Discrete Mathematics

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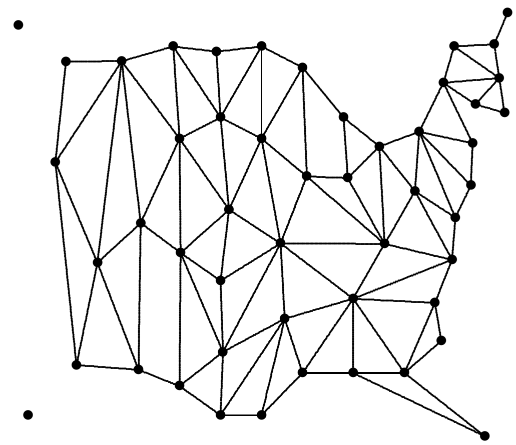

1. A *graph* is a network of dots and lines. What is the meaning of the graph shown at right? What do the dots represent? Why are some dots joined by segments and others are not?

2. Students at Pascal High School have a 25-member student council, which represents the 2000 members of the student body. The class-by-class sizes are: 581 Seniors, 506 Juniors, 486 Sophomores, and 427 Freshmen. How do you think that the seats on the council should be distributed to the classes?

3. Twelve business associates meet for lunch. As they leave to return to their offices a couple of hours later, one of them conducts a small mathematical experiment, asking each one in the group how many times he or she shook hands with someone else in the group. The twelve reported values were 3, 5, 6, 4, 7, 5, 4, 6, 5, 8, 4, and 6. What do you think of this data? Is it believable?

4. If you were President Washington, how would you distribute the 120 seats in the House of Representatives to the fifteen states listed at right? The table shows the results of the 1790 census.

State	**1790 Population**
Connecticut	236841
Delaware	55540
Georgia	70835
Kentucky	68705
Maryland	278514
Massachusetts	475327
New Hampshire	141822
New Jersey	179570
New York	331589
North Carolina	353523
Pennsylvania	432879
Rhode Island	68446
South Carolina	206236
Vermont	85533
Virginia	630560
Total	**3615920**

5. Suppose that consumers prefer Brand X to Brand Y by a 2-to-1 margin, and that consumers also prefer Brand Y to Brand Z by a 2-to-1 margin. Does it follow that consumers prefer Brand X to Brand Z?

6. Three investors control all the stock of Amalgamated Consolidated, Inc. One investor owns 46% of the stock, another owns 37%, and the third owns the remaining 17%. Which stockholder is the most powerful?

7. If you are a voter in a state that has one million other active voters, how likely is it that your vote will be pivotal (in other words, the deciding vote) in the next gubernatorial election?

8. Avery and Cameron want to share a Snickers bar. Describe an *algorithm* (a set of rules that, when followed mechanically, are guaranteed to produce a result) that will create a fair division.

The Apportionment Problem

In every method of apportioning the House of Representatives, the *ideal quota* for a state is now calculated by the formula 435·(state population/total population). Because this is not likely to be an integer, it is necessary to either round up to the *upper quota* or round down to the *lower quota* to obtain a meaningful result. A method of apportionment must specify exactly how this rounding is to be done.

Another quantity of significance in apportionment is the *ideal district size*, which is the total population divided by the total number of representatives. The 2000 census puts this figure at 646 952 = 281 424 177/435. This is how many constituents each representative *should* have (and would have, if Congressional districts were allowed to cross state boundaries).

1. What do you get if you divide a state's population by the ideal district size?

The simplest method of apportionment was proposed in 1790 by Alexander Hamilton, and it is so intuitively appealing that you may have thought of it yourself already: Calculate each state's share of the total number of available seats, based on population proportions, and give each state as many seats as prescribed by the integer part of its ideal quota. The remaining fractional parts of the quotas add up to a whole number of uncommitted seats, which are awarded to those states that have the *largest fractional parts*.

Apply the Hamilton method to the following small, three-state examples. (The names of the states are simply A, B, and C.) You should notice some interesting anomalies.

2. Suppose that the populations are A = 453000, B = 442000, and C = 105000, and that there are 100 delegates to be assigned to these states on the basis of their populations.

3. Suppose that the populations are A = 453000, B = 442000, and C = 105000, and that there are 101 delegates to be assigned to these states on the basis of their populations.

4. Suppose that the populations are A = 647000, B = 247000, and C = 106000, and that there are 100 delegates to be assigned to these states on the basis of their populations.

5. Suppose that the populations are A = 650000, B = 255000, and C = 105000, and that there are 100 delegates to be assigned to these states on the basis of their populations.

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6. A third-grade teacher is arranging a field trip for eight of the boys in his class. These boys do not always behave well together. Adam gets along with Ben, Frank, and Hugh; Ben gets along with Adam, Chris, David, and Hugh; Chris gets along with Ben and Frank; David gets along with Ben, Eric, and Frank; Eric gets along with David, Frank, and Guy; Frank gets along with everyone except Ben; Guy gets along only with Eric and Frank; Hugh gets along with Adam, Ben, and Frank. Draw a graph that models this situation; tell what the vertices and edges represent.

7. Avery, Cameron, and Denis want to divide a rectangular sheet cake fairly. Describe an algorithm for doing so.

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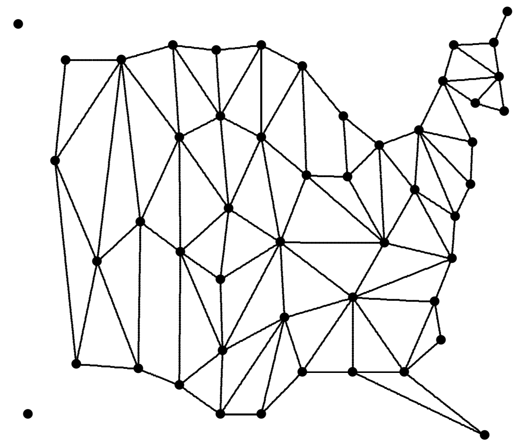


1. Show that it is possible to color a map of the United States using only four colors, so that adjacent states do not receive the same color. The graph at right represents the fifty states and their borders — two states (vertices) are joined when they share a border.

2. Is it possible to do the coloring job described in the preceding item with only three colors? Explain your answer.

Divisor Methods of Apportionment

According to the Hamilton method, quota-rounding decisions are made only after the *entire* list of quotas has been examined (and ranked in order of decreasing fractional parts). There are several other methods for apportionment, each one characterized by a rounding rule that is meant to be applied to individual states, without specific reference to the quotas of other states. These methods are described next.

When an arbitrary (non-ideal) district size is used to divide the state populations, the result is an *adjusted quota* for each state. What happens next depends solely on the rounding rule that is in effect.

The Jefferson method: All adjusted quotas are rounded down. Because all the fractional parts are being discarded, the divisor must be smaller than the ideal district size, if the target number of representatives (435) is to be hit exactly. This method, proposed by Thomas Jefferson, was approved by Washington and applied to the 1790 census, with a House size of 105 and 33000 as the divisor.

A significant amount of trial and error is necessary to carry out this divisor method (or any of the others). If the divisor is too small, the total number of assigned representatives will exceed 435; if the divisor is too large, the total will fall short of 435. For a project of this size (each trial divisor must be divided into *all fifty* state populations), it is desirable to use a computer to carry out the numerical work. The program *Windisc* was written to simplify these investigations. It is described on the next page.

The Adams method: Adjusted quotas are rounded *up*. An acceptable divisor must be larger than the ideal district size. This method was proposed by John Quincy Adams. It has never been adopted.

The Webster method: Adjusted quotas are rounded *in the usual way* — to the nearest whole number. This method was proposed in 1831 by Daniel Webster (PEA 1796), but not used until the 1840 census.

3. Pascal High School has a 25-member student council to represent its 2000 students. There are 581 Seniors, 506 Juniors, 486 Sophomores, and 427 Freshmen. Apply the Jefferson, Adams, and Webster methods to apportion the seats on the council.

Apportionment on the computer

To apply *Windisc* to an apportionment problem, click **Window**|**Apportionment**. You should then see a new window, which contains a partial display of 2000 census data. To see more of the data, use the scrolling buttons. As you scroll, notice that the column totals and headings remain in view, in contrasting color.

The *Quota* column contains the ideal quota for every state. When one of the divisor methods is being used, there is also an *Adjusted Quota* column, which tables the result of dividing each population by the current divisor. Because the initial value of the divisor (when the window first appears) is the ideal district size, the two quota columns contain the same values. The adjusted quota column disappears when the Hamilton method is selected. Another column shows the actual *District size* for each state, which is obtained by dividing the state's population by its assigned number of representatives.

Population data for every US census (there have been 23 of them) is stored in the program. To retrieve another example, use the menu **File**|**New**|**Census**.

The current method is displayed in the window title, and it is also indicated by a check mark in the **Method** menu. When the window first opens, the Hamilton method is selected. To change the method, just click one of the other choices.

Most choices are divisor methods, which require that the divisor be adjusted until an acceptable value is found for it (in other words, until the total number of assigned representatives has the required value). To change the divisor, open the **Divisor** dialog box. You can now type in a new value for the divisor and press *Enter*, or explore a range of possible values with the slider. The display is updated immediately, and the window title tells you whether the new divisor is acceptable, meaning that the total at the bottom of the *Reps* column agrees with the prescribed number (whose default value is 435).

The data is arranged alphabetically by state name. It can also be displayed in order of population. To choose this display mode, click **Edit**|**Arrange by**|**Population**. Check the item **Edit**|**Arrange by**|**Increasing** to put the smallest example first in the list. You can also request that **Row numbers** be displayed.

For some exercises, it might be necessary to edit the population data, and you might also want to alter the names that appear in the left column. To change either kind of data, point at the desired item and left-click it.

The number of states can be changed from its initial value of 50. Click **Edit**|**Number of states**, or **Delete** selected rows. The number of representatives can be assigned a value other than 435 by left-clicking the total at the bottom of the *Quota* column. The program expects to receive a positive whole number, of course.

To print the worksheet for a particular method, or to save it as a text file, or to copy it to the clipboard, click **File**|**Text out**. For more information, click **Help**.

Every state gets a representative: This is mandated by the Constitution. Because the Hamilton, Jefferson, and Webster methods do round down at least some of the time, their rounding rules must be amended to prohibit rounding down to zero. These amended methods are identified in the *Windisc* program by the code *min=1* in the **Methods** menu. From now on, it is understood that the amended methods will be applied.

1. Apply the methods of Jefferson, Adams, and Webster to the 2010 USA census, and compare the results with the Hamilton apportionment (which appears when the window first opens). You will need to use trial and error to find three acceptable divisors, of course. To compare apportionments side-by-side, you need only click **Other|Register apportionment** when the Hamilton apportionment is on the screen, and then click this item again after each divisor search has been successfully completed. To see all four apportionments simultaneously, click **Other|See summary**. For each divisor method, make note of how its result differs from that of the Hamilton method.

2. Repeat the preceding comparison for the 1790 USA census. There were only fifteen states then, and the total number of representatives was 105. The method actually used was Jefferson's. Rediscover his divisor.

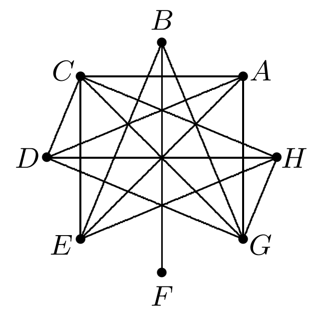
3. The Jefferson method of apportionment, which was used from 1790 through 1820, produced some troubling results. This was the basis for a speech by Daniel Webster (a Senator from Massachusetts) in 1831, when he argued for adoption of his apportionment method. At the urging of former President John Quincy Adams (who proposed his own method as well), Webster was trying to do something about the diminishing representation of the New England states, as well as remind Congress about the specific wording of the apportionment section of the Constitution. Despite the Senator's eloquent defense of the quota concept, the Jeffersonian forces won the debate. To see what the issue was, apply the three methods (Jefferson, Webster, Adams) to the 1830 census.

4. The Hamilton method, which was used from 1850 to 1890, has the undesirable property of not being House-monotonic, meaning that increasing the size of the House can cause a decrease in representation for some state. This phenomenon became known as the *Alabama Paradox* when it appeared after the 1880 census, because representatives were not being assigned to Alabama in a monotonic fashion for House sizes between 298 and 302. This phenomenon had actually been noticed ten years earlier, when the Hamilton method was applied to various House sizes between 268 and 283; the troublesome state then was Rhode Island. The final exasperating appearance of this paradox occurred after the 1900 census. This time, both Colorado and Maine were in fluctuation. A Congressional bill was proposed, which would have enlarged the House to 357 (from 356 in 1890). The resulting debate was bitter and partisan, leading Congress to scrap the Hamilton method. In its place, the Webster method was applied to a House size of 386. This ended the controversy. To see what the fuss was about, compare the Webster apportionment for House size 386 with the Hamilton apportionments for House sizes 356, 357, 358, 385, 386, and 387.

1. When two vertices of a graph are joined by an edge, the vertices are called *adjacent*. A graph in which every pair of vertices is adjacent is called a *complete graph*. How many edges are needed for a 24-vertex graph to be complete?

2. A graph can be *colored with n colors* if each of its vertices can be assigned one of the n colors in such a way that adjacent vertices are not assigned the same color. What does coloring a graph have to do with coloring a map?

3. The smallest value of n for which a graph can be colored with n colors is called the *chromatic number* of the graph. Find the chromatic number of the graph shown at right. Each vertex represents a boy, and two vertices are joined by an edge if the two boys do not get along. For what purpose might you want to color such a graph with a minimal number of colors?



4. What is the chromatic number of a complete graph that has 7 vertices? What is the chromatic number of a complete graph that has n vertices?

5. A *connected* graph has the property that any two of its vertices can be joined by a chain of edges (also called a *path*), so that each edge in the chain has a vertex in common with the edge that follows. For example, the USA graph on page 1 is *not* connected. Give an example of a connected graph with twelve vertices that can be colored with only two colors.

6. You have seen that the Hamilton method of apportionment can produce the Alabama Paradox, which is one of the reasons that this method fell into disuse during the nineteenth century. Perhaps the divisor methods that have taken its place are susceptible to the same flaw, however. What do you think?

7. What happens to the adjusted quota for a state when the trial divisor is made slightly larger? when the trial divisor is made slightly smaller?

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1. The current method of apportionment (used since the 1940 census) is the invention of two mathematicians, Edward Huntington and Joseph Hill. It is a divisor method, whose rounding rule is a bit mysterious. At first glance, one might think that the normal Webster rounding was in effect. To see that this is not the case, consider the 1990 census: Select the Huntington-Hill method, for which an acceptable divisor is 575 000. Look at the adjusted quota entries for Oklahoma and Mississippi; how are they rounded?

2. Show that the *geometric mean* of two different positive numbers is always between the numbers, and is smaller than the *arithmetic mean*. In other words, explain why the inequality $x < \sqrt{xy} < \frac{1}{2}(x+y) < y$ holds whenever x and y are positive numbers, with x smaller than y.

3. A map-coloring algorithm, applied to the USA map: Color the states in alphabetic order, always using the first available color. For instance, Alabama gets color number 1, as do Alaska, Arizona, and Arkansas. California gets color number 2, Colorado gets color number 1, and so on. How close does this algorithm come to doing the best possible job (using only four colors)?

4. A *fair-division scheme* is a set of rules that, when applied, produces a fair division of the object or objects to be divided. Any fair-division scheme should
(a) be decisive;
(b) be internal to the players;
(c) assume that the players have no knowledge about each others' value systems;
(d) assume that the players are rational.

Decide why each of these conditions is important, then describe a fair-division scheme that would allow n people to fairly divide an object or region.

5. Fair-division problems can be classified into three categories: A *continuous* fair-division problem means that the objects under consideration are divisible, like a cake. A *discrete* fair-division problem is one in which the objects under consideration are indivisible, like houses, cars, and jewelry. A *mixed* fair-division problem is one that involves both divisible and indivisible objects — for example, a house, a car, and a parcel of land. Hugo Steinhaus, a 20^{th}-century Polish mathematician, proposed the method of *sealed bids* to deal with discrete fair-division problems like the following:

Suppose that four siblings — Art, Betty, Carla, and Dave — have inherited three discrete objects — a house, a Rolls Royce, and a Picasso. The heirs are instructed to make sealed bids, giving their opinions of what each of the contested objects is worth. The results of these sealed bids are summarized in the table below.

	Art	Betty	Carla	Dave
House	220000	250000	211000	198000
Rolls Royce	40000	30000	47000	52000
Picasso	280000	240000	234000	190000

In other words, the house is worth only \$198000 to Dave, but it is worth \$250000 to Betty.

What do you think Steinhaus would have done with this data?

Which method is fairest?

The most obvious source of dissatisfaction with any apportionment is the *state-to-state variation in district size.* In other words, the ratio of a state's population to its assigned number of representatives is not the same for all states. There are methods of apportionment that seek to *minimize* this variation.

Here is an example: If the Hamilton method were used to apportion the House for the 1990 census, it would give Mississippi 4 representatives (its lower ideal quota) and New Jersey would receive 14 (its upper ideal quota). Why might an apportionment method reverse this lower/upper decision and give 5 to Mississippi and 13 to New Jersey? Let us look at district sizes. Under the Hamilton method, there would be 646 611 citizens for each representative in Mississippi, and 553 474 for each representative in New Jersey; the difference 93 137 is a measure of *inequity* in this two-state apportionment. On the other hand, if New Jersey gave a seat to Mississippi, the New Jersey district size would rise to 596 049 while the Mississippi district size would drop to 517 289, creating an inequity of only $596\,049 - 517\,289 = 78\,760$. The improvement justifies the transfer.

Here is a contrasting example: According to the Huntington-Hill method, Montana gets one representative, creating a district size of 803 655; meanwhile, North Carolina gets twelve representatives and a district size of 554 802. The difference in sizes is 248 853. If North Carolina were to surrender one of its seats to Montana, the district sizes would become 605 239 (N Carolina) and 401 828 (Montana); the difference has dropped to 203 411. It seems that the transfer ought to be made. What justifies *not* making it?

The Huntington-Hill theory equates inequity with the *ratio* of district sizes. The Montana-North Carolina inequity is therefore $803\,655/554\,802 = 1.449$, meaning that the representative in Montana has 44.9% more constituents than each representative in North Carolina has. If North Carolina were to surrender a representative to Montana, the inequity would become $605\,239/401\,828 = 1.506$, meaning that each representative in North Carolina would have 50.6% more constituents than each representative in Montana. Because switching a seat from North Carolina to Montana would therefore *increase* the inequity, it is not done.

The inescapable conflict thus centers on *how* one chooses to measure inequity. It appears to be an arbitrary choice. There are even more possibilities than the two mentioned above. One could instead focus on the fractional part of a representative per person (the reciprocal of the district size, that is), and seek to minimize the state-to-state variation in this quantity. *This is what the Webster method does*. In fact, *any* divisor method can be interpreted (and thus recommended) as a method to minimize state-to-state variation according to some calculated sense of inequity.

In particular, measuring inequity by the *differences* in district size (as in the first example) defines the method that was first proposed in 1831 by James Dean, a professor of mathematics at the University of Vermont. It has never been used, but Montana sued for its adoption following the 1990 census. The second example explains why. The suit was denied by the US Supreme Court in 1991.

1. The Huntington-Hill method of apportionment makes its rounding decisions according to whether the adjusted quota is less than or greater than the geometric mean of the *lower adjusted quota* (L) and *upper adjusted quota* (U). The geometric mean formula is $\sqrt{L \cdot U}$. Explain why this method automatically gives every state at least one representative.

2. If the Webster method were applied to the 1990 census, it would give Massachusetts eleven representatives and Oklahoma five. The Huntington-Hill method, on the other hand, gives Massachusetts ten representatives and Oklahoma six. Give a district-size explanation that justifies this transfer of a representative from Massachusetts to Oklahoma.

3. Compare the configuration formed by the states Nevada, California, Oregon, Idaho, Utah, and Arizona with the configuration formed by the states West Virginia, Ohio, Pennsylvania, Maryland, Virginia, and Kentucky. What do they have in common?

4. The *valence* of a vertex is the number of attached edges. If the graph is *directed*, then the terms *invalence* and *outvalence* are sometimes used as well. What is the largest valence that occurs in the USA graph on page 1? What is the smallest valence?

5. If the Hamilton method were applied to the 2000 census, it would give California 52 representatives and Utah four representatives. The Huntington-Hill method, on the other hand, gives California 53 representatives and Utah receives only three. Give a district-size explanation that justifies this transfer of a representative from Utah to California.

6. Invent an algorithm for coloring a graph, trying to use as few colors as possible.

7. Pascal High School has a 25-member student council to represent its 2000 students. There are 581 Seniors, 506 Juniors, 486 Sophomores, and 427 Freshmen. The Webster method assigns seven representatives to both the Senior and Junior classes. This leaves the Seniors disadvantaged relative to the Juniors, producing a *representational deficiency*, whose formula is $D_{sj} = r_j(p_s/p_j) - r_s$, where $r_j = 7$, $p_s = 581$, $p_j = 506$, and $r_s = 7$.
(a) What is the meaning of the number $r_j(p_s/p_j)$?
(b) Interpret the formula for D_{sj}, and calculate its value.
(c) The Jefferson method transfers one representative from the Juniors to the Seniors, thereby disadvantaging the Juniors. Show that the size of the representational deficiency is decreased by this transfer, which is why the Jefferson method does it.

8. (Continuation) The Webster method assigns seven representatives to the Junior class and five to the Freshman class. This makes the Juniors advantaged relative to the Freshmen, producing a *representational surplus*, whose formula is $S_{jf} = r_j - (p_j/p_f)r_f$, where $r_j = 7$, $p_f = 427$, $p_j = 506$, and $r_f = 5$.
(a) Interpret the formula for S_{jf}, and calculate its value.
(b) The Adams method transfers one representative from the Juniors to the Freshmen, thereby advantaging the Freshmen. Show that the size of the representational surplus is decreased by this transfer, which is why the Adams method does it.

9. (Continuation) The Webster method would undo both of these transfers. Why? Make calculations that justify the Webster approach to apportionment.

How are elections decided?

When there are only two candidates, the winner is necessarily accepted by a *majority* of the voters (more than 50%). When more than two candidates are on the ballot, however, there is no simple, satisfactory answer. Here are some of the traditional approaches:

The Plurality Method: The election is won by the candidate with the most first-place votes, even though the winner might fail to receive a majority of voter support.

Point Count: Each voter is to rank all the candidates from first to last. Each ballot is used to award points to the candidates, on the basis of the voter's rankings: The *Borda system* is to give 0 points for last place, 1 point for next-to-last, 2 points for next-to-next-to-last, and so on. Other systems of awarding points are also possible (5 for first, 3 for second, 1 for third, for example).

Condorcet Method: Each candidate meets every other candidate in a head-to-head majority contest. A candidate who emerges *undefeated* is called the *Condorcet winner*. It is possible to be the tournament champion (have the best won-lost record) without being undefeated, however. A candidate who goes winless is called the *Condorcet loser*.

Winners Runoff: The top two finishers in a plurality contest meet in a head-to-head majority contest.

Eliminate the Losers: After a plurality decision, the last-place candidate is eliminated and a new election is held. This process is repeated until the final two-candidate contest is decided (by a majority).

Pairwise Majority: Arrange the candidates in some (perhaps random) order, then stage a sequence of head-to-head majority contests. The winner of each contest faces the next candidate in the list. The first contest is between the first two candidates in the list.

To analyze these methods, it is expedient to assume that each voter is willing and able to rank the candidates from most preferred to least preferred, and that the ballots collected contain all this information (as in the point-counting method). This enables the election to be evaluated as the result of a single ballot, even when applying a method that seems to call for a sequence of ballots.

For example, if a voter prefers candidate P to candidate Q, and candidate Q to candidate R, the ballot essentially consists of the list PQR. By listing the candidates in order of preference, voters are providing all the information that is necessary to carry out any of the above methods.

The goal of an election is to discern *the will of the voters* — in effect, to produce a *group preference list* that is based on the individual preferential ballots. This is an elusive goal. Not only do the various methods not always agree, but each is susceptible to voting paradoxes and to manipulation.

1. In the 1912 Presidential election, the three candidates were Wilson, Roosevelt, and Taft. Polls showed the following preferences: WRT 45%, RTW 30%, and TRW 25% (the most liked candidate listed first).
(a) Who won the plurality contest?
(b) Was there a Condorcet winner? a Condorcet loser?
(c) Who would have won with the scoring system that assigns the points 2,1,0 (the simple Borda count)? Are there point systems that give different outcomes?
(d) Who would have won if a winners' runoff had been held between the top two finishers in the plurality contest?

2. A slight population shift (or some miscounting) in the 1990 census could have affected the apportionment outcome. For example, calculate the effect of moving 6000 individuals from Washington to Massachusetts. You will need to make a slight change in the divisor.

3. Massachusetts appealed the 1990 apportionment, based on census data. Specifically, the inclusion of overseas military personnel (which has been the law since 1970) alters the data enough to drop Massachusetts to its lower quota (and raise Washington to its upper quota). Use the apportionment program *Windisc* to confirm this result. The special (non-military) data is found in the **File|New|Special** menu. Call it up and apply the Huntington-Hill method to it. You will have to experiment a bit with the divisor, of course. Are any states other than Massachusetts and Washington affected by this change of data?

4. Given a graph, its valences can be listed. Conversely, given a list of nonnegative integers, one wonders whether there is a graph with precisely these valences. Which of the following lists correspond to actual graphs?
(a) 4,4,4,4,4 **(b)** 4,4,2,2,2 **(c)** 4,4,3,3,2 **(d)** 4,4,4,3,2

5. Is it possible for a graph to have exactly one vertex whose valence is odd, the rest of the valences being even?

6. A graph has the following valences: 8,6,6,5,5,3,3,3,3. How many edges does the graph have? Draw such a graph. Is there more than one example?

7. The ballots for a three-candidate contest have been tallied: ABC (38%), CAB (30%), BCA (25%), and BAC (7%). Analyze the results.

8. Fifty-five voters cast ballots for a slate of five candidates: 18 vote ADECB, 12 vote BEDCA, 10 vote CBEDA, 9 vote DCEBA, 4 vote EBDCA, and 2 vote ECDBA. Apply the various methods to determine the winning candidate.

1. The Senior Dance Committee is deciding the format of this year's big event; the options are Caribbean Cruise, Hawaiian Holiday, or Riviera Romance. The nine Committee members rank them as follows: CHR, CHR, HRC, HRC, HRC, RCH, RCH, RCH, RCH.
(a) On the basis of plurality, which theme would prevail?
(b) Apply a pairwise majority approach to determine a winning theme.
(c) Explain how the chairperson of the Committee might (shrewdly) be able to rig the pairwise majority process.
(d) Suppose that the chairperson favors the Riviera theme; what method (or methods) might the Committee find itself using? If the two members who rank the Riviera theme last were clever, how might they gain consolation through strategic (insincere) voting?
(e) Would adopting a point-counting approach clarify the outcome?

2. Because its population did not increase as significantly as that of other states, New York lost 3 representatives from its 1980 apportionment. This could have been offset if Congress had decided to increase the size of the House in 1990. How large a House would be necessary in order to maintain New York's apportionment at its 1980 level? Use the Huntington-Hill method. It is not necessary to use trial and error.

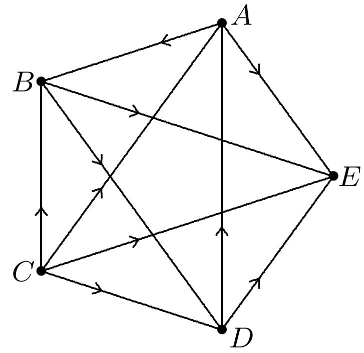
3. If the population of every state had grown by the same fixed percentage during the decade 1980-1990, would the 1990 apportionment have looked any different from the 1980 apportionment? Explain your answer.

4. If the only population change during the decade 1980-1990 had consisted of one state losing residents to another state (the other forty-eight staying the same), what effects could that have had on the 1990 apportionment? Other than the expected change — one state gains a representative from the other — or no change at all, are there any other possible apportionment effects?

5. The Dean method of apportionment makes its rounding decisions according to whether the adjusted quota is less than or greater than the *harmonic mean* of the upper and lower adjusted quota values (the formula is $2UL/(U+L)$). Explain why this method automatically gives every state at least one representative.

6. Suppose that the following medal count occurs in the next Olympic Games: the USA wins 7 Gold, 15 Silver, and 6 Bronze; Japan wins 8 Gold, 12 Silver, and 7 Bronze. Devise point-counting systems that make each team the overall champion.

7. With five candidates in the race, how many head-to-head majority contests need to be examined in order to complete the Condorcet analysis?



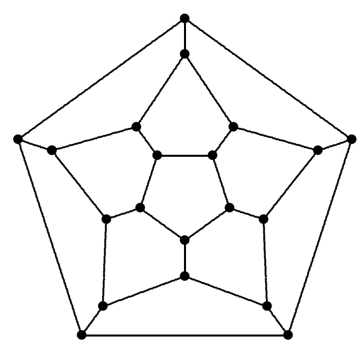
1. The results of a 5-candidate preference poll are shown in the graph at right. Each vertex represents a candidate, and each pair of vertices is joined by an edge, representing a one-on-one comparison. On each edge, the arrow is directed from the winner toward the loser. Because of the arrows, this is an example of a *directed graph* (usually called a *digraph* for short). You can see that there is both a Condorcet winner and a Condorcet loser in this preference poll. Which candidates are these?

2. Produce a similar digraph that represents the results of the 5-candidate preference poll: BEDAC (21%), EBCAD (17%), CAEBD (30%), DCAEB (32%)

3. Twelve members of the mathematics department have been assigned to eight committees, which need to pursue some important end-of-term business. Each committee has to meet for an entire day to do its job. The committees:

A: Coogan, Feng, Geary	E: Braile, Parris, Hardej
B: Feng, Heisey, Wolfson	F: Ibbotson, Mallinson, Braile
C: Mallinson, Wolfson, Holden	G: Hardej, Seidenberg, Heisey
D: Parris, Holden, Coogan	H: Seidenberg, Geary, Ibbotson

Notice that each member is on two of the committees. Is it possible for the committee work to be done in fewer than eight days? If so, how many?



4. What is the chromatic number of the graph shown at right?

5. The graph shown at right has been borrowed from the realm of geometry. Can you identify the source of this diagram?

6. The graph at right is called *trivalent* because every vertex has valence 3. Find some other examples of trivalent graphs.

7. Refer to the 1990 USA census, shift 13 000 persons from New Jersey to Massachusetts. What effect does this have on the Huntington-Hill apportionment?

8. Refer to the 1990 USA census and shift 23 000 persons from Massachusetts to New Jersey. What effect does this have on the Huntington-Hill apportionment?

9. In questions of the preceding sort, where exactly two states redistribute some of their population, it is possible for a third state to become involved in the redistribution of representatives. Is it possible for more than two states to actually see a change in their representation? Explain, or give examples.

1. If there is a Condorcet winner, need there be a Condorcet loser? If there is an Condorcet winner, what will happen if there is a pairwise majority runoff?

2. Given three candidates A, B, and C, there are, in a sense, *eight* possible preferential digraphs. In another sense, there are only *two*. Explain these two statements.

3. Do an analysis of a four-candidate race, similar to the preceding. In particular, decide whether there must be a Condorcet 4-cycle if there is no Condorcet winner.

4. Graphs provide a means of summarizing information. To add to the display, it is sometimes appropriate to label the edges of the network with meaningful numerical data. Give an example of how this might be done.

5. Under what conditions would the ideal district size (the total population divided by the House size) actually serve as a suitable divisor for **(a)** the Webster method; **(b)** the Jefferson method?

6. Apply the Webster method to the four-state census data $A = 70653$, $B = 117404$, $C = 210923$, $D = 1194456$, with house size 35. What do you notice about the results?

7. The following has been proposed as a way of apportioning the House of Representatives: Take the state with the smallest population, and use this as the divisor for a Webster apportionment. The House size is determined by this data. The stipulation that all states be represented is automatically provided for. Prepare an argument for or against adoption of this method.

8. One way to deal with fractional parts of quotas is to leave them as is; if some state's quota is 13.475, then give the state that many votes, to do with as it wishes. For example, that state could send one representative to Congress (with 13.475 as a weighted vote), or it could send fourteen representatives, one of which has a vote weighted at only 0.475. Prepare an argument for or against the adoption of such a system of representative government.

9. As you have seen, the Hamilton method has many flaws, one of which is in coping with the Constitutional requirement that all states be represented. The method begins by rounding all quotas less than 1 up to 1; then the chase to fill remaining seats begins. It is still possible for this process to stall, however. The fractional parts might not add up to enough seats to provide for very small states.
(a) Invent an example to illustrate this phenomenon.
(b) Analyze the following modification of the Hamilton method: Begin by giving every state its one representative; then apply the usual Hamilton process to the remaining seats. It is no longer necessary to give attention to underrepresented states.

10. The rounding rules for the Webster and Huntington-Hill methods are very nearly the same, which suggests that they will usually produce the same apportionments. Give an example that shows that the two methods do not always agree.

1. The Hamilton method is to be applied to the following ideal class quotas: $Q_S = 8.1$, $Q_U = 7.5$, $Q_L = 6.6$, $Q_P = 4.8$.
(a) How many representatives are being apportioned?
(b) How many does each class get?
(c) If the total school population is 990, what is the Senior class size?

2. For each of four divisor methods, give the rounded value that would be produced if the method were applied to the adjusted quota value 5.481.

3. Distinguish clearly among the four similar terms *ideal quota*, *adjusted quota*, *upper quota*, and *lower quota*. Do not use ambiguous terminology.

4. What does it mean for an apportionment method to *respect quota*? Which methods of apportionment are *guaranteed* to respect quota?

5. The Huntington-Hill method for apportioning the House of Representatives has been the official method since 1940.
(a) Give at least two reasons for abandoning it in favor of another method.
(b) Give at least two reasons for continuing to use it.

6. Congress voted to replace the Webster method by the Huntington-Hill method after the 1940 census. Although Hill invented his method in 1911, and Huntington promoted it vigorously from 1921 to 1940, it was not until 1940 that it actually disagreed with the Webster method. The difference was confined to only two states: Michigan (pop 5 256 106) received 18 seats from Webster and 17 from Huntington-Hill; Arkansas (pop 1 949 387) received 6 seats from Webster and 7 from Huntington-Hill. Why does the Huntington-Hill method of measuring fairness force Michigan to give up its 18th seat to Arkansas?

7. Suppose that every voter ranks candidate A ahead of candidate B. Any reasonable procedure for evaluating the votes should, it seems, place A ahead of B in the final summary. Show that this *axiom of unanimity* is not respected by the pairwise-majority approach to elections.

8. Suppose that candidate C withdraws from the race, and that all references to C are removed from every voter's preferential ballot (without affecting the rest of the list, of course). It does not seem that this turn of events should affect how voters, as a group, view A versus B. Is this *axiom of irrelevant alternatives* respected by the various approaches to evaluating an election?

9. To be useful, a method for evaluating voter's ballots ought to produce a group ranking from the individual rankings. Which of the methods we have examined *do* produce such a list? For those that do not, propose ways to fix them.

10. Borda's original thought was to award each candidate a point for every other candidate outranked on a voter's ballot. What point system is this equivalent to?

1. If a graph is connected and has 12 vertices, what is the smallest number of edges it can have? What if the graph is connected and has n vertices?

2. A graph that is connected, with as few edges as its vertex count allows (see the preceding), is called a *tree*.
(a) Explain why a tree can not contain any circuits.
(b) Give some examples of eight-vertex trees.

3. In an n-vertex complete graph, how many edges are there?

4. With a 435-seat House, the 1990 census allows Montana (pop 803 655) only one seat. If the House were enlarged sufficiently, Montana could regain this seat. For example, a House size of 538 would give Montana two seats. What is the *smallest* House size that would restore Montana's lost seat?

5. Examine the Jefferson apportionment based on the 1820 census. Opponents of this method found the data unacceptable. What was the reason for their objections? Mention specific data to justify your answer.

6. If there is a Condorcet winner, does this candidate necessarily win the simple Borda count? (Working with an n-candidate field, Borda would give a candidate $n-1$ points for being first on a ballot, $n-2$ points for being second, and so on, with 0 points awarded for being last.)

7. Seventeen voters have the following preferences regarding A, B, and C: Six rank them ABC, five rank them CAB, four rank them BCA, and two rank them BAC.
(a) Who would win the plurality-with-runoff contest?
(b) The last two voters, swayed by public-opinion polls that suggest A to be the leading candidate, decide at the last minute to vote (insincerely) with the winner. They both change their preferences from BAC to ABC. Does this move actually *help* candidate A?

8. Research the origins of the word *gerrymandering*.

9. Find out who Borda and Condorcet were, and why their names have become part of the theory of voting.

Discrete Mathematics

1. Here is an example from sports, which shows how difficult it is to devise a system of ranking that can be counted on to obey common sense. The problem is to choose the winner of a cross-country race between two teams of seven runners each. The traditional method is fairly simple: A runner who finishes in nth place receives n points, and a team's score is the sum of the points of its first *five* finishers. Low score wins, of course.
(a) For example, suppose that the runners in a race between team A and team B finish in the order AAABBBBBAABABA. Show that team A wins, 25 to 30.
(b) Which team wins the race if the order of finish is ABBAABABBAABAB?
(c) Suppose that three teams take part in a race, and that the twenty-one runners finish in the order CAAABBBBCCCCCBCAABABA. This single race can be scored as three separate 2-way races, by simply ignoring the presence of one team at a time. For instance, verify that the race between team A and team B has already been dealt with in (a). Score the race between team B and team C (ignore A), then score the race between team C and team A (ignore B). Are the results of these three calculations conclusive?

2. Suppose that candidates A, B, and C are ranked by twenty voters, so that A gets 7 firsts, 9 seconds, and 4 thirds; B gets 8 firsts, 5 seconds, and 7 thirds; and C gets 5 firsts, 6 seconds, and 9 thirds.
(a) Show that A wins the simple Borda count election, with B second.
(b) Show that no other system of assigning points (weights) to the three places can possibly improve C's third-place finish.
(c) Show that B can be ranked first with a suitable system of points.

3. Is the data in the preceding question actually *possible*? In other words, is it possible for twenty voters to rank A, B, and C so as to produce the stated number of firsts, seconds, and thirds for each candidate?

4. Show that there are two inequivalent graphs that share the valence list 3, 3, 2, 2, 2.

5. Is it possible for a tournament director to set up a schedule that has each of the nine entered teams playing exactly five games against the others?

6. Is the six-state system consisting of Illinois and its five neighbors equivalent to the six-state system consisting of Nevada and its five neighbors?

Where does the Geometric Mean come from?

In the following, p stands for population, r stands for the number of representatives, and thus p/r stands for a district size.

It is common sense that State 1 is *disadvantaged* compared to State 2 if the district size p_1/r_1 is greater than the district size p_2/r_2. The Huntington-Hill method of measuring this inequity is to calculate the *ratio* of p_1/r_1 to p_2/r_2, which can be written as $\dfrac{p_1 r_2}{p_2 r_1}$.

State 2 should give up one of its representatives to State 1 if it reduces the inequity. (Notice that State 2 must have at least two representatives for this to make sense.) After such a transfer, State 2 would have $r_2 - 1$ representatives and district size $p_2/(r_2-1)$, while State 1 would have $r_1 + 1$ representatives and district size $p_1/(r_1+1)$. If State 1 were *still* disadvantaged after such a move, then it is obvious that this transfer must be made. The ambiguous case occurs when State 2 becomes disadvantaged by the move, meaning that the district size $p_2/(r_2-1)$ is greater than the district size $p_1/(r_1+1)$. The proposed transfer must be rejected if it would increase the inequity. In other words, if

$$\frac{p_1 r_2}{p_2 r_1} < \frac{p_2(r_1+1)}{p_1(r_2-1)}\,,$$

then the Huntington-Hill method leaves the apportionment as is, because *State 2 would be worse off after the transfer than State 1 was before it.* Notice that the left side of this comparison is calculated before State 2 transfers a representative to State 1, and the right side is calculated after the transfer is made. Further algebraic simplification

$$\begin{aligned}\frac{p_1^2}{r_1(r_1+1)} &< \frac{p_2^2}{(r_2-1)r_2}\\ \frac{p_1}{\sqrt{r_1(r_1+1)}} &< \frac{p_2}{\sqrt{(r_2-1)r_2}}\end{aligned}$$

separates the two subscripts from each other. In the final comparison, notice that the geometric-mean formula $\sqrt{L \cdot U}$ has finally appeared. Also notice that each side of this inequality can be viewed as a district size, because each is roughly p/r. Suppose that d is a number such that

$$\frac{p_1}{\sqrt{r_1(r_1+1)}} < d < \frac{p_2}{\sqrt{(r_2-1)r_2}}\,,$$

which is equivalent to

$$\frac{p_1}{d} < \sqrt{r_1(r_1+1)} \quad \text{and} \quad \sqrt{(r_2-1)r_2} < \frac{p_2}{d}\,.$$

If d were a Huntington-Hill divisor, this would say that p_1/d rounds *down* to r_1 and p_2/d rounds *up* to r_2.

This reasoning must of course apply to *every* pair of states. Suppose that the Huntington-Hill method assigns r_1, r_2, r_3, ... representatives to the states whose populations are p_1, p_2, p_3, ... In order that no state-to-state transfers be necessary,

$$\frac{p_n}{\sqrt{r_n(r_n+1)}} < \frac{p_m}{\sqrt{(r_m-1)r_m}}$$

must hold for any states m and n. In particular, the *largest* of the numbers $p_n/\sqrt{r_n(r_n+1)}$ is smaller than the *smallest* of the numbers $p_m/\sqrt{(r_m-1)r_m}$. Let d be any number between these extremes, so that

$$\frac{p}{\sqrt{r(r+1)}} < d < \frac{p}{\sqrt{(r-1)r}}$$

holds for any state. This pair of inequalities can be rewritten

$$\sqrt{(r-1)r} < \frac{p}{d} < \sqrt{r(r+1)}$$

Any such number d serves as a suitable Huntington-Hill divisor, and the inequalities define the Huntington-Hill rounding process. The apportionment for a state with population p is determined by where the value of p/d falls in relation to the list of square roots

$$\sqrt{0}\,, \sqrt{2}\,, \sqrt{6}\,, \sqrt{12}\,, \sqrt{20}\,, \sqrt{30}\,, \sqrt{42}\,, \sqrt{56}\,, \sqrt{72}\,, \ldots$$

In other words, p/d must fall between two successive terms of this sequence — say $\sqrt{(r-1)r}$ and $\sqrt{r(r+1)}$ — and the state is therefore awarded r representatives.

1. It was tacitly assumed above that equalities of the type

$$\frac{p_n}{\sqrt{r_n(r_n+1)}} = \frac{p_m}{\sqrt{(r_m-1)r_m}}$$

cannot occur. Is this a reasonable assumption to make?

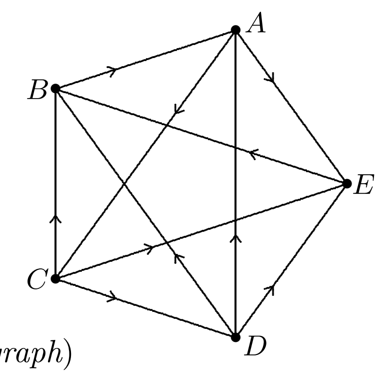
1. Consider the following voter preferences: ABC (29%), ACB (11%), BCA (33%), CAB (23%), and CBA(4%). The voters thus give 40% of the first-place votes to A, 33% to B, and 27% to C. If the point system (3,2,0) were used, C would be declared the winner. If the simple Borda system (2,1,0) were used, however, then A would win. Is there a point system that would make B the winner?

2. In a three-candidate election, suppose that candidate P has more first-place votes than candidate Q, but also more third-place votes. Show that P could outpoint Q or that Q could outpoint P, depending on how points are assigned to the three places.

3. If the Condorcet paradox is present, explain why pairwise majority voting is a totally unsatisfactory way to decide an election.

4. Suppose that the Condorcet (tournament) digraph has been drawn for a field of candidates. If one candidate now withdraws from the race, what changes need to be made in the digraph?

5. If a four-candidate race has no Condorcet winner and no Condorcet loser, you have seen that there must be a four-cycle (the Condorcet paradox). The digraph shown at right is an attempt to find a five-candidate race in which there is no Condorcet winner or loser, but no five-cycle, either. Has the search been successful?



6. (Continuation) In how many ways can arrows be placed on the edges of the tournament graph? (called *directing the graph*)

7. A tennis coach divides her nine players into three teams of three each. Team A has players 1, 6, and 8; team B has players 3, 5, and 7; team C has players 2, 4, and 9. The strongest player is number 1. In general, player i will defeat player j whenever $i < j$. The coach wonders which of the three teams is the strongest. Help her decide.

8. A woman is considering three proposals of marriage, from suitors named A, B, and C. She finds A to be more intelligent than B, and B to be more intelligent than C. She considers B to be physically more attractive than C, and C to be physically more attractive than A. Complicating matters further, C has a greater income than A, who has a greater income than B. If it were a choice between A and B, which would she choose? How about between B and C? Is there a best choice for her?

9. Suppose that a track-and-field meet is to be held among three teams. Each team makes one entry in each of seventeen events (100, 200, 400, 800, 1500, 3000, 4x100 relay, 4x400 relay, shot, javelin, discus, long jump, triple jump, high jump, hurdles, intermediate hurdles, and pole vault). Show that determining the strongest team is like evaluating an election. Are paradoxes possible?

Summary of United States Apportionment

1787: 65 seats arbitrarily apportioned while awaiting first census
1790: 105 seats; Jefferson method chosen over Hamilton method
1800: 141 seats; Jefferson method
1810: 181 seats; Jefferson method
1820: 213 seats; Jefferson method (violates quota)
1830: 240 seats; Jefferson method (violates quota)
1840: 223 seats; Webster method (a)
1850: 234 seats; Hamilton and Webster methods in agreement
1860: 241 seats; modified Hamilton method
1870: 283 seats; Hamilton method; 9 seats added irregularly later (b)
1880: 325 seats; Hamilton and Webster methods in agreement
1890: 356 seats; Hamilton and Webster methods in agreement
1900: 386 seats; Webster method
1910: 433 seats; Webster method
1920: 435 seats; two new states, two new seats, no new apportionment (c)
1930: 435 seats; Webster and Huntington-Hill methods in agreement
1940: 435 seats; Huntington-Hill method
1950: 435 seats; Huntington-Hill method (d)
1960: 435 seats; Huntington-Hill method
1970: 435 seats; Huntington-Hill method
1980: 435 seats; Huntington-Hill method
1990: 435 seats; Huntington-Hill method (e)
2000: 435 seats; Huntington-Hill method
2010: 435 seats; Huntington-Hill method

Notes:
(a) The only time that the House size was diminished.
(b) Irregular apportionment throws 1872 Presidential election to wrong man.
(c) Lack of apportionment violates Constitution.
(d) Two seats temporarily added in 1959 for the new states Hawaii and Alaska.
(e) Montana challenges the constitutionality of Huntington-Hill method.

1. In the 1970 census, California and Oregon had populations of 20098863 and 2110810, respectively. The Huntington-Hill method awarded these states 43 and 4 representatives, respectively. Just as Montana did in 1990, Oregon could have sued to have the apportionment overturned as unconstitutional. Present calculations that support this point of view. Present calculations that explain why the Huntington-Hill method considers the apportionment to be fair as is.

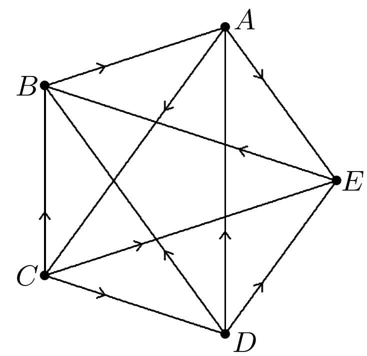
2. The Huntington-Hill method has never violated quota, but it could. In the reference section there is a "mock" census for 1984, along with the corresponding Huntington-Hill apportionment. How many quota violations can you find in it?

Discrete Mathematics

1. Consider the following voter preferences: ABC (29%), ACB (11%), BCA (33%), CAB (23%), and CBA(4%). The voters thus give 40% of the first-place votes to A, 33% to B, and 27% to C. Show that C could still win the election, if suitable points were assigned to the three places.

2. If a graph has a circuit of odd length, then the chromatic number of the graph is at least three. Prove or disprove this statement.

3. The *edge-skeleton* of a polyhedron is the graph that consists of the vertices and the edges of the polyhedron. Show that the edge-skeleton of a cube has chromatic number 2.



4. The digraph at right displays the results of a *round-robin* tournament among five teams, during which each team played one game against each of the other four.
(a) How many games were played during the tournament?
(b) A team can play at most one game per day. How many days must a five-team tournament take?

5. Given the following voter preferences, draw a digraph that shows how each candidate would fare against the other candidates in simple majority elections: CDBA (21%), DACB (2%), DCAB (9%), ACBD (44%), BCDA (20%), CADB (4%)

6. In an original work, Jean-Charles de Borda argued against the plurality method of deciding elections. He proposed the following example: Twenty-one voters have submitted rankings of three candidates. One prefers ABC, seven prefer ACB, seven more prefer BCA, and six prefer CBA. What were Borda's calculations, and what was his argument?

7. The Marquis de Condorcet did not agree with the method of his friend Borda. Because he could find no fault with the preceding example, he proposed the following example instead: Eighty-one voters have submitted rankings of three candidates. Thirty vote ABC, one votes ACB, ten vote CAB, twenty-nine vote BAC, ten vote BCA, and one votes CBA. Why did Condorcet find this data compelling enough to argue against his friend's position?

8. The preceding two examples give the impression that Condorcet's method of analyzing elections produces clear, unambiguous results. Provide an example of your own that shows this is not the case!

9. The population of state X is 6 000 000, the population of the USA is 250 000 000, and the House size is 435.
(a) What is the ideal district size?
(b) What is the quota of state X?
(c) In order for state X to receive its upper quota, the Huntington-Hill method must be applied with a divisor that is less than the ideal divisor. What is the largest whole-number value for this divisor that will give state X its upper quota?

1. Given the following voter preferences among three candidates, determine the winner of the election, according to **(a)** the plurality method; **(b)** the simple Borda count; **(c)** the Condorcet method. Draw the digraph for (c). Notice that there are 150 voters in all.

44 vote ABC, 5 vote ACB, 27 vote BAC, 19 vote BCA, 31 vote CAB, 24 vote CBA

2. The table at right shows the quota calculations for apportioning delegates to ten states. Finish the job, using the Hamilton method.

State	Q
Activity	5.63
Bliss	4.10
Confusion	7.58
Denial	10.52
Euphoria	3.48
Grace	2.85
Hypnosis	8.55
Indecision	9.61
Matrimony	6.17
Panic	6.51

3. If the data at right had been a table of adjusted quotas for the Huntington-Hill method, how many delegates would the state of Euphoria receive? Explain.

4. By assigning 0 points for third, 1 point for second, and x points for first, any of the three candidates in question 1 can be made the winner. How many of the forty-nine values 2, 3, 4, . . . , 50 for x will make A the winner?

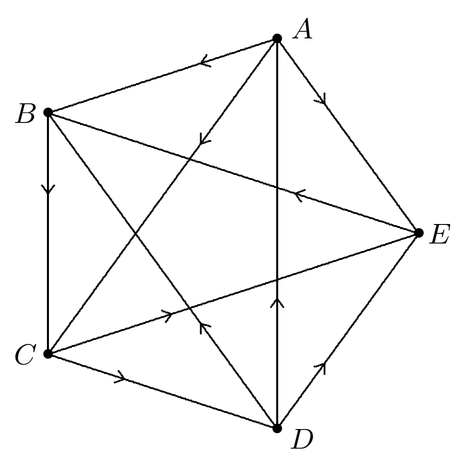
5. In a twenty-team round-robin tournament, how many games must be played?

6. The *Welsh-Powell algorithm* for coloring a graph: Arrange the vertices of a graph by valence, in non-increasing order, breaking ties randomly. Color the vertices in this order, *always using the first available and allowable color.* For example, when applied to the USA map, either Missouri or Tennessee gets color 1, the other gets color 2, Kentucky gets color 3, and so forth.

Try out this algorithm on some simple graphs to see whether it always gives a coloring with the smallest possible number of colors.

7. If the winning team in a track meet were decided solely on the basis of taking the most first places, what method of evaluating elections would be in use?

8. Given the tournament graph shown at right, determine whether there is a Condorcet winner, a Condorcet loser, a Condorcet 5-cycle, or none of the preceding.



9. You have probably already noticed, while using one of the divisor methods, that more than one acceptable divisor can usually be found to solve an apportionment problem. Comment on this phenomenon. Is it ever the case that the acceptable divisor is *unique*?

Discrete Mathematics

1. You have seen that there are $2^{10} = 1024$ possible outcomes for a five-team round-robin tournament. In how many of those 1024 outcomes
(a) is team A the (undefeated) winner?
(b) is there an undefeated team?
(c) is there a winless team?
(d) is team C undefeated and team B winless?
(e) is there neither an undefeated team nor a winless team?

2. One hundred voters rank three candidates as follows: Twenty-five prefer ABC, fifteen prefer ACB, six prefer BAC, twenty-five prefer BCA, seventeen prefer CAB, and twelve prefer CBA. It is clear that A wins the plurality contest with 40% of the vote. Show that any one of the candidates can win the election, however, if a point system is used to make the decision.

3. A round-robin tournament is to be held among six teams. The results will be recorded in the form of a simple digraph.
(a) How many different digraph summaries are possible?
(b) How many games are to be played in all?
(c) If no team can play more than one game per day, how many days will it take to complete the tournament? Give a day-by-day schedule as justification.

4. Show that it is possible for a six-team round-robin tournament to produce no (undefeated) winner, no (winless) loser, yet no Condorcet 6-cycle, either.

5. Using the derivation of the Huntington-Hill rounding formula (based on the geometric mean) as a guide, produce a derivation of the Dean rounding formula.

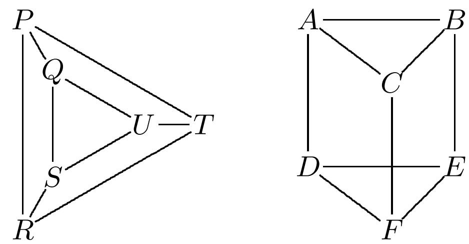
6. To decide a three-candidate election by means of a classic Borda count means to award 2 points for first-place votes, 1 point for second-place votes, and 0 points for third-place votes — (2,1,0) for short.
(a) Show that this method produces the same ranking as would be obtained if points were awarded (3,2,1), or (200,199,198), or $(n, n-1, n-2)$.
(b) What can be said about point systems (5,3,1) and (10,8,6)?
(c) What must be true in order for two point systems (x, y, z) and (p, q, r) to produce rankings that are significantly different?

7. Let k be the largest valence in a graph. Explain why the chromatic number of this graph is at most $k + 1$.

8. Given any apportionment, the range of district sizes (the difference between the largest and the smallest) is a number of interest. It is plausible that increasing the size of the House would have the desirable effect of making the range smaller. Applying the Huntington-Hill method to the 2010 census, verify that the current range of district sizes is 466792, and that the range would drop to 403669 if the House size were increased to 440. Continue this investigation, considering 500, 600, and 620 as potential House sizes.

Discrete Mathematics

1. The graphs shown at right are *isomorphic*. This means that the labels on one figure can be matched with the labels on the other so that adjacent vertices correspond to adjacent vertices. One correspondence that works matches P with A, Q with D, R with C, S with F, T with B, and U with E. Verify that this correspondence works. Then find *another* way of matching the vertices of the two figures so that adjacent vertices always correspond to adjacent vertices.



In the second graph, notice that edge CF intersects edge DE. This creates an *accidental crossing*. In other words, the intersection does not mean anything — it is just a unintended consequence of the way the graph was drawn. In particular, it is not a vertex, so it does not get a label. Notice that the first graph is drawn so that there are no accidental crossings. Graphs that do not have accidental crossings are called *planar*. Graphs that are isomorphic to planar graphs are also called planar, because any accidental crossings are in fact avoidable. Thus the second graph shown above is planar. Make up your own example of a planar graph that has accidental crossings.

2. Is it possible for four countries to each border on all the others? Why or why not?

3. Is it possible for five countries to each border on all the others? Why or why not?

4. Is it possible to find four of the United States whose border graph is isomorphic to the complete graph on four vertices?

5. Consider the 100-voter preference summary shown at right, which displays the number of first-place votes, second-place votes, and third-place votes for each of three candidates. Show that any one of the candidates can outpoint the other two, if suitable points are awarded for the three places.

	1	**2**	**3**
A	40	22	38
B	32	36	32
C	28	42	30

6. In the preceding question, only a place-by-place summary was provided, not an actual tally of preference ballots. Make up data that is consistent with the table. In other words, tell how many voters chose ABC, how many chose BCA, and so forth.

7. Fourteen trustees are in town to conduct their annual business, which is done in subcommittees. There are eight subcommittees that each need to hold a day-long meeting. It is your job to schedule the meetings so that the trustees can get back as soon as possible to their homes, families, and regular jobs. The eight subcommittees are:

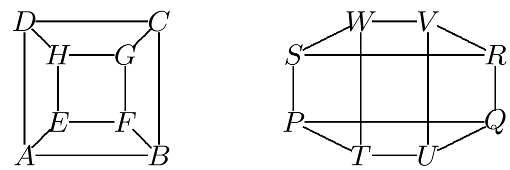
Alumni affairs:	Jones, Bonney, Silberman, Orlov
Budget review:	Bonney, Orlov, Trafelet, Rawson
Capital giving:	Rawson, Michaels, Hallett, Downer
Dining halls:	Trafelet, Panetta, Kurtz, Downer
Environment:	Hallett, Ettinger, Welles
Faculty life:	Andersen, Panetta, Welles
Grand plans:	Jones, Ettinger, Kurtz, Downer
Health and happiness:	Andersen, Michaels, Orlov, Jones

Discrete Mathematics

1. A high school has 1000 students, each of whom is enrolled in five academic courses. Given that the average section size is 12.5 students, estimate how many sections are currently being taught at this school.

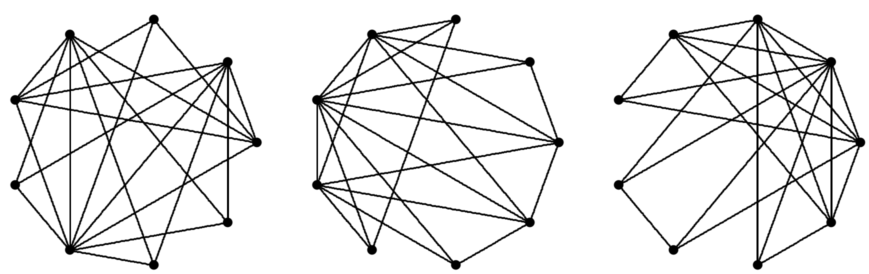
2. The sections described in the previous question had to be scheduled into eight time blocks. Explain how this scheduling problem can be viewed as a coloring question about a suitable graph.

3. By writing out an explicit correspondence between their vertices, explain why the two graphs shown at right are isomorphic.



4. By definition, a planar graph is isomorphic to a graph that can be drawn without accidental crossings. For example, graph $PQRSTUVW$ shown above is planar, even though it has four accidental crossings, because it is isomorphic to $ABCDEFGH$, which has none. It is often difficult to decide whether or not a graph is planar. Consider a complete five-vertex graph — is it planar?

5. Two of the three graphs shown below are isomorphic, meaning that they are connected in essentially the same way. Can you tell *which two* ?



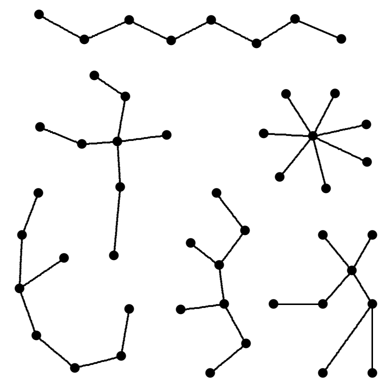
6. Show that the chromatic number of any tree is two.

7. Each of us is associated with a graph that is commonly called an *ancestral tree*. I am at the root of my graph, with my parents adjacent to me, their parents adjacent to them, and so on. What do you think of this terminology? Draw and label a few vertices of your graph, then try estimating its size.

8. What is the conventional name for a graph that has *no circuits* of any size?

1. A graph that has only vertices of valence 3 is called *trivalent*. One of the interesting things about trivalent graphs is that, except for the complete 4-vertex graph, they can all be colored with (at most) three colors. (It has already been shown a couple of pages ago that four colors suffice.) Verify this result experimentally: Invent your own trivalent, ten-vertex graph, then color it minimally.

2. Are the six trees shown at right all different?

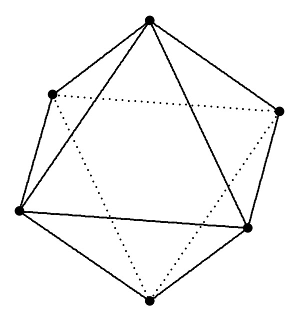


3. *A handshake puzzle*: My wife and I recently attended a party at which there were four other married couples. Various handshakes took place. No one shook hands with one's own spouse, and no two persons shook hands more than once. After all the handshakes were over, I asked each person, including my wife, how many times he/she had shaken hands. To my surprise, each person gave a different answer. How many times did my wife shake hands?

4. Is it possible for the entries in the valence list of a graph to be all different? Explain.

5. I am considering my ancestral graph again.
(a) What are the possible valences for this graph?
(b) Suggest a meaningful way of attaching numerical labels to the edges.
(c) Think of this graph as being *directed*, with arrows leading back in time (in other words, arrows point from children to their parents). Suppose that vertices A and B initiate paths that both lead to a vertex P. If both paths consist of two edges, what does that say about vertices A and B? What if the path from B to P had three edges?

6. The diagram at right shows the edge-skeleton of an octahedron. Show that this graph is planar. What is its chromatic number?



7. Show that the edge-skeleton of *every* convex polyhedron is planar.

Discrete Mathematics

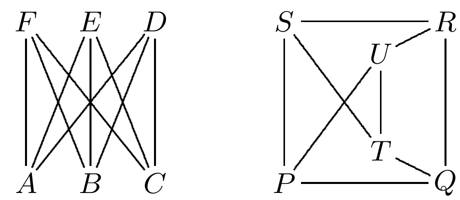
1. The most famous statement about graph coloring is the following *Four-Color Theorem*:

If a graph is planar, then its chromatic number is at most four.

Although it was proposed as a theorem in 1852, this was not proved until 1976. The proof was remarkable, because a computer (at the University of Illinois) was needed to carry out a mind-boggling amount of case-by-case checking. Many mathematicians are still hopeful that a simpler and more elegant proof will be found someday. What is the *converse* of the Four-Color Theorem? Is it a true statement?

2. Which of the United States has valence 2?

3. Decide whether or not the graphs shown at right are isomorphic. Justify your answer.



4. *A large graph*: Consider the acquaintanceship graph for the entire planet! Propose some interesting questions about this graph.

5. Given a graph, it can be converted into a digraph by assigning arrows to each of its edges. This is called *directing a graph.* If the edges of the graph represented city streets, the process would correspond to making each of the streets one-way. An interesting question: For what graphs is it possible to assign directions so that the resulting digraph is connected?
(a) Decide what the question means. In particular, what does it mean for a digraph to be *connected*? Why might this matter (if edges were one-way streets, for instance)?
(b) Find a connected graph that can *not* be made into a connected digraph.

6. *A map-coloring game*: Player A draws a region. Player B colors it and adds a new region to the diagram. Player A colors B's region and adds a third region. Play alternates in this way until one player is forced to use a fifth color; this player loses the game. Formulate a strategy for playing this game.

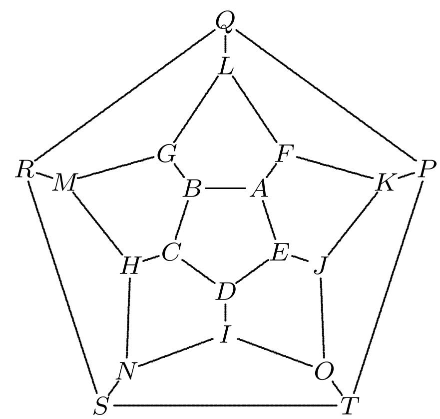
7. A high school has 1000 students. Currently, each student is enrolled in five academic courses, and there are 400 courses being taught. This situation can be modeled by a 1400-vertex graph — one vertex for each student and one vertex for each course. The only edges of this graph are obtained by joining each course to the students who are enrolled in that course. This is an example of a *bipartite* graph.
(a) How many edges does this graph have?
(b) What is the average enrollment per course?
(c) The table below shows the distribution of all the valences for the course vertices.

valence	14	13	12	11	10	9	8	7	6
courses	65	155	128	38	6	x	1	y	z

Find x, y, and z, assuming that they are positive integers.

1. *Another map-coloring algorithm*: Arrange the vertices of a graph by valence, in non-increasing order, breaking ties randomly. Then repeat the following until done: *Select a new color. Go through the list of vertices and apply this color to any uncolored vertex that is not adjacent to a vertex that has already received this color.* How does this algorithm compare to the Welsh-Powell algorithm?

2. In 1859, the Irish mathematician Sir William Rowan Hamilton marketed a puzzle that consisted of a wooden regular dodecahedron (having twelve pentagonal faces), with each of the twenty vertices labeled to represent a famous city. The puzzle was to find a route that traveled along the edges of the solid, visiting each city exactly once and returning to the start. A small nail was driven into each vertex so that the path could be traced with string. Use the graphical representation at right to find a solution to Hamilton's puzzle.



3. A digraph is called *connected* (or *diconnected* or *strongly connected*) if, given any vertices P and Q in the graph, there is a path of edges that leads from P to Q, following the directions specified by the arrows. It is sometimes suggested that any graph in which all valences exceed one can be converted into a connected digraph, by judiciously assigning a direction to each edge. What do you think of this conjecture? Explain.

4. Given a digraph, one can calculate its *invalences* and *outvalences*. Explain this terminology, then propose a version of the Handshake Theorem that applies to digraphs.

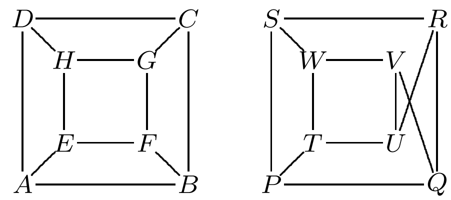
5. Suppose that you want to mail a package, and that you have ample supplies of 1-cent, 13-cent, and 22-cent stamps available. Describe an algorithm for putting the correct postage on the package. Given that there is probably more than one way to put the correct postage on a package, what makes one solution better than another? Does your algorithm always produce the best solution?

6. Eleven council members, whose names are A, B, C, ..., K, have volunteered to serve on seven committees, as follows: $\{B, F\}$, $\{A, D, F, I\}$, $\{D, E, G\}$, $\{E, H, J\}$, $\{C, J, K\}$, $\{A, C, G, H\}$, and $\{B, I, K\}$. These seven committees need to be scheduled.
(a) Represent the given data by a graph of your choosing. Be sure to tell the reader what the vertices and edges of your graph mean.
(b) Schedule the committees into as few time periods as you can.

7. A circuit that includes every vertex of a graph exactly once is called a *Hamiltonian circuit*. (Look above for the origin of this terminology.) Does every graph have a Hamiltonian circuit? Use familiar examples to explain your answer. How many different Hamiltonian circuits does the complete five-vertex graph have? How about the complete n-vertex graph?

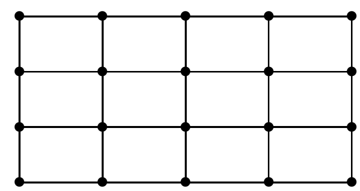
1. Given a list of valences for a graph, can you tell whether the graph is a *tree*? Is it possible for two non-isomorphic trees to have the same valence list?

2. Decide whether the two graphs shown at right are isomorphic, and give your reasons.



3. *Word chains*: The vertices of this graph represent ordinary three-letter English words (as found in my dictionary). Two words are considered adjacent if one can be obtained from the other by replacing a single letter (and doing no rearranging). For example, *bin* and *fin* are adjacent, and so are *fin* and *fun*. Some questions: What are the highest and lowest valences you can find in this graph? Is the graph connected? If not, are *one* and *two* in the same connected component? In other words, is there a path connecting them? How about *ape* and *man*? How about *hen* and *egg*?

4. The graph shown at right is called a *4-by-5 grid*, because there are 4 rows of vertices, and 5 vertices in each row. Does this graph have a Hamiltonian circuit? In general, for what values of m and n does an m-by-n grid have a Hamiltonian circuit?



5. Is it possible to put directions on the edges of the 4-by-5 grid so that the resulting digraph is strongly connected? That means that you should be able to travel from any vertex to any other vertex along an edge-path, *following the direction on each edge used.*

6. Suppose that all the vertices of a *finite* graph have valences that are at least 2. Explain why this graph must have circuits. Why is the assumption "finite" necessary here?

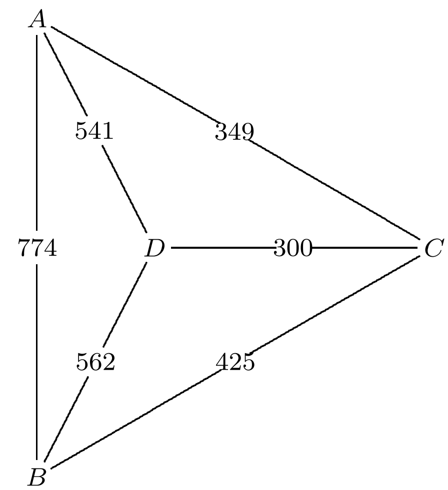
7. A graph is called *Hamiltonian* if it has a Hamiltonian circuit. Suppose that the chromatic number of a Hamiltonian graph is 2. Is it possible for such a graph to have an odd number of vertices? Explain.

8. *The game of Sprouts*: Mark a few dots (two or three will do for beginners) on a piece of paper. A move consists of joining two dots by an arc that does not touch the rest of the drawing, then marking a new dot on the arc (near the middle). No dot is allowed to have valence greater than three, and new dots introduced during the game begin their lives with valence 2. Play alternates between two players until someone is unable to make a move — this player loses.
(a) At most how many moves can the two-dot game last? Who has the advantage?
(b) Do a similar analysis of the three-dot game.
(c) In general, at most how many moves can an n-dot game last?

9. Every complete graph has many Hamiltonian circuits. What if a complete graph is *directed*, so that every edge has an arrow — what does the terminology *Hamiltonian circuit* mean in such a graph? Need there be one?

1. The *Traveling Salesman Problem* (or TSP, for short) is a question about Hamiltonian circuits for complete labeled graphs. Each edge of a complete graph has been labeled with a numerical value, and the objective is to find the Hamiltonian circuit that has the smallest possible sum of edge labels. In the standard application, the vertices represent cities and the labeled edges represent intercity transportation information — mileage, time, or cost, for instance. Other applications: the telephone company wishes to pick up coins from its pay phones; the gas company needs to send out a meter reader; a lobsterman must visit all his traps; an assembly line needs a mechanical device to drill several holes (in succession) in steel plates.

The graph at right displays a four-city TSP, with intercity mileages marked on each edge: $AB = 774$, $AC = 349$, $AD = 541$, $BC = 425$, $BD = 562$, and $CD = 300$. Find the 4-cycle that minimizes the total mileage that our salesman must travel.



2. When a TSP involves only four vertices, examining every conceivable circuit is a manageable approach, for there are essentially only *three* of them. To apply this brute-force approach to a ten-vertex problem requires looking at how many circuits?

3. A manageable TSP algorithm: Select any one of the vertices to start at. Travel to that unvisited vertex that is closest to the current vertex. Repeat the preceding step until every vertex has been reached. Finally, close the circuit by traveling back to the starting vertex. This is called the *nearest-neighbor* algorithm, for obvious reasons. Apply it to the example above. Does the choice of starting vertex affect the outcome?

4. Another TSP algorithm: List the edges by cost, in non-decreasing order, breaking ties randomly. The first edge in the list starts the tour. Work through the edge list, rejecting any edge that would make three edges meet at a vertex, or that would close a circuit that does not include every vertex. A complete circuit will eventually be formed. This is called the *sorted-edges* algorithm, or — less clearly — the *cheapest-link* algorithm. Apply it to the example in question 1.

5. A five-city TSP: The distances are $AB = 60$, $AC = 65$, $AD = 55$, $AE = 70$, $BC = 50$, $BD = 85$, $BE = 95$, $CD = 80$, $CE = 75$, and $DE = 40$. Find the optimal tour.

6. The nearest-neighbor and sorted-edges algorithms are both examples of what are called *greedy* algorithms. Explain this terminology, and give other examples.

7. Neither the sorted-edges algorithm nor the nearest-neighbor algorithm for solving a TSP can be applied to an incomplete graph. Explain why.

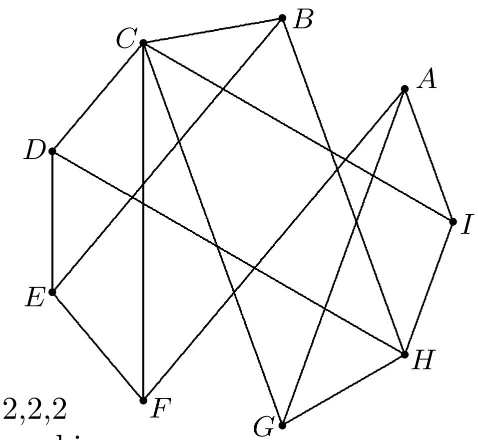
Discrete Mathematics

1. Twelve Seniors have signed up for eight field courses. Schedule these eight sections into as few formats as possible. Explain why your answer cannot be improved.

Art:	Smith, Jones, Lee	*Bio*:	Allen, Smith, Evans
Chem:	Davis, Brown, Evans	*Drama*:	Lee, Miller, Brown
Econ:	Carter, Gray, Jones	*French*:	Carter, Hess, Allen
German:	Hess, Lewis, Davis	*History*:	Gray, Miller, Lewis

2. Write one-sentence definitions for the following: *valence*, *tree*, *accidental crossing*.

3. A hostess needs to seat nine guests around a circular table. The compatibility graph shown at right reveals that every guest is on speaking terms with at least three other persons at the table; in particular, guest C gets along with all except A, H, and E. Is it possible to seat these nine persons around the table so that each person will be on speaking terms with both neighbors? If so, show how to do it. If not, explain why not.



4. Find three non-isomorphic graphs that have 3,3,2,2,2,2 as their valences. Prove that your graphs are not isomorphic.

5. Some questions about a complete graph with 24 vertices: How many edges does the graph have? How many Hamiltonian circuits? What is the chromatic number? Would the chromatic number be affected if one of the edges were erased? If so, how? If not, why not?

6. What are the possible valence lists for a four-vertex tree?

7. What is the chromatic number of a ten-vertex tree?

8. Is it possible for a trivalent graph to have an odd number of vertices? If so, give an example. If not, explain why not.

9. Find a word chain that joins *wet* to *dry*.

10. The edge-skeleton of a cube is a connected, trivalent, eight-vertex graph, whose chromatic number is 2. Are there other eight-vertex graphs that have all of these properties?

11. In graph theory, a complete digraph is called a *tournament*. Show that any tournament has a Hamiltonian path. (A Hamiltonian path visits every vertex once, but need not close like a circuit.)

12. The three-letter word-chain graph is not a *forest*. Explain.

1. Another type of problem that is of interest in network analysis is to find an *Eulerian circuit*. This is a circuit that includes every edge exactly once. It may be necessary that some vertices be visited more than once during such a circuit. Think of some applications that involve looking for an Eulerian circuit in a suitable graph. A graph that has an Eulerian circuit is called *Eulerian*.

2. For what values of n does a complete n-vertex graph have an Eulerian circuit? For what values of m and n does an m-by-n grid have an Eulerian circuit?

3. Consider the complete 4-vertex graph $ABCD$ whose edges are weighted with the numbers $AB = 82$, $BC = 37$, $CD = 79$, $DA = 51$, $DB = 31$, and $AC = 12345$. (The last weight represents an extremely expensive linkage.) By brute force, solve the traveling-salesman problem. Then show that neither the sorted-edges method nor the nearest-neighbor method (check all vertices) gives a reasonable answer.

4. Suppose that a supercomputer can find and evaluate a 20-vertex Hamiltonian circuit every thousandth of a second. To look at every possible Hamiltonian circuit for a complete, labeled 20-vertex graph (to find the one of least cost, say), how much time is needed?

5. At a recent social gathering, the host wanted to seat the entire group at a circular table so that each person knew both neighbors. This problem is equivalent to a well-studied problem in graph theory. Explain.

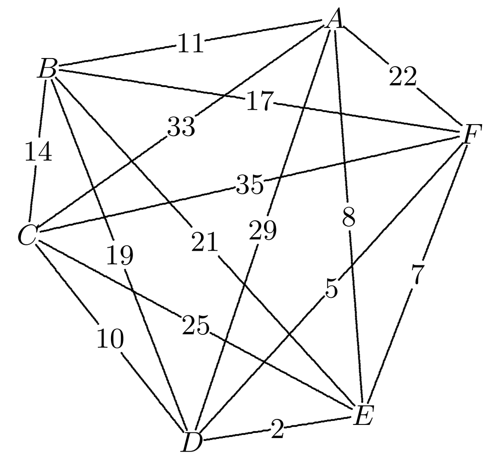
6. Suppose that PEA is about to link six of its buildings — **S**cience, **M**ath, **G**ym, **L**ibrary, **A**dmissions, **B**ookstore — via fiber-optic conduits. The costs (in 1000-dollar units) are as follows: MS = 13, ML = 140, MB = 150, MA = 100, MG = 300, SL = 145, SB = 180, SA = 80, SG = 320, LB = 25, LA = 30, LG = 190, BA = 40, BG = 250, and AG = 220. Notice that it is not necessary to build all fifteen linkages — if station X is connected to Y, and if station Y is connected to Z, then X is then automatically connected to Z. What is the best (cheapest) way to carry out this project? In other words, which conduits should be built? How much will the project cost?

7. Suppose that, at the last minute, it is decided that the **P**ost Office is to be added to the network planned in the preceding question. The linkage costs to the other stations are PM = 70, PS = 38, PL = 220, PB = 200, PA = 150, and PG = 420. How does this affect the plan?

8. Students named A, B, C, ..., K have been assigned to seven committees. The next two lists rank these committees according to the number of conflicts each has.
(a) $\{A, D, F, I\}$, $\{A, C, G, H\}$, $\{C, J, K\}$, $\{B, I, K\}$, $\{E, H, J\}$, $\{D, E, G\}$, $\{B, F\}$
(b) $\{A, C, G, H\}$, $\{A, D, F, I\}$, $\{B, I, K\}$, $\{E, H, J\}$, $\{C, J, K\}$, $\{D, E, G\}$, $\{B, F\}$
For each list, verify that the number of conflicts form a non-decreasing sequence, then schedule the committees by applying the Welsh-Powell coloring algorithm to the list.

1. Show that the octahedral graph is Hamiltonian. That means that the graph has a Hamiltonian circuit.

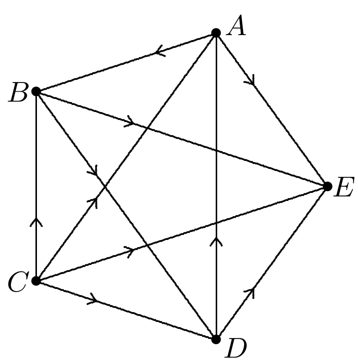
2. Answer the following questions about the TSP shown at right: How many circuits would have to be evaluated in order to find the optimal solution by brute force? By considering vertices A and E, show that the nearest-neighbor algorithm can produce different answers for the same problem. Show also that the sorted-edges algorithm does not have to agree with the nearest-neighbor algorithm.

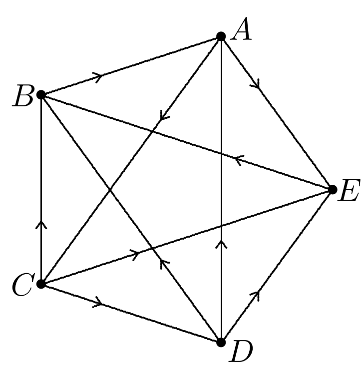


3. Assuming that one generation follows another at 25-year intervals, and assuming that ancestors can be arranged in a tree, calculate how many of your ancestors were walking the Earth 1500 years ago. You should be surprised/dismayed by the size of your answer! Describe what is wrong with this theory.

4. Given a list of positive whole numbers, is there necessarily a graph that has these numbers as its valences? If not, what extra conditions on the list will ensure that such a graph would exist?

5. Do either of the digraphs shown below have Hamiltonian circuits? If not, do they have Hamiltonian *paths*? Explain your reasoning.





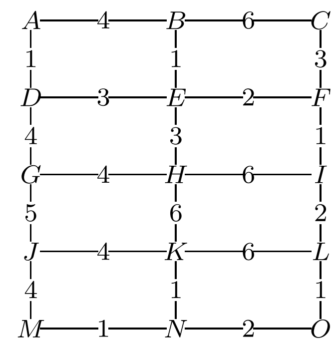
6. The ten-vertex valence list $3, 3, 3, 2, 2, 1, 1, 1, 1, 1$ describes many graphs. How many non-isomorphic examples can you find? How many of your graphs are trees?

7. Consider a round-robin tournament, in which v is a player who was defeated at least once, but who won at least as many games as anyone else. Prove that any player who defeated v must have been defeated by at least one player who was defeated by v.

Discrete Mathematics

1. The solution to the PEA fiber-optics question is a graph with no circuits (there are no redundant edges), hence it is a tree. Because the tree reaches all the vertices of the given graph, it is called a *spanning tree*. Because it minimizes the total cost (the sum of the edge labels), it is called a *minimal-cost spanning tree*, or MST for short.

Find an MST for the graph shown at right. Notice that this is not a complete graph. If you wish, however, you can imagine that the missing edges are prohibitively costly. Placing a label like 1000000 on an edge is a way of removing it from consideration!

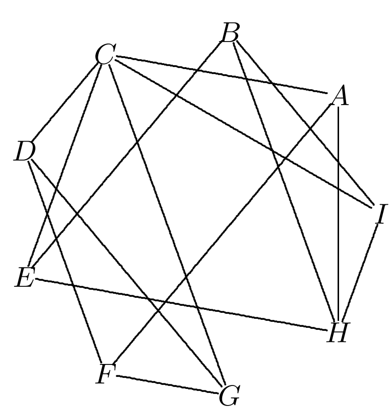


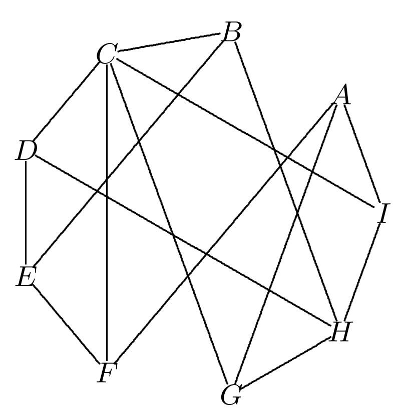
2. Show that there is more than one correct answer to the previous question, by finding an edge *not* in your MST that could be used to replace an edge that *is* in your MST.

3. The sorted-edges method for finding an MST is called *Kruskal's algorithm* because Joseph Kruskal published a proof that it always gives the best possible solution, for any connected, weighted graph. As a result of this theorem, there is no need to consider a brute-force approach to the problem. Nevertheless, you *might* be wondering how many spanning trees would have to be examined for a typical n-vertex graph. For complete graphs, the answer is known. For a 5-vertex complete graph, there are 125 spanning trees; for a 6-vertex complete graph, there are 1296. On the basis of these two facts, guess the simple formula (in terms of n) for this number, then display all the trees for the cases $n = 3$ and $n = 4$ as a check on your formula.

4. Why is the edge-skeleton of a regular dodecahedron sometimes called a *cubic graph*?

5. Show that the graph on the left has a Hamiltonian circuit. Show that the graph on the right does not have a Hamiltonian circuit. What conclusion can you draw from these two results?





1. Give an example of a graph that has a Hamiltonian path but not a Hamiltonian circuit.

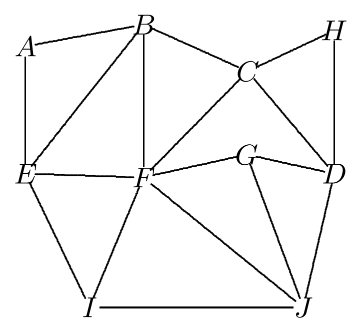
2. Draw a complete 6-vertex graph, then assign a direction (arrow) to each of the edges, trying to avoid the formation of a directed Hamiltonian circuit. When you have finished assigning arrows, look to see whether there are any directed Hamiltonian paths. How many can you find?

3. For any connected, weighted graph, Kruskal's algorithm produces a minimal-weight spanning tree. A connected tree might not emerge until the final step, however. Until then, the unfinished subgraph could at times be in several pieces. Try your hand at inventing an algorithm that keeps the developing tree in one (connected) piece throughout.

4. Adding an edge to a minimal-weight spanning tree will produce a cycle (of some size). Explain why none of the weights that appear in the cycle can be greater than the weight of the edge that was added to the tree.

5. (Continuation) If the weights on the edges of a connected graph are all different, then the graph has a *unique* minimal-weight spanning tree. Prove this statement by assuming, to the contrary, that $\mathcal{T}_1$ and $\mathcal{T}_2$ are different MSTs for the same weighted network; consider the lightest edge that does not belong to both $\mathcal{T}_1$ and $\mathcal{T}_2$.

6. Examine the graph at right, and decide whether an Eulerian circuit is possible. If so, write one out, using the letters given in the diagram. If not, explain why not, and decide whether an Eulerian *path* (which covers every edge exactly once but which does not have to be a closed circuit) is possible. If there is a path, write one out. If not, explain why not.



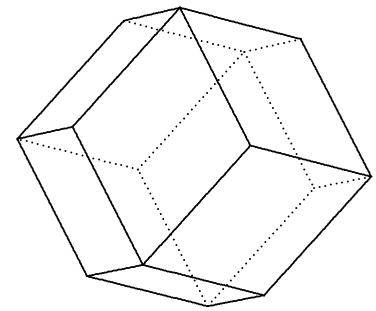
7. A connected graph that has 800 edges can have how many vertices?

8. In order that 1, 1, 1, 1, 1, 2, 2, 2, 3, x be the valences for a *tree*, what must the integer x be? Draw two non-isomorphic trees that illustrate your choice.

9. Suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are different spanning trees for the same network. Prove the *exchange property*: For each edge e_1 that is in $\mathcal{T}_1$ but not in $\mathcal{T}_2$, there is an edge e_2 that is in $\mathcal{T}_2$ but not in $\mathcal{T}_1$, and that has the property that $\mathcal{T}_1$ and $\mathcal{T}_2$ will still be spanning trees if e_1 and e_2 are exchanged.

10. (Continuation) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are different MSTs for the same weighted network, then the edges e_1 and e_2 must have the same weight. Prove this statement. Conclude that $\mathcal{T}_1$ and $\mathcal{T}_2$ must have exactly the same weights.

1. The figure at right shows a *rhombic dodecahedron*. It is a shape often encountered in crystallography. Explain the name. Then explain why it is good thing that Hamilton did not use *this* dodecahedron to market his around-the-world puzzle in 1859.



2. The MST problem is to find a minimal-weight spanning tree for a given set of points. There is a related version of the question, in which one is allowed *to add to the set of vertices*. To be specific, suppose that three towns are situated at the corners of an equilateral triangle, two miles apart. It is proposed that these three towns be linked by a system of roads, with as small a total length as possible. If the only possible roads went straight from town to town, then of course the best possible solution would require building *four* miles of roads. If roads aare allowed to meet at junctions outside the towns, however, then a better solution is possible. Explain how to build the roads.

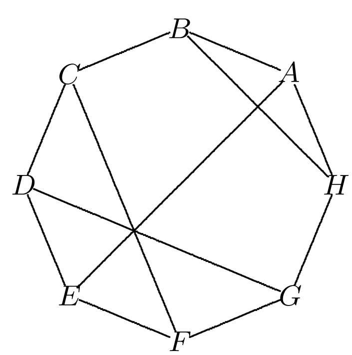
3. As in the preceding, suppose that four towns are situated at the corners of a square, two miles on a side. It is proposed that these four towns be linked by a system of roads, with as small a total length as possible. If roads are only allowed to go straight from town to town, then the best solution would require building *six* miles of roads. If roads are allowed to meet at junctions outside the towns, however, then a better solution is possible.

Optimal solutions to questions such as the preceding are called *Steiner trees*, named for Jakob Steiner, the famous geometer who first studied them. There is much left to be discovered about this topic.

4. By means of an example, show that a trivalent graph can have chromatic number 2.

5. Give an example of a connected graph with 14 vertices that does not have a Hamiltonian path, but each vertex of which belongs to a circuit.

6. Make up a scheduling problem to go with the conflict graph shown at right. Solve the problem.



7. A connected graph with 20 vertices must have at least how many edges?

8. Name three calculable methods that can be used to distinguish graphs that are not isomorphic.

9. The valence list of a tree is 5, 4, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1.
(a) The number of valences is ____.
(b) The ________________ is 22.
(c) The number of edges is ____.
(d) Draw at least two non-isomorphic trees that correspond to this valence list.

In many voting situations, the voters do not exert equal influence on the outcome. A common example is a stockholders' meeting, where each person's vote carries a weight that is proportional to how much stock is owned by that person. Another familiar example is the election of the President of the United States, which takes place in the Electoral College, where states cast blocks of votes that are roughly proportional to their populations.

It is inevitable that questions of fairness arise in situations where votes are weighted. Such questions are difficult to answer without some sort of quantitative measures of the effects of using a weighted voting system.

The central problem is one of measuring the *power* of a voter, be it a stockholder or a block of electors bound to exercise their state's choice in the Electoral College. Some examples to illustrate the possibilities:

1. One investor holds 51% of the stock of Amalgamated Consolidated, and forty-nine other investors hold 1% each. Assuming that decisions are made on the basis of simple majority, the 51% investor is a *dictator*, possessing all the power. The other stockholders are *dummies*; they have no power. This is an extreme example, but it shows that power need not be simply proportional to the weight of one's vote.

2. Three investors control all the stock of RetroFit Industries; the shares are 48%, 47%, and 5%. As before, assume that decisions are made on the basis of simple majority. It may come as a surprise that power is equally divided among the three.

3. Imagine a small version of the United States, consisting of four states A, B, C, and D, which have 7, 5, 3, and 2 electoral votes, respectively. To elect the President, nine votes are needed. It will be seen that states B, C, and D have equal power in this situation, and that state A is three times as powerful as each of them.

How is power quantified in these examples, if not by the weights of the votes? A common approach is the *Banzhaf Index*, proposed in 1968 by Joseph Banzhaf, a lawyer and consumer advocate. The central idea is to count the number of times that a voter can be *pivotal*; that is, the number of times that changing that vote can reverse the results of the election. In other words, power is proportional to the number of situations in which a voter is pivotal.

To illustrate, here are the details for example 3: There are four voters, who vote for or against candidate Bluster — a total of $16 = 2^4$ combinations to analyze. In each combination, the outcome may depend in an essential way on how one or more of the voters voted. The chart on the next page displays all this information. The sixteen rows represent the various combinations of votes; each column displays the number of votes (for Bluster) cast by the individual states (either 7 or 0 by state A, either 5 or 0 by state B, and so on). The votes that are pivotal are in **boldface**. Confirm that changing any of these highlighted votes would change the new President, and that changing any of the other votes would have no effect at all.

Banzhaf Index Calculation (pivotal votes in **boldface**):

A	B	C	D	Total	Bluster
0	0	0	0	0	loses
0	0	0	2	2	loses
0	0	3	0	3	loses
0	**0**	3	2	5	loses
0	5	0	0	5	loses
0	5	**0**	2	7	loses
0	5	3	**0**	8	loses
0	**5**	**3**	**2**	10	wins
7	**0**	**0**	**0**	7	loses
7	0	0	**2**	9	wins
7	0	**3**	0	10	wins
7	0	3	2	12	wins
7	**5**	0	0	12	wins
7	5	0	2	14	wins
7	5	3	0	15	wins
7	5	3	2	17	wins

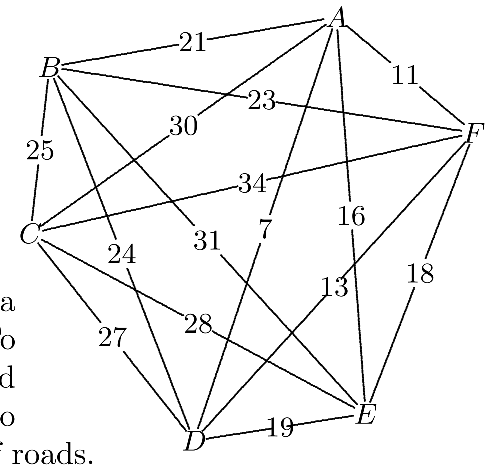
Notice that A casts a pivotal vote in 12 of the 16 combinations, while the other three states cast pivotal votes in only four combinations each. It makes sense, therefore, to regard A as having three times the voting power of state B (or state C or state D). It also makes sense to regard states B, C, and D as having equal power. This is somewhat surprising, given that B casts more than twice as many votes as D!

One aspect of the above table is worth noticing: *Pivotal votes occur in pairs.* This is because every combination where a voter can turn victory into defeat by switching a vote will be matched by a combination where the *same* voter can turn defeat into victory. For example, notice the pair of boldface entries in rows 6 and 8 of the table (look in the C column), and the pair in rows 5 and 13 (look in the A column). It follows that power ratings can be calculated with half the work — simply concentrate on the winning combinations, looking for pivotal votes that would produce defeat if they were changed.

1. Verify the assertion made about equal power in example 2 on the preceding page. Use the illustration shown above to guide your calculations.

2. Given the weights $A = 9$, $B = 5$, $C = 3$, and $D = 2$, calculate the voting power of each voter. Assume that decisions are made based on a simple majority.

1. Apply the Kruskal algorithm to find a minimal-cost spanning tree for the complete labeled graph shown at right.



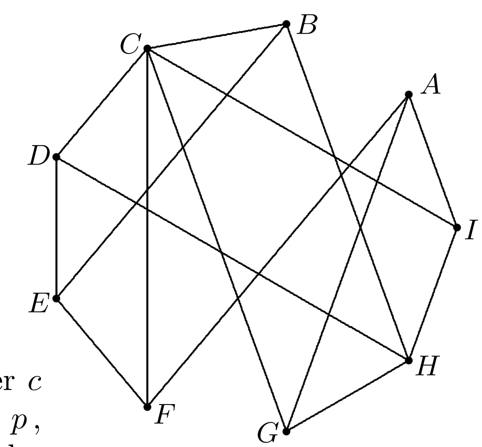
2. Modify the Kruskal algorithm so that it will find a *maximal-cost* spanning tree for the complete labeled graph shown at right.

3. Suppose that three towns form the vertices of a triangle whose sides have lengths 6, 8, and 10 miles. To join the towns, it is clear that it would suffice to build two roads whose total length is 14 miles. Show how to improve the answer by building a different network of roads.

4. Given the following voter preferences among three candidates, determine the winner of the election, according to **(a)** the plurality method; **(b)** the simple Borda count; and **(c)** the Condorcet method. **(d)** Draw the digraph for (c). Notice that 300 ballots were collected.

20 ABC   61 ACB   67 BAC   62 BCA   76 CAB   14 CBA

5. A hostess needs to seat nine guests around a circular table. The acquaintance graph shown at right reveals that every guest knows at least three other persons at the table; in particular, guest C knows everyone except A, H, and E. Is it possible to seat these nine persons around the table so that *no* person will be sitting next to an acquaintance? If so, show how to do it. If not, explain why not.



6. In a round-robin tournament, suppose that player c achieved the following results: For every other player p, either c defeated p, or c defeated another player who defeated p. Such a player c is called a *contender*. You have already shown that any player who wins a maximal number of games is a contender. Prove that any tournament that lacks an undefeated champion must have at least *three* contenders. Prove also that it is possible for one contender to win fewer games than another contender.

7. A 100-tree forest $\mathcal{F}$ has 2011 vertices. How many edges does $\mathcal{F}$ have?

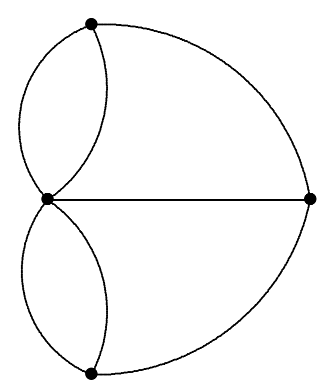
8. Show that the eight words BAD, BAT, BID, BIT, HAD, HAT, HID, and HIT can be arranged in a circle so that each word differs from its neighbors in a single letter.

1. Calculate the voting power of each voter, given the weights below. Assume that decisions are made based on a simple majority. The program *Windisc* can be used to check your answers (or to do calculations for large examples). Click **Window|Banzhaf** to open a calculation window. Click **Edit|Blocs** to change the number of voting blocs, click **Edit|Quota** to set the number of votes needed for a win, and click a bloc name or its number of votes to change the displayed data (which is initially random). Click **Calculate|All cases** to see a summary of pivotal votes and power ratings.
(a) $A = 2$, $B = 2$, $C = 1$
(b) $A = 10$, $B = 10$, $C = 1$
(c) $A = 10$, $B = 10$, $C = 10$, $D = 1$
(d) $A = 10$, $B = 10$, $C = 10$, $D = 10$, $E = 1$
(e) $A = 11$, $B = 5$, $C = 3$, $D = 2$

2. Analyze the power structure of the 1974 British Parliament, given the distribution of 635 seats: Labor 301, Conservative 296, Liberal 14, Irish Union 11, Scottish National 7, Welsh Plaid 2, Irish Catholic 1, Others 3.

3. A corporation has four shareholders, whose stock is distributed as follows: A has 48%, B has 23%, C has 22%, and D has 7%. Simple majority rules at stockholder meetings.
(a) How much stock can A sell to B, without changing the power structure?
(b) How much stock can D sell to A, without changing the power structure?
(c) How much stock can B sell to C, without changing the power structure?

4. In the eighteenth century, the Baltic port of Königsberg (renamed Kaliningrad in the former U.S.S.R.) was divided by the Pregel River into four sections — one on each bank, and one on each of two islands. These sections were made adjacent by seven bridges. The townspeople who liked to go for strolls generated one of the most famous math puzzles of all time. They wondered whether it was possible to make a tour of the town that passed over each of the seven bridges exactly once and then returned to the starting point. The figure shows all the information in skeletal form. The edges of the network represent the bridges, and the four vertices represent the sections of town. *Leonhard Euler*, a brilliant mathematician from Switzerland, heard of this puzzle and proved that a tour was impossible. In the process, he laid the foundation for what is today called *graph theory*. What was Euler's explanation? Notice that the graphical representation of this problem uses more than one edge to join two vertices. Such graphs are sometimes called *multigraphs*.



5. It is given that $A = 7\,000\,000$, $B = 5\,000\,000$, $C = 3\,000\,000$, and $D = 2\,000\,000$ are the populations of four states, and that the electoral votes of these states are 7, 5, 3, and 2, respectively (proportional to their populations). Do you think that the voters in these states have equal power in the electing of a President? Explain.

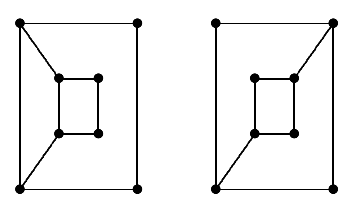
1. Examine the power of the individual states in the *Electoral College*. Each state is given as many votes as it has members in Congress (two senators plus representatives), and the District of Columbia has three votes also. The total number of electors is 538, which makes 270 the quota necessary for winning. Use *Windisc* to do this calculation. First activate an Apportionment window by clicking **Window|Apportionment**, then set the correct **Method** (Huntington-Hill) and a correct **Divisor**. Then click the menu item **Other|Electoral Power**. To do a complete power calculation is out of the question here, for there are 2^{51} combinations of votes to process, which would take decades of computer time. Instead of clicking **Calculate|All cases**, click **Calculate|Monte Carlo**. The program has been taught to randomly sample as many of the 2^{51} combinations as you allow it to — the window's title bar tells you to press a key when you have waited long enough. The resulting display will show the results of the poll, which ought to agree closely with the (unknowable) correct answer. In particular, look to see which states (if any) have power ratings that are not proportional to their voting weights. On the basis of your data, can you decide where it is best to live, in order that your vote have its greatest influence on the outcome of a Presidential election?

2. Another *Windisc* activity: Time how long it takes for the computer to do a complete power calculation, using examples ranging from 25 states to 30 states. The actual results do not matter — just the time it takes to do them. You should notice a predictable pattern developing. Account for the pattern, then use it to investigate the "decades of computer time" assertion made in the preceding question.

3. Find the simplest example of a graph that is connected but that does not have a Hamiltonian path.

4. You have ample supplies of 3-cent, 8-cent, and 10-cent stamps, and a small package that requires $1.07 worth of postage. Show that it is possible to put the correct postage on the package, but that a greedy approach to the problem does not work.

5. The two graphs shown at right are not isomorphic. Demonstrate this by finding a suitable distinguishing characteristic.

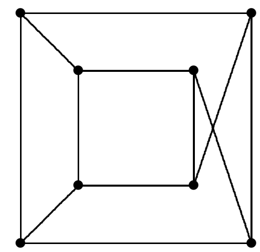


6. *Given two vertices in a tree, there is exactly one path that leads from one vertex to the other.* Is this a true statement?

7. It *is* possible for a connected graph to have a *bridge* and an Eulerian path, and for all its valences to be greater than 1. Give an example of such a graph.

8. Suppose that a connected graph has exactly four vertices of odd valence. Need the following statement be true? There are two paths that do not have any edges in common and that — between them — use every edge of the graph exactly once.

1. A strategy for coloring the graph shown at right: Run through a list of its vertices, assigning to each the first color that is acceptable. This is an instance of a *greedy* algorithm. Show that this approach to coloring the graph can produce a result that is less than optimal.



2. Given a list of 100 distinct positive integers, describe an *algorithm* for rewriting the list in decreasing order.

3. If a graph is Eulerian, then it can be converted into a strongly connected digraph. Explain this remark.

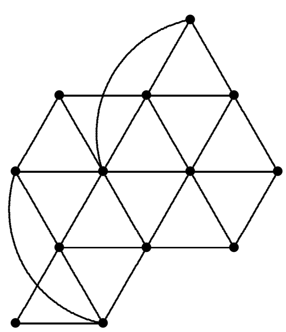
4. A connected graph has e edges and v vertices. A spanning tree for the graph can be obtained by erasing some of the edges. How many?

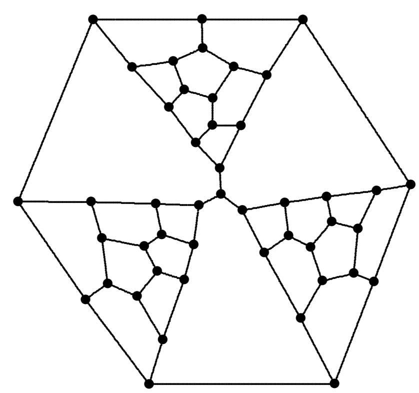
5. Within an acquaintanceship graph, what might the term *clique* refer to?

6. Given five labeled vertices, there are $125 = 5^3$ trees that span the five points. If isomorphic trees are *not* viewed as being different, however, then how many different five-vertex trees are there really?

7. What is the chromatic number of the graph shown below left?

8. What is the chromatic number of the graph shown below right? This is a more difficult question than the preceding, and not just because there are more vertices. Why?





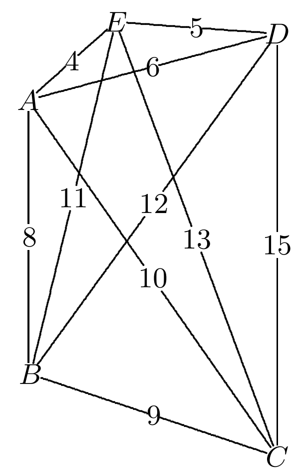
9. Given eight vertices $ABCDEFGH$, a graph can be defined by listing its edges (at most 28 of them). While inspecting such a list, you notice that the graph is trivalent. What can you deduce about the graph from this information?

10. There are exactly six different (non-isomorphic) trivalent, eight-vertex graphs. Draw pictures of them, and find distinguishing characteristics.

11. How many Hamiltonian circuits does a labeled cube skeleton have? How many Hamiltonian circuits does an unlabeled cube skeleton have?

Discrete Mathematics

1. After a sleep-over party at her house Saturday night, Evelyn had to drive her friends Ashley, Bea, Courtney, and Dana home. The driving distances from one house to another are summarized in the complete graph shown at right.



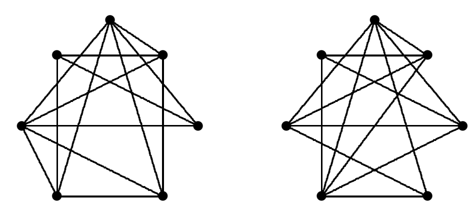
(a) Being a student of discrete mathematics, Evelyn knew about the nearest-girlfriend algorithm, and followed it. How many miles did Evelyn drive in all?
(b) As she was putting the car away, Evelyn remembered something called the sorted-edges method, and wondered if that would have been a better way to get her friends home. Would it have been? Explain your answer.
(c) To satisfy her curiosity (and to be ready for the next party), Evelyn used brute force to figure out the best circuit. *How many* circuits did she have to consider to arrive at her answer?

2. A connected graph has 96 edges and v vertices. What are the possible values for v?

3. Give an example of a connected, ten-vertex graph, whose chromatic number is 2, and all of whose valences are at least 3.

4. The valences for a connected graph are 1, 1, 1, 1, 1, 1, 2, 3, y. What are the possible values for y? If the graph is a tree, what must y be?

5. Decide whether the graphs shown at right are isomorphic, and give clear evidence for your answer.



6. What was the Kruskal algorithm designed to do? Given that it always produces optimal results, why did R. C. Prim decide to design his own algorithm to accomplish the same task?

7. Suppose that you are presented with a connected graph that has many vertices, all with even valence. The problem is to find an Eulerian circuit. Consider the procedure

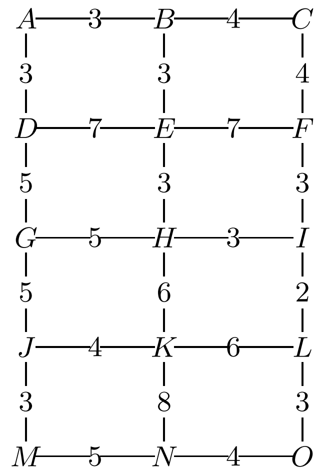
> *Start at any vertex. Choose randomly from among the available (unused) edges and move to the next vertex. Repeat the preceding instruction as long as it is possible to do so.*

Is this an effective procedure for finding an Eulerian circuit? Explain, using an example if necessary.

8. It took *Windisc* 87.73s and 178.69s to do complete power calculations on my computer for 21- and 22-state examples, respectively. The doubling pattern that was so conspicuous for smaller examples seems to be breaking down. Can you offer any explanation for this development?

Discrete Mathematics

1. You have now done several complete power calculations, in which you tallied all 2^n ways in which n votes can be cast. Perhaps you have noticed that the pivotal count for each voter has always been an *even* number? For example, in the 1974 British Parliament example, the counts were 152, 104, 80, 48, 40, 8, 8, and 16. Explain this phenomenon.



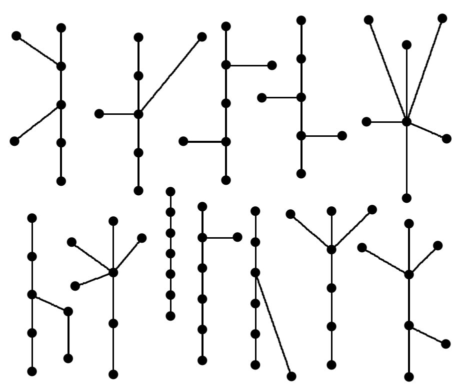
2. The graph shown at right represents the installation costs associated with linking fifteen nodes by a telecommunications network. Consider any edge that is *not* shown — AE or FN, for example — to be prohibitively expensive. Find a minimal-cost spanning tree (and its cost) for these fifteen nodes.

3. Show that there is more than one correct answer to the previous question, by finding an edge *not* in your MST that could be used to replace an edge that *is* in your MST.

4. Suppose that a nine-member board of governors makes all its decisions by simple majority. Each of the governors casts one (unweighted) vote. Three of the governors have formed a secret coalition, however, agreeing to always vote the same way. How they vote in front of the board is determined by a secret vote beforehand, also decided by simple majority. Does this coalition affect the power structure of the nine-member board? Explain.

5. Examine the power structure in the 1983 Italian Chamber of Deputies, which consisted of the following ten groups: Christian Democrats (225), Communist (198), Socialist (73), Italian Social Movement (42), Republican (29), Democratic Socialist (23), Liberal (16), Radical (11), Proletarian Democrats (7), and Others (6). The figures in parentheses are the number of votes each party casts.

6. Give an example of a graph that has a Hamiltonian circuit but not an Eulerian circuit. Give an example of a graph that has an Eulerian circuit but not a Hamiltonian circuit.



7. To calculate the number of different trees (that means non-isomorphic) with a specified number of vertices is a difficult problem. It is easy to see that, given either two or three vertices, there is essentially only *one* tree that can be drawn. Thereafter, one finds that, given 4 vertices, there are 2 trees; given 5 vertices, there are 3 trees; given 6 vertices, there are 6 trees; given 7 vertices, there are 11 trees; given 8 vertices, there are 23 trees. In the figure at right, there are a *dozen* 7-vertex trees. At least two of them must therefore be isomorphic. Which ones?

8. Given its list of valences, is it possible to tell whether a graph is connected? Explain.

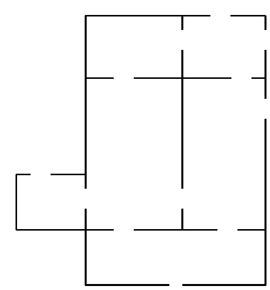
Discrete Mathematics

1. *Debut of the Banzhaf Index*: To support his 1965 lawsuit against the Board of Supervisors of Nassau County (NY), John Banzhaf introduced a method of measuring voting power. The county is divided into six districts. The 115 votes on the Board of Supervisors were allocated in 1964 to the districts as follows: 31 for Hempstead A, 31 for Hempstead B, 28 for Oyster Bay, 21 for North Hempstead, 2 for Long Beach, and 2 for Glen Cove. Assuming that decisions were made on the basis of simple majority — 58 votes or better needed to win — analyze the distribution of power to see what caused the controversy.

2. As the result of various lawsuits brought against the Board of Supervisors of Nassau County, the numbers of votes allocated to the six districts and the number of votes (the *quota*) needed to pass a measure have all been changed. In 1994, a total of 108 votes were allocated as follows: 30 for Hempstead A, 28 for Hempstead B, 22 for Oyster Bay, 15 for North Hempstead, 7 for Long Beach, and 6 for Glen Cove. The quota was 65. Analyze the distribution of power on this Board of Supervisors.

3. Given its list of valences, is it possible to tell whether a graph is a tree? Explain.

4. The figure at right shows the floor plan of a house. Decide whether it is possible to make a walking tour of the building that takes you through each doorway exactly once. The exterior of the house (the outdoors) should be thought of as a room.

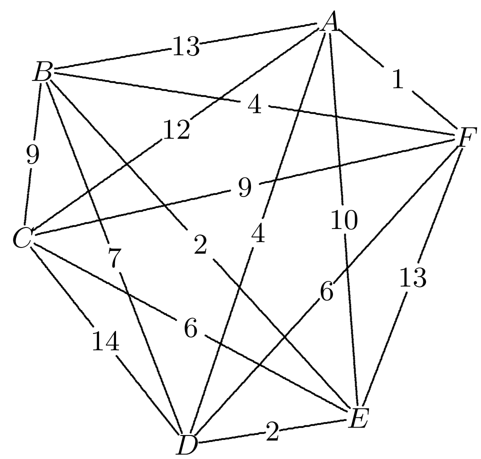


5. Suppose that a graph can be colored with two colors. Is this information sufficient to conclude that the graph is planar? Explain.

6. If a connected graph has vertices of odd valence, then an Eulerian circuit is out of the question. There is still the possibility of an Eulerian path, however (which finishes at a vertex different from the starting vertex). Suppose that a graph has two vertices of odd valence. Introducing an extra edge that joins these two vertices is called *Eulerizing the graph*. What effect does that have on the Eulerian problem? After the new version has been solved, the extra edge can be discarded. What happens to the circuit?

7. It is possible for the Electoral College to choose a President who does not win the popular vote. One wonders, therefore, how *small* a percentage of the popular vote a candidate can take and still win the election! Try your hand at inventing an extreme two-candidate example of this sort. To simplify your calculations, assume that all of a state's electoral votes are committed to the candidate that wins the state popularity contest. To make the figures even more dramatic, assume that the winning candidate (the President) gets *no* popular votes in the states that he loses, and that he gets only *half* the popular votes (plus 1) in the states that he carries. A further assumption, if you do not wish to consult population tables, is that a state's apportionment of representatives is roughly proportional to its population.

1. Find a solution to the traveling-salesman problem shown at right, by applying the sorted-edges algorithm to the given numerical data. Explain why there are two possible results.

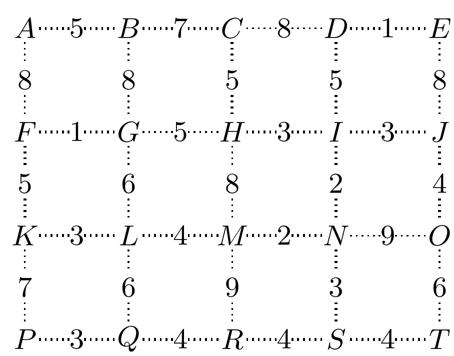


2. By applying the nearest-neighbor algorithm to vertex A, find a solution to the traveling-salesman problem shown at right.

3. There are $1296 = 6^4$ ways to find a tree that spans six given vertices, but most of these are isomorphic duplicates. How many non-isomorphic six-vertex trees are there?

4. In a three-candidate race for President, the Electoral College could elect someone who won a *very* low percentage of the popular vote. Following the guidelines described on the preceding page, find just how small a percentage of the popular vote is sufficient for victory.

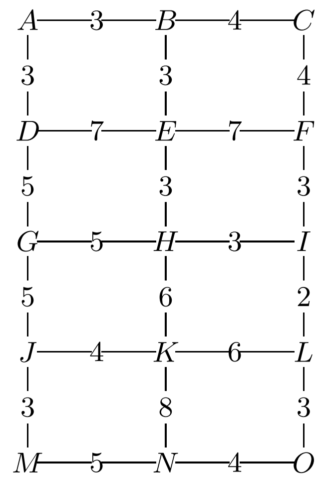
5. The Kruskal algorithm provides a minimal-weight spanning tree for any finite connected weighted graph. Such a tree, however, does not provide minimal-weight paths between specific vertices. In the network shown at right, it is evident that the spanning tree does not include the path of lowest weight between vertex A and vertex F, nor does it include the path of lowest weight between vertex A and vertex I. Devise an algorithm that will find the path of lowest weight between two given vertices in a network.



6. If a certain voting combination is pivotal for state A, and if the combination is reversed (in other words, each state switches its vote), will the resulting combination still be pivotal for state A? Explain.

7. *A Hamiltonian game*: Start with a complete n-vertex graph. Two players take turns marking directions on the edges of the graph, the objective being *to avoid forming a directed Hamiltonian path* — a path of $n-1$ edges that visits every vertex. The person whose arrow creates such a path is the loser. It is not hard to see that the first player should lose the 3-vertex game. Which player has the advantage in the 4-vertex game? Is it possible for this game to end in a tie?

8. Make up an example of an eight-room house with lots of doors that allows a tour that goes through every door exactly once, finishing and starting in the kitchen. Diagram your house as was done in the example on the preceding page.



1. Each edge in the graph at right has a associated numerical *capacity*, with which the edge is labeled. Keeping this in mind, define the *capacity of a path* to be the smallest of the capacities encountered along the path. Find a path from C to L that has the largest capacity that such a path can have.

2. (Continuation) Find a *maximal-capacity* spanning tree for the graph shown at right.

3. (Continuation) In your tree, there is a unique path from C to L. What is its capacity? Hmm...

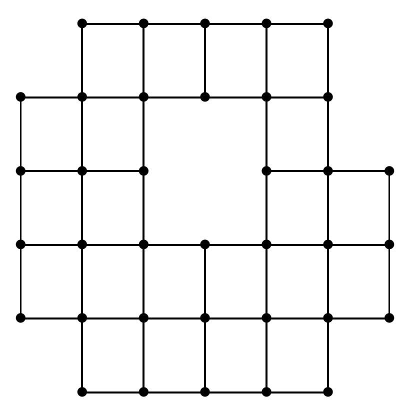
4. Refer to the question about the Italian Chamber of Deputies. Fundamentally incompatible, the two largest parties will *never* vote the same way. In other words, they will never be found in the same coalition. Use *Windisc* to see the effect that this has on the power structure. Open the **Misc|Opposition** dialog box, select the two opposed parties, and click the **conflict** button. Then request the complete power calculation, which will ignore all combinations in which the Communists are aligned with the Christian Democrats. Which party is the strongest?

5. A combination of voting blocs that has enough votes to carry an issue is called a *winning coalition.* A winning coalition in which every member is pivotal is called a *minimal* winning coalition. Return to the introductory example (the election of President Bluster) and enumerate the minimal winning coalitions.

6. Using M as the root vertex, apply Dijkstra's algorithm to the 4-by-5 grid found on the preceding page. The resulting tree should include minimal-weight paths from M to every other vertex in the network.

7. What if the seating problem (question 3 on page 32) had only involved seating the guests along one side of a long straight table? How would the answer be affected?

8. Is it possible to find an example of a voting situation where a group of voters is better off voting singly? In other words, the individual voters would have more power than that which comes from sharing the collective power of their voting block.

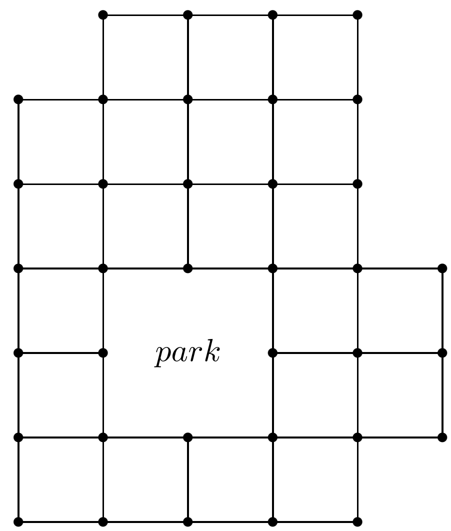


9. The graph shown at right does not have an Eulerian circuit. If it is Eulerized, however, a circuit can be found. When an extra edge duplicates an existing edge (joining edges already adjacent), think of it as a *reused edge.* Show that this graph can be Eulerized by means of reused edges, and do it using as few duplicates as possible.

10. Eulerizing a graph requires that the vertices of odd valence be paired up. It therefore seems that this technique will not work on a graph that has an *odd* number of odd vertices. What do you think?

1. You have seen that a Hamiltonian tour of the vertices of the rhombic dodecahedron is impossible. What about a Hamiltonian *path*, however — is this a possibility?

2. While searching for new three-bloc voting-power examples, I often found myself considering two possibilities: either one of the three blocs has dictatorial status, or else all three share power equally. Are there in fact any other distributions of power for a three-bloc voting system?



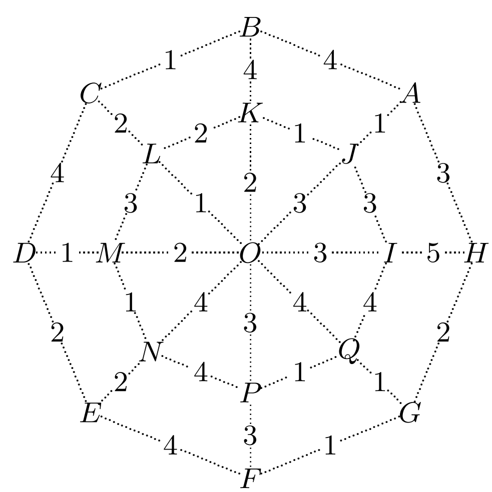
3. Find a minimal Eulerization for the network shown at right. In other words, enable an Eulerian circuit by reusing as few edges as possible.

4. The graph at right can be viewed as a network of city streets. Suppose that the Mayor has decided to make all the streets *one way*. Is it possible to do this and still have a connected network? This means, for clear practical reasons, that it should be possible to drive legally from any intersection to any other intersection.

5. Does the network at right have a Hamiltonian path, or perhaps even a Hamiltonian circuit?

6. In a connected graph, a *bridge* is an edge whose removal would disconnect the graph. For instance, every edge of a tree is a bridge (although that does sound strange). Consider the following proposition: *If a graph is connected and has no bridges, then every edge is part of a circuit.* Is this statement true? Explain your reasoning.

7. The chromatic number of a complete 12-vertex graph is 12. How many edges must be removed so that the chromatic number will drop to 11?



8. Given the graph shown at right, use Dijkstra's algorithm to find a spanning tree of minimal-weight paths from vertex O to the other sixteen vertices.

9. Consider a voting system where the parties have sizes 1, 2, 3, 4, and 5. Find the probability that
(a) the single voter is pivotal;
(b) the five-member party is pivotal;
(c) a designated member of the party of five is pivotal within that group;
(d) a designated member of the party of five is pivotal in the full assembly.
As usual, assume that all issues are decided by simple majority.

Discrete Mathematics

1. To calculate the power of a citizen voting for President of the United States, it is necessary to calculate two probabilities — that the citizen will cast a pivotal vote in the state election, and that the citizen's state will cast pivotal votes in the Electoral College. The product of these two numbers measures the power of the citizen in the Presidential election.

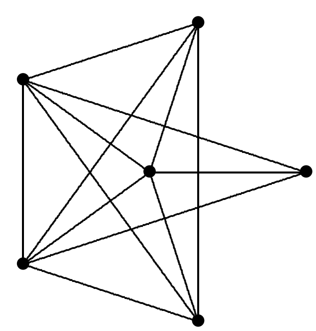
The first probability requires special treatment. It concerns an election system that could consist of *millions* of voters (who carry equal weight). The probability that the state result will depend on the vote of a *single* person is small, of course, but it is calculable, and it depends in a simple way on the number of voters in the state. It is also larger than you might think.

Let n be the number of voters in the state. By examining simple special cases, show that the probability of a single voter being pivotal is *not* proportional to $1/n$. In particular, show that *doubling* the voting population of a state does not simply *halve* the probability.

2. By examining more cases (perhaps with the help of the computer), try to find a simple description of what the effect is of doubling the size of the voting population in the state. The probability of a single voter being pivotal gets smaller, of course, but according to what rule? Make use of data for small n-values to discern a rule that can be applied to very large n-values.

3. Suppose that 40 coins are tossed. What is the probability that they land showing 20 heads and 20 tails?

4. In the preceding questions on voting power, there has been a tacit assumption that each vote is just as likely to be *for* the candidate as it is to be *against* the candidate. In other words, in a two-candidate race, we have assumed that the candidates are evenly matched. What effect would candidates of unequal strength have, however? What would happen to the probability of a single voter being pivotal?

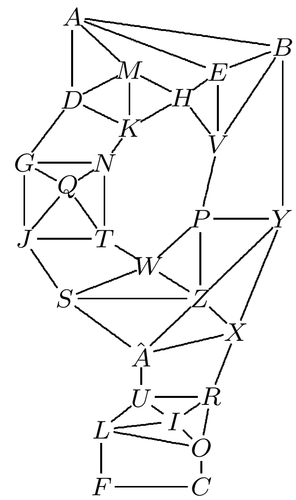


5. Decide whether or not the graph at right is planar, and give your reasons.

6. Suppose that a cashier has only quarters, dimes, and pennies to work with (the tray of nickels is empty). Show that a greedy approach to giving 30 cents change does not produce an optimal result.

7. A student proposed this "reversed sorted-edges" algorithm for solving TSPs: First put the edges in order of *decreasing* desirability (most expensive first). Then — starting with the complete graph — work down the list, *removing* an edge whenever the valences of its endpoints are both at least 3. What do you think of this algorithm?

1. Some applications of Eulerian circuit theory are to situations in which it is necessary to traverse every edge twice, once in each direction. Give an example of such a situation, and describe how to model it with a graph. Then think about whether it is likely that such a problem has a solution.



2. It is occasionally necessary to convert a connected graph into a *connected digraph* — each edge is given a direction, so that you can still travel from any vertex to any other. Is this conversion possible for the network shown at right?

3. Complete the analysis of individual voting power in Presidential elections that was outlined on the preceding page. Calculate pivotal probabilities for individual voters in a representative cross-section of states (eight or so is enough), including the state that has the strongest voters (most likely to cast pivotal votes in a state-wide election) and the state that has the weakest. Then multiply each of these state-by-state probabilities times the probability that the state is pivotal in the Electoral College — data that you have already obtained by using Monte Carlo methods in *Windisc*.

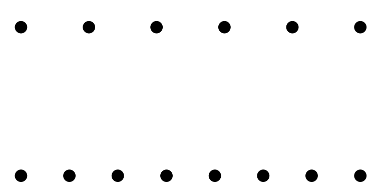
4. What effect did Montana's loss of a representative after the 1990 census have on the voting power of its residents? In other words, was the probability that a typical Montanan could cast the deciding vote for President significantly affected by the loss of the second seat? Show your calculations.

5. Suppose that the President were elected by direct popular vote. What is the probability that such an election would be decided by a single vote? Take the voting population to be 120000000 (which is approximately the number who voted in the 2004 Presidential election), and assume that the two candidates are evenly matched.

6. A graph is called *bipartite* if it can be colored with two colors. In other words, such graph has two types of vertices, and every edge is adjacent to one vertex of each type. A bipartite diagram can therefore be arranged so that all the edges run conspicuously from one type of vertex to the other.

(a) Explain why the sum of the valences of one type of vertex must equal the sum of the valences of the other type.

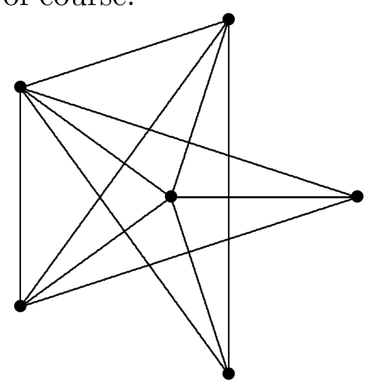
(b) By adding edges to the diagram at right, show that it is possible for a bipartite graph to have eight trivalent vertices and six tetravalent vertices.



7. Some authors define "bipartite" to mean a graph that has no circuits of odd length. Explain why this definition is equivalent to one given in the preceding question.

1. Suppose that the plurality results of a three-candidate race are $C = 43$, $B = 38$, and $P = 19$. Investigate how the results of this contest might vary, depending on what method is used to evaluate the election. In particular, could either B or P win a point-counting method (traditional Borda or other variant)? Could either B or P emerge as a Condorcet winner? Could either B or P be a runoff winner? Create distributions of preference ballots (how many of each type — CBP, BPC, and so on) to define your examples. Each example should be consistent with the given 100-vote plurality data, of course.

2. Decide whether or not the graph at right is planar, and give your reasons.



3. The final result of a tournament is a complete graph all of whose edges have been directed. (An arrow that runs from vertex A to vertex B is usually taken to mean that A has defeated B.) What is the significance of a tournament being connected?

4. A tournament is called *transitive* if, whenever arrows run from A to B and from B to C, the edge AC is always directed from A to C. What features must any transitive tournament have?

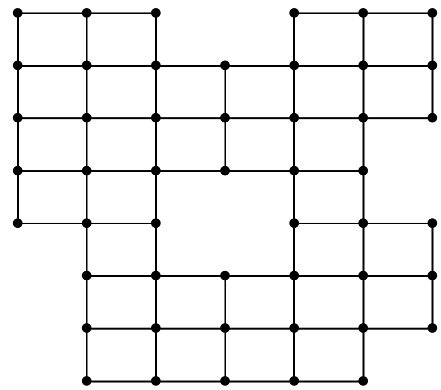
5. Which of the following properties of a digraph is the most restrictive? the least restrictive? (1) *strongly connected*, (2) *has a Hamiltonian path*, (3) *has a Hamilitonian circuit.*

6. If it is possible at all, there are usually many ways to carry out the conversion of a connected graph into a connected digraph. The inevitable question then arises: Are some of these solutions better than others? Propose a definition of what it means for one solution to be better than another. Illustrate by doing a simple example in two different ways and comparing them.

7. The ancient Egyptians had a peculiar way of expressing fractional quantities — as sums of positive reciprocals, all terms required to be different. An acceptable way of writing $\frac{3}{5}$ was $\frac{1}{2} + \frac{1}{10}$, for example, and $\frac{4}{5}$ might have been written $\frac{1}{2} + \frac{1}{4} + \frac{1}{20}$.
(a) Show that these two representations can be obtained by means of a greedy approach to the problem, and try this approach on a proper fraction of your choosing.
(b) By considering the example $\frac{4}{17} = \frac{1}{6} + \frac{1}{15} + \frac{1}{510}$, show that the greedy approach does not always produce optimal results. (You will have to decide what "not optimal" means in this instance.)

1. It is given that $A = 7\,000\,000$, $B = 5\,000\,000$, $C = 3\,000\,000$, and $D = 2\,000\,000$ are the populations of four states, and that the electoral votes of these states are 7, 5, 3, and 2, respectively. Which state has the most powerful voters?

2. Does the graph shown at right have a Hamiltonian circuit? If not, does it have a Hamiltonian path? If not, why not?

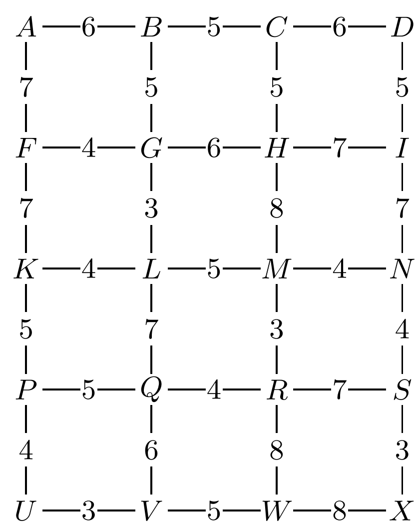


3. Given a digraph, a *source* is a vertex whose invalence is 0, and a *sink* is a vertex whose outvalence is 0. If a digraph has a source or a sink, then it cannot be connected. Explain. Then give an example of a connected graph that you have converted (by assigning arrows to all the edges) into a disconnected, sourceless, and sinkless digraph.

4. Three towns A, B, and C are situated in the desert, at three corners of a square that is 6 miles on a side. It is possible to link the towns by paving 12 miles of straight roads. Show that it is in fact possible to link the towns by paving less than 11.7 miles of roads. For extra credit, find a way that requires paving less than 11.6 miles of roads.

5. Which would a Californian prefer — to vote for President in a state of 15 000 000 voters that has a 48% chance of being pivotal in the Electoral College, or to vote for President in a national popular-vote election in which there are 125 000 000 voters? Assume that the two candidates are equally matched, and justify your answer with some calculations.

6. Given the graph shown at right, use Dijkstra's algorithm to find a spanning tree of minimal-weight paths from vertex A to the other nineteen vertices.

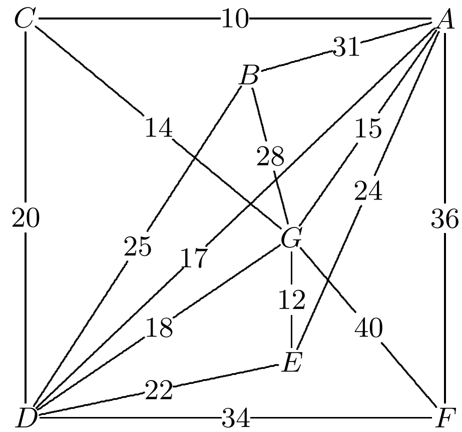


7. Give an example of a planar graph that has seven vertices, an Eulerian path, and no Eulerian circuit.

8. A graph that has 500 edges must have at least how many vertices?

9. A piece of wire is to be bent into the shape of a square, two inches on a side. How long must the wire be? The answer is of course eight inches. Now for a more difficult question: How long must the wire be, in order that it be bent to form the twelve-edge framework of a two-inch cube? The wire may not be cut, and only 90-degree bends are allowed. It will be necessary to double some edges.

1. The graph shown at right represents the installation costs associated with linking the seven nodes by a communications network. Find a minimal-cost tree (and its cost) that spans the seven nodes.



2. Give an example of a connected planar graph that has seven vertices and chromatic number 4. Explain why your graph can not be colored with 3 colors.

3. Suppose that the Webster method is applied with a divisor of 500000 to the hypothetical four-state population data $A = 745\,000$, $B = 1\,245\,000$, $C = 2\,245\,000$, and $D = 12\,765\,000$. What is the resulting size of the House of Representatives, and how are the seats allocated to the four states? What would Webster himself find objectionable about the results?

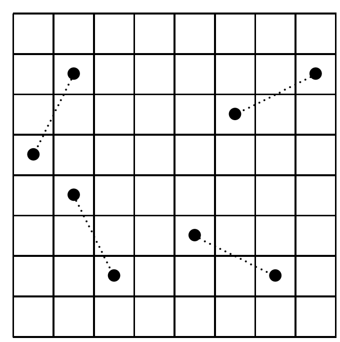
4. Given the data of the previous question, the Huntington-Hill method would require that state D give up one of its Webster-allocated seats to state A. According to the proponents of this method, what are the logical grounds for making this change?

5. Exactly one of the valence lists given below can be modeled by an actual graph:

9,8,7,6,5,4,3,2   3,3,3,3,2,2,2,2   4,3,3,2,2,2,1

Find two non-isomorphic graphs that this list describes. State a reason why you are sure that your graphs are non-isomorphic.

6. *A graph from the game of chess*: Let the vertices of the graph represent the 64 squares of the chessboard, and join any two vertices whenever it is possible for a knight to jump from one square to the other. Each of the four dotted segments in the figure at right illustrates the unusual L-shaped move of the knight. What are the valences of this graph? Explain why there is no Eulerian circuit for this graph. The *knight's tour* is one of the best-known of all recreational Hamiltonian circuit questions. It is solvable but difficult. Try it only if you have the time.



7. Suppose that the valences of a connected graph are 6, 6, 6, 6, 3, 3, 3, 3, 3, 3, 3, and 3. Is it possible that such a graph could fail to have a Hamiltonian path? Explain.

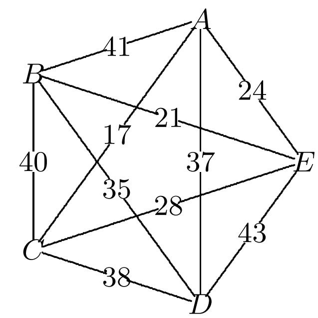
8. Show that all the valences of a planar graph can be 5.

9. Given the ten distances $AB = 40$, $AC = 38$, $AD = 45$, $AE = 41$, $BC = 22$, $BD = 54$, $BE = 43$, $CD = 24$, $CE = 20$, and $DE = 25$, solve the TSP — first using the nearest-neighbor algorithm based at D, then using the sorted-edges method. Can you carry out these algorithms *without making a diagram*? While you are at it, try applying Kruskal's and Prim's algorithms to find an MST for this graph.

1. A *leaf* is a vertex whose valence is 1. Suppose that a ten-vertex tree has m leaves. What are all the possible values for m?

2. Is it possible for a graph to have 24 vertices and 280 edges? Explain.

3. One algorithm for coloring the vertices of a graph begins by arranging the vertices by valence, in *non-increasing order*. Why is it better to begin the coloring process with the vertices of highest valence? In other words, what might happen if the vertex of highest valence were left uncolored until late in the process?



4. Given the ten weights shown in the graph at right, find a solution to the TSP by applying **(a)** the nearest-neighbor algorithm based at A; **(b)** the sorted-edges algorithm.

5. Given the ten weights shown in the graph at right, use either Kruskal's algorithm or Prim's algorithm to find an MST.

6. In chemistry, an *alkane* is an acyclic, saturated hydrocarbon molecule. In terms of graph theory, this simply means that any alkane can be described by a *tree* in which the valence of each vertex is either 1 (the vertex represents a hydrogen atom) or 4 (the vertex represents a carbon atom). If an alkane has six carbon atoms, then how many hydrogen atoms must it have?

7. Most graphs do not have Eulerian paths. To trace *all* the edges of a network, using each edge *exactly once*, several separate paths may be required. In other words, to trace the network with a pencil and paper, it may be necessary to lift the pencil off the paper occasionally. Given the valence list for a connected network, it is possible to tell *how many times* the pencil must be lifted (to begin a new path) in the process of covering every edge. Explain how this number can be found. In particular, how many separate paths are needed for the edge-skeleton of a cube? Make a diagram that illustrates your answer.

8. The vertices of any tree can be colored using only two colors. Suppose that a ten-vertex tree has been colored so that r vertices are red and b vertices are blue. What are the possible values for r?

9. Just as a planar graph can be drawn without accidental crossings on a plane surface, a *spherical* graph can be drawn without accidental crossings on a spherical surface, and a *toroidal* graph can be drawn without accidental crossings on a *torus*, which is the surface of a doughnut, or an inner tube.
(a) Explain why — despite the terminology — there is no essential difference between the terms *planar* and *spherical* when they are applied to graphs.
(b) Provide a simple example of a graph that is toroidal but not planar.

10. Show that the eight words LAD, LAG, LED, LEG, BAD, BAG, BED, and BEG can be arranged in a circle so that each word differs from its neighbors in a single letter.

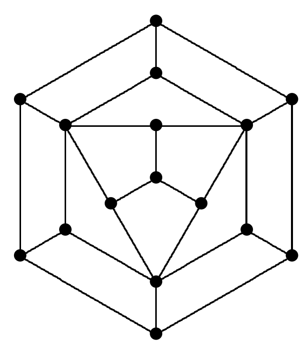
1. Given the following voter preferences among three candidates, show that each of the candidates could emerge as the winner of the election, depending on what method is used to evaluate the results.

45 ABC 33 ACB 85 BAC 23 BCA 70 CAB 44 CBA

Notice that 300 ballots were collected.

2. What is the probability that a Student Council election between two evenly matched candidates will end in a tie? Assume that all 970 students vote.

3. *An old puzzle*: A farmer has to get a wolf, a goat, and a cabbage across a river, using a boat that is only big enough to accommodate the farmer and one other object (the cabbage is quite big). The project will therefore take several trips, if it can be done at all. As you have probably guessed, the farmer can not leave the goat alone with the wolf — the wolf would eat the goat — nor can the farmer leave the goat alone with the cabbage — goats like to eat cabbage. It is safe to leave the wolf alone with the cabbage, however, because wolves do not like cabbage. The puzzle can be represented by means of a ten-vertex graph, whose vertices are named WGCF, WCF, WGF, CGF, WC, GF, W, C, G, and $\emptyset$. These ten vertices represent the combinations that are *safe* to have on the side of the river where the party of four is waiting. The last symbol $\emptyset$ represents the so-called *empty set*, which means that nothing is left on that side of the river — the puzzle has been solved.
(a) Explain why the vertices WG, CG, WF, FC, F, and WGC were left out of the graph.
(b) Continue this graphical interpretation of the story by drawing ten edges, using these ten vertices. One of the edges should join WCF to C. What action does this edge represent? Find similar meanings for the other nine edges you draw.
(c) The graph you have just drawn should be *connected*. Use it to find a solution to the puzzle. By the way, is the graph a tree?



4. Does the graph shown at right have a Hamiltonian circuit? If so, make a diagram of such a circuit. If not, does the graph have a Hamiltonian path? If so, make a diagram of such a path. If not, explain why not.

5. It is unusual for the Webster and Huntington-Hill methods to disagree, but it happened during the 1960 reapportionment. The states in question were Massachusetts and New Hampshire, whose populations were 5148578 and 606921, respectively. The Webster method would have awarded Massachusetts 13 representatives and New Hampshire only 1. What justification would Huntington and Hill have offered for taking one of the representatives from Massachusetts and giving it to New Hampshire? Show all calculations, please.

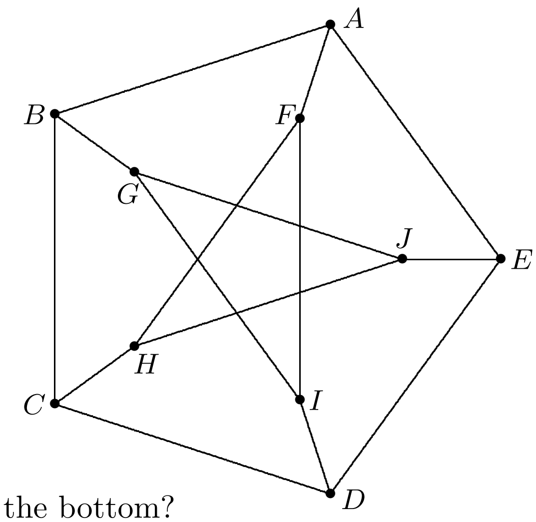
6. The five participants in a weighted voting system are A, B, C, D, and E, which have been given 2, 3, 4, 6, and 8 votes, respectively. Issues are decided by a simple 12-vote majority. There are two winning and two losing combinations in which A plays a pivotal role. What are they? What is the probability that A will be pivotal on an issue?

The greedy coloring algorithm. Given a list of the vertices of the graph, go through the list once, assigning each vertex the first available color.

An interesting way of creating the initial list is to arrange the vertices by valence, in non-increasing order, breaking ties randomly. Because of valence duplications, the outcome of this algorithm may depend on chance.

1. Apply the greedy, decreasing-valence approach to the United States map. You will have to break a tie immediately, because Missouri and Tennessee both have valence 8. One will get color number 1, the other will get color number 2. Why do Hawaii and Alaska both get color number 1?

2. Notice that the valence of every vertex in the graph shown at right is 3. To apply the greedy coloring algorithm, the vertex list can be chosen entirely at random. Apply the algorithm,
(a) using the list $ABCDEFGHIJ$;
(b) using the list $ABCDEIJFGH$;
(c) using the list $ABCDEHGFJI$.
(d) Is it possible to color this graph using two colors? Explain your answer.



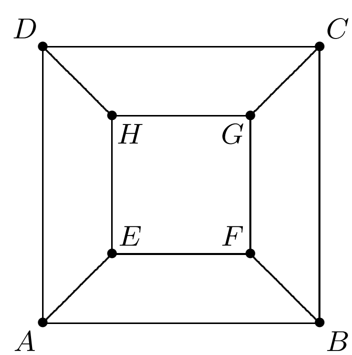
3. For coloring purposes, why does it make sense to put large valences at the top of the list, not at the bottom?

4. Explain the use of the word "greedy" to describe this algorithm.

5. Randomness is introduced into this coloring algorithm because of duplicate valences. Is it possible for a list of valences to contain *no* duplicates? Explain.

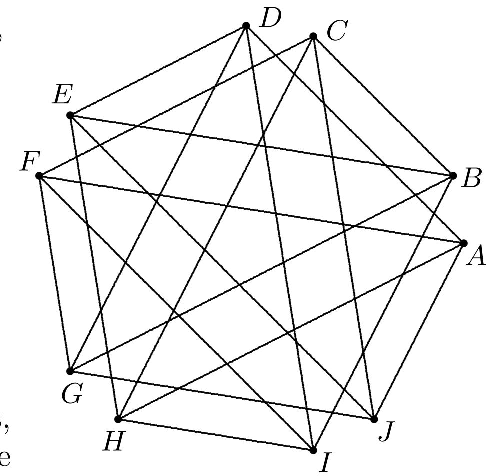
6. Given any graph, is it possible to put the vertices of the graph into a list in such a way that the greedy algorithm produces an optimal coloring of that graph? Explain.

7. The diagram shows another graph, all of whose vertices have valence 3. Apply the greedy coloring algorithm to the vertex list $DFEHCGBA$. Does this produce an optimal coloring?



8. The assignment of frequencies to radio stations can be viewed as yet another illustration of coloring the vertices of a graph. Explain. In particular, what do the vertices represent, what do the edges represent, and what is the meaning of a color?

1. The ten-vertex graph shown at right is *tetravalent*, meaning that that the valence of every vertex is 4.
(a) How many edges does this graph have?
(b) Apply the greedy coloring algorithm to the alphabetic list of vertices $ABCDEFGHIJ$.
(c) What is the chromatic number of this graph?
(d) Find a listing of vertices that will produce an optimal coloring (using fewest possible colors) when the greedy algorithm is applied to it.



2. Given a division of an island nation into states, each of which consists of one connected piece (unlike Michigan), you have seen that there is an associated "border" graph that represents the arrangement of states. In this model, vertices represent states, and edges represent states that share a nontrivial border. One can ask the converse question: Given a planar graph, does it necessarily correspond in this way to a map?

3. No matter how many states cover an island nation, the entire map can be colored with only four colors, according to the Four-Color Theorem. Suppose that you have just finished such a coloring, but then notice that the island nation is surrounded by water, which needs to be colored as well. Does this situation require five colors?

4. Given a list of vertices for a graph, there are two basic plans for creating a coloring from the list. The *single-pass* approach goes through the list only once, giving each vertex the first allowable color. The *multiple-pass* approach goes through the list several times, applying a single color on each pass to every vertex that admits it. Prove that these two approaches produce exactly the same coloring.

5. The greedy coloring algorithm can produce dramatically bad results! For example, invent a graph and make a list of its vertices, in such a way that the greedy algorithm will require 2005 colors when applied to the list, even though the chromatic number of the graph is actually 2.

6. To avoid mid-air jetliner collisions, each flight is assigned a cruising altitude. Explain how this might be viewed as a graph-coloring problem.

7. Graphs sometimes give rise to *edge-coloring* problems. The goal is to color the edges of a given graph using as few colors as possible, so that adjacent edges (which are edges that meet at a vertex) are never assigned the same color. Give an example of how this concept can be applied to a real problem.

Brooks' Theorem

The greedy coloring algorithm has been applied to show that the chromatic number of any graph is at most 1 more than the largest valence in the graph. There are two types of graph that make the phrase "1 more than" necessary — odd circuits and complete graphs. One wonders if there is third type of graph (neither complete nor an odd circuit), whose chromatic number is 1 more than its largest valence. Such a graph might have many vertices, but — if there were one — there would have to be a *smallest* example. For the next two pages, assume that the connected graph $\mathcal{G}$ is that minimal example, whose largest valence is m, and whose chromatic number is $m+1$.

1. Analyze the cases $m = 1$ and $m = 2$. It will be assumed that $2 < m$ from now on.

2. All the vertices of $\mathcal{G}$ must have the same valence m. Otherwise, some vertex V would have a valence smaller than m. Consider the graph $\mathcal{G} - V$ obtained by deleting V and the edges that meet at V. By assumption, the chromatic number of $\mathcal{G} - V$ is at most m. Explain. Why does this contradict our assumptions about $\mathcal{G}$?

3. Choose an arbitrary vertex V of $\mathcal{G}$, and consider the graph $\mathcal{G} - V$, as above. It will be useful to assume that $\mathcal{G} - V$ has been colored with m colors, and that the vertices $V_1, V_2, \ldots, V_m$ that are adjacent to V in $\mathcal{G}$ are assigned different colors $c_1, c_2, \ldots, c_m$, respectively. Justify this assumption.

4. Let $\mathcal{H}_{12}$ be the subgraph of $\mathcal{G} - V$ that contains only those vertices that are colored either c_1 or c_2, and that contains only those edges that join vertices colored either c_1 or c_2. Recognizing that $\mathcal{H}_{12}$ might not be a connected graph, consider the component of $\mathcal{H}_{12}$ to which V_1 belongs. Explain why swapping the colors c_1 and c_2 on V_1 and on all other vertices in this component will produce an allowable coloring of $\mathcal{G} - V$. Then explain how our assumptions about $\mathcal{G}$ require that V_2 and V_1 belong to the same component of $\mathcal{H}_{12}$.

5. It follows that V_1 and V_2 must lie in the same component $\mathcal{C}_{12}$ of $\mathcal{H}_{12}$. This implies that there are paths in $\mathcal{H}_{12}$ that lead from V_1 to V_2. In fact, there is only one such path. To see this, show first that V_1 is adjacent to only one vertex W of $\mathcal{G} - V$ that is colored c_2. Show, in fact, that the vertices adjacent to V_1 in $\mathcal{G} - V$ must use the colors $c_2, \ldots, c_m$ once each, lest our assumptions about $\mathcal{G}$ be contradicted.

6. It is possible that $W = V_2$, in which case a path from V_1 to V_2 has been found! If not, then there are at least two vertices adjacent to W that are colored c_1 — one of them being V_1. Recall what the valence of W is, and explain why the vertices adjacent to W must use *all* the colors $c_3, \ldots, c_m$. Conclude that there are exactly two vertices adjacent to W that are colored c_1. Call the new one X.

7. In a similar fashion, explain why there is only one vertex other than W that is adjacent to X and that is colored c_2. This vertex might be V_2. In any event, it should now be clear that this path in $\mathcal{H}_{12}$ is unique, and that it must eventually reach V_2. Moreover, the component $\mathcal{C}_{12}$ consists *only* of this path. Explain.

Brooks' Theorem (continued)

1. Here is a summary of the current assumptions and deductions: the valence of every vertex of $\mathcal{G}$ is m, where $2 < m$, the chromatic number of $\mathcal{G}$ is $m+1$, and no graph smaller than $\mathcal{G}$ has these properties, unless it is complete or an odd cycle. Given an arbitrary vertex V of $\mathcal{G}$, the subgraph $\mathcal{G} - V$ can therefore be colored with m colors. What has just been established for colors c_1 and c_2 can of course be established for any two colors c_i and c_j. What has been learned about the paths $\mathcal{C}_{ij}$?

2. Suppose that i, j, and k are different indices. The paths $\mathcal{C}_{ij}$ and $\mathcal{C}_{ik}$ of course intersect at V_i. Show that they cannot intersect anywhere else.

3. The colors c_i and c_j can be interchanged along any path $\mathcal{C}_{ij}$, without affecting the validity of the coloring of $\mathcal{G} - V$. Explain.

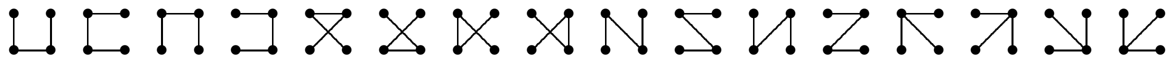
4. Select indices i and j for which the path $\mathcal{C}_{ij}$ consists of more than the single edge V_iV_j. Let X be the vertex adjacent to V_i that has color c_j. Next, select an index k that is different from i and j. Explain why this is possible. Then swap the colors c_i and c_k all along the path $\mathcal{C}_{ik}$. Explain why this is still a valid coloring of $\mathcal{G} - V$.

5. Because this new coloring must have the same properties (deduced above) as does every other coloring of $\mathcal{G} - V$, it follows that there is a unique path from V_j to V_k (which now is colored c_i), along which colors alternate c_j, c_i, c_j, $\ldots$, c_i. It also follows that there is a unique path from V_i (which is now colored c_k) to V_j, along which colors alternate c_k, c_j, c_k, $\ldots$, c_j. Explain why both of these paths go through X. Explain why this is a contradiction.

6. The contradiction shows that it is impossible to select indices i and j for which the path $\mathcal{C}_{ij}$ consists of more than the single edge V_iV_j. What does this say about $\mathcal{G}$?

Counting Labeled Trees

Given four vertices, which are distinguishable by virtue of being labeled or being in fixed locations, there are 16 possible trees that span these vertices, as shown in the diagram:



1. In general, let T_n be the number of different spanning trees determined by n labeled vertices. The display above illustrates the case $T_4 = 16$. Draw similar (but simpler) diagrams that illustrate the cases $T_3 = 3$, $T_2 = 1$ and $T_1 = 1$.

In a similar fashion, one could make a large diagram to illustrate the case $T_5 = 125$. Rather than do this, however, it is interesting to learn how to calculate the number $T_6 = 1296$ by using known values for T_1, T_2, T_3, T_4, and T_5. To this end, mark six dots on a piece of paper. The actual arrangement does not matter, but a roughly circular arrangement works well (so that no three dots line up). The next questions concern these six dots.

2. Choose *four* of the dots and draw one of the sixteen possible trees that span them. Then draw the tree that spans the *other* two dots. (It is of course unique, which is what $T_2 = 1$ means.)
(a) By drawing *one more edge*, these two trees can be combined to form a tree that spans all six dots. Explain why there are 8 different ways of drawing this fifth edge.
(b) Using the fact $T_4 = 16$, explain how your four-dot selection ultimately leads to 128 different trees that span the six dots.

3. In a similar fashion, choosing a *different* set of four dots will lead to another 128 trees that span the six dots.
(a) In how many ways can this set of four dots be chosen?
(b) How many six-dot spanning trees can be produced by splitting the six dots into a set of four dots and a set of two dots?

4. Now choose *three* of the dots and draw one of the three trees that span them. Draw one of the three trees that span the other three dots.
(a) By drawing one more edge, these two trees can be combined to form a tree that spans all six dots. Explain why there are 9 different ways of drawing this fifth edge.
(b) Using the fact $T_3 = 3$, explain how a single three-dot selection ultimately leads to 81 different trees that span the six dots.

5. How many six-dot spanning trees can be produced by splitting the six dots into two sets of three dots?

Counting Labeled Trees (continued)

1. How many six-dot spanning trees can be produced by splitting the six dots into a set of five dots and a set of one dot?

2. How many six-dot spanning trees can be produced by splitting the six dots into a set of two dots and a set of four dots?

3. How many six-dot spanning trees can be produced by splitting the six dots into a set of one dot and a set of five dots?

4. Now select a specific tree $\mathcal{T}$ that spans your six dots. It of course has five edges. Consider what the diagram would look like if you were to erase one of these edges. Then explain why $\mathcal{T}$ has actually been counted 10 times in the preceding inventories. Summarize the calculation of $T_6 = 1296$, using the five values $T_1 = 1$, $T_2 = 1$, $T_3 = 3$, $T_4 = 16$, and $T_5 = 125$.

5. Explain how the four values $T_1 = 1$, $T_2 = 1$, $T_3 = 3$, and $T_4 = 16$ can be combined to calculate $T_5 = 125$.

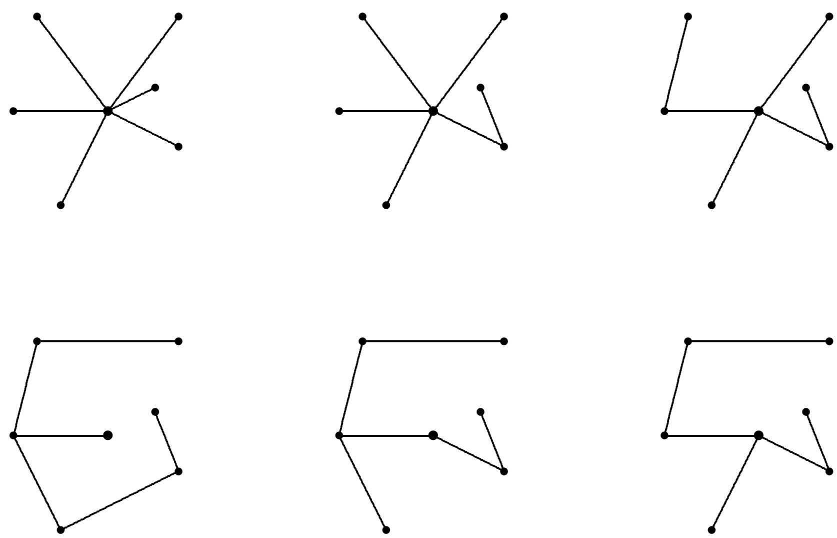
6. Use the six values $T_1 = 1$, $T_2 = 1$, $T_3 = 3$, $T_4 = 16$, $T_5 = 125$, and $T_6 = 1296$ to confirm that $T_7 = 16807$.

7. Use the preceding investigations to explain the recursive equation

$$T_n = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \binom{n}{k} \cdot k \cdot (n-k) \cdot T_k \cdot T_{n-k}\,.$$

Counting Labeled Trees

Given n distinguishable (labeled) vertices, there are n^{n-2} different trees that span the vertices. One way to establish this formula is to choose one of the vertices — call it A — and classify the spanning trees according to the valence of A. When the valence of A is k, the tree is said to be of "type k". The diagram shows six of the many possible trees in the case $n = 7$, one of each possible type. Vertex A is the large dot near the center of each example.



1. Let T_k be the number of spanning trees of type k. Explain why T_k is nonzero only for $0 < k < n$, and why $T_{n-1} = 1$.

2. The diagram shows a representative 7-vertex spanning tree for each of the types $k = 1$ to $k = 6$. The second tree is obtained by *thinning* the first tree at A, which means that one of the edges incident at A is detached from A and re-attached somewhere else. All other edges are preserved. Explain why there are 30 ways to thin the $k = 6$ example shown, and why the resulting trees are all different examples for $k = 5$. This should enable you to explain why $T_{n-2} = (n-2)(n-1)$ in general.

3. Explain why there are 24 ways to thin the second tree at A, and why each of them produces a different example of type 4. The third tree is one of the examples.

4. Because there are 30 trees of type 5, the preceding suggests that there are 720 trees of type 4. Explain why each tree of type 4 is actually the result of thinning *two* different trees of type 5. This implies that there are $T_4 = 360$ spanning trees of type 4. Explain the logic.

5. Explain why there are $(n-3)(n-1)$ ways to thin any n-vertex spanning tree of type $n-2$. You might think that there are $(n-2)(n-2)$, but ....

Counting Labeled Trees (continued)

1. Any n-vertex tree $\mathcal{T}$ of type $k-1$ can be obtained as the result of thinning a tree of type k. In fact, there are $n-k$ different trees that produce $\mathcal{T}$ when thinned. Explain.

2. The case $n=7$ of the formula $T_{n-3} = \binom{n-2}{2}(n-1)^2$ has already been confirmed. The established formula $T_{n-2} = (n-2)(n-1)$ and the preceding two questions should help you explain the derivation of this general formula.

3. Return to the $n=7$ example, and consider the 360 trees of type 4, one of which is shown in the diagram. Show that there are $18 = 4 \cdot 5 - 2$ ways to thin any such tree, to obtain a tree of type 3.

4. It has already been established that each tree of type 3, including the one in the diagram, can be obtained in *three* ways by thinning a tree of type 4. Review this reasoning, then conclude that there are $2160 = 360 \cdot 18/3$ trees of type 3 that span 7 vertices.

5. In general, each n-vertex spanning tree of type $n-3$ can be thinned in $(n-3)(n-2)-2$ ways to obtain a tree of type $n-4$. Notice that this count can be rewritten as $(n-4)(n-1)$. Recall also that each type $n-4$ tree is counted three times in this way.

6. Conclude that there are

$$T_{n-3} \cdot \frac{(n-4)(n-1)}{3} = \binom{n-2}{2}(n-1)^2 \cdot \frac{(n-4)(n-1)}{3} = \binom{n-2}{3}(n-1)^3$$

trees of type $n-4$.

7. In general, each n-vertex spanning tree of type k can be thinned in $k(n-2)-(n-1-k)$ ways to obtain a tree of type $k-1$. Notice that this count can be rewritten as $(k-1)(n-1)$. Recall also that each type $k-1$ tree is counted $n-k$ times in this way. Conclude that

$$T_{k-1} = T_k \cdot \frac{(k-1)(n-1)}{n-k} \ .$$

In particular, verify that $T_2 = 6480$ and $T_1 = 7776$ in the $n=7$ case.

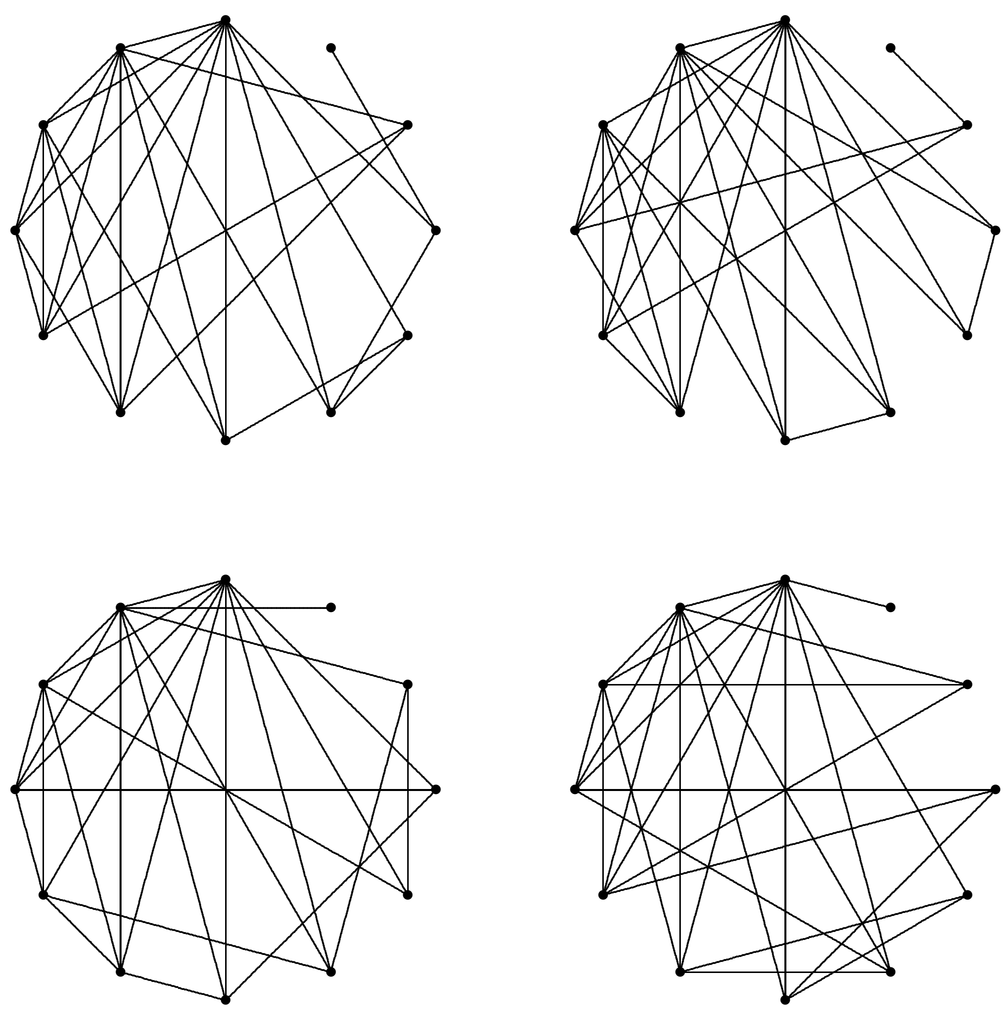
8. The preceding recursion implies that $T_k = \binom{n-2}{k-1}(n-1)^{n-k-1}$. Verify this formula when $n=7$ on the six types of trees, then explain why it is correct for all values of n.

9. The total number of n-vertex spanning trees is $T_1 + T_2 + \cdots + T_{n-1}$. Use the Binomial Theorem to show that this sum is n^{n-2}.

1. Let $v_1 \geq v_2 \geq \ldots \geq v_n > 0$ be the valences for an n-vertex graph. Show that
(a) $v_1 \leq n-1$;
(b) $v_1 + v_2 \leq 2 + min\{2, v_3\} + \cdots + min\{2, v_n\}$;
(c) $v_1 + v_2 + v_3 \leq 6 + min\{3, v_4\} + \cdots + min\{3, v_n\}$; and, for any $1 \leq p \leq n$,
(d) $v_1 + \cdots + v_p \leq p(p-1) + min\{p, v_{p+1}\} + \cdots + min\{p, v_n\}$.

Conversely, given a sequence $v_1 \geq v_2 \geq \ldots \geq v_n > 0$ that satisfies all the inequalities above, and for which the sum $v_1 + v_2 + \cdots + v_n$ is even, there is at least one n-vertex graph whose valences are $v_1, v_2, \ldots, v_n$. The *Havel-Hakimi algorithm* builds such a graph in $n-1$ stages: The first stage joins the first vertex to the v_1 vertices that follow it in the list. This leaves at most $n-1$ vertices to be processed. Succeeding stages apply the same plan to the revised valence list (re-ordered if necessary).

2. The four graphs shown below all share the vertex list 9, 8, 6, 5, 5, 5, 4, 4, 3, 3, 3, 1.
(a) Show that the Havel-Hakimi algorithm produces only one of the four graphs.
(b) Show that no two of the graphs are isomorphic.



3. Show that there is no graph whose valences are 10, 9, 7, 7, 6, 6, 5, 3, 2, 1, 1, and 1.

Edge Colorings

Two edges of a graph are called *adjacent* if they share a vertex. A graph can be *edge-colored with n colors* if each of its edges can be assigned one of the n colors in such a way that adjacent edges are not assigned the same color. The smallest value of n for which the edges a graph can be colored with n colors is called the *chromatic index* of the graph (not to be confused with the *chromatic number*).

1. The edge-skeleton of a cube is a connected, trivalent, eight-vertex graph. What is its chromatic index?

2. By finding a couple of examples, show that it is possible for the chromatic number of a graph to be smaller than the chromatic index. Is it possible for the chromatic index to be smaller than the chromatic number?

3. What does edge-coloring a graph have to do with scheduling a tournament?

4. If a graph has a circuit of odd length, then the chromatic index of the graph is at least three. Prove or disprove this statement.

5. Let d be the largest valence that occurs in graph $\mathcal{G}$. Then the chromatic index of $\mathcal{G}$ must be at least d. Explain.

6. Consider a complete graph $\mathcal{G}$ that has n vertices, where n is an *odd* number greater than 1. Show that the chromatic index of $\mathcal{G}$ is n. (*Hint*: Visualize $\mathcal{G}$ as a regular polygon together with all its diagonals; a coloring algorithm should present itself.)

7. Consider a complete graph $\mathcal{G}$ that has n vertices, where n is an *even* number greater than 1. Show that the chromatic index of $\mathcal{G}$ is $n-1$. (*Hint*: Apply your edge-coloring knowledge for complete graphs with an odd number of vertices.)

8. *Vizing's Theorem*. Let d be the largest valence that occurs in graph $\mathcal{G}$. Then the chromatic index of $\mathcal{G}$ is either d or $d+1$.

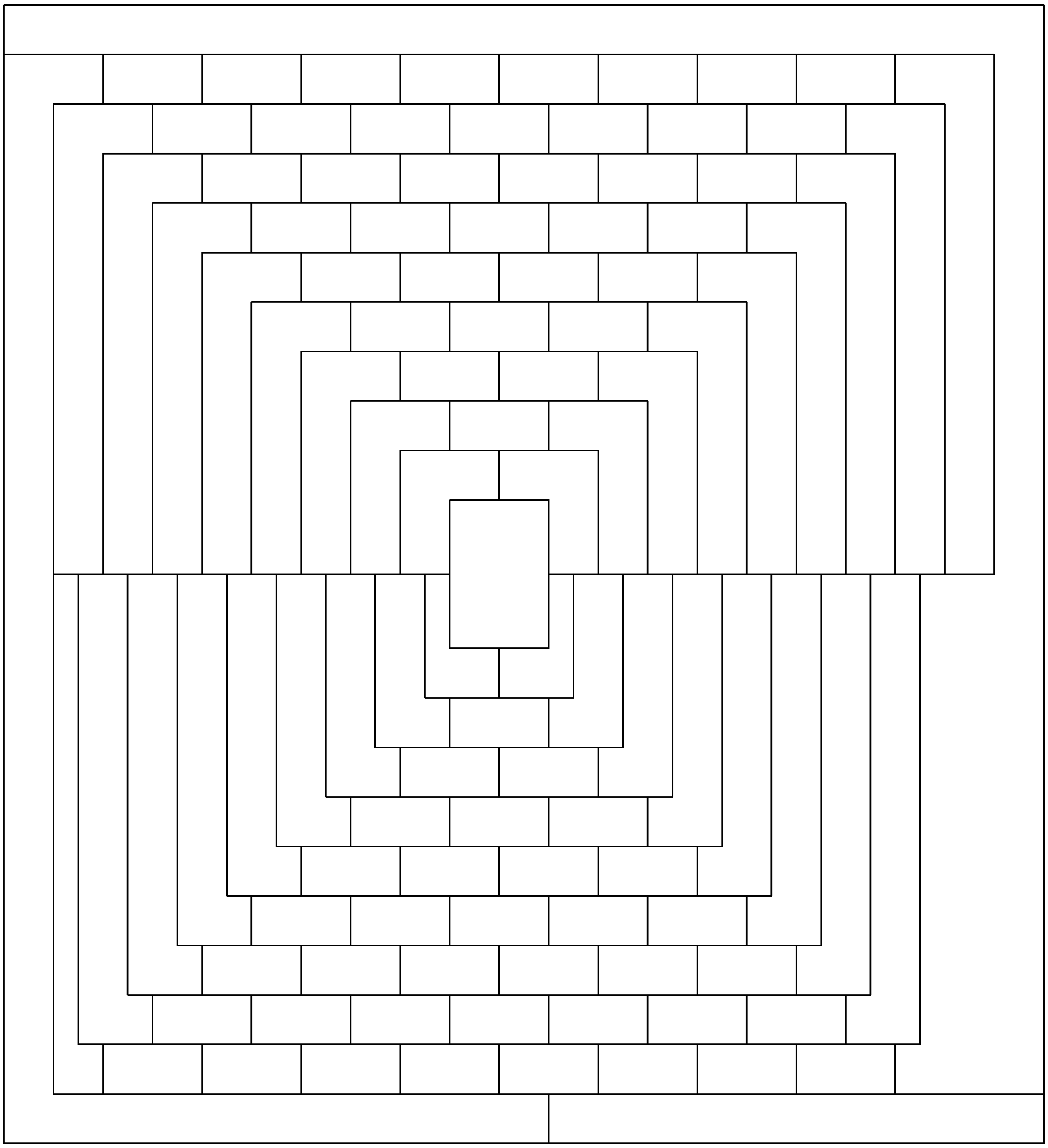
Explain how the conclusion of this theorem is supported by the three preceding problems.

9. According to Vizing's Theorem, there are two types of graphs. Let d be the largest valence in graph $\mathcal{G}$. If the chromatic index of $\mathcal{G}$ is d, then $\mathcal{G}$ is a graph of *class 1*. If the chromatic index of $\mathcal{G}$ is $d+1$, then $\mathcal{G}$ is a graph of *class 2*. It is not easy to determine the class of an arbitrary graph, but special methods can be devised for special graphs (as you have done for complete graphs). Show that any tree belongs to class 1.

10. (Continuation) Another special example is a *complete* bipartite graph: Suppose that the vertices of $\mathcal{G}$ are $A_1, \ldots, A_m$, and $B_1, \ldots, B_n$, where $m \le n$, and that every vertex A_i is adjacent to every vertex B_j. Show that $\mathcal{G}$ belongs to class 1, by showing that the chromatic index of $\mathcal{G}$ is n.

Discrete Mathematics

1. In April 1975, Martin Gardner published this 110-country map in *Scientific American*, asserting that it could not be colored with fewer than five colors. What do you think?



Matchings

Let $\mathcal{G}$ be a bipartite graph, which means that its vertices are partitioned into two sets, $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, so that every edge of $\mathcal{G}$ joins a vertex in A to a vertex in B. A typical application: the vertices of A represent workers, the vertices of B represent tasks, and the edges represent capability. A *complete matching* from A to B is a collection of m edges of $\mathcal{G}$, no two having a vertex in common. It is evident that this is possible only if $m \le n$. In the application, this matching represents the assignment of a suitable task to each worker, with no two workers being assigned the same task. Two edges that do not have a vertex in common are called *disjoint*, or *independent*.

1. In the proposed application, what would a complete matching from B to A mean?

2. Invent an application of bipartite graphs, and interpret the matching concept.

3. For any subset S of A, its set $N(S)$ of *neighbors* consists of those vertices in B that are adjacent to vertices in S. In order for there to exist a complete matching from A to B, it is necessary that the size $|N(S)|$ of every set of neighbors satisfy $|S| \le |N(S)|$. Explain.

4. Let $A = \{a, b, c, d\}$ and $B = \{w, x, y, z\}$. A bipartite graph is defined by the set of ten edges $\{aw, ax, ay, az, bw, bx, cw, cx, dw, dx\}$. Explain why there is no complete matching from A to B for this graph.

5. Suppose that every vertex of $\mathcal{G}$ has the same valence d (such a graph is called *regular*) and that $\mathcal{G}$ is bipartite, with vertex sets A and B. Explain why A and B must have the same number of vertices. Find an example that shows that $\mathcal{G}$ need not be connected.

6. Given a regular bipartite graph $\mathcal{G}$, and any subset S of vertices taken from A, justify the neighbor-set inequality $|S| \le |N(S)|$.

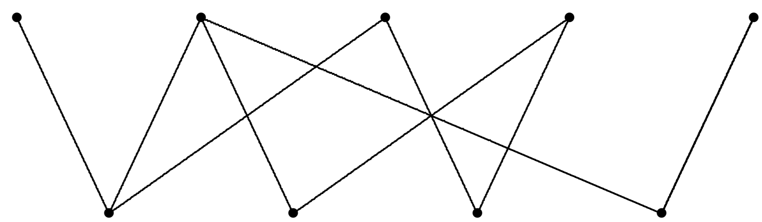
7. (Continuation) It is plausible that $\mathcal{G}$ has a complete matching from $\mathcal{A}$ to $\mathcal{B}$. In proving that this is so, it can be assumed that $1 < d$. Explain why.

8. (Continuation) Let $\mathcal{M}$ be an *incomplete matching*, consisting of fewer than n independent edges of $\mathcal{G}$, where $|A| = n = |B|$. It suffices to show that the size of $\mathcal{M}$ can be increased. Let a be an *uncovered* vertex of A, meaning that none of the d edges at a belong to $\mathcal{M}$. If one of the corresponding d endpoints in B is also uncovered, then it is clear how to increase the size of $\mathcal{M}$ — just add that edge. Otherwise, every edge at a is adjacent to an edge in $\mathcal{M}$, which in turn is adjacent to $d-1$ edges that are not in $\mathcal{M}$. This illustrates how *alternating paths* in $\mathcal{G}$ are built, whereby edges not in $\mathcal{M}$ lead from A to B, and edges of $\mathcal{M}$ lead from B to A. Show that $\mathcal{M}$ can be enlarged whenever an alternating path starting at a terminates (after an odd number of edges) at an uncovered vertex of B.

9. (Continuation) Let A_a (resp. B_a) consists of all those vertices in A (resp. B) that can be reached from a by an alternating path. Put a in A_a. Explain why $B_a = N(A_a)$. Expecting a contradiction, assume that $\mathcal{M}$ cannot be enlarged (which implies that no member of B_a is uncovered). Show that $|N(A_a)| = |A_a| - 1$. Explain why this is a contradiction, and why it proves that regular bipartite graphs have complete matchings.

1. Show that the chromatic index of a d-regular bipartite graph $\mathcal{G}$ is d. In other words, show that the edges of $\mathcal{G}$ can be colored with d colors. This application of matchings establishes another special case of Vizing's Theorem.

2. The diagram shows a non-regular bipartite graph in which the largest valence is 3. Show that this graph can be enlarged (by adding vertices and edges) to form a 3-regular (cubic) bipartite graph, in such a way that no new edge joins two of the given nine vertices.



3. Let $\mathcal{G}$ be a bipartite graph, with vertex sets A and B, whose largest valence is d. If $\mathcal{G}$ is not regular, then it is possible to build a bipartite graph $\mathcal{G}'$ that is d-regular, and that has $\mathcal{G}$ as a subgraph. One way to build such a $\mathcal{G}'$ is to add a dummy vertex for each edge that $\mathcal{G}$ is "missing." For example, if b is a vertex in B whose valence is some number v smaller than d, then put $d-v$ vertices in A' (which already has the vertices of A), and add to $\mathcal{G}'$ the corresponding $d-v$ edges from b. This process continues until the vertices of A and B all have valence d. The extra vertices created to form sets A' and B' will of course all have valence 1. At this time, it will probably also be necessary to add some dummy vertices of valence 0, at least to make the sizes of A' and B' agree (as regularity requires). If the number of zero-valence vertices is carefully chosen, it is possible to draw additional edges so that all valences of $\mathcal{G}'$ are d. Provide the remaining details of this argument.

The approach outlined here actually makes $\mathcal{G}$ an *induced* subgraph of $\mathcal{G}'$. This means that A' and B' are enlargements of A and B, respectively, and that every edge of $\mathcal{G}'$ that joins a vertex in A to a vertex in B is already an edge of $\mathcal{G}$.

4. Let d be the largest valence that occurs in an arbitrary bipartite graph $\mathcal{G}$. Show that the edges of $\mathcal{G}$ can be colored with d colors, thus proving that every bipartite graph is of class 1. This establishes another special case of Vizing's Theorem.

Discrete Math Reference

accidental crossing: when two edges of a graph intersect at a non-vertex — an accident of the drawing. [25,26,55]

Adams method: a *divisor method* for solving the apportionment problem; favors small states. [3,5]

adjacent: vertices of a graph that are joined by an edge, or edges of a graph that intersect at a vertex. [6,66].

adjusted quota: the number you get when you divide a state's population by an arbitrary divisor. [3]

Alabama paradox: an anomaly produced by the *Hamilton method* when the House size is changed. [5,6]

algorithm: a prescription for carrying out some task mechanically; it need not produce an optimal result. [1,2,6,7,23,29,31]

alkane: in chemistry, an acyclic, saturated hydrocarbon molecule, which can be described by a *tree* in which the valence of each vertex is either 1 (hydrogen) or 4 (carbon). [55]

apportionment: a discrete fair-division problem; the best-known example being to fairly assign each state some of the 435 seats in the House of Representatives. [2,3,4,21]

around-the-world: a puzzle marketed by Hamilton in 1859, this was the first appearance of the Hamiltonian-circuit concept. [37]

Banzhaf index: measures the power of a voter (or voting bloc), by counting pivotal ballots. [38,39,46]

bipartite: describes a graph whose vertices can be separated into two groups so that adjacent vertices never belong to the same group. This is is equivalent to saying that the chromatic number of the graph is 2. [28,37,48,51,58]

Borda count: see the *point-count method.*

Brooks' Theorem: the chromatic number of a graph is at most equal to its largest valence, unless the graph is complete or an odd circuit. [59]

bridge: an edge that — if it were removed — would disconnect an otherwise connected graph. [42,49]

census: a decennial event, mandated by the Constitution for the purpose of apportionment. [1,2]

chromatic number: how many colors are needed to color the vertices of a graph. [6]

chromatic index: how many colors are needed to color the edges of a graph. [66]

circuit: a path that starts and finishes at the same vertex. [16,22,26,29,30]

color a graph: assign colors (or numbers) to the vertices of a graph, so that adjacent vertices receive different colors. [3,6]

complete graph: a graph in which every pair of vertices is joined by an edge. [6,16,25]

Condorcet method: a *round-robin* method of deciding elections that involve more than two candidates. [10,11,12,13]

Condorcet paradox: when all the candidates can be arranged in a cyclic fashion, each one preferred to the next one in the list. [14,20]

Condorcet cycle: see *Condorcet paradox.*

Condorcet winner: a candidate who is preferred to all other candidates; a source in the tournament digraph. [10,11,13,14]

connected graph: a graph in which each pair of vertices can be joined by a chain of adjacent edges. [6,16]

connected digraph: a directed graph in which, given any starting vertex and any destination vertex, a (directed) path can be found that leads from the starting vertex to the destination vertex. [28]

cost: a numerical label applied to an edge of a graph. [31,35,36]

cubic: when referring to graphs, this is synonymous with *trivalent.* [35]

cycle: synonymous with *circuit.* [14,20,31]

Dean method: a *divisor method* for solving the apportionment problem. [8,12,17]

dictator: in a voting-block situation, a voter who has all the power.[38]

digraph: an abbreviation of "directed graph"; this is a graph in which each edge is assigned a direction. [9,13,14]

Dijkstra's algorithm: given a starting vertex in a weighted, connected graph, this method produces a spanning tree that includes minimum-weight paths from the starting vertex to every other vertex in the graph. [48,49,53]

direct a graph: apply a direction to each of the edges of a graph, converting it into a digraph. [20,28,29]

directed graph: see *digraph.*

disconnected: a graph that is not connected.

district size: the size of the Congressional districts within a given state; obtained by dividing a state's population by its assigned number of representatives. [2]

divisor method: any method of apportionment that applies a rounding rule to *adjusted quotas*, obtained by dividing a fixed but arbitrary number into state populations. [3]

dummy: in a voting-block situation, a voter who has no power. [38]

edge-skeleton: given a polyhedron, this is the graph that results when the faces are removed. [22,27,32]

Electoral College: where the President of the United States is chosen; a state's electors typically vote as a bloc. [38,41,46]

Eulerian circuit: a circuit in which each edge of a graph appears exactly once. [33,36,42]

Eulerian graph: a graph that has an Eulerian circuit. [33,43]

Eulerian path: a path in which each edge of a graph appears exactly once. [33,36,42]

Eulerize: to make a graph Eulerian by adding extra edges. [48]

forest: a disconnected graph that has edges but no circuits. [32,45,61,63]

Four-Color Theorem: the *chromatic number* of a planar graph is at most four. [28]

geometric mean: the geometric mean of two positive numbers a and b is $\sqrt{ab}$. [7,9,18]

graph: a network of dots and lines [1,6]

greedy: an algorithm that looks only one move ahead. [31,42,43]

grid: a graph whose vertices represent the intersections of a rectangular system of streets and avenues. [30]

Hamilton method: an algorithm for solving the apportionment problem; rounds up those ideal quotas with the largest fractional parts. [2,3,5]

Hamiltonian circuit: a circuit in which each vertex of a graph appears exactly once. [29-32]

Hamiltonian graph: a graph that has a Hamiltonian circuit. [30]

Hamiltonian path: a path in which each vertex of a graph appears exactly once. [34]

Handshake Theorem: the total of all the valences of a graph must be even; for a digraph, the sum of the invalences must equal the sum of the outvalences. [27,29]

harmonic mean: the harmonic mean of two positive numbers a and b is $2ab/(a+b)$. [12]

Havel-Hakimi algorithm: a method that determines whether a given list of valences corresponds to an actual graph. [65]

Huntington-Hill method: a *divisor method* for solving the apportionment problem; in current use. [7,8,9,11,18]

ideal district size: the total population divided by the House size. [2]

ideal quota: a state's fair share of the size of the House. [2]

invalence: counts the number of edges attached to and directed toward a vertex of a digraph. [9,29,53]

isomorphism: an equivalence between two graphs, which consists of matching their vertices in such a way that adjacent vertices in one graph correspond to adjacent vertices in the other graph. [25,26,28]

Jefferson method: a *divisor method* for solving the apportionment problem; favors large states. [3,5]

knight's tour: a chessboard recreation that is one of the best-known of all Hamiltonian problems. [54]

Königsberg bridges: there were seven of them, and they became the source of a famous Eulerian puzzle. [41]

Kruskal's algorithm: a sorted-edges method that produces a minimal-cost spanning tree in any (complete) labeled graph. [35]

lower adjusted quota: the largest integer that does not exceed the adjusted quota. [9]

lower quota: the largest integer that does not exceed the ideal quota. [2]

nearest-neighbor algorithm: a strategy for solving the *Traveling-Salesman Problem*; it occasionally produces optimal results. [31]

outvalence: counts the number of edges attached to and directed away from a vertex of a digraph. [9,29,53]

path: in a graph, a sequence of edges, with the property that successive edges have exactly one vertex in common; in a digraph, the directions of successive edges also have to agree. [6,27]

pivotal: a situation where a voter's choice determines the outcome of an election. [38,39]

planar: a graph that is *isomorphic* to a graph that can be drawn in the plane without accidental crossings. [25-28,46]

plurality method: awards an election to the candidate who has the most first-place votes. [10]

point-count method: each candidate receives points from each voter based on that voter's preferences; also called the Borda method. [10,12,52]

Prim's algorithm: a multiple-pass method that produces a minimal-cost spanning tree in any (complete) labeled graph; the tree is kept in one connected piece throughout the process. [44,55]

quota: depending on the context, this can refer to the *ideal quota* or an *adjusted quota*; in the former case, it is the fair share.

regular graph: a graph in which each vertex has the same valence d is called d-regular; for example, a *trivalent* graph is 3-regular. [68]

representational deficiency: the number $r_1(p_2/p_1) - r_2$, where p_i and r_i are the population and the appportionment for State i, respectively, assuming that $r_2/p_2 < r_1/p_1$ (in other words, State 2 has a deficit because it is disadvantaged relative to State 1). [9]

representational surplus: the number $r_2 - (p_2/p_1)r_1$, where p_i and r_i are the population and the appportionment for State i, respectively, assuming that $r_1/p_1 < r_2/p_2$ (in other words, State 2 has a surplus because it is advantaged relative to State 1). [9]

respect quota: an apportionment method does this if it always awards each state either its lower quota or its upper quota, regardless of the census. [15]

round-robin: a tournament in which each competitor faces every other competitor. [22,23,24]

sink: in a digraph, a vertex whose outvalence is 0. [53]

sorted-edges algorithm: a strategy for solving the *Traveling-Salesman Problem*; it occasionally produces optimal results. [31]

source: in a digraph, a vertex whose invalence is 0. [53]

spanning tree: given a graph, this is a tree that includes every vertex of the graph. [34]

Sprouts: a pencil-and-paper game that creates networks. [30]

Steiner tree: given a set of vertices in the plane, this is a tree that spans the points and that has minimal total length. [37]

tournament: a complete directed graph. [10,17,20,52]

Traveling Salesman Problem: to find the minimal-cost Hamiltonian circuit in a complete labeled graph. [31]

tree: a connected graph that has no circuits. [16,26,27]

trivalent: a graph in which every the valence of every vertex is 3. [13,27]

upper adjusted quota: the smallest integer that is not less than the *adjusted quota*. [9]

upper quota: the smallest integer that is not less than the ideal quota. [2,11]

valence: the number of edges attached to a vertex of a graph; see also *invalence* and *outvalence*. [9,11]

violate quota: this happens when an apportionment awards a state more than its upper quota or fewer than its lower quota. [21]

Vizing's Theorem: the *chromatic index* of a graph is either d or $d + 1$, where d is the largest valence that occurs in the graph. [66,69]

Webster method: a *divisor method* for solving the apportionment problem. [3,5,7]

weight: like a *cost*, a numerical label applied to an edge of a graph [33,35,36]; the size of a voting bloc [38,39,41]; number used to evaluate a candidate's position on a preferential ballot. [17]

Welsh-Powell algorithm: a method for coloring graphs. [23,29]

word chain: a puzzle (invented by Lewis Carroll) that is equivalent to looking for paths in a large graph. [30,32]

Discrete Math Reference

Mock 1984 census data, apportionment by the Huntington-Hill method (divisor 529300):

State	**1984 Pop**	**Quota**	**Rep**	**District**	**Adj Q**
Alabama	3608877	7.099	7	515554	6.818
Alaska	329928	0.649	1	329928	0.623
Arizona	1885014	3.708	4	471253	3.561
Arkansas	1948052	3.832	4	487013	3.680
California	21944556	43.167	41	535233	41.460
Colorado	2410663	4.742	5	482133	4.554
Connecticut	3438575	6.764	7	491225	6.496
Delaware	802199	1.578	2	401099	1.516
Florida	7671182	15.090	15	511412	14.493
Georgia	5053140	9.940	10	505314	9.547
Hawaii	840834	1.654	2	420417	1.589
Idaho	804232	1.582	2	402116	1.519
Illinois	12290721	24.177	23	534379	23.221
Indiana	5570655	10.958	11	506423	10.525
Iowa	2958171	5.819	6	493028	5.589
Kansas	2456416	4.832	5	491283	4.641
Kentucky	3465010	6.816	7	495001	6.546
Louisiana	3968799	7.807	8	496100	7.498
Maine	1298870	2.555	3	432957	2.454
Maryland	3978966	7.827	8	497371	7.517
Massachusetts	6086136	11.972	12	507178	11.498
Michigan	9489634	18.667	18	527202	17.929
Minnesota	4003368	7.875	8	500421	7.564
Mississippi	2421339	4.763	5	484268	4.575
Missouri	5108552	10.049	10	510855	9.652
Montana	773730	1.522	2	386865	1.462
Nebraska	1834178	3.608	4	458544	3.465
Nevada	534291	1.051	1	534291	1.009
New Hampshire	842868	1.658	2	421434	1.592
New Jersey	7676299	15.100	15	511753	14.503
New Mexico	1313105	2.583	3	437702	2.481
New York	19842029	39.031	37	536271	37.487
North Carolina	5552320	10.922	11	504756	10.490
North Dakota	755938	1.487	2	377969	1.428
Ohio	11735587	23.085	22	533436	22.172
Oklahoma	2901743	5.708	6	483624	5.482
Oregon	2385753	4.693	5	477151	4.507
Pennsylvania	12779259	25.138	24	532469	24.144
Rhode Island	1316663	2.590	3	438888	2.488
South Carolina	2905301	5.715	6	484217	5.489
South Dakota	752887	1.481	2	376443	1.422
Tennessee	4031836	7.931	8	503979	7.617
Texas	12228700	24.055	23	531683	23.104
Utah	1360383	2.676	3	453461	2.570
Vermont	485488	0.955	1	485488	0.917
Virginia	5026197	9.887	10	502620	9.496
Washington	3519403	6.923	7	502772	6.649
West Virginia	1854004	3.647	4	463501	3.503
Wisconsin	4519866	8.891	9	502207	8.539
Wyoming	376698	0.741	1	376698	0.712
	221138415	**435**	**435**	**508364**	

The 2010 Census data as it appears in *Windisc*, with the Hamilton method selected:

State	2010 Pop	Quota	Rep	District
Alabama	4802982	6.757	7	686140
Alaska	721523	1.015	1	721523
Arizona	6412700	9.022	9	712522
Arkansas	2926229	4.117	4	731557
California	37341989	52.538	53	704566
Colorado	5044930	7.098	7	720704
Connecticut	3581628	5.039	5	716326
Delaware	900877	1.267	1	900877
Florida	18900773	26.592	27	700029
Georgia	9727566	13.686	14	694826
Hawaii	1366862	1.923	2	683431
Idaho	1573499	2.214	2	786749
Illinois	12864380	18.099	18	714688
Indiana	6501582	9.147	9	722398
Iowa	3053787	4.296	4	763447
Kansas	2863813	4.029	4	715953
Kentucky	4350606	6.121	6	725101
Louisiana	4553962	6.407	6	758994
Maine	1333074	1.876	2	666537
Maryland	5789929	8.146	8	723741
Massachusetts	6559644	9.229	9	728849
Michigan	9911626	13.945	14	707973
Minnesota	5314879	7.478	8	664360
Mississippi	2978240	4.190	4	744560
Missouri	6011478	8.458	8	751435
Montana	994416	1.399	1	994416
Nebraska	1831825	2.577	3	610608
Nevada	2709432	3.812	4	677358
New Hampshire	1321445	1.859	2	660722
New Jersey	8807501	12.392	12	733958
New Mexico	2067273	2.909	3	689091
New York	19421055	27.324	27	719298
North Carolina	9565781	13.458	13	735829
North Dakota	675905	0.951	1	675905
Ohio	11568495	16.276	16	723031
Oklahoma	3764882	5.297	5	752976
Oregon	3848606	5.415	5	769721
Pennsylvania	12734905	17.917	18	707495
Rhode Island	1055247	1.485	2	527623
South Carolina	4645975	6.537	7	663711
South Dakota	819761	1.153	1	819761
Tennessee	6375431	8.970	9	708381
Texas	25268418	35.551	36	701900
Utah	2770765	3.898	4	692691
Vermont	630337	0.887	1	630337
Virginia	8037736	11.309	11	730703
Washington	6753369	9.502	10	675337
West Virginia	1859815	2.617	3	619938
Wisconsin	5698230	8.017	8	712279
Wyoming	568300	0.800	1	568300
	309183463	**435**	**435**	**710767**

The 2010 census data, apportionment by the Huntington-Hill method (divisor 710000):

State	2010 Pop	Quota	Rep	District	Adj Q
Alabama	4802982	6.757	7	686140	6.763
Alaska	721523	1.015	1	721523	1.016
Arizona	6412700	9.022	9	712522	9.030
Arkansas	2926229	4.117	4	731557	4.121
California	37341989	52.538	53	704566	52.583
Colorado	5044930	7.098	7	720704	7.104
Connecticut	3581628	5.039	5	716326	5.043
Delaware	900877	1.267	1	900877	1.269
Florida	18900773	26.592	27	700029	26.615
Georgia	9727566	13.686	14	694826	13.698
Hawaii	1366862	1.923	2	683431	1.925
Idaho	1573499	2.214	2	786749	2.216
Illinois	12864380	18.099	18	714688	18.115
Indiana	6501582	9.147	9	722398	9.155
Iowa	3053787	4.296	4	763447	4.300
Kansas	2863813	4.029	4	715953	4.033
Kentucky	4350606	6.121	6	725101	6.126
Louisiana	4553962	6.407	6	758994	6.413
Maine	1333074	1.876	2	666537	1.877
Maryland	5789929	8.146	8	723741	8.153
Massachusetts	6559644	9.229	9	728849	9.237
Michigan	9911626	13.945	14	707973	13.957
Minnesota	5314879	7.478	8	664360	7.484
Mississippi	2978240	4.190	4	744560	4.194
Missouri	6011478	8.458	8	751435	8.465
Montana	994416	1.399	1	994416	1.400
Nebraska	1831825	2.577	3	610608	2.579
Nevada	2709432	3.812	4	677358	3.815
New Hampshire	1321445	1.859	2	660722	1.861
New Jersey	8807501	12.392	12	733958	12.402
New Mexico	2067273	2.909	3	689091	2.911
New York	19421055	27.324	27	719298	27.348
North Carolina	9565781	13.458	13	735829	13.470
North Dakota	675905	0.951	1	675905	0.952
Ohio	11568495	16.276	16	723031	16.290
Oklahoma	3764882	5.297	5	752976	5.301
Oregon	3848606	5.415	5	769721	5.419
Pennsylvania	12734905	17.917	18	707495	17.933
Rhode Island	1055247	1.485	2	527623	1.486
South Carolina	4645975	6.537	7	663711	6.542
South Dakota	819761	1.153	1	819761	1.154
Tennessee	6375431	8.970	9	708381	8.978
Texas	25268418	35.551	36	701900	35.582
Utah	2770765	3.898	4	692691	3.902
Vermont	630337	0.887	1	630337	0.888
Virginia	8037736	11.309	11	730703	11.318
Washington	6753369	9.502	10	675337	9.510
West Virginia	1859815	2.617	3	619938	2.619
Wisconsin	5698230	8.017	8	712279	8.024
Wyoming	568300	0.800	1	568300	0.800
	309183463	**435**	**435**	**710767**	

The 2000 Census data as it appears in *Windisc*, with the Hamilton method selected:

State	2000 Pop	Quota	Rep	District
Alabama	4461130	6.896	7	637304
Alaska	628933	0.972	1	628933
Arizona	5140683	7.946	8	642585
Arkansas	2679733	4.142	4	669933
California	33930798	52.447	52	652515
Colorado	4311882	6.665	7	615983
Connecticut	3409535	5.270	5	681907
Delaware	785068	1.213	1	785068
Florida	16028890	24.776	25	641156
Georgia	8206975	12.686	13	631306
Hawaii	1216642	1.881	2	608321
Idaho	1297274	2.005	2	648637
Illinois	12439042	19.227	19	654686
Indiana	6090782	9.415	9	676754
Iowa	2931923	4.532	5	586385
Kansas	2693824	4.164	4	673456
Kentucky	4049431	6.259	6	674905
Louisiana	4480271	6.925	7	640039
Maine	1277731	1.975	2	638865
Maryland	5307886	8.204	8	663486
Massachusetts	6355568	9.824	10	635557
Michigan	9955829	15.389	15	663722
Minnesota	4925670	7.614	8	615709
Mississippi	2852927	4.410	4	713232
Missouri	5606260	8.666	9	622918
Montana	905316	1.399	1	905316
Nebraska	1715369	2.651	3	571790
Nevada	2002032	3.095	3	667344
New Hampshire	1238415	1.914	2	619207
New Jersey	8424354	13.022	13	648027
New Mexico	1823821	2.819	3	607940
New York	19004973	29.376	29	655344
North Carolina	8067673	12.470	13	620590
North Dakota	643756	0.995	1	643756
Ohio	11374540	17.582	18	631919
Oklahoma	3458819	5.346	5	691764
Oregon	3428543	5.300	5	685709
Pennsylvania	12300670	19.013	19	647404
Rhode Island	1049662	1.622	2	524831
South Carolina	4025061	6.222	6	670843
South Dakota	756874	1.170	1	756874
Tennessee	5700037	8.811	9	633337
Texas	20903994	32.312	32	653250
Utah	2236714	3.457	4	559178
Vermont	609890	0.943	1	609890
Virginia	7100702	10.976	11	645518
Washington	5908684	9.133	9	656520
West Virginia	1813077	2.802	3	604359
Wisconsin	5371210	8.302	8	671401
Wyoming	495304	0.766	1	495304
	281424177	**435**	**435**	**646952**

Discrete Math Reference

The 2000 census data, apportionment by the Huntington-Hill method (divisor 645800):

State	2000 Pop	Quota	Rep	District	Adj Q
Alabama	4461130	6.896	7	637304	6.908
Alaska	628933	0.972	1	628933	0.974
Arizona	5140683	7.946	8	642585	7.960
Arkansas	2679733	4.142	4	669933	4.149
California	33930798	52.447	53	640204	52.541
Colorado	4311882	6.665	7	615983	6.677
Connecticut	3409535	5.270	5	681907	5.280
Delaware	785068	1.213	1	785068	1.216
Florida	16028890	24.776	25	641156	24.820
Georgia	8206975	12.686	13	631306	12.708
Hawaii	1216642	1.881	2	608321	1.884
Idaho	1297274	2.005	2	648637	2.009
Illinois	12439042	19.227	19	654686	19.261
Indiana	6090782	9.415	9	676754	9.431
Iowa	2931923	4.532	5	586385	4.540
Kansas	2693824	4.164	4	673456	4.171
Kentucky	4049431	6.259	6	674905	6.270
Louisiana	4480271	6.925	7	640039	6.938
Maine	1277731	1.975	2	638865	1.979
Maryland	5307886	8.204	8	663486	8.219
Massachusetts	6355568	9.824	10	635557	9.841
Michigan	9955829	15.389	15	663722	15.416
Minnesota	4925670	7.614	8	615709	7.627
Mississippi	2852927	4.410	4	713232	4.418
Missouri	5606260	8.666	9	622918	8.681
Montana	905316	1.399	1	905316	1.402
Nebraska	1715369	2.651	3	571790	2.656
Nevada	2002032	3.095	3	667344	3.100
New Hampshire	1238415	1.914	2	619207	1.918
New Jersey	8424354	13.022	13	648027	13.045
New Mexico	1823821	2.819	3	607940	2.824
New York	19004973	29.376	29	655344	29.429
North Carolina	8067673	12.470	13	620590	12.493
North Dakota	643756	0.995	1	643756	0.997
Ohio	11374540	17.582	18	631919	17.613
Oklahoma	3458819	5.346	5	691764	5.356
Oregon	3428543	5.300	5	685709	5.309
Pennsylvania	12300670	19.013	19	647404	19.047
Rhode Island	1049662	1.622	2	524831	1.625
South Carolina	4025061	6.222	6	670843	6.233
South Dakota	756874	1.170	1	756874	1.172
Tennessee	5700037	8.811	9	633337	8.826
Texas	20903994	32.312	32	653250	32.369
Utah	2236714	3.457	3	745571	3.463
Vermont	609890	0.943	1	609890	0.944
Virginia	7100702	10.976	11	645518	10.995
Washington	5908684	9.133	9	656520	9.149
West Virginia	1813077	2.802	3	604359	2.807
Wisconsin	5371210	8.302	8	671401	8.317
Wyoming	495304	0.766	1	495304	0.767
	281424177	**435**	**435**	**646952**	

Discrete Math Reference

The 1990 Census data as it appears in *Windisc*, with the Hamilton method selected:

State	1990 Pop	Quota	Rep	District
Alabama	4062608	7.097	7	580373
Alaska	551947	0.964	1	551947
Arizona	3677985	6.425	6	612997
Arkansas	2362239	4.126	4	590560
California	29839250	52.124	52	573832
Colorado	3307912	5.778	6	551319
Connecticut	3295669	5.757	6	549278
Delaware	668696	1.168	1	668696
Florida	13003362	22.715	23	565364
Georgia	6508419	11.369	11	591674
Hawaii	1115274	1.948	2	557637
Idaho	1011986	1.768	2	505993
Illinois	11466682	20.030	20	573334
Indiana	5564228	9.720	10	556423
Iowa	2787424	4.869	5	557485
Kansas	2485600	4.342	4	621400
Kentucky	3698969	6.461	6	616495
Louisiana	4238216	7.403	7	605459
Maine	1233223	2.154	2	616611
Maryland	4798622	8.382	8	599828
Massachusetts	6029051	10.532	11	548096
Michigan	9328784	16.296	16	583049
Minnesota	4387029	7.663	8	548379
Mississippi	2586443	4.518	4	646611
Missouri	5137804	8.975	9	570867
Montana	803655	1.404	1	803655
Nebraska	1584617	2.768	3	528206
Nevada	1206152	2.107	2	603076
New Hampshire	1113915	1.946	2	556957
New Jersey	7748634	13.536	14	553474
New Mexico	1521779	2.658	3	507260
New York	18044505	31.521	31	582081
North Carolina	6657630	11.630	12	554802
North Dakota	641364	1.120	1	641364
Ohio	10887325	19.018	19	573017
Oklahoma	3157604	5.516	5	631521
Oregon	2853733	4.985	5	570747
Pennsylvania	11924710	20.830	21	567843
Rhode Island	1005984	1.757	2	502992
South Carolina	3505707	6.124	6	584284
South Dakota	699999	1.223	1	699999
Tennessee	4896641	8.554	9	544071
Texas	17059805	29.801	30	568660
Utah	1727784	3.018	3	575928
Vermont	564964	0.987	1	564964
Virginia	6216568	10.859	11	565143
Washington	4887941	8.538	9	543105
West Virginia	1801625	3.147	3	600542
Wisconsin	4906745	8.571	9	545194
Wyoming	455975	0.797	1	455975
	249022783	**435**	**435**	**572466**

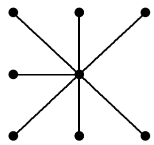
Discrete Math Reference

The 1990 census data, apportionment by the Huntington-Hill method (divisor 575000):

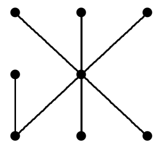
State	1990 Pop	Quota	Rep	District	Adj Q
Alabama	4062608	7.097	7	580373	7.065
Alaska	551947	0.964	1	551947	0.960
Arizona	3677985	6.425	6	612997	6.396
Arkansas	2362239	4.126	4	590560	4.108
California	29839250	52.124	52	573832	51.894
Colorado	3307912	5.778	6	551319	5.753
Connecticut	3295669	5.757	6	549278	5.732
Delaware	668696	1.168	1	668696	1.163
Florida	13003362	22.715	23	565364	22.615
Georgia	6508419	11.369	11	591674	11.319
Hawaii	1115274	1.948	2	557637	1.940
Idaho	1011986	1.768	2	505993	1.760
Illinois	11466682	20.030	20	573334	19.942
Indiana	5564228	9.720	10	556423	9.677
Iowa	2787424	4.869	5	557485	4.848
Kansas	2485600	4.342	4	621400	4.323
Kentucky	3698969	6.461	6	616495	6.433
Louisiana	4238216	7.403	7	605459	7.371
Maine	1233223	2.154	2	616611	2.145
Maryland	4798622	8.382	8	599828	8.345
Massachusetts	6029051	10.532	10	602905	10.485
Michigan	9328784	16.296	16	583049	16.224
Minnesota	4387029	7.663	8	548379	7.630
Mississippi	2586443	4.518	5	517289	4.498
Missouri	5137804	8.975	9	570867	8.935
Montana	803655	1.404	1	803655	1.398
Nebraska	1584617	2.768	3	528206	2.756
Nevada	1206152	2.107	2	603076	2.098
New Hampshire	1113915	1.946	2	556957	1.937
New Jersey	7748634	13.536	13	596049	13.476
New Mexico	1521779	2.658	3	507260	2.647
New York	18044505	31.521	31	582081	31.382
North Carolina	6657630	11.630	12	554802	11.578
North Dakota	641364	1.120	1	641364	1.115
Ohio	10887325	19.018	19	573017	18.934
Oklahoma	3157604	5.516	6	526267	5.491
Oregon	2853733	4.985	5	570747	4.963
Pennsylvania	11924710	20.830	21	567843	20.739
Rhode Island	1005984	1.757	2	502992	1.750
South Carolina	3505707	6.124	6	584284	6.097
South Dakota	699999	1.223	1	699999	1.217
Tennessee	4896641	8.554	9	544071	8.516
Texas	17059805	29.801	30	568660	29.669
Utah	1727784	3.018	3	575928	3.005
Vermont	564964	0.987	1	564964	0.983
Virginia	6216568	10.859	11	565143	10.811
Washington	4887941	8.538	9	543105	8.501
West Virginia	1801625	3.147	3	600542	3.133
Wisconsin	4906745	8.571	9	545194	8.533
Wyoming	455975	0.797	1	455975	0.793
	249022783	**435**	**435**	**572466**	

Discrete Math Reference

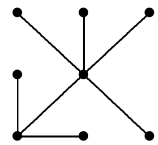
The 23 non-isomorphic 8-vertex spanning trees and their valence lists:



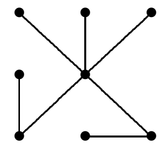
$7,1,1,1,1,1,1,1$



$6,2,1,1,1,1,1,1$



$5,3,1,1,1,1,1,1$



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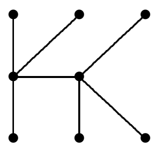
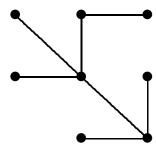
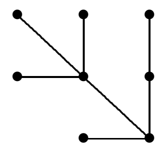


$4,4,1,1,1,1,1,1$



$4,3,2,1,1,1,1,1$



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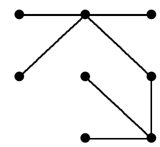

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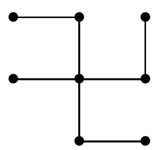

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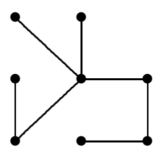

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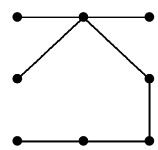

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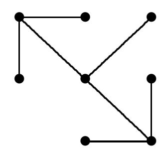
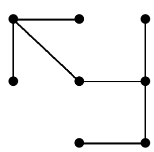
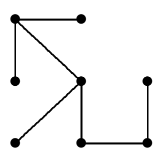


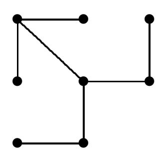
$3,3,3,1,1,1,1,1$



$3,3,2,2,1,1,1,1$



$3,3,2,2,1,1,1,1$



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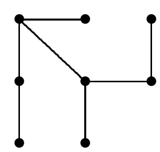

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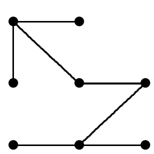

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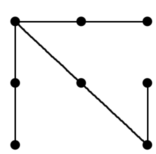
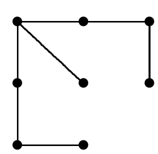


$3,2,2,2,2,1,1,1$



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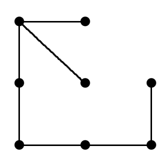
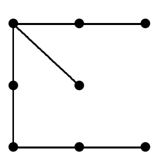
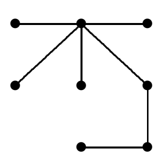


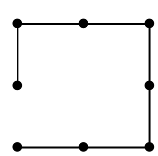
$3,2,2,2,2,1,1,1$



$3,2,2,2,2,1,1,1$



$5,2,2,1,1,1,1,1$

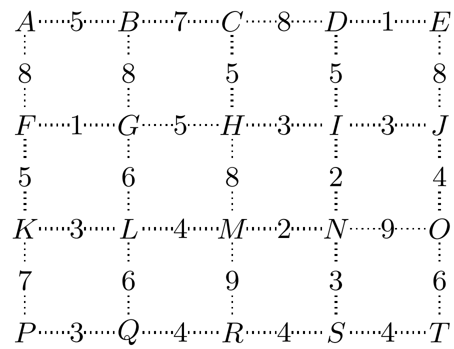


$2,2,2,2,2,2,1,1$

There are 11 different valence lists. The list $3,3,2,2,1,1,1,1$ corresponds to five trees; the list $3,2,2,2,2,1,1,1$ corresponds to four trees; the list $4,2,2,2,1,1,1,1$ corresponds to three trees, as does the list $4,3,2,1,1,1,1,1$; and the list $5,2,2,1,1,1,1,1$ corresponds to two trees.

Dijkstra's algorithm

Given a finite connected weighted graph, and a *root* vertex A, the goal is to find minimum-weight paths from A to *every* other vertex in the network. It so happens that these paths collectively form a special type of spanning tree $\mathcal{T}$. The following algorithm for producing $\mathcal{T}$ was invented by E.W. Dijkstra. The vertices of the network are labeled (and relabeled) during the procedure, so that they always display the current total weight of the best path known so far from A. The label ∞ is used to mean that no such path has been formed yet. The procedure grows $\mathcal{T}$ in one connected piece throughout.



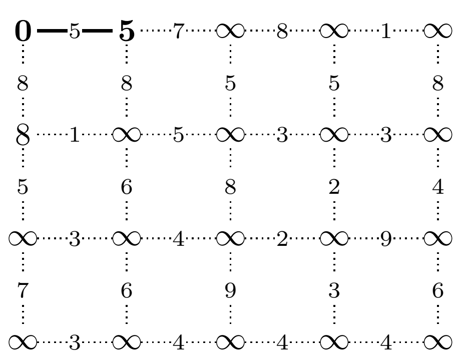
Step 1: Label vertex A with 0 and all other vertices with ∞. Let $\mathcal{T}$ initially consist of just the vertex A.

Step 2: For each vertex P that is adjacent to a vertex of $\mathcal{T}$, adjust the label on P (if necessary) as follows: Consider each edge that joins P to a vertex Q in $\mathcal{T}$, and calculate the sum of the label on Q and the weight on the edge. Use this sum to relabel P if the sum is lower than the label already on P.

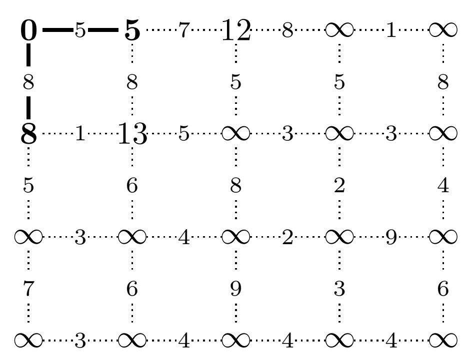
Step 3: From among all the vertices adjacent to $\mathcal{T}$ (but not in $\mathcal{T}$), choose the one with lowest label, then find the edge that produced this label. Add this edge to $\mathcal{T}$.

Repeat steps 2 and 3 until $\mathcal{T}$ becomes a spanning tree.

As $\mathcal{T}$ grows, the set of vertices that are adjacent to $\mathcal{T}$ also grows, making each round of calculations take slightly longer than the preceding round. The first two rounds are shown below. The next five rounds are shown on the next page, along with the finished tree, which takes nineteen rounds to complete.

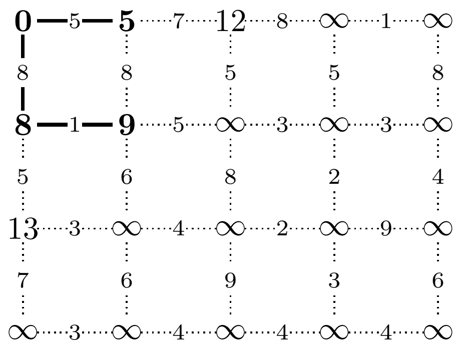


round 1

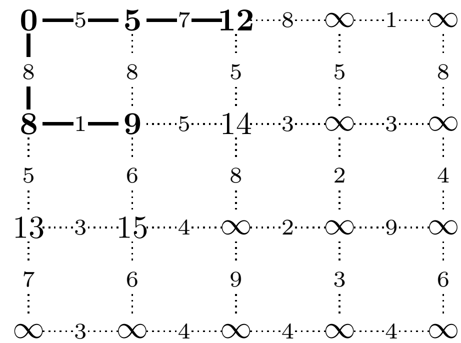


round 2

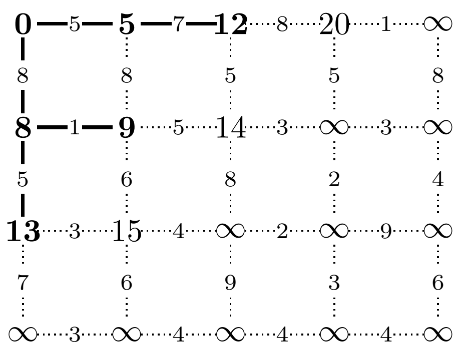
Dijkstra's algorithm



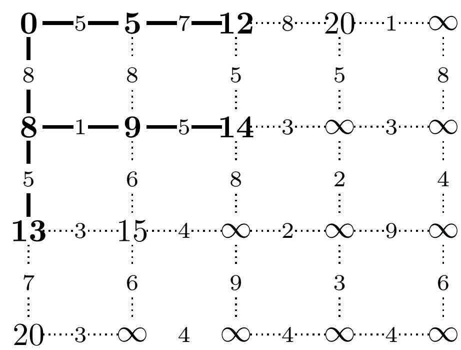
round 3



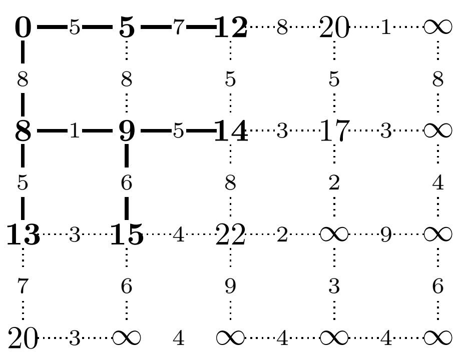
round 4



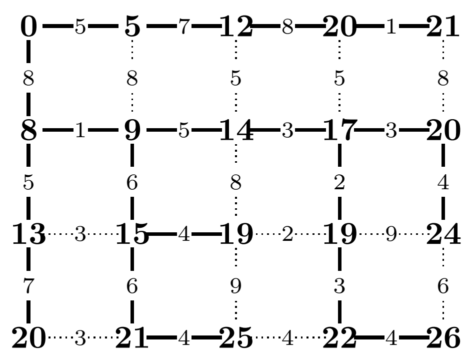
round 5



round 6



round 7



round 19

The finished tree $\mathcal{T}$ includes minimal-weight paths from A (the upper left corner) to every vertex in the network. The label on each vertex is the total weight of the minimal-weight path to that vertex.

Electoral College simulation (119350980 mock elections)

state	elec votes	% total	pivotal	% piv	rating	% total
Alabama	9	1.67	8175220	6.85	3.01	1.64
Alaska	3	0.56	2720216	2.28	1.00	0.55
Arizona	11	2.04	10004304	8.38	3.68	2.01
Arkansas	6	1.12	5444856	4.56	2.00	1.09
California	55	10.22	56656196	47.47	20.84	11.36
Colorado	9	1.67	8173985	6.85	3.01	1.64
Connecticut	7	1.30	6355612	5.33	2.34	1.27
Delaware	3	0.56	2720075	2.28	1.00	0.55
Dist of Columbia	3	0.56	2721139	2.28	1.00	0.55
Florida	29	5.39	26935322	22.57	9.91	5.40
Georgia	16	2.97	14607443	12.24	5.37	2.93
Hawaii	4	0.74	3627356	3.04	1.33	0.73
Idaho	4	0.74	3628846	3.04	1.33	0.73
Illinois	20	3.72	18331130	15.36	6.74	3.68
Indiana	11	2.04	10002270	8.38	3.68	2.01
Iowa	6	1.12	5442536	4.56	2.00	1.09
Kansas	6	1.12	5443265	4.56	2.00	1.09
Kentucky	8	1.49	7261557	6.08	2.67	1.46
Louisiana	8	1.49	7263648	6.09	2.67	1.46
Maine	4	0.74	3625914	3.04	1.33	0.73
Maryland	10	1.86	9085869	7.61	3.34	1.82
Massachusetts	11	2.04	9999779	8.38	3.68	2.01
Michigan	16	2.97	14605333	12.24	5.37	2.93
Minnesota	10	1.86	9084727	7.61	3.34	1.82
Mississippi	6	1.12	5442919	4.56	2.00	1.09
Missouri	10	1.86	9085463	7.61	3.34	1.82
Montana	3	0.56	2719018	2.28	1.00	0.55
Nebraska	5	0.93	4538953	3.80	1.67	0.91
Nevada	6	1.12	5445093	4.56	2.00	1.09
New Hampshire	4	0.74	3627894	3.04	1.33	0.73
New Jersey	14	2.60	12756700	10.69	4.69	2.56
New Mexico	5	0.93	4534580	3.80	1.67	0.91
New York	29	5.39	26934646	22.57	9.91	5.40
North Carolina	15	2.79	13676932	11.46	5.03	2.74
North Dakota	3	0.56	2719757	2.28	1.00	0.55
Ohio	18	3.35	16460109	13.79	6.05	3.30
Oklahoma	7	1.30	6354705	5.32	2.34	1.27
Oregon	7	1.30	6353806	5.32	2.34	1.27
Pennsylvania	20	3.72	18331436	15.36	6.74	3.68
Rhode Island	4	0.74	3623367	3.04	1.33	0.73
South Carolina	9	1.67	8176220	6.85	3.01	1.64
South Dakota	3	0.56	2719643	2.28	1.00	0.55
Tennessee	11	2.04	10001500	8.38	3.68	2.01
Texas	38	7.06	35940365	30.11	13.22	7.21
Utah	6	1.12	5442815	4.56	2.00	1.09
Vermont	3	0.56	2719302	2.28	1.00	0.55
Virginia	13	2.42	11835945	9.92	4.35	2.37
Washington	12	2.23	10921265	9.15	4.02	2.19
West Virginia	5	0.93	4533653	3.80	1.67	0.91
Wisconsin	10	1.86	9091179	7.62	3.34	1.82
Wyoming	3	0.56	2718803	2.28	1.00	0.55
	538		119350980			100.00

Electoral College probabilities (119350980 mock elections)

state	voters	state %	e votes	elec %	natl %	chances
Alabama	1777103	0.0599	9	6.85	0.00410	1/24392
Alaska	266964	0.1544	3	2.28	0.00352	1/28412
Arizona	2372699	0.0518	11	8.38	0.00434	1/23031
Arkansas	1082705	0.0767	6	4.56	0.00350	1/28586
California	13816536	0.0215	55	47.47	0.01019	1/9814
Colorado	1866624	0.0584	9	6.85	0.00400	1/25002
Connecticut	1325202	0.0693	7	5.33	0.00369	1/27094
Delaware	333324	0.1382	3	2.28	0.00315	1/31750
Dist of Columbia	222638	0.1691	3	2.28	0.00386	1/25938
Florida	6993286	0.0302	29	22.57	0.00681	1/14686
Georgia	3599199	0.0421	16	12.24	0.00515	1/19427
Hawaii	505739	0.1122	4	3.04	0.00341	1/29326
Idaho	582195	0.1046	4	3.04	0.00318	1/31452
Illinois	4759821	0.0366	20	15.36	0.00562	1/17803
Indiana	2405585	0.0514	11	8.38	0.00431	1/23195
Iowa	1129901	0.0751	6	4.56	0.00342	1/29215
Kansas	1059611	0.0775	6	4.56	0.00354	1/28288
Kentucky	1609724	0.0629	8	6.08	0.00383	1/26136
Louisiana	1684966	0.0615	8	6.09	0.00374	1/26732
Maine	493237	0.1136	4	3.04	0.00345	1/28973
Maryland	2142274	0.0545	10	7.61	0.00415	1/24097
Massachusetts	2427068	0.0512	11	8.38	0.00429	1/23304
Michigan	3667302	0.0417	16	12.24	0.00510	1/19613
Minnesota	1966505	0.0569	10	7.61	0.00433	1/23090
Mississippi	1101949	0.0760	6	4.56	0.00347	1/28849
Missouri	2224247	0.0535	10	7.61	0.00407	1/24554
Montana	367934	0.1315	3	2.28	0.00300	1/33370
Nebraska	677775	0.0969	5	3.80	0.00369	1/27131
Nevada	1002490	0.0797	6	4.56	0.00364	1/27506
New Hampshire	488935	0.1141	4	3.04	0.00347	1/28831
New Jersey	3258775	0.0442	14	10.69	0.00472	1/21168
New Mexico	764891	0.0912	5	3.80	0.00347	1/28850
New York	7185790	0.0298	29	22.57	0.00672	1/14887
North Carolina	3539339	0.0424	15	11.46	0.00486	1/20576
North Dakota	250085	0.1595	3	2.28	0.00364	1/27504
Ohio	4280343	0.0386	18	13.79	0.00532	1/18802
Oklahoma	1393006	0.0676	7	5.32	0.00360	1/27782
Oregon	1423984	0.0669	7	5.32	0.00356	1/28093
Pennsylvania	4711915	0.0368	20	15.36	0.00565	1/17713
Rhode Island	390441	0.1277	4	3.04	0.00388	1/25796
South Carolina	1719011	0.0609	9	6.85	0.00417	1/23987
South Dakota	303312	0.1449	3	2.28	0.00330	1/30291
Tennessee	2358909	0.0519	11	8.38	0.00435	1/22971
Texas	9349315	0.0261	38	30.11	0.00786	1/12726
Utah	1025183	0.0788	6	4.56	0.00359	1/27827
Vermont	233225	0.1652	3	2.28	0.00376	1/26565
Virginia	2973962	0.0463	13	9.92	0.00459	1/21795
Washington	2498747	0.0505	12	9.15	0.00462	1/21651
West Virginia	688132	0.0962	5	3.80	0.00365	1/27370
Wisconsin	2108345	0.0550	10	7.62	0.00419	1/23891
Wyoming	210271	0.1740	3	2.28	0.00396	1/25229