Exam \#1
$\qquad$
1.) Let our "universe" be $U=\{a, b, c, d, e, f, g, h, i, j, k\}$. Let $A=\{a, b, c, d\}, B=\{a, e, i\}$, and $\mathrm{C}=\{\mathrm{a}, \mathrm{b}, \mathrm{e}, \mathrm{i}\}$. Evaluate each of the following.
a.) Evaluate $(\mathrm{A} \cup \mathrm{C}) \cap \mathrm{B}$.
b.) Evaluate $\overline{A \cup B} \cap C$.
c.) Evaluate $(\mathrm{B} \cap \mathrm{C})$ - A .
2.) Let $\mathbf{Z}$ denote the set of integers. Which of the following quantified predicate statements are true? Justify your answers.
a.) $\quad \forall n \in \mathbf{Z}, \exists k \in \mathbf{Z}, n+k=0$
b.) $\quad \exists n \in \mathbf{Z}, \quad \forall k \in \mathbf{Z}, n+k=0$
3.) a.) Using a Venn diagram, represent the set $\overline{A \cup B}-\mathrm{C}$.
b.) Using appropriate set notation, describe the shaded region represented in the following Venn diagram.

4.) Find a counterexample to show that the following statement is false. Clearly illustrate that it is a counter example.

$$
A \cap(B \cup C)=(A \cap B) \cup C
$$

5.) Let $A=\{a, b\}$ and $B=\{2,3,4\}$. Write out all elements of $A \times B$.
6.) Using A from \#5, find $\wp(\wp(A))$ (where $\wp$ denotes "power set").
7.) Prove that $(A \cup B) \cap C=(A \cap C) \cup(B \cap C)$.
8.) Prove that the difference of the squares of any two even numbers is even.
9.) Use a truth table to determine if the following is a valid argument; depending on your approach, you may have to add columns. Give your conclusion using complete sentences.

$$
\begin{aligned}
& p \rightarrow r \wedge q \\
& \frac{\neg r \vee \neg q}{\therefore \neg p}
\end{aligned}
$$

10.) Construct a truth table for:

$$
\neg(\mathrm{P} \vee \mathrm{Q}) \Rightarrow \neg \mathrm{Q}
$$

## Exam \#2

1.) Use the contrapositive to prove the following.

Suppose $n$ is a positive integer. If $7 n+4$ is even, then $n$ is even.
2.) Prove the following.

For every natural number $n, n^{3}+n$ is even. (Hint: Try proof by cases.)
3.) Prove the following by mathematical induction.

For every natural number, $\mathrm{n}, 1^{2}+2^{2}+\cdots+n^{2}=\frac{n(n+1)(2 n+1)}{6}$.
4.) Use proof by contradiction to prove the following.

If $n$ is an odd integer, then $n^{2}+n$ is even.
5.) Compute the following:
a.) $23 \bmod 5$
b.) $6 \bmod 9$
c.) $\quad-35 \bmod 8$
d.) [3.7]
e.) 【-3.7]
f.) โ3.7〕
g.) $\quad\lceil-3.7\rceil$
6.) For each of the following, give an example. If an example is not possible, explain why not. Assume all of the following sets are non-empty.
a.) A set of real numbers containing no minimum number.
b.) A set of positive integers containing no minimum number.
c.) An infinite set of positive numbers containing no minimum number.
7.) Consider the following definitions.
$A=\{a, b, c, d, e\} \quad B=\{1,3,4,5,6\} \quad C=\{x, y, z, w\}$
$D=\{B o b$, Mary, Dave, Joseph, Martha $\}$
$f$ is a relation defined by $\{(x, a),(y, b),(z, b),(w, d)\}$
$g$ is a relation defined by $\{($ Bob, $a),($ Mary, $b),($ Dave, $c),(J o s e p h, d)$, (Martha, e\}
$h$ is a relation defined by $\{(a, 1),(b, 4),(c, 3),(d, 5),(e, 4)\}$
Each relation is defined with one of $A, B, C$ or $D$ as its domain and one as its codomain.
a.) Fill in the chart.

|  | $f$ | $g$ | $h$ |
| :--- | :--- | :--- | :--- |
| Is it a function? Yes or No |  |  |  |
| What is its domain? |  |  |  |
| What is its codomain? |  |  |  |
| Is it one-to-one? Yes or No |  |  |  |
| Is it onto? Yes or No |  |  |  |

b.) Which composition(s) would be defined?
8.) Let A be the set of positive integers. Consider the function $f: \mathrm{A} \times \mathrm{A} \rightarrow \mathrm{A}$ defined by $f(x, y)=x+y$.
a.) Determine if $f$ is one-to-one. If so, prove it is one-to-one. If not, provide a counterexample.
b.) Determine if $f$ is onto. If so, prove it is onto. If not, provide a counterexample.
9.) Use the contrapositive to prove the following.

Suppose $n$ is a non-negative integer. If $a$ is an irrational number, then $a / n$ is irrational.
10.) Prove the following.

For all integers $n>4$, if $n$ is a perfect square, then $n-1$ is not a prime number.
11.) Prove or disprove each of the following.
a.) The sum of a rational number and an irrational number is irrational.
b.) The sum of a rational number and a rational number is rational.
c.) The sum of an irrational number and an irrational number is irrational.
d.) For all real numbers $a$ and $b$, if $b^{2}>a^{2}$, then $b>a$.
12.) Prove the following by mathematical induction.
a.) $3+11+19+\cdots+(8 n-5)=4 n^{2}-n$ for all positive integers n .
b.) For all integers $n>6,5 n+5 \leq n^{2}$

## Exam \#3

1.) a.) Consider the sequence $a_{n}=2 n+3$.
(i) Write out the first five terms of the sequence.
(ii) Determine a recursive formula for the sequence.
b.) Consider the sequence $a_{1}=3, a_{n}=a_{n-1}+2 n+1, n>1$.
(i) Write out the first five terms of the sequence.
(ii) Determine a closed form (non-recursive) formula for the sequence.
2.) Consider the relation on $R^{2}$ defined as follows.
$(x, y) \mathrm{R}(z, w)$ if and only if $x=z$
a.) Determine whether or not this is an equivalence relation.
b.) If it is an equivalence relation, give a geometric description of the partition of $R^{2}$ formed by the equivalence classes.
3.) Consider $\wp(S)$ (the power set of $S$ ) if $S=\{1,2,3\}$. Define a relation on $\wp(S)$ as follows.
$A R B$ if and only if $A \cap B \neq \varnothing$
Determine whether or not R defines an equivalence relation $\wp(S)$.
4.) Suppose $a, b$ and $c$ are integers. Show that if $a \mid b$ and $a \mid c$, then, for any integers, $m$ and $n, a \mid(m b+n c)$.
5.) List all members of the SMALLEST set of possible divisors one would use to determine whether 647 is prime or composite. Give the set and briefly explain why you chose the set.
6.) Find the prime factorization of 4158000.
7.) Do the following.
a.) Convert $1235_{\text {ten }}$ to base 8
b.) Convert $1 \mathrm{ABO} 1_{\text {nex }}$ to base 10
c.) Suppose we are doing work with various number bases. You are given the number 123456 but the base was accidentally erased. What can you say about the base? Explain.
8.) Consider the numbers $a=670824$ and $b=858000$. [Note: $670824=2^{3} \times 3^{2} \times 7^{1}$ $\times 11^{3}$ and $858000=2^{4} \times 3^{1} \times 5^{3} \times 11^{1} \times 13^{1}$.]
a.) Use the prime factorization to find $\operatorname{gcd}(\mathrm{a}, \mathrm{b})$. Show your work.
a.) Use the prime factorization to find $\operatorname{Icm}(\mathrm{a}, \mathrm{b})$. Show your work.
9.) Compute the following:
a.) $23 \bmod 5$
b.) $6 \bmod 9$
c.) $\quad-35 \bmod 8$
d.) $35!\bmod 26$
g.) $5^{37} \bmod 12$
11.) Do the following problems.
a.) List three integers, one of which must be negative, in the congruence class of 1 modulo 6 .
b.) Is 7 congruent to $-13 \bmod 5$ ? Justify your answer.
11.) Consider the numbers $\mathrm{a}=8568$ and $\mathrm{b}=84084$.
a.) Use the Euclidean algorithm to find the gcd(a, b). Show your work. You must use this method to receive credit. You may use your calculator but you MUST show all details. Do NOT use the prime factorization of the numbers.
b.) Use your answer from part a.) to find the Icm(a, b). Do NOT use the prime factorization of the numbers.

## Exam \#4

1.) Suppose license plates are being made with three letters followed by 3 characters that can be either letters or numbers. BE SURE TO SHOW YOUR WORK.
a.) How many license plates can be made?
b.) How many license plates can be made if letters may not be repeated?
c.) How many license plates can be made if two of the last three characters must be numbers?
2.) A baseball team's roster consists of 20 players. Bob and Dave are the only two who can pitch and neither can play any other position. The other players can all play every other position. How many different defensive arrangements (pitcher, catcher, 1st baseman, 2nd baseman, 3rd baseman, shortstop, left fielder, center fielder, right fielder) are possible? BE SURE TO SHOW YOUR WORK.
3.) At Bubba's Pizzeria, the possible toppings are sausage, Canadian bacon, pineapple, chitterlings, head cheese, pickled pig's feet and grits. You can have up to 4 toppings on a pizza. Assuming the same topping cannot be used twice on the same pizza, how many pizzas are possible? BE SURE TO SHOW YOUR WORK.
4.) A club consists of 10 men and 8 women. Answer the following. BE SURE TO SHOW YOUR WORK.
a.) How many different lines can be formed?
b.) How many different lines can be formed if we only consider whether the person is male or female?
c.) The club creates a 5 -person committee.
(i) How many committees are possible?
(ii) How many committees are possible if the committee must include exactly three men?
5.) An integer, $n$, between 100 and 999 inclusive is selected at random. If $n=100 a+$ $10 b+c$ has digits a (must be non-zero), $b$, and $c$, find the probability of the following. BE SURE TO SHOW YOUR WORK.:
a.) the digits of $n$ are all distinct.
b.) The digits of $n$ are all equal.
c.) $\quad a=c$.
d.) $\quad b \neq c$.
6.) a.) Use the binomial theorem to expand $(2 x+3 y)^{4}$.
b.) Consider $(x-3 y)^{24}$. What is the coefficient of the term containing $x^{17}$ once it is expanded?
7.) There are 35 students in a class.
a.) Show that at least two students must have last names beginning with the same letter of the alphabet?
b.) Is it necessary that four students will share the same month of birth? If so, explain why. If not, how many more students would there need to be?
8.) Prove that $\binom{n}{r}=\binom{n}{n-r}$.
9.) There are 69 students in a class. Every one of them is at least one of the following: mathematics major, piano player, Wyoming resident. Forty are mathematics majors. Forty play the piano. Thirty-five are from Wyoming. Twenty are mathematics majors from Wyoming. Eighteen are piano players from Wyoming. Nineteen are piano playing mathematics majors. How many are piano playing mathematics majors from Wyoming? BE SURE TO SHOW YOUR WORK. [Hint: This is NOT a pigeon hole principle problem.]

## Final Exam

1. Consider a group of 10 men and 12 women.
a. How many five person committees can be chosen?
b. How many five person committees can be chosen if the committee is to have at least three women?
c. How many five person committees can be chosen if the committee has a president, vice-president, secretary and treasurer?
2. Consider the letters in the word MISSISSIPPI.
a. How many distinguishable arrangements of the letters are there (using ALL the letters)?
b. How many distinguishable arrangements are there if all letters are used and the Ps end up adjacent to each other?
3. In how many ways can four non-negative numbers add up to 20?
4. In how many ways can 21 pennies be distributed among 5 children so that each child gets at least one penny?
5. Construct a truth table for the following.

$$
(\neg p) \rightarrow(p \rightarrow q)
$$

6. Use a Venn diagram to represent the set $\overline{A \cup B}-\mathrm{C}$.
7. Find a counterexample to show that the following statement is false. Clearly illustrate that it is a counter example.

$$
A \cap(B \cup C)=(A \cap B) \cup C
$$

8. Using an element-wise proof, prove that $(A \cup B) \cap C=(A \cap C) \cup(B \cap C)$. You must use an element-wise proof for credit.
9. Prove $\sqrt{2}$ is irrational.
10. Complete exactly two of the following proofs. Indicate if your proof is a direct proof, proof by contrapositive, proof by contradiction or proof by cases. If you do more than two proofs, only the first two will be graded.
a. If $n$ is odd, then $n^{3}-n$ is divisible by 4 .
b. If $3 n$ is odd, then $n$ is odd.
c. If $n$ has the form $4 k+3$ for some integer $k$, then $n^{2}-n$ is not divisible by 4 .
11. Prove the following statement is true for all integers $n \geq 1$, using mathematical induction.

$$
\frac{1}{1 \cdot 2}+\frac{1}{2 \cdot 3}+\frac{1}{3 \cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}
$$

12. Let $R$ be a relation on $\mathbb{R}$ (the set of real numbers) given by the rule $(x, y) \in R$ iff $x-y$ is divisible by 3 . Prove that $R$ is an equivalence relation by proving it is reflexive, symmetric and transitive.
13. Use the Binomial Theorem for each of the following.
a. Expand $(x-2 y)^{4}$.
b. Determine the coefficient of $x^{4} y^{7}$ in the expansion of $(2 x-y)^{11}$.
14. In how many ways can eight people be seated around a circular table?
15. Let our "universe" be $\mathrm{U}=\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g}, \mathrm{h}, \mathrm{i}, \mathrm{j}, \mathrm{k}\}$. Let $\mathrm{A}=\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}, \mathrm{B}=\{\mathrm{a}, \mathrm{e}, \mathrm{i}\}$, and $\mathrm{C}=\{\mathrm{a}, \mathrm{b}, \mathrm{e}, \mathrm{i}\}$. Evaluate each of the following.
a. Evaluate $(\mathrm{A} \cup \mathrm{C}) \cap \mathrm{B}$.
b. Evaluate $\overline{A \cup B} \cap C$.
c. Evaluate $(B \cap C)$ - $A$.
16. Compute the following:
a. $\quad 23 \bmod 5$
b. $\quad 6 \bmod 9$
c. $\quad-35 \bmod 8$
d. [3.7]
e. $\quad$-3.7]
f. โ3.7]
g. $\quad\lceil-3.7\rceil$
17. Suppose you have a class of 23 students.
a. Explain why it must be true that at least two students must share the same birth month.
b. How many more students must be present to be guaranteed that three share the same birth month?
c. How many more students must be present to be guaranteed that seven share the same birth month?
