if a selected model is appropriate.

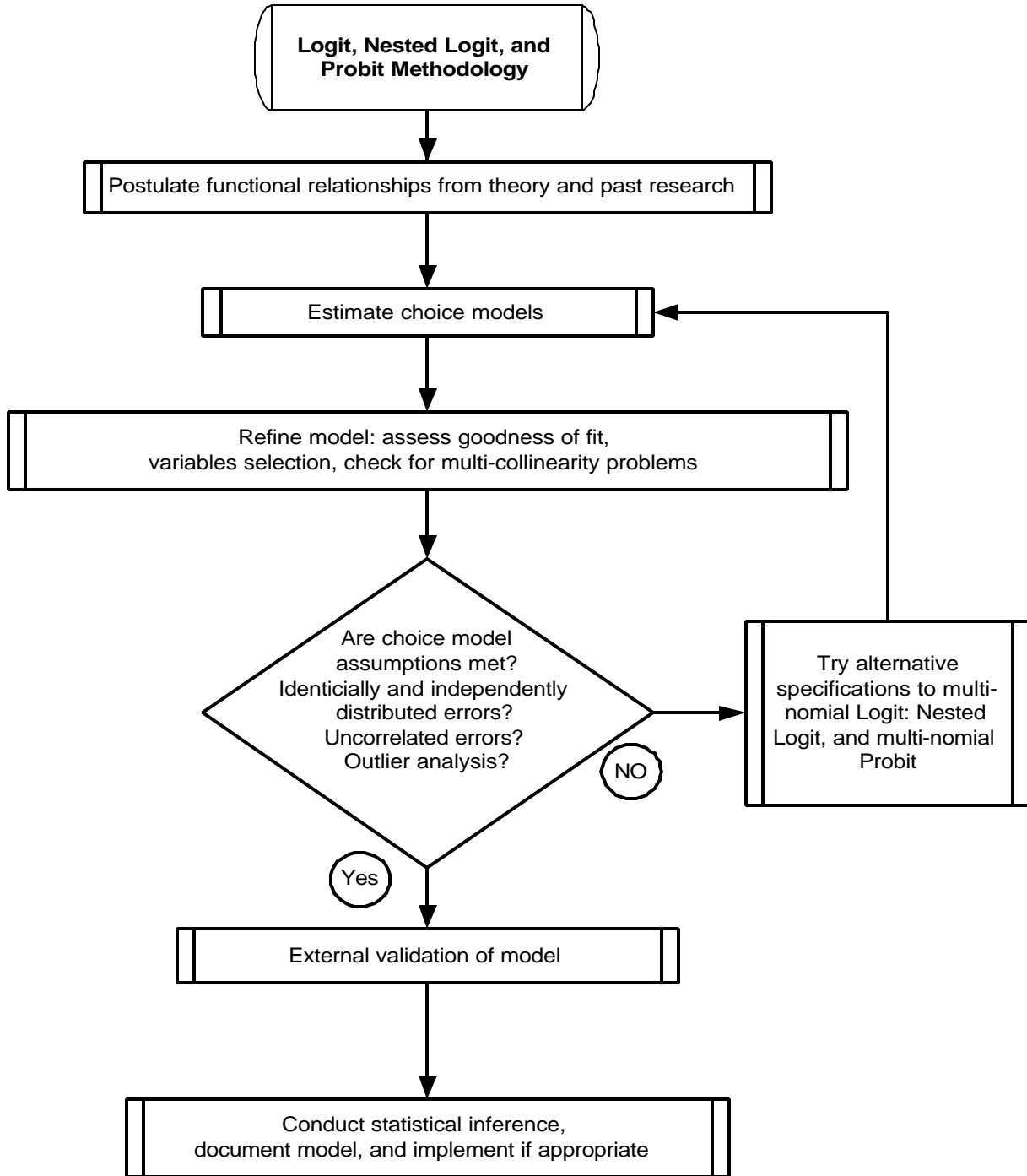## Inputs for Logit, Nested Logit, and Probit Models:

Discrete variable Y is the observed choice or classification, such as brand selection, transportation mode selection, etc. For grouped data, where choices are observed for homogenous experimental units or observed multiple times per experimental unit, the dependent variable is proportion of choices observed.

One or more continuous and/or discrete variables X, which describe the attributes of the choice maker or event and/or various attributes of the choices thought to be causal or influential in the decision or classification process.

Outputs of Logit, Nested Logit, and Probit Models:

- Functional form of relation between Y and X's.

- Strength of association between Y and X's (individual X's and collective set of X's).

- Proportion of choice or classification uncertainty explained by hypothesized relation.

- Confidence in predictions of future/other observations on Y given X.

Logit, Nested Logit, and Probit Methodology:

Logit, Nested Logit, and Probit Methodology

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Postulate functional relationships from theory and past research

↓

Estimate choice models

↓

Refine model: assess goodness of fit,
variables selection, check for multi-collinearity problems

↓

Are choice model
assumptions met?
Identicially and independently
distributed errors?
Uncorrelated errors?
Outlier analysis?

NO → Try alternative specifications to multi-nomial Logit: Nested Logit, and multi-nomial Probit

Yes

↓

External validation of model

↓

Conduct statistical inference,
document model, and implement if appropriate

Examples of Logit, Nested Logit, and Probit:

Pavements
Koehne, Jodi, Fred Mannering, and Mark Hallenbeck (1996). <u>Analysis of Trucker and Motorist Opinions Toward Truck-lane Restrictions</u>. Transportation Research Record #1560 pp. 73-82. National Academy of Sciences.

Traffic
Mannering, Fred, Jodi Koehne and Soon-Gwan Kim. (1995). <u>Statistical Assesssment of Public Opinion Toward Conversion of General-Purpose Lanes to High-Occupancy Vehicle Lanes.</u> Transportation Research Record #1485 pp. 168-176. National Academy of Sciences.

Planning
Koppelman, Frank S., and Chieh-Hua Wen (1998). <u>Nested Logit Models: Which Are You Using?</u> Transportation Research Record #1645 pp. 1-9. National Academy of Sciences.

Yai, Tetsuo, and Tetsuo Shimizu (1998). <u>Multinomial Probit with Structured Covariance for Choice Situations with Similar Alternatives.</u> Transportation Research Record #1645 pp. 69-75. National Academy of Sciences.

McFadden, Daniel. <u>Modeling the Choice of Residential Location</u>. (1978). Transportation Research Record #673 pp. 72-77. National Academy of Sciences.

Horowitz, Joel L. (1984) <u>Testing Disaggregate Travel Demand Models by Comparing Predicted and Observed Market Shares</u>. Transportation Research Record #976 pp. 1-7. National Academy of Sciences.

Interpretation of Logit, Nested Logit, and Probit:

How is a choice model equation interpreted?
How do continuous and indicator variables differ in the choice model?
How are beta coefficients interpreted?
How is the Likelihood Ratio Test interpreted?
How are t-statistics interpreted?
How are phi and adjusted phi interpreted?
How are confidence intervals interpreted?
How are degrees of freedom interpreted?
How are elasticities computed and interpreted?
When is the independence of irrelevant alternatives (IIA) assumption violated?

Troubleshooting: Logit, Nested Logit, and Probit:

Should interaction terms be included in the model?
How many variables should be included in the model?
What methods can be used to specify the relation between choice and the X's?
What methods are available for fixing heteroscedastic errors?

What methods are used for fixing serially correlated errors?
What can be done to deal with multi-collinearity?
What is endogeneity and how can it be fixed?
How does one know if the errors are Gumbel distributed?

Logit, Nested Logit, and Probit References:

- Ben Akiva, Moshe and Steven R. Lerman. Discrete Choice Analysis: Theory and Application to Predict Travel Demand. The MIT Press, Cambridge MA. 1985.

- Greene, William H. Econometric Analysis. MacMillan Publishing Company, New York, New York. 1990.

- Ortuzar, J. de D. and L. G. Willumsen. Modelling Transport. Second Edition. John Wiley and Sons, New York, New York. 1994.

- Train, Kenneth. Qualitative Choice Analysis: Theory, Econometrics, and an Application to Automobile Demand. The MIT Press, Cambridge MA. 1993.

Logit, Nested Logit, and Probit Methodology:

Postulate mathematical models from theory and past research.

Discrete choice models (logit, nested logit, and probit) are used to develop models of behavioral choice or of event classification. It is accepted *a priori* that the analyst doesn't know the complexity of the underlying relationships, and that any model of reality will be wrong to some degree. Choice models estimated will reflect the *a priori* assumptions of the modeler as to what factors affect the decision process. Common applications of discrete choice models include choice of transportation mode, choice of travel destination choice, and choice of vehicle purchase decisions. There are many potential applications of discrete choice models, including choice of residential location, choice of business location, and transportation project contractor selection.

In order to postulate meaningful choice models, the modeler should review past literature regarding the choice context and identify factors with potential to affect the decision making process. These factors should drive the data-collection process—usually a survey instrument given to experimental units, to collect the information relevant in the decision making process. There is much written about survey design and data collection, and these sources should be consulted for detailed discussions of this complex and critical aspect of choice modeling

> *Transportation Planning Example: An analyst is interested in modeling the mode choice decision made by individuals in a region. The analyst reviews the literature and develops the following list of potential factors influencing the mode choice decision for most travelers in the region.*
> *1. Trip maker characteristics (within the household context):*
> *Vehicle availability, possession of driver's license, household structure (stage of life-cycle), role in household, household income (value of time)*
> *2. Characteristics of the journey or activity:*
> *journey or activity purpose; work, grocery shopping, school, etc.*

> *time of day, accessibility and proximity of activity destination*
> *3. Characteristics of transport facility:*
> *Qualitative Factors; comfort and convenience, reliability and regularity, protection, security*
> *Quantitative Factors; in-vehicle travel times, waiting and walking times, out-of-pocket monetary costs, availability and cost of parking, proximity/accessibility of transport mode*

Estimate choice models

Qualitative choice analysis methods are used to describe and/or predict discrete choices of decision-makers or to classify a discrete outcome according to a host of regressors. The need to model choice and/or classification arises in transportation, energy, marketing, telecommunications, and housing, to name but a few fields. There are, as always, a set of assumptions or requirements about the data that need to be satisfied. The response variable (choice or classification) must meet the following three criteria.

1. The set of choices or classifications must be finite.

2. The set of choices or classifications must be mutually exclusive; that is, a particular outcome can only be represented by one choice or classification.

3. The set of choices or classifications must be collectively exhaustive, that is all choices or classifications must be represented by the choice set or classification.

Even when the 2nd and 3^{d} criteria are not met, the analyst can usually re-define the set of alternatives or classifications so that the criteria are satisfied.

> *Planning Example: An analyst wishing to model mode choice for commute decisions defines the choice set as AUTO, BUS, RAIL, WALK, and BIKE. The modeler observed a person in the database drove her personal vehicle to the transit station and then took a bus, violating the second criteria. To remedy the modeling problem and similar problems that might arise, the analyst introduces some new choices (or classifications) into the modeling process: AUTO-BUS, AUTO-RAIL, WALK-BUS, WALK-RAIL, BIKE-BUS, BIKE-RAIL. By introducing these new categories the analyst has made the discrete choice data comply with the stated modeling requirements.*

Deriving Choice Models from Random Utility Theory

Choice models are developed from economic theories of random utility, whereas classification models (classifying crash type, for example) are developed by minimizing classification errors with respect to the X's and classification levels Y. Because most of the literature in transportation is focused on choice models and because mathematically choice models and classification models are equivalent, the discussion here is based on choice models. Several assumptions are made when deriving discrete choice models from random utility theory:

1. An individual is faced with a finite set of choices from which only one can be chosen.

2. Individuals belong to a homogenous population, act rationally, and possess perfect information and always select the option that maximizes their net personal utility.

3. If C is defined as the universal choice set of discrete alternatives, and **J** the number of elements in C, then each member of the population has some subset of C as his or her choice set. Most decision-makers, however, have some subset C_n, that is considerably smaller than C. It should be recognized that defining a subset C_n, that is the feasible choice set for an individual is not a trivial task; however, it is assumed that it can be determined.

4. Decision-makers are endowed with a subset of attributes $x_n \in X$, all measured attributes relevant in the decision making process.

Planning Example: In identifying the choice set of travel mode the analyst identifies the universal choice set C to consist of the following:
1. *driving alone*
2. *sharing a ride*
3. *taxi*
4. *motorcycle*
5. *bicycle*
6. *walking*
7. *transit bus*
8. *light rail transit*

The analyst identifies a family whose choice set is fairly restricted because the do not own a vehicle, and so their choice set C_n is given by:
1. *sharing a ride*
2. *taxi*
3. *bicycle*
4. *walking*
5. *transit bus*
6. *light rail transit*

The modeler, who is an OBSERVER of the system, does not possess complete information about all elements considered important in the decision making process by all individuals making a choice, so Utility is broken down into 2 components, V and ε:

$$U_{in} = (V_{in} + \varepsilon_{in});$$
where;
U_{in} is the overall utility of choice i for individual n,
V_{in} is the systematic or measurably utility which is a function of x_n and i for individual n and choice i
ε_{in} includes idiosyncrasies and taste variations, combined with measurement or observations errors made by modeler, and is the random utility component.

The error term allows for a couple of important cases: 1) two persons with the same measured attributes and facing the same choice set make different decisions; 2) some individuals do not select the best alternative (from the modelers point of view it demonstrated irrational behavior).

The decision maker n chooses the alternative from which he derives the greatest utility. In the binomial or two-alternative case, the decision-maker chooses alternative 1 if and only if:

$$U_{1n} \geq U_{2n}$$
or when:

$$V_{1n} + \varepsilon_{1n} \geq V_{2n} + \varepsilon_{2n}.$$

In probabilistic terms, the probability that alternative 1 is chosen is given by:

$$\text{Pr}\,(1) = \text{Pr}\,(U_1 \geq U_2)$$
$$= \text{Pr}\,(V_1 + \varepsilon_1 \geq V_2 + \varepsilon_2)$$
$$= \text{Pr}\,(\varepsilon_2 - \varepsilon_1 \leq V_1 - V_2).$$

Note that this equation looks like a cumulative distribution function for a probability density. That is, the probability of choosing alternative 1 (in the binomial case) is equal to the probability that the difference in random utility is less than or equal to the difference in deterministic utility.

If $\varepsilon = \varepsilon_2 - \varepsilon_1$, which is the difference in unobserved utilities between alternatives 2 and 1 for travelers 1 through N (subscript not shown), then the probability distribution or density of ε, $f(\varepsilon)$, can be specified to form specific classes of models.

$$F(V_1 - V_2) = \int_{-\infty}^{V_1 - V_2} f(e)\,de$$
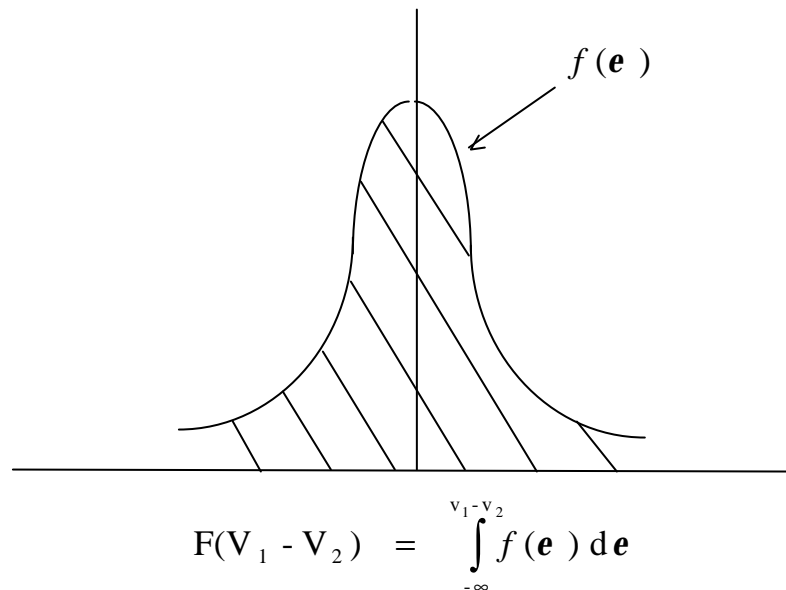
A couple of important observations about the probability density given by $F(V_1 - V_2)$ can be made.

1. The error ε is small when there are large differences in systematic utility between alternatives one and two.
2. Large errors are likely when differences in utility are small, thus decision makers are more likely to choose an alternative on the 'wrong' side of the indifference line ($V_1 - V_2 = 0$).
   Alternative 1 is chosen when $V_1 - V_2 > 0$ (or when $\varepsilon > 0$), and alternative 2 is chosen when $V_1 - V_2 < 0$.
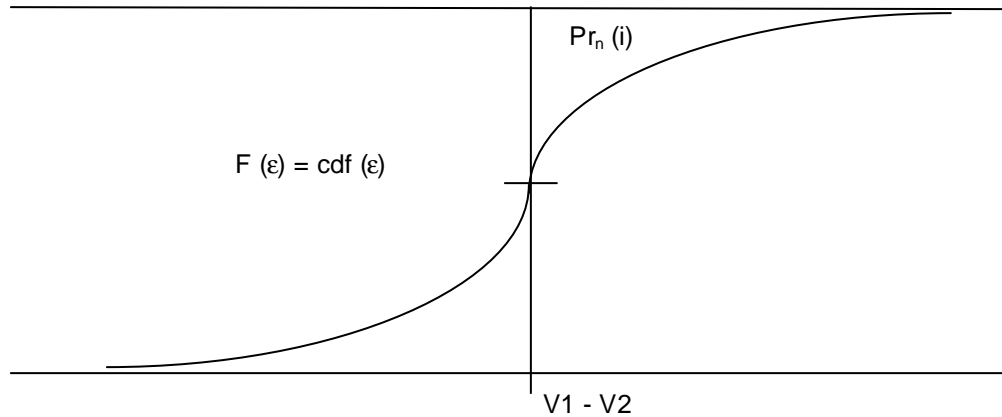
Thus, for binomial models of discrete choice:

$$\text{Pr}_n(1) = \int_{-\infty}^{V_1 - V_2} f(e)\,de$$

The cumulative distribution function, or CDF, typically looks like:

$$\Pr_n(i)$$

$$F(\varepsilon) = cdf(\varepsilon)$$

$$V1 - V2$$

This structure for the error term is a general result for binomial choice models. By making assumptions about the probability density of the residuals, the modeler can choose between several different binomial choice model formulations. Two types of binomial choice models are most common and found in practice: the logit and the probit models. The logit model assumes a logistic distribution of errors, and the probit model assumes a normal distributed errors. These models, however, are not practical for cases when there are more than two cases, and the probit model is not easy to estimate (mathematically) for more than 4 to 5 choices.

Mathematical Estimation of Choice Models

Recall that choice models involve a response Y with various levels (a set of choices or classification), and a set of X's that reflect important attributes of the choice decision or classification. Usually the choice or classification of Y is a modeled as a linear function or combination of the X's. Maximum likelihood methods are employed to solve for the betas in choice models.

Consider the likelihood of a sample of N independent observations with probabilities p_1, p_2,......,p_n. The likelihood of the sample is simply the product of the individual likelihoods. The product is a maximum when the most likely set of p's is used.

$$\text{i.e. Likelihood } \mathbf{L}^* = p_1 p_2 p_3 \ldots \ldots p_n = \prod_{i=1}^{n} p_i$$

For the binary choice model:

$$\mathbf{L}^* = (\beta_1, \ldots\ldots, \beta_K) \quad \prod_{i=1}^{n} \Pr_n(i)^{y_{in}} \Pr_n(j)^{y_{jn}}, \quad =$$

where, $\Pr_n(i)$ is a function of the betas, and i and j are alternatives 1 and 2 respectively. It is generally mathematically simpler to analyze the logarithm of $\mathbf{L}^*$, rather than the likelihood function itself. Using the fact that $\ln(z_1 z_2) = \ln(z_1) + \ln(z_2)$, $\ln(z)^x = x \ln(z)$, $\Pr(j)=1-\Pr(i)$, and $y_{jn} = 1 - y_{in}$, the equation becomes:

$$L = \log\left(L^*\right) = \log\left(\prod_{i=1}^{n} Pr_n(i)^{y_{in}} \, Pr_n(j)^{y_{jn}}\right)$$

$$= \sum_{i=1}^{n} \log[Pr_n(i)^{y_{in}} \, Pr_n(j)^{y_{jn}}]$$

$$= \sum_{i=1}^{n} (y_{in} \log[Pr_n(i)] + y_{jn} \log[Pr_n(j)])$$

$$= \sum_{i=1}^{n} (y_{in} \log[Pr_n(i)] + (1 - y_{in}) \log[1 - Pr_n(i)])$$

The maximum of **L** is solved by differentiating the function with respect to each of the beta's and setting the partial derivatives equal to zero, or the values of $\beta_1, \ldots, \beta_K$ that provides the maximum of **L** . In many cases the log likelihood function is globally concave, so that if a solution to the first order conditions exist, they are unique. This does not always have to be the case, however. Under general conditions the likelihood estimators can be shown to be consistent, asymptotically efficient, and asymptotically normal.

In more complex and realistic models, the likelihood function is evaluated as before, but instead of estimating one parameter, there are many parameters associated with X's that must be estimated, and there are as many equations as there are X's to solve. In practice the probabilities that maximize the likelihood function are likely to be different across individuals (unlike the simplified example above where all individuals had the same probability).

Because the likelihood function is between 0 and 1, the log likelihood function is negative. The maximum to the log-likelihood function, therefore, is the smallest negative value of the log likelihood function given the data and specified probability functions.

Planning Example. Suppose 10 individuals making travel choices between auto (A) and transit (T) were observed. All travelers are assumed to possess identical attributes (a really poor assumption), and so the probabilities are not functions of betas but simply a function of p, the probability of choosing Auto. The analyst also does not have any alternative specific attributes—a very naive model that doesn't reflect reality. The likelihood function will be:

*$L* = p^x (1-p)^{n-x} = p^7 (1-p)^3$*

where;

p = probability that a traveler chooses A,
1-p = probability that a traveler chooses T,
n = number of travelers = 10
x = number of travelers choosing A.

Recall that the analyst is trying to estimate p, the probability that a traveler chooses A. If 7 travelers were observed taking A and 3 taking T, then it can be shown that the maximum likelihood estimate of p is 0.7, or in other words, the value of L* is maximized when p=0.7 and 1-p=0.3. All other combinations of p and 1-p result in lower values of L*. To see this, the analyst plots numerous values of L* for all integer values of P (T) from 0.0 to 10.0. The following plot is obtained:
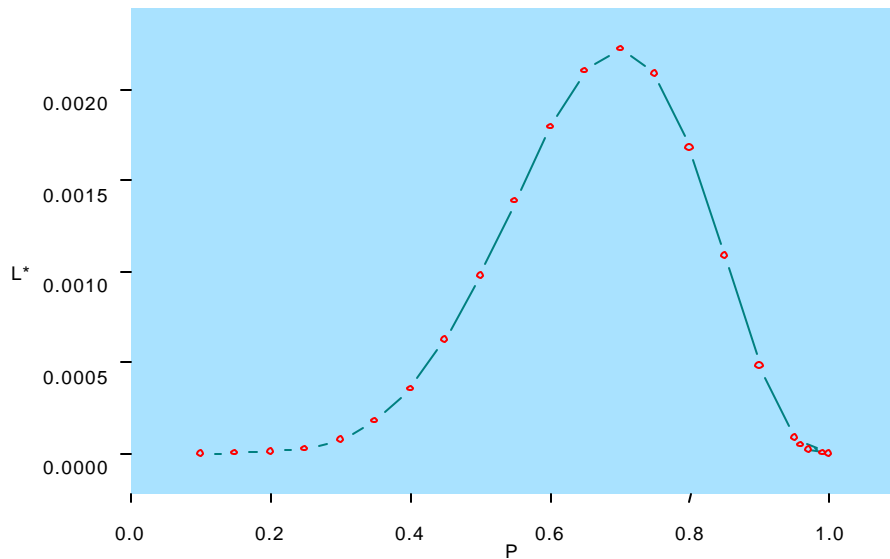
Similarly (and in practice), one could use the log likelihood function to derive the maximum likelihood estimates, where L = log (L*) = Log [p^7 (1-p)^3] = Log p^7 + Log (1-p)^3 = 7 Log p + 3 Log (1-p).

LogLikehood Function
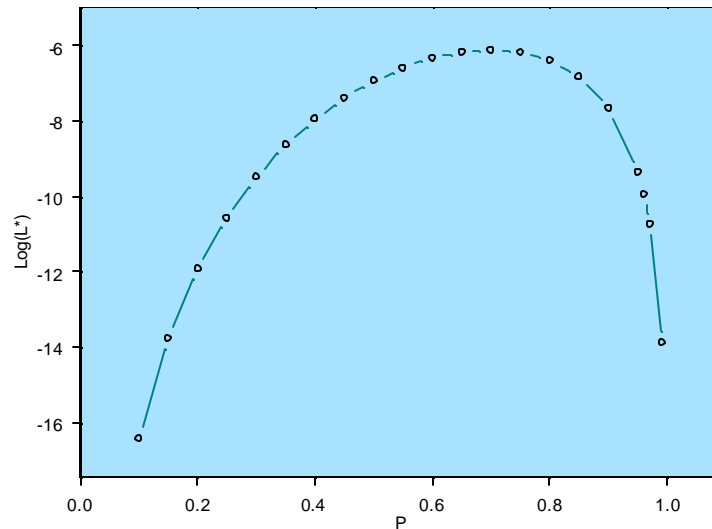
Note that in this simple model p is the only parameter being estimated, so maximizing the likelihood function L* or the log (L*) only requires one first order condition, the derivative of p with respect to log (L*).

The Multinomial Logit Model

The multinomial logit (MNL) model is the most commonly applied model to explain and forecast discrete choices due to its ease of estimation and foundation in utility theory. The MNL model is a general extension of the binomial choice model to more than two alternatives. The universal choice set is C, which contains j elements, and a subset of C for each individual C_n, defines their restricted choice sets. It should be noted that it is not a trivial task to define restricted choice sets for individuals. In most cases J_n for decision maker n is less than or equal to J, the total number of alternatives in the universal choice set, however it is often assumed that all decision makers face the same set of universal alternatives.

Without showing the derivation, which can be found in the references for this chapter, the MNL model is expressed as:

$$\Pr_n(i) = \frac{e^{V_{in}}}{\sum_{j \in C_n} e^{V_{jn}}}$$

Where;

1. Utility for traveler n and mode i $= U_{in} = V_{in} + \varepsilon_{in}$

2. $P_n(i)$ is the probability that traveler n chooses mode i

3. Numerator is utility for mode i for traveler n, denominator is the sum of utilities for all alternative modes C_n for traveler n

4. The disturbances e_{in} are independently distributed

5. The disturbances e_{in} are identically distributed

6. The disturbances are Gumbel distributed with location parameter $\boldsymbol{h}$ and a scale parameter $\boldsymbol{m}$ > 0.

The MNL model expresses the probability that a specific alternative is chosen is the exponent of the utility of the chosen alternative divided by the exponent of the sum of all alternatives (chosen and not chosen). The predicted probabilities are bounded by zero and one. There are several assumptions embedded in the estimation of MNL models.

Linear in parameters restriction:

The linear in parameters restriction is made for convenience of estimation, which enables simple and efficient estimation of parameters. When the functional form of the systematic component of the utility function is linear in parameters, the MNL model can be written as:

$$\Pr_n(i) = \frac{e^{\boldsymbol{b}'x_{in}}}{\sum_{j=1}^{J_n} e^{\boldsymbol{b}'x_{jn}}}$$

where x_{in} and x_{jn} are vectors describing the attributes of alternatives i and j as well as attributes of traveler n.

Independence from Irrelevant Alternatives Property (IIA)

Succinctly stated, the IIA property states that for a specific individual the ratio of the choice probabilities of any two alternatives is entirely unaffected by the systematic utilities of any other alternatives. This property arises from the assumption in the derivation of the logit model that the error terms ε_n across individuals are independent. In other words, it is assumed that un-observed attributes (error terms) of alternatives are independent. In many cases this is an unrealistic assumption, and creates some difficulties. For example, if driver n has an unobserved (error term) preference for public transit, then public transit mode error terms will not be independent.

Another way to express IIA is that the ratio of choice probabilities of any two alternatives for a specific individual is entirely unaffected by the systematic utilities of any other alternatives.

$$\frac{\Pr_n(i)}{\Pr_n(l)} = \frac{\dfrac{e^{b'x_{in}}}{\sum\limits_{j=1}^{J_n} e^{b'x_{jn}}}}{\dfrac{e^{b'x_{ln}}}{\sum\limits_{j=1}^{J_n} e^{b'x_{jn}}}} = \frac{e^{b'x_{in}}}{e^{b'x_{ln}}} = e^{b'x_{in} - b'x_{ln}}$$

Note that the ratio of probabilities of modes i and j for individual n are unaffected by 'irrelevant' alternatives in C_n.

One way to pose the IIA problem is to explain the red bus/blue bus paradox. Assume that the initial choice probabilities for an individual are as follows:

P (auto) = P (A) = 70%

P (blue bus) = P (BB) = 20%

P (rail) = P(R) = 10%

By the IIA assumption:

P (A)/P (BB) = 70 / 20 = 3.5, and

P(R)/P (BB) = 10 / 20 = .5.

Assume that a red bus is introduced with all the same attributes as those of the blue bus (i.e. it is indistinguishable from blue bus except for color, an unobserved attribute). So, in order to retain constant ratio's of alternatives (IIA), the original share of blue bus probability, the following is obtained:

$$P(A) \quad = \quad \frac{7}{12} \quad = \quad 58.33\%$$

$$P(BB) \quad = \quad \frac{2}{12} \quad = \quad 16.67\%$$

$$P(RB) \quad = \quad \frac{2}{12} \quad = \quad 16.67\%$$

$$P(R) \quad = \quad \frac{1}{12} \quad = \quad 8.33\%$$

since the probability of the red bus and blue bus must be equal, and the total probability of all choices must sum to one. If one attempts an alternate solution where the original 'bus' share is split between RB and BB, and the correct ratios are retained, one obtains the same answer as previously.

This is an unrealistic forecast by the logit model, since the individual is forecast to use buses more than before, and auto and rail less, despite the fact that a new mode with new attributes has not been introduced. In reality, one would not expect the probability of auto to decline,

because for traveler n a 'new' alternative has not been introduced. In estimating MNL models, the analyst must be cautious of cases similar to the red-bus/blue-bus problem, in which mode share should decrease by a factor for each of the 'similar' alternatives.

If one attempts an alternate solution where the original 'bus' share is split between RB and BB, and the correct ratios are retained, one obtains the same answer as previously.

$$P(A) \quad = \frac{3.5}{6} \quad = \quad 58.33\%$$

$$P(BB) \quad = \frac{1}{6} \quad = \quad 16.67\%$$

$$P(RB) \quad = \frac{1}{6} \quad = \quad 16.67\%$$

$$P(R) \quad = \frac{0.5}{6} \quad = \quad 8.33\%$$

This is an unrealistic forecast by the logit model, since the individual is forecast to use buses more than before, and auto and rail less, despite the fact that a new mode with new attributes has not been introduced. In reality, one would not expect the probability of auto to decline, because for traveler n a 'new' alternative has not been introduced. In estimating MNL models, the analyst must be cautious of cases similar to the red-bus/blue-bus problem, in which mode share should decrease by a factor for each of the 'similar' alternatives.

The IIA restriction does not apply to the population as a whole. That is, it does not restrict the shares of the population choosing any two alternatives to be unaffected by the utilities of other alternatives. The key in understanding this distinction is that for homogenous market segments IIA holds, but across market segments unobserved attributes vary, and thus the IIA property does not hold for a population of individuals.

A MNL therefore is an appropriate model if the systematic component of utility accounts for heterogeneity across individuals. In general, models with many socio-economic variables have a better chance of not violating IIA.

When IIA does not hold, there are various methods that can be used to 'get around' the problem, such as nested logit and probit models.

Elasticities of MNL

The analyst can use coefficients estimated in logit models to determine both disaggregate and aggregate elasticities, as well as cross-elasticities.

Disaggregate Direct Elasticities

An example of dissaggregate direct elasticity is given by the following. An analyst wants to know the effect of a unit change in the value of some attribute on alternative mode utilities, or mode preferences for traveler n. The elasticity of traveler n for parameter k on alternative i is given by:

$$E_{x_{ink}}^{P_n(i)} = [1 - \Pr_n(i)]x_{ink}\boldsymbol{b}_k.$$

For example, assume that individual 18 (an observation in observed data) has an auto travel time of 51.0 minutes and transit travel time of 85.0 minutes. For this individual, the probability of choosing auto is given by plugging auto travel time and transit travel time into the MNL estimated on the sample of data:

$$\Pr_{18}(auto) = \frac{e^{-0.2375-0.0531(51.0)}}{e^{-0.2375-0.0531(51.0)} + e^{0.0531(85.0)}} = \frac{0.0526}{0.0526+0.0089} = .8553$$

This individual's direct elasticity of auto travel time with respect to auto choice probability is calculated to obtain:

$$E_{x_{bus,n,dist.}}^{P_n(bus)} = [1 - .8553](51)(-0.0531) = -.3919\%$$

Thus for an additional minute of travel time in the auto, there would be a decrease of 0.39% in auto usage for this individual. Of course this is the statistical result, which suggests that over repeated choice occasions, the decision maker would use auto 1 time less in 100 per 3 minutes of additional travel time.

Disaggregate Cross-Elasticities

If the analyst was instead interested in the effect that travel time had on choosing mode j, say auto, where P_n (Auto) = .45, the analyst could simply set up a cross elasticity as follows:

$$E_{x_{jnk}}^{P_n(i)} = -\Pr_n(j)x_{jnk}\boldsymbol{b}_k = -(.45)(10)(-5) = 22.5$$

which suggests that the deterministic utility of the Auto for traveler n increases by 22.5 given a unit increase of 1 minute of travel time to the bus stop.

Aggregate Direct Elasticities

This type of elasticity is simply the weighted average of individual elasticities across some subgroup of individuals who chose alternative i. This is useful for predicting the change in the expected mode share across the group who chose alternative i. The elasticity is given by:

$$E_{x_{jk}}^{\overline{P}(i)} = \frac{\sum_{n=1}^{N} \Pr_n(i)E_{x_{jnk}}^{\Pr_n(i)}}{\sum_{n=1}^{N} \Pr_n(i)}$$

The dis-aggregate elasticity is the probability that a subgroup will choose mode i with respect to a unit or incremental change in variable k. For example, this could be used to predict the change in mode share for group of transit if transit fare increased by 1 unit.

The process of Estimating Multinomial Logit Models

Specification of the MNL requires several distinct steps to be taken by the analyst.

1. Identify the choice set, C of alternatives. This will be different depending upon the geographical location, population, socio-economic characteristics, attributes of the alternatives, and factors that influence the choice context.

2. Identify the feasible choice subsets C_n for individuals in the sample. Note that there are two choice sets; one 'universal' choice set C for the entire population, and choice sets C_n for individuals in the population. It is important that choice sets do not include modes that are not considered, and conversely, that all considered modes are represented. In practice it can be difficult to forecast with restricted choice sets, but the resulting model will be improved if restricted choice sets are known for individuals.

3. Next, the analyst must identify which variables influence the decision process, which characteristics of individuals are important in the choice process, and how to measure and collect them.

4. Design and administer a survey instrument (includes devising a sampling scheme) to collect the necessary information (the topic of another course), or observe/record the choice being made by individuals.

5. Finally, MNL models are estimated and refined to select the 'best' using all of the data gathered in previous steps.

Interpretation of MNL Model Results:

Estimation of MNL models leads to fairly standard output from estimation programs. In general program output can be obtained showing coefficient estimates, model goodness of fit, elasticities, and various other aspects of model fitting.

Model Coefficients

There are several rules to consider when interpreting the coefficients in the MNL model.

1. Alternative specific constants can only be included in MNL models for n-1 alternatives. Characteristics of decision-makers, such as socio-economic variables, must be entered as alternative specific. Characteristics of the alternative decisions themselves, such as costs of different choices, can be entered in MNL models as "generic" or as alternative specific.

2. Variable coefficients only have meaning in relation to each other, i.e., there is no 'absolute' interpretation of coefficients. In other words, the absolute magnitude of coefficients is not interpretable like it is in ordinary least squares regression models.

3. Alternative Specific Constants, like regression, allow some flexibility in the estimation process and generally should be left in the model, even if they are not significant.

4. Like model coefficients in regression, the probability statements made in relation to t statistics are conditional. For example, most computer programs provide t-statistics that provide the probability that the data were observed given that true coefficient value is zero. This is different than the probability that the coefficient is zero, given the data.

Planning Example: A binary logit model was estimated on data from Washington, D.C. (see Ben-Akiva and Lerman, 1985). The following table (adapted from Ben-Akiva et. al.) shows the model results, specifically coefficient estimates, asymptotic standard errors, and the asymptotic t-statistic. These t-statistics represent the t-value (which corresponds to some probability) that the true model parameter is equal to zero. Recall that the critical values of for a two-tailed test are ± 1.65 and ± 1.96 for the 0.90 and 0.95 confidence levels respectively.

Variable Name	Coef. Estimate	Standard Error	t statistic
Auto Constant	*1.45*	*0.393*	*3.70*
In-vehicle time (min)	*-0.00897*	*0.0063*	*-1.42*
Out-of-vehicle time (min)	*-0.0308*	*0.0106*	*-2.90*
Out of pocket cost*	*-0.0115*	*0.00262*	*-4.39*
Transit fare **	*-0.00708*	*0.00378*	*-1.87*
Auto ownership*	*0.770*	*0.213*	*3.16*
Downtown workplace* (indicator variable)	*-0.561*	*0.306*	*-1.84*

* auto specific variable; **transit specific variable

Inspection of the estimation results suggests that all else being equal, the auto is the preferred alternative, since the alternative specific constant for auto is positive. Note that only one alternative specific constant is entered in the model. Also, all but one of the variables is statistically significant at the 10% level of significance.

The model shows that for an additional minute of in-vehicle travel time, the utility of that mode decreases. Since the variable is entered as "generic", it reflects the effect of in vehicle travel time in either transit or auto. It might be believed that travelers do not have the same response to travel time by mode, and so this variable could be entered as alternative specific.

The model shows that out-of-vehicle time, entered as "generic", is by a factor of about 3 more influential on utility than is in-vehicle time.

The model shows that travelers are sensitive to travel costs; utility for transit decreases as transit fare increases, and utility for auto decreases as out-of-pocket costs increase. Notice that auto riders are approximately doubly sensitive to travel costs as are transit riders.

Owning a vehicle provides greater utility for taking auto, as one would expect. Working in the downtown actually reduces the utility of the auto—presumably the downtown is easily accessed via transit, and the impedance to downtown via auto is great. Although part of the impedance may partly be cost and travel time, which has already been captured, there may be additional impedance due to availability and cost of parking, safety, and other factors.

Refine models

Assess Goodness of Fit

There are several goodness of fit measures available for testing 'how' well a MNL model fits the data on which it was estimated.

The likelihood ratio test is a generic test that can be used to compare models with different levels of complexity. Let $L(\beta')$ be the maximum log likelihood attained with the estimated parameter vector β', on which no constraints have been imposed. Let (β'_c) be the maximum log likelihood attained with constraints applied to a subset of coefficients in β'. Then, asymptotically (i.e. for large samples) $-2(L(\beta'_c) - L(\beta'))$ has a chi-square distribution with degrees of freedom equaling the number of constrained coefficients. Thus the above statistic, called the "likelihood ratio", can be used to test the null hypothesis that two different models perform approximately the same (in explaining the data). If there is insufficient evidence to support the more complex model, then the simpler model is preferred. For large differences in log likelihood there is evidence to support preferring the more complex model to the simpler one.

In the context of discrete choice analysis, two standard tests are often provided. The first test compares a model estimated with all variables suspected of influencing the choice process to a model that has no coefficients whatsoever—a model that predicts equal probability for all choices. The test statistic is given by:

$$-2(L(0) - L(\beta')) \approx \chi^2, \text{ df = total number of coefficients}$$

The null hypothesis, H_0 is that all coefficients are equal to 0, or, all alternatives are equally likely to be chosen. $L(0)$ is the log likelihood computed when all coefficients including alternative specific constants are constrained to be zero, and $L(\beta')$ is the log likelihood computed with no constraints on the model.

A second test compares a complex model with another naïve model, however this model contains alternative specific constants for $n-1$ alternatives. This naïve model is a model that predicts choice probabilities based on the observed market shares of the respective alternatives. The test statistics is given by:

$$-2(L(C) - L(\beta')) \approx \chi^2, \text{ df = total number of coefficients}$$

The null hypothesis, H_0 is that coefficients are zero except the alternative specific constants. $L(C)$ is the log likelihood value computed when all slope coefficients are constrained to be equal to zero except alternative specific constants.

In general the analyst can conduct specification tests to compare 'full' models versus reduced models using the chi-square test as follows:

$$-2(L(F) - L(R)) \approx \chi^2, \text{ df = total number of restrictions, } = K_F - K_R$$

In logit models there is a statistic similar to R-Squared in regression called the Pseudo Coefficient of Determination. The Psuedo coefficient of determination is meant to convey similar information as does R-Squared in regression—the larger is r^2 the larger the proportion of log-likelihood 'explained' by the parameterized model.

$$r^2 = 1 - \frac{L(\boldsymbol{b}')}{L(0)}, \qquad \text{or} \qquad r^2 = 1 - \frac{L(\boldsymbol{b}')}{L(C)}, \quad \text{where} \ \ 0 \le r^2 \le 1$$

The r_c^2 statistic is the r^2 statistic corrected for the number of parameters estimated, and is given by:

$$\overline{r}^2 = 1 - [\frac{L(\boldsymbol{b}) - K}{L(0)}]$$

Planning Example Continued: Consider again the binary logit model estimated on Washington, D.C., data. The summary statistics for the model are provided below (adapted from Ben-Akiva and Lerman, 1985).

<u>Summary Statistics for Washington D.C. Binary Logit Model</u>
number of model parameters = 7
L (0) = - 1023
L (b) = -374.4
-2[L (0) – L (b)] = 1371.7
c^2 (7, 0.95) = 14.07
r = 0.660
r_c = 0.654

The summary statistics show the log likelihood values for the naïve model with zero parameters, and the value for the model with the seven parameters discussed previously. Clearly, the model with seven parameters has a larger log likelihood than the naïve model, and in fact the likelihood ratio test suggests that this difference is statistically significant. The r and r_c suggest that about 65% of the log likelihood is "explained" by the seven parameter model. This interpretation of r should be used loosely, as this interpretation is not strictly correct. A more useful application of r_c would be to compare it to a competing model estimated on the same data. This would provide one piece of objective criterion for comparing alternative models.

Variables Selection

There are asymptotic t-statistics that are evaluated similarly to t-statistics in regression save for the restriction on sample sizes. That is to say that as the sample size grows the sampling distribution of estimated parameters approaches the t-distribution. Thus, variables entered into the logit model can be evaluated statistically using t-statistics and jointly using the log likelihood ratio test (see goodness of fit). Of course variables should be selected *a priori* based upon their theoretical or material role in the decision process being modeled.

Tests of Non-Linear Specifications

As in classical regression, there may be times when a linear specification is an inappropriate one. Thus, there are a couple of approaches to employ: a piece-wise linear approach, and/or variable transformations and/or power expansions. Overall model tests can be conducted to determine if the additional variables or variable transformation offers an improvement.

Test of Taste Variations

Choice theory is essentially a dis-aggregate modeling approach. Specifically, it states that individuals have utility functions that include attributes of alternatives and of individuals. However, the current model has fixed parameter estimates, that is, all individuals are expected to respond (linearly) to the same coefficients. This might not be an altogether reasonable assumption, however, since individuals may have different parameter values, or even utility functions themselves.

There are two methods employed to get around this problem. The first is to treat the socio-economic variables differently that enter into the models. The second is to introduce what is called a random coefficients logit model, which is technically difficult and computationally burdensome. In addition, for forecasting purposes it is not that useful to have random coefficients in a model.

Market segmentation allows for different Beta values across market segments. In this approach G market segments are defined, and a vector of parameters for each of the G market segments is estimated. Then, the null hypothesis that: $\beta^1 = \beta^2 = \beta^3 = \ldots\ldots = \beta^G$ is tested using the fact that:

$$-2[\mathrm{L}(\boldsymbol{b}) - \sum_{g=1}^{G} \mathrm{L}(\boldsymbol{b}^g)] \quad \text{is chi-square distributed with} \quad \sum_{g=1}^{G} \mathrm{K}_g - \mathrm{K} \quad \text{degrees of freedom.}$$

Are choice model assumptions met?

Identically and independently distributed errors

If the IIA assumption does not hold, there are alternative methods for estimating choice models. To determine if alternative models are necessary, there are some useful tests to determine if IIA is violated.

Since the ratio of choice probabilities between two modes is expected to remain unchanged relative to 'other' choices, other choices could feasibly be added to the 'choice set' and the original choice probability ratios should remain unchanged.

A test proposed by Hausman and McFadden (1984) incorporates the use of a test conducted between the restricted choice set model (r), which is the model estimated without one of the choice alternatives, and a full model (f), estimated on the full set of alternatives. If IIA is not violated, then the coefficient estimates should only be affected by random fluctuation caused by statistical sampling. The test statistic $q=[b_u - b_r]'\ [V_u - V_r]\ [b_u - b_r]$ is asymptotically chi-square distributed with K_r degrees of freedom, where K_r is the number of coefficients in the restricted choice set model, b_u and b_r are the coefficient vectors estimated for the unrestricted and restricted choice sets respectively, and V_u and V_r are the variance-covariance matrices for the unrestricted and restricted choice sets respectively. This test can be found in textbooks on discrete choice and some software programs.

Uncorrelated errors

Correlated errors occur when either unobserved attributes of choices are shared across alternatives (the IIA assumption), or when panel data are used and choices are correlated over time. Violation of the IIA assumption has been dealt with in a previous section. Panel data need to be handled with more sophisticated methods incorporating both cross-sectional and panel data.

Outlier Analysis

Similar to regression, the analyst should perform outlier analysis. In doing so, the analyst should inspect the predicted choice probabilities with the chosen alternative. An outlier can arbitrarily be defined as a case where a decision was chosen even though it only had a 1 in 100 chance of being selected. When these cases are identified, the analyst looks for miscoding and measurement errors made on variables.

If an observation is influential, but is not erroneous, then the analyst must search for ways to investigate. One way is to estimate the model without the observation included, and then again without the observation.

Alternative choice model specifications

When the IIA property of MNL is violated, the modeler should consider alternative specifications. Recall that IIA is violated when alternatives share unobserved attributes. When there are shared unobserved components associated with different choices or alternatives, the utilities of the elements of the corresponding multidimensional choice set cannot be independent. There are two common strategies of dealing with violations of the IIA assumption in the MNL model— nested logit and multi-nomial probit models.

Nested logit

One may think of a multi-dimensional choice context as one with inherent structure, or hierarchy. This notion helps the analyst to visualize the nested logit model, although the nested logit model is not inherently a hierarchical model. Consider the case of four travel alternatives, auto, bus with walk access, bus with auto access, and carpool. This might be thought of as a nested choice structure, the first decision is made between public transit and auto, and then between which alternative given that public or private has been selected. Mathematically, this nested structure allows subsets of alternatives to share unobserved components of utility, which is a strict violation of the IIA property in the MNL model. For example, if transit alternatives are nested together, then it is feasible that these alternatives shared unobserved utility components such as comfort, ride quality, safety, and other attributes of transit that were omitted from the systematic utility functions. This 'work-around' solution to the IIA assumption in MNL model is a feasible and relatively easy solution. In a nutshell, the analyst groups alternatives that share unobserved attributes at different levels of a nest, so as to allow error terms within a nest to be correlated. For more detailed discussions on nested logit models consult the references in listed in this chapter.

Multi-nomial probit

Multinomial probit is an extension of probit models to more than two alternatives. Unfortunately, they are difficult to estimate for more than 4 or 5 alternatives due to the mathematical

complexity of the likelihood function as the number of alternatives increases. As computers become faster and/or computational methods become improved, multinomial probit models may be used to estimate models for reasonable sized choice sets.

External validation of model

Prediction Tests

Like in regression, perhaps the most powerful test of any model comes from the use of external 'validation' data. External data can test that the model does not over-fit the estimation data, and can also be used to assess the generalizeability of model results across space and time. To externally validate a model, new data are collected and used to assess the predictive ability of the model. It is important to understand that MNL models are difficult to interpret for individual predictions or decisions, so validation should be consistent with the true interpretation of the model, that is, the Pr_n (i) is a long-term notion of probability, and therefore could best be validated preferably over panels of observations on individuals or through observations on 'apparently' homogenous groups of individuals.

Conduct Statistical Inference, Document Model, and Implement

The modeler is now ready to conduct statistical inference, document the model, and implement. Statistical inference is the process by which inferences are made about the population, or process being modeled, based on the model estimated on the sample of data. Statistical inference is the cornerstone of statistical theory and allows the modeler to make statements about the population.

There are several estimated parameters that are used to make inferences about the population, the betas, which represent the mean change in Y (choice probability) given a unit change in the X's, and Y_{hat}, which reflects the mean response given a combination of X values.

The interpretation of a confidence interval is very explicit and should be treated with caution. A 1- α confidence interval on β_1 indicates that the true value of β_1 will fall within the confidence limits given repeated samples taken on the same X levels α times out of 100. Recall that β_1 is the mean change in the mean of the distribution in Y with a unit change in x.

One-sided versus Two-sided Hypothesis Testing

The confidence interval given by $b_1 +- t \{1 - \alpha / 2; n - 2\} s [b_1]$ is a two-sided confidence interval. This means that the analyst is concerned with how probable events are on either side of the true parameter β_1. Often, the engineer wants to ask the question "does some confidence interval of β_1 contain the value 0?" The alternative hypotheses then are:

$$H_0: \beta_1 = 0, \text{ and } H_a: \beta_1 \neq 0.$$

If the confidence interval for a given confidence level 1 - α (i.e. 1 - .05 = .95) does not contain 0, then one can conclude that at that confidence level, $\alpha*100$ times out of 100 when repeat samples are drawn at the same x levels the confidence interval will not contain 0.

The analyst could alternatively test whether β_1 is positive. In this case the test hypotheses are:

$$H_{0:}\ \beta_1 \leq 0, \text{ and } H_{a:}\ \beta_1 > 0.$$

In this case one side of the distribution is considered such that all the error is assigned to one side of the probability distribution.

Model Documentation

Once a model has been estimated and selected to be the best among competing models it needs to be thoroughly documented, so that others may learn from the modeler's efforts. It is important to recognize that a model that performs below expectations is still a model worth reporting. This is because the accumulation of knowledge is based on objective reporting of findings—and only presenting success stories is not objective reporting. It is just as valuable to learn that certain variables don't appear effectual on a certain response than vice versa.

When reporting the results of models, enough information should be provided so that another researcher could replicate your results. Not reporting things like sample sizes, manipulations to the data, estimated variance, etc., could render follow-on studies difficult.

Perhaps the most important aspect of model documentation is the theory behind the model. That is, all the variables in the model should be accompanied by a material explanation for being there. Why are the X's important in their influence on Y? What is the mechanism by which X influences Y? Would one suspect an underlying causal relation, or is the relationship merely associative? These are the types of questions that should be answered in the documentation accompanying a modeling effort. In addition, the model equations, the t-statistics, R-square, MSE, and F-ratio tests results should be reported. Thorough model documentation will allow for future enhancements to an existing model.

Model Implementation

Model implementation, of course, should have been considered early on in the planning stages of any research investigation. There are a number of considerations to take into account during implementation stages:

1) Are the variables needed to run the model easily accessible?

2) Is the model going to be used within the domain with which it was intended?

3) Has the model been validated?

4) Will the passage of time render model predictions invalid?

5) Will transferring the model to another geographical location jeopardize model accuracy?

These questions and other carefully targeted questions about the particular phenomenon under study will aid in an efficient and scientifically sound implementation plan.

Interpretation of Logit, Nested Logit, and Probit:

How is a choice model equation interpreted?

A MNL model equation represents the association between a dependent variable Y, which represents the probability of a particular choice being made, and one or more independent variables (X's) that reflect attributes of the choices and the choice-maker. Unlike the linear regression model, the coefficients in choice models are multiplicative on the response. The model parameters, or partial slope coefficients, represent the change in Y given a unit change in X, all else held constant. If the model was estimated using experimental data, then the parameters may represent the change in Y caused by a unit change in a particular X. If the model was estimated using quasi-experimental or observational data, then the regression parameters represent the change in Y associated with a unit change in a particular X, and do not necessarily represent causal effects.

The choice model equation is meant to model as accurately as possible the relationships in the true population, in as simple an equation as is possible. The regression model is known *a priori* not to capture all the structure in the real data, and is known to be wrong to some degree. The model represents a convenient way to explain relationships or predict future events given known inputs, or value of the independent variables (X's).

How do continuous and indicator variables differ in the choice model?

Continuous variables are usually interval or ratio scale variables, whereas indicator variables are usually nominal or ordinal scale variables. Indicator variables can be entered in the choice model or can be interacted with a continuous variable and effect the slope coefficient of the interacted variable. Indicator variables are somewhat analogous to testing the difference in means between two groups as in ANOVA. Indicator variables in the choice model can only take on one of two values— 0 or 1.

How are beta coefficients interpreted?

The betas in a logit model are called the model coefficients. The coefficient with the variable X_1, β_1, indicates the change in the mean of the probability distribution of Y, the probability that a choice is made, per unit increase in the multiplicative exponent of X_1. Thus, the interpretation is not straightforward as is the interpretation for linear regression.

How is the Likelihood Ratio Test interpreted?

The likelihood ratio test is similar to the F test in regression. Under the null hypothesis that all coefficients are zero (or some other null hypothesis that represents a restricted model), that is $\beta_1 = \beta_2 = \ldots = \beta_K = 0$, the test statistic $-2\{L(0) - L(\mathbf{b})\}$ is χ^2 distributed with K degrees of freedom.

Often a more useful test compares a model with alternative specific constants in the model only, instead of a model with all coefficients equal to zero. Alternatively, the analyst can compare any full model with a restricted model using the likelihood ratio test.

A large value of the likelihood ratio test statistic provides evidence against the restricted model. For a likelihood ratio observed at an alpha of 5%, the likelihood ratio test statistic as large as the one observed would be obtained in 5 samples in 100.

How are t-statistics interpreted?

A t-statistic is similar to a likelihood ratio test, except the test is for a single variable in the model. The standard t-test provided by most standard statistical software packages is used to determine the probability that an individual variable's parameter is equal to zero. In actuality the test is conditional on the variable's parameter equaling zero, and provides the probability of the data having arisen under this constraint.

In the theory of discrete choice models t-tests are not exact results, and are instead asymptotic results. This implies that as the sample size approaches infinity, the estimated model coefficients are distributed as t.

How are phi and adjusted phi interpreted?

Phi, or the likelihood ratio index, is analogous to R^2 in linear regression. An adjusted phi, compensating for models with different numbers of explanatory variables, adjusts for the fact that phi can only increase or stays the same with additional explanatory variables. In general, the larger is phi, the greater is the explanatory power of the model.

How are confidence intervals interpreted?

A confidence interval is interpreted as follows: If samples were repeatedly drawn at the same X-levels as were drawn in the original sample, then alpha (α) times out of 100 the mean of the sample Y's will fall within the $(1-\alpha)$% confidence interval. In simpler but less technically correct terms, the analyst is $(1-\alpha)$% confident that the mean of a new sample falls in the confidence interval. Confidence intervals might be constructed around parameter values in discrete choice models using asymptotic t-distribution results. In this case, the analyst will make inferences about values of the true model parameters, which are estimated by the model coefficients.

How are degrees of freedom interpreted?

Degrees of freedom are associated with sample size. Every time a statistical parameter is estimated on a sample of data the ability to computer additional parameters decreases. Degrees of freedom are the number of independent data points used to estimate a particular parameter.

How are elasticities computed and interpreted?

An elasticity is the change in some response due to an independent variable. For instance an analyst may want to know the effect of a unit change in the value of some attribute, say travel time, on alternative mode utilities, or mode preferences for traveler n.

When is the independence of irrelevant alternatives (IIA) assumption violated?

Succinctly stated, the IIA property states that for a specific individual the ratio of the choice probabilities of any two alternatives is entirely unaffected by the systematic utilities of any other alternatives. This property arises from the assumption in the derivation of the logit model that

the error terms ε_n across individuals are independent. In other words, it is assumed that un-observed attributes (error terms) of alternatives are independent. In many cases this is an unrealistic assumption, and creates some difficulties. For example, if driver n has an unobserved (error term) preference for public transit, then public transit mode error terms will not be independent.

TroubleShooting: Logit, Nested Logit, and Probit

Should interaction terms be included in the model?

Interactions represent synergistic effects of two or more variables. Interaction terms represent potentially real relationships embedded in data. Most often they arise in quasi-experimental and observational data. An interaction that is important should be included in the model, despite the fact that it might not contribute much to model explanatory power. In general, third and higher order interactions (that are real in the population) can be ignored without much detriment to the model.

How many variables should be included in the model?

The objective of most modeling efforts is to economize the model, since it is known *a priori* that the great complexity underlying the data cannot be modeled exactly. In other words, the analyst generally wishes to explain as much of the data complexity with as few variables as practicable. Generally seven variables plus or minus two variables covers most models, although smaller and larger models can be found. It is generally better to favor a simpler model to a more complex one, simply because interpretation and implementation are simplified also. On the other hand, if the phenomenon is sufficiently complex, then making too simple a model may sacrifice too much explanatory or predictive power.

What methods can be used to specify the relation between choice and the X's?

Unlike linear regression, which represents a linear relation between a continuous variable and one or more independent variables, it is difficult to develop a useful plot between explanatory variables and the response used in discrete choice models. An exception to this occurs when repeated observations are made on an individual, or data are grouped (aggregate). For instance, a plot of proportion choosing alternative A by group (or individual across repeat observations) may reveal some differences across experimental groups (or individuals).

What methods are available for fixing heteroscedastic errors?

Heteroscedasticity in discrete choice models is a violation of the IIA property. It occurs when there are shared unobserved components associated with different choice dimensions. The most common procedure for dealing with heteroscedasticity is by employing the nested logit model.

What methods are used for fixing serially correlated errors?

Serially correlated errors occur when observations are taken over time. The primary reason is that there are unobserved attributes that affect the decision process for an individual over time, such as built in biases, experiences, etc. These kinds of data, often referred to as panel or

time-series data, represent more complicated models of choice behavior. Consult Greene for additional details on dealing with serially correlated data in discrete choice models.

What can be done to deal with multi-collinearity?

Multi-collinearity is when two variables co-vary, and is commonly found in non-experimental data. It is not a violation of choice models explicitly, however it does cause problems in the mathematics of solving for the regression parameters. In essence, highly collinear variables (e.g. a correlation coefficient of 0.7 or higher) cause regression parameters to be inefficient (high standard errors), and can cause the signs of the regression coefficients to be counter-intuitive.

There are few remedies to the multi-collinearity problem. First, highly correlated variables can be left in the model and assumed to reflect the natural state of those variables in reality. In this case the analyst must rely on the collinearity being ever-present in future observed data. A second option is to remove the less important of the two collinear variables and keep only one in the model. This is usually the preferred option. A third option is to employ a biased estimation technique such as ridge regression to opt for more biased but precise estimates, which are not influenced by the presence of multicollinearity.

What is endogeneity and how can it be fixed?

Endogeneity is fancy term for having an independent variable that is directly influenced by the dependent variable Y. It is presumed *a priori* that all independent variables are exogenous—that they are determined by influences outside of the modeling system. When a particular X is endogenous, the model errors are correlated with the variable, and problems in the regression occur, such as biased estimates, etc. Some remedies include the use of instrumental variables approaches, proxy variables, and structural equations models.

How does one know if the errors are Gumbel distributed?

The Gumbel distribution is the assumed distribution of the MNL and nested logit models. The Gumbel distribution has two parameters, a scale parameter η and a location parameter, μ. It is conveniently assumed in MNL and nested logit models that the scale parameter is equal to 1, since it is not directly estimable.

Unlike linear regression, it is not easy to determine if model errors are Gumbel distributed. In the case or grouped (aggregate) or repeated individual observation data, an analyst could plot the distribution of errors by computing the choice probabilities and comparing them to observed proportions of choices. However, this is often impractical, and due to the lack of alternative distributional forms offered in discrete choice models, is often not performed.

CHAPTER 5; SECTION B: POISSON AND NEGATIVE BINOMIAL MODELS

Purpose of Poisson and Negative Binomial Models:

Poisson and negative binomial models are used most frequently in transportation for modeling count data (ordinal scale variable).

> ***Examples:***
> 1. ***Crash occurrence at a road section, intersection, often follows a Poisson or negative binomial distribution.***
> 2. ***Number of failures during a specified time period, can often be modeled as a Poisson process.***

Basic Assumptions/Requirements of Poisson and Negative Binomial Models:

1) Response or dependent variable follows a Poisson or negative binomial process, characterized by: low probability of success in a given time period or duration and a large number of trials (time periods), fixed probability of "success" in any given time period.

2) A negative binomial process results from a mixture of Poisson process, such as sampling from multiple Poisson process with varying means, such that the resulting distribution (negative binomial) is said to be an over dispersed Poisson distribution.

3) Data can be censored or truncated, but this is not an essential characteristic of the process.

Inputs for Poisson and Negative Binomial Models:

- Ordinal variable Y, which represents the number of events observed during intervals of time or space,

- Sample of observations on Y for a vector of explanatory variables, X.

Outputs of Poisson and Negative Binomial Models:

- Goodness of fit statistics of observed count data,

- Estimated effects of covariates, X, on count during intervals of time or space.

Overview of Poisson and Negative Binomial Models:

There are several characteristics of the modeling of count data that warrant mention here. Note that modeling count data is not covered in detail in this manuscript, and so only some of the main highlights are provided. The references provided at the end of this chapter should be consulted for detailed guidance for estimating models based on count data.

1) A random variable Y that follows a Poisson process is characterized by a Poisson mean λ, where $E[Y] = Var[Y] = \lambda$. When $E[Y] > Var[Y]$, the Poisson process is said to be over-dispersed.

2) Over-dispersion of the Poisson process occurs in a number of ways. It can occur when a Poisson process is observed where time intervals are random instead of fixed. It can also occur when the means of Poisson random variables are thought to vary across subjects, such that the distribution of λ is gamma distributed. This occurs often in crash investigations, as sights being investigated have different underlying safety as reflected by their Poisson process means.

3) The Poisson regression model produces probabilities of an event occurring Y=y times, given a host of covariates thought to effect the Poisson mean. Thus, the model is used to compute the long-run probability that an experimental unit with covariates X has Y=y occurrences of the event of interest. For instance, the Poisson regression model can be used to model crash occurrence on two-lane rural roads with differing crash probabilities as a function of site characteristics.

Examples of Poisson and Negative Binomial Regression Models:

Safety
Hadi, Mohammed A., Aruldhas Jacob, Chow Lee-Fang and Wattlewort Joseph A. (1995). Estimating Safety Effects of Cross-Section Design for Various Highway types Using Negative Binomial Regression. Transportation Research Record #1500 pp. 169-177. National Academy of Sciences.

Poisson and Negative Binomial Model References:

- Greene, William, (1990). *Econometric Analysis.* Macmillan Publishing Company. Hillsdale, NJ.

- McCullagh, P. and J.A. Nelder, (1983). *Generalized Linear Models.* Second Edition. Chapman and Hall, New York, New York.

- Venables, W.N., and B.D. Ripley, (1994). Modern Applied Statistics with S-Plus, Second Edition. Springer-Verlag, New York, New York.

CHAPTER 5; SECTION C: LOGISTIC REGRESSION MODELS

Purpose of Logistic Regression Models:

Logistic regression extends the analysis of multi-way tables into the regression modeling framework, where binomial outcome probabilities are modeled (choices with multi-nomial outcomes are modeled with discrete choice models). For modeling data, logistic regression is probably more useful than log-linear models.

> **Examples of logistic regression model applications include:**
> 1. **Modeling the probability of seat-belt use by automobile drivers.**
> 2. **Modeling the probability of choosing between transit and auto.**
> 3. **Modeling the probability that a bridge deck becomes delaminated.**

Basic Assumptions/Requirements of Logistic Regression Models:

1) For logistic regression models, the response variable must be binomial, or result in one of two outcomes. Many multinomial distributions can be made binomial by collapsing across multiple categories.

2) A set of predictors (independent variables) thought to affect the outcome of the binomial outcome.

Inputs for Logistic Regression Models:

A binomial outcome variable Y measured on each experimental unit (e.g. 0 or 1), binomial proportions measured on groups of individuals (e.g. 0.45, 0.55), or repeated measurements data, where an experimental unit has been observed/measured over repeated trials (e.g. 0.33, 0.67).

Sample of observations on binomial outcome Y for vector of explanatory variables, X. These can be grouped or individual X's.

Outputs of Logistic Regression Models:

- Goodness of fit statistics of model to binomial outcome data,

- Estimated effects of covariates, X, on binomial outcome or proportions.

Overview of Logistic Regression Models:

There are numerous characteristics of the modeling of binomial outcome data that warrant mention here. Note that modeling binomial outcome data is not covered in detail in this manuscript, and so only some of the main highlights are provided. The references provided at

the end of this chapter should be consulted for estimating binomial outcome, or logistic regression models.

1) Logistic regression models can be used to model fundamentally different response variables, those that are truly binomial, such as 0, 1, and proportions data, which are continuous data within the interval (0, 1). Binomial data are individual level observations on a binomial outcome, whereas proportions data could be obtained from grouped data (multiple experimental units observed on the binary outcome variable), or panel data (multiple observations on the same experimental unit over time).

2) Logistic regression models make use of the logistic transformation, given by $\log[\, p\,/(1-p)]$, where p is the binomial probability of "success". The logistic transformation, employed as the response variable in the logistic regression model, ensures that the model cannot predict outside the range of (0, 1).

3) Graphical plots of logistic regression data are only useful for grouped or panel data, since the predicted response lies in the interval (0, 1), whereas the binomial outcome takes on 0 or 1 discretely.

4) Outliers in logistic regression are determined by observing too many 0's or 1's when the model predicts an extremely low probability of observing this outcome.

5) Similar to linear regression, the probability of observing a 0 or 1 is influenced by an exogenous set of predictor variables, or X's.

6) Standard statistics such as parameter estimates, standard errors, and t-statistics are provided in the modeling of binary outcome data via logistic regression models.

Logistic Regression Examples:

Safety
Knuiman, Matthew W., Council Forrest M, and Reinfurt Donald W. (1993). <u>Association of Median Width and Highway Accident Rates</u>. Transportation Research Record #1401 pp. 70-82. National Academy of Sciences.

Council, Forrest M., and Stewart J. Richard. (1993). <u>Attempt to Define the Relationship Between Forces to Crash-Test Vehicles and Occupant Injury in Similar Real-World Crashes</u>. Transportation Research Record #1419 pp. 78-85. National Academy of Sciences.

Tzuoo-Ding, Lin, and Paul P. Jovanis and Chun-Zin Yang. (1993). <u>Modeling the Safety of Truck Driver Service Hours Using Time Dependent Logistic Regression.</u> Transportation Research Record #1407 pp. 1-10. National Academy of Sciences.

Pavements/Materials
Heydinger, Andrew G., Xie Qinglu, Randolph Brian W., and Gupta Jiwan D. (1996). <u>Analysis of Resilient Modulus of Dense and Open Graded Aggregates.</u> Transportation Research Record #1547 pp. 1-6. National Academy of Sciences.

Traffic

Chang, Gang-Len, Chien-Yu Chen, and Cesar Perez (1996). <u>Hybrid Model for Estimating Permitted Left-Turn Saturation Flow Rate.</u> Transportation Research Record #1566 pp. 54-63. National Academy of Sciences.

Logistic Regression Model References:

- Christensen, Ronald (1990). *Log-Linear Models.* Springer-Verlag. New York, New York.

- McCullagh, P. and J.A. Nelder, (1983). *Generalized Linear Models.* Second Edition. Chapman and Hall, New York, New York.