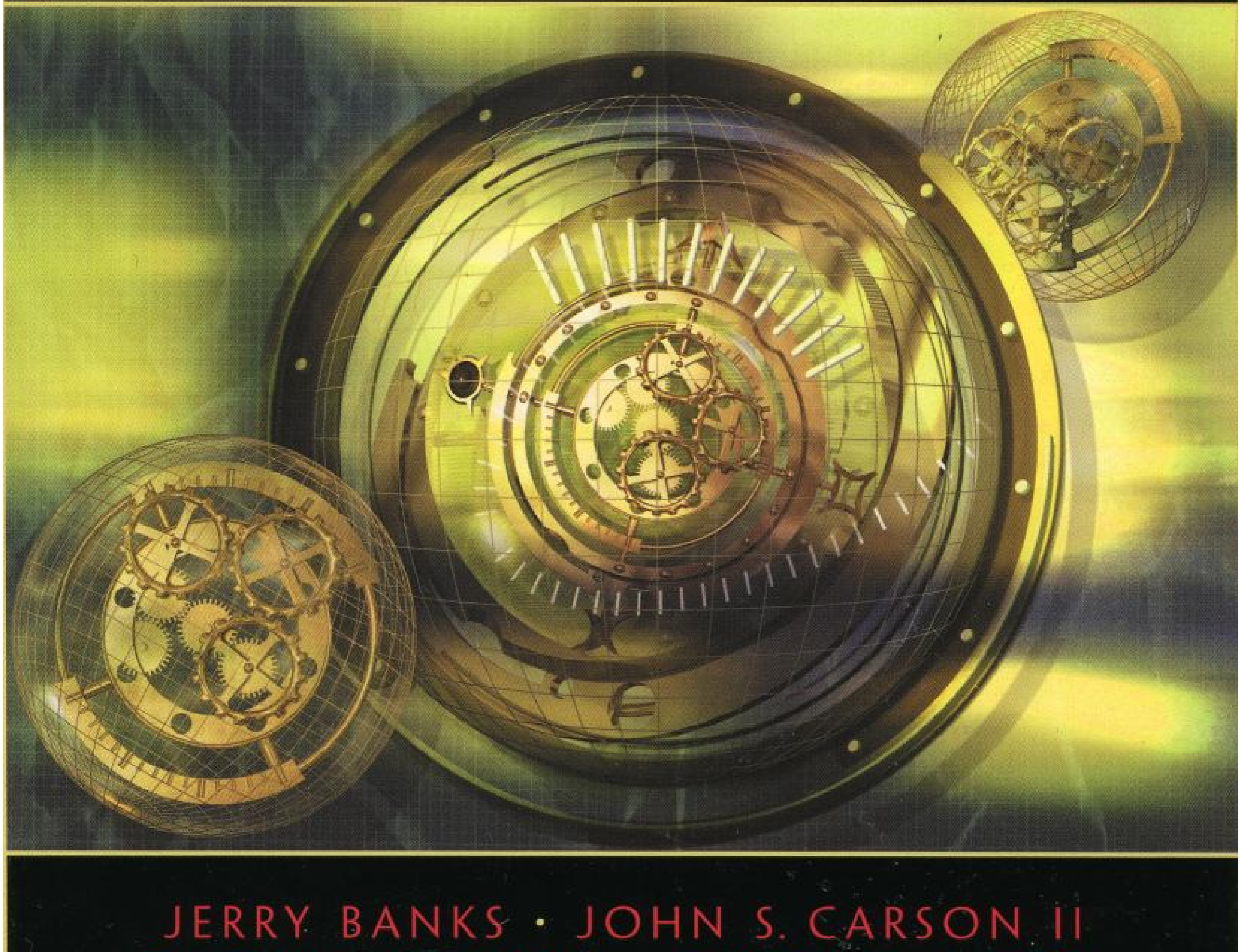
DISCRETE-EVENT SYSTEM SIMULATION

FOURTH EDITION



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Prentice Hall International Series in Industrial and Systems Engineering
W. J. Fabrycky and J. H. Mize, Series Editors

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CHAPTER – 1

INTRODUCTION TO SIMULATION

Simulation

A Simulation is the imitation of the operation of a real-world process or system over time.

Brief Explanation

- The behavior of a system as it evolves over time is studied by developing a simulation model.
- This model takes the form of a set of assumptions concerning the operation of the system.

  The assumptions are expressed in
  - Mathematical relationships
  - Logical relationships
  - Symbolic relationships

  Between the entities of the system.

Measures of performance

The model solved by mathematical methods such as differential calculus, probability theory, algebraic methods has the solution usually consists of one or more numerical parameters which are called measures of performance.

1.1 When Simulation is the Appropriate Tool

- Simulation enables the study of and experimentation with the internal interactions of a complex system, or of a subsystem within a complex system.

- Informational, organizational and environmental changes can be simulated and the effect of those alternations on the model's behavior can be observer.

- The knowledge gained in designing a simulation model can be of great value toward suggesting improvement in the system under investigation.

- By changing simulation inputs and observing the resulting outputs, valuable insight may be obtained into which variables are most important and how variables interact.

- Simulation can be used as a pedagogical device to reinforce analytic solution methodologies.

- Simulation can be used to experiment with new designs or policies prior to implementation, so as to prepare for what may happen.

- Simulation can be used to verify analytic solutions.

- By simulating different capabilities for a machine, requirements can be determined.

- Simulation models designed for training, allow learning without the cost and disruption of on-the-job learning.

- Animation shows a system in simulated operation so that the plan can be visualized.

- The modern system(factory, water fabrication plant, service organization, etc) is so complex that the interactions can be treated only through simulation.

1.2 When Simulation is Not Appropriate

- Simulation should be used when the problem cannot be solved using common sense.

- Simulation should not be used if the problem can be solved analytically.

- Simulation should not be used, if it is easier to perform direct experiments.

- Simulation should not be used, if the costs exceeds savings.

- Simulation should not be performed, if the resources or time are not available.

- If no data is available, not even estimate simulation is not advised.

- If there is not enough time or the person are not available, simulation is not appropriate.

- If managers have unreasonable expectation say, too much soon – or the power of simulation is over estimated, simulation may not be appropriate.

- If system behavior is too complex or cannot be defined, simulation is not appropriate.

1.3 Advantages of Simulation

- Simulation can also be used to study systems in the design stage.

- Simulation models are run rather than solver.

- New policies, operating procedures, decision rules, information flow, etc can be explored without disrupting the ongoing operations of the real system.

- New hardware designs, physical layouts, transportation systems can be tested without committing resources for their acquisition.

- Hypotheses about how or why certain phenomena occur can be tested for feasibility.

- Time can be compressed or expanded allowing for a speedup or slowdown of the phenomena under investigation.

- Insight can be obtained about the interaction of variables.

- Insight can be obtained about the importance of variables to the performance of the system.

- Bottleneck analysis can be performed indication where work-in-process, information materials and so on are being excessively delayed.

- A simulation study can help in understanding how the system operates rather than how individuals think the system operates.

- "what-if" questions can be answered. Useful in the design of new systems.

1.4 Disadvantages of simulation

- Model building requires special training.
- Simulation results may be difficult to interpret.
- Simulation modeling and analysis can be time consuming and expensive.
- Simulation is used in some cases when an analytical solution is possible or even preferable.

1.5 Applications of Simulation

Manufacturing Applications

1. Analysis of electronics assembly operations
2. Design and evaluation of a selective assembly station for high-precision scroll compressor shells.
3. Comparison of dispatching rules for semiconductor manufacturing using large facility models.
4. Evaluation of cluster tool throughput for thin-film head production.
5. Determining optimal lot size for a semiconductor backend factory.
6. Optimization of cycle time and utilization in semiconductor test manufacturing.
7. Analysis of storage and retrieval strategies in a warehouse.
8. Investigation of dynamics in a service oriented supply chain.
9. Model for an Army chemical munitions disposal facility.

Semiconductor Manufacturing

1. Comparison of dispatching rules using large-facility models.
2. The corrupting influence of variability.
3. A new lot-release rule for wafer fabs.
4. Assessment of potential gains in productivity due to proactive retied management.
5. Comparison of a 200 mm and 300 mm X-ray lithography cell.
6. Capacity planning with time constraints between operations.
7. 300 mm logistic system risk reduction.

Construction Engineering

1. Construction of a dam embankment.
2. Trench less renewal of underground urban infrastructures.

3. Activity scheduling in a dynamic, multiproject setting.
4. Investigation of the structural steel erection process.
5. Special purpose template for utility tunnel construction.

Military Applications

1. Modeling leadership effects and recruit type in a Army recruiting station.
2. Design and test of an intelligent controller for autonomous underwater vehicles.
3. Modeling military requirements for nonwarfighting operations.
4. Multitrajectory performance for varying scenario sizes.
5. Using adaptive agents in U.S. Air Force retention.

Logistics, Transportation and Distribution Applications

1. Evaluating the potential benefits of a rail-traffic planning algorithm.
2. Evaluating strategies to improve railroad performance.
3. Parametric Modeling in rail-capacity planning.
4. Analysis of passenger flows in an airport terminal.
5. Proactive flight-schedule evaluation.
6. Logistic issues in autonomous food production systems for extended duration space exploration.
7. Sizing industrial rail-car fleets.
8. Production distribution in newspaper industry.
9. Design of a toll plaza
10. Choosing between rental-car locations.
11. Quick response replenishment.

Business Process Simulation

1. Impact of connection bank redesign on airport gate assignment.
2. Product development program planning.
3. Reconciliation of business and system modeling.
4. Personal forecasting and strategic workforce planning.

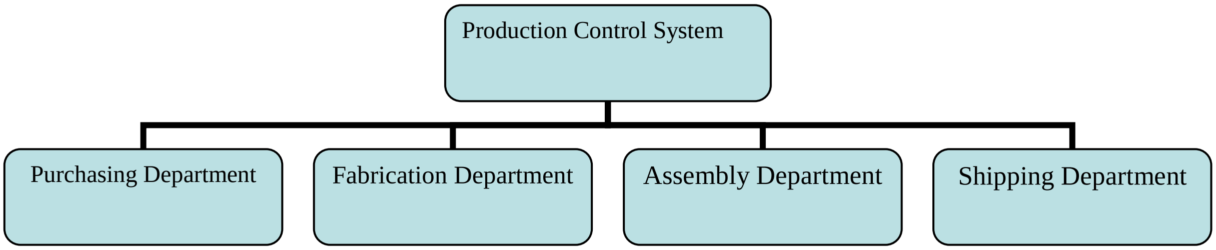
Human Systems

1. Modeling human performance in complex systems.
2. Studying the human element in out traffic control.

1.6 **Systems**

A system is defined as an aggregation or assemblage of objects joined in some regular interaction or interdependence toward the accomplishment of some purpose.

Example : Production System



In the above system there are certain distinct objects, each of which possesses properties of interest. There are also certain interactions occurring in the system that cause changes in the system.

1.7 Components of a System

Entity

An entity is an object of interest in a system.

Ex: In the factory system, departments, orders, parts and products are The entities.

Attribute

An attribute denotes the property of an entity.

Ex: Quantities for each order, type of part, or number of machines in a Department are attributes of factory system.

Activity

Any process causing changes in a system is called as an activity.

Ex: Manufacturing process of the department.

State of the System

The state of a system is defined as the collection of variables necessary to describe a system at any time, relative to the objective of study. In other words, state of the system mean a description of all the entities, attributes and activities as they exist at one point in time.

Event

An event is define as an instaneous occurrence that may change the state of the system.

1.8 System Environment

The external components which interact with the system and produce necessary changes are said to constitute the system environment.

In modeling systems, it is necessary to decide on the boundary between the system and its environment. This decision may depend on the purpose of the study.

Ex: In a factory system, the factors controlling arrival of orders may be considered to be outside the factory but yet a part of the system environment. When, we consider the demand and supply of goods, there is certainly a relationship between the factory output and arrival of orders. This relationship is considered as an activity of the system.

Endogenous System

The term endogenous is used to describe activities and events occurring within a system. Ex: Drawing cash in a bank.

Exogenous System

The term exogenous is used to describe activities and events in the environment that affect the system. Ex: Arrival of customers.

Closed System

A system for which there is no exogenous activity and event is said to be a closed. Ex: Water in an insulated flask.

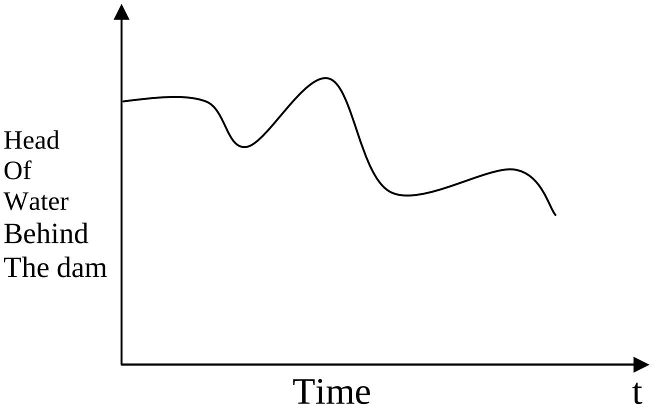
Open system

A system for which there is exogenous activity and event is said to be a open. Ex: Bank system.

Discrete and Continuous Systems

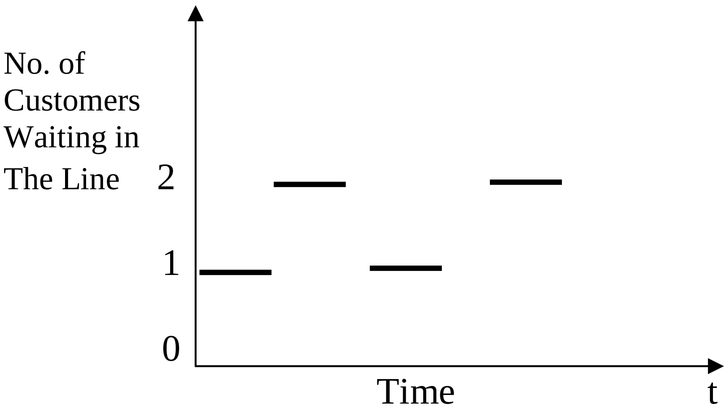
Continuous Systems

Systems in which the changes are predominantly smooth are called continuous system. Ex: Head of a water behind a dam.



Discrete Systems

Systems in which the changes are predominantly discontinuous are called discrete systems. Ex: Bank – the number of customers changes only when a customer arrives or when the service provided a customer is completed.



1.10 Model of a system

A model is defined as a representation of a system for the purpose of studying the system. It is necessary to consider only those aspects of the system that affect the problem under investigation. These aspects are represented in a model, and by definition it is a simplification of the system.

1.11 Types of Models

The various types models are

- Mathematical or Physical Model
- Static Model
- Dynamic Model
- Deterministic Model
- Stochastic Model
- Discrete Model
- Continuous Model

Mathematical Model

Uses symbolic notation and the mathematical equations to represent a system.

Static Model

Represents a system at a particular point of time and also known as Monte-Carlo simulation.

Dynamic Model

Represents systems as they change over time. Ex: Simulation of a bank

Deterministic Model

Contains no random variables. They have a known set of inputs which will result in a unique set of outputs. Ex: Arrival of patients to the Dentist at the scheduled appointment time.

Stochastic Model

Has one or more random variable as inputs. Random inputs leads to random outputs. Ex: Simulation of a bank involves random interarrival and service times.

Discrete and Continuous Model

Used in an analogous manner. Simulation models may be mixed both with discrete and continuous. The choice is based on the characteristics of the system and the objective of the study.

1.12 Discrete-Event System Simulation

Modeling of systems in which the state variable changes only at a discrete set of points in time. The simulation models are analyzed by numerical rather than by analytical methods.

Analytical methods employ the deductive reasoning of mathematics to solve the model. Eg: Differential calculus can be used to determine the minimum cost policy for some inventory models.

Numerical methods use computational procedures and are ‘runs’, which is generated based on the model assumptions and observations are collected to be analyzed and to estimate the true system performance measures.

Real-world simulation is so vast, whose runs are conducted with the help of computer. Much insight can be obtained by simulation manually which is applicable for small systems.

1.13 Steps in a Simulation study

1. **Problem formulation**

   Every study begins with a statement of the problem, provided by policy makers. Analyst ensures its clearly understood. If it is developed by analyst policy makers should understand and agree with it.

2. **Setting of objectives and overall project plan**

   The objectives indicate the questions to be answered by simulation. At this point a determination should be made concerning whether simulation is the appropriate methodology. Assuming it is appropriate, the overall project plan should include

   - A statement of the alternative systems
   - A method for evaluating the effectiveness of these alternatives
   - Plans for the study in terms of the number of people involved

- Cost of the study
- The number of days required to accomplish each phase of the work with the anticipated results.

3. **Model conceptualization**

The construction of a model of a system is probably as much art as science. The art of modeling is enhanced by an ability

- To abstract the essential features of a problem
- To select and modify basic assumptions that characterize the system
- To enrich and elaborate the model until a useful approximation results

Thus, it is best to start with a simple model and build toward greater complexity. Model conceptualization enhance the quality of the resulting model and increase the confidence of the model user in the application of the model.

4. **Data collection**

There is a constant interplay between the construction of model and the collection of needed input data. Done in the early stages. Objective kind of data are to be collected.

5. **Model translation**

Real-world systems result in models that require a great deal of information storage and computation. It can be programmed by using simulation languages or special purpose simulation software. Simulation languages are powerful and flexible. Simulation software models development time can be reduced.

6. **Verified**

It pertains to he computer program and checking the performance. If the input parameters and logical structure and correctly represented, verification is completed.

7. **Validated**

It is the determination that a model is an accurate representation of the real system. Achieved through calibration of the model, an iterative process of comparing the model to actual system behavior and the discrepancies between the two.

8. Experimental Design

The alternatives that are to be simulated must be determined. Which alternatives to simulate may be a function of runs. For each system design, decisions need to be made concerning

- Length of the initialization period
- Length of simulation runs
- Number of replication to be made of each run

9. Production runs and analysis

They are used to estimate measures of performance for the system designs that are being simulated.

10.More runs

Based on the analysis of runs that have been completed. The analyst determines if additional runs are needed and what design those additional experiments should follow.

11.Documentation and reporting

Two types of documentation.

- Program documentation
- Process documentation

Program documentation

Can be used again by the same or different analysts to understand how the program operates. Further modification will be easier. Model users can change the input parameters for better performance.

Process documentation

Gives the history of a simulation project. The result of all analysis should be reported clearly and concisely in a final report. This enable to review the final formulation and alternatives, results of the experiments and the recommended solution to the problem. The final report provides a vehicle of certification.

12.Implementation

Success depends on the previous steps. If the model user has been thoroughly involved and understands the nature of the model and its outputs, likelihood of a vigorous implementation is enhanced.

The simulation model building can be broken into 4 phases.

I Phase

- Consists of steps 1 and 2
- It is period of discovery/orientation
- The analyst may have to restart the process if it is not fine-tuned
- Recalibrations and clarifications may occur in this phase or another phase.

II Phase

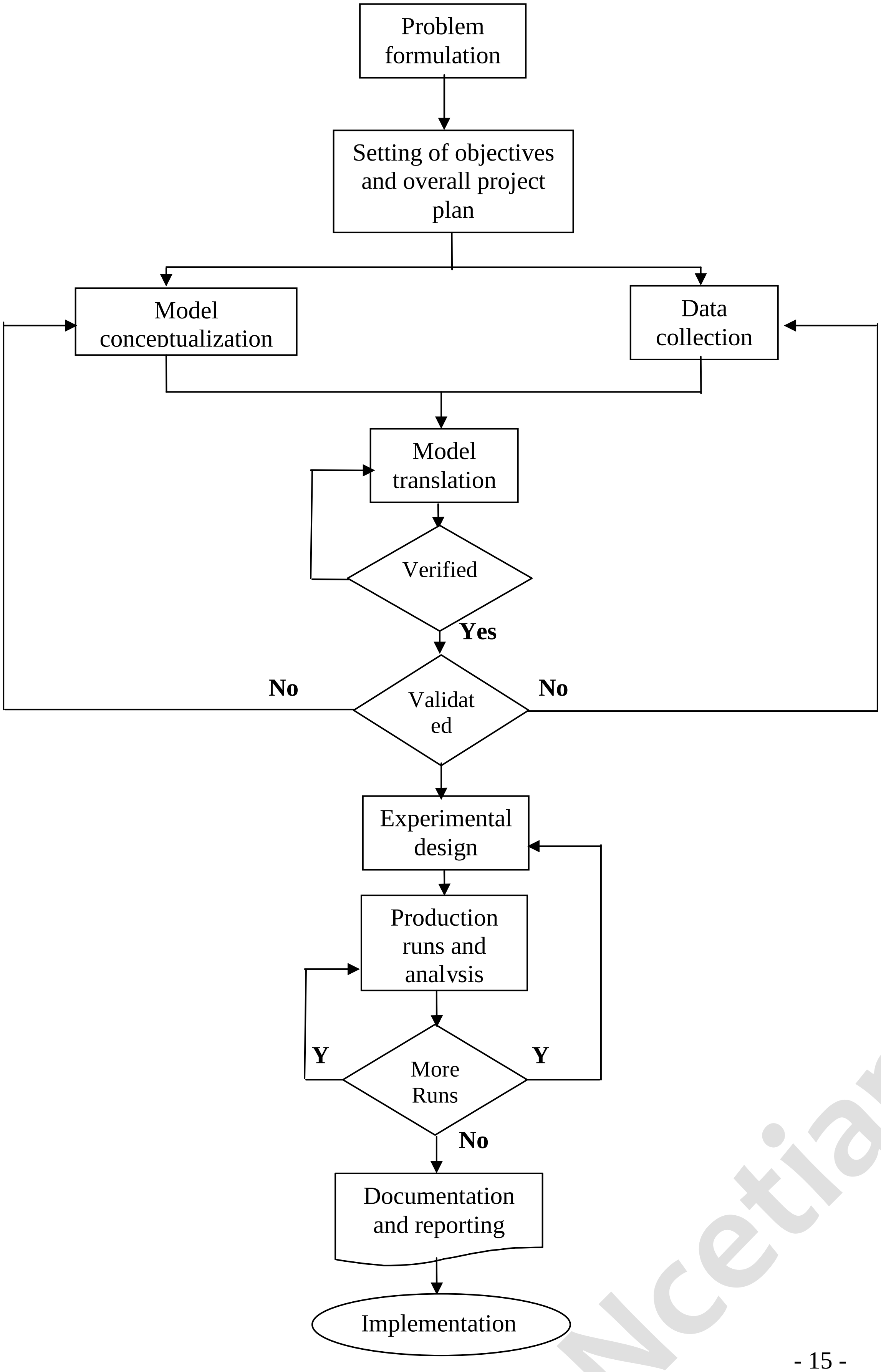
- Consists of steps 3,4,5,6 and 7
- A continuing interplay is required among the steps
- Exclusion of model user results in implications during implementation

III Phase

- Consists of steps 8,9 and 10
- Conceives a thorough plan for experimenting
- Discrete-event stochastic is a statistical experiment
- The output variables are estimates that contain random error and therefore proper statistical analysis is required.

IV Phase

- Consists of steps 11 and 12
- Successful implementation depends on the involvement of user and every steps successful completion.


Problem formulation
Setting of objectives and overall project plan
Model conceptualization
Data collection
Model translation
Verified
Yes
No
Validated
No
Experimental design
Production runs and analysis
Y
More Runs
Y
No
Documentation and reporting
Implementation

Chapter 2
Simulation Examples

- Simulation is often used in the analysis of queueing models. In a simple typical queueing model, shown in fig 1, customers arrive from time to time and join a queue or waiting line, are eventually served, and finally leave the system.

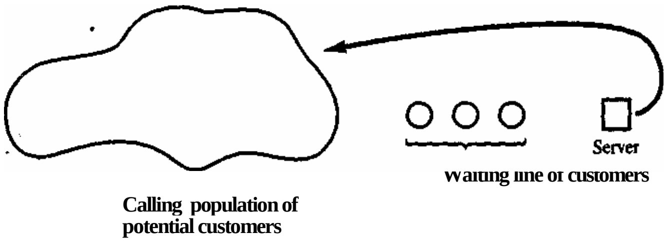

Server
Waiting line of customers
Calling population of potential customers

fig 1: Simple Queuing Model

- The term "customer" refers to any type of entity that can be viewed as requesting "service" from a system.

2.1 Characteristics of Queueing Systems

- The key elements, of a queueing system are the customers and servers. The term **"customer"** can refer to people, machines, trucks, mechanics, patients—anything that arrives at a facility and requires service
- The term **"server"** might refer to receptionists, repairpersons, CPUs in a computer, or washing machines…. any resource (person, machine, etc. which provides the requested service.
- Table 1 lists a number of different queueing systems.

System	*Customers*	*Server(s)*
Reception desk	People	Receptionist
Repair facility	Machines	Repairperson
Garage	Trucks	Mechanic
Tool crib	Mechanics	Tool-crib clerk
Hospital	Patients	Nurses
Warehouse	Pallets	Crane
Airport	Airplanes	Runway
Production line	Cases	Case packer
Warehouse	Orders	Order picker
Road network	Cars	Traffic light
Grocery	Shoppers	Checkout station
Laundry	Dirty linen	Washing machines/dryers
Job shop	Jobs	Machines/workers
Lumberyard	Trucks	Overhead crane
Saw mill	Logs	Saws
Computer	Jobs	CPU, disk, tapes
Telephone	Calls	Exchange
Ticket office	Football fans	Clerk
Mass transit	Riders	Buses, trains

Table 1: Examples of Queueing Systems

➢ The elements of a queuing system are:-

- ***<u>The Calling Population</u>:-***

✓ The population of potential customers, referred to as the ***calling population****, m*ay be assumed to be finite or infinite.

✓ For example, consider a bank of 5 machines that are curing tires. After an interval of time, a machine automatically opens and must be attended by a worker who removes the tire and puts an uncured tire into the machine. The machines are the **"customers",** who "arrive" at the instant they automatically open. The worker is the **"server",** who "serves" an open machine as soon as possible. The calling population is finite, and consists of the five machines.

✓ In systems with a large population of potential customers, the calling population is usually assumed to be finite or infinite. Examples of infinite populations include the potential customers of a restaurant, bank, etc.

✓ The main difference between finite and infinite population models is how the arrival rate is defined. In an infinite-population model, the arrival rate is not affected by the number of customers who have left the calling population and joined the queueing system. On the other hand, for finite calling population models, the arrival rate to the queueing system does depend on the number of customers being served and waiting.

- ***<u>System Capacity:-</u>***

✓ In many queueing systems there is a limit to the number of customers that may be in the waiting line or system. For example, an automatic car wash may have room for only 10 cars to wait in line to enter the mechanism.

✓ An arriving customer who finds the system full does not enter but returns immediately to the calling population.

✓ Some systems, such as concert ticket sales for students, may be considered as having unlimited capacity. There are no limits on the number of students allowed to wait to purchase tickets.

✓ When a system has limited capacity, a distinction is made between the arrival rate (i.e., the number of arrivals per time unit) and the effective arrival rate (i.e., the number who arrive and enter the system per time unit).

- ***<u>The Arrival Process:-</u>***

✓ Arrival process for infinite-population models is usually characterized in terms of interarrival times of successive customers. Arrivals may occur at scheduled times or at random times. When at random times, the interarrival times are usually characterized by a probability distribution

✓ The most important model for random arrivals is the Poisson arrival process. If A_n represents the interarrival time between customer *n-1* and customer *n* $(A_1$ *is* the actual arrival time of the first customer), then for a Poisson arrival process. A_n is exponentially distributed with mean I/λ time Units. The arrival rate is λ customers per time unit. The number of arrivals in a time interval of length t, say $N(t)$, has the Poisson distribution with mean *λt customers*.

✓ The Poisson arrival process has been successfully employed as a model of the arrival of people to restaurants, drive-in banks, and other service facilities.

✓ A second important class of arrivals is the scheduled arrivals, such as patients to a physician's office or scheduled airline flight arrivals to an airport. In this case, the interarrival times $[A_n,\ n = 1,2,\dots\}$ may be constant, or constant plus or minus a small random amount to represent early or late arrivals.

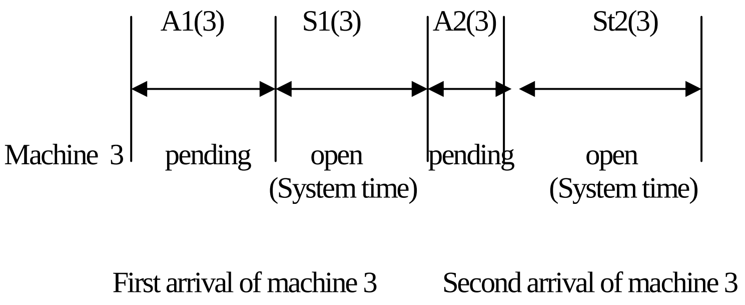
✓ A third situation occurs when at least one customer is assumed to always be present in the queue, so that the server is never idle because of a lack of customers. For example, the "customers" may represent raw material *for a* product, and sufficient raw material is assumed to be always available.

✓ For finite-population models, the arrival process is characterized in a completely different fashion. Define a customer as *pending* when that customer is outside the queueing system and a member of the potential calling population.

✓ ***Runtime of a given customer*** is defined as the length of time from departure from the queueing system until that customer's next arrival to the queue.

✓ Let $A_i^{(i)}$, $A_2^{(i)}$,… be the successive runtimes of customer /, and let $S^{(i)}_{1,}$ $S^{(i)}_{2.....}$be the corresponding successive system times; that is, $S^{(i)}_n$ is the total time spent in the system by customer i during the nth visit. Figure 2 illustrates these concepts for machine 3 in the tire-curing example. The total arrival process is the superposition of the arrival times of all customers.
Fig 2 shows the first and second arrival of machine 3.

Fig 2: Arrival process for a finite-population model.



✓ One important application of finite population models is the machine repair problem. The machines are the customers and a runtime is also called time to failure. When a machine fails, it "arrives" at the queuing

g system (the repair facility) and remains there until it is "served" (repaired). Times to failure for a given class of machine have been characterized by the exponential, the

Weibull, and the gamma distributions. Models with an exponential runtime are sometimes analytically tractable.

- ***Queue Behavior and Queue Discipline:-***

✓ **Queue behavior** refers to customer actions while in a queue waiting for service to begin. In some situations, there is a possibility that incoming customers may balk (leave when they see that the line is too long), renege (leave after being in the line when they see that the line is moving too slowly), or jockey (move from one line to another if they think they have chosen a slow line).

✓ **Queue discipline** refers to the logical ordering of customers in a queue and determines which customer will be chosen for service when a server becomes free.

- Common queue disciplines include **first-in, first-out (FIFO); last-in first-out (LIFO); service in random order (SIRO); shortest processing time first |(SPT)** and **service according to priority (PR)**.

- In a job shop, queue disciplines are sometimes based on due dates and on expected processing time for a given i type of job. Notice that a FIFO queue discipline implies that services begin in the same order as arrivals, but that customers may leave the system in a different order because of different-length service times.

- ***Service Times and the Service Mechanism:-***

✓ The service times of successive arrivals are denoted by $S_{1,}$ S_2, $S_{3...}$They may be constant or of random duration. The exponential,Weibull, gamma, lognormal, and truncated normal distributions have all been used successfully as models of service times in different situations.

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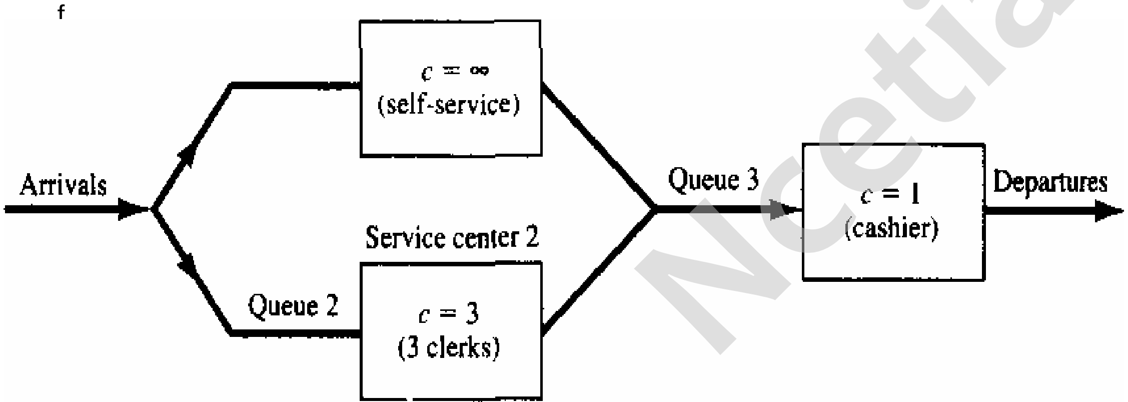
✓ Sometimes services may be identically distributed for all customers of a given type or class or priority, while customers of different types may have completely different service-time distributions. In addition, in some systems, service times depend upon the time of day or the length of the waiting line. For example, servers may work faster than usual when the waiting line is long, thus effectively reducing the service times.

✓ A queueing system consists of a number of service centers and interconnecting queues. Each service center consists of some number of servers, *c,* working in parallel; that is, upon getting to the head of the line, a customer takes the first available server. Parallel service mechanisms are either single server *(c* = 1), multiple server (1 < *c* < ∞), or unlimited servers *(c*= ∞). (A self-service facility is usually characterized as having an unlimited number of servers.)

- ***Example 1:-***

Consider a discount warehouse where customers may either serve themselves; or wait f



of three clerks, and finally leave after paying a single cashier. The system is represented by the flow diagram in figure 1 below:

Figure 1: Discount warehouse with three service centers

The subsystem, consisting of queue 2 and service center 2, is shown in more detail in figure 2 below. Other variations of service mechanisms include batch service (a server *serving* several customers simultaneously) or a customer requiring several servers simultaneously.

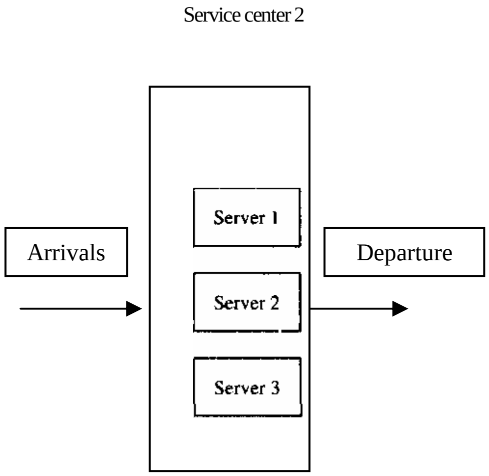


Figure 2: Service center 2, with c = 3 parallel servers.

- ***Example 2:-***A candy manufacturer has a production line which consists of three machines separated by inventory-in-process buffers. The first machine makes and wraps the individual pieces of candy, the second packs 50 pieces in a box, and the third seals and wraps the box. *The* two inventory buffers have capacities of 1000 boxes each. As illustrated by Figure 3, the system is modeled as having three service centers, each center having c = 1 server (a machine), with queue-capacity constraints between machines. It is assumed that a sufficient supply of raw material is always available at the first queue. Because of the queue-capacity constraints, machine 1 shuts down whenever the inventory buffer fills to capacity, while machine 2 shuts down whenever the buffer empties. In brief, the system consists of three single-server queues in series with queue-capacity constraints and a continuous arrival stream at the first queue.

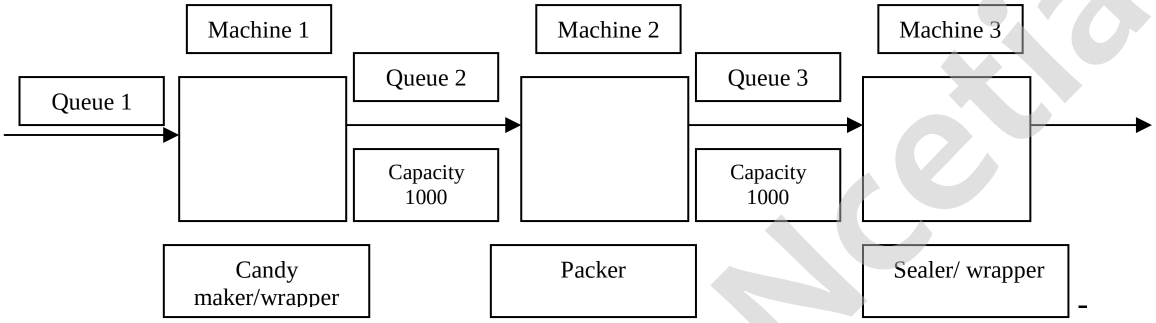


Figure 3: Candy production line

2.2 Queueing Notation:-

- Recognizing the diversity of queueing systems, Kendall [1953] proposed a notational system for parallel server systems which has been widely adopted. An abridged version of this convention is based on the format *A /B / c / N / K*. These letters represent the following system characteristics:
    - ✓ *A* represents the interarrival time distribution.
    - ✓ *B* represents the service-time distribution.
    - ✓ [Common symbols for *A* and *B* include *M* (exponential or Markov), *D* (constant or deterministic), E_k (Erlang of order *k), PH* (phase-type), *H* (hyperexponential), G (arbitrary or general), and *GI* (General independent).]
    - ✓ c represents the number of parallel servers.
    - ✓ *N* represents the system capacity.
    - ✓ *K* represents the size of the calling population

- For example, *M / M / 1* / ∞ / ∞ indicates a single-server system that has unlimited queue capacity and an infinite population of potential arrivals. The interarrival times and service times are exponentially distributed. When *N* and *K* are infinite, they may be dropped from the notation. For example, M / M / 1 / ∞ / ∞ is often shortened to *M/M/l.*

- Additional notation used for parallel server systems is listed in Table 1 given below. The meanings may vary slightly from system to system. All systems will be assumed to have a FIFO queue discipline.

Table 1. Queueing Notation for Parallel Server Systems

P_n	Steady-state probability of having *n* customers in system
$P_n,(t)$	Probability of *n* customers in system at time *t*
λ	Arrival rate
λ_e	Effective arrival rate
μ	Service rate of one server
ρ	Server utilization
A_n	Interarrival time between customers *n* — 1 and *n*
S_n,	Service time of the nth arriving customer
W_n	Total time spent in system by the nth arriving customer
W_n^Q	Total time spent in the waiting line by customer *n* .
L(t)	The number of customers in system at time /
$L_Q(t)$	The number of customers in queue at time *t*
L	Long-run time-average number of customers in system
L_Q	Long-run time-average number of customers in queue

$\acute{\omega}$ Long-run average time spent in system per customer

$\acute{\omega}_Q$ Long-run average time spent in queue per customer

2.3 Simulation of Queuing systems

- A queueing system is described by its calling population, the nature of the arrivals, the service mechanism, the system capacity, and the queueing discipline. A single-channel queueing system is portrayed in figure1.

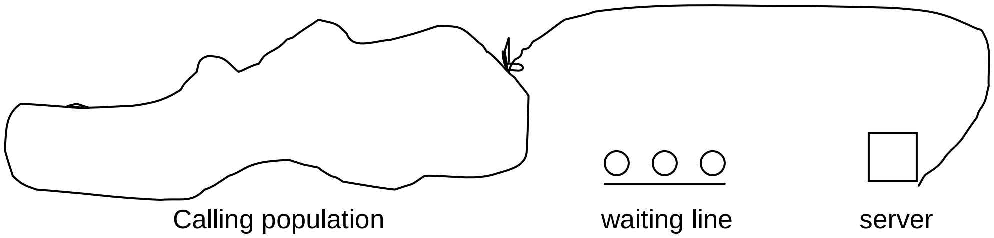


Figure 1: Queueing System

- In the single-channel queue, the calling population is infinite; that is, if a unit leaves the calling population and joins the waiting line or enters service, there is no change in the arrival rate of other units that may need service.

- Arrivals for service occur one at a time in a random fashion; once they join the waiting line, they are eventually served. In addition, service times are of some random length according to a probability distribution which does not change over time.

- The system capacity; has no limit, meaning that any number of units can wait in line.

- Finally, units are served in the order of their arrival by a single server or channel.

- Arrivals and services are defined by the distributions of the time between arrivals and the distribution of service times, respectively.

- For any simple single or multi-channel queue, the overall effective arrival rate must be less than the total service rate, or the waiting line will grow without bound. When queues grow without bound, they are termed **“explosive”** or **unstable**.

- The state of the system is the number of units in the system and the status of the server, busy or idle.

- An event is a set of circumstances that cause an instantaneous change in the state of the system. In a single –channel queueing system there are only two possible events that can affect the state of the system.

    - They are the entry of a unit into the system.
    - The completion of service on a unit.

- The queueing system includes the server, the unit being serviced, and units in the queue. he simulation clock is used to track simulated time. If a unit has just completed service, the simulation proceeds in the manner shown in the flow diagram of figure.2. Note that the server has only two possible states: it is either busy or idle.

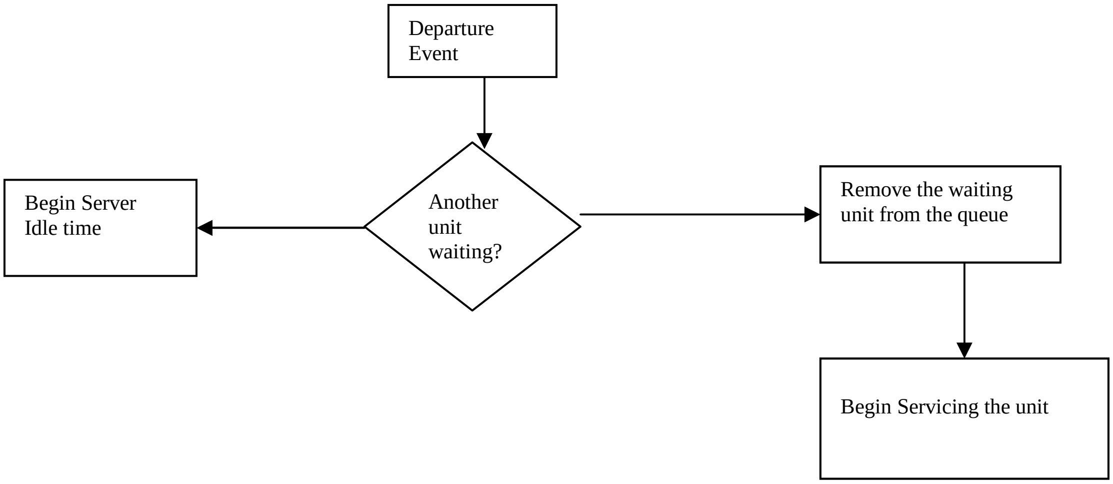


Figure 2: Service-just-completed flow diagram

- The arrival event occurs when a unit enters the system. The flow diagram for the arrival event is shown in figure 3. The unit may find the server either idle or busy; therefore, either the unit begins service immediately, or it enters the queue for the server. The unit follows the course of action shown in fig 4.

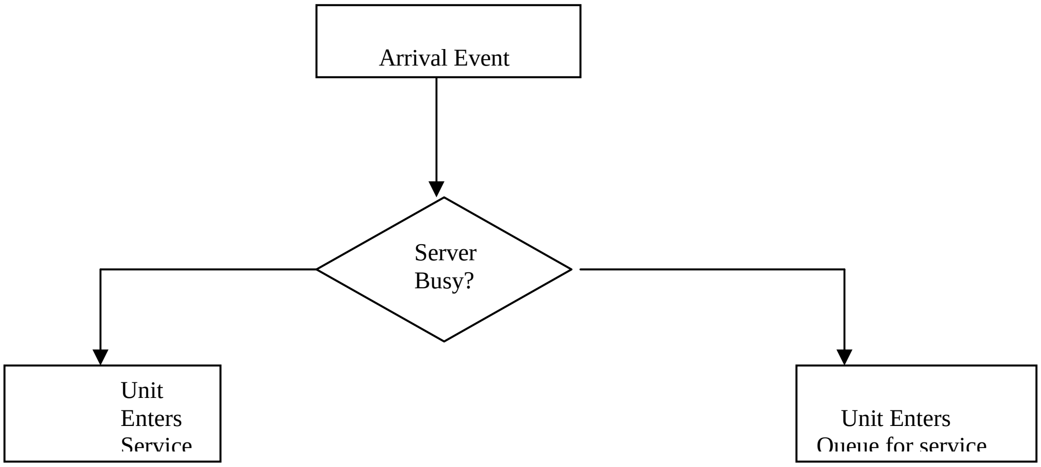


Figure 3: Unit-Entering system flow diagram

- If the server is busy, the unit enters the queue. If the server is idle and the queue is empty, the unit begins service. It is not possible for the server to be idle and the queue to be nonempty.

		Queue Status	
		Not Empty	Empty
Server Status	Busy	Enter Queue	Enter Queue
	Idle	Impossible	Enter Service

Figure 4: Potential unit actions upon arrival

❖ After the completion of a service the service may become idle or remain busy with the next unit. The relationship of these two outcomes to the status of the queue is shown in fig 5. If the queue is not empty, another unit will enter the server and it will be busy. If the queue is empty, the server will be idle after a service is completed. These two possibilities are shown as the shaded portions of fig 5. It is impossible for the server to become busy if the queue is empty when a service is completed. Similarly, it is impossible for the server to be idle after a service is completed when the queue is not empty.

		Queue Status	
		Not Empty	Empty
Server Status	Busy	■	Impossible
	Idle	Impossible	■

Figure 5: Server outcomes after service completion

Simulation clock times for arrivals and departures are computed in a simulation table customized for each problem. In simulation, events usually occur at random times. In these cases, a statistical model of the data is developed from either data collected and analyzed, or subjective estimates and assumptions.

❖ Random numbers are distributed uniformly and independently on the interval (0, 1). Random digits are uniformly distributed on the set {0, 1, 2… 9}. Random digits can be used to form random numbers by selecting the proper number of digits for each random number and placing a decimal point to the left of the value selected. The proper number of digits is dictated by the accuracy of the data being used for input purposes. If the input distribution has values with two decimal places, two digits are taken from a random-digits table and the decimal point is placed to the left to form a random number.

- When numbers are generated using a procedure, they are often referred to as pseudo-random numbers. Since the method is known, it is always possible to know the sequence of numbers that will be generated prior to the simulation.

- In a single-channel queueing system, interarrival times and service times are generated from the distributions of these random variables. The examples that follow show how such times are generated. For simplicity, assume that the times between arrivals were generated by rolling a die five times and recording the up face. Table 1 contains a set of five interarrival times are used to compute the arrival times of six customers at the queuing system.

Table 1: Interarrival and Clock Times

Customer	Interarrival Time	Arrival Time on Clock
1	--	0
2	2	2
3	4	6
4	1	7
5	2	9
6	6	15

- The first customer is assumed to arrive at clock time 0. This starts the clock in operation. The second customer arrives two time units later, at a click time of 2. The third customer arrives four time units later, at a clock time of 6; and so on.

- The second time of interest is the service time. The only possible service times are one, two, three, and four time units. Assuming that all four values are equally likely to occur, these values could have been generated by placing the numbers one through four on chips and drawing the chips from a hat with replacement, being sure to record the numbers selected.

- Now, the interarrival times and service times must be meshed to simulate the single-channel queueing system. As shown in table 2, the first customer arrives at clock time 0 and immediately begins service, which requires two minutes. Service is completed at clock time 2. The second customer arrives at clock time 2 and is a finished at clock time 3. Note that the fourth customer arrived at clock time 7, but service could not begin until clock time 9. This occurred because customer 3 did not finish service until clock time 9.

- Table 2 was designed specifically for a single-channel queue which serves customers on a first-in, first-out (FIFO) basis. It keeps track of the clock time at which each event occurs. The second column of table 2 records the clock time of each arrival event, while the last column records the clock time of each departure event.

Table 2: Simulation Table emphasizing Clock Times

A	B	C	D	E
Customer No.	Arrival Time (Clock)	Time Service Begins (Clock)	Service Time (Duration)	Time Service Ends (Clock)
1	0	0	2	2
2	2	2	1	3
3	6	6	3	9
4	7	9	2	11
5	9	11	1	12
6	15	15	4	19

<u>EXAMPLE 1:</u> Single-Channel Queue

- A small grocery store has only one checkout counter. Customers arrive at this checkout counter at random from 1 to 8 minutes apart. Each possible value of interarrival time has the same probability of occurrence. The service times vary from 1 to 6 minutes with the probabilities shown in table 5. The problem is to analyze the system by simulating the arrival and service of 20 customers.

Table 5: Service Time Distribution

Service Time (Min)	Probability	Cumulative Frequency	Random Digit Assignment
1	0.10	0.10	01-10
2	0.20	0.30	11-12
3	0.30	0.60	31-60
4	0.25	0.85	61-85
5	0.10	0.95	86-95
6	0.05	1.00	96-00

- A simulation of a grocery store that starts with an empty system is not realistic unless the intention is to model the system from startup or to model until steady state operation is reached.

- A set of uniformly distributed random numbers is needed to generate the arrivals at the checkout counter. Random numbers have the following properties:

  1. The set of random numbers is uniformly distributed between 0 and 1.

2. Successive random numbers are independent.

- ❖ The time-between-arrival determination is shown in table 6. Note that the first random digits are 913. To obtain the corresponding time between arrivals, enter the fourth column of table 4 and read 8 minutes from the first column of the table. Alternatively, we see that 0.913 is between the cumulative probabilities 0.876 and 1.000, again resulting in 8 minutes as the generated time

Table 6: Time between Arrivals Determination

Customers	Random Digits	Time Between Arrivals (Min)	Customers	Random Digits	Time Between Arrivals (Min)
1	--	--	11	109	1
2	913	8	12	093	1
3	727	6	13	607	5
4	015	1	14	738	6
5	948	8	15	359	3
6	309	3	16	888	8
7	922	8	17	106	1
8	753	.7	18	212	2
9	235	2	19	493	4
10	302	3	20	535	5

- ❖ Service times for all 20 customers are shown in table 7. These service times were generated based on the methodology described above, together with the aid of table 5. The first customer's service time is 4 minutes because the random digits 84 fall in the bracket 61-85, or alternatively because the derived random number 0.84 falls between the cumulative probabilities 0.61 and 0.85.

Table 7: Service Times Generated

Customer	Random Digits	Service Time (Min)	Customer	Random Digits	Service Time (Min)
1	84	4	11	32	3
2	10	1	12	94	5
3	74	4	13	79	4
4	53	3	14	05	1
5	17	2	15	79	5
6	79	4	16	84	4
7	91	5	17	52	3
8	67	4	18	55	3
9	89	5	19	30	2
10	38	3	20	50	3

- ❖ The essence of a manual simulation is the simulation table. These tables are designed for the problem at hand, with columns added to answer the questions posed. The simulation table for the single-channel queue, shown, in table 8 that is an extension of table 2. The first step is to initialize the table by filling in cells for the first customer.

- ❖ The first customer is assumed to arrive at time 0. Service begins immediately and finishes at time 4. The customer was in the system for 4 minutes. After the first customer, subsequent rows in the table are based on the random numbers for interarrival time and service time and the completion time of the previous customer. For example, the second customer arrives at time 8. Thus, the server was idle for 4 minutes. Skipping down to the fourth customer, it is seen that this customer arrived at time 15 but could not be served until time 18. This customer had to wait in the queue for 3 minutes. This process continues for all 20 customers.

Table 8: Simulation Table for the queueing problem

A	B	C	D	E	F	G	H	I
Customers	Time since last Arrival (Min)	Arrival Time	Service Time	Time Service Begins	Time customer waits in queue	Time Service Ends	Time customer spends in system	Idle Time of Server
1	--	0	4	0	0	4	4	0
2	8	8	1	8	0	9	1	4
3	6	14	4	14	0	18	4	5
4	1	15	3	18	3	21	6	0
5	8	23	2	23	0	25	2	2
6	3	26	4	26	0	30	4	1
7	8	34	5	34	0	39	5	4
8	7	41	4	41	0	45	4	2
9	2	43	5	45	2	50	7	0
10	3	46	3	50	4	53	7	0
11	1	47	3	53	6	56	9	0
12	1	48	5	56	8	61	13	0
13	5	53	4	61	8	65	12	0
14	6	59	1	65	6	66	7	0
15	3	62	5	66	4	71	9	0
16	8	70	4	71	1	75	5	0
17	1	71	3	75	4	78	7	0
18	2	73	3	78	5	81	8	0
19	4	77	2	81	4	83	6	0
20	5	82	3	83	1	86	4	0
			68		56		124	18

1. The average waiting time for a customer is 2.8minutes. this is determined in the following manner:

$$\text{Average Waiting Time} = \frac{\text{Total time customers wait in queue (min)}}{\text{Total no. of customers}}$$

$$= \frac{56}{20} = 2.8 \text{ minutes}$$

2. The probability that a customer has to wait in the queue is 0.65. This is determined in the following manner:

$$\text{Probability (wait)} = \frac{\text{number of customers who wait}}{\text{Total number of customers}}$$

$$= \frac{13}{20} = 0.65$$

3. The fraction of idle time of the server is 0.21. This is determined in the following manner:

Total idle time of server (minutes)

$$\text{Probability of idle server} = \frac{}{\text{Total run time of simulation (minutes)}}$$

$$= \frac{18}{86} = 0.21$$

The probability of the server being busy is the complement of 0.21, or 0.79.

4. The average service time is 3.4 minutes, determined as follows:

$$\text{Average service time (minutes)} = \frac{\text{Total service time}}{\text{Total number of customers}}$$

$$= \frac{68}{20} = 3.4 \text{ minutes}$$

This result can be compared with the expected service time by finding the mean of the service-time distribution using the equation

$$E(s) = \sum sp(s)$$

Applying the expected-value equation to the distribution in table 2.7 gives an expected service time of:

$$= 1(0.10) + 2(0.20) + 3(0.30) + 4(0.25) + 5(0.10) + 6(0.50)$$

$$= 3.2 \text{ minutes}$$

The expected service time is slightly lower than the average time in the simulation. The longer simulation, the closer the average will be to E (S).

5. The average time between arrivals is 4.3 minutes. This is determined in the following manner:

$$\text{Average time between arrivals (minutes)} = \frac{\text{Sum of all times between arrivals (minutes)}}{\text{Number of arrivals - 1}}$$

$$= \frac{82}{19} = 4.3 \text{ minutes}$$

One is subtracted from the denominator because the first arrival is assumed to occur at time 0. This result can be compared to the expected time between arrivals by finding the mean of the discrete uniform distribution whose endpoints are a = 1 and b = 8. The mean is given by

$$E(A) = \frac{a+b}{2} = \frac{1+8}{2} = 4.5 \text{ minutes}$$

The expected time between arrivals is slightly higher than the average. However, as the simulation becomes longer, the average value of the time between arrivals will approach the theoretical mean, E (A).

6. The average waiting time of those who wait is 4.3 minutes. This is determined in the following manner:

$$\text{Average waiting time of Those who wait (minutes)} = \frac{\text{total time customers wait in queue}}{\text{Total number of customers who wait}}$$

$$= \frac{56}{13} = 4.3 \text{ minutes}$$

7. The average time a customer spends in the system is 6.2 minutes. This can be determined in two ways. First, the computation can be achieved by the following relationship:

$$\text{Average time customer Spends in the system} = \frac{\text{total time customers spend in the system}}{\text{Total number of customers}}$$

$$= \frac{124}{20} = 6.2 \text{ minutes}$$

The second way of computing this same result is to realize that the following relationship must hold:

Average time customer Spends in the system = average time customer spends waiting in the queue + average time customer spends in service

From findings 1 and 4 this results in:

Average time customer spends in the system = 2.8+ 3.4 = 6.2 minutes.

EXAMPLE 2:- <u>The Able Baker Carhop Problem</u>

- ❖ This example illustrates the simulation procedure when there is more than one service channel. Consider a drive-in restaurant where carhops take orders and bring food to the car. Cars arrive in the manner shown in table 1. There are two carhops-Able and Baker. Able is better able to do the job and works a bit faster than Baker. The distribution of their service times are shown in tables 2 and 3.

Table 1: Interarrival distribution of Cars

Time Between arrivals (Min)	Probability	Cumulative Probability	Random Digit Assignment
1	0.25	0.25	01-25
2	0.40	0.65	26-65
3	0.20	0.85	66-85
4	0.15	1.00	86-00

- ❖ The simulation proceeds in a manner similar to example 1, except that it its more complex because of the two servers. A simplifying rule is that Able gets the customer if both carhops are idle. Perhaps, Able has seniority. (The solution would be different if the decision were made at random or by any other rule.)

Table 2: Service Distribution of Able

Service Time	Probability	Cumulative Probability	Random-Digit Assignment
2	0.30	0.30	01-30
3	0.28	0.58	31-58
4	0.25	0.83	59-83
5	0.17	1.00	84-00

Table 3: Service Distribution of Baker

Service Time	Probability	Cumulative Probability	Random-Digit Assignment
3	0.35	0.35	01-35
4	0.25	0.60	36-60
5	0.20	0.80	61-80
6	1.00	1.00	81-00

- ❖ Here there are more events: a customer arrives, a customer begins service from able, a customer completes service from Able, a customer begins service from Baker, and a customer completes service from Baker. The simulation table is shown in table 4.

- ❖ After the first customer, the cells for the other customers must be based on logic and formulas. For example, the "clock time of arrival" in the row for the second customer is computed as follows:

  D2 = D1+C2

- ❖ .The logic to compute who gets a given customer, and when that service begins, is more complex. The logic goes as follows when a customer arrives: if the customer finds able idle, the customer begins service immediately with able. If able is not idle but baker is, then the customer begins service immediately with baker. If both are busy, the customer begins service with the first server to become free.

<u>The analysis of table 4 results in the following:</u>

1. Over the 62-minute period able was busy 90% of the time.
2. Baker was busy only 69% of the tome. The seniority rule keeps baker less busy.
3. Nine of the 26 arrivals had to wait. The average waiting time for all customers was only about 0.42 minute, which is very small.
4. Those nine who did have to wait only waited an average of 1.22 minutes, which is quite low.
5. In summary, this system seems well balanced. One server cannot handle all the diners, and three servers would probably be too many. Adding an additional server would surely reduce the waiting time to nearly zero. However, the cost of waiting would have to be quite high to justify an additional server.

Table 4: Simulation Table for the Carhop Example

A Customer No.	B Random Digits for Arrival	C Time between Arrivals	D Clock time Arrival	E Random Digits for Service	F Time Service Begins	G Service Time	H Time Service Ends	I Time Service Begins	J Service Time	K Time Service Ends	L Time In Queue

1	--	--	0	95	0	5	5				0
2	26	2	2	21				2	3	5	0
3	98	4	6	51	6	3	9				0
4	90	4	10	92	10	5	15				0
5	26	2	12	89				12	6	18	0
6	42	2	14	38	15	3	18				1
7	74	3	17	13	18	2	20				1
8	80	3	20	61	20	4	24				0
9	68	3	23	50				23	4	27	0
10	22	1	24	49	24	3	27				0
11	48	2	26	39	27	3	30				1
12	34	2	28	53				28	4	32	0
13	45	2	30	88	30	5	35				0
14	24	1	31	01				32	3	35	1
15	34	2	33	81	35	4	39				2
16	63	2	35	53				35	4	39	0
17	38	2	37	81	39	4	43				2
18	80	3	40	64				40	5	45	0
19	42	2	42	01	43	2	45				1
20	56	2	44	67	45	4	49				1
21	89	4	48	01				48	3	51	0
22	18	1	49	47	49	3	52				0
23	51	2	51	75				51	5	56	0
24	71	3	54	57	54	3	57				0
25	16	1	55	87				56	6	62	1
26	92	4	59	47	59	3	62				0
						56			43		11

2.4 Simulation of Inventory Systems

An important class of simulation problems involves inventory systems.

- ❖ A simple inventory system is shown in fig 1. This inventory system has a periodic review of length N, at which time the inventory level is checked. An order is made to bring the inventory up to the level M. At the end of the review period, an order quantity, Q1, is placed. In this inventory system the lead time is zero. Demand is shown as being uniform over the time period in fig 1. In actuality, demands are not usually uniform and do fluctuate over time. One possibility is that demands all occur at the beginning of the cycle. Another is that the lead time is random of some positive length.

- ❖ Notice that in the second cycle, the amount in inventory drops below zero, indicating a shortage. In fig 1, these units are backordered. When the order arrives, the demand for the backordered items is satisfied first. To avoid shortages, a buffer, or safety, stock would need to be carried.

- ❖ Carrying stock in inventory has an associated cost attributed to the interest paid on the funds borrowed to buy the items. Other costs can be placed in the carrying or holding cost column: renting of storage space, hiring guards, and so on.

- ❖ An alternative to carrying high inventory is to make more frequent reviews, and consequently, more frequent purchases or replenishments. This has an associated cost: the ordering cost. Also, there is a cost in being short. Larger inventories decrease the possibilities of shortages. These costs must be traded off in order to minimize the total cost of an inventory system.

- ❖ The total cost of an inventory system is the measure of performance. This can be affected by the policy alternatives. For example, in fig 1, the decision maker can control the maximum inventory level, M, and the cycle, N.

- In an (M, N) inventory system, the events that may occur are: the demand for items in the inventory, the review of the inventory position, and the receipt of an order at the end of each review period. When the lead time is zero, as in fig 1, the last two events occur simultaneously.

- **The Newspaper Seller's Problem**

  - A classical inventory problem concerns the purchase and sale of newspapers. The paper seller buys the papers for 33 cents each and sells them for 50 cents each. Newspapers not sold at the end of the day are sold as scrap for 5 cents each. Newspapers can be purchased in bundles of 10. Thus, the paper seller can buy 50, 60, and so on.

  - There are three types of Newsday's, "good," "fair," and "poor," with probabilities of 0.35, 0.45, and 0.25, respectively. The distribution of papers demanded on each of these days is given in table 2.15. The problem is to determine the optimal number of papers the newspaper seller should purchase. This will be accomplished by simulating demands for 20 days and recording profits from sales each day.

<u>The profits are given by the following relationship:</u>

$$\text{Profit} = \left[\begin{matrix}\text{Revenue}\\ \text{form sales}\end{matrix} - \begin{matrix}\text{cost of}\\ \text{newspapers}\end{matrix} - \begin{matrix}\text{lost profit form}\\ \text{excess demand}\end{matrix} + \begin{matrix}\text{salvage from sale}\\ \text{of scrap papers}\end{matrix} \right]$$

Table 5: Distribution of Newspaper Demanded

	Demand Probability Distribution		
Demand	Good	Fair	Poor
40 50 60	0.03 0.05 0.15	0.10 0.18 0.40	0.44 0.22 0.16

- Form the problem statement, the revenue from sales is 50 cents for each paper sold. The Cost 9 of newspapers is 33 cents for each paper purchased. The lost profit from excess demand is 17 cents for each paper demanded that could not be provided. Such a shortage cost is somewhat controversial but makes the problem much more interesting. The salvage value of scrap papers is 5 cents each.

- Tables 6 and 7 provide the random-digit assignments for the types of Newsday's and the demands for those Newsday's.

Table 6: Random Digit Assignment for Type of Newsday

Type of Newsday	Probability	Cumulative probability	Random Digit Assignment
Good	0.35	0.35	01 – 35
Fair	0.45	0.80	36 – 80
Poor	0.20	1.00	81 - 00

Table 7: Random Digit Assignment for Newspapers Demanded

	Cumulative Distribution			Random Digit Assignment		
Demand	Good	Fair	Poor	Good	Fair	Poor
40	0.03	0.10	0.44	01 – 03	01 – 10	01 – 44
50	0.08	0.28	0.66	04- 08	11 – 28	45 – 66
60	0.23	0.68	0.82	09 - 23	29 - 68	67 - 82

- ❖ The simulation table for the decision to purchase 70 newspapers is shown in table 8.On day 1 the demand is for 60 newspapers. The revenue from the sale of 60 newspapers is $30.00. Ten newspapers are left over at the end of the day. The salvage value at 5 cents each is 50 cents. The profit for the first day is determined as follows:

  Profit = $30.00 - $ 23.10 – 0 + $.50 = $7.40

Table 8: Simulation table fro purchase of 70 newspapers

Day	Random digits for type of Newsday	Type of Newsday	Random digits for demand	Demand	Revenue from sales	Lost profit from excess demand	Salvage from sale of scrap	Daily profit
1	94	Poor	80	60	$30	-	$0.50	$7.40
2	77	Fair	20	50	$25	-	$1.0	$2.90
3	49	Fair	15	50	$25	-	$1.0	$2.90

- ❖ On the fifth day the demand is greater than the supply. The revenue from sales is $35.00, since only 70 papers are available under this policy. An additional 20 papers could have been sold. Thus, a lost profit of $3.40 (20*17 cents) is assessed. The daily profit is determined as follows:

  Profit = $35.00 - $23.10 - $3.40 + 0 = $8.50

- ❖ The profit for the 20-day period is the sum of the daily profits, $174.90. it can also be computed from the totals for the 20 days of the simulation as follows:

  Total profit = $645 - $462 - $13 .60 + $5.50 = $174.90

- ❖ In general, since the results of one day are independent of those of previous days, inventory problems of this type are easier than queueing problems.

Simulation of an (M, N) Inventory System

- Suppose that the maximum inventory level, M, is 11 units and the review period, N, is 5 days. The problem is to estimate, by simulation, the average ending units in inventory and the number of days when a shortage condition occurs. The distribution of the number of units demanded per day is shown in table 9. In this example, lead-time is a random variable, as shown in table 10. Assume that orders are placed at the close of business and are received for inventory at the beginning as determined by the lead-time.

Table 9: Random digits assignments for daily demand

Demand	Probability	Cumulative Probability	Random digits assignments
0 1 2	0.10 0.25 0.35	0.10 0.35 0.70	01 – 10 11 – 35 36 - 70

Table 10: Random digit assignments for lead time

Lead Time (Days)	Probability	Cumulative Probability	Random digits assignments
1 2 3	0.6 0.3 0.1	0.6 0.9 1.0	1 – 6 7 – 9 0

Table 11: Simulation table for (M, N) Inventory System

Cycle	Day	Beginning Inventory	Random digits for demand	Demand	Ending Inventory	Shortage quantity	Order quantity	Random digits For lead time	Days order
1	1	3	24	1	2	0	-	-	
	2	2	35	1	1	0	-	-	
	3	9	65	2	7	0	-	-	
	4	7	81	3	4	0	-	-	
	5	4	54	2	2	0	9	5	

Note : Refer cycle 2,3,4,5 from Text book page no 47.

- To make an estimate of the mean units in ending inventory, many cycles would have to be simulated. For purposes of this example, only five cycles will be shown. The reader is asked to continue the example as an exercise at the end of the chapter.

- The random-digit assignments for daily demand and lead time are shown in the rightmost columns of tables 9 and 10. The resulting simulation table is shown in table 11.The simulation has been started with the inventory level at 3 units and an order of 8 units scheduled to arrive in 2 days time.

- Following the simulation table for several selected days indicates how the process operates. The order for 8 units is available on the morning of the third day of the first cycle, raising the inventory level from 1 unit to 9 units;

demands during the remainder of the first cycle reduced the ending inventory level to 2 units on the fifth day. Thus, an order for 9 units was placed. The lead time for this order was 1 day. The order of 9 units was added to inventory on the morning of day 2 of cycle 2.

- Notice that the beginning inventory on the second day of the third cycle was zero. An order for 2 units on that day led to a shortage condition. The units were backordered on that day and the next day;; also on the morning of day 4 of cycle 3 there was a beginning inventory of 9 units that were backordered and the 1 unit demanded that day reduced the ending inventory to 4 units.

- Based on five cycles of simulation, the average ending inventory is approximately 3.5 (88/25) units. On 2 of 25 days a shortage condition existed.

3

General Principles

Introduction

- This chapter develops a common framework for the modeling of complex systems using discrete-event simulation.
- It covers the basic building blocks of all discrete-event simulation models: **entities and attributes, activities and events**.
- In discrete-event simulation, a system is modeled in terms of its state at each point in time; the entities that pass through the system and the entities that rep-j resent system resources; and the activities and events that cause system state to change.
- This chapter deals exclusively with dynamic, stochastic system (i.e. involving time and containing random elements) which changes in a discrete manner.

3.1 Concepts in Discrete-Event Simulation

- The concept of a system and a model of a system were discussed briefly in earlier chapters.
- This section expands on these concepts and develops a framework for the development of a discrete-event model of a system.
- The major concepts are briefly defined and then illustrated with examples:
    - **System**: A collection of entities (e.g., people and machines) that ii together over time to accomplish one or more goals.
    - **Model:** An abstract representation of a system, usually containing structural, logical, or mathematical relationships which describe a system in terms of state, entities and their attributes, sets, processes, events, activities, and delays.
    - **System state:** A collection of variables that contain all the information necessary to describe the system at any

time.

- **Entity:** Any object or component in the system which requires explicit representation in the model (e.g., a server, a customer, a machine).

  - **Attributes:** The properties of a given entity (e.g., the priority of a v customer, the routing of a job through a job shop).
  - **List:** A collection of (permanently or temporarily) associated entities ordered in some logical fashion (such as all customers currently in a waiting line, ordered by first come, first served, or by priority).
  - **Event:** An instantaneous occurrence that changes the state of a system as an arrival of a new customer).
  - **Event notice:** A record of an event to occur at the current or some future time, along with any associated data necessary to execute the event; at a minimum, the record includes the event type and the event time.
  - **Event list:** A list of event notices for future events, ordered by time of occurrence; also known as the future event list (**FEL**).
  - **Activity:** A duration of time of specified length (e.g., a service time or arrival time), which is known when it begins (although it may be defined in terms of a statistical distribution).
  - **Delay: A** duration of time of unspecified indefinite length, which is not known until it ends (e.g., a customer's delay in a last-in, first-out waiting line which, when it begins, depends on future arrivals).
  - **Clock:** A variable representing simulated time**.**
- The future event list is ranked by the event time recorded in the event notice.
- An activity typically represents a service time, an interarrival time, or any other processing time whose duration has been characterized and defined by the modeler.
- An activity’s duration may be specified in a number of ways:

1. Deterministic-for example, always exactly 5 minutes.
2. Statistical-for example, as a random draw from among 2,5,7 with equal probabilities.
3. A function depending on system variables and/or entity attributes.

- The duration of an activity is computable from its specification at the instant it begins .
- A delay's duration is not specified by the modeler ahead of time, but rather is determined by system conditions.

- A delay is sometimes called a *conditional wait,* while an activity is called un*conditional wait.*

- The completion of an activity is an event, often called *primary event.*
- The completion of a delay is sometimes called a conditional or secondary event.

EXAMPLE 3.1 (Able and Baker, Revisited)

Consider the Able-Baker carhop system of Example 2.2. A discrete- event model has the following components:

System state

L_Q(t), the number of cars waiting to be served at time *t*

$L_A(t)$, 0 or 1 to indicate Able being idle or busy at time *t*

L_B *(t)*, 0 or 1 to indicate Baker being idle or busy at time *t*

Entities

Neither the customers (i.e., cars) nor the servers need to be explicitly represented, except in terms of the state variables, unless certain customer averages are desired (compare Examples 3.4 and 3.5)

Events

Arrival event

Service completion by Able

Service completion by Baker

Activities

Interarrival time, defined in Table 2.11
Service time by Able, defined in Table 2.12
Service time by Baker, defined in Table 2.13

Delay

A customer's wait in queue until Able or Baker becomes free.

3.2 The Event-Scheduling/Time-Advance Algorithm

- The mechanism for advancing simulation time and guaranteeing that all events occur in correct chronological order is based on the future event list (FEL).

- This list contains all event notices for events that have been scheduled to occur at a future time.

- At any given time t, the FEL contains all previously scheduled future events and their associated event times

- The FEL is ordered by event time, meaning that the events are arranged chronologically; that is, the event times satisfy

  $t < t1 <= t2 <= t3 <= \ldots, <= tn$

  t is the value of CLOCK, the current value of simulated time. The event dated with time *t1* is called the imminent event; that is, it is the next event will occur. After the system snapshot at simulation time CLOCK == *t has b*een updated, the CLOCK is advanced to simulation time CLOCK _= t1 and the imminent event notice is removed from the FEL and the event executed.. This process repeats until the simulation is over.

- The sequence of actions which a simulator must perform to advance the clock system snapshot is called the event-scheduling/time-advance algorithm

Old system snapshot at time t

ClK	System State		Future Event List	
T	(5,1,6)		(3, t1)— Type 3 event to occur at time *t1* (1, t2)— Type 1 event to occur at time *t2* (1, t3)- Type 1 event to occur at time t_3 (2, *tn)*— Type 2 event to occur at time t_n	

Event-scheduling/time-advance algorithm

Step 1. Remove the event notice for the imminent event
(event 3, time t\) from PEL

Step 2. Advance CLOCK to imminent event time
(i.e., advance CLOCK from r to t1).

Step 3. Execute imminent event: update system state,
change entity attributes, and set membership as needed.

Step 4. Generate future events (if necessary) and
place their event notices on PEL ranked by event time.

(Example: Event 4 to occur at time t*, where t2 < t* < t3.)

Step 5. Update cumulative statistics and counters.

New system snapshot at time *t1*

OCK	System State		Future Event List	
t	(5,1,5)		(1, *t2)*— Type 1 event to occur at time *t1* (4, t*)— Type 4 event to occur at time t* (1, t_3)— Type 1 event to occur at time t_3 (2, tn)— Type 2 event to occur at time *tn*	

Figure *3.2* Advancing simulation time and updating system image

- The management of a list is called list processing
- The major list processing operations performed on a FEL are removal of the imminent event, addition of a new event to the list, and occasionally removal of some event (called cancellation of an event).
- As the imminent event is usually at the top of the list, its removal is as efficient as possible. Addition of a new event (and cancellation of an old event) requires a search of the list.

The removal and addition of events from the PEL is illustrated in Figure 3.2.

  - When event 4 (say, an arrival event) with event time t* is generated at step 4, one possible way to determine its correct position on the FEL is to conduct a top-down search:

    If *t* < t2*,*place event 4 at the top of the FEL.
    If t2 < *t** < t3, place event 4 second on the list.
    If *t3,* < *t** < t4, place event 4 third on the list.

    If tn < t*, event 4 last on the list.
  - Another way is to conduct a bottom-up search.

- The system snapshot at time 0 is defined by the initial conditions and the generation of the so-called exogenous events.
- The method of generating an external arrival stream, called *bootstrapping.*
- Every simulation must have a stopping event, here called *E,* which defines how long the simulation will run. There are generally two ways to stop a simulation:

1. At time 0, schedule a stop simulation event at a specified future time T_E. Thus, before simulating, it is known that the simulation will run over the time interval [0, *TE]*. *Example:* Simulate a job shop for *TE* = 40 hours.
2. Run length *TE* is determined by the simulation itself. Generally, *TE* is the time of occurrence of some specified event *E*. *Examples: TE* is the time of the 100th service

completion at a certain service center. *TE* is the time of breakdown of a complex system.

3.3 World Views

- When using a simulation package or even when using a manual simulation, a modeler adopts a world view or orientation for developing a model.
- Those most prevalent are the event scheduling world view, the process-interaction worldview, and the activity-scanning world view.
- When using a package that supports the process-interaction approach, a simulation analyst thinks in terms of processes .
- When using the event-scheduling approach, a simulation analyst concentrates on events and their effect on system state.
- The process-interaction approach is popular because of its intuitive appeal, and because the simulation packages that implement it allow an analyst to describe the process flow in terms of high-level block or network constructs.
- Both the event-scheduling and the process-interaction approaches use a / variable time advance.
- The activity-scanning approach uses a fixed time increment and a rule-based approach to decide whether any activities can begin at each point in simulated time.
- The pure activity scanning approach has been modified by what is called the three-phase approach.
- In the three-phase approach, events are considered to be activity duration-zero time units. With this definition, activities are divided into two categories called B and C.
  - **B activities:** Activities bound to occur; all primary

events and unconditional activities.

- **C activities:** Activities or events that are conditional upon certain conditions being true.

- With the. three-phase approach the simulation proceeds with repeated execution of the three phases until it is completed:
    - **Phase** A: Remove the imminent event from the FEL and advance the clock to its event time. Remove any other events from the FEL that have the event time.
    - **Phase B:** Execute all B-type events that were removed from the FEL.
    - **Phase C:** Scan the conditions that trigger each C-type activity and activate any whose conditions are met. Rescan until no additional C-type activities can begin or events occur.
- The three-phase approach improves the execution efficiency of the activity scanning method.

EXAMPLE 3.2 (Able and Baker, Back Again)

. Using the three-phase approach, the conditions for beginning each activity in Phase C are:

Activity	*Condition*
Service time by Able	A customer is in queue and Able is idle,
Service time by Baker	A customer is in queue, Baker is idle, and Able is busy.

3.4 Manual Simulation Using Event Scheduling

- In an event-scheduling simulation, a simulation table is used to record the successive system snapshots as time advances.
- Lets consider the example of a grocery shop which has only one checkout counter.

Example 3.3 (Single-Channel Queue)

- The system consists of those customers in the waiting line plus the one (if any) checking out.
- The model has the following components:

- **System state** *(LQ(i),* Z,S(r)), where *LQ(([* is the number of customers in the waiting line, and *LS(t)* is the number being served (0 or 1) at time *t.*
- **Entities** The server and customers are not explicitly modeled, except in terms of the state variables above.
- **Events**

  Arrival (A)
  Departure *(D)*
  Stopping event (£"), scheduled to occur at time 60.
- **Event notices**

  (A, i). Representing an arrival event to occur at future time *t*
  (D, t), representing a customer departure at future time t
  (£, 60), representing the simulation-stop event at future time 60
- **Activities**

  Inlerarrival time, denned in Table 2.6
  Service time, defined in Table 2.7
- **Delay**

  Customer time spent in waiting line.

- In this model, the FEL will always contain either two or three event notices.
- The effect *of* the arrival and departure events was first shown in Figures
  below

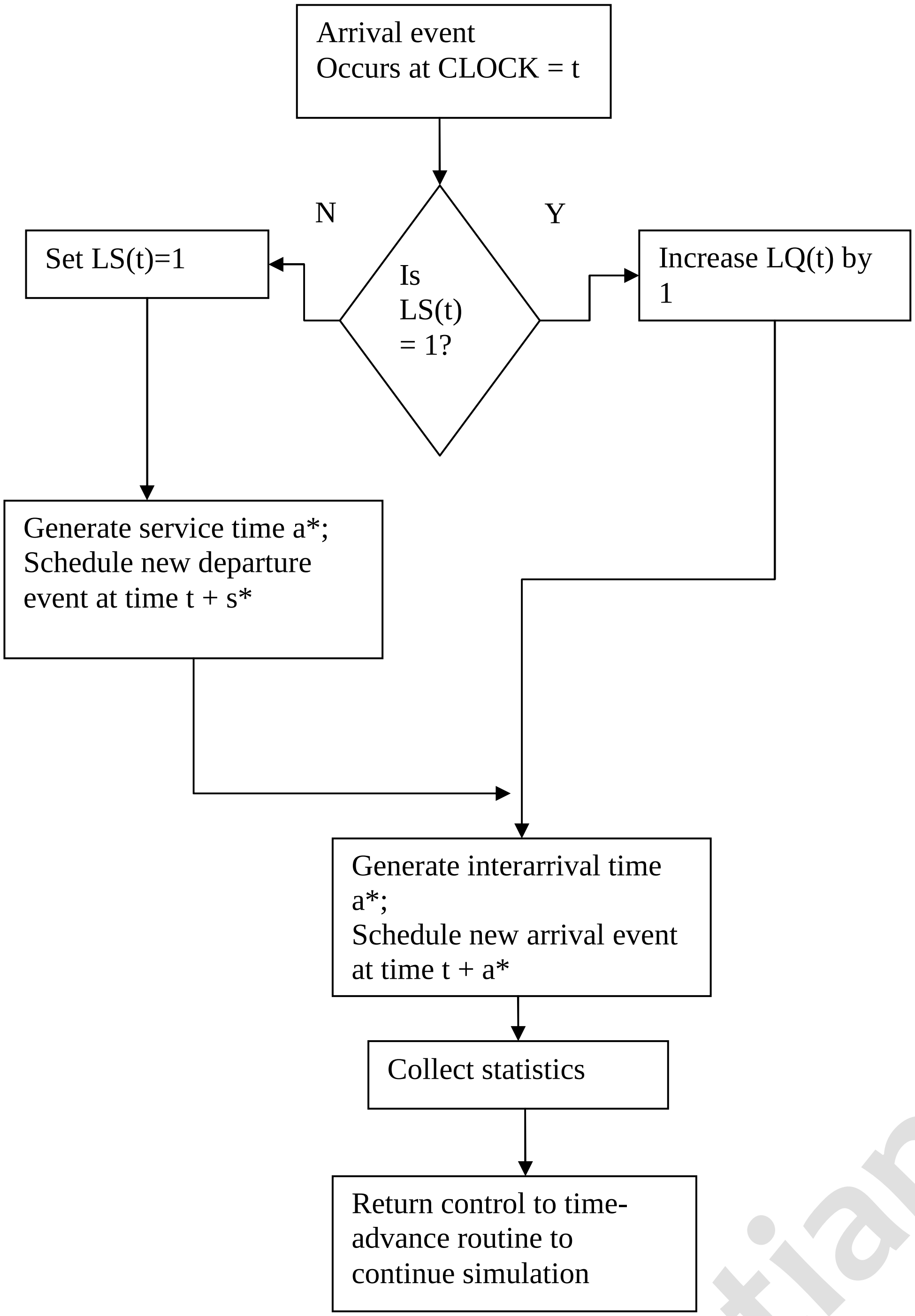

Arrival event
Occurs at CLOCK = t
N
Y
Set LS(t)=1
Is LS(t) = 1?
Increase LQ(t) by 1
Generate service time a*;
Schedule new departure event at time t + s*
Generate interarrival time a*;
Schedule new arrival event at time t + a*
Collect statistics
Return control to time-advance routine to continue simulation

Fig 1(A): Execution of the arrival event.

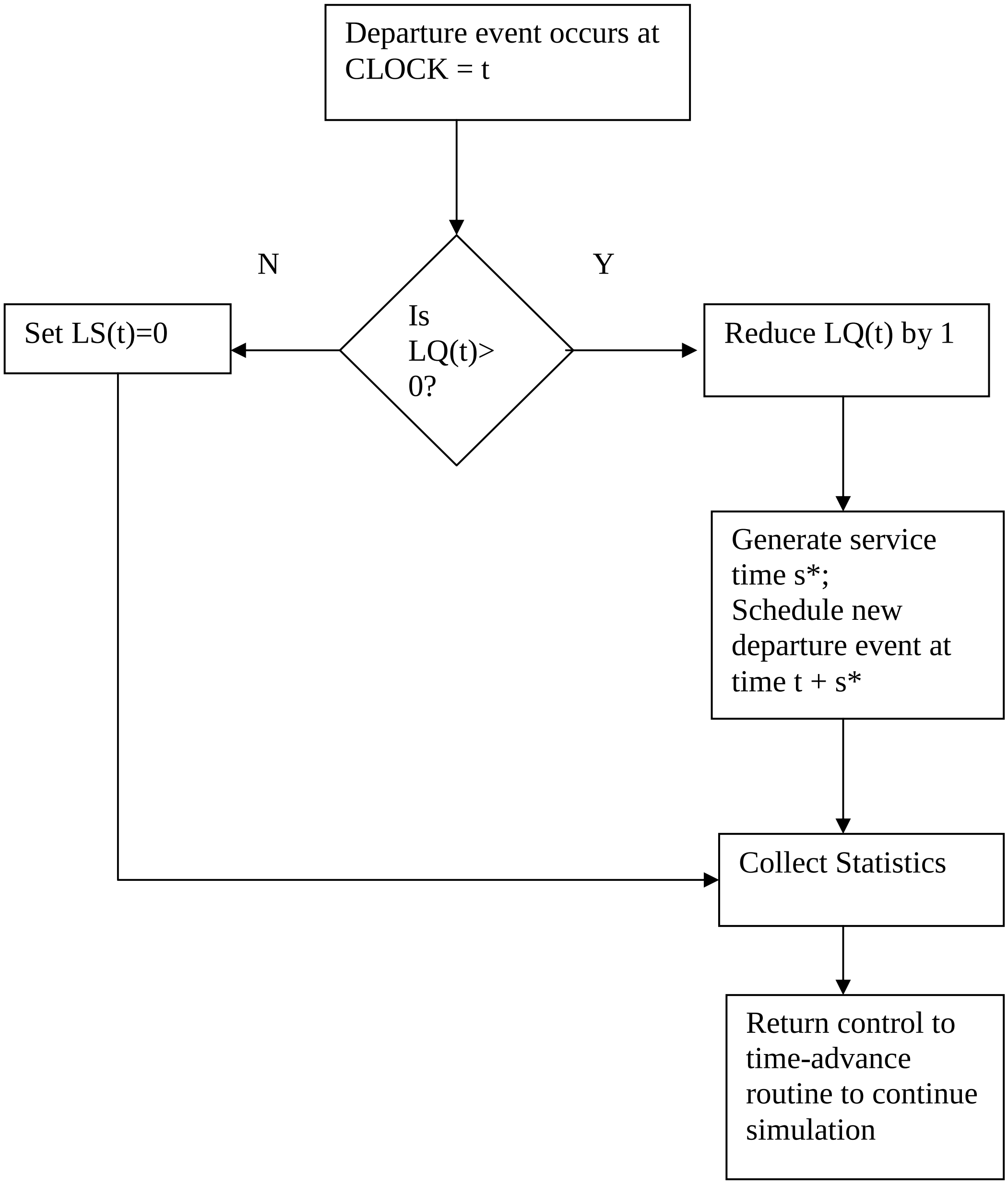

Departure event occurs at CLOCK = t
N
Y
Is LQ(t)> 0?
Set LS(t)=0
Reduce LQ(t) by 1
Generate service time s*;
Schedule new departure event at time t + s*
Collect Statistics
Return control to time-advance routine to continue simulation

Fig 1(B): Execution of the departure event.

- Initial conditions are that the first customer arrives at time 0 and begins service.
- This is reflected in Table below by the system snapshot at time zero (CLOCK = 0), with LQ (0) = 0, LS (0) = 1, and both a departure event and arrival event on the FEL.
- The simulation is scheduled to stop at time 60.
- Two statistics, server utilization and maximum queue length, will be collected.
- Server utilization is defined by total server busy time .(B) divided by total time*(T_e).*
- Total busy time, B, and maximum queue length MQ, will be accumulated as the simulation progresses.
- As soon as the system snapshot at time CLOCK = 0 is complete, the simulation begins.
- At time 0, the imminent event is (D, 4).
- The CLOCK is advanced to time 4, and (D, 4) is removed from the FEL.
- Since *LS(t*) = 1 for 0 <= t <= 4 (i.e., the server was busy for 4 minutes), the cumulative busy time is Increased from B = 0 to B = 4.
- By the event logic in Figure 1(B), set LS (4) = 0 (the server becomes idle).
- The FEL is left with only two future events, (A, 8) and (E, 0).
- The simulation CLOCK is next advanced to time 8 and an arrival event is executed.
- The simulation table covers interval [0,9].

Simulation table for checkout counter.

Clock	LQ(t)	LS(t)	FEL	Comment	B	MQ
0	0	1	(D,4) (A,8) (E,60)	First A occurs (a* = 8) schedule next A (s* = 4) schedule next D	0	0
4	0	0	(A,8) (E,60)	First D occurs;(D,4)	4	0
8	0	1	(D,9) (A,14) (E,60)	Second A occurs;(A,8) (a* = 6) schedule next A (s* = 1) schedule next D	4	0
9	0	0	(A,14) (E,60)	Second D occurs;(D,9)	5	0

Example 3.4 (The Checkout-Counter Simulation, Continued)

- Suppose the system analyst desires to estimate the mean response time and mean proportion of customers who spend 4 or more minutes in the system the above mentioned model has to be modified.
  - **Entities** (Ci, t), representing customer Ci who arrived at time t.
  - **Event notices** *(A, t,* Ci), the arrival of customer Ci at future time t

    (D, f, Cj), the departure of customer Cj at future time t.
  - **Set** "CHECKOUT LINE," the set of all customers currently at the checkout Counter (being served or waiting to be served), ordered by time of arrival

- Three new statistics are collected: S, the sum of customers response times for all customers who have departed by the current time; F, the total number of customers who spend 4 or more minutes at the check out counter; N_D the total number of departures up to the current simulation time.
- These three cumulative statistics are updated whenever the departure event occurs.
- The simulation table is given below

Simulation Table for Example 3.4

System state Cumulative statistics

Clock	LQ(t)	LS(t)	Checkout line	FEL	S	N_D	F
0	0	1	(C 1,0)	(D,4,C1) (A,8,C2) (E,60)	0	0	0
4	0	0		(A,8,C2) (E,60)	4	1	1
8	0	0	(C2,8)	(D,9,C2) (A,14,C3) (E,60)	4	1	1
9	0	0		(A,14,C3) (E,60)	5	2	1

Example 3.5 (The Dump Truck Problem)

- Six dump truck are used to haul coal from the entrance of a small mine to the railroad.
- Each truck is loaded by one of two loaders.
- After loading, a truck immediately moves to scale, to be weighted as soon as possible.
- Both the loaders and the scale have a first come, first serve waiting line(or queue) for trucks.
- The time taken to travel from loader to scale is considered negligible.
- After being weighted, a truck begins a travel time and then afterward returns to the loader queue.
- The model has the following components:
  - **System state**

    [LQ(0, L(f), WQ(r), W(r)], where

LQ(f) = number of trucks in loader queue
L(t) = number of trucks (0,1, or 2)being Loaded
WQ(t*)*= number of trucks in weigh queue
W(t) = number of trucks (0 or 1) being weighed, all at simulation time *t*

- **Event notices**

  (ALQ,, t, DTi), dump truck arrives at loader queue (ALQ) at time t
  (EL, t, *DTi),* dump truck i ends loading *(EL)* at time t
  (EW, t, *DTi),* dump truck i ends weighing *(EW)* at time t
- **Entities** The six dump trucks *(DTI,..., DT6)*
- **Lists**

  Loader queue, all trucks waiting to begin loading, ordered on a first-come, first-served basis
  Weigh queue, all trucks waiting to be weighed, ordered on a first-come, first-serve basis.
- **Activities** Loading time, weighing time, and travel time.
- **Delays** Delay at loader queue, and delay at scale.

Distribution of Loading for the Dump Truck

Loading time	Probability	Cumulative probability	Random-Digit Assignment
5	0.30	0.30	1-3
10	0.50	0.80	4-8
15	0.20	1.00	9-0

Distribution of Weighing Time for the Dump Truck

Weighing time	Probability	Cumulative probability	Random-Digit Assignment
12	0.70	0.70	1-7
16	0.30	1.00	8-0

Distribution of Travel Time for the Dump Truck

Travel time	Probability	Cumulative probability	Random-Digit Assignment
40	0.40	0.40	1-4
60	0.30	0.70	5-7
80	0.20	0.90	8-9
100	0.10	1.00	0

- The activity times are taken from the following list

Loading time	10	5	5	10	15	10	10
Weighing time	12	12	12	16	12	16	
Travel time	60	100	40	40	80		

- Simulation table for Dump Truck problem

System state Lists
cumulative stat

Clock t	LQ(t)	L(t)	WQ(t)	W(t)	Loader queue	Weigh queue	FEL	B_L	B_S
0	3	2	0	1	DT4 DT5 DT6		(EL,5,DT3) (EL,10,DT2) (EL,12,DT1)	0	0
5	2	2	1	1	DT5 DT6	DT3	(EL,10,DT2) (EL,5 + 5 ,DT4) (EW,12,DT1)	10	5
10	1	2	2	1	DT6	DT3 DT2	(EL,10,DT4) (EW,12,DT1) (EL,10+10,DT5)	20	10
10	0	2	3	1		DT3 DT2 DT4	(EW,12,DT1) (EL,20,DT5) (EL,10+15,DT6)	20	10
12	0	2	2	1		DT2 DT4	(EL,20,DT5) (EW,12+12,DT3) (EL,25,DT6) (ALQ,12+60,DT1)	24	12
20	0	1	3	1		DT2 DT4 DT5	(EW,24,DT3) (EL,25,DT6) (ALQ,72,DT1)	40	20
24	0	1	2	1		DT4 DT5	(EL,25,DT6) (EW,24+12,DT2) (ALQ,72,DT1) (ALQ,24+100,DT3)	44	24

Average Loader Utilization = 44/2 = 0.92

24

Average Scale Utilization=24/24 = 1.00
****

4. Random-Number Generation

Random numbers are a necessary basic ingredient in the simulation of almost all discrete systems. Most computer languages have a subroutine, object, or function that will generate a random number. Similarly simulation languages generate random numbers that arc used to generate event limes and other random variables.

4.1 Properties of Random Numbers

A sequence of random numbers, W1, W2, .. , must have two important statistical properties, uniformity and independence. Each random number *Ri*, is an independent sample drawn from a continuous uniform distribution between zero and 1. That is, the pdf is given by

$$\mathbf{F(x)}=\begin{cases} \mathbf{1,} & \mathbf{0 \leq x \leq 1} \\ \mathbf{0,\ otherwise} \end{cases}$$

This density function is shown in Figure 7.1. The expected value of each *Ri*, is:

$$E(R\} = \int_0^1 x\,dx = \frac{x2}{2}\Big|_0^1$$

and the variance is given by

$$V(R)=\int_0^1 x^2dx - [E(R)]^2 = \frac{x3}{3}\Big|_0^1 - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

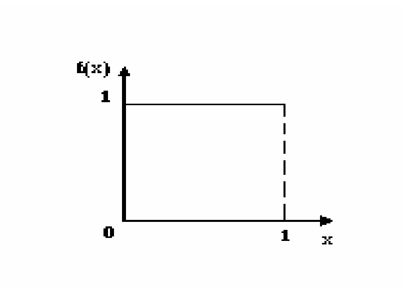


Figure 7.*1** The: pdf for random numbers.

Some consequences of the uniformity and independence properties are the following:

1. If the interval (0,1) is divided into *n* classes, or subintervals of equal length, the expected number of observations m each interval ii *N/n* where A' is the total number of observations.

2- The probability of observing a value in a particular interval is of the previous values drawn

4.2 Generation of Pseudo-Random Numbers

Pseudo means false,so false random numbers are being generated.The goal of any generation scheme, is to produce a sequence of numbers belween zero and 1 which simulates, or initates, the ideal properties of uniform distribution and independence as closely as possible.

When generating pseudo-random numbers, certain problems or errors can occur. These errors, or departures from ideal randomness, are all related to the properties stated previously. Some examples include the following

1. The generated numbers may not be uniformly distributed.
2. The generated numbers may be discrete -valued instead continuous valued
3. The mean of the generated numbers may be too high or too low.
4. The variance of the generated numbers may be too high or low
5. There may be dependence. The following are examples:
   (a) Autocorrelation between numbers.
   (b) Numbers successively higher or lower than adjacent numbers.
   (c) Several numbers above the mean followed by several numbers below the mean.

Usually, random numbers are generated by a digital computer as part of the simulation. Numerous methods can be used to generate the values. In selecting among these methods, or routines, there are a number of important considerations.

1. The routine should be fast. . The total cost can be managed by selecting a computationally efficient method of random-number generation.
2. The routine should be portable to different computers, and ideally to different programming languages .This is desirable so that the simulation program produces the same results wherever it is executed.
3. The routine should have a sufficiently long cycle. The cycle length, or period, represents the length of the random-number sequence before previous numbers begin to repeat themselves in an earlier order. Thus, if 10,000 events are to be generated, the period should be many times that long,

   A special case cycling is degenerating. A routine degenerates when the same random numbers appear repeatedly. Such an occurrence is certainly unacceptable. This can happen rapidly with some methods.
4. The random numbers should be replicable. Given the starting point (or conditions), it should be possible to generate the same set of random numbers, completely independent of the system that is being simulated. This is helpful for debugging purpose and is a means of facilitating comparisons between systems.
5. Most important, and as indicated previously, the generated random numbers should closely approximate the ideal statistical properties of uniformity and independences

4.3 Techniques for Generating Random Numbers

4.3.1 Linear Congruential Method

The linear congruential method, initially proposed by Lehmer [1951], produces a sequence of integers, *X*, *X2*,... between zero and *m* — 1 according to the following recursive relationship:

$$X_{i+1} = (a\ Xi + c) \bmod m, \quad i = 0,1,2,\ldots(7.1)$$

The initial value X_0 is called the seed, *a* is called the constant multiplier, c is the increment, and *m* is the modulus.

If $c \neq 0$ in Equation (7.1), the form is called the ***mixed congruential method***. When $c = 0$, the form is known as the ***multiplicative congruential method***. The selection of the values for a, *c*, m and Xo drastically affects the statistical properties and the cycle length. . An example will illustrate how this technique operates.

EXAMPLE 4.1

Use the linear congruential method to generate a sequence of random numbers with $X_0 = 27$, $a= 17$, $c = 43$, and $m = 100$. Here, the integer values generated will all be between zero and 99 because of the value of the modulus . These random integers should appear to be uniformly distributed the integers zero to 99.Random numbers between zero and 1 can be generated by

$R_i = X_i/m$, i= 1,2,…… **(7.2)**

The sequence of X_i and subsequent R_i values is computed as follows:

$X_0 = 27$

$X_1 = (17.27 + 43) \bmod 100 = 502 \bmod 100 = 2$

$R_{1=}2/100=0.\ 02$

$X_2 = (17 \bullet 2 + 43) \bmod 100 = 77 \bmod 100 = 77$

$R_{2=}77\ /100=0.\ 77$

*[3pt] $X_3 = (17 \bullet 77+ 43) \bmod 100 = 1352 \bmod 100 = 52$

$R_3=52/100=0.\ 52$

First, notice that the numbers generated from Equation (7.2) can only assume values from the set I = {0,1 /m, 2/m,..., (m — l)/m), since each *Xi* is an integer in the set {0,1,2,..., m — 1}. Thus, each *Ri* is discrete on I, instead of continuous on the interval [0, 1], This approximation appears to be of little consequence, provided that the modulus m is a very large integer. (Values such as $m = 2^{31} — 1$ and $m = 2^{48}$ are in common use in generators appearing in many simulation languages.) By maximum density is meant that the values assumed by *Ri* = 1, 2,..., leave no large gaps on [0,1]

Second, to help achieve maximum density, and to avoid cycling (i.e., recurrence of the same sequence of generated numbers) in practical applications, the generator should have the largest possible period. Maximal period can be achieved by the proper choice of *a*, c, m, and X_0.

- For m a power of 2, say $m = 2^b$ and $c \neq 0$, the longest possible period is $P = m = 2^b$, which is achieved provided that *c* is relatively prime to m (that is, the greatest common factor of c and m is l), and =a = l+4k, where *k* is an integer.
- For m a power of 2, say $m = 2^b$ and $c = 0$, the longest possible period is $P = m/4 = 2^{b-2}$, which is achieved provided that the seed X_0 is odd and the multiplier ,a, is given by a=3+8K , for some

K=0,1,..

- For m a prime number and c=0, the longest possible period is P=m-1, which is achieved provided that the multiplier , a, has the property that the smallest integer k such that

a ^{k}- 1is divisible by m is k= m-1.

EXAMPLE 4.3

Let $m = 10^2 = 100$, $a = 19$, $c = 0$, and $X_0 = 63$, and generate a sequence c random integers using Equation (7.1).

$X_0 = 63$

$X_1 = (19)(63) \bmod 100 = 1197 \bmod 100 = 97$

$X_2 = (19)(97) \bmod 100 = 1843 \bmod 100 = 43$

$X_3 = (19)(43) \bmod 100 = 817 \bmod 100 = 17$

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When m is a power of 10, say $m = 10^b$, the modulo operation is accomplished by saving the *b* rightmost (decimal) digits.

EXAMPLE 4.4

Let $a = 7^5 = 16{,}807$, $m = 2^{31}-1 = 2{,}147{,}483{,}647$ (a prime number), and c= 0. These choices satisfy the conditions that insure a period of $P = m— 1$. Further, specify a seed, $X_Q = 123{,}457$. The first few numbers generated are as follows:

$X^1= 7^5(123{,}457) \bmod (2^{31} - 1) = 2{,}074{,}941{,}799 \bmod (2^{31} - 1)$

$X^1 = 2{,}074{,}941{,}799$

$R^1= X^1/2^{31}$

$X_2 = 7^5(2{,}074{,}941{,}799) \bmod(2^{31} - 1) = 559{,}872{,}160$

$R_2 = X^2/2^{31}= 0.2607$

$X_3 = 7^5(559{,}872{,}160) \bmod(2^{31} - 1) = 1{,}645{,}535{,}613$

$R_3 = X^3/2^{31}= 0.7662$

Notice that this routine divides by *m* + 1 instead of *m* ; however, for sucha large value of *m* , the effect is negligible.

4.3.2 Combined Linear Congruential Generators

As computing power has increased, the complexity of the systems that we are able to simulate has also increased.

. One fruitful approach is to combine two or more multiplicative congruen-tial generators in such a way that the combined generator has good statistical properties and a longer period. The following result from L'Ecuyer [1988] suggests how this can be done:

If $W_{i,1}$, $W_{i,2}$ . . . , $W_{i,k}$ are any independent, discrete-valued random variables (not necessarily identically distributed), but one of them, say $W_{i,1}$, is uniformly distributed on the integers 0 to mi — 2, then

$$Wi= \left(\sum_{j=1}^{k} W_{i,j}\right) \bmod m_1 - 1$$

is uniformly distributed on the integers 0 to mi — 2.

To see how this result can be used to form combined generators, let $X_{i,1}$, $X_{i,2}$,..., $X_{i,k}$ be the i th output from *k* different multiplicative congru-ential generators, where the *j* th generator has prime modulus $m_{j,}$ and the multiplier a_j is chosen so that the period is m_j — 1. Then the j'th generator is producing integers $X_{i,j}$ that are approximately uniformly distributed on 1 to m_j - 1, and $W_{i,j} = X_{i,j}$ — 1 is approximately uniformly distributed on 0 to m*j - 2.* L'Ecuyer [1988] therefore suggests combined generators of the form

$$\sum_{j=1}^{k}(-1)^{j-1} X_{i.i} \quad \bmod \quad m1 - 1$$

with

$$R_i = \begin{cases} X_i/m_1 , & Xi > 0 \\ m_{1-1}/m_1 & Xi = 0 \end{cases}$$

Notice that the " $(-1)^{j-1}$ "coefficient implicitly performs the subtraction $X_{i,1}$-1; for example, if *k* = 2, then $(-1)°(X_{i1} - 1) - (-l)^l (X_{i2} - 1) = \sum^2_{i=1(-1)}{}^{j-1} X_{i,i}$

The maximum possible period for such a generator is

$$\frac{(m_1 -1)(m_2 - 1) - - - (m_k - 1)}{2^{k-1}}$$

which is achieved by the following generator:

EXAMPLE 4.5

For 32-bit computers, L'Ecuyer [1988] suggests combining *k = 2* generators with m_1 = 2147483563, a_1= 40014, m_2 = 2147483399, and a_2 = 40692. This leads to the following algorithm:

1. Select seed $X_{1,0}$ in the range [1, 2147483562] for the first generator, and seed X_2.o in the range [1, 2147483398].

Set $j = 0$.

2. Evaluate each individual generator.

$X_{1,j+1} = 40014X_{1,j}$ mod 2147483563

X_2,j+i = 40692X_2,j mod 2147483399

3 Set

$X_{j+1} = (X_{1,j+1} - X_{2\ j+1})$ mod 2147483562

4. Return

$R_{j+1} = X_{j+1/}$ 2147483563 , $X_{j+1} > 0$

2147483563 / 2147483563. $X_{i+1} = 0$

5. Set $j = j + 1$ and go to step 2.

4.4 Tests for Random Numbers

The desirable properties of random numbers — uniformity and independence To insure that these desirable properties are achieved, a number of tests can be performed (fortunately, the appropriate tests have already been conducted for most commercial simulation software}. The tests can be placed in two categories according to the properties of interest,

The first entry in the list below concerns testing for uniformity. The second through fifth entries concern testing for independence. The five types of tests

1. **Frequency test** Uses the Kolmogorov-Smirnov or the chi- square test to compare the distribution of the set of numbers generated to a uniform distribution.

2. **Runs test**. Tests the runs up and down or the runs above, and below the mean by comparing the actual values to expected values. The statistic for comparison is the chi-square.

*3.***Autocorrelation test** Tests the correlation between numbers and compares the sample correlation to the expected correlation of zero.

4. **Gap test**. Counts the number of digits that appear between repetitions of particular digit and then uses the Kolmogorov-Smirnov test to compare with the expected size of gaps,

5 **Poker test** . Treats numbers grouped together as a poker hand. Then the hands obtained are compared to what is expected using the chi-square test.

In testing for uniformity, the hypotheses are as follows:

$$H_0: R_i \sim U/[0,1]$$

$$H_1: R_i \sim U/[0,l]$$

The null hypothesis, H_0 reads that the numbers are distributed uniformly on the interval [0,1]. Failure to reject the null hypothesis means that no evidence of nonuniformity has been detected on the basis of this test. This does not imply that further testing of the generator for uniformity is unnecessary.

In testing for independence, the hypotheses are as follows:

H_0: Ri ~ independently

H_1:Ri~independently

This null hypothesis H_0 reads that the numbers are independent. Failure to reject the null hypothesis means that no evidence of dependence has been detected on the basis of this test. This does not imply that further testing of the generator for independence is unnecessary.

For each test, a level of significance *a* must be stated. The level *a* is the probability of rejecting the null hypothesis given that the null hypothesis is true, or

$$a = P \text{ (reject } H_0 | H_0 \text{ true)}$$

The decision maker sets the value of & for any test. Frequently, *a* is set to 0.01 orO.05.

If several tests are conducted on the same set of numbers, the probability of rejecting the null hypothesis on at least one test, by chance alone [i.e., making a Type I (a) error], increases. Say that *a*= 0.05 and that five different tests are conducted on a sequence of numbers. The probability of rejecting the null hypothesis on at least one test, by chance alone, may be as large as 0.25.

4.4.1 **Frequency Tests**

A basic test that should always be performed to validate a new generator is the test of uniformity. Two different methods of testing are available. They are the **Kolmogorov-Smirnov** and the **chi-square test**. Both of these tests measure the degree of agreement between the distribution of a sample of generated random numbers and the theoretical uniform distribution. Both tests are based on the null hypothesis of no significant difference between the sample distribution and the theoretical distribution.

1. **The Kolmogorov-Smirnov test**. This test compares the continuous cdf, F(X), of the uniform distribution to the empirical cdf, $S_N(x)$, of the sample of N observations. By definition,

$$F(x) = x, \quad 0 <= x <= 1$$

If the sample from the random-number generator is $R_1\ R_2, ,\bullet \bullet \bullet, RN$, then the empirical cdf, $S_N(X)$, is defined by

$$S_N(X) = \frac{\text{number of } R_1\ R_2, ,\bullet \bullet \bullet, \text{Rn which are} <= x}{N}$$

As N becomes larger, $S_N(X)$ should become a better approximation to F(X) , provided that the null hypothesis is true.

The **Kolmogorov-Smirnov test** is based on the largest absolute deviation between F(x) and $S_N(X)$ over the range of the random variable. That is.it is based on the statistic

$$D = \max | F(x) - S_N(x)| \qquad (7.3)$$

For testing against a uniform cdf, the test procedure follows these steps:

Step 1. Rank the data from smallest to largest. Let $R_{(i)}$ denote the i th smallest observation, so that

$$R_{(1)} <= R_{(2)} <= \bullet \bullet \bullet <= R_{(N)}$$

Step 2. Compute

$$D^+ = {}^{\max} \left\{ \mathbf{i/N} - R_{(i)} \right\}$$

$$^{1}\!\!<= \mathbf{i} <=^{N}$$

$$D^{-} = \max_{^{1}<= \mathbf{i} <=^{N}} \left\{ \mathbf{i/N} \ - \ R_{(i)} \right\}$$

Step3. Compute D = max(D^+, D^-).
Step 4. Determine the critical value, D_a, from Table A.8 for the specifiedsignificance level a and the given sample size N.
Step 5. If the sample statistic D is greater than the critical value, D_a, the null hypothesis that the data are a sample from a uniform distribution is rejected.

If D <= D_a, conclude that no difference has been detected between
the true distribution of **{** R_1 $R_{2,}$,• • •, Rn **}** and the uniform distribution.

EXAMPLE 4.6

Suppose that the five numbers 0.44,0.81,0.14,0.05,0.93 were generated, and it is desired to perform a test for uniformity using the Kolmogorov-Smirnov test with a level of significance a of 0.05.

First, the numbers must be ranked from smallest to largest. The calculations can be facilitated by use of Table 7.2. The top row lists the numbers from smallest ($R_{(1)}$) to largest ($R_{(n)}$) .The computations for D^+, namely i / N -$R_{(i\,)}$ and for D^-, namely $R_{(i\,)}$ - (i - l) / N, are easily accomplished using Table 7.2. The statistics are computed as D^+ = 0.26 and D^- = 0.21. Therefore, D = max{0.26, 0.21} = 0.26. The critical value of **D,** obtained from Table A.8 for a = 0.05 and N = 5, is 0.565. Since the computed value, 0.26, is less than the tabulated critical value, 0.565, the hypothesis of no difference between the distribution of the generated numbers and the uniform distribution is not rejected.

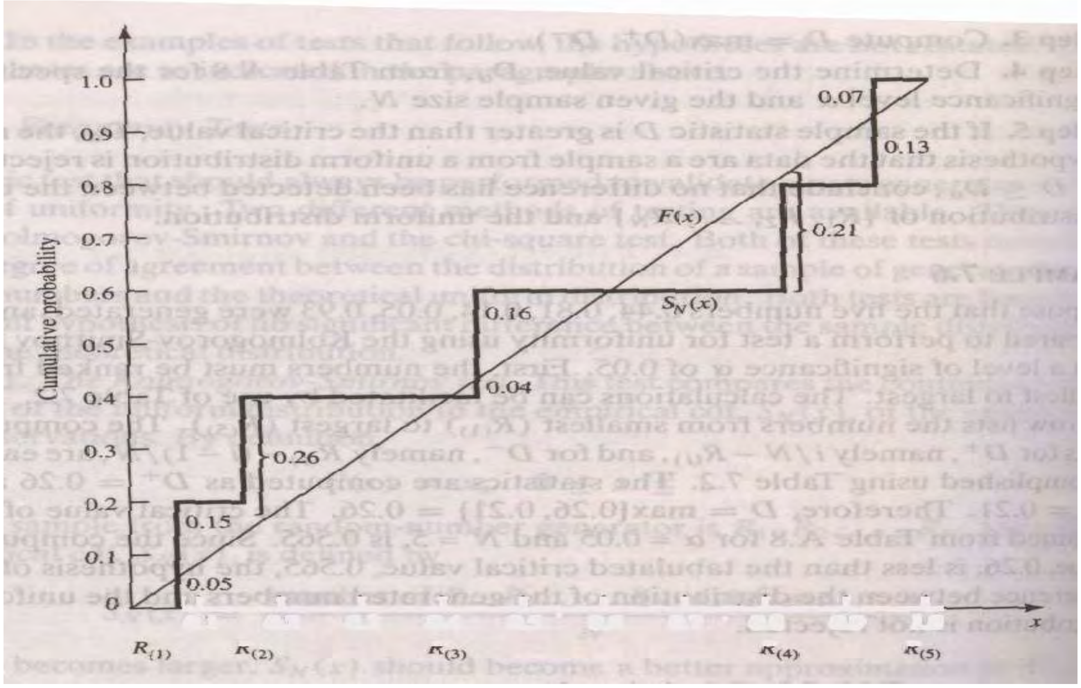
Table 7.2. Calculations for
Kolmogorov-Smirnov Test

$R_{(i)}$	0.05	0.14	0.44	0.81	0.93
i/N	0.20	0.40	0.60	0.80	1.00
i / N - $R_{(i)}$	0.15	0.26	0.16	—	0.07
$R_{(i) – (i-1)/N}$	V 0.05	—	0.04	0.21	0.13

The calculations in Table 7.2 are illustrated in Figure 7.2, where the empirical cdf, S_N(X), is compared to the uniform cdf, F(x). It can be seen that D^+ is the largest deviation of S_N(x) above F(x), and that D^- is the largest deviation of S_N(X) below F(x). For example, at $R_{(3)}$ the value of D^+ is given by 3/5 - $R_{(3)}$ = 0.60 - 0.44 =0.16 and of D^- is given by $R_{(3)}$ = 2/5 = 0.44 - 0.40 = 0.04. Although the test statistic D is defined by Equation (7.3) as the maximum deviation over all x, it can be seen from Figure 7.2 that the maximum deviation will always occur at one of the jump points $R_{(1)}$, $R_{(2)}$ . . . , and thus the deviation at other values of x need not be considered.

2. The chi-square test. The chi-square test uses the sample statistic

$$X_0^{\,2} = \sum^{n} \ (O_i - E_i)^2/Ei$$



i=1

Figure 7.2. Comparison of *F(x)* and *SN(X)*,

where Oi; is the observed number in the *i* th class, *Ei* is the expected number in the ith class, and *n* is the number of classes. For the uniform distribution, *Ei* the expected number in each class is given by

Ei = N/n

for equally spaced classes, where *N* is the total number of observations. It can be shown that the sampling distribution of X_0^2 is approximately the chi-square distribution with *n* - 1 degrees of freedom

EXAMPLE 4.7

Use the chi-square test with a = 0.05 to test whether the data shown below are uniformly distributed. Table 7.3 contains the essential computations. The test uses n = 10 intervals of equal length, namely

[0, 0.1), [0.1, 0.2), . . . , [0.9, 1.0). The value of X_0^2 is 3.4. This is compared with the critical value $X^2_{0.05,9}$=16.9.Since X_0^2 is much smaller than the tabulated value of $X^2_{0.05,9}$, the null hypothesis of a uniform distribution is not rejected.

0.34	0.90	0.25	0.89	0.87	0.44	0.12	0.21	0.46	0.67
0.83	0.76	0.79	0.64	0.70	0.81	0.94	0.74	0.22	0.74
0.96	0.99	0.77	0.67	0.56	0.41	0.52	0.73	0.99	0.02
0.47	0.30	0.17	0.82	0.56	0.05	0.45	0.31	0.78	0.05
0.79	0.71	0.23	0.19	0.82	0.93	0.65	0.37	0.39	0.42
0.99	0.17	0.99	0.46	0.05	0.66	0.10	0.42	0.18	0.49
0.37	0.51	0.54	0.01	0.81	0.28	0.69	0.34	0.75	0.49
0.72	0.43	0.56	0.97	0.30	0.94	0.96	0.58	0.73	0.05
0.06	0.39	0.84	0.24	0.40	0.64	0.40	0.19	0.79	0.62
0.18	0.26	0.97	0.88	0.64	0.47	0.60	0.11	0.29	0.78

Both the Kolmogorov-Smirnov and the chi-square test are acceptable for testing the uniformity of a sample of data, provided that the sample size is large. However, the Kolmogorov-Smirnov test is the more powerful of the two and is recommended. Furthermore, the Kolmogorov-Smirnov test can be applied to small sample sizes, whereas the chi-square is valid only for large samples, say N>=50.

Imagine a set of 100 numbers which are being tested for independence where the first 10 values are in the range 0.01-0.10, the second 10 values are in the range 0.11-0.20, and so on. This set of numbers would pass the frequency tests with ease, but the ordering of the numbers produced by the generator would not be random. The tests in the remainder of this chapter are concerned with the independence of random numbers which are generated. The presentation of the tests is similar to that by Schmidt and Taylor [1970].

Table 7.3. Computations for Chi-Square Test

Interval	O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$(O_i - E_i)^2/E_i^2$
1	8	10	-2	4	0.4
2	8	10	—2	4	0.4
3	10	10	0	0	0.0
4	9	10	-1	1	0.1
5	12	10	2	4	0.4
6	8	10	-2	4	0.4
7	10	10	0	0	0.0
8	14	10	4	16	1.6
9	10	10	0	0	0.0
10	11	10	1	1	0.1
	100	100	0		34

4.4.2 ***Runs Tests***

1. ***Runs up and runs down***. Consider a generator that provided a set of 40 numbers in the following sequence:

0.08	0.09	0.23	0.29	0.42	0.55	0.58	0.72	0.89	0.91
0.11	0.16	0.18	0.31	0.41	0.53	0.71	0.73	0.74	0.84
0.02	0.09	0.30	0.32	0.45	0.47	0.69	0.74	0.91	0.95
0.12	0.13	0.29	0.36	0.38	0.54	0.68	0.86	0.88	0.91

Both the Kolmogorov-Smirnov test and the chi-square test would indicate that the numbers are uniformly distributed. However, a glance at the ordering shows that the numbers are successively larger in blocks of 10 values. If these numbers are rearranged as follows, there is far less reason to doubt their independence

0.41	0.68	0.89	0.84	0.74	0.91	0.55	0.71	0.36	0.30
0.09	0.72	0.86	0.08	0.54	0.02	0.11	0.29	0.16	0.18
0.88	0.91	0.95	0.69	0.09	0.38	0.23	0.32	0.91	0.53
0.31	0.42	0.73	0.12	0.74	0.45	0.13	0.47	0.58	0.29

The runs test examines the arrangement of numbers in a sequence to test the hypothesis of independence.
Before defining a run, a look at a sequence of coin tosses will help with some terminology. Consider the following sequence generated by tossing a coin 10 times:

H T T H H T T T H T

There are three mutually exclusive outcomes, or events, with respect to the sequence. Two of the possibilities are rather obvious. That is, the toss can result in a head or a tail. The third possibility is "no event." The first head is preceded by no event and the last tail is succeeded by no event. Every sequence begins and ends with no event.

A run is defined as a succession of similar events preceded and followed by a different event. The length of the run is the number of events that occur in the run. In the coin-flipping example above there are six runs. The first run is of length one, the second and third of length two, the fourth of length three. and the fifth and sixth of length one.

There are two possible concerns in a runs test for a sequence of number The number of runs is the first concern and the length of runs is a second concern. The types of runs counted in the first case might be runs up and runs down. An up run is a sequence of numbers each of which is succeeded by a larger number. Similarly, a down run is a sequence of numbers each of which is succeeded by a smaller number. To illustrate the concept, consider the following sequence of 15 numbers:

-0.87 +0.15 +0.23 $^{+}$0.45 -0.69 -0.32 -0.30 +0.19 -.24 +0.18 +0.65 +0.82 -0.93
+0.22 0.81

The numbers are given a " + " or a " — " depending on whether they are followed by a larger number or a smaller number. Since there are 15 numbers, and they are all different, there will be 14 +'s and — 's. The last number is followed by "no event" and hence will get neither a + nor a — . The sequence of 14 +s and — 's is as follows:

- + + + - - - + - + + - +

Each succession of + 's and — 's forms a run. There are eight runs. The first run is of length one. the second and third are of length three, and so on. Further, there are four runs up and four runs down.

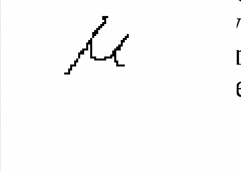
There can be too few runs or too many runs. Consider the following sequence of numbers:

0.08 0.18 0.23 0.36 0.42 0.55 0.63 0.72 0.89 0.91

This sequence has one run, a run up. It is unlikely that a valid random-number generator would produce such a sequence. Next, consider the following sequence

0,08 0.93 0.15 0.96 0.26 0.84 0.28 0.79 0.36 0.57

This sequence has nine runs, five up and four down. It is unlikely that a sequence of lO numbers w ' runs. What is more likely is that the number of runs will be somewhere be mes. These two extremes can be formalized as follows: if *N* is the number of ence, the maximum number of runs is *N* — *I* and the minimum number of ru



If *a* is the total number of runs in a truly random sequence, the mean and variance of *a* are given by

$\mu_a = 2N - 1 / 3$ (7.4)

and

$\sigma_a^2 = 16N - 29 / 90$ (7.5)

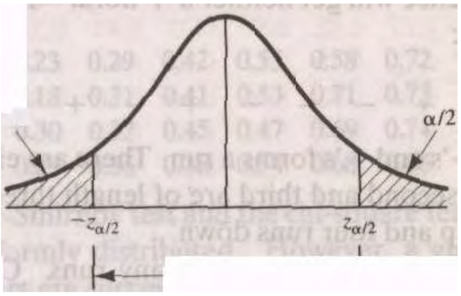
For $N > 20$, the distribution of a is reasonably approximated by a normal distribution, N(μ_a, σ_a^2) This approximation can be used to test the independence of numbers from a generator. In that case the standardized normal test statistic is developed by subtracting the mean from the observed n and dividing by the standard deviation. That is, the test statistic is

$$Z_0 = a - \mu_a / \sigma_a$$

Substituting Equation (7,4) for μ_a and the square root of Equation (7.5) for σ_a yields

$$Z_0 = a - [(2N - 1)/3] / \text{sqrt}((16N - 29) / 90)$$

where Zo ~ $N(0\ 1)$. Failure to reject the hypothesis of independence occur when $— z_a / 2 <=$ Zo $<= z_a / 2$ where a is the level of significance. The critical values and rejection region are shown in Figure 7.3.



Failure to reject **Figure** 7.3.

2. Runs above and below the mean. The test for runs up and runs down is not completely adequate to assess the independence of a group of numbers. Consider the following 40 numbers:

0.63	0.72	0.79	0.81	0.52	0.94	0.8.1	0.93	0.87	0.67
0.54	0.83	0.89	0.55	0.88	0.77	0.74	0.95	0.82	0.86
0.43	0.32	0.36	0.18	0.08	0.19	0.18	0.27	0.36	0.34
0.31	0.45	0.49	0.43	0.46	0.35	0.25	0.39	0.47	0.41

The sequence of runs up and runs down is as follows:

+ + + - + - + - - - + + - + - - + - + - - + - - + - + + - - + + - + - - + + -

This sequence is exactly the same as that in Example 7.8. Thus, the numbers would pass the runs-up and runs-down test. However, it can be observed that the first 20 numbers are all above the mean [(0.99 + O.OO)/2 = 0.495] and the last 20 numbers are all below the mean. Such an occurrence is highly unlikely. The previous runs analysis can be used to test for this condition, if the definition of a run is changed. Runs will be described as being above the mean or below the mean. A " + " sign will be used to denote an observation above the mean, and a "-" sign will denote an observation below the mean.

For example, consider the following sequence of 20 two-digit random numbers;

0.40 0.84 0.75 0.18 0.13 0.92 0.57 0.77 0.30 0.71
0.42 0.05 0.78 0.74 0.68 0.03 0.18 0.51 0.10 0.37

The pluses and minuses are as follows:

- + + - - + + + - + - - + + + - - + - -

In this case, there is a run of length one below the mean followed by a run of length two above the mean, and so on. In all. there are 11 runs, five of which are above the mean and six of which are below the mean. Let n_1 and n_2 be the number of individual observations above and below the mean and let *b* be the total number of runs. Notice that the maximum number of runs is $N = n_1 + n_2$ and the minimum number of runs is one. Given n_1 and n_2. the mean — with a continuity correction suggested by Swed and Eisenhart [1943] —and the variance of *b* for a truly independent sequence are given by

$$\mu_b = 2\, n_1\, n_{2\,/\,N\;+1/2} \qquad (7.6)$$

and

$$\sigma_b^2 = 2n_1n_2(2n_1n_2-N) \;/\; N_2(N-1) \qquad (7.7)$$

For either n_1 or n_2 greater than 20, *b* is approximately normally distributed. The test statistic can be formed by subtracting the mean from the number of runs and dividing by the standard deviation, or

$$\mathbf{Z_0 = (b-(2n_1n_2/N)-1/2) / [2n_1n_2(2n_1n_2-N/N^2(N-1)]^{1/2}}$$

Failure to reject the hypothesis of independence occurs when $—z_a/2 <= Z_0 < =z_a/2$-, where *a* is the level of significance.

EXAMPLE 4.9

Determine whether there is an excessive number of runs above or below the mean for the sequence of numbers given in Example 7.8. The assignment of + 's and — 's results in the following:

- + + + + + + + - - - + + - + - - - - - - - + +
- - - - + + - - + - + - - + + -

The values of $n_1, n_2,$ and *b* are as follows:

$$n_1 = 18$$

$n_2 = 22$

$N = n_1 + n_2 = 40$

$b = 17$

Equations (7.6) and (7.7) are used to determine μ_b and σ_b^2 as follows:

$\mu_b = 2(18)(22)/40 + 1/2 = 20.3$

$\sigma_b^2 = 2(18)(22)[(2)(18)(22)-40] /(40)^2 (40-1)=9.54$

Since n_2 is greater than 20, the normal approximation is acceptable, resulting in a Z_0 value of

$Z_0 = 17-20.3/\sqrt{9.54} = -1.07$

Since $Z_{0.025} = 1.96$ the hypothesis of independence cannot be rejected on the basis of this test

3. ***Runs test: length of runs***. Yet another concern is the length of runs. As an example of what might occur, consider the following sequence of numbers

0.16, 0.27, 0.58, 0.63, 0.45, 0.21, 0.72, 0.87, 0.27, 0.15, 0.92, 0.85,...

Assume that this sequence continues in a like fashion: two numbers below the mean followed by two numbers above the mean. A test of runs above and below the mean would detect no departure from independence. However, it is to be expected that runs other than of length two should occur.

Let *Y,* be the number of runs of length ***i*** in a sequence of *N* numbers. For an independent sequence, the expected **value** of *Yj* for runs up and down is given by

$E(Yi) = 2/(i+3)!\,[\, N(i^2+3i+1)-(i^3+3i^2-i-4) \,] \quad ,i \;<=\; N - 2 \quad (7.8)$

$E(Yi) =2/N! \quad i = N – 1 \qquad (7.9)$

For runs above and below the mean, the expected value of y, is approximately given by

$E(Yi) = N\, w_i / E(I\,), \quad N>20 \qquad (7.10)$

where w_i the approximate probability that a run has length i, is given by

$w_i=(n_1 /N)^I (n_2/N) + (n_1/N)(n_2/N)^I \quad N>20 \qquad (7.11)$

and where E(I) , the approximate expected length of a run, is given by

$E(I\,)= n_1 / n_2 + n_2/ n_1 \quad N>20 \qquad (7.12)$

The approximate expected total number of runs (of all lengths) in a sequence of length *N*, E(A), is given by

$E(A)=N/ E(I) \quad N>20 \qquad (7.13)$

The appropriate test is the chi-square test with Oi, being the observed number of runs of length i. Then the test statistic is

$$X_0^2=\sum_{i=1}^{L} [\ Oi - E(Yi)^2\ ]\ /\ E(Yi)$$

where $L = N - 1$ for runs up and down and $L = N$ for runs above and below the mean. If the null hypothesis of independence is true, then $X_0^2$is approximately chi-square distributed with L — 1 degrees of freedom.

EXAMPLE 4.10

Given the following sequence of numbers, can the hypothesis that the numbers are independent be rejected on the basis of the length of runs up and down at
a =0.05?

0.30	0.48	0.36	0.01	0.54	0.34	0.96	0.06	0.61	0.85
0.48	0.86	0.14	0.86	0.89	0.37	0.49	0.60	0.04	0.83
0.42	0.83	0.37	0.21	0.90	0.89	0.91	0.79	0.57	0.99
0.95	0.27	0.41	0.81	0.96	0.31	0.09	0.06	0.23	0.77
0.73	0.47	0.13	0.55	0.11	0.75	0.36	0.25	0.23	0.72
0.60	0.84	0.70	0.30	0.26	0.38	0.05	0.19	0.73	0.44

For this sequence the +'s and —'s are as follows

+ - - + - + - + + - + - + + - + + - + - + - - + - + - - + - - + + + - - - + + - - - + -
+ - - - + - + - - - + - + + -

The length of runs in the sequence is as follows:

1,2,1,1,1,1,2,1,1,1,2,1,2,1,1,1,1,2,1,1, 1,2,1,2,3,3,2,3,1,1,1,3,1,1,1, 3,1,1,2,1

The number of observed runs of each length is as follows:

Run length ,i	1	2	3
Observe Run **0i**	26	9	5

The expected numbers of runs of lengths one, two, and three are computed from Equation (7.8) as

$$E(Yi) = 2/4![60(1 + 3 + 1) - (1 + 3 - 1 - 4)]= 25.08$$

$$E(Y_2) = 2/5![60(4 + 6 + 1) - (8 + 12 - 2- 4)] = 10.77$$

$$E(Y_3) =2/6![60(9 + 9 + 1) – (27 + 27 - 3 – 4)] =3.04$$

The mean total number of runs (up and down) is given by Equation (7.4) as

$\mu a = 2(60)-1/3=39.67$

Thus far, the E(Yi) for i = 1, 2, and 3 total 38.89. The expected number of runs of length 4 or more is the difference $\mu a –\sum^{3}_{i=1} E(Yi)$ or 0.78

Table 7.4. Length of Runs Up and Down: x^2 Test

Rim Length, i	*Observed Number of Runs, Oj*	*Expected Number of Runs, $E(Y,)$*	$\frac{[0, - E(Y,)\}^2}{E(Yi)}$
1	26	25.08	0.03
2	9 }	10.77 }	
>=3	5 } **14**	3.82 } **14.5 9**	0.02
	40	39.67	0.05

4.4.3 Tests for Autocorrelation

The tests for autocorrelation are concerned with the dependence between numbers in a sequence. As an example, consider the following sequence of numbers:

0.12	0.01	0.23	0.28	0.89	0.31	0.64	0.28	0.83	0.93
0.99	0.15	0.33	0.35	0.91	0.41	0.60	0.27	0.75	0.88
0.68	0.49	0.05	0.43	0.95	0.58	0.19	0.36	0.69	0.87

From a visual inspection, these numbers appear random, and they would probably pass all the tests presented to this point. However, an examination of the 5th, 10th, 15th (every five numbers beginning with the fifth), and so on. indicates a very large number in that position. Now, 30 numbers is a rather small sample size to reject a random-number generator, but the notion is that numbers in the sequence might be related. In this particular section, a method for determining whether such a relationship exists is described. The relationship would not have to be all high numbers. It is possible to have all low numbers in the locations being examined, or the numbers may alternately shift from very high to very low.

The test to be described below requires the computation of the autocorrelation between every *m* numbers *(m* is also known as the lag) starting with the ith number. Thus, the autocorrelation $p,_m$ between the following numbers would be of interest: /?,-, *Rj+m, Ri+2m,* • • •, *Ri+(M+\)m-* The value M is the largest integer such that / + (M + l)m < *N,* where *N* is the total number of values in the sequence. (Thus, a subsequence of length M + 2 is being tested.)

Since a nonzero autocorrelation implies a lack of independence, the following two-tailed test is appropriate:

$$H_0: \rho_{im} = 0$$

$$Hi: \rho_{im} \neq 0$$

For large values of M, the distribution of the estimator of ρ_{im} denoted $\rho^{\wedge}_{im}$ is approximately normal if the values R_i, R_{i+m}, R_{i+2m},…….$R_{i+(M+1)m}$ are un-correlated. Then the test statistic can be formed as follows:

$$Z_0 = \rho^{\wedge}_{im} / \sigma_{\rho^{\wedge}im}$$

which is distributed normally with a mean of zero and a variance of 1, under the assumption of independence, for large M.

The formula for $\rho^{\wedge}_{im}$ in a slightly different form, and the standard deviation of the estimator, $\sigma_{\rho^{\wedge}im}$ are given by Schmidt and Taylor [1970] as follows:

$$\rho^{\wedge}_{im} = 1/\,M+1\,[\,\textstyle\sum^{M}_{k=0} R_{i+km} R_{i+(k+1)m}\,] - 0.25$$

and

$$\sigma_{\hat{\rho}_{im}} = \sqrt{(13M + 7) / 12(M +1)}$$

After computing Z_0, do not reject the null hypothesis of independence if **$-z_a/2$** <= **Z_0** <= **$z_a/2$**, where *a* is the level of significance

If $\rho_{im} > 0$, the subsequence is said to exhibit positive autocorrelation. In this case, successive values at lag *m* have a higher probability than expected of being close in value (i.e., high random numbers in the subsequence followed by high, and low followed by low). On the other hand, if $\rho_{im} < 0$, the subsequence is exhibiting negative autocorrelation, which means that low random numbers tend to be followed by high ones, and vice versa. The desired property of independence, which implies zero autocorrelation, means that there is no discernible relationship of the nature discussed here between successive, random numbers at lag *m*.

EXAMPLE 4.12

Test whether the 3rd, 8th, 13th, and so on, numbers in the sequence at the beginning of this section are autocorrelated. (Use *a* = 0.05.) Here, *i* = 3 (beginning with the third number), *m* = 5 (every five numbers), *N* = 30 (30 numbers in the sequence), and *M* = 4 (largest integer such that 3 + (M +1)5 < 30). Then,

$\hat{\rho}_{35}$ = 1/ 4 + 1**[** (0.23)(0.28) + (0.28)(0.33) + (0.33)(0.27) + (0.27)(0.05) + (0.05)(0.36) **]** = -0.1945

and

$\sigma_{\hat{\rho}_{35}} = \sqrt{(13(4) + 7) / 12(4 + 1)} = 0.1280$

Then, the test statistic assumes the value

$Z_0 = -0.1945/0.1280 = -1.516$

Now, the critical value is

$$Z0.025 = 1.96$$

Therefore, the hypothesis of independence cannot be rejected on the basis of this test.

It can be observed that this test is not very sensitive for small values of M, particularly when the numbers being tested are on the low side. Imagine what would happen if each of the entries in the foregoing computation of $\hat{\rho}_{im}$ were equal to zero. Then, $\hat{\rho}_{im}$ would be equal to —0.25 and the calculate would have the value of —1.95, not quite enough to reject the hypothesis of independence.

Many sequences can be formed in a set of data, given a large value of *N*. For example, beginning with the first number in the sequence, possibilities include

1) the sequence of all numbers,

(2) the sequence formed from the first. third, fifth,..., numbers,

(3) the sequence formed from the first, fourth, numbers, and so on. If *a* — 0.05, there is a probability of 0.05 of rejecting a true hypothesis. If 10 independent sequences are examined, the probability of finding no significant autocorrelation, by chance alone, is $(0.95)^{10}$ or 0.60. Thus, 40% of the time significant autocorrelation would be detected when it does not exist. If *a* is 0.10 and 10 tests are conducted, there is a 65% chance of finding autocorrelation by chance alone. In conclusion, when "fishing" for autocorrelation, upon performing numerous tests, autocorrelation may eventually be detected, perhaps by chance alone, even when no autocorrelation is present.

4.4.4 Gap Test

The gap test is used to deter.nine the significance of the interval between the recurrences of the same digit. A gap of length *x* occurs between the recurrences of some specified digit. The following example illustrates the length of gaps associated with the digit 3:

```
4, 1,  3,  5,  1,  7.  2.  8.  2.  0,  7.  9.  1.  3.  5,  2,  7,  9.  4.  1.  6.  3
3, 9.  6,  3,  4.  8,  2.  3,  1,  9,  4.  4,  6.  8.  4,  1,  3.  8.  9.  5.  5.  7
3, 9,  5,  9.  8,  5.  3.  2,  2,  3,  7.  4,  7.  0.  3.  6.  3,  5,  9.  9.  5.  5
5, 0,  4.  6.  8.  0,  4.  7.  0,  3.  3,  0,  9,  5.  7,  9.  5.  1.  6.  6.  3.  8
8, 8,  9,  2,  9.  1.  8.  5,  4.  4.  5,  0,  2.  3,  9,  7.  1.  2.  0.  3,  6.  3
```

To facilitate the analysis, the digit 3 has been underlined. There are eighteen 3's in the list. Thus, only 17 gaps can occur. The first gap is of length 10. the second gap is of length 7, and so on. The frequency of the gaps is of interest. The probability of the first gap is determined as follows:

10 of these terms

$$P(gapof\ l0) = P(no3)---P(no3)P(3) = (0.9)^{10}\ (0.1) \qquad (7.12)$$

since the probability that any digit is not a 3 is 0.9, and the probability

that any digit is a 3 is 0.1. In general,

$P(t$ followed by exactly x non-r digits) = $(0.9)^{x}$ (0.1), $x = 0.1.2..$

The theoretical frequency distribution for randomly ordered digits is given by

$P(gap <=x) = F(x) = 0.1 \sum^{x}_{n=0} (0.9)^{n} = 1 - 0.9^{x+1}$

The procedure for the test follows the steps below. When applying the test to random numbers, class intervals such as [0, 0.1), [0.1,0.2),. . . play the role of random digits.

Step 1. Specify the cdf for the theoretical frequency distribution given by Equation (7.14) based on the selected class interval width.

Step2.Arrange the observed sample of gaps in a cumulative distribution with these same classes.

Step 3. Find D, the maximum deviation between *F(x)* and *SN(X)* as in Equation (7.3).

Step 4. Determine the critical value, D_a, from Table A.8 for the specified value of *a* and the sample size *N.*

Step 5. If the calculated value of *D* is greater than the tabulated value of D_a, the null hypothesis of independence is rejected.

EXAMPLE 4.13

Based on the frequency with which gaps occur, analyze the 110 digits above to test whether they are independent. Use *a* = 0.05. The number of gaps is given by the number of data values minus the number of distinct digits, or 110 —10 = 100 in the example. The number of gaps associated with the various digits are as follows:

Digit	0	1	2	3	4	5	6	7	8	9
Number of gaps	7	8	8	17	10	13	7	8	9	13

The gap test is presented in Table 7.6. The critical value of D is given by

$D_{0.05}$ = 1.36 / √ 100 = 0.136

Since D = max $|F(x) - S_N(x)|$ = 0.0224 is less than $D_{0.05}$ do not reject the hypothesis of independence on the basis of this test.

Table 7.6. Gap-Test Example

Gap Length	*Frequency*	*Relative Frequency*	*Cumulative Relative Frequency*		F(x)
		$\lvert F(x)-s_N(x)\rvert$			
0-3	35	0.35	0.35	0.3439	0.0061
4-1	22	0.22	0.57	0.5695	0.0005
8-11	17	0.17	0.74	0.7176	0.0224
12-15	9	0.09	0.83	0.8147	0.0153
16-19	5	0.05	0.88	0.8784	0.0016
20-23	6	0.06	0.94	0.9202	0.0198
24-27	3	0.03	0.97	0.9497	0.0223
28-31	0	0.0	0.97	0.9657	0.0043
32-35	0	0.0	0.97	0.9775	0.0075
36-39	2	0.02	0.99	0.9852	0.0043
40-43	0	0.0	0.99	0.9903	0.0003
44-47	1	0.01	1.00	0.9936	0.0064

4.4.5 Poker Test

The poker test for independence is based on the frequency with which certain digits are repeated in a series of numbers. The following example shows an unusual amount of repetition:

0.255, 0.577, 0.331, 0.414, 0.828, 0.909, 0.303,0.001, …

In each case, a pair of like digits appears in the number that was generated. In three-digit numbers there are only three possibilities, as follows:

1. The individual numbers can all be different.
2. The individual numbers can all be the same.
3. There can be one pair of like digits.

The probability associated with each of these possibilities is given by the following

P(three different digits) = *P*(second different from the first) x P(third different from the first and second) = (0.9)(0.8) = 0.72

P(three like digits) = P(second digit same as the first) x *P* (third digit same as the first) = (0.1)(0.1) = 0.01

P(exactly one pair) = 1 - 0.72 - 0.01 = 0.27

Alternatively, the last result can be obtained as follows:

P(exactly one pair) = $\binom{3}{2}$ (0.1)(0.9) = 0.27

The following example shows how the poker test (in conjunction with the chi-square test) is used to ascertain independence.

EXAMPLE 4.14

A sequence of 1000 three-digit numbers has been generated and an analysis indicates that 680 have three different digits, 289 contain exactly one pair of like digits, and 31 contain three like digits. Based on the poker test, are these numbers independent? Let *a* = 0.05. The test is summarized in Table 7.7.

The appropriate degrees of freedom are one less than the number of class intervals. Since 47.65 > $X^2_{0.05.2}$ = 5.99. the independence of the numbers is rejected on the basis of this test.

Table 7.7. Poker-Test Results

Combination, i	*Observed Frequency,* O,	*Expected Frequency,* £,	$\frac{(0,-Ei)^2}{E,}$
Three different digits	680	720	2.22
Three like digits	31	10	44.10
Exactly one pair	289	270	1.33
	1000	1000	47.65

*****

5 . Random - Variate Generation

INTRODUCTION :

This chapter deals with procedures for sampling from a variety of widely used continuous and discrete distributions. Here it is assumed that a distribution has been completely specified, and ways are sought to generate samples from this distribution to be used as input to a simulation model. The purpose of the chapter is to explain and illustrate some widely used techniques for generating random variates, not to give a state-of-the-art survey of the most efficient techniques.

TECHNIQUES :

- **INVERSE TRANSFORMATION TECHNIQUE**
- **ACCEPTANCE-REJECTION TECHNIQUE**

All these techniques assume that a source of uniform (0,1) random numbers is available R1,R2….. where each R1 has probability density function pdf $$fR(X)= \begin{array}{ll} 1 & , 0<= X <=1 \\ 0 & , \text{otherwise} \end{array}$$

cumulative distribution function

cdf $$fR(X)= \begin{array}{ll} 0 & , X<0 \\ X & , 0<= X <= 1 \\ 1 & , X>1 \end{array}$$

The random variables may be either discrete or continuous.

5.1 Inverse Transform Technique :

The inverse transform technique can be used to sample from exponential, the uniform, the Weibull, and the triangular distributions and empirical distri-Jbutions. Additionally, it is the underlying principle for sampling from a wide variety of discrete distributions. The technique will be explained in detail for the exponential distribution and then applied to other distributions. It is the most straightforward, but always the most efficient., technique computationally.

5.1.1 EXPONENTIAL DISTRIBUTION :

The exponential distribution, discussed as before has probability density function (pdf) given by

$$f(X)= \begin{array}{ll} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{array}$$

and cumulative distribution function (cdf) given by

$$f(X)= \int_{-\infty}^{x} f(t)\,dt = \begin{array}{ll} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{array}$$

The parameter can be interpreted as the mean number of occurrences per time unit. For example, if interarrival times Xi , X_2, X_3, . . . had an exponential distribution with rate , then could be interpreted as the mean number of arrivals per time unit, or the arrival rate| Notice that for any j

$$E(Xi)= 1/\lambda$$

so that is the mean interarrival time. The goal here is to develop a procedure for generating values X1,X2,X3, …... which have an exponential distribution.

The inverse transform technique can be utilized, at least in principle, for any distribution. But it is most useful when the cdf. F(x), is of such simple form that its inverse, F , can be easily computed. A step-by-step procedure for the inverse transform technique illustrated by me exponential distribution, is as follows:

Step 1. Compute the cdf of the desired random variable *X*. For the exponential distribution, the cdf is $F(x) = 1 - e \quad , x > 0$.

Step 2. Set $F(X) = R$ on the range of X. For the exponential distribution, it becomes

$1 - e^{-\lambda X} = R$ on the range $x >= 0$. Since X is a random variable (with the exponential distribution in this case), it follows that 1 -   is also a random variable, here called *R*. As will be shown later, *R* has a uniform distribution over the interval (0,1).,

Step 3. Solve the equation $F(X) = R$ for *X* in terms of *R*. For the exponential distribution, the solution proceeds as follows:

$$1 - e^{-\lambda x} = R$$

$$e^{-\lambda x} = 1 - R$$

$$-\lambda X = ln(1 - R)$$

$$x = -1/\lambda \, ln(1 - R) \qquad (5.1)$$

Equation (5.1) is called a random-variate generator for the exponential distribution. In general, Equation (5.1) is written as $X=F^{-1}(R\ )$. Generating a sequence of values is accomplished through steps 4.

Step 4. Generate (as needed) uniform random numbers *R1*, R2, R3,... and compute the desired random variates by

$$Xi = F^{-1} \ (Ri)$$

For the exponential case, $F \quad (R) = (-1/\lambda)\ln(1- R)$ by Equation (5.1), so that

$$Xi = -1/\lambda \ln (1 - Ri) \qquad (5.2)$$

for i = 1,2,3,.... One simplification that is usually employed in Equation (5.2) is to replace 1 – Ri by Ri to yield

$$Xi = -1/\lambda \ln Ri \qquad (5.3)$$

which is justified since both Ri and 1- Ri are uniformly distributed on (0,1).

Table 5.1 Generation of Exponential Variates X, with Mean 1, Given Random Numbers Ri,

I	1	2	3	4	5
Ri	0.1306	0.0422	0.6597	0.7965	0.7696
Xi	0.1400	0.0431	1.078	1.592	1.468

Table 5.1 gives a sequence of random numbers from Table A.1 and the computed exponential variates, Xi, given by Equation (5.2) with a value of = 1. Figure 5.1(a) is a histogram of 200 values, R1,R2,…R200 from the uniform distrubition and figure 5.1(b)

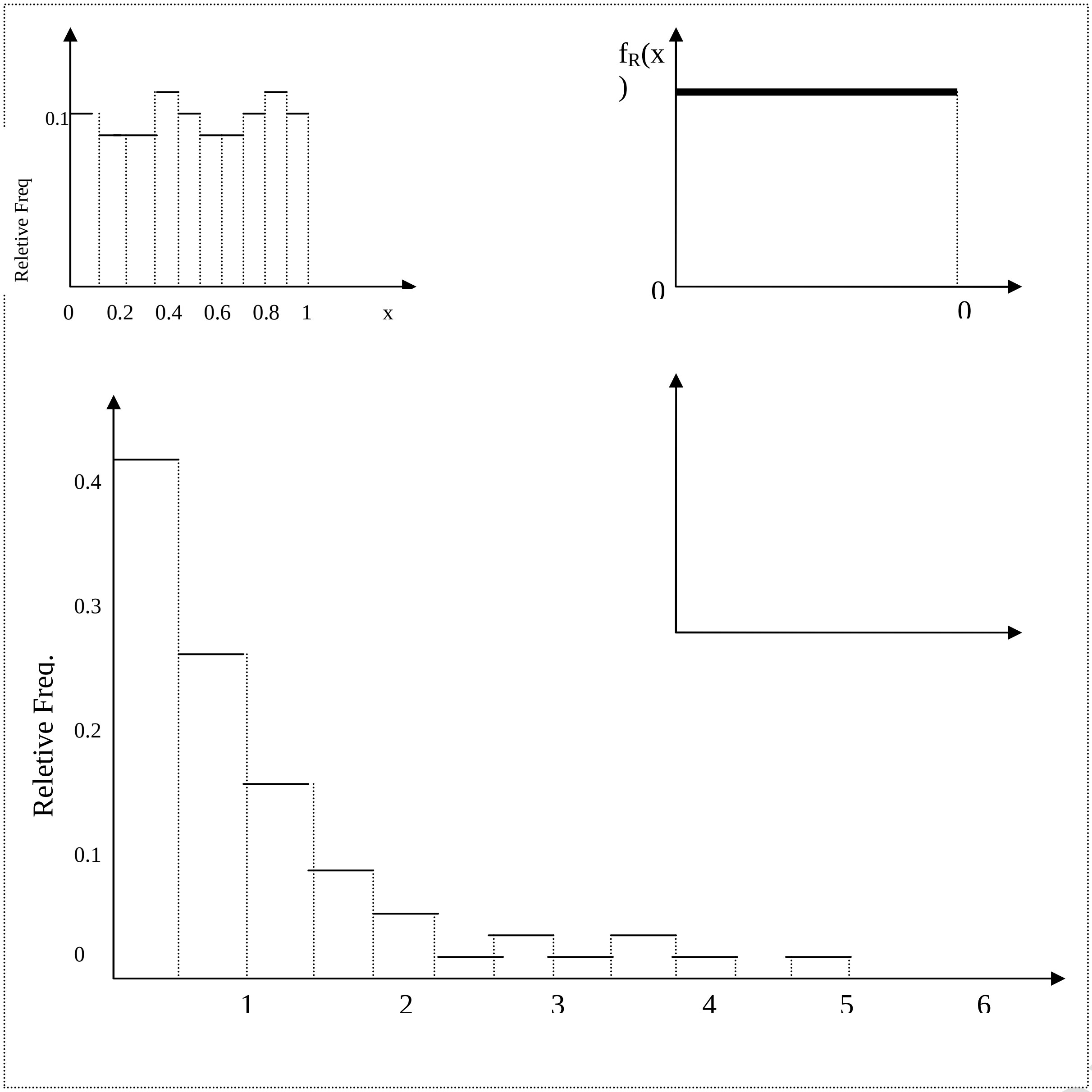


Figure 5.1. (a) Empirical histogram of 200 uniform random numbers; (b) empirical histogram of 200 exponential variates; (c) theoretical uniform density on (0,1); (d) theoretical exponential density with mean 1.

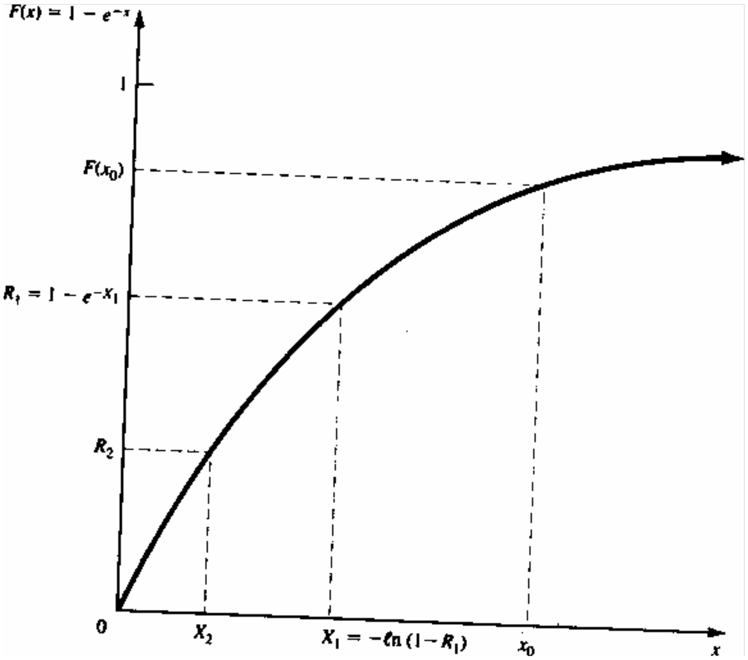


Figure 5.2. Graphical view of the inverse transform technique.

and Figure 5.1(b) is a histogram of the 200 values, X1,X2,…,X200, computed by Equation (5.2). Compare these empirical histograms with the theoretical density functions in Figure 5.1(c) and (d). As illustrated here, a histogram is an estimate of the underlying density function. (This fact is used in Chapter 9 as a way to identify distributions.)

Figure 5.2 gives a graphical interpretation of the inverse transform technique. The cdf shown is F(x) = 1- e an exponential distribution with rate . To generate a valueX1 with cdf F(X), first a random number Ri between 0 and 1 is generated, a horizontal line is drawn from R1 to the graph of the cdf, then a vertical line is dropped to the x -axis to obtain X1, the desired result. Notice the inverse relation between R1 and X1, namely

$$Ri = 1 - e^{-X1}$$

and

$$X_1 = -\ln (1 - R_1)$$

In general, the relation is written as

$$R_1 = F (X_1)$$

and

$$X_1 = F^{-1} (R1)$$

Why does the random variable X1 generated by this procedure have the desired distribution? Pick a value JCQ and compute the cumulative probability

$$P(X1 < x0) = P(R1 < F(x_0)) = F(X0) \qquad (5.4)$$

To see the first equality in Equation (8.4), refer to Figure 5.2, where the fixed numbers x0 and F(X0) are drawn on their respective axes. It can be seen that X1 < xo when and only when R1 < F(X0). Since 0 < F(xo) < 1, the second equality in Equation (8.4) follows immediately from the fact that *R* is uniformly distributed on (0,1). Equation (8.4) shows that the cdf of X1 is F; hence, X1 has the desired distribution.

5.1.2 Uniform Distribution :

Consider a random variable X that is uniformly distributed on the interval *[a, b].* A reasonable guess for generating X is given by

$$X = a + (b - a)R \qquad (5.5)$$

[Recall that *R* is always a random number on (0,1). The pdf of X is given by

$$f(x) = \begin{cases} 1/(b-a), & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

The derivation of Equation (5..5) follows steps 1 through 3 of Section 5.1.1:

Step 1. The cdf is given by

$$F(x) = \begin{cases} 0, & x < a \\ (x - a)/(b - a), & a \le x \le b \\ 1, & x > b \end{cases}$$

Step 2. Set $F(X) = (X - a)/(b - a) = R$

Step 3. Solving for X in terms of *R* yields $X = a + (b - a)R$, which agrees with Equation (5.5).

5.1.3 Discrete Distribution

All discrete distributions can be generated using the inverse transform technique, either numerically through a table-lookup procedure, or in some cases algebraically with the final generation scheme in terms of a formula. Other techniques are sometimes used for certain distributions, such as the convolution technique for the binomial distribution. Some of these methods are discussed in later sections. This subsection gives examples covering both empirical distributions and two of the standard discrete distributions, the (discrete) uniform and the geometric. Highly efficient table-lookup procedures for these and other distributions are found in Bratley, Fox, and Schrage [1987] and Ripley [1987].

Table 5.5. Distribution of Number of Shipments, *X*

X	*PM*	F(x)
0	0.50	0.50
1	0.30	0.80
2	0.20	1.00

Example 1 (An Empirical Discrete Distribution) :

At the end of the day, the number of shipments on the loading dock of the IHW Company (whose main product is the famous, incredibly huge widget) is either 0, 1, or 2, with observed relative frequency of occurrence of 0.50, 0.30, and 0.20, respectively. Internal consultants have been asked to develop a model to improve the efficiency of the loading and hauling operations, and as part of this model they will need to be able to generate values, X, to represent the number of shipments on the loading dock at the end of each day. The consultants decide to model X as a discrete random variable with distribution as given in Table 5.5 and shown in Figure 5.6.

The probability mass function (pmf), P(x) is given by

$$p(0) = P(X = 0) = 0.50$$
$$p(1) = P(X = 1) = 0.30$$
$$p(2) = P(X = 2) = 0.20$$

and the cdf, $F(x) = P(X < x)$, is given by

$$F(x) = \begin{array}{ll} 0, & x < 0 \\ 0.5, & 0 \le x < 1 \\ 0.8, & 1 \le x < 2 \\ 1.0 & 2 \le x \end{array}$$

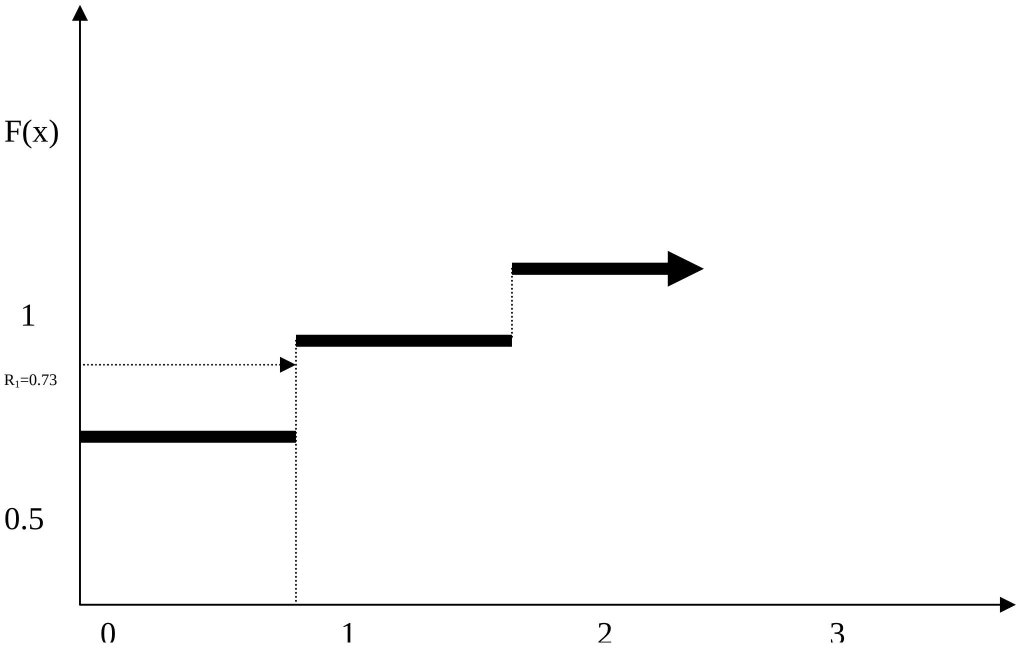


Figure 5.6. The cdf of number of shipments, *X*.

Table 5.6. Table for Generating the Discrete Variate X

I	*Input, n*	*Output, Xi*
1	0.50	0
2	0.80	1
3	1.00	2

Recall that the cdf of a discrete random variable always consists of horizontal line segments with jumps of size p(x) at those -points, x, which the random variable can assume. For example, in Figure 8.6 there is a jump of size = 0.5 at x = 0, of size p(l)=0.3 at x=1, and of size p(2) = 0.2 at x=2.

For generating discrete random variables, the inverse transform technique becomes a table-lookup procedure, but unlike the case of continuous variables, interpolation is not required. To illustrate the procedure, suppose that R1 = 0.73 is generated. Graphically, as illustrated in Figure 8.6, first locate R1 = 0.73 on the vertical axis, next draw a horizontal line segment until it hits a "jump" in cdf, and then drop a perpendicular to the horizontal axis to get the generated variate. Here R1 = 0.73 is transformed to X1 = 1. This procedure is analogous to the procedure used for empirical continuous distributions,except that the final step of linear interpolation is eliminated.

The table-lookup procedure is facilitated by construction of a table such as table 5.6. When R1 == 0.73 is generated, first find the interval in which R1 lies. In general, for R = R1, if

$$F\ (xi\text{-}1) = ri\text{-}1 < R \le ri = F(xi) \qquad (5.13)$$

then set X1 = xi. Here ro = 0, x0 =   , while x1,x2,….,xn are the possible values of the random variable, and rk = p(x1) +….. + p(xk), k = 1,2,..., n. For this example, n = 3, x1 = 0, x2 = 1, x3 = 2, and hence r1 = 0.5, r2 = 0.8, and r3 = 1.0. (Notice that rn = 1.0 in all cases.)

Since n = 0.5 < R1 = 0.73 < r2=0.8, set X1 = x2 = 1. The generation scheme is summarized as follows:

$$X = \begin{array}{ll} 0, & R \le 0.5 \\ 1, & 0.5 < R \le 0.8 \\ 2, & 0.8 < R \le 1.0 \end{array}$$

Example 8.4 illustrates the table-lookup procedure, while the next example illustrates an algebraic approach that can be used for certain distributions.

Example 2 (A Discrete Uniform Distribution) :

Consider the discrete uniform distribution on {1,2,..., k} with pmf and cdf given by

p(x) = 1/k, x = 1,2, . ..k,

and

$$F\ (x) = \begin{cases} 0, & x < 1 \\ 1/k, & 1 \le x < 2 \\ 2/k, & 2 \le x < 3 \\ . & \\ . & \\ . & \\ k\text{-}1/k, & k\text{-}1 \le x < k \\ 1, & k \le x \end{cases}$$

Let xi=i and ri = p(l) + ……. + p(xi) —F(xi) = i/k for i=1,2,..., k. Then by using Inequality (5.13) it can be seen that if the generated random number R satisfies

$$R_{i-1} = i-1 / k < R \le r_i = i/k \qquad (5.14)$$

then X is generated by setting X = i . Now, Inequality (5.14) can be solved for j:

$$i - 1 < Rk \le i$$

$$Rk \le i < Rk + 1 \qquad (5.15)$$

Let [y] denote the smallest integer > y. For example, [7.82] = 8, [5.13] = 6 and [-1,32] = -1. For $y > 0$, [y] is a function that rounds up. This notation and Inequality (5.15) yield a formula for generating X, namely

$$X = \lceil R\,k \rceil$$

For example, consider generating a random variate X, uniformly distributed on {1,2,..., 10}. The variate, X, might represent the number of pallets to be loaded onto a truck. Using Table A.I as a source of random numbers, R, and Equation (5.16) with $k = 10$ yields

$$\begin{array}{lll} R1 = 0.78, & Xi = [7.8] & = 8 \\ R_2 = 0.03, & X_2 = [0.3] & = 1 \\ R_3 = 0.23, & X_3 = [2.3] & = 3 \\ R_4 = 0.97, & X_4 = [9.7] & = 10 \end{array}$$

The procedure discussed here can be modified to generate a discrete uniform random variate with any range consisting of consecutive integers. Exercise 13 asks the student to devise a procedure for one such case.

Example 3 (The Geometric Distribution)

Consider the geometric distribution with pmf

$$p(x) = p\,(l - p)^x, \quad x = 0,1,2,\ldots$$

where $0 < p < 1$. Its cdf is given by

$$\begin{aligned} F(x) &= \textstyle\sum_{j=0}^{x} p(1 - p)^j \\ &= p\{1- (1- p)^{x+1}\} / 1 - (1 - p) \\ &= 1 - (1 - p)^{x+1} \end{aligned}$$

for x = 0, 1, 2, ... Using the inverse transform technique [i.e., Inequality (5.13)], recall that a geometric random variable X will assume the value x whenever

$$F(x - 1) = 1 - (1 - p)^x < R < 1 - (1 - p)^{x+1} = F(x) \qquad (5.19)$$

where R is a generated random number assumed 0 < R <1. Solving Inequality (5.19) for x proceeds as follows:

$$(1-p)^{x+1} \le 1 - R < (1 - p)^x$$
$$(x + 1)\ln(1 - p) \le \ln(1 - R) < x\ln(1 - p)$$

But $1 - p < 1$ implies that $\ln(1-p) < 0$. so that

$$\text{Ln}\,(1 - R) / \ln(1 - p) - 1 \le x < \ln(1 - R) / \ln(1 - p) \qquad (5.20)$$

Thus, X=x for that integer value of x satisfying Inequality (5.20) or in brief using the round-up function ┌.┐

$$X = \ln(1 - R) / \ln(1 - p) - 1 \qquad (5.21)$$

Since p is a fixed parameter, let = —1/£n(l — p). Then > 0 and, by Equation (5.21), X = [-]. By Equation (5.1), is an exponentially distributed random variable with mean , so that one way of generating a geometric variate with parameter p is to generate (by any method) an exponential variate with parameter, subtract one, and round up.Occasionally, a geometric variate X is needed which can assume values {q, q + 1, q + 2,...} with pmf p(x) = p(1 - p) (x = q, q +1,...). Such a variate, X can be generated,
using Equation (5.21), by

$$X = q + \ln(1-R) / \ln(1 - p) - 1$$

One of the most common cases is q = 1.

5.2Acceptance-Rejection Technique :

Suppose that an analyst needed to devise a method for generating random variates, X, uniformly distributed between ¼ and 1. One way to proceed would be to follow these steps:

Step1 : Generate a random number R.

Step 2a.: If R > 1/4, accept X = /?, then go to step 3.

Step 2b.: If R < 1/4, reject /?, and return to step 1.

Step 3.: If another uniform random variate on [1/4,1] is needed, repeat the procedure beginning at step 1. If not, stop.

Each time step 1 is executed, a new random number R must be generated. Step 2a is an “acceptance” and step 2b is a "rejection" in this acceptance-rejection technique. To summarize the technique, random variates (R) with some distribution (here uniform on [0,1]) are generated until some condition (R > 1/4) is satisfied. When the condition is finally satisfied, the desired random variate, X (here uniform on [1/4,1]), can be computed (X = R). This procedure can be shown to be correct by recognizing that the accepted values of R are conditioned values; that is, R itself does not have the desired distribution, but R conditioned on the event {R > 1/4} does have the desired distribution. To show this, take 1/4 < a < b < 1; then

$$P (a < R \le b \mid ¼ \le R \le 1) = P (a < R \le b) / P (¼ \le R \le 1) = b – a / 3/4 \quad (5.28)$$

which is the correct probability for a uniform distribution on [1/4, 1]. Equation (5.28) says that the probability distribution of /?, given that *R* is between 1/4 and 1 (all other values of *R* are thrown out), is the desired distribution. Therefore, if 1/4 < *R* < 1, set X = *R.*

5.2.1 Poisson Distribution :

A Poisson random variable, N, with mean *a* > 0 has pmf

$$p(n) = P(N = n) = e^{-a} a^n/ n! \quad , \qquad n = 0,1,2,…….$$

but more important, *N* can be interpreted as the number of arrivals from a Poisson arrival process in one unit of time. Recall that the inter- arrival times, *A1*, A2,... of successive customers are exponentially distributed

with rate *a* (i.e., *a* is the mean number of arrivals per unit time); in addition, an exponential variate can be generated by Equation (5.3). Thu/there is a relationship between the (discrete) Poisson distribution and the (continuous) exponential distribution, namely

$$N = n \qquad (5.29)$$

if and only if

$$A1 + A2 + \dots\dots\dots + An \le 1 < A1 + \dots.. + An + An{+}1 \quad (5.30)$$

Equation (5.29)/*N* = *n,* says there were exactly *n* arrivals during one unit of time; but relation (8.30) says that the *nth* arrival occurred before time 1 while the *(n* + l)st arrival occurred after time l) Clearly, these two statements are equivalent. Proceed now by generating exponential interarrival times until some arrival, say *n* + 1, occurs after time 1; then set *N* = *n.*

For efficient generation purposes, relation (5.30) is usually simplified by first using Equation (5.3), Ai= (—1/ α)ln Ri, to obtain

$$\sum_{i=1}^{n} – 1/\alpha \ln Ri \le 1 < \sum^{n+1}_{i=1} -1/\alpha \ln Ri$$

Next multiply through by reverses the sign of the inequality, and use the fact that a sum of logarithms is the logarithm of a product, to get

$$\ln \prod^{n}_{i=1} Ri = \sum^{n}_{i=1} \ln Ri \ge - \alpha > \sum^{n+1}_{i=1} \ln Ri = \ln \prod^{n+1}_{i=1} Ri$$

Finally, use the relation $e^{lnx}=x$ for any number *x* to obtain

$$\prod^{n}_{i=1} Ri \ge e^{-\alpha} > \prod^{n+1}_{i=1} Ri \qquad (8.31)$$

which is equivalent to relation (8.30).The procedure for generating a Poisson random variate, *N,* is given by the following steps:

Step 1. Set $n = 0$, $P = 1$.
Step 2. Generate a random number R_n+*i* and replace *P* by *P* •R_n+*i*.
Step 3. If $P < e^{-\alpha}$, then accept $N = n$. Otherwise, reject the current n, \ increase *n* by one, and return to step 2.

Notice that upon completion of step 2, *P* is equal to the rightmost expression in relation (5.31). The basic idea of a rejection technique is again exhibited; if $P > e^{\sim a}$ in step 3, then *n* is rejected and the generation process must proceed through at least one more trial.

How many random numbers will be required, on the average to generate one poisson variate, N? if N=n, then n+1 random numbers are required so the average number is given by

$$E(N+1) = \alpha \ +1$$

Which is quite large if the mean , alpha, of the poisson distribution is large.

Example 4 :

Generate three Poisson variates with mean *a =-02*. First compute £"" = $e^{\wedge\circ\prime 2}$ = 0.8187. Next get a sequence of random numbers *R* from Table A.I and follow steps 1 to 3 above:

Step l. Set n = 0, P = 1.
Step 2. R1 = 0.4357, P = 1 • R1 = 0.4357.
Step 3. Since P = 0.4357 < $e^{-\alpha}$ = 0.8187, accept N = 0.
Step 1-3. (Ri =0.4146 leads to N = 0.)
Step l. Set n = 0, P = 1.

n	R_{n+1}	p	accept/reject	Result
0	0.4357	0.4357	$P<e^{-\alpha}$ (accept)	N=0
0	0.4146	0.4146	$P<e^{-\alpha}$ (accept)	N=0
0	0.8353	0.8353	$P\geq e^{-\alpha}$ (reject)	
1	0.9952	0.8313	$P\geq e^{-\alpha}$ (reject)	
2	0.8004	0.6654	$p < e^{-\alpha}$ (accept)	N=2

Step 2. R_1= 0.8353, P = 1 - R1 = 0.8353.
Step 3. Since P > $e^{-\alpha}$, reject n = 0 and return to step 2 with n = 1.
Step 2. R_2 = 0.9952, P = R_1R_2 = 0.8313.)
Step 3. Since P > $e^{-\alpha}$, reject n = 1 and return to step 2 with n = 2.
Step 2. R3 = 0.8004, P = $R_1R_2R_3$ = 0.6654.
Step 3. Since P < $e^{-\alpha}$, accept N = 2.

five random numbers, to generate three Poisson variates here *(N* = 0, and *N = 2),* but in the long run to generate, say, 1000 Poisson variates = 0.2 it would require approximately 1000 *(a* +1) or 1200 random numbers.

6 : Input Modeling

- Input data provide the driving force for a simulation model. In the simulation of a queuing system, typical input data are the distributions of time between arrivals and service times.
- For the simulation of a reliability system, the distribution of time-to=failure of a component is an example of input data.

There are four steps in the development of a useful model of input data:

- Collect data from the real system of interest. This often requires a substantial time and resource commitment. Unfortunately, in some situations it is not possible to collect data
- Identify a probability distribution to represent the input process. When data are available, this step typically begins by developing a frequency distribution, or histogram, of the data.
- Choose parameters that determine a specific instance of the distribution family. When data are available, these parameters may be estimated from the data.
- Evaluate the chosen distribution and the associated parameters for good-of-fit. Goodness-of-fit may be evaluated informally via graphical methods, or formally via statistical tests. The chi-square and the Kolmo-gorov-Smirnov tests are standard goodness-of-fit tests. If not satisfied that the chosen distribution is a good approximation of the data, then the analyst returns to the second step, chooses a different family of distributions, and repeats the procedure. If several iterations of this procedure fail to yield a fit between an assumed distributional form and the collected data

6.1 Data Collection

- Problems are found at the end of each chapter, as exercises for the reader, in mathematics, physics, chemistry, and other technical subject texts.
- Data collection is one of the biggest tasks in solving real problem. It is one of the most important and difficult problems in simulation. And even if when data are available, they have rarely been recorded in a form that is directly useful for simulation input modeling.
- "GIGO," or "garbage-in, garbage-out," is a basic concept in computer science and it applies equally in the area of discrete system simulation. -Many are fooled by a pile of computer output or a sophisticated animation, as if these were the absolute truth.
- Many lessons can be learned from an actual experience in data collection. The first five exercises at the end of this chapter suggest some situations in which the student can gain such experience.

  The following suggestions may enhance and facilitate data collection, although they are not all – inclusive.

  1. A useful expenditure of time is in planning. This could begin by a practice or pre observing session. Try to collect data while preobserving.

2. Try to analyze the data as they are being collected. Determine if any data being collected are useless to the simulation. There is no need to collect superfluous data.

3. Try to combine homogeneous data sets. Check data for homogeneity in successive time periods and during the same time period on successive days.

4. Be aware of the possibility of data censoring, in which a quantity of interest is not observed in its entirety. This problem most often occurs when the analyst is interested in the time required to complete some process (for example, produce a part, treat a patient, or have a component fail), but the process begins prior to, or finishes after the completion of, the observation period.

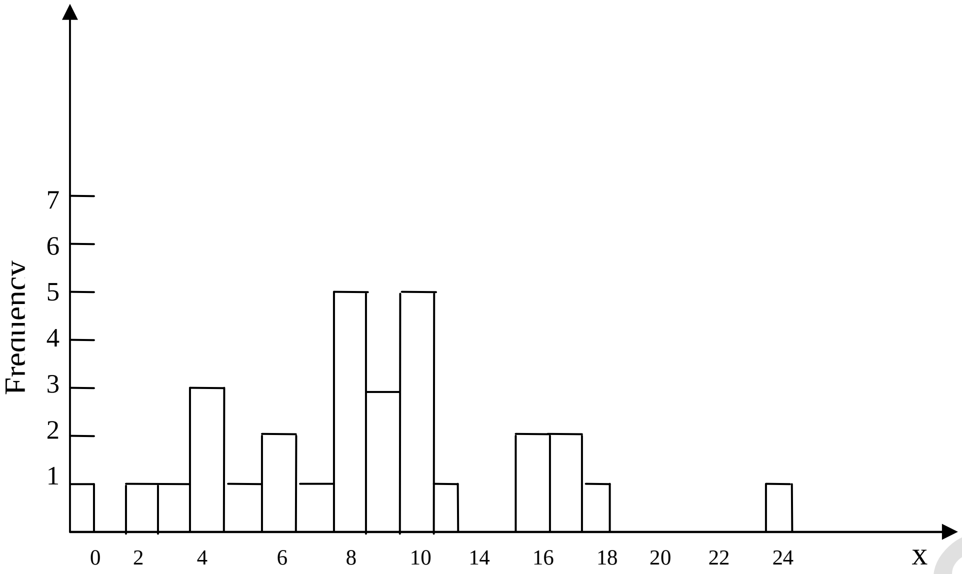
5. To determine whether there is a relationship between two variables, build a scatter diagram.

6. Consider the possibility that a sequence of observations which appear to be independent may possess autocorrelation. Autocorrelation may exist in successive time periods or for successive customers.

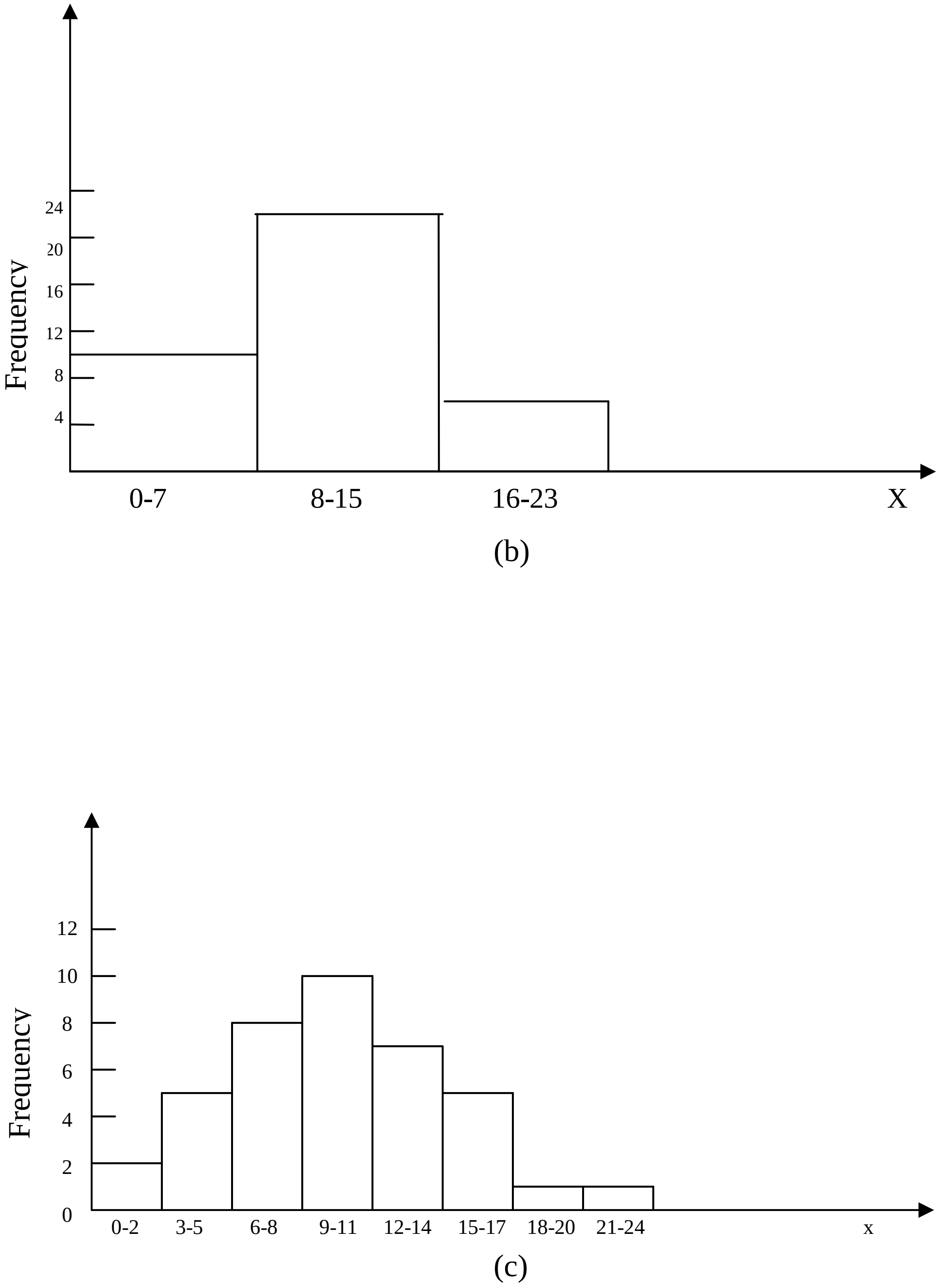
7. Keep in mind the difference between input data and output or performance data, and be sure to collect input data. Input data typically represent the uncertain quantities that are largely beyond the control of the system and will not be altered by changes made to improve the system.

Example 6.1 (The Laundromat)

- As budding simulation students, the first two authors had assignments to simulate the operation of an ongoing system. One of these systems, which seemed to be a rather simple operation, was a self-service Laundromat with 10 washing machines and six dryers.



(a)



(b)

(c)

Fig 6.1 Ragged, coarse, and appropriate histogram :
(a) original data- too ragged; (b) combining adjacent cells – too coarse;
(c) combining adjacent cells - appropriate

6.2 Identifying the Distribution with Data.

- In this section we discuss methods for selecting families of input distributions when data are available.

6.2.1 Histogram

- A frequency distribution or histogram is useful in identifying the shape of a distribution. A histogram is constructed as follows:

1. Divide the range of the data into intervals (intervals are usually of equal width; however, unequal widths however, unequal width may be used if the heights of the frequencies are adjusted).
2. Label the horizontal axis to conform to the intervals selected.
3. Determine the frequency of occurrences within each interval.
4. Label the vertical axis so that the total occurrences can be plotted for each interval.
5. Plot the frequencies on the vertical axis.

- If the intervals are too wide, the histogram will be coarse, or blocky, and its shape and other details will not show well. If the intervals are too narrow, the histogram will be ragged and will not smooth the data.
- The histogram for continuous data corresponds to the probability density function of a theoretical distribution.

Example 6.2 : The number of vehicles arriving at the northwest corner of an intersection in a 5 min period between 7 A.M. and 7:05 A.M. was monitored for five workdays over a 20-week period. Table shows the resulting data. The first entry in the table indicates that there were 12:5 min periods during which zero vehicles arrived, 10 periods during which one vehicles arrived, and so on,

Table 6:1 Number of Arrivals in a 5 Minute period

Arrivals Per period	Frequency	Arrivals Per Period	Frequency
0	12	6	7
1	10	7	5
2	19	8	5
3	17	9	3
4	10	10	3
5	8	11	1

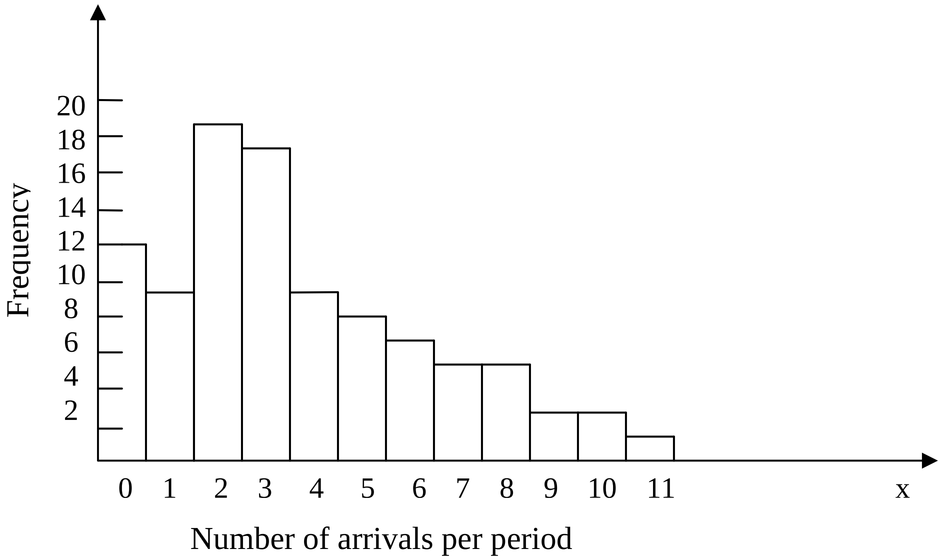
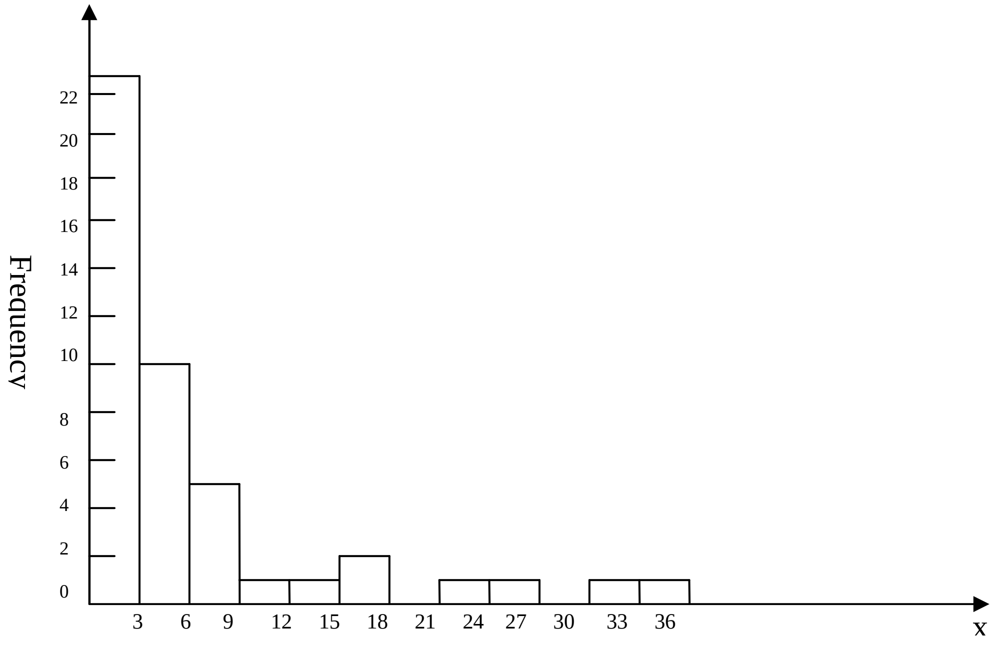


Fig 6.2 Histogram of number of arrivals per period.

Example 6.3 : Life tests were performed on a random sample of electronic chips at 1.5 times the nominal voltage, and their lifetime (or time to failure) in days was recorded:

79.919	3.081	0.062	1.961	5.845
3.027	6.505	0.021	0.013	0.123
6.769	59.899	1.192	34.760	5.009
18.387	0.141	43.565	24.420	0.433
144.695	2.663	17.967	0.091	9.003
0.941	0.878	3.371	2.157	7.579
0.624	5.380	3.148	7.078	23.96
0.590	1.928	0.300	0.002	0.543
7.004	31.764	1.005	1.147	0.219
3.217	14.382	1.008	2.336	4.562

Lifetime, usually considered a continuous variable, is recorded here to three decimal-place accuracy. The histogram is prepared by placing the data in class intervals.



Clip life

Fig 6.3 Histogram of chip life

6.2.2 Selecting the Family of Distributions

- Additionally, the shapes of these distributions were displayed. The purpose of preparing histogram is to infer a known pdf or pmf. A family of distributions is selected on the basis of what might arise in the context being investigated along with the shape of the histogram.

- Thus, if interarrival-time data have been collected, and the histogram has a shape similar to the pdf in Figure 5.9.the assumption of an exponential distribution would be warranted.

- Similarly, if measurements of weights of pallets of freight are being made, and the histogram appears symmetric about the mean with a shape like that shown in Fig 5.12, the assumption of a normal distribution would be warranted.
- The exponential, normal, and Poisson distributions are frequently encountered and are not difficult to analyze from a computational standpoint. Although more difficult to analyze, the gamma and Weibull distributions provide array of shapes, and should not be overlooked when modeling an underlying probabilistic process. Perhaps an exponential distribution was assumed, but it was found not to fit the data. The next step would be to examine where the lack of fit occurred.
- If the lack of fit was in one of the tails of the distribution, perhaps a gamma or Weibull distribution would more adequately fit the data.

- Literally hundreds of probability distributions have been created, many with some specific physical process in mind. One aid to selecting distributions is to use the physical basis of the distributions as a guide. Here are some examples:

Binomial : Models the number of successes in *n* trials, when the trials are independent with common success probability, *p;* for example, the number of defective computer chips found in a lot of *n* chips.

Negative Binomial (includes the geometric distribution) : Models the number of trials required to achieve *k* successes; for example, the number of computer chips that we must inspect to find 4 defective chips.

Poisson : Models the number of independent events that occur in a fixed amount of time or space: for example, the number of customers that arrive to a store during 1 hour, or the number of defects found in 30 square meters of sheet metal.

Normal : Models the distribution of a process that can be thought of as the sum of a number of component processes; for example, the time to assemble a product which is the sum of the times required for each assembly operation. Notice that the normal distribution admits negative values, which may be-impossible for process times.

Lognormal : Models the distribution of a process that can be thought of as the product of (meaning to multiply together) a number of component processes; for example, the rate of return on an investment, when interest is compounded, is the product of the returns for a number of periods.

Exponential : Models the time between independent events, or a process time which is memoryless (knowing how much time has passed gives no information about how much additional time will pass before the process is complete); for example, the times between the arrivals of a large number of customers who act independently of each other.

The exponential is a highly variable distribution and is sometimes overused because it often leads to mathematically tractable models. Recall that, if the time between events is exponentially distributed, then the number of events in a fixed period of time is Poisson.

Gamma : An extremely flexible distribution used to model nonnegative random variables. The gamma can be shifted away from 0 by adding a constant.

Beta : An extremely flexible distribution used to model bounded (fixed upper and lower limits) random variables. The beta can be shifted away from 0 by adding a constant and can have a larger range than [0,1] by multiplying by a constant.

Erlang : Models processes that can be viewed as the sum of several exponentially distributed processes; for example, a computer network fails when a computer and two backup computers fail, and each has a time to failure that is exponentially distributed. The Erlang is a special case of the gamma.

Weibull : Models the time to failure for components; for example, the time to failure for a disk drive. The exponential is a special case of the Weibull.

Discrete or Continuous Uniform Models complete uncertainty , since all outcomes are equally likely. This distribution is often overused when there are no data.

Triangular Models a process when only the minimum, most-likely, and maximum values of the distribution are known; for example, the minimum, most- likely, and maximum time required to test a product.

Empirical Resamples from the actual data collected; often used when no theoretical distribution seems appropriate.

- Do not ignore physical characteristics of the process when selecting distributions. Is the process naturally discrete or continuous valued? Is it bounded or is there no natural bound? This knowledge, which does not depend on data, can ,help narrow the family of distributions from which to choose.

6.2.3 Quantile-Quantile Plots

- Further, our perception of the *fit* depends on widths of the histogram intervals. But even if the intervals are well chosen, grouping of data into cells makes it difficult to compare a histogram to a continues probability density function

- If X is a random variable with cdf F, then the q-quintile of X is that y such that $F(y) = P(X < y) = q$, for $0 < q < 1$. When F has an invererse, we write $y = F^{-1}(q)$.
- Now let $\{Xi, i = 1, 2,\ldots,n\}$ be a sample of data from X. Order the observations from the smallest to the largest, and denote these as $\{yj, j = 1,2 ,,,n\}$, where $y1 < y2 < \ldots.. < y_n$- Let j denote the ranking or order number. Therefore, $j = 1$ for the smallest and $j = n$ for the largest. The q-q plot is based on the fact that y1 is an estimate of the $(j — 1/2)/n$ quantile of X other words,

$$\text{Yj is approximately } F^{-1}\left(\frac{J - ½}{n}\right)$$

- Now suppose that we have chosen a distribution with cdf F as a possible representation of the distribution of X. If F is a member of an appropriate family of distributions, then a plot of y_j versus $F^{-1}((j — 1/2)/n)$ *will be approximately a straight line.*

6.3 Parameter Estimation

- After a family of distributions has been selected, the next step is to estimate the parameters of the distribution. Estimators for many useful distributions are described in this section. In addition, many software packages—some of them integrated into simulation languages—are now available to compute these estimates.

6.3.1 Preliminary Statistics: Sample Mean and Sample Variance

- In a number of instances the sample mean, or the sample mean and sample variance, are used to estimate of the parameters of hypothesized distribution; see Example 9.4. In the following paragraphs, three sets of equations are given for computing the sample mean and sample variance, -Equations (9.1) and (9.2). Equations (9.3}.and (9.4}. are used when the data are discrete and have been grouped in frequency distribution. Equations (9.5) and (9.6) are used when the data are discrete or continuous and-have been placed in class intervals. Equations (9.5) and (9.6) are approximations and should be used only when the raw data are unavailable.

- If the observations in a sample of size *n* are $X_1, X_2, ..., X_n$, the sample mean (*X*) is defined by

$$\overline{X} = \frac{\sum^{n}_{i=1} Xi}{n} \qquad 9.1$$

and the sample variance, s^2 is defined by

$$S^2 = \frac{\sum^{n}_{i=1} Xi^2 - n\overline{X}^2}{n - 1} \qquad 9.2$$

If the data are discrete and grouped in frequency distribution, Equation (9.1) and (.2) can be modified to provide for much greater computational efficiency, The sample mean can be computed by

$$\overline{X}^2 = \frac{\sum^{n}_{j=1} fjXj}{n} \qquad 9.3$$

And the sample variance by

$$X^2 = \frac{\sum^{k}_{j=1} fjXj^2 - n\overline{X}^2}{n - 1} \qquad 9.4$$

where *k* is the number of distinct values of *X* and fj is the observed frequency of the value Xj, of *X*.

6.3.2 *Suggested Estimators*

- Numerical estimates of the distribution parameters are needed to reduce the family of distributions to a specific distribution and to test the resulting hypothesis.

- These estimators are the maximum-likelihood estimators based on the raw data. (If the data are in class intervals, these estimators must be modified.)

- The triangular distribution is usually employed when no data are available, with the parameters obtained from educated guesses for the minimum, most likely, and maximum possible value's; the uniform distribution may also be used in this way if only minimum and maximum values are available.

- Examples of the use of the estimators are given in the following paragraphs. The reader should keep in mind that a parameter is an unknown constant, but the estimator is a statistic or random variable because it depends on the sample values. To distinguish the two clearly, if, say, a parameter is denoted by a, the estimator will be denoted by α.

6.4 Goodness-of-Fit Tests

- These two tests are applied in this section to hypotheses about distributional forms of input data.
  Goodness-of-fit tests provide help full guidance for **evaluating** the suitability of a potential input model.
- However, since there is no single correct distribution in a real application, you should not be a slave to the verdict of such tests.
- It is especially important to understand the effect of sample size. If very little data are available, then a goodness-of-fit test is unlikely to reject any candidate distribution; but if a lot of data are available, then a goodness-of-fit test will likely reject all candidate distribution.

6.4.1 Chi-Square Test

- One procedure for testing the hypothesis that a random sample of size *n* of the random variable *X* follows a specific distributional form is the chi-square goodness-of-fit test.
- This test formalizes the intuitive idea of comparing the histogram of the data to the shape of the candidate density or mass function, The test is valid for large sample sizes, for both discrete and continuous distribution assumptions, When parameters are estimated by maximum likelihood.

$$X_0^2 = \sum_{I=1}^{k} \frac{(Oi - Ei)^2}{Ei} \qquad 9.16$$

- where 0, is the observed frequency in the ith class interval and Ei, is the expected frequency in that class interval. The expected frequency for each class interval is computed as Ei=npi, where pf is the theoretical, hypothesized probability associated with the ith class interval.
- It can be shown thatX02 approximately follows the chi-square distribution with k-s-1 degrees of freedom, where s represents the number of parameters of the hypothesized distribution estimated by sample statistics. The hypotheses are :

  H0: the random variable, X, conforms to the distributional assumption with the parameter(s) given by the parameter estimate(s)

H1 : the random variable X does not conform

- If the distribution being tested is discrete, each value of the random variable should be a class interval, unless it is necessary to combine adjacent class intervals to meet the minimum expected cell-frequency requirement. For the discrete case, if combining adjacent cells is not required,

$$Pi = P(X_I) = P(X = X_i)$$

Otherwise, pi, is determined by summing the probabilities of appropriate adjacent cells.

- If the distribution being tested is continuous, the class intervals are given by $[a_{i-1},ai)$, , where ai-1 and ai, are the endpoints of the *ith* class interval. For the continuous case with assumed pdf f(x), or assumed cdf *F(x),* pi, can be computed By

$$Pi = \int_{ai-1}^{ai} f(x)\,dx = F(a_i) - F(a_{i-1})$$

6.4.2 Chi-Square Test with Equal Probabilities

- If a continuous distributional assumption is being tested, class intervals that are equal in probability rather than equal in width of interval should be used.

- Unfortunately, there is as yet no method for deter mining the; probability associated with each interval that maximize the; power of a test o f a given size.

$$Ei = n\,p\,i \geq 5$$

- Substituting for p i yields $n/k \geq 5$

- and solving for k yields $k \leq n/5$

6.4.3 Kolmogorov - Smirnov Goodness-of-Fit Test

- The chi-square goodness-of-fit test can accommodate the estimation of parameters from the data with a resultant decrease in the degrees of freedom (one for J each parameter estimated). The chi-square test requires that the data be placed in class intervals, and in the case of continues distributional assumption, this grouping is arbitrary.

- Also, the distribution of the chi-square test statistic is known only approximately, and the power of the test is sometimes rather low. As a result of these considerations, goodness-of-fit tests, other than the chi-square, are desired.

- The Kolmogorov-Smirnov test is particularly useful when sample sizes are small and when no parameters have been estimated from the data.

- (Kolmogoro-Smirnov Test for Exponential Distribution)

- Suppose that 50 interarriaval times (in minutes) are collected over the following 100 minute interval (arranged in order of occurrence) :

0.44	0.53	2.04	2.74	2.00	0.30	2.54	0.52	2.02	1.89	1.53	0.21
2.80	0.04	1.35	8.32	2.34	1.95	0.10	1.42	0.46	0.07	1.09	0.76
5.55	3.93	1.07	2.26	2.88	0.67	1.12	0.26	4.57	5.37	0.12	3.19
1.63	1.46	1.08	2.06	0.85	0.83	2.44	2.11	3.15	2.90	6.58	0.64

Ho : the interarrival times are exponentially distributed
H1: the interarrival times are not exponentially distributed

- The data were collected over the interval 0 to T = 100 min. It can be shown that if the underlying distribution of interarrival times { T1, T2, … } is exponential, the arrival times are uniformly distributed on the interval (0,T).
- The arrival times T1, T1+T2, T1+T2+T3,…..,T1+…..+T50 are obtained by adding interarrival times.
- On a (0,1) interval, the points will be [T1/T, (T1+T2)/T,…..,(T1+….+T50)/T].

0.0044	O.0097	0.0301	0.0575	0.0775	0.0805	0.1059	0.1111	0.1313	0.1502
0.1655	0.1676	0.1956	0.1960	0.2095	0.2927	0.3161	0.3356	0.3366	0.3508
0.3553	0.3561	0.3670	0.3746	0.4300	0.4694	0.4796	0.5027	0.5315	0.5382
0.5494	0.5520	0.5977	0.6514	0.6526	0.6845	0.7008	0.7154	0.7262	0.7468
0.7553	0.7636	0.7880	0.7982	0.8206	0.8417	0.8732	0.9022	0.9680	0.9744

6.4.4 p-Values and "Best Fits"

- To apply a goodness-of-fit test a significance level must be chosen. The traditional significance levels are 0.1,0.05, and 0.01. Prior to the availability of high-speed computing, having a small set of standard values made it possible to produce tables of useful critical values. Now most statistical software computes critical values as needed, rather than storing them in tables. Thus, if the analyst prefers a level of significance of, say, 0.07, then he or she can choose it.
- However, rather than require a prespecified significance level, many software packages compute a *p-value* for the test statistic. The p-value is the significance level at which one would just reject H0 for the given value of the test statistic. while a small p-value suggests a poor fit (to accept we would have to insist on almost no risk).
- The p-value can be viewed as a measure of fit, with larger values being better. This suggests that we could fit every distribution at our disposal, compute a test statistic for each fit, and then choose the distribution that yields the largest p-value.
- While we know of no input modeling software that implements this specific algorithm, many such packages do include a "best-fit'" option in which the software recommends an input model to the user based on evaluating all feasible models.
- In the end some summary measure of fit, like the p-value, is used to rank the distributions. There is nothing wrong with this, but there are several things to keep in mind:
  1. The software may know nothing about the physical basis of the data and that information can suggest distribution families that are appropriate.

2. Recall that both the Erlang and the exponential distributions are special cases of the gamma, while the exponential is also a special case of the more flexible Weibull. Automated best-fit procedures tend to choose the more flexible distributions (gamma and Weibull over Erlang and exponential) because the extra flexibility allows closer conformance to the data and a better summary measure of fit. But again, close conformance to the data may not always lead to the most appropriate input model.

3. A summary statistic, like the p-value, is just that, a summary measure. It says little or nothing about where the lack of fit occurs (in the body of the distribution, in the right tail or in the left tail). A human, using graphical tools, can see where the lack of fit occurs and decide whether or not it is important for the application at hand.

6.5 Selecting Input Models without Data

- Unfortunately. it is often necessary in practice to develop a simulation model for demonstration purposes or a preliminary study—before any i data are available.) In this case the modeler must be resourceful in choosing input models and must carefully check the sensitivity of results to the models.

Engineering data : Often a product or process has performance ratings pro vided by the manufacturer.

Expert option : Talk to people who are experienced with the procesws or similar processes. Often they can provide optimistic, pessimistic and most likely times.

Physical or conventional limitations : Most real processes have physical limit on performance. Because of company policies, there may be upper limits on how long a process may take. Do not ignore obvious limits or bound: that narrow the range of the input process.

The nature of the process It can be used to justify a particular choice even when no data are available.

6.6 Multivariate and Time-Series Input Models

The random variables presented were considered to be independent of any other variables within the context of the problem. However, variables may be related, and if the variables appear in a simulation model as inputs, the relationship should be determined and taken into consideration.

7. Verification and Validation of Simulation Models

- One of the most important and difficult tasks facing a model developer is the verification and validation of the simulation model.

- It is the job of the model developer to work closely with the end users throughout the period (development and validation to reduce this skepticism and to increase the credibility.

- The goal of the validation process is twofold:

1: To produce a model that represents true system behavior closely enough for the model to be used as a substitute for the actual system for the purpose of experimenting with system.

2: To increase an acceptable, level the credibility of the model ,so that the model will be used by managers and other decision makers. |

- The verification and validation process consists of the following components:-

1:Verification is concerned with building the model right. It is utilized in comparison of the conceptual model to the computer representation that implements that conception. It asks the questions: Is the model implemented correctly in the computer? Are the input parameters and logical structure of the model correctly represented?

2: Validation is concerned with building the right model. It is utilized to determine that a model is an accurate representation of the real system. It is usually achieved through the calibration of the model.

7.1 Model Building, Verification, and Validation

- The first step in model building consists of observing the real system and the interactions among its various components and collecting data on its behavior. Operators, technicians, repair and maintenance personnel, engineers, supervisors, and managers under certain aspects of the system which may be unfamiliar to others. As model development proceeds, new questions may arise, and the model developers will return, to this step of learning true system structure and behavior.

- The second step in model building is the construction of a conceptual model - a collection of assumptions on the components and the structure of the system, plus hypotheses on the values of model input parameters, illustrated by the following figure.

- The third step is the translation of the operational model into a computer recognizable form- the computerized model.

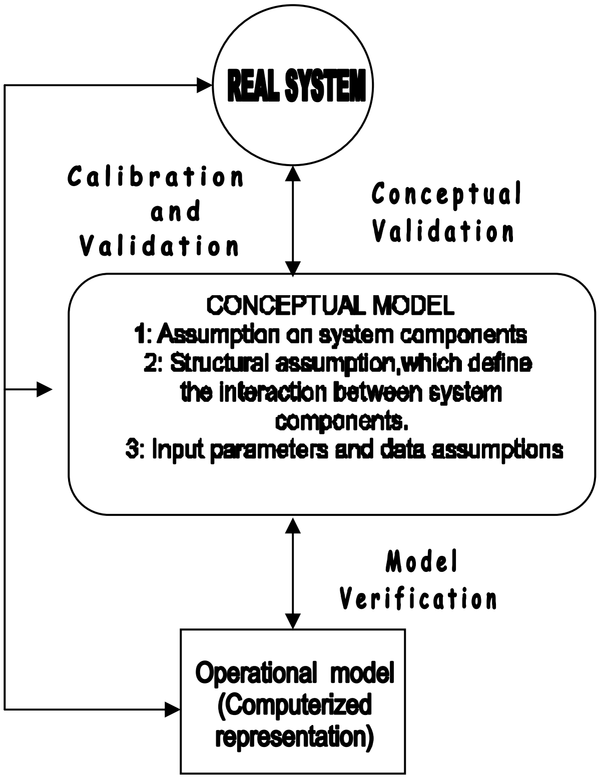

REAL SYSTEM
Calibration and Validation
Conceptual Validation
CONCEPTUAL MODEL
1: Assumption on system components
2: Structural assumption,which define the interaction between system components.
3: Input parameters and data assumptions
Model Verification
Operational model (Computerized representation)

Fig. 7.1 **Model building, Verification and Validation**

7.2 Verification of Simulation Models

The purpose of model verification is to assure that the conceptual model is reflected accurately in the computerized representation.

The conceptual model quite often involves some degree of abstraction about system operations, or some amount of simplification of actual operations.

Many common-sense suggestions can be given for use in the verification process:-

1: Have the computerized representation checked by someone other than its developer.

2: Make a flow diagram which includes each logically possible action a system can take when an event occurs, and follow the model logic for each a for each action for each event type.

3: Closely examine the model output for reasonableness under a variety of settings of input parameters.

4. Have the computerized representation print the input parameters at the end of the simulation to be sure that these parameter values have not been changed inadvertently.

5. Make the computerized representation of self-documenting as possible.

6. If the computerized representation is animated, verify that what is seen in the animation jmitates the actual system.

7. The interactive run controller (IRC) or debugger is an essential component of successful simulation model building. Even the best of simulation analysts makes mistakes or commits logical errors when building a model. The IRC assists in finding and correcting those errors in the follow ways:

(a) The simulation can be monitored as it progresses.

(b) Attention can be focused on a particular line of logic or multiple lines of logic that constitute a procedure or a particular entity.

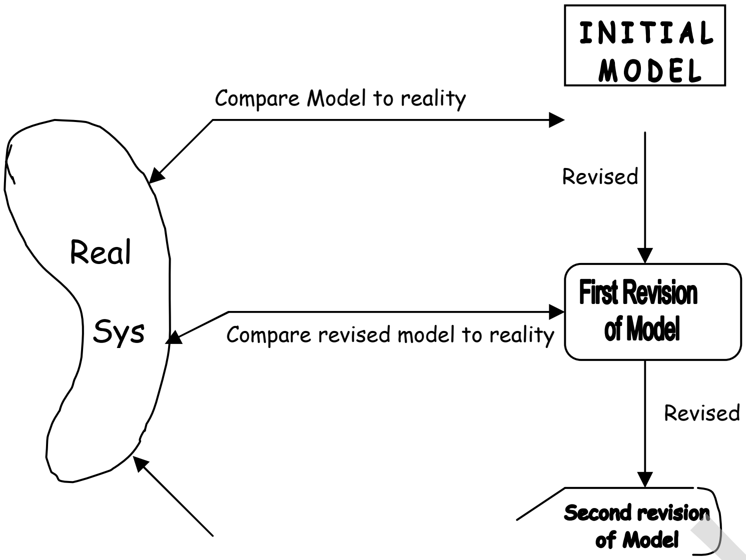
(c) Values of selected model components can be observed. When the simulation has paused, the current value or status of variables, attributes, queues, resources, counters, etc., can be observed.

(d) The simulation can be temporarily suspended, or paused, not only to view information but also to reassign values or redirect entities.

8. Graphical interfaces are recommended for accomplishing verification & validation .

7.3 Calibration and Validation of Models

- Verification and validation although are conceptually distinct, usually are conducted simultaneously by the modeler.
- Validation is the overall process of comparing the model and its behavior to the real system and its behavior.
- Calibration is the iterative process of comparing the model to the real system, making adjustments to the model, comparing again and so on.
- The following figure 7.2 shows the relationship of the model calibration to the overall validation process.
- The comparison of the model to reality is carried out by variety of test
- Test are subjective and objective.
- Subjective test usually involve people, who are knowledgeable about one or more aspects of the system, making judgments about the model and its output.
- Objective tests always require data on the system's behavior plus the corresponding data produced by the model.



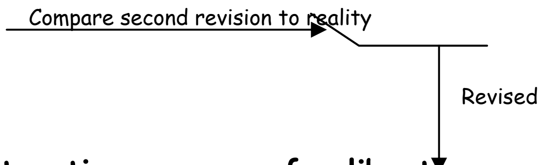


Fig. 7.2 Iterative process of calibration a model

- A possible criticism of the calibration phase, were it to stop at point, ie., the model has been validated only for the one data set used; that is, the model has been "fit" to one data set

- Validation is not an either/or proposition—no model is ever totally representative of the system under study. In addition, each revision of the model, as in the Figure above involves some cost, time, and effort.

As an aid in the validation process, Naylor and Finger [1967] formulated a three step approach which has been widely followed:-

1. Build a model that has high face validity.
2. Validate model assumptions.
3. Compare the model input-output transformations to corresponding input-output transformations for the real system.

7.3.1 FACE VALIDITY

- The first goal of the simulation modeler is to construct a model that appears reasonable on its face to model users and others who are knowledgeable about the real system being simulated.

- The users of a model should be involved in model construction from its conceptualization to its implementation *to* ensure that a high degree of realism is built into the model through reasonable assumptions regarding system structure, and reliable data.

- Another advantage of user involvement is the increase in the models perceived validity or credibility without which manager will not be willing to trust simulation results as the basis for decision making.

- Sensitivity analysis can also be used to check model's face validity.

- The model user is asked if the model behaves in the expected way when one or more input variables is changed.

- Based on experience and observations on the real system the model user and model builder would probably have some notion at least of the direction of change in model output when an input variable is increased or decreased.

- The model builder must attempt to choose the most critical input variables for testing if it is too expensive or time consuming to vary all input variables.

7.3.2 Validation of Model Assumptions

***Model assumptions fall into two general classes: structural assumptions and data assumptions.**

*Structural assumptions involve questions of how the system operates and usually involve simplification and abstractions of reality.

*For example, consider the customer queuing and service facility in a bank. Customers may form one line, or there may be an individual line for each teller. If there are many lines, customers may be served strictly on a first-come, first-served basis, or some customers may change lines if one is moving faster. The number of tellers may be fixed or variable. These structural assumptions should be verified by actual observation during appropriate time periods together with discussions with managers and tellers regarding bank policies and actual implementation of these policies.

*Data assumptions should be based on the collection of reliable data and correct statistical analysis of the data.

*data were collected on:

1. Inter arrival times of customers during several 2-hour periods of peak loading ("rush-hour" traffic)
2. Inter arrival times during a slack period
3. Service times for commercial accounts
4. Service times for personal accounts.

*The procedure for analyzing input data consist of three steps:-

1: Identifying the appropriate probability distribution.

2: Estimating the parameters of the hypothesized distribution .

3: Validating the assumed statistical model by goodness – of – fit test such as the chi-square test, KS test and by graphical methods.

10.3.3 Validating Input-Output Transformation:-

- In this phase of validation process the model is viewed as input – output transformation .
- That is, the model accepts the values of input parameters and transforms these inputs into output measure of performance. It is this correspondence that is being validated.
- Instead of validating the model input-output transformation by predicting the future ,the modeler may use past historical data which has been served for validation purposes that is, if one set has been used to develop calibrate the model, its recommended that a separate data test be used as final validation test.
- Thus accurate " prediction of the past" may replace prediction of the future for purpose of validating the future.
- A necessary condition for input-output transformation is that some version of the system under study exists so that the system data under at least one set of input condition can be collected to compare to model prediction.
- If the system is in planning stage and no system operating data can be collected, complete input-output validation is not possible.
- Validation increases modeler's confidence that the model of existing system is accurate.
- Changes in the computerized representation of the system, ranging from relatively minor to relatively major include :

  1: Minor changes of single numerical parameters such as speed of the machine, arrival rate of the customer etc.

  2: Minor changes of the form of a statistical distribution such as distribution of service time or a time to failure of a machine.

  3: Major changes in the logical structure of a subsystem such as change in queue discipline for waiting-line model, or a change in the scheduling rule for a job shop model.

  4: Major changes involving a different design for the new system such as computerized inventory control system replacing a non computerized system .

If the change to the computerized representation of the system is minor such as in items one or two these change can be carefully verified and output from new model can be accepted with considerable confidence.

Partial validation of substantial model changes in item three and four may be possible.

7.3.4: Input-Output Validation: Using Historical Input Data

When using artificially generated data as input data the modeler expects the model produce event patterns that are compatible with, but not identical to, the event patterns that occurred in the real system during the period of data collection.

- Thus, in the bank model, artificial input data $\{X\backslash_{n}, X_{2n}, n = 1,2, , .\}$ for inter arrival and service times were generated and replicates of the output data Y*2 w*ere compared to what was observed in the real system
- An alternative to generating input data is to use the actual historical record, $\{A_n, S_n, n = 1,2,...\}$, to drive simulation model and then to compare model output to system data.
- To implement this technique for the bank model, the data $Ai, A_2,..., S1\ S2$ would have to be entered into the model into arrays, or stored on a file to be read as the need arose.
- To conduct a validation test using historical input data, it is important that all input data $(A_n, S_n,...)$ and all the system response data, such as average delay(Z2), be collected during the same time period.
- Otherwise, comparison of model responses to system responses, such as the comparison of average delay in the model (Y_2) to that in the system (Z2), could be misleading.
- responses $(Y_2$ and $22)$ depend on the inputs $(A_n$ and $S_n)$ as well as on the structure of the system, or model.
- Implementation of this technique could be difficult for a large system because of the need for simultaneous data collection of all input variables and those response variables of primary interest.

7.3.5: Input-Output Validation: Using a Turing Test

*In addition to statistical tests, or when no statistical test is readily applicable

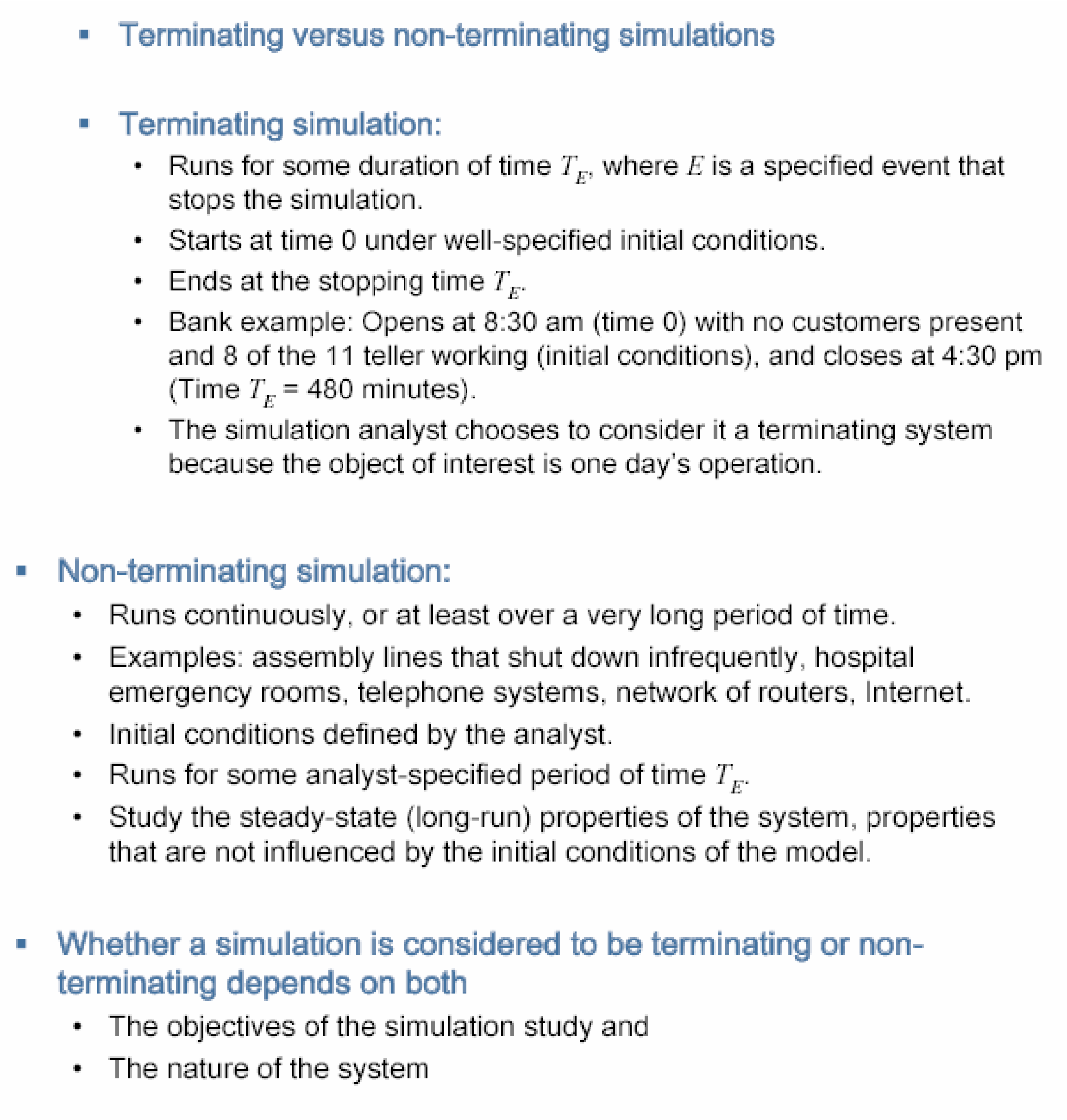
persons knowledgeable about system behavior can be used to compare model output to system output.

*For example, suppose that five reports of system performance over five different days are prepared, and simulation output are used to produce five "fake" reports. The 10 reports should all be in exactly in the same format and should contain information of the type that manager and engineer have previously seen on the system.

- The ten reports are randomly shuffled and given to the engineers , who is asked to decide which report are fake and which are real.
- If engineer identifies substantial number of fake reports the model builder questions the engineer and uses the information gained to improve the model.
- If the engineer cannot distinguish between fake and real reports with any consistency ,the modeler will conclude that this test provides no evidence of model inadequacy .
- This type of validation test is called as TURING TEST.

8. OUTPUT ANALYSIS OF A MODEL

8.1 Types of simulation w.r.t Output Analysis



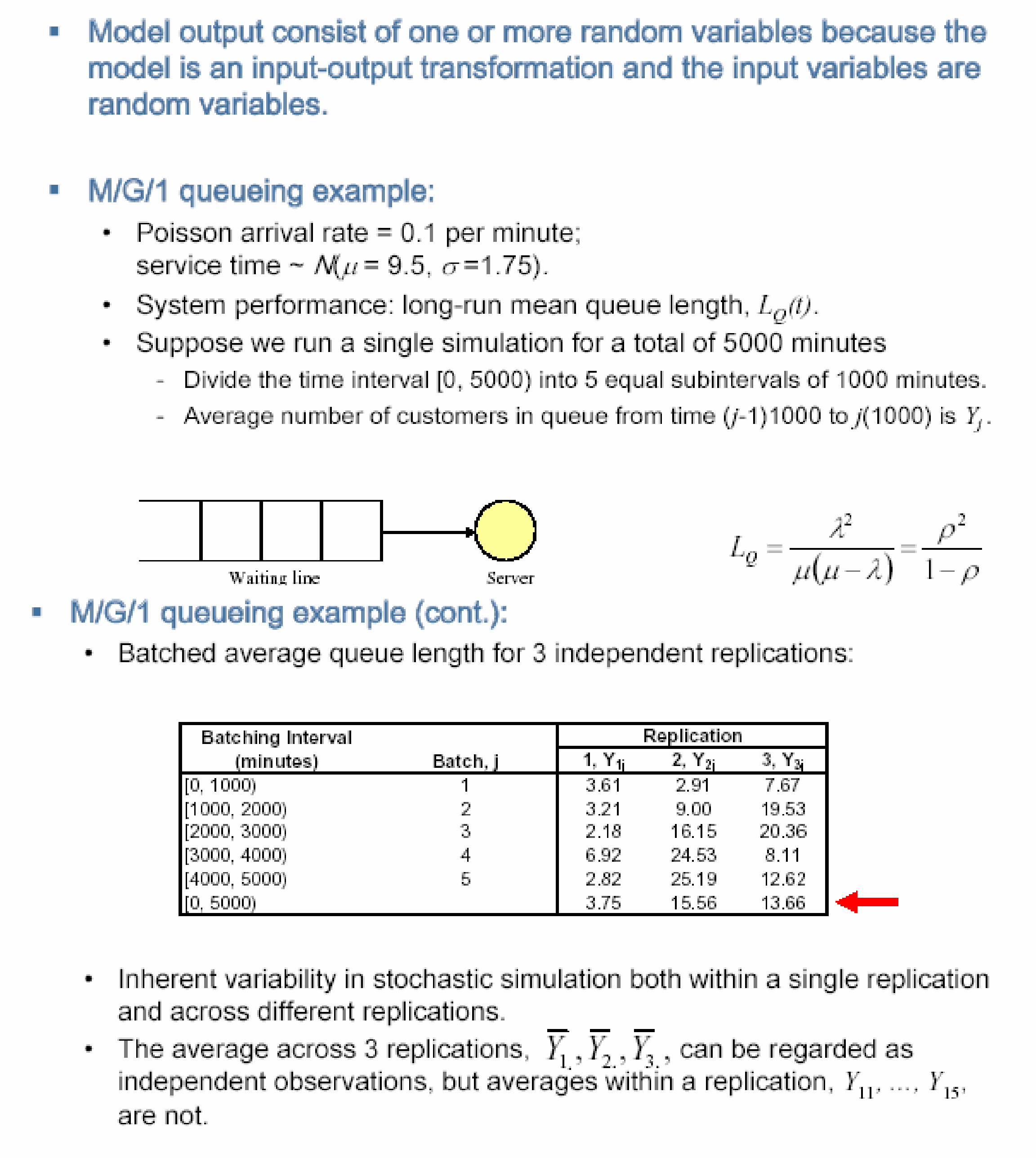
- Terminating versus non-terminating simulations

- Terminating simulation:
  - Runs for some duration of time T_E, where E is a specified event that stops the simulation.
  - Starts at time 0 under well-specified initial conditions.
  - Ends at the stopping time T_E.
  - Bank example: Opens at 8:30 am (time 0) with no customers present and 8 of the 11 teller working (initial conditions), and closes at 4:30 pm (Time T_E = 480 minutes).
  - The simulation analyst chooses to consider it a terminating system because the object of interest is one day's operation.

- Non-terminating simulation:
  - Runs continuously, or at least over a very long period of time.
  - Examples: assembly lines that shut down infrequently, hospital emergency rooms, telephone systems, network of routers, Internet.
  - Initial conditions defined by the analyst.
  - Runs for some analyst-specified period of time T_E.
  - Study the steady-state (long-run) properties of the system, properties that are not influenced by the initial conditions of the model.

- Whether a simulation is considered to be terminating or non-terminating depends on both
  - The objectives of the simulation study and
  - The nature of the system

8.2 Stochastic Nature of output data



- Model output consist of one or more random variables because the model is an input-output transformation and the input variables are random variables.

- M/G/1 queueing example:
  - Poisson arrival rate = 0.1 per minute; service time ~ $N(\mu = 9.5, \sigma = 1.75)$.
  - System performance: long-run mean queue length, $L_Q(t)$.
  - Suppose we run a single simulation for a total of 5000 minutes
    - Divide the time interval [0, 5000) into 5 equal subintervals of 1000 minutes.
    - Average number of customers in queue from time $(j-1)1000$ to $j(1000)$ is Y_j.

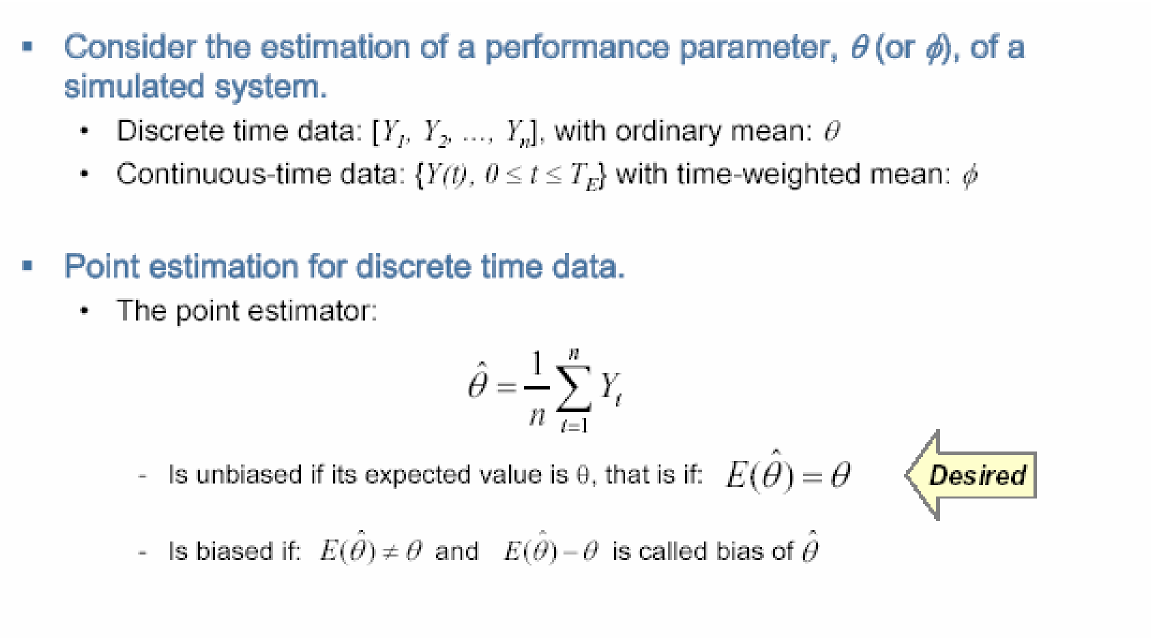
$$L_Q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$$

- M/G/1 queueing example (cont.):
  - Batched average queue length for 3 independent replications:

Batching Interval (minutes)	Batch, j	Replication 1, Y_{1j}	2, Y_{2j}	3, Y_{3j}
[0, 1000)	1	3.61	2.91	7.67
[1000, 2000)	2	3.21	9.00	19.53
[2000, 3000)	3	2.18	16.15	20.36
[3000, 4000)	4	6.92	24.53	8.11
[4000, 5000)	5	2.82	25.19	12.62
[0, 5000)		3.75	15.56	13.66

  - Inherent variability in stochastic simulation both within a single replication and across different replications.
  - The average across 3 replications, $\bar{Y}_{1.}, \bar{Y}_{2.}, \bar{Y}_{3.}$, can be regarded as independent observations, but averages within a replication, $Y_{11}, \ldots, Y_{15}$, are not.

8.3 Measures of Performance and their Estimation

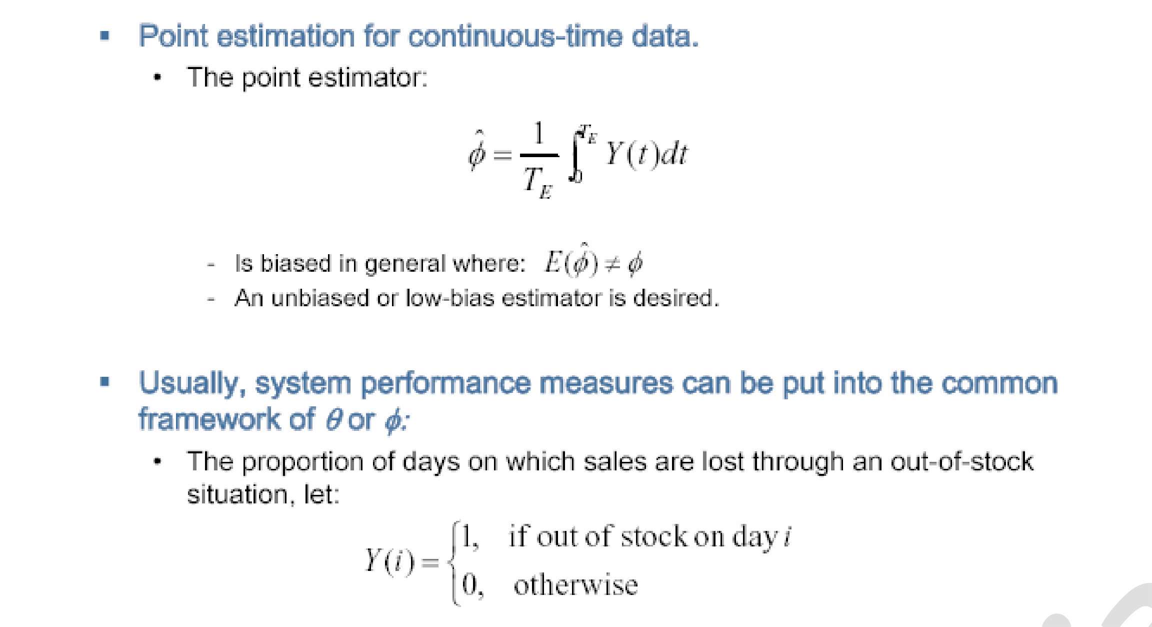


- Consider the estimation of a performance parameter, θ (or ϕ), of a simulated system.
  - Discrete time data: $[Y_1, Y_2, \ldots, Y_n]$, with ordinary mean: θ
  - Continuous-time data: $\{Y(t), 0 \le t \le T_E\}$ with time-weighted mean: ϕ
- Point estimation for discrete time data.
  - The point estimator:

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} Y_i$$

- Is unbiased if its expected value is θ, that is if: $E(\hat{\theta}) = \theta$ Desired
- Is biased if: $E(\hat{\theta}) \neq \theta$ and $E(\hat{\theta}) - \theta$ is called bias of $\hat{\theta}$

8.3.1 Point Estimation



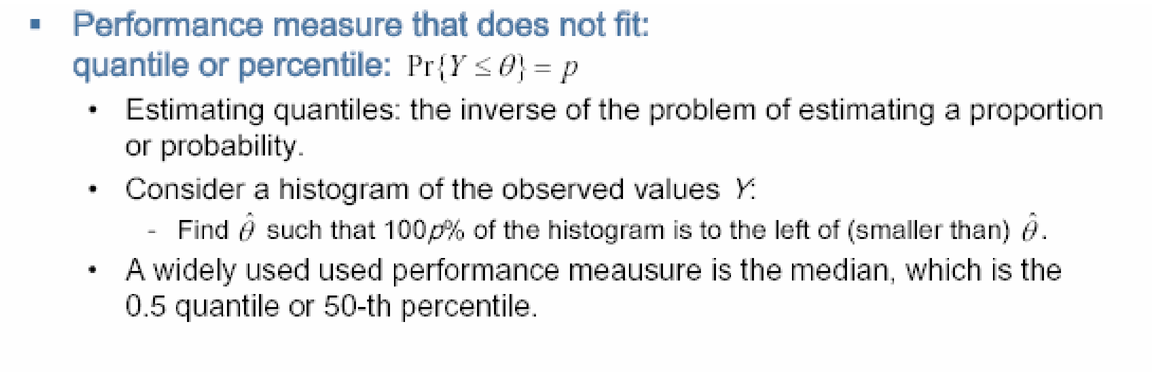
- Point estimation for continuous-time data.
  - The point estimator:

$$\hat{\phi} = \frac{1}{T_E} \int_0^{T_E} Y(t)dt$$

- Is biased in general where: $E(\hat{\phi}) \neq \phi$
- An unbiased or low-bias estimator is desired.

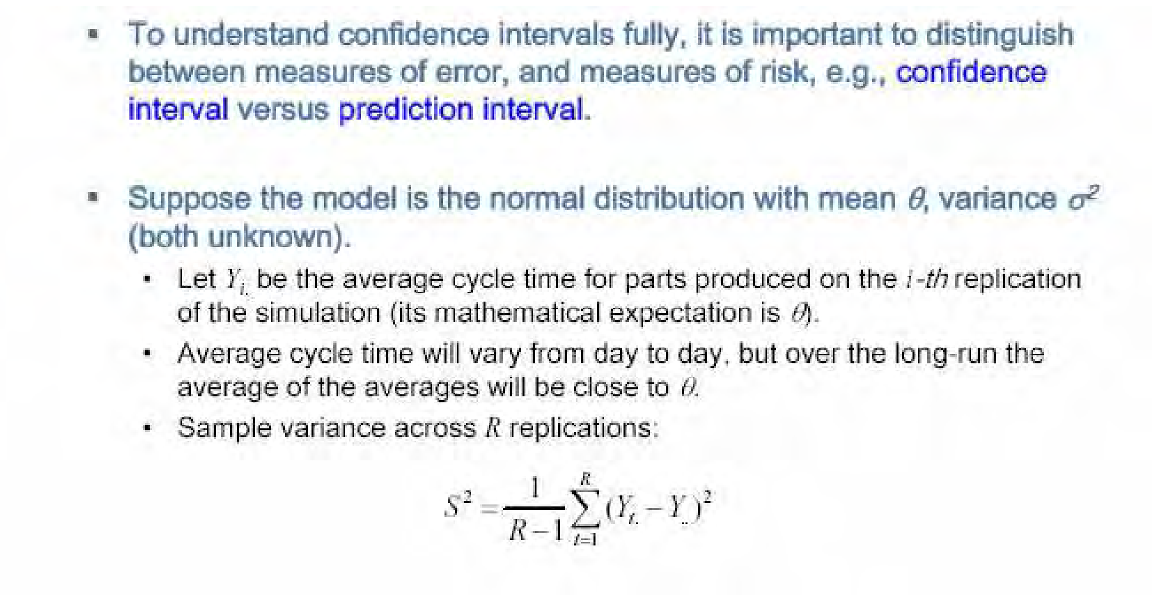
- Usually, system performance measures can be put into the common framework of θ or ϕ:
  - The proportion of days on which sales are lost through an out-of-stock situation, let:

$$Y(i) = \begin{cases} 1, & \text{if out of stock on day } i \\ 0, & \text{otherwise} \end{cases}$$



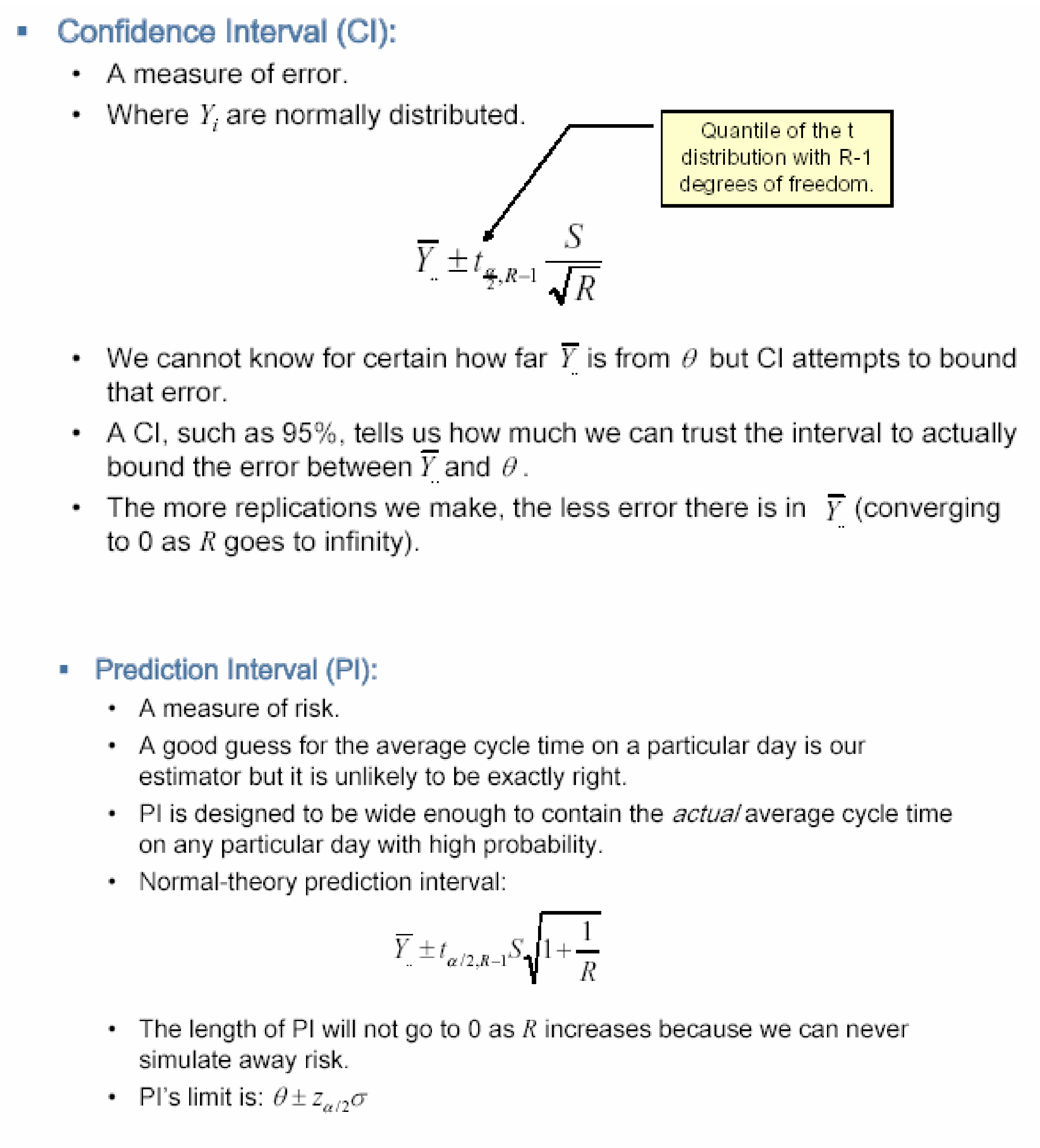
- Performance measure that does not fit: quantile or percentile: $\Pr\{Y \le \theta\} = p$
  - Estimating quantiles: the inverse of the problem of estimating a proportion or probability.
  - Consider a histogram of the observed values Y:
    - Find $\hat{\theta}$ such that $100p$% of the histogram is to the left of (smaller than) $\hat{\theta}$.
  - A widely used used performance meausure is the median, which is the 0.5 quantile or 50-th percentile.

8.3.2 Interval Estimation



- To understand confidence intervals fully, it is important to distinguish between measures of error, and measures of risk, e.g., confidence interval versus prediction interval.
- Suppose the model is the normal distribution with mean θ, variance σ^2 (both unknown).
  - Let $Y_{i.}$ be the average cycle time for parts produced on the i-*th* replication of the simulation (its mathematical expectation is θ).
  - Average cycle time will vary from day to day, but over the long-run the average of the averages will be close to θ.
  - Sample variance across R replications:

$$S^2 = \frac{1}{R-1}\sum_{i=1}^{R}(Y_{i.} - Y_{..})^2$$



- **Confidence Interval (CI):**
  - A measure of error.
  - Where Y_i are normally distributed.

Quantile of the t distribution with R-1 degrees of freedom.

$$\bar{Y}_{..} \pm t_{\frac{\alpha}{2},R-1}\frac{S}{\sqrt{R}}$$

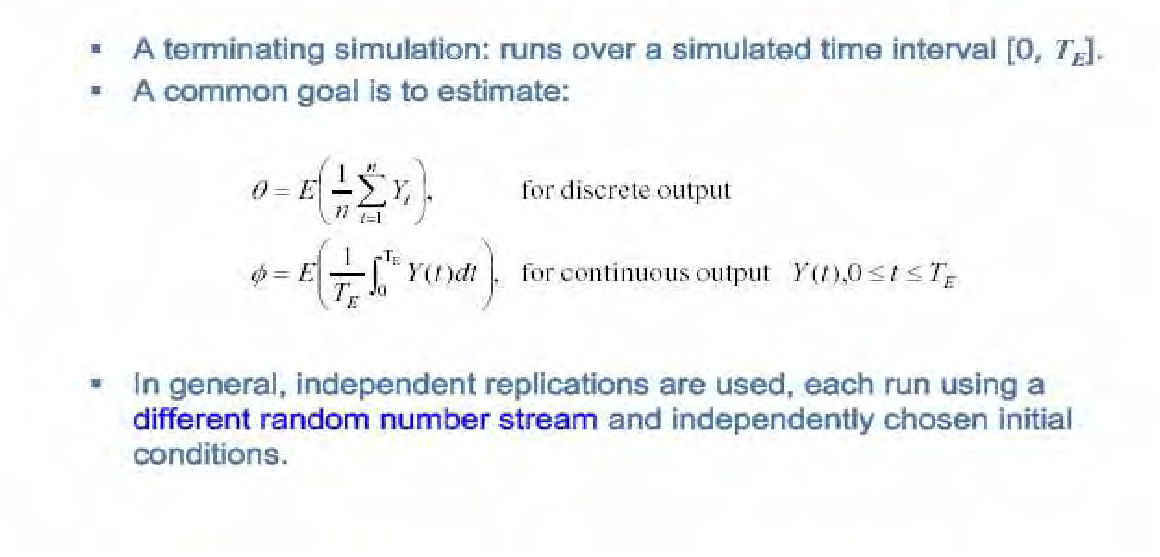
  - We cannot know for certain how far $\bar{Y}_{..}$ is from θ but CI attempts to bound that error.
  - A CI, such as 95%, tells us how much we can trust the interval to actually bound the error between $\bar{Y}_{..}$ and θ.
  - The more replications we make, the less error there is in $\bar{Y}_{..}$ (converging to 0 as R goes to infinity).

- **Prediction Interval (PI):**
  - A measure of risk.
  - A good guess for the average cycle time on a particular day is our estimator but it is unlikely to be exactly right.
  - PI is designed to be wide enough to contain the *actual* average cycle time on any particular day with high probability.
  - Normal-theory prediction interval:

$$\bar{Y}_{..} \pm t_{\alpha/2,R-1}S\sqrt{1+\frac{1}{R}}$$

  - The length of PI will not go to 0 as R increases because we can never simulate away risk.
  - PI's limit is: $\theta \pm z_{\alpha/2}\sigma$

8.4 Output Analysis for Terminating simulations



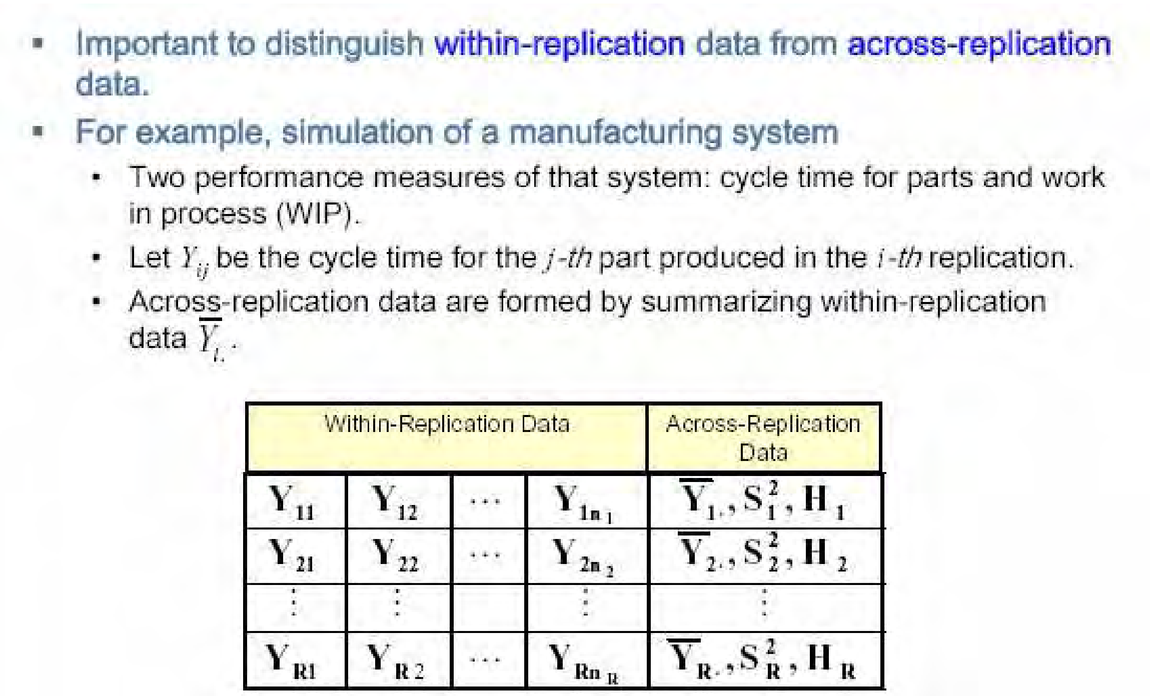
- A terminating simulation: runs over a simulated time interval [0, T_E].
- A common goal is to estimate:

$$\theta = E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right), \quad \text{for discrete output}$$

$$\phi = E\left(\frac{1}{T_E}\int_0^{T_E} Y(t)dt\right), \quad \text{for continuous output} \quad Y(t), 0 \le t \le T_E$$

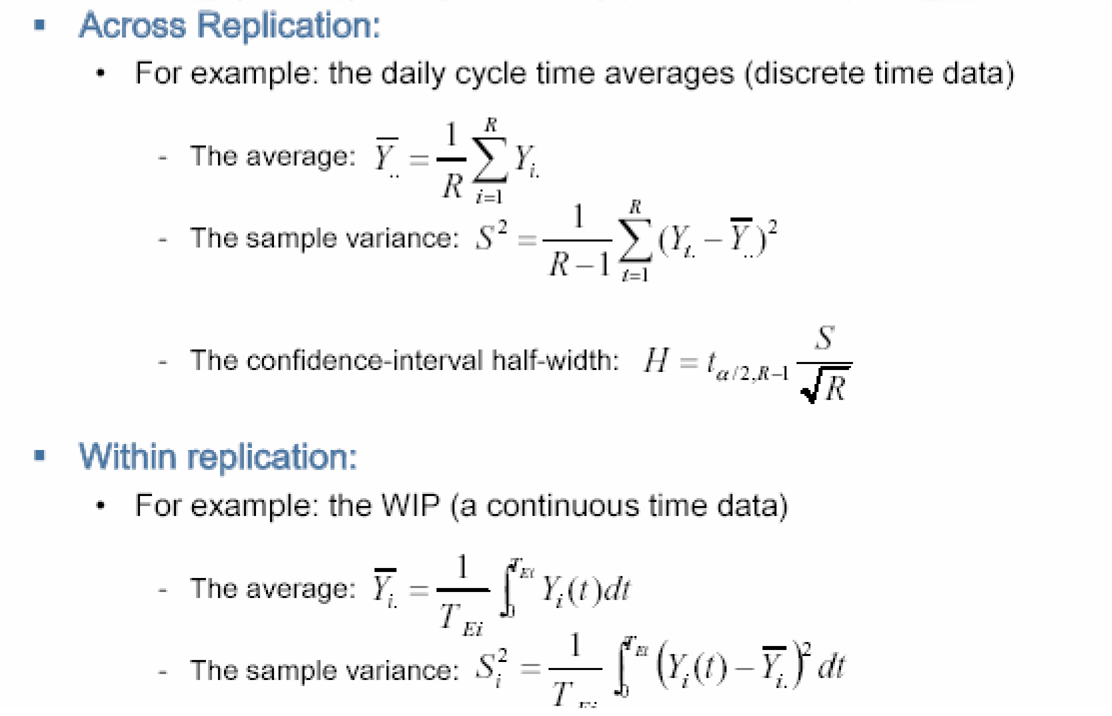
- In general, independent replications are used, each run using a different random number stream and independently chosen initial conditions.

8.4.1 Statistical background

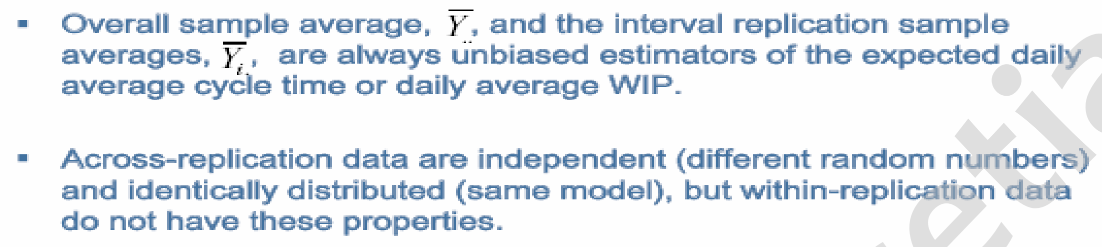


- Important to distinguish within-replication data from across-replication data.
- For example, simulation of a manufacturing system
  - Two performance measures of that system: cycle time for parts and work in process (WIP).
  - Let Y_{ij} be the cycle time for the *j-th* part produced in the *i-th* replication.
  - Across-replication data are formed by summarizing within-replication data $\bar{Y}_{i.}$.

Within-Replication Data				Across-Replication Data
Y_{11}	Y_{12}	$\cdots$	Y_{1n_1}	$\bar{Y}_{1.}, S_1^2, H_1$
Y_{21}	Y_{22}	$\cdots$	Y_{2n_2}	$\bar{Y}_{2.}, S_2^2, H_2$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
Y_{R1}	Y_{R2}	$\cdots$	Y_{Rn_R}	$\bar{Y}_{R.}, S_R^2, H_R$

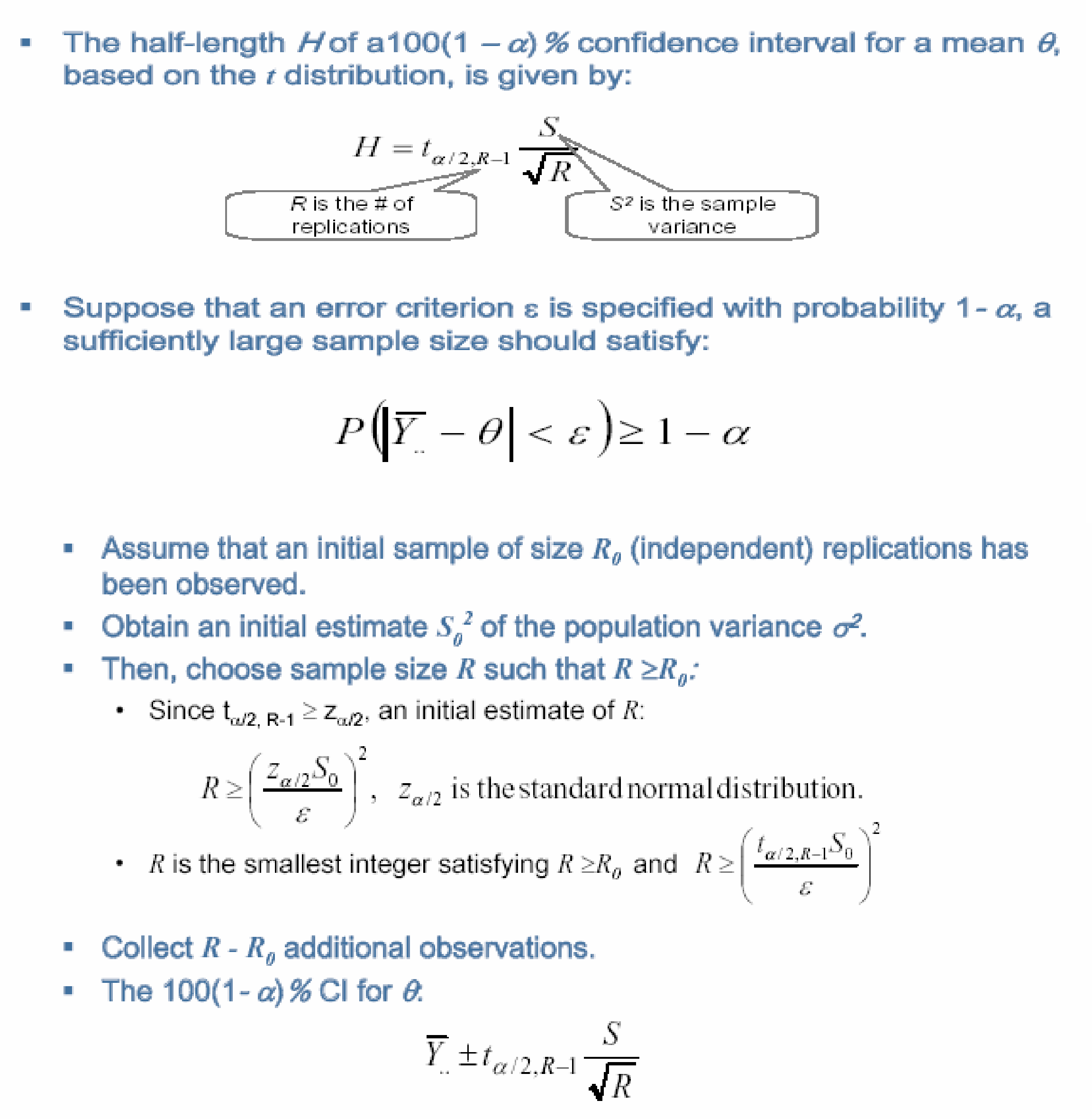


- Across Replication:
  - For example: the daily cycle time averages (discrete time data)
    - The average: $\bar{Y}_{..} = \frac{1}{R}\sum_{i=1}^{R} Y_{i.}$
    - The sample variance: $S^2 = \frac{1}{R-1}\sum_{i=1}^{R}(Y_{i.} - \bar{Y}_{..})^2$
    - The confidence-interval half-width: $H = t_{\alpha/2,R-1}\frac{S}{\sqrt{R}}$
- Within replication:
  - For example: the WIP (a continuous time data)
    - The average: $\bar{Y}_{i.} = \frac{1}{T_{Ei}}\int_0^{T_{Ei}} Y_i(t)dt$
    - The sample variance: $S_i^2 = \frac{1}{T_{Ei}}\int_0^{T_{Ei}}\left(Y_i(t) - \bar{Y}_{i.}\right)^2 dt$



- Overall sample average, $\bar{Y}_{..}$, and the interval replication sample averages, $\bar{Y}_{i.}$, are always unbiased estimators of the expected daily average cycle time or daily average WIP.
- Across-replication data are independent (different random numbers) and identically distributed (same model), but within-replication data do not have these properties.

8.4.3 Confidence Intervals specified precision



- The half-length *H* of a100(1 – *α*) % confidence interval for a mean *θ*, based on the *t* distribution, is given by:

$$H = t_{\alpha/2,R-1}\frac{S}{\sqrt{R}}$$

R is the # of replications

*S*² is the sample variance

- Suppose that an error criterion ε is specified with probability 1 - *α*, a sufficiently large sample size should satisfy:

$$P\left(\left|\bar{Y}_{..} - \theta\right| < \varepsilon\right) \geq 1 - \alpha$$

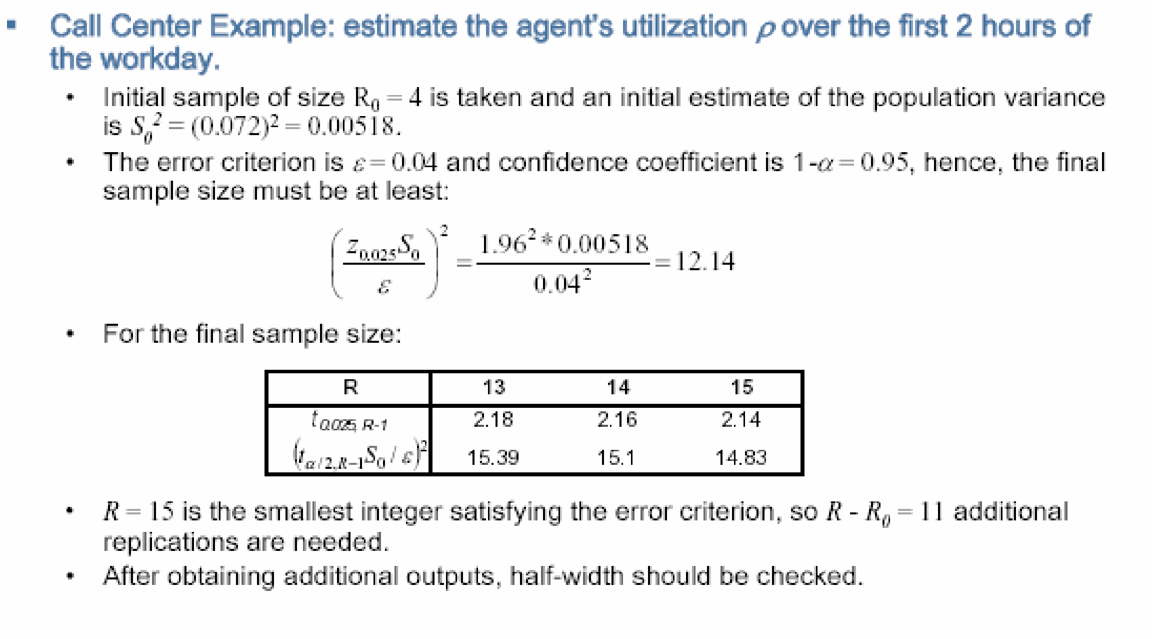
  - Assume that an initial sample of size R_0 (independent) replications has been observed.
  - Obtain an initial estimate S_0^2 of the population variance σ^2.
  - Then, choose sample size R such that $R \geq R_0$:
    - Since $t_{\alpha/2,\,R-1} \geq z_{\alpha/2}$, an initial estimate of *R*:

$$R \geq \left(\frac{z_{\alpha/2}S_0}{\varepsilon}\right)^2, \quad z_{\alpha/2} \text{ is the standard normal distribution.}$$

    - *R* is the smallest integer satisfying $R \geq R_0$ and $R \geq \left(\frac{t_{\alpha/2,R-1}S_0}{\varepsilon}\right)^2$

  - Collect $R - R_0$ additional observations.
  - The 100(1 - *α*) % CI for *θ*:

$$\bar{Y}_{..} \pm t_{\alpha/2,R-1}\frac{S}{\sqrt{R}}$$



- **Call Center Example: estimate the agent's utilization ρ over the first 2 hours of the workday.**
  - Initial sample of size $R_0 = 4$ is taken and an initial estimate of the population variance is $S_0^2 = (0.072)^2 = 0.00518$.
  - The error criterion is $\varepsilon = 0.04$ and confidence coefficient is $1\text{-}\alpha = 0.95$, hence, the final sample size must be at least:

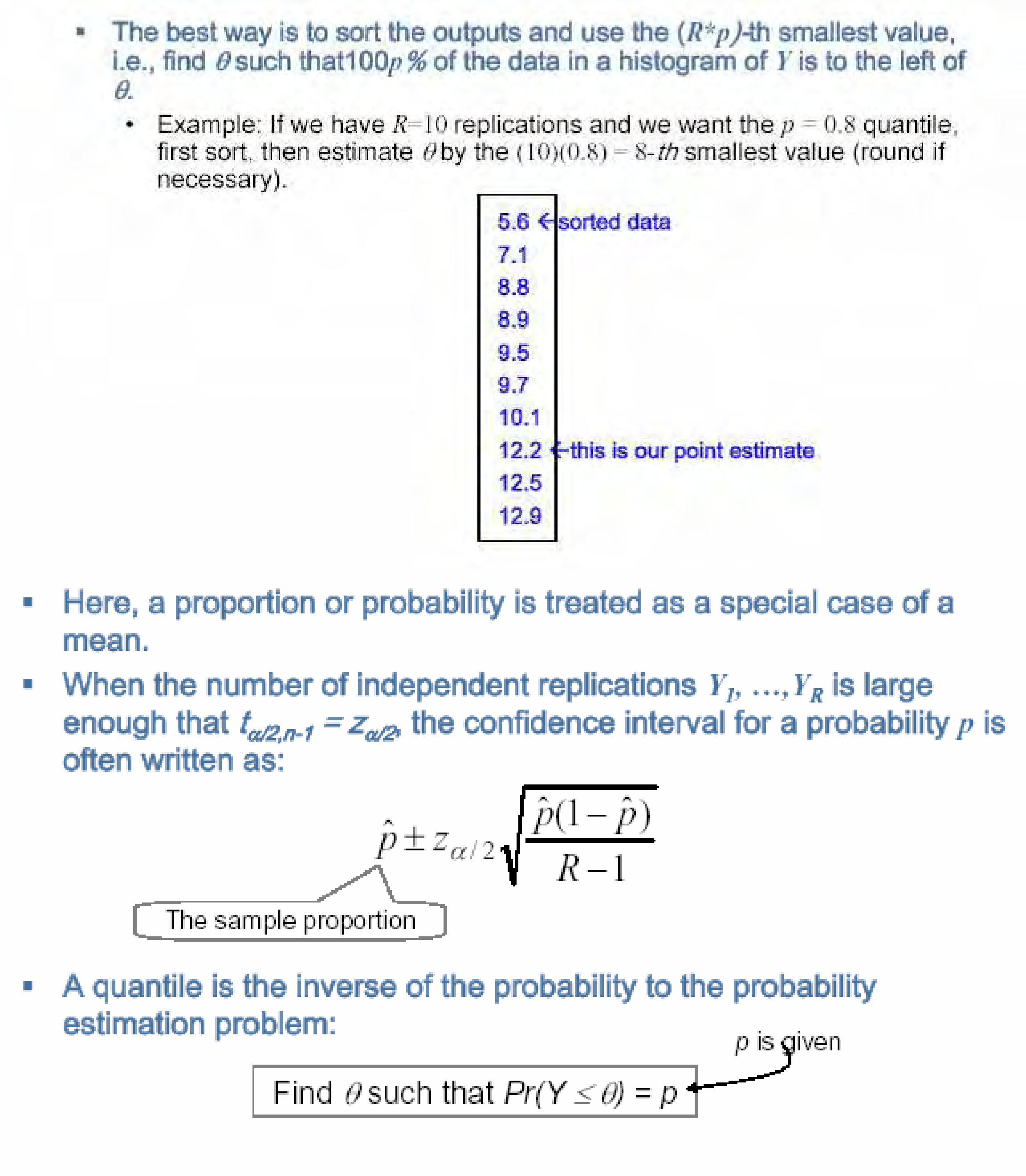
$$\left(\frac{z_{0.025}S_0}{\varepsilon}\right)^2 = \frac{1.96^2 * 0.00518}{0.04^2} = 12.14$$

  - For the final sample size:

R	13	14	15
$t_{0.025, R-1}$	2.18	2.16	2.14
$(t_{\alpha/2,R-1}S_0/\varepsilon)^2$	15.39	15.1	14.83

  - $R = 15$ is the smallest integer satisfying the error criterion, so $R - R_0 = 11$ additional replications are needed.
  - After obtaining additional outputs, half-width should be checked.

8.4.4 Confidence Intervals for Quantiles



- The best way is to sort the outputs and use the $(R*p)$-th smallest value, i.e., find θ such that100p % of the data in a histogram of Y is to the left of θ.
  - Example: If we have R=10 replications and we want the $p = 0.8$ quantile, first sort, then estimate θ by the $(10)(0.8) = 8$-*th* smallest value (round if necessary).

5.6 ← sorted data
7.1
8.8
8.9
9.5
9.7
10.1
12.2 ← this is our point estimate
12.5
12.9

- Here, a proportion or probability is treated as a special case of a mean.
- When the number of independent replications $Y_1, \ldots, Y_R$ is large enough that $t_{\alpha/2,n-1} = z_{\alpha/2}$, the confidence interval for a probability p is often written as:

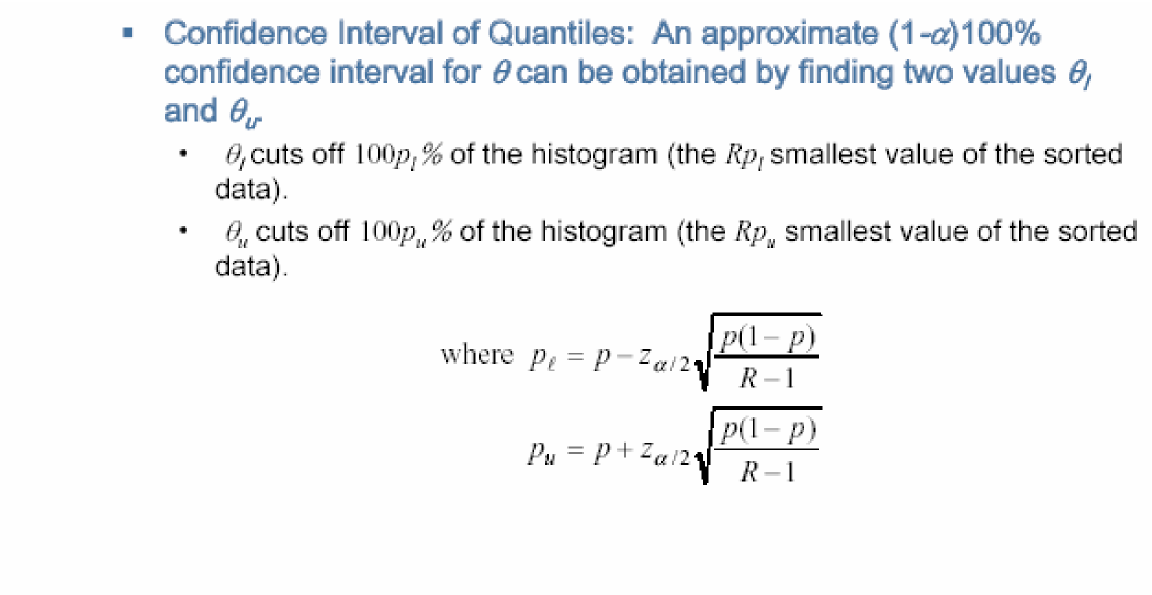
$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{R-1}}$$

The sample proportion

- A quantile is the inverse of the probability to the probability estimation problem:

p is given

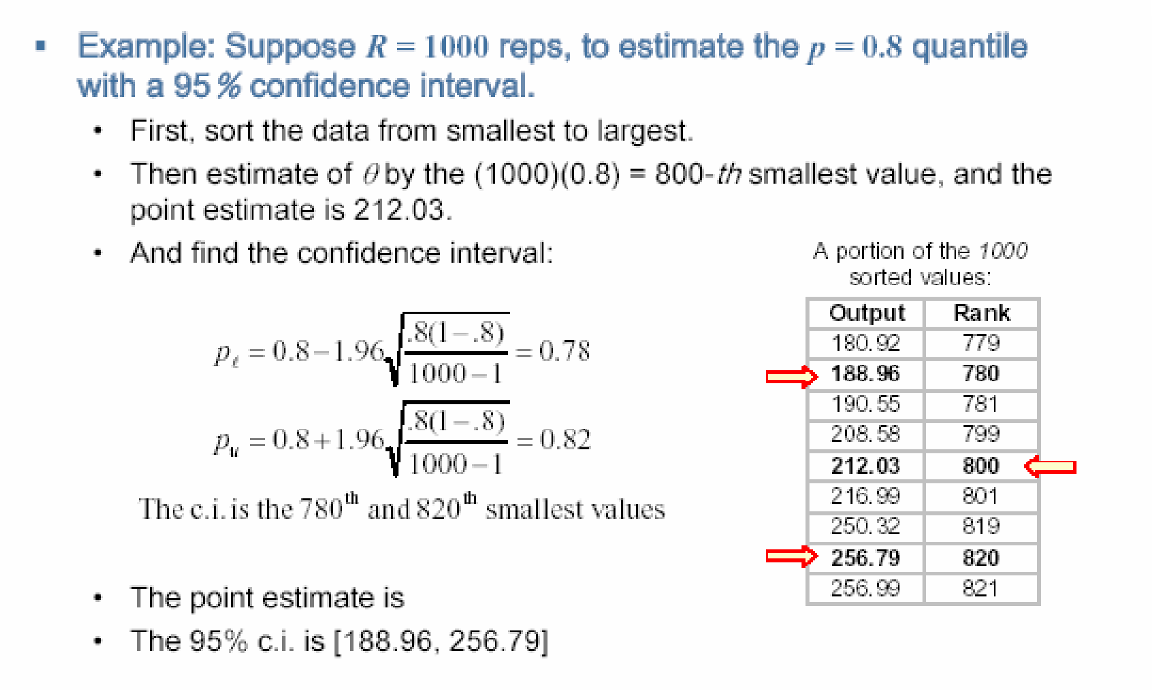
$$\boxed{\text{Find } \theta \text{ such that } Pr(Y \le \theta) = p}$$



- **Confidence Interval of Quantiles: An approximate (1-α)100% confidence interval for θ can be obtained by finding two values θ_l and θ_u.**
  - θ_l cuts off $100p_l$% of the histogram (the Rp_l smallest value of the sorted data).
  - θ_u cuts off $100p_u$% of the histogram (the Rp_u smallest value of the sorted data).

$$\text{where } p_\ell = p - z_{\alpha/2}\sqrt{\frac{p(1-p)}{R-1}}$$

$$p_u = p + z_{\alpha/2}\sqrt{\frac{p(1-p)}{R-1}}$$



- **Example: Suppose $R = 1000$ reps, to estimate the $p = 0.8$ quantile with a 95% confidence interval.**
  - First, sort the data from smallest to largest.
  - Then estimate of θ by the (1000)(0.8) = 800-*th* smallest value, and the point estimate is 212.03.
  - And find the confidence interval:

$$p_\ell = 0.8 - 1.96\sqrt{\frac{.8(1-.8)}{1000-1}} = 0.78$$

$$p_u = 0.8 + 1.96\sqrt{\frac{.8(1-.8)}{1000-1}} = 0.82$$

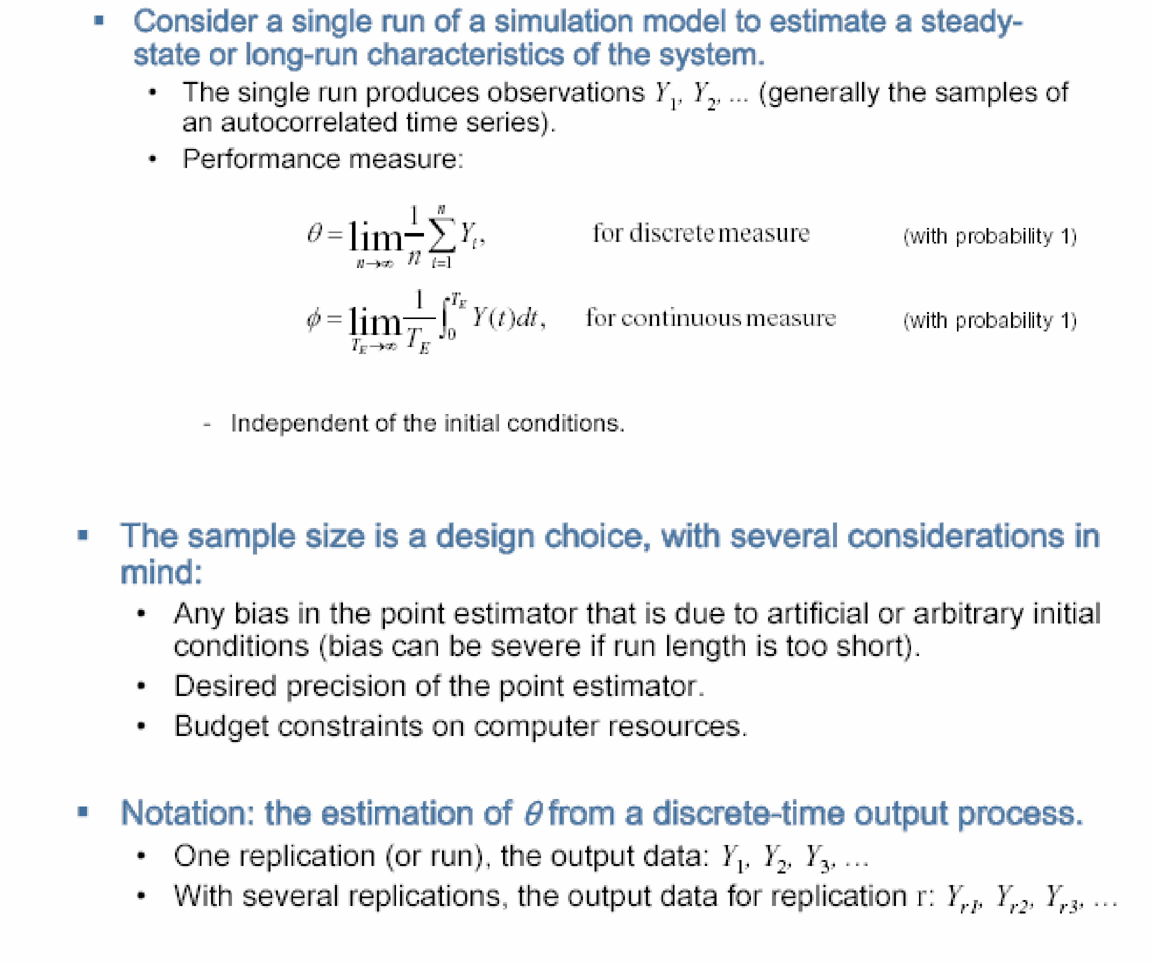
The c.i. is the 780th and 820th smallest values

A portion of the *1000* sorted values:

Output	Rank
180.92	779
188.96	**780**
190.55	781
208.58	799
212.03	**800**
216.99	801
250.32	819
256.79	**820**
256.99	821

  - The point estimate is
  - The 95% c.i. is [188.96, 256.79]

8.5 Output analysis for Steady state simulation



- Consider a single run of a simulation model to estimate a steady-state or long-run characteristics of the system.
  - The single run produces observations Y_1, Y_2, ... (generally the samples of an autocorrelated time series).
  - Performance measure:

$$\theta = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Y_i, \qquad \text{for discrete measure} \qquad \text{(with probability 1)}$$

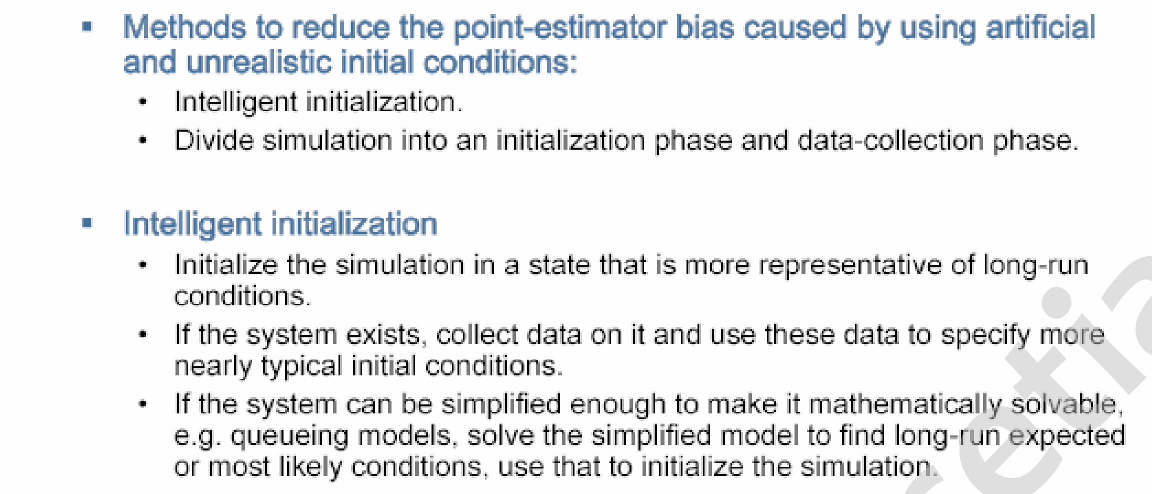
$$\phi = \lim_{T_E\to\infty} \frac{1}{T_E} \int_0^{T_E} Y(t)dt, \qquad \text{for continuous measure} \qquad \text{(with probability 1)}$$

    - Independent of the initial conditions.

- The sample size is a design choice, with several considerations in mind:
  - Any bias in the point estimator that is due to artificial or arbitrary initial conditions (bias can be severe if run length is too short).
  - Desired precision of the point estimator.
  - Budget constraints on computer resources.

- Notation: the estimation of θ from a discrete-time output process.
  - One replication (or run), the output data: Y_1, Y_2, Y_3, …
  - With several replications, the output data for replication r: Y_{r1}, Y_{r2}, Y_{r3}, …

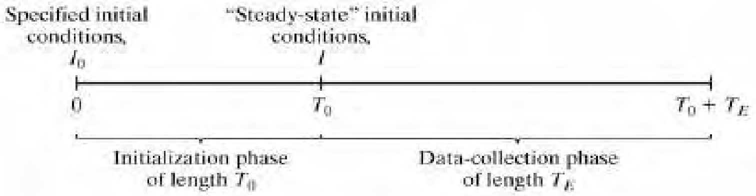
8.5.1 Initialization Bias



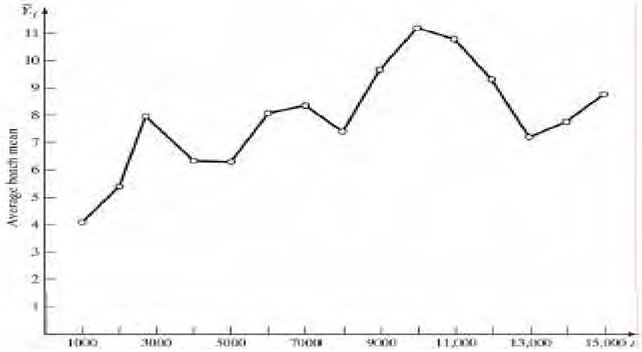
- Methods to reduce the point-estimator bias caused by using artificial and unrealistic initial conditions:
  - Intelligent initialization.
  - Divide simulation into an initialization phase and data-collection phase.

- Intelligent initialization
  - Initialize the simulation in a state that is more representative of long-run conditions.
  - If the system exists, collect data on it and use these data to specify more nearly typical initial conditions.
  - If the system can be simplified enough to make it mathematically solvable, e.g. queueing models, solve the simplified model to find long-run expected or most likely conditions, use that to initialize the simulation.

- **Divide each simulation into two phases:**
  - An initialization phase, from time 0 to time T_0.
  - A data-collection phase, from T_0 to the stopping time T_0+T_E.
  - The choice of T_0 is important:
    - After T_0, system should be more nearly representative of steady-state behavior.
  - System has reached steady state: the probability distribution of the system state is close to the steady-state probability distribution (bias of response variable is negligible).


Specified initial conditions, I_0
"Steady-state" initial conditions, I
0
T_0
$T_0 + T_E$
Initialization phase of length T_0
Data-collection phase of length T_E

- **A plot of the ensemble averages, $\overline{Y}..(n,d)$, versus 1000*j*, for $j = 1,2, \ldots,15$.**

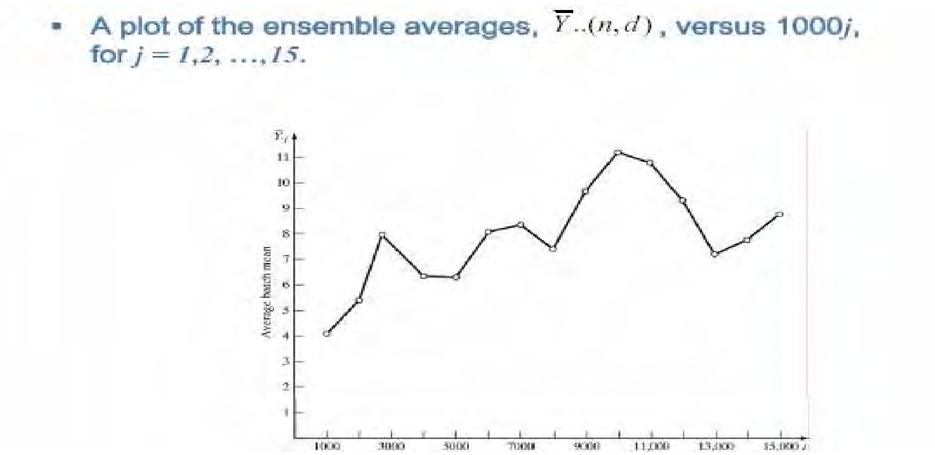


- **M/G/1 queueing example: A total of 10 independent replications were made.**
  - Each replication beginning in the empty and idle state.
  - Simulation run length on each replication was T_0+T_E = 15000 minutes.
  - Response variable: queue length, $L_Q(t,r)$ (at time t of the r-th replication).
  - Batching intervals of 1000 minutes, batch means
- **Ensemble averages:**
  - To identify trend in the data due to initialization bias
  - The average corresponding batch means *across* replications:

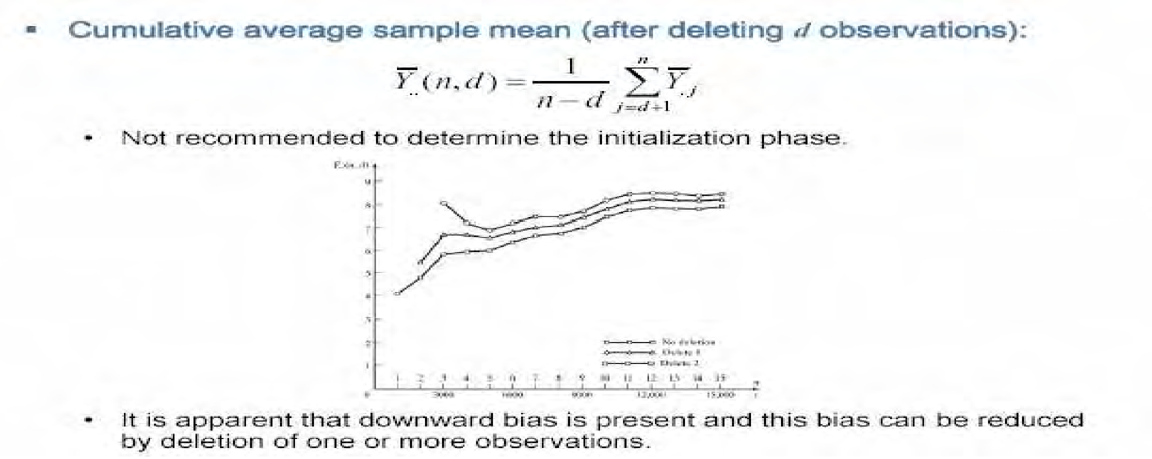
$$\overline{Y}_{.j} = \frac{1}{R}\sum_{r=1}^{R} Y_{rj}$$

R replications

  - The preferred method to determine deletion point.



- A plot of the ensemble averages, $\overline{Y}_{..}(n,d)$, versus 1000*j*, for $j = 1,2, \ldots, 15$.

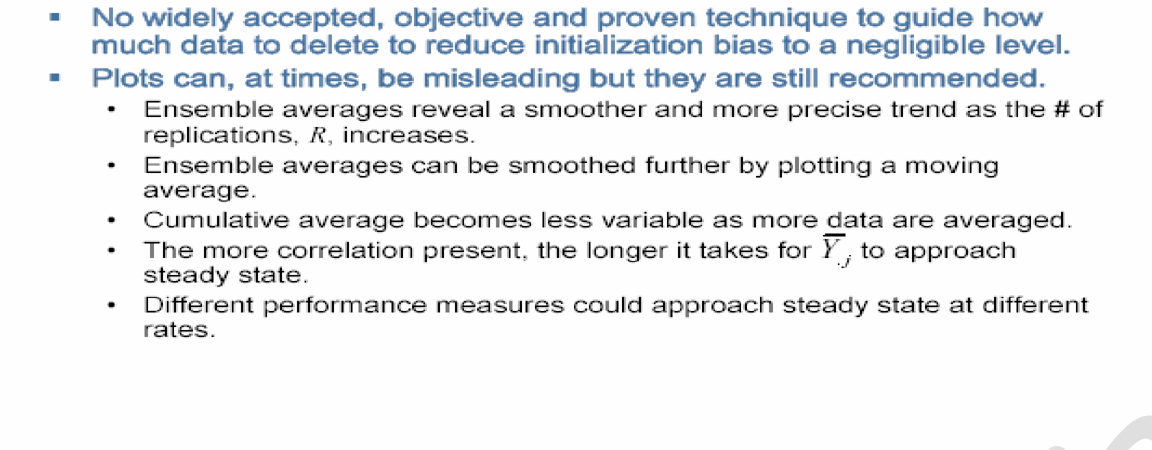


- Cumulative average sample mean (after deleting *d* observations):

$$\overline{Y}_{..}(n,d) = \frac{1}{n-d} \sum_{j=d+1}^{n} \overline{Y}_{.j}$$

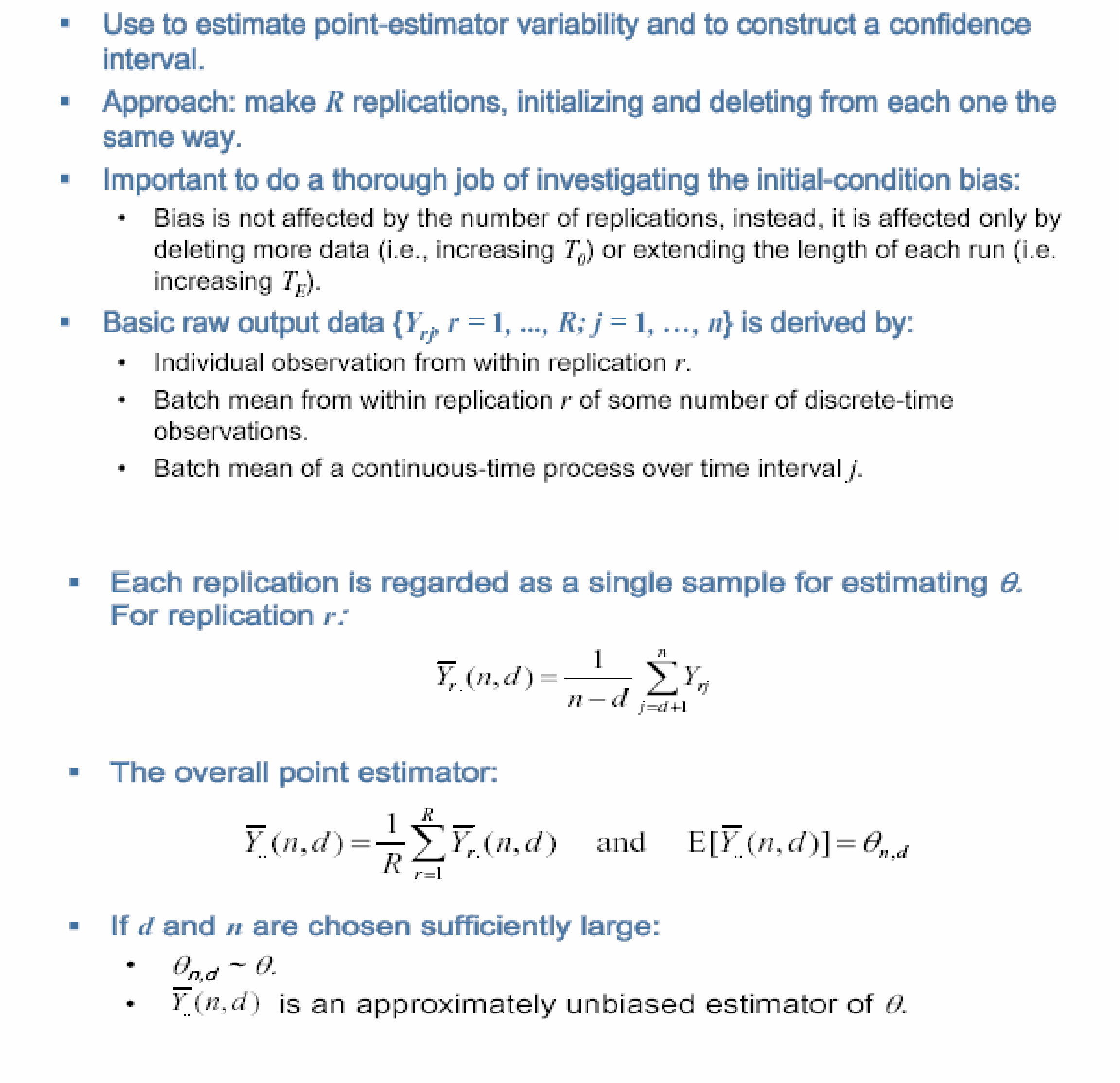
  - Not recommended to determine the initialization phase.

  - It is apparent that downward bias is present and this bias can be reduced by deletion of one or more observations.



- No widely accepted, objective and proven technique to guide how much data to delete to reduce initialization bias to a negligible level.
- Plots can, at times, be misleading but they are still recommended.
  - Ensemble averages reveal a smoother and more precise trend as the # of replications, *R*, increases.
  - Ensemble averages can be smoothed further by plotting a moving average.
  - Cumulative average becomes less variable as more data are averaged.
  - The more correlation present, the longer it takes for $\overline{Y}_{.j}$ to approach steady state.
  - Different performance measures could approach steady state at different rates.

8.5.2 Replication Method



- Use to estimate point-estimator variability and to construct a confidence interval.
- Approach: make R replications, initializing and deleting from each one the same way.
- Important to do a thorough job of investigating the initial-condition bias:
  - Bias is not affected by the number of replications, instead, it is affected only by deleting more data (i.e., increasing T_0) or extending the length of each run (i.e. increasing T_E).
- Basic raw output data $\{Y_{rj}, r = 1, ..., R; j = 1, \ldots, n\}$ is derived by:
  - Individual observation from within replication r.
  - Batch mean from within replication r of some number of discrete-time observations.
  - Batch mean of a continuous-time process over time interval j.

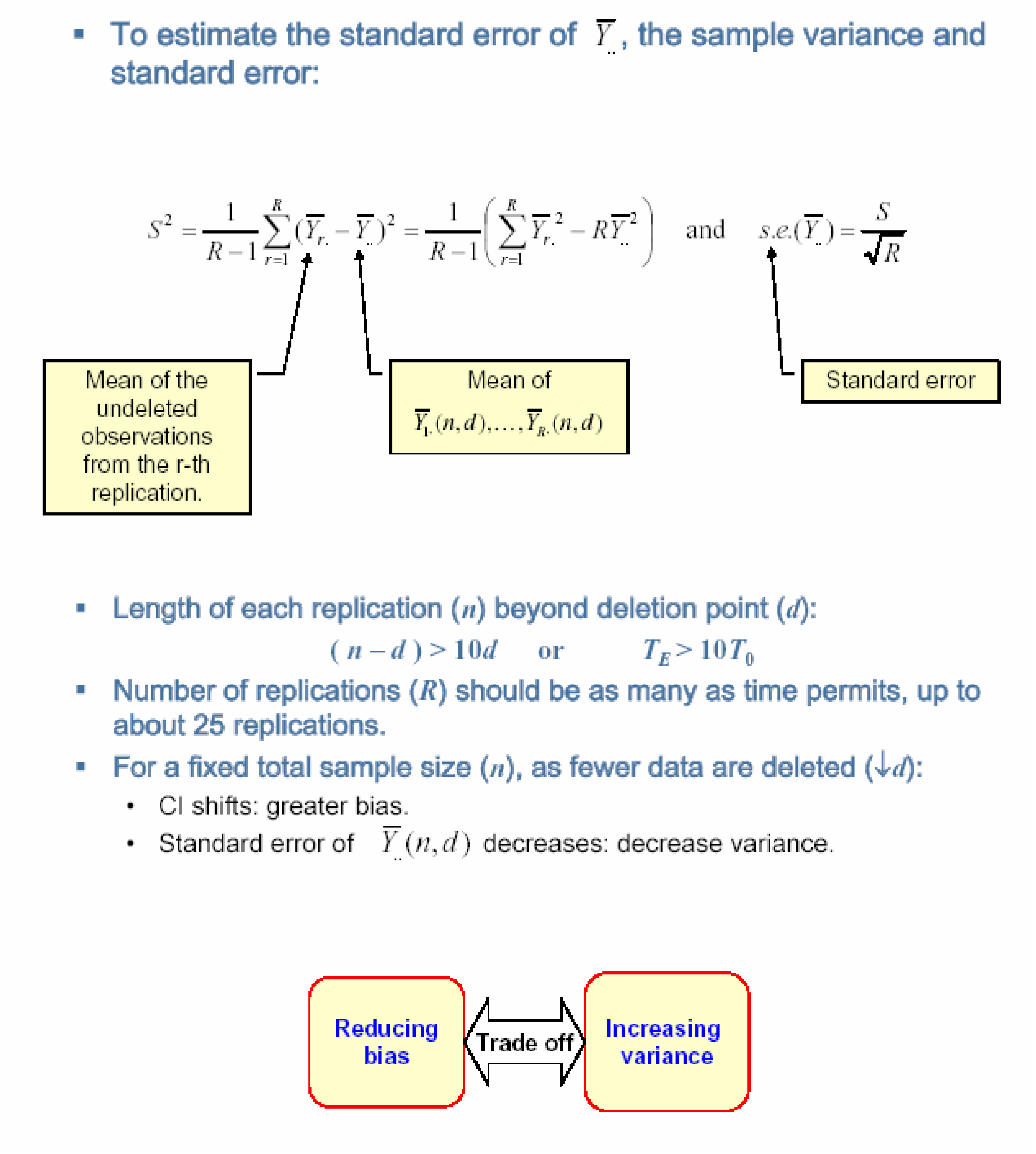
- Each replication is regarded as a single sample for estimating θ. For replication r:

$$\overline{Y}_{r\cdot}(n,d) = \frac{1}{n-d}\sum_{j=d+1}^{n} Y_{rj}$$

- The overall point estimator:

$$\overline{Y}_{\cdot\cdot}(n,d) = \frac{1}{R}\sum_{r=1}^{R}\overline{Y}_{r\cdot}(n,d) \quad \text{and} \quad \mathrm{E}[\overline{Y}_{\cdot\cdot}(n,d)] = \theta_{n,d}$$

- If d and n are chosen sufficiently large:
  - $\theta_{n,d} \sim \theta$.
  - $\overline{Y}_{\cdot\cdot}(n,d)$ is an approximately unbiased estimator of θ.



- To estimate the standard error of $\overline{Y}_{..}$, the sample variance and standard error:

$$S^2 = \frac{1}{R-1}\sum_{r=1}^{R}(\overline{Y}_{r.} - \overline{Y}_{..})^2 = \frac{1}{R-1}\left(\sum_{r=1}^{R}\overline{Y}_{r.}^{\,2} - R\overline{Y}_{..}^{\,2}\right) \quad \text{and} \quad s.e.(\overline{Y}_{..}) = \frac{S}{\sqrt{R}}$$

Mean of the undeleted observations from the r-th replication.

Mean of $\overline{Y}_{1.}(n,d),\ldots,\overline{Y}_{R.}(n,d)$

Standard error

- Length of each replication (n) beyond deletion point (d):

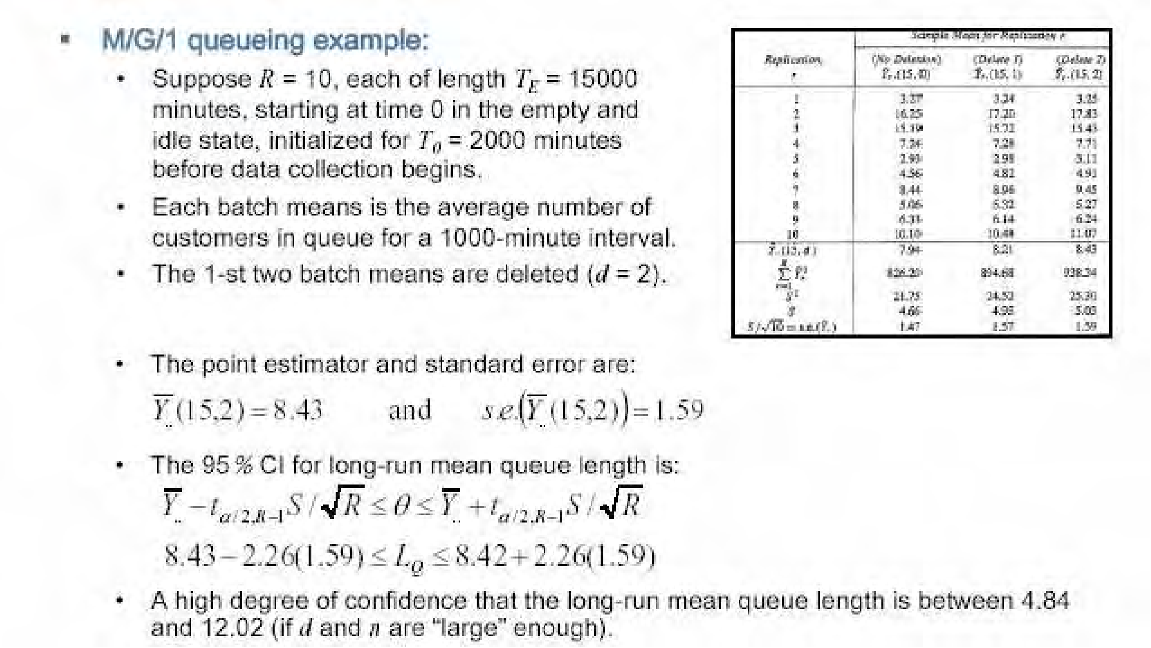
$$(n-d) > 10d \quad \text{or} \quad T_E > 10T_0$$

- Number of replications (R) should be as many as time permits, up to about 25 replications.
- For a fixed total sample size (n), as fewer data are deleted (↓d):
  - CI shifts: greater bias.
  - Standard error of $\overline{Y}_{..}(n,d)$ decreases: decrease variance.

Reducing bias

Trade off

Increasing variance



- **M/G/1 queueing example:**
  - Suppose R = 10, each of length T_E = 15000 minutes, starting at time 0 in the empty and idle state, initialized for T_0 = 2000 minutes before data collection begins.
  - Each batch means is the average number of customers in queue for a 1000-minute interval.
  - The 1-st two batch means are deleted (d = 2).

	Sample Means for Replication r		
Replication, r	(No Deletion) $\bar{Y}_{r\cdot}(15, 0)$	(Delete 1) $\bar{Y}_{r\cdot}(15, 1)$	(Delete 2) $\bar{Y}_{r\cdot}(15, 2)$
1	3.27	3.24	3.25
2	16.25	17.20	17.83
3	15.19	15.72	15.43
4	7.24	7.28	7.71
5	2.93	2.98	3.11
6	4.56	4.82	4.91
7	8.44	8.96	9.45
8	5.06	5.32	5.27
9	6.33	6.14	6.24
10	10.10	10.48	11.07
$\bar{Y}_{\cdot\cdot}(15, d)$	7.94	8.21	8.43
$\sum_{r=1}^{R} \bar{Y}_{r\cdot}^2$	826.20	894.68	938.24
S^2	21.75	24.52	25.30
S	4.66	4.95	5.03
$S/\sqrt{10} = \text{s.e.}(\bar{Y}_{\cdot\cdot})$	1.47	1.57	1.59

  - The point estimator and standard error are:

$$\bar{Y}_{\cdot\cdot}(15,2) = 8.43 \qquad \text{and} \qquad s.e.\left(\bar{Y}_{\cdot\cdot}(15,2)\right) = 1.59$$

  - The 95 % CI for long-run mean queue length is:

$$\bar{Y}_{\cdot\cdot} - t_{\alpha/2,R-1} S/\sqrt{R} \le \theta \le \bar{Y}_{\cdot\cdot} + t_{\alpha/2,R-1} S/\sqrt{R}$$

$$8.43 - 2.26(1.59) \le L_Q \le 8.42 + 2.26(1.59)$$

  - A high degree of confidence that the long-run mean queue length is between 4.84 and 12.02 (if d and n are "large" enough).

8.5.3 Sample Size

- **To estimate a long-run performance measure, θ, within $\pm\varepsilon$ with confidence $100(1-\alpha)\%$.**
- **M/G/1 queueing example (cont.):**
  - We know: R_0 = 10, d = 2 and S_0^2 = 25.30.
  - To estimate the long-run mean queue length, L_Q, within ε = 2 customers with 90 % confidence (α = 10 %).
  - Initial estimate:

$$R \geq \left(\frac{z_{0.05}S_0}{\varepsilon}\right)^2 = \frac{1.645^2(25.30)}{2^2} = 17.1$$

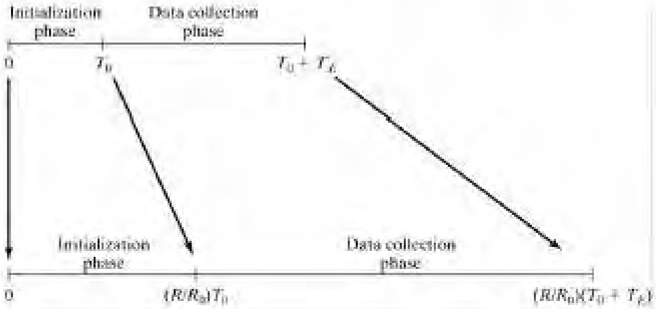
  - Hence, at least 18 replications are needed, next try R = 18,19, … using $R \geq (t_{0.05,R-1}S_0/\varepsilon)^2$. We found that:

$$R = 19 \geq (t_{0.05,19-1}S_0/\varepsilon)^2 = (1.73^2 * 25.3/4) = 18.93$$

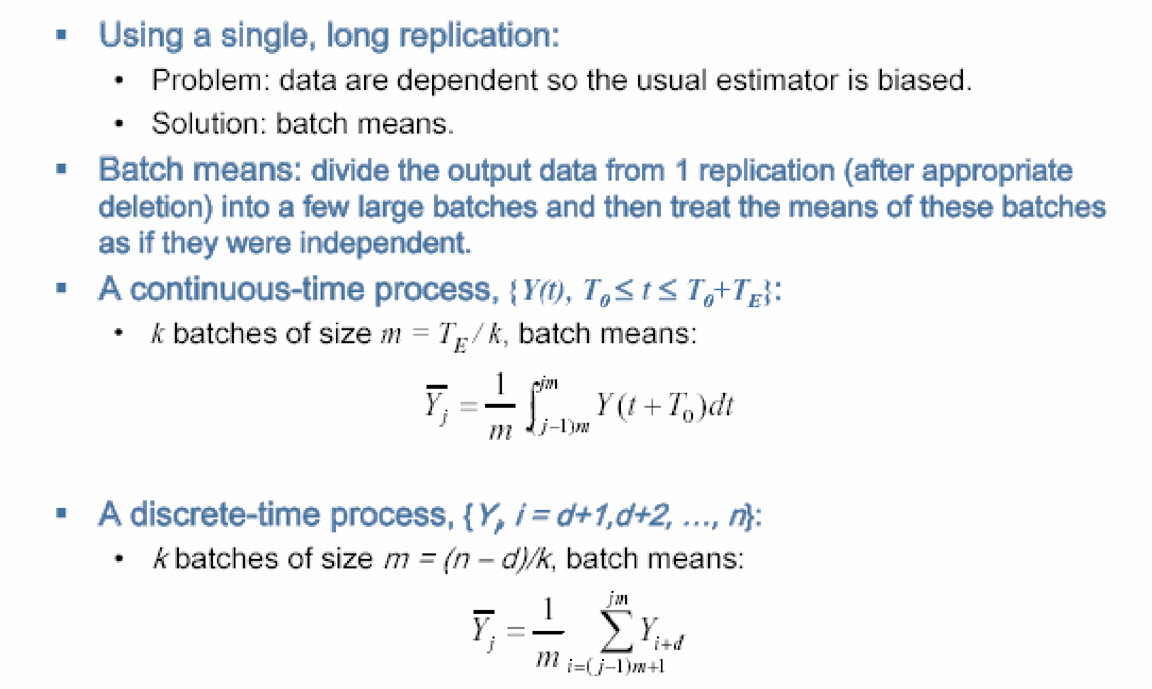
  - Additional replications needed is $R - R_0$ = 19-10 = 9.

An alternative to increasing R is to increase total run length T_0+T_E within each replication.

- Approach:
  - Increase run length from (T_0+T_E) to $(R/R_0)(T_0+T_E)$, and
  - Delete additional amount of data, from time 0 to time $(R/R_0)T_0$.
- Advantage: any residual bias in the point estimator should be further reduced.
- However, it is necessary to have saved the state of the model at time T_0+T_E and to be able to restart the model.



8.5.4 Batch Means for Interval estimation

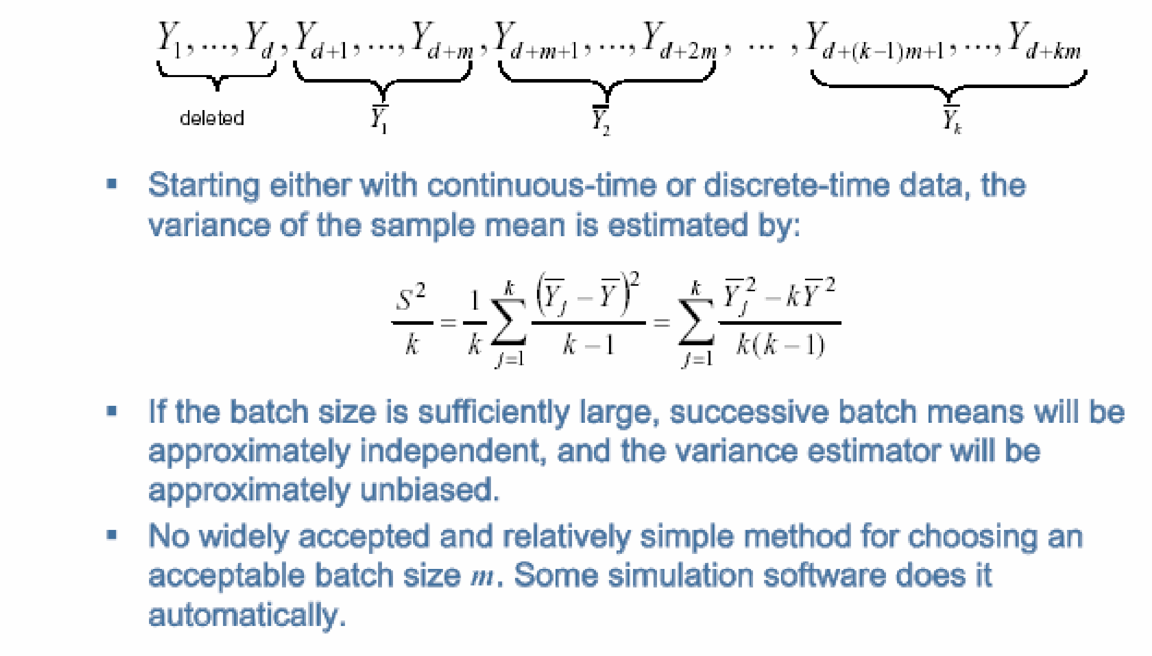


- Using a single, long replication:
  - Problem: data are dependent so the usual estimator is biased.
  - Solution: batch means.
- Batch means: divide the output data from 1 replication (after appropriate deletion) into a few large batches and then treat the means of these batches as if they were independent.
- A continuous-time process, $\{Y(t),\ T_0 \leq t \leq T_0+T_E\}$:
  - k batches of size $m = T_E/k$, batch means:

$$\overline{Y}_j = \frac{1}{m}\int_{(j-1)m}^{jm} Y(t+T_0)\,dt$$

- A discrete-time process, $\{Y_i,\ i = d+1, d+2, \ldots, n\}$:
  - k batches of size $m = (n-d)/k$, batch means:

$$\overline{Y}_j = \frac{1}{m}\sum_{i=(j-1)m+1}^{jm} Y_{i+d}$$



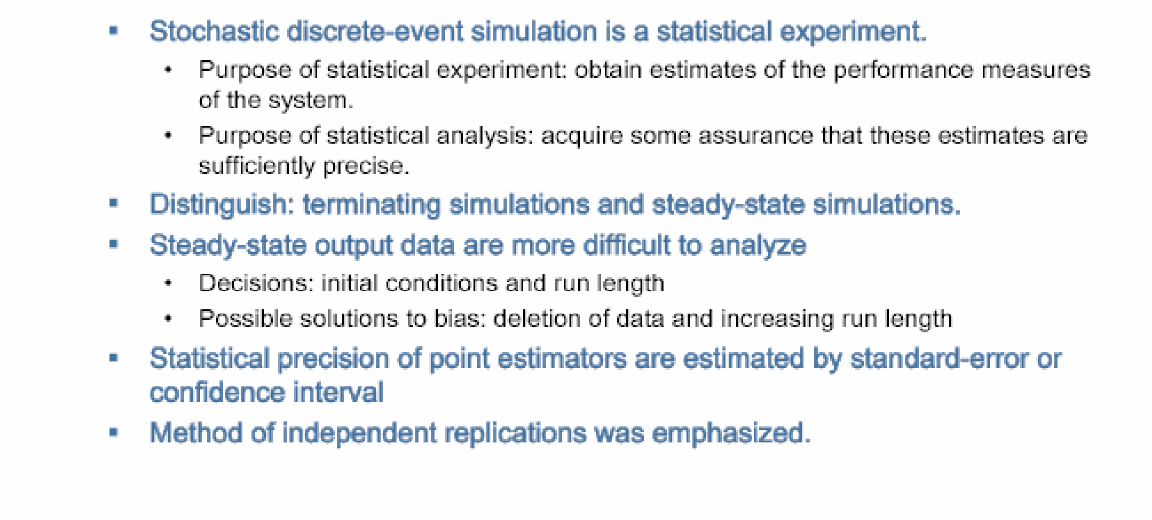
$$\underbrace{Y_1,\ldots,Y_d}_{\text{deleted}},\underbrace{Y_{d+1},\ldots,Y_{d+m}}_{\overline{Y}_1},\underbrace{Y_{d+m+1},\ldots,Y_{d+2m}}_{\overline{Y}_2},\ \cdots\ ,\underbrace{Y_{d+(k-1)m+1},\ldots,Y_{d+km}}_{\overline{Y}_k}$$

- Starting either with continuous-time or discrete-time data, the variance of the sample mean is estimated by:

$$\frac{S^2}{k} = \frac{1}{k}\sum_{j=1}^{k}\frac{\left(\overline{Y}_j - \overline{Y}\right)^2}{k-1} = \sum_{j=1}^{k}\frac{\overline{Y}_j^2 - k\overline{Y}^2}{k(k-1)}$$

- If the batch size is sufficiently large, successive batch means will be approximately independent, and the variance estimator will be approximately unbiased.
- No widely accepted and relatively simple method for choosing an acceptable batch size m. Some simulation software does it automatically.

8.14 Summary



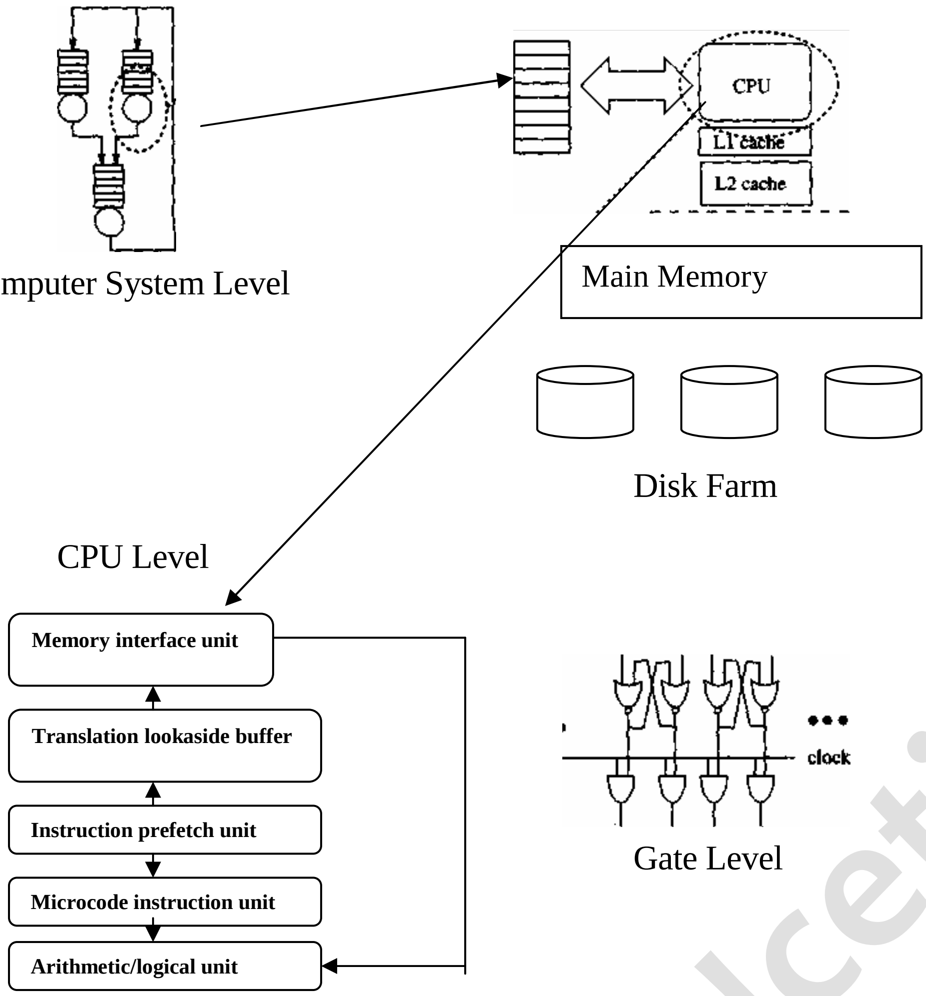
- Stochastic discrete-event simulation is a statistical experiment.
  - Purpose of statistical experiment: obtain estimates of the performance measures of the system.
  - Purpose of statistical analysis: acquire some assurance that these estimates are sufficiently precise.
- Distinguish: terminating simulations and steady-state simulations.
- Steady-state output data are more difficult to analyze
  - Decisions: initial conditions and run length
  - Possible solutions to bias: deletion of data and increasing run length
- Statistical precision of point estimators are estimated by standard-error or confidence interval
- Method of independent replications was emphasized.

9. Simulation of Computer Systems

Simulation is used extensively to simulate computer systems, because of their great importance to the everyday operations of business, industry, government and universities. The motivations for simulating computer systems, the different types of approaches used, and the interplay between characteristics of the model and implementation strategies are discussed below.

9.1 Introduction

Computer systems are incredibly complex. A computer system exhibits complicated behavior at time scales from the time it takes to "flip" a transistor’s state (on the order of 10^{-11} seconds) to the time it takes a human to interact with it (on the order of seconds or minutes). **Computer systems are designed hierarchically, in an effort to manage this complexity.** Figure below illustrates the point.



Different levels of abstraction

- At a high level of abstraction (the system level) one might view computational activity in terms of tasks circulating among servers, queueing for service when a server is busy.
- A lower level in the hierarchy can view the activity among components of a given processor (its registers, its memory hierarchy).
- A lower level still one views the activity of functional units that together make up a central processing unit.
- At an even lower level one can view the logical circuitry that makes it all happen.

Simulation is used extensively at every level of this hierarchy, with **results from one level being used at another.** For instance, engineers working on designing a new chip will begin by partitioning the chip functionally, establish interfaces between the subsystems, then design and test the subsystems individually. Given a subsystem design, the electrical properties of the circuit are first studied, using a circuit simulator that solves differential equations describing electrical behavior. At this level engineers work to ensure that signal timing is correct throughout the circuit, and that the electrical properties fall within the parameters intended by the design. Once this level of validation has been achieved, the electrical behavior is abstracted into logical behavior; e.g., signals formerly thought of as thought of as logical 1's or 0's. A different type of simulator is next used to test the correctness of the circuit's logical behavior. A common testing technique is to present the design with many different sets of logical inputs for which the desired logical outputs are known. Discrete-event simulation is used to evaluate the logical response of the circuit to each test vector, and is also used to evaluate timing. Once a chip’s subsystems are designed and tested, the designs are integrated, and then the whole system is subjected to testing, again by simulation.

At a higher level one simulates using functional abstractions. For instance, a memory chip could be modeled simply as an array of numbers and a reference to memory as just an indexing operation. A special type of description language exists for this level, called “register-transfer language”. This is like a programming language, with preassigned names for registers and other hardware-specific entities, and with assignment statements used to indicate data transfer between hardware entities. For example, the sequence below loads into register R3 the data whose memory address is in register R6, subtracts one from it, and writes the result into the memory location that is word-adjacent (a word in this example is 4 bytes in size) to the location first read.

```
R3=M [R6];   R3=R3-1;
R6=R6+4;     M [R6] =R3;
```

The abstraction makes sense when one is content to assume that the memory works, and that the time to put a datum in or out is a known constant. **The "known constant" is a**

value resulting from analysis at a lower level of abstraction. Functional abstraction is also commonly used to simulate subsystems of a central processing unit (CPU), when studying how an executing program exercises special architectural features of the CPU.

At higher level still one might study how an input-output (I/O) system behave in response to execution of a computer program. The program's behavior may be abstracted to the point of being *modeled* but with some detailed description of I/O demands, e.g., with a **Markov-chain** that with some specificity describes an I/O operation as the Markov chain transitions. The behavior of the I/O devices may be abstracted to the point that all that is considered is how long it takes to complete a specified I/O operation. Because of these abstractions one can simulate larger systems, and simulate them more quickly. The execution of a program is modeled with a randomly sampled CPU service interval; its I/O demand is modeled as a randomly sampled service time on a randomly sampled I/O device.

9.2 Simulation Tools

Hand-in-hand with different abstraction levels one finds different tools used to perform and evaluate simulations.

An important **characteristic of a tool is how it supports model building.** In many tools one constructs networks of components whose local behavior is already known and already programmed into the tool. This is a powerful paradigm for complex model construction. At the low end of the abstraction hierarchy, electrical circuit simulators and gate-level simulators are driven by network descriptions. Likewise, at the high end of the abstraction hierarchy, tools that sirnulate queueing networks and Petri nets are driven by network descriptions, as are sophisticated commercial communication-system simulators that have extensive libraries of preprogrammed protocol behaviors.

VHDL Language

- A very significant player in computer-system design at lower levels of abstraction is the VHDL language.
- VHDL is the result of a U.S. effort in the 1980s to standardize the languages used to build electronic systems for the government.
- VHDL serves both as a design specification and as a simulation specification.
- VHDL is a rich language, full of constructs specific to digital systems, as well as the normal constructs one finds in a procedural programming language.
- It achieves its dual role by imposing a clear separation between system topology and system behavior.

- Design specification is a matter of topology, whereas simulation specification is a matter of behavior.
- The language promotes user-defined programmed behavior.
- VHDL is also innovative in its use of abstract interfaces which different "architectures" at different levels of abstraction may be attached.
- VHDL is widely used in the electrical engineering community but is hardly used outside of it.

Disadvantage of VHDL

- VHDL is a big language, requiring a substantive VHDL compiler, and vendors typically target the commercial market at prices that exclude academic research.

Advantage of VHDL

- The relative rigidity of the programming model makes possible **graphical model building**, thereby raising the whole model-building endeavor up to a higher level of abstraction. Some tools have so much preprogrammed functionality that it is to design and run a model without writing a single line of computer code.

9.2.1 Process Orientation

- A process-oriented view implies that the tool must support separately schedulable "user threads."
- A "user thread" is a separately schedulable thread of execution control implemented as part of a single executing process.
- A group of user threads operate in the same process memory space, with each thread having allocated to it a relatively small portion of that space for its own use.
- The private space is used to store variables that are local to the thread, and information needed to support the suspension and resumption of that thread.

- In **process-oriented simulations,** a thread is executed at a particular point in simulation time, say *t,* as a result of some event being scheduled at time t; the thread execution is in some sense the event handler.
- The thread yields by executing a statement whose semantics mean **"suspend this thread until the following condition occurs"**
- The **hold** statement is a classic example of this, it specifies how long in simulated time the thread suspends.
- The CSIM model has examples of all of these.

Execution of CSIM Model

- The model is compiled and linked with CSIM libraries, creating an executable which, when run, simulates the model.
- The main procedure associated with any C++ program—where program execution begins—is buried inside the CSIM kernel.
- Beginning execution, control enters inside the CSIM kernel where initialization is done; then the kernel calls procedure **sim**.
- CSIM requires a modeler to declare a procedure called **sim**, and that **sim** call create to turn the call into a thread. This call creates the first thread.
- In the specific example of interest that thread generates customers, 1000 of them. It enters a loop where it first suspends for an exponentially distributed period of time (an interarrival delay), and after that delay simply creates a new **customer**, and returns to the top of the loop. The call to **customer** creates a new thread.
- **sim** calls **customer**, which calls **create. create** allocates the internal data structures for the new thread, **schedules** that new thread to execute at the current simulation time, but before executing the thread further, control is returned to sim.
- On receiving back control, **sim** immediately suspends through a **hold** call, and the scheduler selects the thread whose invocation time is least among all executable threads.
- Almost always this will be the **customer** thread just created. This thread begins execution at the call to **use**, which suspends the thread until all customers generated before it have finished their executions, and then an implicit hold is performed on behalf of the thread, modeling its service delay.

CSIM Example

Threads access variables that are globally visible to all threads, and variables that are local to a specific thread. Variables **Total Customers** and **NumberOf Customers** are examples of global variables; local variables **i** (in sim) and **service** (in customer) are examples of local variables. Every instance of **customer** has its own copy of **service**. Local variables, the location of where a thread resumes execution, and the contents of machine registers at the time a thread suspended are all part of the thread *state*. To start a thread's execution, the scheduler must restore all of a thread's state to the appropriate places. There are different ways to store and restore a thread's state. The most common involve the **<u>program's runtime stack</u>**. When a program makes a procedure call, the runtime system pushes a new **"stack frame"** onto the stack. A stack frame contains the return address of the calling routine, copies of input arguments to the procedure, and space for local variables used by the called procedure. A machine register called the stack-frame pointer contains the address of the first byte of the frame; machine code that references local variables does so using addressing modes that specify an offset from the stack-frame pointer and cause the hardware to compute the actual address by adding the offset to the contents of the stack-frame pointer. When the called procedure returns control to the caller, the runtime system pops its frame off the stack.

```
Void procA(int a)
{
create("thread");
int x;
procB(x=2*a);
procB(2*x);
}
void procB(int b)
{
int x;
procC(x=2*b);
procC(2*x);
}
void procC(int c)
{
hold(c);
}
```

rtn adrs in scheduler	
a	procA stack frame
x	
rtn adrs in procA	
b	procB stack frame
x	
rtn adrs in procB	procC stack frame
c	
rtn adrs in procC	hold stack frame
c	

Fig. 9.2 Runtime stack in threaded simulation

Figure 9.2 illustrates a sequence of procedures and the stack of frames present on the first call to **hold**. The call to **hold** suspends the thread, and when the thread executes once again it will require all the information shown on the stack. Consequently the threading mechanism save this portion of the stack, and restore it before the thread is executed again. CSIM's approach is to have **all threads use the same stack space as they execute, with the frame for the first procedure in the thread body always starting at the same memory location.** When the thread suspends, the CSIM kernel copies the used stack into separate storage associated with the thread. To restore a thread, the CSIM kernel first copies the previously stored stack into the fixed stack position, then lets the normal runtime control take over. The **CSIM kernel also saves and restores machine registers.** An **alternative approach to thread stack management avoids the copying costs by preallocating memory for the stack of each different thread.** Machine registers must still be saved and restored, but a switch between threads is faster because only the stack pointer must be changed, not the stack itself. The key limitation of such an approach is that one must declare ahead of time the maximum stack size a thread may require, and there is always the danger of the thread's pushing the stack beyond its allocation.

Process-oriented simulators are known to be slower than event-oriented simulators, stack management being a key contributing factor. Depending on the threading implementation, there may be additional factors. CSIM's threads are tailored to simulation, and the internal scheduling mechanisms are purely temporal ones. However, threading is a concept that extends well beyond simulation and threads exist in more general forms.

The **problem for a simulator** is that it wants to select the next thread to run, based on simulation time. Most thread systems have ways of explicitly making a blocked thread runnable, and of having a running thread block itself. The simulation scheduler can itself be a thread. When it runs, it identifies the next thread to run, as a function of an ordinary event-list, puts that thread in the runnable set, and blocks itself. The one thread that can now be run does, and as a part of its suspension sequence causes the scheduler to be runnable, and blocks itself.

9.2.2 Event Orientation

- An event-oriented simulator is much simpler to design and implement using a general programming language.
- The semantics of the program expressing a model are the semantics of the programming language; there is no hidden activity akin to stack management or thread scheduling.
- It is natural to use object oriented techniques and express a simulation in terms of messages passed between simulation objects.
- Languages like C++ and Java are outstanding for developing abstract base classes to express the general structure of the simulator.
- In such an approach model building entails development of concrete classes derived from the base classes, which includes expression of their methods.

Example

We suppose a simulation system has an abstract base class for all simulation objects, and for all events comrnunicated below objects. Figure displays a simple example of such base classes, in C++. The **SimEvt** class defines an interface for all event data structures a user would define. The **SimEvt** base class contains a "private" data member that stores the time-stamp of a delivered event, and contains a "public" method that allows any code to see what time-stamp is. Class **SimScheduler** is declared to be a "friend class" which means that its methods have the same access privileges to an event's data as does the **SimEvt** class. The last bit of mystery is the functions **~SimEvt** and **~SimObj** prefaced by the keyword virtual.

A modeler may tailor to the application, by definition, new event classes that include the entire interface of the **SimEvt** class. **PktEvt** is an example of this. The user wished for the event to carry a data packet of general length, and so endows the class with the infrastructure necessary to store and access that packet. The user creates a “derived” instance of the class by calling the **PktEvt** constructor, which initializes its specialized data structures. The critical feature is that pointer to an **PktEvt** instance can b passed to any routine with a type declaration of **SimEvt**, and that routine can call the **SimEvt** methods and deal with the event in those terms. This is exactly what happens when a simulation object schedules an event.

```
class SimEvt {
  friend bool operator<(const SimEvt  &el,  const SimEvt &e2); friend bool
  operator==(const SimEvt &el,  const SimEvt &e2); friend class
  SimScheduler;
  private;
     stime _time;
  public:
  SimEvt();
  Virtual –SimEvt();
   Void set-time (stime when) {_time =when ;}
   stime get_timeQ {return _time ;}
  };

  class SimObj {
    protected:
    public:
        SimObj();
  Virtual –SimObj();
  Virtual void acceptEvt(SimEvt *)=0;
  };

  class SimScheduler{
   public:
     SimScheduler();
     Int ScheduleEvt(SimObj *, SimEvt *, stime delay);
     Stime now();
  };

  class PktEvt : public SimEvt {
   private:
    int pkt_length;
    void *packet;
   public:
    PktEvt();
    PktEvt(int len, void*pkt) {
      Packet=(void *)new char[len];
      Memcpy(packet, pkt, len);
       Pkt_length=len;
  }
  ~PktEvt() {delete packet;}
  void * get_pkt()  {return pkt;}
  int get_pkt_length {return pkt_length; }
  };
```

Class **SimScheduler** gives the interface for a scheduler object. Class **SimObj** is the abstract base class for a simulation object. Its interface specifies a "virtual" method **acceptEvt**; any schedulable object the modeler may wish to define must include this method as part of its interface. **acceptEvt** is called by the simulation scheduler to pass an event to the object. The action where the object's **acceptEvt** method is the simulation of the event; a pointer to the event that causes the execution is passed as a parameter in that call. This event was **ScheduleEvt** method associated with an instance of the **SimScheduler** class. The code scheduling the event has first created the event, acquired a pointer to the **SimObj** instance to receive the event, and decided on the delay into the future when the event ought to be received and acted upon.

Flexibility is the key requirement in computer-systems simulation. Flexibility in most contexts means the ability to use the full power of a general programming language. This requires a level of programming expertise that is not needed for the use of commercial graphically oriented modeling packages. The implementation requirements of an object-oriented event-oriented approach are much less delicate than for a threaded simulator, and the amount of simulator overhead involved in delivering an event to an object is considerably less than the cost of a context switch in a threaded system. The choice of using a process-oriented or event-oriented simulator-or writing one's own-is a function of the level of modeling ease, versus execution speed. It also happens that the process-oriented view is more naturally employed at the higher levels of the abstraction hierarchy.

Typical System	**Model Results**	**Tools**
CPU network	Job throughput, Job response time	Queueing network, Petri net simulators, scratch
Processor	Instruction throughput, Time/instruction	VHDL, scratch
Memory system	Miss rates, response time	VHDL, scratch
ALU	Timing, correctness	VHDL, scratch
Logic network	Timing, correctness	VHDL, scratch

The table lists different levels of abstraction in computer-systems simulation, the sorts of questions whose answers are sought from the models, and the sorts of tools typically used for modeling.

9.3 Model Input

There are different means of providing input to a model. The model might be driven by

- Stochastically generated input, using either simple or complicated operational assumptions, used at the high end of the abstraction. Useful for the study the system behavior over a range of scenarios.
- Trace input, measured from actual systems used at lower levels.

- High-level systems simulations accept a stream of job descriptions; CPU simulations accept a stream of instruction descriptions, memory simulations accept a stream of memory references; and gate-level simulations accept a stream of logical signals.
- Computer systems modeled as queueing networks typically interpret "customers" as computer programs; servers typically represent services such as attention by the CPU or an input/output system.
- Random sampling generates customer interarrival times, and it may also be used to govern routing and time-in-service.
- However, it is common in computer systems contexts to have routing and service times be state dependent; e.g., the next server visited is already specified in the customer's description, or may be the attached server with least queue length.

9.3.1 Modulated Poisson Process

- An input model that is sometimes used to retain a level of tractability is a **modulated Poisson process, or MPP.**
- The underlying framework is a continuous-time Markov chain (CTMC).
- A CTMC is always in some *state;* for descriptive purposes states are named by the integers: 1, 2,....
- The CTMC remains in a state for a random period of time, transitions randomly to another state, stays there for a random period of time, transitions again, and so on.
- The CTMC behavior is completely determined by its *generator matrix, Q = {q,j}.* For states i<> *j,* entry q,j describes the rate at which the chain transitions from state i into state *j.*
- The rate describes how quickly the transition is made; its units are transitions per unit simulation time.
- Diagonal element *q,i* is the negated sum of all rates out of state *i : q,i* = - € $q_{i,j}$
- An operational view of the CTMC is that upon entering a state *i* , it remains in that state for an exponentially distributed period of time, the exponential having rate –$q_{i,i}$ Making the transition, it chooses state *j* with probability –$q_{i,j}$ / $q_{i,i}$.
- Many CTMCs are *ergodic,* meaning that, left to run forever, every state is visited infinitely often.
- In an ergodic chain Π_I, denotes state i's stationary *probability, which* we can interpret as the long-term average fraction of time the CTMC is in state i.
- A critical relationship exists between stationary probabilities and transition rates: for every state i,

$$\pi_i \sum q_{i,j} = \sum \pi_j q_{j,i}$$

9.3.2 Virtual Memory Referencing

- Randomness can also be used to drive models in the middle levels of abstraction.
- In such a system, the data and instructions used by the program are organized in units called *pages*.
- All pages are the same size, typically 2^{10} to 2^{12} bytes in size.
- The physical memory of a computer is divided into *page frames*, each capable of holding exactly one page.
- The decision of which page to map to which frame is made by the operating system.
- As the program executes, it makes memory references to the "virtual memory," as though it occupied a very large memory starting at address 0, and is the only occupant of the memory.
- On every memory reference made by the program, the hardware looks up the identity of the page frame containing the reference, and translates the virtual address into a physical address.
- The hardware may discover that the referenced page is not present in the main memory; this is called *& page fault*
- When a page fault occurs, the hardware alerts the operating system, which then takes over to bring in the referenced page from a disk and decides which page frame should contain it.
- The operating system may need to evict a page from a page frame to make room for the new one.
- The policy the operating system uses to decide which page to evict is called the "replacement policy”.
- The quality of a replacement policy is often measured in terms of the fraction of references made whose page frames are found immediately, the *hit ratio.*
- Virtual memory systems are used in computers that support concurrent execution of multiple programs.
- In order to study different replacement policies one could simulate the memory-referencing behavior of several different programs, simulate the replacement policy, and count the number of references that page-fault.
- Virtual memory works well precisely because programs do tend to exhibit a certain type of behavior, so-called locality of reference.
- The program references tend to cluster in time and space, that when a reference to a new page is made and the page is brought in from the disk, it is likely that the other data or instructions on the page will also soon be referenced.
- In this way the overhead of bringing in the page is amortized over all the references made to that page before it is eventually evicted.
- A program's referencing behavior can usually be separated into a sequence of "phases," where during each phase the program makes references to a relatively small collection of pages called its *working set.*
- Phase transitions essentially change the program's working set. The challenge for the operating system is to recognize when the pages used by a program are

no longer in its working set, for these are the pages it can safely evict to make room for pages that *are* in some program's working set.

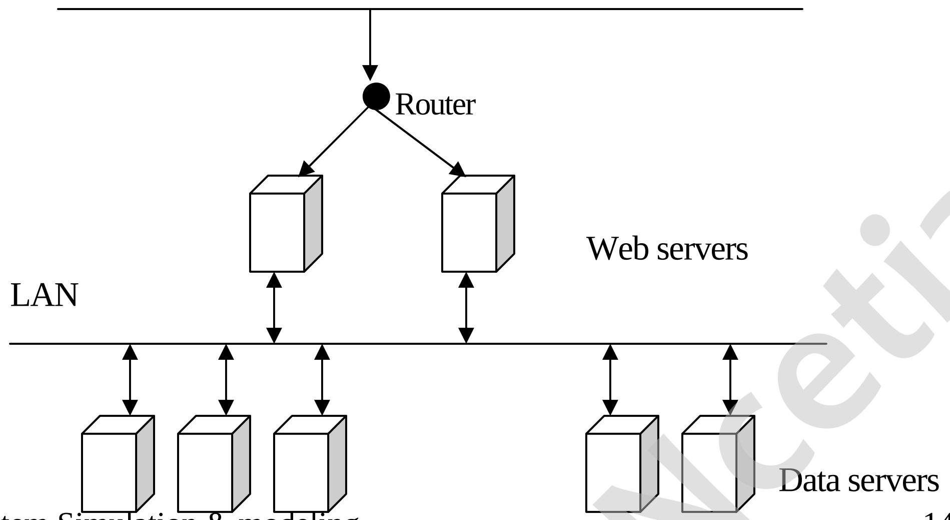
9.4 High-Level Computer System Simulation

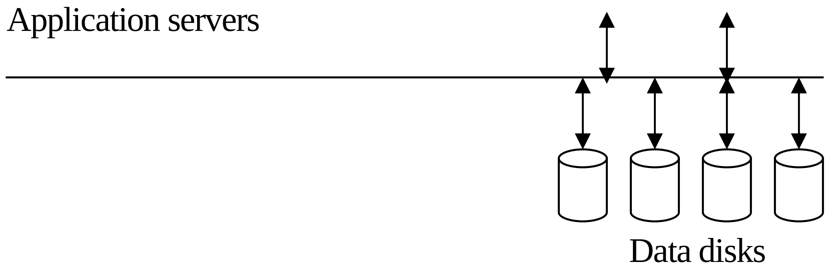
Here we illustrate concepts typical of high-level computer simulations, by sketching a simulation model of a computer system that services requests from the World Wide Web.

EXAMPLE

- A company that provides a major web site for searching and links to sites for Travel, commerce, entertainment, and the like wishes to conduct a capacity planning study.
- The overall architecture of their system is shown in Figure.
- At the back end one finds data servers responsible for all aspects of handling specific queries and updating databases.
- Data servers receive requests for service from application servers—machines dedicated to running specific applications supported by the site.
- In front of applications are web servers that manage the interaction of applications with the World Wide Web, and the portal to the whole system is a load-balancing router that distributes requests directed to the web site among web servers.

- The goal of the study is to determine the site's ability to handle load at peak periods.
- The desired output is an empirical distribution of the access response time.
- Thus, the high-level simulation model should focus on the impact of timing at each level that is used, system factors that affect that timing, and the effects of timing on contention for resources.





- To understand where those delays occur let us consider the processing associated with a typical query.
- All entries into the system are through a dedicated router, which examines the request and forwards it to some web server.
- It is reasonable to assume one switching time for a preexisting request and a different time for a new request.
- The result of the first step is selection of a web server, and enqueueing there of a request for service.
- A web server can be thought of as having a queue of threads of new requests, a queue of threads that are suspended awaiting a response from an application server, and a queue of threads "ready" to process responses from application servers.
- An accepted request from the router creates a new request thread.
- The servicing of a new request amounts to identification of an application and associated application server.
- A request for service is formatted and forwarded to an application server, and the requesting thread joins the suspended queue.
- At an application server, requests for service are organized along application types.
- A new request creates a thread that joins a new request queue associated with the identified application.
- An application request is modeled as a sequence of sets of requests from data servers, interspersed with computational bursts — e.g.,

burst 1
request data from Dl, D3 and D5
burst 2
request data from Dl and D2
burst 3

- An application server will implement a scheduling policy over sets of ready application threads.
- A data server creates a new thread to respond to a data request and places it in a queue of ready threads.

- A data server may implement memory-management policies, and may require further coordination with the application server to know when to release used memory.
- Upon receiving service the thread requests data from a disk, and suspends until the disk operation completes, at which point the thread is moved from the suspended list to the ready list, and when executed again reports back to the application server associated with the request.
- The thread suspended at the application server responds; eventually the application thread finishes and reports its completion back to the web-server thread that initiated it, which in turn communicates the results back over the Internet.

The web-site model is an excellent candidate for a threaded (process-oriented) approach to modeling. There are two natural approaches for defining a process. One emulates our CSIM example, defining each request as a process. The model would be expressed from the point of view of a request going through the router, to a web server, and so on.

Disadvantage of query centered modeling

One complication with equating a query with a process is that it is not an exact fit to what happens in this model. A query passed from application server to data server may actually cause multiple concurrent requests to the data disks—it is insufficient to "push" a query process from router to disks and back. A query process can spawn concurrent supporting processes, which implies encoding a fork-and-join synchronization mechanism (a process spawns new processes, and waits for them to return). The CSIM example illustrated this by having the main thread suspend, once all job arrival processes had been generated, until the last one finished.

Advantage of query centered modeling

The advantage to a query-centered modeling approach is clarity of expression, and the ease with which query-oriented statistics can be gathered. A given query process can simply measure its delays at every step along the way, and upon departing the system include its observations in the statistical record the simulation maintains for the system.

An alternative process-oriented approach is to associate processes with servers. The simulation model is expressed from an abstracted point of view of the servers' operating system. Individual queries become messages that are passed between server processes. An advantage of this approach is that it explicitly exposes the scheduling of query processing at the user level. The modeler has both the opportunity and the responsibility to provide the logic of scheduling actions that model processing done on behalf of a query. It is a modeling viewpoint that simplifies analysis of server behavior — an overloaded server is easily identified by the length of its queue of runnable queries. However, it is a modeling viewpoint that is a bit lower in abstraction than the first one, and requires more modeling and coding on the part of the user.

An event-oriented model of this system need not look a great deal different than the second of our process-oriented models. Queries passed as messages between servers have an obvious event-oriented expression. A modeler would have to add to the logic, events, and event handlers that describe the way a CPU passes through simulation time. In an event-oriented model one would need to define events that reflect "starting" and "stopping" the processing of a query, with some scheduling logic interspersed. Additional events and handlers need to be defined for any "signaling" that might be done between servers in a process-oriented model. A process-oriented approach, even one focused on servers rather than queries, lifts model expression to a higher level of abstraction and reduces the amount of code that must be written. In a system as complex as the web site, one must factor complexity of expression into the overall model-development process.

9.5 CPU Simulation

The main challenge to making effective use of a CPU is to avoid stalling it, which happens whenever the CPU commits to executing an instruction whose inputs are not all present. A leading cause of stalls is the latency delay between CPU and main memory, which can be tens of CPU cycles.

One instruction may initiate a read,

e.g.,

```
load $2, 4($3)
```

which is an assembly-language statement that instructs the CPU to use the data in register 3 (after adding value 4 to it) as a memory address, and to put the data found at that address in register *2*. If the CPU insisted on waiting for that data to appear in register *2* before further execution, the instruction could stall the CPU for a long time if the referenced address was not found in the cache. High-performance CPUs avoid this by recognizing that additional instructions can be executed, up to the point where the CPU attempts to execute an instruction that reads the contents of register 2,

e.g.,

add $4, $2, $5

This instruction adds the contents of registers *2* and 5 and places the result in register 4. If the data expected in register *2* is not yet present, the CPU will stall.

To allow the CPU to continue past a memory load it is necessary to

- mark the target register as being unready
- allow the memory system to asynchronously load the target register while the CPU continues on in the instruction stream
- stall the CPU if it attempts to read a register marked as unready
- Clear the unready status when the memory operation completes.

The sort of arrangement described above was first used in the earliest supercomputers, designed in the 1960s. Modern microprocessors add some additional capabilities to exploit **instruction-level parallelism (ILP).**

Pipelining

- The technique of pipelining has long been recognized as a way of accelerating the execution of computer instructions.
- Pipelining exploits the fact that each instruction goes sequentially through several stages in the course of being processed; separate hardware resources are dedicated to each stage, permitting multiple instructions to be in various stages of processing concurrently
- A typical sequence of stages in an ILP CPU are as follows:

Instruction fetch : the instruction is fetched from the memory.

Instruction decode : the memory word holding the instruction is interpreted to determine what operation is specified; the registers involved are identified.

Instruction issue : an instruction is "issued" when there are no constraints holding it back from being executed. Constraints that keep an instruction from being issued include data not yet being ready in an input register, and unavailability of a functional unit (e.g., arithmetic-logical-unit) needed to execute the instruction.

Instruction execute : the instruction is performed.

Instruction complete : results of the instruction are stored in the destination register.

Instruction graduate : executed instructions are graduated in the order that they appear in the order that they appear in the instruction stream.

9.6 Memory Simulation

- One of the great challenges of computer architecture is finding ways to effectively deal with the increasing gap in operation speed between CPUs and main memory.
- The main technique that has evolved is to build hierarchies of memories.

- A relatively small memory—the L1 cache—operates at CPU speed. A larger memory—the L2 cache— is larger, and operates more slowly.
- The main memory is larger still, and slower still. The smaller memories hold data that was referenced recently.
- Data moves up the hierarchy on demand, and ages out as it becomes disused, to make room for the data in current use.
- For instance, when the CPU wishes to read memory location 100000, hardware will look for it in the L1 cache, and failing to find it there, look for it in the L2 cache.
- If found there, an entire block containing that reference is moved from the L2 cache into the L1 cache.
- If not found in the L2 cache, a block of data containing location 10000 is copied from the main memory to the L2 cache, and part of that block is copied into the LI cache.
- It may take 50 cycles or more to accomplish this.
- After this cost has been suffered, the hope and expectation is that the CPU will continue to make references to data in the block brought in, because accesses to LI data are made at CPU speeds.
- After a block ceases to be referenced for a time, it is ejected from the LI cache.
- It may remain in the L2 cache for a while, and be brought back into the LI cache if any element of the block is referenced again.
- Eventually a block remains unreferenced long enough so that it is ejected also from the L2 cache.

An alternative method copies back a block from one memory level to the lower level, at the point the block is being ejected from the faster level. The write-through strategy avoids writing back blocks when they are ejected, whereas the write-back strategy requires that an entire block be written back when ejected, even if only one word of the block was modified, once.

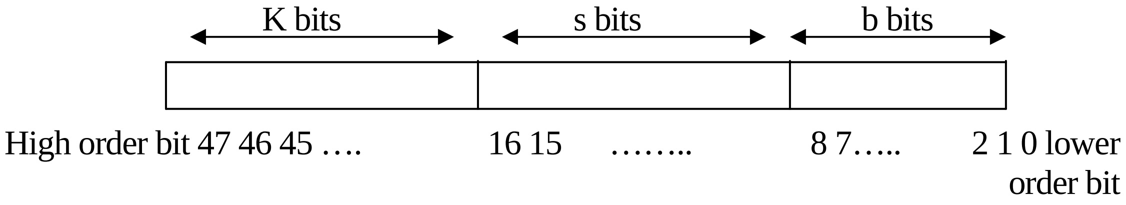
The principal measure of the quality of a memory hierarchy is its hit ratio at each level. As with CPU models, to evaluate a memory hierarchy design one must study the design in response to a very long string of memory references. Direct-execution simulation can provide such reference stream, as can long traces measured reference traffic. Nearly every caching system is a demand system, which means that a new block is not brought into a cache before a reference is made to a word in that block.

Set Associativity

- The concept of set associativity arises in response to the cost of the mechanism used to look for a match.
- Imagine we have an L2 cache with 2 million memory words.
- The CPU references location 10000—the main memory has, say, 2^{12} words, so the L2 cache holds but a minute fraction of the main memory.
- How does the hardware determine whether location 10000 is in the L2 cache? It uses what is called an associative memory, one that associates search keys with data.
- One queries an associative memory by providing some search key. If the key is found in the memory, then the data associated with the key is returned, otherwise indication of failure is given.

Fully Associative

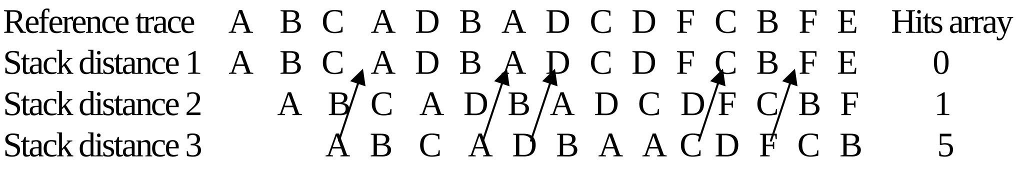
- A fully associative cache is one where any address can appear anywhere in the cache.
- The idea is to partition the address space into sets. Figure given illustrates how a 48-bit memory address might be partitioned in key, set id, and block offset.



- Any given memory address is mapped to the set identified by its set id address bits.
- This scheme assigns the first block of 2^b addresses to set 0, the second block of 2^b addresses to set 1, and so on, wrapping around back to set 0 after 2^s blocks have been assigned.
- Each set is given a small portion of the cache—the set size—which typically is 2 or 4 or 8 words. Only those addresses mapped to the same set compete for storage in that space.
- Given an address, the hardware uses the set id bits to identify the set number, and the key bits to identify the key.
- The hardware matches the keys of the blocks already in the identified set to comparator inputs, and also provides the key of the sought address as input to all the comparators.
- Comparisons are made in parallel; in the case of a match, the block offset bits are used to index into the identified block to select the particular address being referenced.

Least-recently-used

- LRU is the replacement policy most typically used. When a reference is made and is not found in a set, some block in the set is ejected to make room for the one containing the new reference.
- LRU is one of several replacement policies known as *stack* policies. These are characterized by the behavior that for any reference in any reference string, if that reference misses in a cache of size n, then it also misses in every cache of size m <*n*, and if it hits in a cache of size m, then it hits in every cache of size n> m
- The stack distance *of* a block in this list is its distance from the front; the most recently referenced block has stack distance 1, the next most recently referenced block has stack distance 2, the LRU block has stack distance 64.
- Presented with a reference, the simulation searches the list of cache blocks for a match.
- If no match is found, then by the stack property no match will be found in any cache of a size smaller than 64, on this reference, for this reference string. If a match is found and the block has stack distance k, then no match will be found in any cache smaller than size k, and a match will always be found in a cache of size larger than k.
- Rather than record a hit or miss, one increment the kth element of a 64-element array that records the number of matches at each LRU level.
- To determine how many hits occured in a cache of size *n*, one sums up the counts of the first *n* elements of the array.
- Thus, with a little arithmetic at the end of the run, one can determine (for each set cache) the number of hits for every set of every size between 1 and 64.



Reference trace	A	B	C	A	D	B	A	D	C	D	F	C	B	F	E	Hits array
Stack distance 1	A	B	C	A	D	B	A	D	C	D	F	C	B	F	E	0
Stack distance 2		A	B	C	A	D	B	A	D	C	D	F	C	B	F	1
Stack distance 3			A	B	C	A	D	B	A	A	C	D	F	C	B	5

Figure illustrates the evolution of an LRU list in response to a reference string.

- Under each reference is the state of the LRU stack *after* the reference is processed.
- The horizontal direction from left to right symbolizes the trace, reading from left to right.
- A hit is illustrated by a circle, with an arrow showing the migration of the symbol to the top of the heap.
- The "hits" array counts the number of hits found at each stack distance.

- Thus we see that a cache of size 1 will have a hit ratio of 0/15, a cache of size 1 will have a hit ratio of 1/15, and a cache of size 3 will have a hit ratio of 6/15.
- In one pass one can get hit ratios for varying set sizes, but it is important to note that each change in set size corresponds to a change in the overall size of the entire cache

I. INTRODUCCIÓN A LA SIMULACIÓN DEL SISTEMA DE EVENTOS DISCRETOS.

1. Introducción a la simulación.

- Cuando la simulación es la herramienta adecuada.
- Cuando la simulación no es apropiado.
- Ventajas y desventajas de la simulación.
- Las áreas de aplicación. Sistemas y Medio Ambiente del Sistema.
- Componentes de un sistema.
- Discretos y sistemas continuos.
- Modelo de un sistema.
- Tipos de modelos.
- Simulación de eventos discretos del sistema.
- Pasos en un estudio de simulación.

2. Ejemplos de simulación.

- Simulación de Sistemas de colas.
- Simulación de Sistemas de Inventario.
- Otros ejemplos de simulación.

3. Principios generales.

- Conceptos de simulación de eventos discretos.
- Lista de procesamiento.

4. Software de simulación.

- Historia del software de simulación.
- Selección de Software de Simulación.
- Un ejemplo de simulación.
- Simulación en C + +.
- Simulación en GPSS.
- Simulación en CSIM.
- Los paquetes de simulación.
- Experimentación y Herramientas de Análisis Estadístico.
- Tendencias en Software de simulación.

II. MATEMÁTICA Y MODELOS ESTADÍSTICOS.

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- Revisión de la Terminología y conceptos.
- Útiles Modelos Estadísticos.
- Distribuciones discretas.
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- Proceso de Poisson.
- Las distribuciones empíricas.

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- Características de los sistemas de colas.
- Colas de notación.
- Las medidas a largo plazo del rendimiento de los sistemas de colas.
- En estado estable el comportamiento de los modelos de Markov Infinita-Población.
- En estado estable el comportamiento de los modelos finitos de Población.

- Las redes de colas.

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- Generación de números pseudo-aleatorios.
- Técnicas para generar números aleatorios.
- Las pruebas de números aleatorios.

8. La generación aleatoria-variable.
- Transformada inversa de la técnica.
- La transformación directa de las distribuciones Normal y Lognormal.
- Método de convolución.
- La aceptación-rechazo de la técnica.

IV. ANÁLISIS DE LOS DATOS DE SIMULACIÓN.

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- Recopilación de datos.
- La identificación de la distribución con datos.
- Estimación de parámetros.
- Pruebas de ajuste.
- Selección de entrada Modelos sin datos.
- Multivariado y modelos de series temporales de entrada.

10. Verificación y Validación de Modelos de Simulación.
- La construcción de modelos, verificación y validación.
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- Calibración y Validación de Modelos.

11. Resultados del análisis de un modelo único.
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13. Simulación de Sistemas de Gestión de Manufactura y Materiales.
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- Problemas en simulaciones de fabricación y manipulación de materiales.
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14. Simulación de Sistemas Informáticos.

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- Simulación de la CPU.
- Memoria de simulación.

Apéndice

Tablas.