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LEARNING GOAL

Students use discrete structures to model and understand mathematical relationships and solve problems.

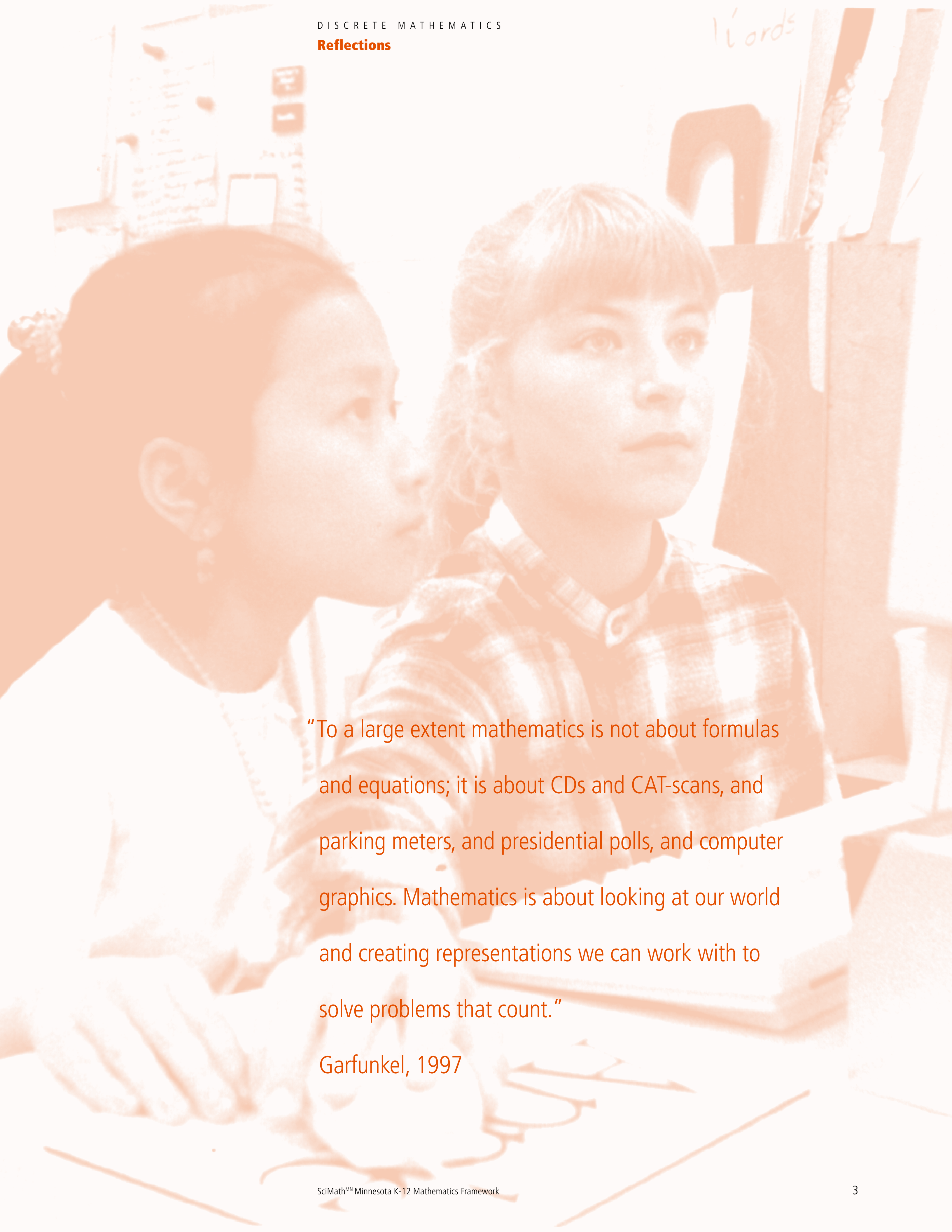
Components

With this learning goal in mind, Minnesota students will have the opportunity to:

1. investigate and apply systematic counting techniques, set relationships, and principles of logic to represent, analyze, and solve problems.
2. use charts, vertex-edge graphs, and matrices to model and solve problems.
3. explore, develop, and analyze algorithmic thinking to accomplish a task or solve a problem.
4. analyze, extend, and model iterative and recursive patterns.

(For more detailed information, see pages 19-27 in this content section.)

Reflections



"To a large extent mathematics is not about formulas and equations; it is about CDs and CAT-scans, and parking meters, and presidential polls, and computer graphics. Mathematics is about looking at our world and creating representations we can work with to solve problems that count."

Garfunkel, 1997

Reflections

Computers have changed our culture in fundamental ways, including the way we learn mathematics, the way we do mathematics, the kinds of problems we can consider. Even our imaginations, our creative visions, and our sense of what is possible have been altered. One way to think about computers is as discrete machines, capable of dealing only with finite information. **Discrete mathematics**, with its many real-world applications and its close ties to computer science, has grown rapidly over the last thirty years. In the words of John Dossey, it is the **"math for our time."** (Dossey quoted in Kenney, 1991).

Applications of discrete mathematics are found in a variety of settings, including project management, communication networks, systems analysis, social decision making, population growth, and finance. Discrete mathematics is used to design efficient computer networks, optimally assign frequencies to cellular phones, track pollution, fairly rank competitors in a tournament, accurately represent public opinion in political elections, efficiently schedule large projects, plan optimal routes, and solve many other problems, both applied and abstract. These applications oblige us to provide students the knowledge and skills of discrete mathematics to prepare them for life-work in the twenty-first century.

"…the nonmaterial world of information processing requires the use of discrete (discontinuous) mathematics."

NCTM, 1989

Discrete mathematics is concerned with finite processes and phenomena. It **involves the study of objects and ideas that can be divided into 'separate' or 'discontinuous' parts.** While discrete mathematics is sometimes contrasted with calculus, which focuses on infinite processes and continuous phenomena, it is more a complement to calculus than a competitor.

The concepts and techniques of discrete mathematics can be used to model and solve problems involving enumeration (determining a count), decision making in finite settings, relationships among a finite number of elements, and sequential change. It is used to investigate settings in which functions are defined on discontinuous sets of numbers, such as the positive integers (Dossey in Kenney, 1991, p. 1).

Given the relative novelty of discrete mathematics to average citizens as well as many teachers of mathematics, it may be more helpful to consider some characteristics of discrete mathematics than to examine its definitions.

Problems in discrete mathematics can be classified into three broad categories:

1. Existence problems deal with whether or not a solution exists for a given problem.

A familiar context for this type of situation is the eighteenth-century problem that intrigued the Swiss mathematician Leonard Euler (1707-1783). In the 1700s, seven bridges connected two islands in the river to the rest of the city of Königsberg (see Figure 1).

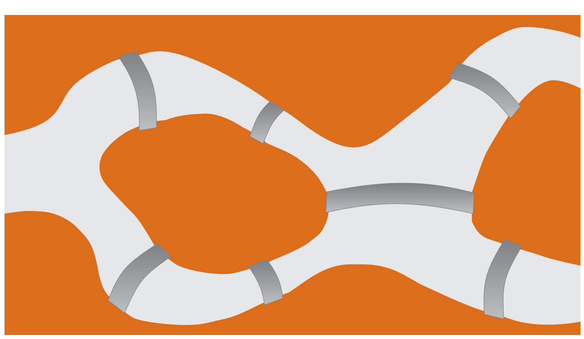
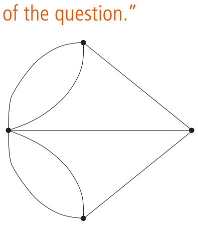


Figure 1. Representation of the seven bridges of Königsberg (now known as Kaliningrad, Russia)

After describing ways in which the existence and traceability of paths are used in a variety of cultures, Ascher & Ascher (1981, p. 164) conclude: "In the Western history of mathematics, Euler's paper on the Königsberg bridge is considered the birth of the subject called graph theory.…the achievements of Euler…might even be more impressive because of the more universal nature of the question."



This is a vertex-edge graph representing the seven bridges of Königsberg. Vertices are the land masses connected and the edges are the bridges.

Is it possible to walk through the city by crossing each bridge exactly once and return to the original starting point? Using a vertex-edge graph in which the vertices represented the landmasses of the city and the edges represented the bridges, Euler found that there was no such walk possible. His investigation of this problem, however, led Euler to make a number of generalizations about the traceability of vertex-edge graphs.

To learn more about what Euler discovered, try to traverse the following graphs without lifting your pencil or tracing the edges more than once. When can you draw the figures without retracing any edge and still end up at your starting point? When can you draw the figure without retracing any edge but end up at a point different from where you started? When can you NOT draw the figure without retracing an edge? (Crisler et al., 1994, pp. 174-5).

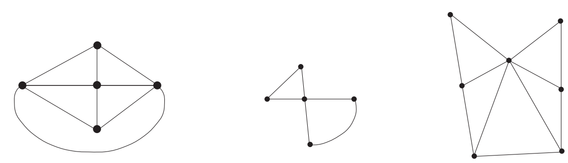


FIGURE 2

2. Counting problems explore how many solutions may exist for a given problem.

A familiar application is the number of phone numbers that can exist for a given area code. A seven-digit telephone number cannot begin with a 0 or a 1. A common strategy is to apply the multiplication principle: multiply the number of possible choices for the first digit, second digit, etc.:

8 x 10 x 10 x 10 x 10 x 10 x 10. There are 8,000,000 possible telephone numbers for a given area code.

" The educational value of much simple discrete mathematics lies precisely in the fact that it forces students to think about very elementary things..."

Kenney, 1991

Prior to 1996, Minnesota was served by three area codes: 218, 612, and 507. In 1996, the 320 area code was added. What information would you need to know in order to predict whether or not the state will need a new area code in the near future?

3. Optimization problems focus on finding a "best" solution to a particular problem, "best" being defined by the context of the problem—the most efficient method, or the shortest path, or the fairest decision, etc.

Fair division conflicts are common. One context is voting apportionment schemes. Central High School has 464 sophomores, 240 juniors and 196 seniors. The problem involves dividing the 20 seats on the student council among the three classes. An ideal ratio (total population divided by the number of seats) results in 45 students per seat. The calculated quotas for each class (the class size divided by the ideal ratio) result in decimal values. That presents a dilemma, as a single seat cannot be split to give part of it to the seniors, part of it to the juniors, and part of it to the sophomores. How could the 20 seats be distributed fairly? (Crisler et al., 1994, pp. 59-60)

These problems illustrate the **characteristic ways of thinking** in discrete mathematics that can be developed over the entire mathematics curriculum. The basic questions that should be asked at every level are:

1. Is there a solution to this problem?
2. How can we solve this problem? How many solutions are there?
3. Which of these solutions is the "best" within the context of the problem?

"At the introductory level discrete mathematics should be based on a handful of important ideas, not on theories or standard algorithms. It should concentrate on problems that require students to think in contexts that are familiar and natural, and it should not resort to poorly understood routines."
Gardiner in Kenney, 1991

Discrete mathematics is especially *rich in the variety of applications* it can treat. Some typical examples include:

routing problems—design an efficient plan for city-wide snow removal

scheduling problems—design an optimal schedule for legislative subcommittee meetings

matching problems—fill necessary jobs with capable applicants at minimal cost

sorting problems—describe an efficient method for alphabetizing 100,000 names

searching problems—describe a method for locating a particular item in a data base

There are some *content areas that are typically identified* with the field of discrete mathematics. These include:

combinatorics—the application of systematic counting techniques

graph theory—the use of vertex-edge diagrams to study relationships among a finite number of elements

game theory—the mathematics of voting, fair division, apportionment, and cooperation and competition

recursion—the method of describing sequential change by indicating how the next stage of a process is determined by previous stages

algorithmic thinking—the development and analysis of a rule system to solve a problem or a class of problems

"The development and analysis of algorithms lie at the heart of computer methods of solving problems."
NCTM, 1989

This last notion, algorithmic thinking, is a particularly important concept in discrete mathematics. Problems precede algorithms, but once the problem has been stated, the focus is on how it might be solved. Our interest in algorithms is in their development and analysis; computers can usually carry out the rote steps of algorithms. Examples of algorithmic thinking include whether all the proposed solution procedures are correct and which are most efficient. It is in the design of algorithms that new insights into the original problem are often found.

Discrete mathematics is accessible to students at all levels. Arithmetic offers a fertile field for interesting problems in discrete mathematics. Many practical everyday problems can be modeled as graphs. Almost any puzzle or challenging problem, even (and especially) those of a recreational nature, will involve discrete mathematics in some form. There is a growing literature of excellent materials that can be used to promote discrete thinking. (See the *Sample Problems*, the *Sample Tasks*, and the *Teaching Resources* in this section.)

"This standard neither advocates nor describes a separate course in discrete mathematics at the secondary school level; rather, it identifies those topics from discrete mathematics that should be integrated throughout the high school curriculum."
NCTM, 1989

There are **workable and practical ways to include discrete mathematics** in an already overcrowded curriculum (Hart, 1991, pp. 76-77).

- Many topics in discrete mathematics, including matrices, counting techniques, induction, sets, and sequences overlap with other content strands. A teacher can emphasize these topics that do "double duty" in the curriculum.
- The tools and techniques of discrete mathematics can be used to approach traditional mathematics in new ways. For example, recursive formulas can represent sequential change, vertex-edge graphs can model relationships in mathematical problems, or matrices can be used to solve systems of linear equations.

- Short units can be taught on discrete topics, such as graph theory or game theory.
- Mathematics courses can be integrated in such a way that discrete mathematics occurs amidst algebra, trigonometry, geometry, statistics and probability topics. Many new standards-based curriculum projects, developed with sponsorship from the National Science Foundation, take this approach to content organization, and hold promise for delivering the *Minnesota Graduation Standards* in mathematics in efficient and meaningful ways.

Discrete mathematics is both *important and relevant to many real-world situations.* It is, in fact, the mathematics used by many decision-makers in our society, including workers in health care, transportation, telecommunications, and a variety of government agencies. Introducing topics from discrete mathematics thus serves to broaden students' knowledge of the range of mathematics while making the "school to work" connection.

The inclusion of discrete mathematics in the K-12 curriculum has other payoffs for teachers and students:

" Discrete mathematics is core mathematics. Why?

- it is used widely in the real world
- it provides a broader view of mathematics
- it reinforces mathematical thinking
- it uses powerful mathematical tools
- it is active and alive
- it is pedagogically powerful and accessible"

Hart, 1997

1. Discrete mathematics is full of unsolved problems and unique strategies. A focus on discrete mathematics reinforces the central theme of problem solving in mathematics education while communicating to students the contemporary and dynamic nature of mathematics. It increases students' understanding of what it means to DO mathematics by encouraging them to formulate and test conjectures.

2. The study of discrete mathematics also has the potential to promote students' critical thinking, mathematical reasoning, and visualization skills while making important connections between mathematics and unique problem situations. Discrete mathematics often utilizes geometric ideas in ways that complement symbolic manipulation, providing students with multiple ways to think about and approach problems.

3. Discrete mathematics does not have extensive prerequisites, yet it poses challenging problems to all students. These challenges are engaging and accessible, and can broaden and enrich the other content strands. Discrete mathematics has the potential to stimulate greater interest in mathematics among students of all abilities at all grade levels.

4. Discrete mathematics has the potential to provide students with an array of new and powerful models for thinking about and doing mathematics, including vertex-edge graphs and matrices.

We will need to renew ourselves as learners of mathematics to give adequate time and attention to this relatively recent field of discrete mathematics. The *Curriculum and Evaluation Standards for School Mathematics* of the National Council of Teachers of Mathematics and the *Minnesota Graduation Standards* recommend that our teaching style should reflect a problem solving approach that permeates the entire mathematics program. Discrete mathematics provides rich and motivating contexts and challenges in which both teachers and students can become engaged, thereby increasing the chances that both groups will continue their study of mathematics and improve their problem solving skills.

Vignette: Discrete Mathematics in the Classroom

Business and industry use counting and optimization to solve problems, such as finding a shortest path to route a telephone call.

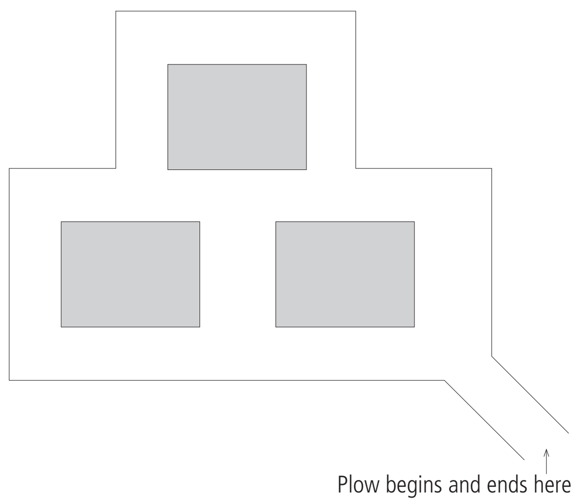
In Minnesota this is a very real life situation: is there a best way to clear snow? Shortest path may be one of many factors influencing plowing patterns.

Mr. K. wanted his fourth grade students to explore discrete mathematics, and more specifically, algorithmic thinking. This lesson, related to finding a 'shortest path,' was to precede a more detailed investigation of vertex-edge graphs, including characteristics that determined their traceability. Mr. K. wanted his students to learn that although there may be many approaches to solving a real-life problem, there might be one best (optimal) solution.

It has just snowed. Each of the three towns of Tri-Square Village, Quad-Square Town, and Cobblestone City need to have their streets plowed. The plow driver would like to plow with the least amount of driving. Beginning and ending at the road to the garage, what route should the snowplow take? How many blocks does the plow travel for the shortest route? Can you find a strategy to find the length of the shortest route?

The map of the first town was distributed and students were assigned to work with partners.

Tri-Square Village



MR. K: Are there any questions before you begin to explore this problem?

STUDENT: How do you know how long a path is? Should we count blocks?

MR. K: Yes, that seems to be the best unit of measure for this situation.

STUDENT: If you only go halfway down the block and turn onto another street, do you only count that as half?

MR. K: Right. Don't forget you have to plow both sides of the street.

STUDENT: Should we count the blocks a plow travels to get out of the garage and back in?

MR. K: If you consider those blocks part of the plowing route, then you should count them.

The focus is on organized counting. The teacher is careful to clarify students' questions.

Mr. K. circulated among the students.

STUDENT: Sometimes we are going over a street more than twice. I think I need two different colored pencils.

Students know where the supplies are that may used for solving problems. They have been encouraged to use a variety of tools.

STUDENT: Well, if you see two lines, you have done both sides of the street. One line means you have to go back over the street to plow the other side.

STUDENT: I have so many lines, I need another paper.

STUDENT: I have so many lines, I can't tell which is the route I want. I'm going to get a transparency and some pens.

Students begin counting by adding ones, twos and sometimes halves. Some begin to connect the problem with multiplication as a short way to add.

STUDENT: The first time we tried, we got a route that was 24 blocks long, but some of the streets were covered three times. Then we got it down to 20 blocks.

STUDENT: How could you get it down to 20? I think it has to be 24 because there are 3 blocks with 4 sides each that have to be plowed both ways so 3 blocks multiplied by 4 sides multiplied by 2 is 24.

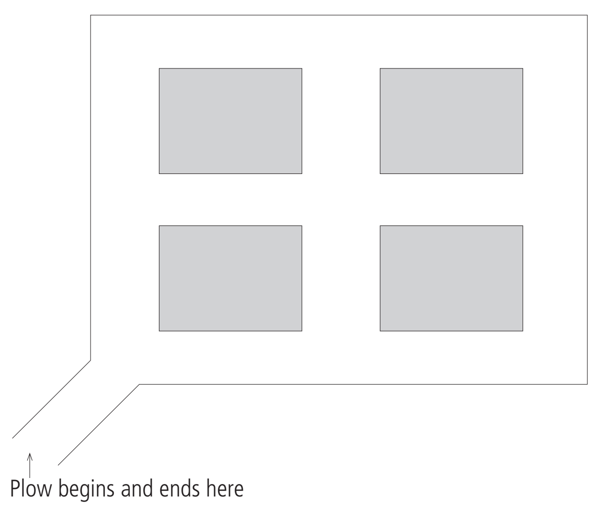
STUDENT: Well, we got it down to 20 because we noticed that the streets or parts of streets that are between two blocks get counted twice. So in this problem you have to subtract 4 blocks. That is the number of blocks that get counted twice.

Solutions may be optimal, though not unique.

After groups shared their solutions, the class agreed that although the route could be more than 20 blocks, 20 blocks was the shortest snow plowing route for Tri-Square Village. The students observed that there were many different ways to plow this route. (A few groups felt it was important to count the two blocks to and from the garage, so their shortest route was 22 blocks.)

Mr. K. passed out another map and the students again worked in partners.

Quad-Square Town



MR. K: What about Quad-Square Town? Some of you found different paths for Tri-Square Village. What do you expect the shortest route for Quad-Square Town will be?

Why is this the answer? In this age of computers, the "why" is more important than ever. Is the answer meaningful? Will the algorithm mean that you can actually find a physically feasible shortest path?

Often students expect a problem to be more difficult because it involves more things, in this case four blocks instead of three. Sometimes factors other than quantity influence the difficulty, in this case whether the blocks are "staggered" or directly on top of each other.

Mr. K. distributes the different maps one at a time so students don't become so focused on finding the "right" answer that they miss developing, implementing, and analyzing a strategy, evaluating multiple solutions, and recognizing optimal solutions.

STUDENT: I think if we did 4 blocks multiplied by 4 sides multiplied by 2 plow lines, we would get the answer. That's 32 for the route because 4 x 2 x 4 = 32.

STUDENT: Wait. That wasn't right before. You have to subtract when you are in the middle, plowing 2 blocks at the same time, remember? In this problem, that is 8 blocks so 24 is the shortest route. 32 - (4 x 2) = 24.

STUDENT: We looked at the problem this way. There are 6 streets that go north to south, and 6 streets that go east to west. That is 12 streets in all, and they all have to be plowed on both sides. So that is 12 multiplied by 2, or 24.

MR. K: Those are good explanations of your strategies. Can you find a path that is actually 24 blocks long? Will your answer really work? What are your strategies for finding a route that works?

These were some strategies that were shared after some time to explore the new town's configuration.

STUDENT: This is what I did—I thought about doing figure eight's. When I got to a street that had already been plowed once, I didn't take that street. But sometimes that left me in a place where I had to go over a street 3 times in order to get back. Then I got more than 24 blocks.

STUDENT: I tried doing all the inside streets and then going back around the outside.

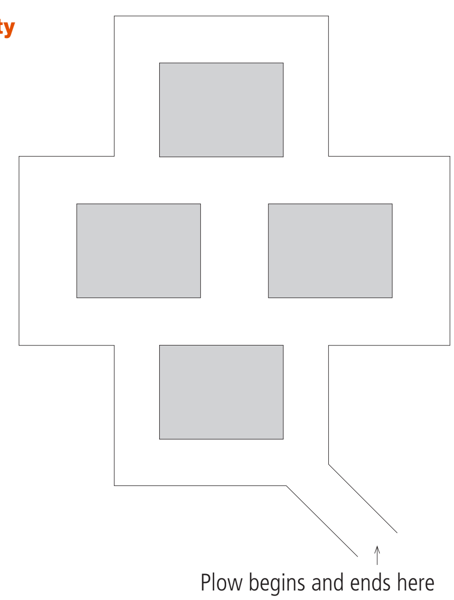
STUDENT: I tried going to the furthest place and then working my way back so that I wouldn't get stuck far from the exit. If I could start and end in different places, it would be easier.

STUDENT: I tried going over every street once. I thought that once I had that path, I could just re-do it a second time and every street would be plowed twice. But it didn't work because to get around once, you have to cover some of the blocks twice.

The students looked over each other's solutions. They saw that many strategies had drawbacks. Eventually they found a route that gave them a total of 24 blocks, reinforcing the earlier predictions of some of the students.

Mr. K. then distributed the map for Cobblestone City.

Cobblestone City



Mr. K. wrote these questions on the board:

- How is the layout similar to or different from the other cities?
- Will your answer be similar to the previous maps or different?
- Can you use your previous strategies here?
- What do you think the shortest route is?
- How will you find it?

STUDENT: I think that there are four blocks, so 24 blocks is the shortest route, like the last one.

STUDENT: But the blocks are arranged differently. They're not touching on as many sides. My prediction is 26. I thought 4 blocks multiplied by 4 sides multiplied by 2 plowings a side. Then I counted the sides that were connected—3— and since those would be repeats, I subtracted 3 x 2 which is 6. So, 32 subtract 6 is 26.

STUDENT: I also got 26 but I thought about it differently. I thought it looked just like Tri-Square City but it had an extra block at the bottom. So I plowed Tri-Square City (which was a 20-block route) and then plowed 3 sides of the extra block twice. 20 plus 6 is 26. And my work matches my prediction, so I know there is a path of 26.

This student is using another reasoning method. The student views Cobblestone City as an earlier figure to which a new element has been attached. The student solves for the new town by combining the new element with the earlier solution. This is an example of using **recursion** to think about the problem.

MR. K: That's an interesting way to add blocks. So you might be able to add blocks in set patterns like modules and know the best route to take without trial and error.

STUDENT: But with a new block in the right place, you might take a different path that is even shorter.

STUDENT: Do the plows really find the shortest route this way?

MR. K: If they are looking for the shortest route and the problem is not too complicated, it can be worked out the way you are solving it. Large municipalities will have more than one plow, and would have to divide the territory into multiple routes, making the problem more complicated. People who solve those and other routing problems probably use sophisticated computer programs to help in the analysis. However, you cannot just push a button and come up with the best answer.

Creating routes for street sweeping and sanitation trucks have different requirements than for meter readers and letter carriers who can walk with or against traffic.

STUDENT: I found a problem. I live on a one-way street. What if the plow tries to go in the wrong direction?

MR. K: People who design the routes have to consider information such as traffic direction, number of lanes, time restrictions, and divided routes. There are also other considerations such as making good use of the available workers and avoiding long delays in plowing all streets. (Is it still snowing or has the snow stopped? How much snow needs to fall before the plows begin?) There is more to consider than just the efficiency of the plow routes.

Real world considerations affect what is an optimal solution. Once solutions are proposed, the problem can be restated to include more real world criteria (e.g., Can a plow make a u-turn? Do blocks differ in length?)

STUDENT: It's interesting to know that math can help people solve problems like this. I think I'd like to go back and include some one-way streets to see how that might affect our solutions.

Later, with help from their parents, a pair of students visited the Public Works department. The students saw a demonstration of the computer program that the department used to map out the optimal street routes for plowing. They were told, though, that the snow plow drivers often improvised based on weather and parking conditions that were not in the program.

Reality Check: Discrete Mathematics in the Workplace

Winning a game and running a successful business both require the right combination of skill and luck. When you are running a business and using *real* money, you need to minimize the role that luck plays. Like playing a game, you must examine the rules and the situation in order to create a strategy that will work.

In business, luck is called risk.

Warren B. manages a successful shopping mall, the Galleria in Edina, Minnesota. Many of the situations he must address to make the mall a success involve using what mathematicians call discrete mathematics, but Warren calls "careful strategizing." This is a mathematics of finite solutions, where he must generate lists of possible alternative approaches to his problems and choose the best solution. To help make a decision, he uses models and simulations similar to those used in game theory and risk analysis.

Does a solution exist? How many solutions are there? Is there an optimum solution? These are familiar questions asked in business and industry which discrete math addresses.

Ten years ago Warren learned that the Mall of America was planning to open not far from his mall. He decided that he would make a major change in his mall so it would remain successful and would not be in direct competition with the new mall. He needed to appeal to different customer needs and to make better use of his property. A major change like this involved many alternative solutions. Warren needed to consider them all and then to choose one with the greatest return (profit) for the least risk.

In trying to appeal to a different group of customer, Warren had several options. The mall could be "downscaled" with discount stores as tenants. But this did not seem to be an appropriate decision for a property located in an area with high income and employment. Alternatively, he could "upscale" his tenant roster. This would require renovating the property. While renovating, it might be feasible to build more square footage on the same amount of land, and thereby get a higher concentration of high-end retailers in one location. This concentration would make his property the preeminent high-end fashion mall in the Twin Cities—and remain outside the competitive realm of the Mall of America.

Reformulate a complex question into a simpler one. Solve that, then introduce some complication.

Though renovating and expanding the property would be a good marketing strategy, would it be economically profitable, as well as physically feasible? To determine its economic viability, Warren used spreadsheets to develop multiple projections (**sensitivity analyses**) in which he could adjust different assumptions, such as lease-up period, rental rate, retail sales, inflation, consumer economics conditions, and new competition. Each projection could employ varying assumptions. For every small change in an assumption, Warren wanted to know how much the worth of the property would change. For example, if rental rates increased by 5%, but expenses increased by 5% also, how much would his **operating income** (revenue minus expenses) increase or decrease?

Matrices and computers let you see many quantities and how changes will affect those quantities.

The answer depends, in part, on the ratio of revenue to expenses.

Warren had a general expectation of the level of return (profit on his cash investment) he needed to justify the risk he perceived he would be taking. Return is a function of the level of risk. To increase his return, he needed to take a greater risk in developing and operating his property. Given his desired level of return and risk, he selected an appropriate strategy.

Warren did not expect to make up the renovation expenses in the first year. Rather, he financed the cost with a mortgage and paid for the rest with cash (equity). For example, say the total renovation cost was $5,000,000 and $1,000,000 was paid in cash with a $4,000,000 mortgage at 10% annual interest rate covering the balance. If the property earned an operating income of $600,000 a year, he could then pay the $400,000 in interest and have $200,00 as a return (profit) on the cash.

Example of a spreadsheet analysis, similar to those Warren used. A formula embedded in a cell of the spreadsheet expresses relationships between/among other cells. Thus changing one of the values in a cell changes values in the cells which relate to it. The return on investment (ROI) can be viewed as it changes when other factors change.

Assumptions	
Total Property Size (Sq. Ft.)	450,000
Retail Property (%)	60%
Renovation Cost ($/Sq. Ft.)	$ 110.00
Expansion Cost ($/Sq. Ft.)	$ 145.00
Initial Equity ($)	$ 5,000,000
Mortgage Rate (APR)	12%
Inflation Rate (APR)	5
Rental Rate ($/Sq. Ft.)	$ 115.00
Base Occupancy Rate (%)	95%
Market Size (Units)	1,000,000
Market Share (%)	12%
Retail Sales ($/Market Unit)	$ 2,000.00

Economic Conditions	Probability
Depression	5%
Recession	40%
Normal	35%
Boom	15%

	Mortgage ($000)		**Operating Income ($000)**					**Operating Expenses**	
Proposed Expansion	Principal	Interest	Depression	Recession	Normal	Boom	Expected Value	Projected	ROI
-	$ 44,500	$ 5,340	$ 29,187	$ 33,845	$ 36,950	$ 38,502	$ 33,704.78	$ 30,250.00	(0.36)
5,000	$ 45,225	$ 5,427	$ 29,511	$ 34,221	$ 37,360	$ 38,930	$ 34,079.27	$ 30,262.50	(0.31)
10,000	$ 45,950	$ 5,514	$ 29,836	$ 34,597	$ 37,771	$ 39,358	$ 34,453.77	$ 30,275.00	(0.25)
15,000	$ 46,675	$ 5,601	$ 30,160	$ 34,973	$ 38,181	$ 39,785	$ 34,828.27	$ 30,287.50	(0.20)
20,000	$ 47,400	$ 5,688	$ 30,484	$ 35,349	$ 38,592	$ 40,213	$ 35,202.77	$ 30,300.00	(0.15)
25,000	$ 48,125	$ 5,775	$ 30,809	$ 35,725	$ 39,002	$ 40,641	$ 35,577.26	$ 30,312.50	(0.10)
30,000	$ 48,850	$ 5,862	$ 31,133	$ 36,101	$ 39,413	$ 41,069	$ 35,951.76	$ 30,325.00	(0.04)
35,000	$ 49,575	$ 5,949	$ 31,099	$ 36,477	$ 39,823	$ 41,497	$ 36,308.33	$ 30,337.50	0.00
40,000	$ 50,300	$ 6,036	$ 31,057	$ 36,853	$ 40,234	$ 41,924	$ 36,664.53	$ 30,350.00	0.05
45,000	$ 51,025	$ 6,123	$ 31,008	$ 37,229	$ 40,644	$ 42,352	$ 37,020.36	$ 30,362.50	0.10
50,000	$ 51,750	$ 6,210	$ 30,951	$ 37,605	$ 41,055	$ 42,780	$ 37,375.82	$ 30,375.00	0.15
55,000	$ 52,475	$ 6,297	$ 30,888	$ 37,458	$ 41,466	$ 43,208	$ 37,521.84	$ 30,387.50	0.16
60,000	$ 53,200	$ 6,384	$ 30,816	$ 37,301	$ 41,876	$ 43,636	$ 37,663.35	$ 30,400.00	0.17
65,000	$ 53,925	$ 6,471	$ 30,738	$ 37,134	$ 42,287	$ 44,063	$ 37,800.36	$ 30,412.50	0.17
70,000	$ 54,650	$ 6,558	$ 30,652	$ 36,956	$ 42,697	$ 44,491	$ 37,932.85	$ 30,425.00	0.18
75,000	$ 55,375	$ 6,645	$ 30,558	$ 36,768	$ 42,202	$ 44,919	$ 37,743.86	$ 30,437.50	0.13
80,000	$ 56,100	$ 6,732	$ 30,458	$ 36,570	$ 41,690	$ 45,347	$ 37,544.33	$ 30,450.00	0.07
85,000	$ 56,825	$ 6,819	$ 30,349	$ 36,361	$ 41,160	$ 45,775	$ 37,334.25	$ 30,462.50	0.01
90,000	$ 57,550	$ 6,906	$ 30,234	$ 36,142	$ 40,613	$ 46,202	$ 37,113.62	$ 30,475.00	(0.05)
95,000	$ 58,275	$ 6,993	$ 30,111	$ 35,913	$ 40,049	$ 46,630	$ 36,882.45	$ 30,487.50	(0.11)
100,000	$ 59,000	$ 7,080	$ 29,981	$ 35,673	$ 39,468	$ 47,058	$ 36,640.73	$ 30,500.00	(0.18)

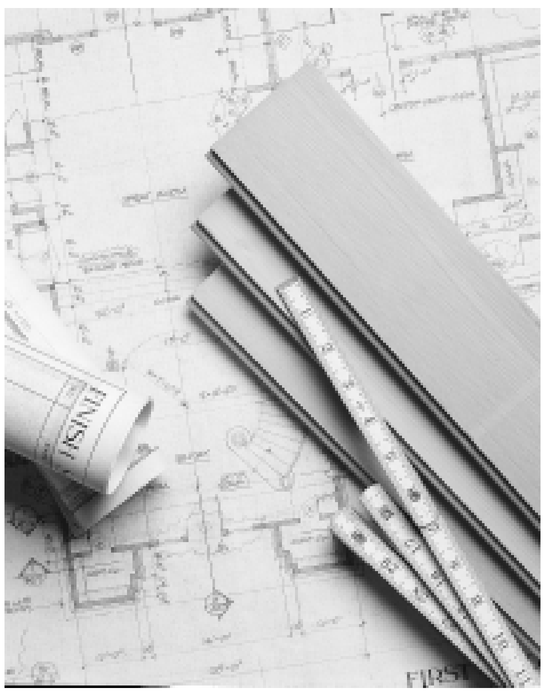
Assumptions are based on projections, probability and/or other industry practices. Assumptions, data, and results need to be tested for reasonableness in a particular context.

The mathematics of finance is an application of iteration. It is a type of constant refiguring known as **recursion**. The compounding of interest is one example.

Risk analysis aids decision making in fields such as insurance, product liability, and medical research.

This $200,000 would represent a 20% return on the $1,000,000 cash investment—a fine return if the risk were not too great. However, if the property earned only $350,000, he would not even have enough return to pay the mortgage interest and there would be no return on the cash. This projection of $350,000 would represent a financially unviable strategy.

Before the shovel went into the ground, Warren adjusted most of the variables in his projections until he felt comfortable with the plan for the property—its size, number of stores, rental rates, tenants and so on. Once the project was underway his flexibility became more limited. Getting his projections as close to "right" on paper was very important. Of course, there continue to be variables which he cannot control, like the inflation rate and new competition. Here, he must make sure that he has built enough "cushion" (or sensitivity) into his projections to accommodate some variation of expectations in the future.



With satisfactory projections in hand based upon his experiences and insight, Warren found a building contractor, a mortgage lender, equity, tenants and customers to shop in the mall. Though much can still go wrong—some within his control and other events outside his control—only by doing this careful planning could he have been able to understand the amount and variability of his risk and return.

How did Warren learn to make these kinds of decisions backed by sensitivity analysis and probabilities? In college, he majored in English and history. He went into real estate shortly after college and began reading the *Wall Street Journal, Forbes,* reports of the Urban Land Institute and reports of real estate forecasting companies. He also reads *Women's Wear Daily,* considers future retail trends and looks at neighboring store development. He talks to tenants and customers and uses good old common sense based on experience. Warren assembles information from many sources to apply to the property he manages, always with the goal of finding the "best" strategy. He says, "As financing and leasing options have become more complicated, no longer can this figuring be done on the back of an envelope. Computers and spreadsheet programs have made it possible for me to use mathematics. I need a broad understanding of how the assumptions interact and I need to know if what I get makes sense." In his work, Warren makes a positive contribution to the quality of life in the community, as well as minimizing his risk and maximizing his profit.

(These Reality Checks are included to remind us that it is our responsibility as teachers to prepare students for the real world — specifically for life-work.)

FOCUS

Students entering the primary grades have a natural curiosity about people and things in their world. This curiosity leads to questions about how things fit together or connect. Primary students display their natural need to organize things. They sort, compare, and label objects in a collection according to similarities and differences in a single attribute. They recognize that some things change over time, which leads to an intuitive sense for prediction.

PRIMARY

The focus of instruction at the primary level is to blend purposeful play with simple analysis to help students begin to develop early concepts in discrete mathematics. Students should be encouraged to recognize and work with repetitive patterns and processes involving numbers and shapes. They should investigate ways of sorting objects/people into sets according to attributes and ways of arranging data into charts and tables. Students should pursue activities involving systematic listing, counting, and arranging of people/objects in very concrete ways.

Students entering the intermediate grades identify, apply, and create patterns. They use visual representations to explore counting arrangements. They can follow, describe, and create practical lists of instructions (like how to make a peanut butter sandwich). These students can sort and classify persons/objects, visually represent these sorts, and use *all, some, none* to describe them.

INTERMEDIATE

The focus of instruction at the intermediate level is on students' systematic reflection on their work with patterns, sorting, arranging, and graphing. Intermediate students interpret statements involving the language *all, some, none* to determine when such statements are true and when they are false. They explore vertex-edge graphs to depict or describe mathematical or real-world relationships. They investigate number sequences related to patterns in nature. These students invent, describe and discuss simple algorithmic procedures and evaluate their efficiency. They begin to analyze games and problems, determining and listing all possible solutions, and should in simple cases determine and discuss what is the best solution to a problem.

Students entering the middle grades have used visual representation to explore problems involving arrangements or systematic counting. They are familiar with the notion of connected graphs and recognize situations in which graphs can be an appropriate model. They have explored iterative and recursive patterns in Logo programming activities or in following flow chart directions. These students can describe algorithms for solving a variety of problems.

MIDDLE SCHOOL

The focus of instruction at the middle school level is to make connections between topics in number, shape, chance, data, and change and problem solving situations in discrete mathematics. Middle school students devise, describe, and test algorithms. They apply number theory and the multiplication principle to problems involving permutations and combinations. They explore the properties of vertex-edge graphs and design them for specific situations, including listing all possible outcomes in a probability experiment. These students also explore methods for storing, processing, and communicating information.

Students entering the high school grades have applied discrete processes in a variety of different contexts. They are comfortable using vertex-edge graphs (and other visual representations), systematic counting techniques, and recursive approaches to model and solve problems.

HIGH SCHOOL

The focus of instruction at the high school graduation standard level is on the application of discrete mathematics as a tool to solve practical, real-world problems. High school students use vertex-edge graphs and matrices to represent and solve problems involving conflicts, optimization, and scheduling. They apply discrete mathematical models to solve problems involving the concept of fairness in matters of social choice and decision making. These students use recursive formulas to solve problems involving permutations, combinations, and growth patterns. They use calculators and computers to solve problems involving iterative and recursive processes. High school students apply algebraic concepts and skills to enhance their understanding of discrete mathematics and to generalize solution approaches to discrete problems.

Students working at the high school graduation standard level are comfortable representing and analyzing finite graphs using matrices. They have developed and analyzed algorithms using computer methods. They have investigated combinatorics, recursion, and sequences at an appropriate level.

BEYOND HS STANDARD

The focus of instruction beyond high school graduation standard level is to represent and solve more complex problems using methods of discrete mathematics. Students doing advanced work apply discrete techniques to analyze a variety of algorithms, including sorting and backtracking algorithms. They investigate and solve problems using linear programming and difference equations. These students apply the concepts and methods of computer technology, including set theory, the rules of logic, and computer validation to illustrate and apply the major ideas of discrete mathematics.

Components

LEARNING GOAL

Students use discrete structures to model and understand mathematical relationships and solve problems.

Components

With this learning goal in mind, Minnesota students will have the opportunity to:

1. investigate and apply systematic counting techniques, set relationships, and principles of logic to represent, analyze, and solve problems.
2. use charts, vertex-edge graphs, and matrices to model and solve problems.
3. explore, develop, and analyze algorithmic thinking to accomplish a task or solve a problem.
4. analyze, extend, and model iterative and recursive patterns.

1

Investigate and apply systematic counting techniques, set relationships, and principles of logic to represent, analyze, and solve problems.

Learning Goal: Use discrete structures to model and understand mathematical relationships and to solve problems.

PRIMARY K-2

- use manipulatives, diagrams, and lists to explore problems involving counting and arranging objects (see *Sample Problems* P1-1)
- sort and classify objects (see *Sample Problems* P1-2)
- understand logical terms such as *and, or, not, some*, and *all*, and use them correctly (see *Sample Problems* P1-3)

INTERMEDIATE 3-5

- make systematic lists of permutations and combinations for small sets (see *Sample Problems* I 1-1)
- sort and classify objects using two or more attributes (see *Sample Problems* I 1-2)
- express a verbal rule to describe a given set of objects; sort objects to follow a given rule (see *Sample Problems* I 1-3)
- use Venn diagrams or other diagrams to interpret *and, or,* and *not* terminology (see *Sample Problems* I 1-4)

Excerpts from MN Standards Related to Component #1

MN State Excerpts

Primary Level: K-2

Mathematics: Number Sense
- use whole numbers to represent numbers in more than one way *(e.g., manipulatives, pictures, diagrams, symbols),* count and order
- solve problems and justify thinking…use concrete objects, diagrams or maps to solve simple problems involving counting, arrangements or routes

Intermediate Level: 3-5

Mathematics: Number Sense
- use lists or diagrams to solve counting and arrangement problems

MIDDLE SCHOOL 6-8

- use manipulatives, diagrams, or systematic lists to develop counting strategies and apply them in appropriate situations (see *Sample Problems* M1-1)
- interpret set relationships and notation from real-world contexts to represent algebraic concepts (see *Sample Problems* M1-2)

HIGH SCHOOL 9-12

- use concepts of order and repetition to differentiate situations involving permutations and combinations or other techniques of counting (see *Sample Problems* H1-1)
- apply permutation and combination formulas and the multiplication principle to solve multi-step counting problems (see *Sample Problems* H1-2)
- explore the combinatorial interpretation of Pascal's Triangle (see *Sample Problems* H1-3)
- explore the pattern of coefficients in the expansion of binomials leading to the binomial theorem (see *Sample Problems* H1-4)
- interpret logic relationships represented in algebraic notation (see *Sample Problems* H1-5)
- understand and use conditional logic statements (see *Sample Problems* H1-6)

BEYOND HS STANDARD

- apply the inclusion/exclusion principle and the pigeonhole principle to solve sorting problems (see *Sample Problems* B1-1)
- extend knowledge of set theory to include partitioning of sets (see *Sample Problems* B1-2)
- understand and apply identities involving binomial coefficients, e.g.,

  $$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$$

  for k such that $0 \le k \le n$ (see *Sample Problems* B1-3)
- use combinatorial arguments to justify binomial identities (see *Sample Problems* B1-4)
- extend logic to include predicate calculus (see *Sample Problems* B1-5)
- develop and interpret truth tables and computer logic networks (see *Sample Problems* B1-6)

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Sample Problems aligned with the grid indicators begin on page 29. A Brief Glossary for Discrete Mathematics begins on page 59.

Mathematics: Patterns & Functions

- recognize, analyze and generalize patterns found in…data from lists, graphs and tables
- translate algebraic expressions into equivalent forms…in problem situations
- use properties of mathematics to informally justify reasoning in a logical argument

Middle Level: 6-8

Mathematics: Discrete Mathematics

- solve problems by… permutations, combinations and other principles of systematic counting
- use properties of mathematics to justify reasoning in a logical argument

Mathematics: Chance & Data Analysis

- demonstrate understanding of concepts related to uncertainty of randomness, permutations, combinations

Minnesota High School Graduation Standard

2

Use charts, vertex-edge graphs, and matrices to model and solve problems.

Learning Goal:
Use discrete structures to model and understand mathematical relationships and to solve problems.

PRIMARY K-2

- explore the use of diagrams and simple charts to represent all possible outcomes of an event or experiment (see *Sample Problems* P2-1)
- use maps and vertex-edge graphs to study simple routing and tracing problems (see *Sample Problems* P2-2)

INTERMEDIATE 3-5

- use diagrams, vertex-edge graphs, and simple charts to represent all possible outcomes of an event or experiment (see *Sample Problems* I 2-1)
- use tree diagrams to represent steps in a decision or choices in a problem situation (see *Sample Problems* I 2-2)
- use a vertex-edge graph to interpret relationships between or among objects (e.g., persons, events, locations) (see *Sample Problems* I 2-3)
- explore the conditions for the traceability of a vertex-edge graph (see *Sample Problems* I 2-4)

Excerpts from MN Standards Related to Component #2

MN State Excerpts

Primary Level: K-2

Inquiry: Data Categorization, Classification & Recording
- display gathered information using the appropriate format *(e.g., graphs, diagram, maps)*

Mathematics: Number Sense
- solve problems and justify thinking…organize data using pictures and charts; use concrete objects, diagrams or maps to solve simple problems involving counting, arrangements or routes

Intermediate Level: 3-5

Mathematics: Number Sense
- represent real-life situations mathematically
- use lists or diagrams to solve counting and arrangement problems

Mathematics: Chance & Data Handling
- conduct experiments involving uncertainty; list possible outcomes

MIDDLE SCHOOL 6-8

- extend tree diagrams to represent factors of a number and outcomes in a probability experiment
(see *Sample Problems* M2-1)
- use a systematic approach to identify, represent, and record all possible paths or outcomes in an experiment
(see *Sample Problems* M2-2)
- design vertex-edge graphs to represent relationships between or among objects
(see *Sample Problems* M2-3)
- investigate some classical graph theory problems (for example: highway inspector problem, garbage collection problem, traveling salesperson problem)
(see *Sample Problems* M2-4)

HIGH SCHOOL 9-12

- use vertex-edge graphs to solve discrete optimization problems (e.g., critical path problems)
(see *Sample Problems* H2-1)
- use matrices to represent connectivity or adjacency in vertex-edge graphs
(see *Sample Problems* H2-2)
- operate on matrices to obtain information about a vertex-edge graph and the situation it models
(see *Sample Problems* H2-3)
- use a systematic approach to identify, represent, and record all possible paths or outcomes in complex experiments
(see *Sample Problems* H2-4)

BEYOND HS STANDARD

- apply graph theory to solve complex problems
(see *Sample Problems* B2-1)

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Sample Problems aligned with the grid indicators begin on page 29. A Brief Glossary for Discrete Mathematics begins on page 59.

Mathematics:
Shape, Space & Measurement

- use vertex-edge graphs to solve problems

Middle Level: 6-8

Mathematics: Discrete Mathematics

- describe the difference between discrete and continuous models of data
- translate between real world situations and discrete mathematical models using vertex-edge graphs, matrices, verbal descriptions, and sequences
- analyze and solve problems by building discrete mathematical models, developing and comparing algorithms or sequences of procedures, and determining whether solutions exist, the number of possible solutions, and the best solutions

Minnesota High School Graduation Standard

3

Explore, develop, and analyze algorithmic thinking to accomplish a task or solve a problem.

Learning Goal: Use discrete structures to model and understand mathematical relationships and to solve problems.

PRIMARY K-2

- follow a simple set of directions to accomplish a task (see *Sample Problems* P3-1)
- develop and communicate directions for a simple task verbally and in writing (see *Sample Problems* P3-2)
- develop and communicate directions for solving a problem involving computation (see *Sample Problems* P3-3)
- compare and evaluate multiple solution strategies (see *Sample Problems* P3-4)

INTERMEDIATE 3-5

- develop lists, illustrations, or flow charts to describe a sequence of events (see *Sample Problems* I 3-1)
- invent algorithms to accomplish a task or solve a problem involving computation; describe how they work and evaluate which are most useful (see *Sample Problems* I 3-2)
- follow a flow chart to accomplish a task (see *Sample Problems* I 3-3)

Excerpts from MN Standards Related to Component #3

MN State Excerpts

Primary Level: K-2

Mathematics: Number Sense

- solve problems and justify thinking by selecting appropriate numbers and representations…generate multiple solutions

Mathematics: Shape, Space & Measurement

- demonstrate an understanding of patterns…represent spatial patterns pictorially, numerically, or both

Intermediate Level: 3-5

Mathematics: Number Sense

- generate and describe more than one method to solve problems

Mathematics: Shape, Space & Measurement

- use maps or graphs to determine the most efficient routes

MIDDLE SCHOOL 6-8

- use manipulatives, drawings, or descriptions to demonstrate algorithmic thinking (see *Sample Problems* M3-1)
- explain the output of an algorithm described by a flow chart (see *Sample Problems* M3-2)
- create, communicate, and defend a strategy for winning a game (see *Sample Problems* M3-3)
- create and communicate algorithms to solve problems (see *Sample Problems* M3-4)

HIGH SCHOOL 9-12

- investigate algorithms related to problems in graph theory (e.g., planning and scheduling problems) (see *Sample Problems* H3-1)
- investigate algorithmic thinking to solve problems involving social choice and decision making (e.g., weighted voting, fair division, apportionment) (see *Sample Problems* H3-2)
- investigate applications of information coding in real world contexts (e.g., bar codes, zip codes) (see *Sample Problems* H3-3)
- develop algorithms for solving mathematical puzzles (see *Sample Problems* H3-4)

BEYOND HS STANDARD

- explore and analyze a variety of algorithms (e.g., simple sorting algorithms, data coding, backtracking) (see *Sample Problems* B3-1)
- translate algorithms into simple calculator or computer programs to investigate the interaction of variables (see *Sample Problems* B3-2)

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Sample Problems aligned with the grid indicators begin on page 29. A Brief Glossary for Discrete Mathematics begins on page 59.

Mathematics: Number Sense

- analyze and justify operations and methods used and evaluate the reasonableness of computed results to problems with proposed solutions

Mathematics: Patterns & Functions

- represent and interpret cause and effect relationships using…verbal descriptions
- use properties of mathematics to informally justify reasoning in a logical argument

Middle Level: 6-8

Mathematics: Discrete Mathematics

- analyze and solve problems by building discrete mathematical models, developing and comparing algorithms or sequences of procedures, and determining whether solutions exist, the number of possible solutions, and the best solutions

Minnesota High School Graduation Standard

4

Analyze, extend, and model iterative and recursive patterns.

Learning Goal: Use discrete structures to model and understand mathematical relationships and to solve problems.

PRIMARY K-2

- explore simple patterns and sequences (see *Sample Problems* P4-1)
- recognize patterns and symmetries in designs and artwork (see *Sample Problems* P4-2)

INTERMEDIATE 3-5

- represent iterative patterns in words, pictures, and numbers (see *Sample Problems* I 4-1)

Excerpts from MN Standards Related to Component #4

MN State Excerpts

Primary Level: K-2

Mathematics: Number Sense
- use whole numbers to… describe and extend patterns
- solve problems and justify thinking…use operations, patterns and estimation

Mathematics: Shape, Space & Measurement
- demonstrate an understanding of patterns…identify and/or create symmetrical patterns

Intermediate Level: 3-5

Mathematics: Number Sense
- represent patterns using words, pictures and numbers

Mathematics: Chance & Data Handling
- describe patterns, trends or relationships in data displayed in graphs, tables or charts

Mathematics: Shape, Space & Measurement
- extend or create geometric patterns to solve problems

MIDDLE SCHOOL 6-8

- explore patterns in more complex geometric and numerical problems (e.g., Fibonacci sequence, Pascal's triangle) (see *Sample Problems* M4-1)
- examine patterns in an iteration to predict successive terms and long-range trends (see *Sample Problems* M4-2)
- employ recursion in counting contexts (e.g., moves in Tower of Hanoi puzzle) (see *Sample Problems* M4-3)

HIGH SCHOOL 9-12

- apply inductive reasoning to determine the formula for a general term in a sequence (see *Sample Problems* H4-1)
- compute the first few terms of a recursively-defined sequence (see *Sample Problems* H4-2)

BEYOND HS STANDARD

- use mathematical induction to construct a proof (see *Sample Problems* B4-1)
- write computer code to generate the first few terms of a recursively-defined sequence (see *Sample Problems* B4-2)

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Sample Problems aligned with the grid indicators begin on page 29. A Brief Glossary for Discrete Mathematics begins on page 59.

Mathematics: Patterns & Functions
- recognize, analyze and generalize patterns found in linear and non-linear phenomena…sequences
- connect verbal, symbolic and graphical representations…in problem situations

Middle Level: 6-8

Mathematics: Discrete Mathematics
- analyze and model iterative and recursive patterns

Minnesota High School Graduation Standard

Sample Problems

Grid Alignment

Primary Sample Problems

P1-1

use manipulatives, diagrams, and lists to explore problems involving counting and arranging objects

Tom has three different shirts (red, green, and blue) and two different pairs of pants (black, tan). How many different outfits can he make?

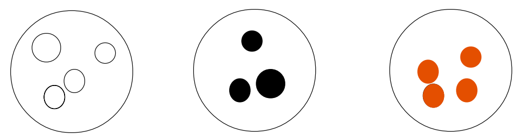
[Have identical pictures of Tom so the children can color the different outfits or have actual paper cut outs of shirts and pants for students to physically manipulate as they explore this problem.]

How many possible outfits are there when Tom wears black pants?

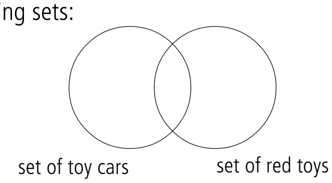
P1-2

sort and classify objects

Use yarn or plastic hoops to sort buttons by one characteristic (color, size, texture, number of holes, etc.)



Later, to explore two attributes, students can discuss an appropriate way to express the sorting of two intersecting sets:



P1-3

understand logical terms such as *and, or, not, some* and *all*, and use them correctly.

Create projects that allow for discussion of the data using the words *and, or, not, all, some, none*. For example, have every child contribute his/her right shoe. Ask them how they might sort them: color, type of shoe (tennis, sneaker, boot, etc.), type of fastener (laces, Velcro, buckle), color of sole. The discussion would yield statements such as: "These shoes have rubber or leather soles," and "Some of the shoes are brown," and "None of the shoes are yellow."

P2-1

explore the use of diagrams and simple charts to represent all possible outcomes of an event or experiment

I have two coins in my pocket. How much money might I have? A chart might begin:

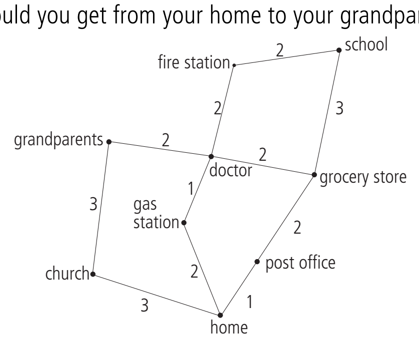
1	5	10	25	AMOUNT
x x				2¢
x	x			6¢
x		x		11¢

[Have actual money available for students to use to solve this problem.]

P2-2

use maps and vertex-edge graphs to study simple routing and tracing problems

How would you get from your home to your grandparents' house? What is the shortest route?

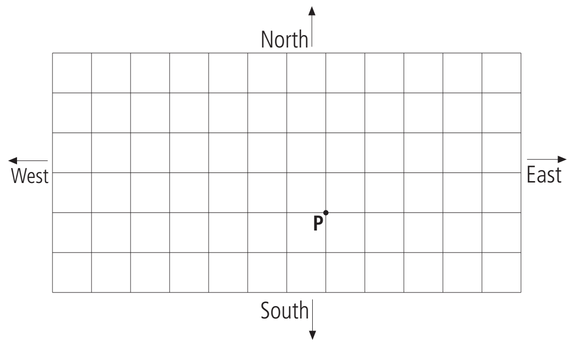


You need to go to the doctor and stop at school. What is the shortest distance if you are at home and want to return home?

P3-1

follow a simple set of directions or simple flow chart to accomplish a task

On the map shown below, where will you be if you start at point P, move two spaces east, then four spaces north



P3-2

develop and communicate directions for a simple task verbally and in writing

Give directions for making your favorite kind of sandwich.

Draw a map for getting to a location in the school if you start at the classroom door.

Explain to someone how to find the total of 36¢ and 45¢.

P3-3

develop and communicate directions for solving a problem involving computation

Candy mints cost 4 cents apiece, bubble gum cost 5 cents, and suckers cost 6 cents. Explain how you would determine how much three mints, two bubble gums, and three suckers would cost.

P3-4

compare and evaluate multiple solution strategies

Carla and Deng solved the candy problem above (P3-3) differently. Here is their work. Explain what each student was thinking. Which strategy would you have used? Why? Can you find another way to solve it?

Carla	Deng
4¢ + 4¢ + 4¢ + 5¢ + 5¢ + 6¢ + 6¢ + 6¢ = 40¢	4¢ + 6¢ = 10¢
	3 tens = 30¢
	5¢ + 5¢ = 10¢
	30¢ + 10¢ = 40¢

P4-1

explore simple patterns and sequences

Play the "Guess My Pattern" game. Think of a "rule" that will establish a pattern, such as a sequence of girl, boy, girl, boy, girl, boy…Have four children that fit the pattern stand in a row. Add children to the line and indicate whether or not they fit the pattern. If they do, they stay; if not, they sit down. Have children try to guess the pattern, but not to divulge it till after everyone has had a chance to think about it. Rules might involve length of hair, color of shirt, etc.

P4-2

recognize patterns and symmetries in designs and artwork

Students take a "pattern walk" through the school, searching for patterns in the bricks, the play equipment, the shapes in the classrooms, the pattern of numbering classrooms, patterns of placement and design of classrooms, patterns in the floors and ceilings, etc.; the purpose is to create an awareness of all the patterns and symmetries which surround them daily.

Grid Alignment

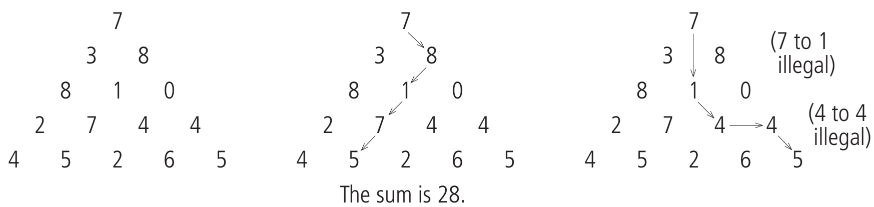
Intermediate Sample Problems

I 1-1

make systematic lists of permutations and combinations for small sets

A pizza parlor offers six toppings: sausage, anchovies, onions, olives, mushrooms, and tomatoes. Make a list of all the different pizzas having two different toppings.

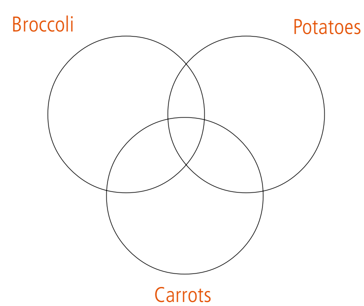
The following figure shows a number triangle. What is the largest sum of numbers on a path that starts at the top and ends somewhere on the base? Each step can go either diagonally down to the left or diagonally down to the right.



I 1-2

sort and classify objects using two or more attributes

Have each student indicate whether they like three items: e.g., broccoli, carrots, and potatoes. Clarify that students can like more than one of the items. Compile the results and indicate the numbers in each part of a 3-attribute Venn diagram. Write about the results.



I 1-3

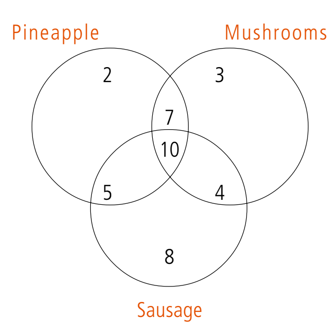
express a verbal rule to describe a given set of objects; sort objects to follow a given rule

Have students write a "clue card" for a given object in a set. Test the clues by seeing if other students can use them to identify the same object. "My mystery shape has five faces and five vertices. Four of the faces are triangles and one is a square. Which is my object? What is its name?"

I 1-4

use Venn diagrams or other diagrams to interpret *and, or,* and *not* terminology

The following Venn diagram shows the number of students who enjoy various combinations of sausage, mushrooms, and pineapple.



T/F Questions:

- All students who like sausage also like mushrooms or pineapple.
- At least one student likes mushrooms, but neither sausage nor pineapple.
- At least one student likes all three.

Other questions:

How many students like sausage and mushrooms, but not pineapple?

How many students like only pineapple, only sausage, or only mushrooms?

I 2-1

use diagrams, vertex-edge graphs, and simple charts to represent all possible outcomes of an event or experiment

Extend this tree diagram to represent and analyze the outcomes if a coin is flipped three times. In how many outcomes are at least two heads obtained?

I 2-2

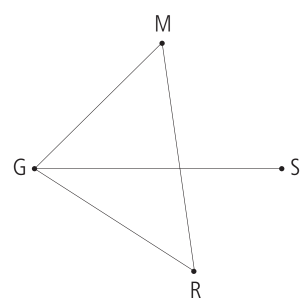
use tree diagrams to represent steps in a decision or choices in a problem situation

Abby and Michael play the following game. Before them is a pile of five pennies. Abby begins, and can take either one or two pennies. Michael follows and can take either one or two pennies from those that remain. They alternate turns until there are no coins left. The winner is the player who takes the last coin. Draw a tree diagram to represent the game possibilities and determine who wins in each case. Generalize by repeating for 6 pennies or using different rules for removing pennies.

I 2-3

use a vertex-edge graph to interpret relationships between or among objects (e.g., persons, events, locations)

Mr. Butler bought four different kinds of fish—guppies (G), mollies (M), swordfish (S), and gold rams (R). Some of the fish can live in the same tank, but others cannot. The vertices on the vertex-edge graph represent the type of fish. An edge connecting two vertices means those two fish types can live in the same tank. Answer the questions by using the information in the graph.



T/F Questions:

- All the fish could be put in the same tank.
- Swordfish can be placed in the same tank as mollies.
- Guppies can share a tank with any other fish.

Other Questions:

If you have three tanks, can you house all the fish? Explain. If there is more than one possible placement of fish in the tanks, list the possibilities.

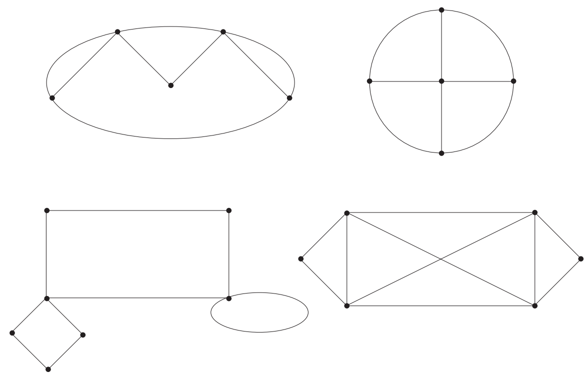
If you have two tanks, can you house all the fish? Explain. If there is more than one possible placement of fish in the tanks, list the possibilities.

[You might use paper cutouts representing the fish and drawings of tanks on paper to help students solve these problems.]

I 2-4

explore the conditions for the traceability of a vertex-edge graph

State whether you can trace (or draw) each graph below without lifting your pencil and without repeating any edge.



Draw some of your own and see if you can find a rule about when it will be possible to trace a graph. Does it make a difference if you must return to where you started?

I 3-1

develop lists, illustrations, or flow charts to describe a sequence of events

Work in your small group to write a flow chart or list to describe organizing a checkers tournament that has 16 participants for your school. The flowchart should account for the possibility of a draw.

Teachers: Select a cartoon strip from the newspaper. Cut the frames apart and mix them up. Have students arrange the frames in order to make a reasonable story. Have them explain the reasons for the order they chose. Ask whether there is more than one reasonable order. A variation of this is to have the students draw a panel cartoon or story of their own, cut the frames apart, and exchange them with another student to reorganize.

I 3-2

invent algorithms to accomplish a task or solve a problem involving computation; describe how they work and evaluate which are most useful

Working in groups, students create and explain a fair way of sharing a bagful of similar candies or cookies. For example, if the bag has 30 brownies and there are 20 children, then they might suggest that each child gets one whole brownie and that the teacher divides each of the remaining brownies in half. Or they might suggest that each pair of children figure out how to share one brownie. *What if there were 30 hard candies instead of brownies? What if there were 25 brownies? What if there were 15 brownies and 15 chocolate chip cookies?* The purpose of the activity is for students to brainstorm possible solutions in the situations where there may be no solution that *everyone* perceives as fair.
–adapted from *New Jersey Mathematics Curriculum Framework,* 1996, p. 459

You are given two rulers, but with no marks on either one. One ruler has a length of 5 units and the other a length of 8 units. Explain how you can mark off a length of 1 unit using these two rulers. If one ruler has a length of 6 units and another 16 units, what is the smallest unit that can be marked off using only these two rulers?

I 3-3

follow a flow chart to accomplish a task

Follow the directions in the flowchart:

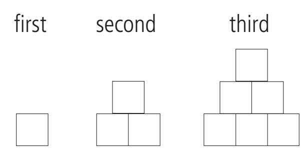
Think of a number
↓
Double it
↓
Add 8
↓
Multiply by 5
↓
Subtract 30
↓
Double the result
↓
Subtract 20
↓
Multiply by 5

Make a table comparing *Start* numbers with their *End* numbers. What pattern do you see? Explain your pattern.

I 4-1

represent iterative patterns in words, pictures, and numbers

Show what the next three tile structures will look like.



Explain how you will build the tenth structure in the pattern.

Grid Alignment

M1-1

use manipulatives, diagrams, or systematic lists to develop counting strategies and apply them in appropriate situations

M1-2

interpret set relationships and notation from real-world contexts to represent algebraic concepts

M2-1

extend tree diagrams to represent factors of a number and outcomes in a probability experiment

M2-2

use a systematic approach to identify, represent, and record all possible paths or outcomes in an experiment

Middle School Sample Problems

Sequences of zeroes and ones are used to represent messages in computer code. A bit string of length 4 is a sequence of four characters (bits) each of which is 0 or 1.
How many such bit strings are there?
How many of these begin and end in 1?
How many contain exactly one 1?

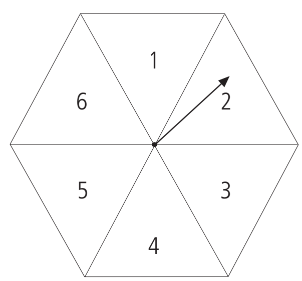
During the winter months, Mrs. G's comfort zone is between temperatures of 68°F to 75°F. Mr. G is comfortable at temperatures of 65°F to 70°F.

a) Write each person's comfort zone as an inequality.

b) Write the set of temperatures for which both Mr. and Mrs. G. are comfortable.

c) How is the set in part (b) related to the sets in part (a)

The spinner shown has 6 congruent regions. Use a tree diagram to help you list all the outcomes that can occur, if you spin twice.



Fifty-five delegates to a convention must vote on their choice of five candidates A, B, C, D, E. The following chart shows how many prefer particular rankings. (These rankings were agreed on in party caucuses.)

	Number of Delegates					
	18	12	10	9	4	2
First Choice	A	B	C	D	E	E
Second Choice	D	E	B	C	B	C
Third Choice	E	D	E	E	D	D
Fourth Choice	C	C	D	B	C	B
Fifth Choice	B	A	A	A	A	A

a) Who wins the election if everyone votes for her/his favorite candidate?

b) Who wins if everyone votes for her/his preferred candidate, and a runoff election is held between the top two finishers?

c) Who wins if everyone votes for her/his preferred candidate, but the candidate with the fewest votes is eliminated and another election is held? The candidate with the lowest total is eliminated each time until only one candidate remains.

d) Can some candidates beat all the others in a two-way race?

M2-3

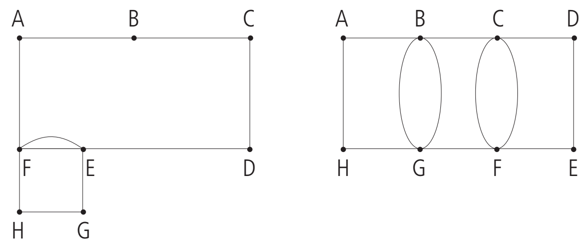
design vertex-edge graphs to represent relationships between or among objects

Five girls played in a round-robin tennis tournament, with every girl playing every other girl. The results were the following: Adria beat Diana; Bobbi beat Adria, Chandra, and Diana; Chandra beat Adria, Ellen, and Diana; Diana beat Ellen; Ellen beat Adria and Bobbi. Represent the tournament results with a graph, by letting the vertices be the players and drawing an arrow from one player to a second player if the first player beat the second. Who is the dominant player? Explain your answer.

M2-4

investigate some classical graph theory problems (for example: highway inspector problem, garbage collection problem, traveling salesperson problem)

Shown below are graph models for two sections of a town. Street corners are represented by the vertices with parking meters placed along the sidewalks (edges).



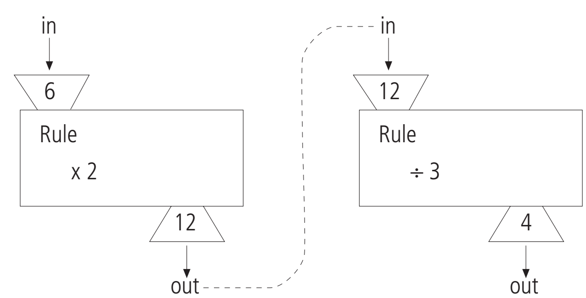
a) Why would it be helpful for a parking control officer to know if these graphs had Euler circuits?

b) Does each graph have an Euler circuit? Explain.

M3-1

use manipulatives, drawings, or descriptions to demonstrate algorithmic thinking

When Sam sees a problem like: $2/3 \times 6$, he thinks about function machines like the ones below. The (x 2) function machine followed by the (÷3) function machine can be used to transform 6 into 4.



Draw the multiplication and division function machines that Sam could use to transform:

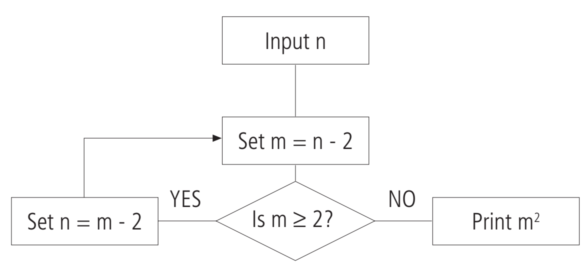
1) 15 → 10
2) 4 → 6
3) 9 → 12

What fraction multiplication problem is represented by each drawing?

M3-2

explain the output of an algorithm described by a flow chart

Explain the output for a given positive integer n.



What is the printed output when the input is 1? 2? 3? 4? 5? n?

M3-3

create, communicate, and defend a strategy for winning a game

Play the "Guess My Number" game. I have a number between 1 and 1000, and you are to guess it by asking 10 or fewer questions of the form:

Is it more than ______?

Is it less than ______?

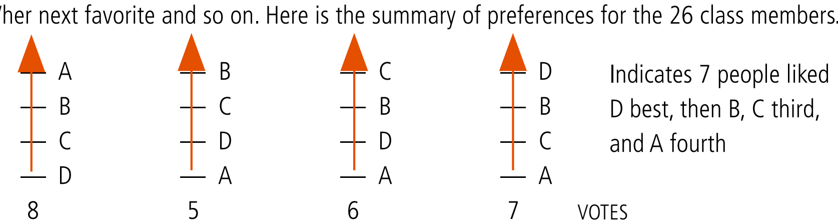
Is it equal to ______?

Describe a strategy that will allow you to always guess the number with 10 or fewer questions. How many questions are needed if the number is between 1000 and 2000? How many questions are needed if the number is between 1 and 2000?

M3-4

create and communicate algorithms to solve problems

Each student ranked four soft drinks A, B, C, and D by writing a 1 next to his/her favorite, a 2 by his/her next favorite and so on. Here is the summary of preferences for the 26 class members.



Develop an algorithm for determining a class ranking of the four soft drinks.

How would you handle ties?

—adapted from Crisler, N., Fisher, P., & Froelich, G., 1994, pp. 1-8.

M4-1

explore patterns in more complex geometric and numerical problems (e.g., Fibonacci sequence, Pascal's triangle)

Generate the possibilities of coin tossing experiments. The lists will look like this for 1, 2, and 3 coins.

For 1 coin:	For 2 coins:	For 3 coins:
1. H	1. HH	1. HHH
2. T	2. HT	2. HHT
	3. TH	3. HTH
	4. TT	4. HTT
		5. THH
		6. THT
		7. TTH
		8. TTT

Now replace each H with a zero and each T with a one. How do the results relate to binary numbers? Use this pattern to predict the results of tossing 4, 5, and 6 coins.

Extension: Tabulate the occurrences of H in decreasing order.
Thus, 1 coin becomes
1 Head *once*
0 Heads *once* → 1 1

2 coins gives
2 Heads *once*
1 Head *twice*
0 Heads *once* → 1 2 1

3 coins gives
3 Heads *once*
2 Heads *three times*
1 Head *three times*
0 Heads *once* → 1 3 3 1

How are the results related to Pascal's triangle?

M4-2

examine patterns in an iteration to predict successive terms and long-range trends

Think of a number with two or more digits. Add the digits. If the sum is more than 9, add the digits of the resulting number. Repeat this until the sum of the digits is less than or equal to 9. Then go back and divide your original number by 9. What is the remainder?

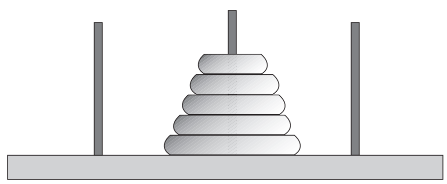
For example, if your number is 42,317 then add all the digits: 4 + 2 + 3 + 1 + 7 = 17 and then 1 + 7 = 8. Now divide 42,317 by 9. What is the remainder?

Try this with several different numbers. Make a conjecture based on several trials of this procedure.

M4-3

employ recursion in counting contexts (e.g., moves in Tower of Hanoi puzzle)

Common throughout India and Asian countries, this puzzle consists of three pegs and several disks of gradually increasing diameters with holes in their centers. The disks are stacked from largest on the bottom to smallest on top as shown below. If the disks are moved one at a time to another peg and if a larger disk cannot be placed on a smaller disk, what is the fewest number of moves needed to move the entire stack of disks to another peg, so that they are arranged from largest on the bottom to smallest on top? Explore the minimum number of moves needed to transfer various stacks from 1 to 5 disks tall.



Learn how to solve this puzzle and then describe a solution in terms of recursion.

(See also Sample Activity for Middle School)

Grid Alignment

High School (Graduation Standard Level) Sample Problems

H1-1

use concepts of order and repetition to differentiate situations involving permutations and combinations or other techniques of counting

A committee of three is to be formed from among four men (Adam, Dan, Eric, Fong) and four women (Becca, Cat, Gena, and Heather).

In how many ways can this be done:

a) if Becca is on the committee?

b) if Cat and Dan are not on the committee?

c) if the committee has exactly 2 men on it and Becca is not on it?

d) if Adam and Becca refuse to serve on it together?

H1-2

apply permutation and combination formulas and the multiplication principle to solve multi-step counting problems

Recently the phone company had to divide the Twin Cities area into more area codes. This has happened recently in many larger cities. If an area code cannot begin with zero, how many area codes are possible? Determine the largest number of different phone numbers that can be served using one area code if the following rules are given for selecting numbers:

a) Numbers have 7 digits (in addition to the area code).

b) Numbers cannot begin with 0 or 1.

c) Numbers cannot begin with 555.

H1-3

explore the combinatorial interpretation of Pascal's Triangle

Use Pascal's triangle to explore the identity

$$\begin{bmatrix} n \\ k-1 \end{bmatrix} + \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n+1 \\ k \end{bmatrix} \quad \text{where } 1 \le k \le n$$

H1-4

explore the pattern of coefficients in the expansion of binomials leading to the binomial theorem

Evaluate $(x+y)^n$ for integral values of $n \geq 0$. Look at the coefficients of the resulting values to find a pattern. Try to predict the coefficients of any power of $(x+y)$.

H1-5

interpret logic relationships represented in algebraic notation

Rewrite the following formal statements into equivalent sentences that a sixth grader could understand.

a) For all real numbers x, $x^2 \geq 0$.

b) For all real numbers x, $x^2 \neq -1$.

c) There exists an integer m, such that $m^2 = m$.

d) There exists a rational number p/a which solves $2x^2 + 7x - 30 = 0$

H1-6

understand and use conditional logic statements

A teacher makes this promise at the beginning of the course: "If a student's test scores total more than 700, then that student will get an A for the course." At the end of the course, John, who is a student enrolled in this course, has test scores which total 685. The teacher gives John an A. Has the teacher kept the promise? Explain.

H2-1

use vertex-edge graphs to solve discrete optimization problems (e.g., critical path problems)

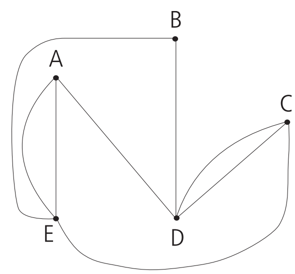
Determine the six largest cities in Minnesota. (If you live in one of the six largest cities, find and use the seventh-largest city also.) Consult a road atlas or other source to construct a graph that represents the road distance from your hometown to these six cities. Pick the minimum tour route to visit each city and return home.

–based on COMAP, *For All Practical Purposes*, 1997, p. 74.

H2-2

use matrices to represent connectivity or adjacency in vertex-edge graphs

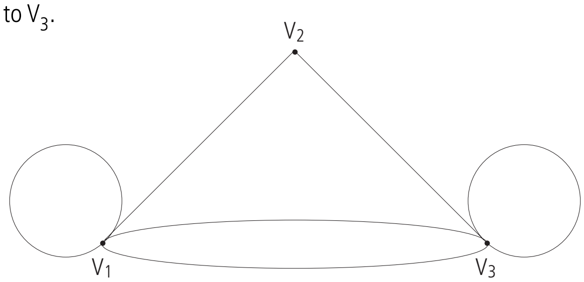
The graph represents five cities in the county: Ada, Beaumont, Canby, Danube, and Elgin. An edge between any two vertices indicates the cities are connected by a paved road. Use an adjacency matrix to represent the same information.



H2-3

operate on matrices to obtain information about a vertex-edge graph and the situation it models

Set up and use the adjacency matrix for this graph and find the number of walks of length 3 from V_1 to V_3.



H2-4

use a systematic approach to identify, represent, and record all possible paths or outcomes in complex experiments

Two teams are playing each other in a tournament. A team is declared the winner if it wins three games in a row or a total of 4 games. Determine how many outcomes there are for the tournament. List the outcomes.

H3-1

investigate algorithms related to problems in graph theory (e.g., planning and scheduling problems)

Three college roommates together decided to cook a special dinner for their friends. They broke down the project into five component activities with time estimates as follows:

K	clean house	30 minutes
D	decide on menu	15 minutes
P	purchase food	60 minutes
C	cook food	50 minutes
S	set table	10 minutes
F	place prepared food on table	4 minutes

Decide on a reasonable activity analysis for this job. You may decide which activities precede others but be prepared to justify your choices. Determine the optimal scheduling and the minimum time corresponding to the optimal scheduling.

H3-2

investigate algorithmic thinking to solve problems involving social choice and decision making (e.g., weighted voting, fair division, apportionment)

Six supervisors in Nassau County, NY were given weighted votes to compensate for the unequal populations in their counties. A simple majority of 16 votes is needed to pass a measure. Find all the minimal winning coalitions given the following table and the county charter provision which says support must include supervisors from at least two different municipalities. (Hempstead 1 and 2 are parts of the same municipality.)

Weighted Voting, Nassau County Board of Supervisors, 1958

Municipality	Number of Votes
Hempstead 1	9
Hempstead 2	9
North Hempstead	7
Oyster Bay	3
Glen Cove	1
Long Beach	1
Total	30

–adapted from: COMAP, *For All Practical Purposes*, 1997, p. 445

H3-3

investigate applications of information coding in real world contexts (e.g., bar codes, zip codes)

A check digit is the last digit entered in a data code. Companies use check digits for error detection. In the case of United Parcel Service (UPS), it is the remainder when the number (without the check digit) is divided by 7. Evaluate the accuracy for a package identified by the UPS number 873345672.

–Source: COMAP, *For All Practical Purposes*, 1997, p. 373

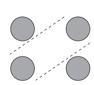
H3-4

develop algorithms for solving mathematical puzzles

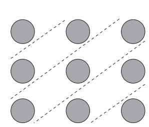
Find all the ways in which six queens can be placed on a 6 by 6 chessboard so that none can attack any other.

H4-1

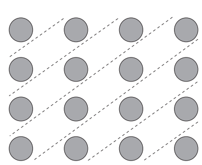
apply inductive reasoning to determine the formula for a general term in a sequence



$1 + 2 + 1 = 2^2$



$1 + 2 + 3 + 2 + 1 = 3^2$



$1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$

Extend the sequence of pictures to help you decide the sum of $1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1$. Generalize your result to find a formula for $1 + 2 + 3 + \ldots + n + \ldots + 3 + 2 + 1$

–adapted from Nelson, *Proof Without Words*, p. 74.

H4-2

compute the first few terms of a recursively-defined sequence

A sequence of positive integers is defined as follows:

$a_1 = 2,\ a_{n+1} = \left\lfloor \frac{3}{2} a_n \right\rfloor \qquad n = 1,2,3 \ldots$

Are there infinitely many odd numbers in this sequence? Are there infinitely many even numbers? Note: $\lfloor x \rfloor$ is called the floor function which is a function f defined on the real numbers such that $f(x)$ is the greatest integer less than or equal to x. It is also called the greatest integer function frequently represented on a calculator as INT (x). For example, in our sequence if n = 1,

$a_2 = \left\lfloor \frac{3(2)}{2} \right\rfloor = 3$ and if n = 2, $a_3 = \left\lfloor \frac{3(3)}{2} \right\rfloor = \left\lfloor \frac{9}{2} \right\rfloor = \lfloor 4.5 \rfloor = 4.$

Grid Alignment

Beyond Graduation Standard Level Sample Problems

B1-1

apply the inclusion/exclusion principle and the pigeonhole principle to solve sorting problems

a) Show that, given any three integers, at least two of them must have the property that their sum is even.

b) For n such that $n \in$ integers, $1 \le n \le \infty$, how many integers between 0 and 2n must you pick in order to be certain that at least one of them is odd?

B1-2

extend knowledge of set theory to include partitioning of sets

Suppose a public opinion poll found that from a sample of 1200 adults:

- 675 are married,
- 682 are from 20 to 30 years old,
- 684 are female,
- 195 are married and are from 20 to 30 years old,
- 467 are married females,
- 318 are females from 20 to 30 years old, and
- 165 are married females from 20 to 30 years old.

a) How many married males were from 20 to 30 years old?

b) How many unmarried males were questioned?

B1-3

understand and apply identities involving binomial coefficients

e.g., $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$

for k such that $0 \le k \le n$

Use algebra to verify:

a) $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n}{k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ b) $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$

where $1 \le k \le n$

B1-4

use combinatorial arguments to justify binomial identities

Let $S = \{1, 2, 3, \ldots, n\}$ and $1 \le k \le n$.

a) How many subsets of size k can be formed from S?

b) How many subsets of size k can be formed from S that do not contain the element 3?

c) How many subsets of size k can be formed from S that contain the element 3?

d) Explain how the results of (a), (b), and (c) show that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

B1-5

extend logic to include predicate calculus

Give the contrapositive, converse, and inverse of each of the following statements in colloquial language.

a) $\forall$ real numbers x, if $x > 2$, then $x^2 > 4$.

b) $\forall$ real numbers x, if $x(x+1) > 0$, then $x > 0$ or $x < -1$.

c) $\forall$ integers n, if n is prime then n is odd or $n = 2$.

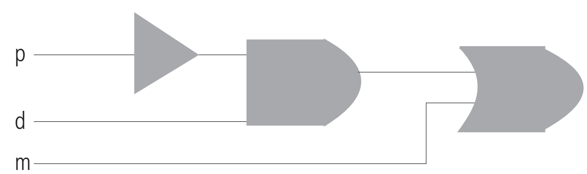
d) $\forall$ real numbers x, x satisfies $|x| < 3$.

B1-6

develop and interpret truth tables and computer logic networks

In order to print, a particular computer printer requires the "out of paper" switch *not* to be on and the "data ready" switch to be on. There is also a "manual paper feed" switch which overrides these. We can use a logic network to represent this information: p: Paper is out. d: Data is ready to print. m: Manual feed is being used. Write a logic proposition that describes when the printer will be activated. Draw the related logic network.

Answer: When $(\sim p \vee d) \wedge m$ is true, the printer will be activated. The circuit looks like this:



(based on Wiitala, 1987, p.123)

B2-1

apply graph theory to solve complex problems

Suppose a zoo wishes to provide enclosed areas to accommodate 10 species of animals. Animals that are natural enemies cannot be in the same enclosure, and the X's in the following table indicate such incompatibilities. No entry in a box means the two species are compatible and can live in the same enclosure. What is the minimum number of different enclosures required to house these animals, and how should the animals be grouped in this minimum arrangement?

	1	2	3	4	5	6	7	8	9	10
1			X							
2				X						
3	X			X						
4		X	X							
5						X				X
6					X		X			
7						X		X		X
8							X		X	
9								X		X
10					X		X		X	

B3-1

explore and analyze a variety of algorithms (e.g., simple sorting algorithms, data coding, backtracking)

Given an even number of boxes (2n) in a line, side by side: two adjacent boxes are empty, and the other boxes contain (n – 1) symbols "A" and (n – 1) symbols "B". For example, for n = 5

A	B	B	A			A	B	A	B

The contents of any two adjacent nonempty boxes can be moved into the two empty ones, preserving their order. Can you obtain a configuration where all A's are placed to the left of all B's, no matter where the empty boxes are initially located? If so, describe an algorithm for doing this, and describe those cases where it cannot be done, if there are any.

B3-2

translate algorithms into simple calculator or computer programs to investigate the interaction of variables

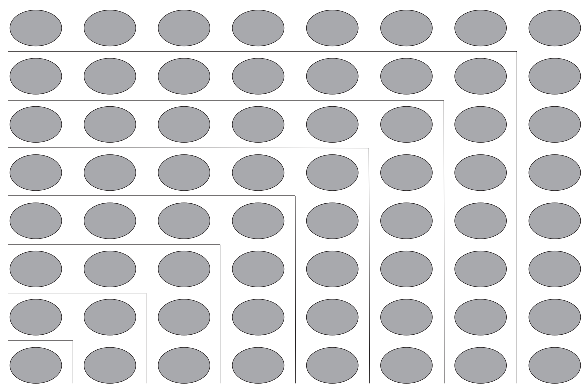
Write a program for your graphing calculator (or set up a spreadsheet on the computer) to evaluate earning compound interest at 7% per year on a given amount of money *m* over various periods of time. Analyze various amounts over different time periods using your program (spreadsheet).

When Sonja deposits a series of equal annual payments in an account, she is creating an annuity. If she deposits $100 each year in an account that offers 9% yearly interest, what is the value of the annuity after a certain number of years? Develop a recursive definition for A(n), the amount in the annuity at the beginning of year n, and use a spreadsheet to gain information about the annuity.

B4-1

use mathematical induction to construct a proof

Somewhere around 100 A.D., Nicholas of Gerasa presented the following visual evidence that the sum of the first n consecutive odd integers is equal to n^2.



$1 + 3 + 5 + \ldots + (2n-1) = n^2$

Use the Principle of Mathematical Induction to construct a formal proof that
$1 + 3 + 5 + 7 + \ldots + (2n-1) = n^2$.

– adapted from Nelson, *Proof Without Words*, p. 71.

B4-2

write computer code to generate the first few terms of a recursively-defined sequence

Write a program that uses the recursive formula to produce the sequence

2, 4, 8, 16,…

Write a program that uses the recursive formula to produce the Fibonacci sequence.

Sample Activities

Sample Activity: Primary

Create a House Number

The house number has three different digits.

The digits used are 2, 5 and 8.

What could the house number be? List all the possible answers.

Give the students multiple copies of the house. Have them list the numbers they can think of. After a few minutes, have students share their numbers, taking turns giving one of their solutions while the teacher lists them on the board or overhead. When they have run out of different numbers, ask "Do we have them all? How do you know?" If all six were not listed, have them work in partners to get all the numbers.

When students have identified all the combinations possible and have discussed how they know they have identified all possible house numbers, give them the street (drawn on a piece of paper for each pair of students). Ask them to place their houses along the street in the way they would be organized on real streets.

Have the student pairs compare their order with students next to them. Discuss the placement of each house and the reason for placement as a class, having each group place a house in its correct spot on the board or wall chart. (Or you might have each group assemble their street and post these.) Use the suggested teacher questions in the "About This Activity" section to discuss patterns.

A new town rule allows houses to have two digits alike. What new house numbers does this add to our group? Have students make a list and place them on the street. Discuss the changes this makes in the street.

These activities reinforce many of the key ideas in the K-12 Components at the primary level, especially:

- use manipulatives, diagrams and lists to explore problems involving counting and arranging objects
- understand logical terms such as *and, or, not, some* and *all,* and use them correctly
- explore the use of diagrams and simple charts to represent all possible outcomes of an event or experiment
- compare and evaluate multiple solution strategies
- discover simple patterns and sequences

About this Activity

- This activity is appropriate for the end of first grade or for second grade.
- The essence of this activity is not the manipulation of the numbers; it is knowing when you have all the house numbers and communicating how you know.
- Have the students predict how many house numbers they think can be made with the three digits before they begin to make their lists.
- Help students build an organized list of their combinations.

Teacher questions might include:

- How would you order the houses if they were all placed on a street?
- Would all the houses be on the same block? Would all the houses be on the same side of the street?
- How would you decide where to place the houses?
- What would happen if you could use a digit twice? Three times? How many more house numbers do you think you would get? Make a list to check your guess.

Where do we go from here?

Organize a new set (e.g. 1, 3, 6) which has two odd digits and one even digit. Students list the possible house numbers and place them on a street. Compare this street with the original street. How many house numbers could be made if some or all the digits could be the same?

Suppose one digit is added to the original group to give the set 3, 2, 5 and 8. Estimate how many house numbers would be possible. Make all the possible house numbers.

Form all possible house numbers given these rules to follow:

a) the digits 0-9 can be selected,

b) no house number begins with 0,

c) the house number has three different digits, and

d) the sum of the digits in the house number is 6.

Suggested Teaching Resources

Burns, M. (1992). *About teaching mathematics, a K-8 resource.* Sausalito, CA: Marilyn Burns Educational Associates.

Burns, M. (1992). "A three hat day." *Math and literature (K-3).* Sausalito, CA: Math Solutions Publications.

Cook, M. (1989). "Ideas for teachers." *Arithmetic Teacher,* 36(5).

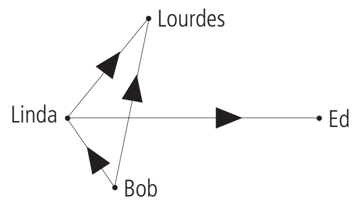
Merriam, E. & Karlin, B. (1993). *Twelve ways to get to eleven.* New York, NY: Simon & Schuster.

Sample Activity: Intermediate

The Checkers Tournament

Adapted with permission from: Mathematical Science Education Board (1993). *Measuring up: Prototypes for mathematics assessment.* National Academy Press, p. 31-41.

If students are not already familiar with the directed graph notation, the teacher should introduce it (without the terminology "directed graph") as a means of displaying information about four students in a Tic-Tac-Toe tournament. In the tournament each pair of players plays one game. On the digraph, an edge connects two players who have played a game and the arrow indicates the winner.



The teacher should explain the situation and ask questions such as:

Which students has Linda played? (Lourdes, Ed, and Bob.)

Which games did she win? (Lourdes and Ed)

Which games did she lose? (Bob)

Find two students who have not yet played each other. (Lourdes and Ed; Ed and Bob)

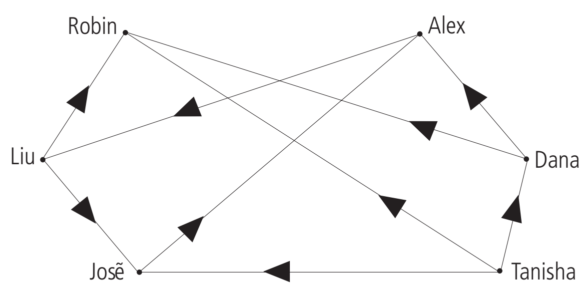
Who has played the fewest games? (Ed has played only one game.)

The teacher should pass out copies of the student sheet and read through the introduction and question 1 to be sure everyone has an understanding of the meaning of the dots and arrows.

Student Sheet

Six children are in a checkers tournament.
The figure below shows the results of the games played so far.

1. What information is shown by the graph?



2. How many games have been played by the children so far?

3. Make a table showing the current standings of the six children.

4. The tournaments will be over when everybody has played everybody else exactly once.
   Tell how many more games need to be played to finish the tournament. Explain your reasoning.

5. Dana and Liu have not played yet. Explain who you think will win when they play and your reasons.

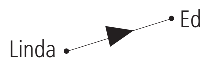
These activities reinforce the following key idea in the K-12 Components at the intermediate level:

- use vertex-edge graphs to interpret relationships between or among objects (e.g., persons, events, locations)

About this Activity

This task assumes children have had prior experience in translating information from one form to another.

- This task was chosen for its richness in the use of vertex-edge graphs to represent a game situation.
- The teacher needs to explain that, in the picture, an arrow like that below means that Linda won the game against Ed. Here the arrow points from the winner to the loser.



One possible way to assess this activity would be to use a rubric where the characteristics are:

High response: understands various components of the graph; all answers are correct; all arguments are justified.

Medium response: may make some incorrect conclusions; does not present a complete argument; may have difficulty connecting the directed graph notation with the standings table.

Low response: little awareness of the relationships between the directed graph and the games played; answers may be incorrect; arguments lack justification.

Where do we go from here?

Several variations of this problem can be explored. One can add players, add or change the directions of the arrows, and ask other kinds of questions - for example, *If there were 7 players, how many games would be needed if everyone played everyone else exactly once?*

One can extend the setting to relations that are transitive - for example, an arrow pointing from A to B means that *A is taller than B*. This relation, unlike the one in this task, is transitive, and so one can infer that if there are arrows from A to B and from B to C, then there must be an arrow from A to C.

Set up a tournament in the classroom or keep track of the results of a neighborhood sports league. Challenge the students to use a directed graph to summarize the results. What information is not included in a directed graph?

Recommended Teaching Resources

Erickson, T. (1986). *Off & running: The computer offline activities book.* University of California, Berkeley, CA: EQUALS, Lawrence Hall of Science.

Fraser, S., Stenmark, J., Downie, D., Joseph, H., Kaseberg, A., Campbell, C., Gilliland, K., & Thompson, V. (1982). *SPACES: Solving problems of access to careers in engineering and science.* Palo Alto, CA: Dale Seymour Publications.

Kenney, M. (Ed.). (1991). *Discrete mathematics across the curriculum K - 12: 1991 yearbook.* Reston, VA: NCTM.

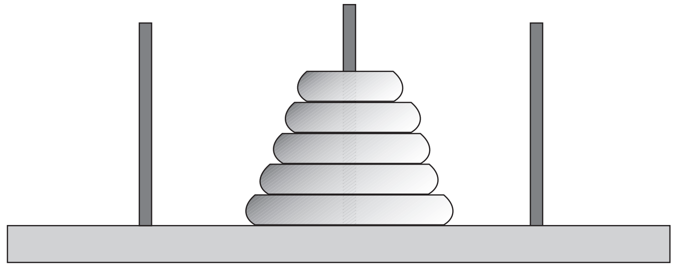
Maletsky, E. (Ed.). (1987). *Teaching with* Student math notes (pp. 63-68). Reston, VA: NCTM.

Mathematical Sciences Education Board (MSEB). (1993). *Measuring up: Prototypes for mathematics assessment.*

Sample Activity
Middle School

The Tower of Hanoi

Adapted with permission from: Phillips, E., Gardella, T., Kelly, C., & Stewart, J. (1991). *Patterns and functions: Curriculum and evaluation standards for school mathematics addenda series, grades 5-8.* National Council of Teachers of Mathematics, p. 11.



This puzzle consists of three pegs and several disks of graduated size with holes in their centers. The disks are stacked from largest on the bottom to smallest on top as shown. If the disks are moved one at a time to another peg and if a larger disk cannot be placed on a smaller disk, what is the fewest numbers of moves needed to move the disks from one peg to another peg, so that they are arranged from largest on the bottom to smallest on top? Explore the minimum number of moves needed to transfer stacks from 1 to 5 disks tall.

Have students make a table with class data. A possible table is:

No. of disks	No. of moves	Recursive pattern
1	1	
2	3	= 2(1) + 1 = 3
3	7	= 2(3) + 1 = 7
4	15	= 2(7) + 1 = 15
5	31	= 2(15) + 1 = 31

Have students graph their results and predict what will happen with 6 disks and beyond. Then have them try the puzzle with six or more disks.

Students can be guided to view the task recursively, that is, to express the number of moves needed to solve the puzzle with one more disk in terms of the number of moves needed for the puzzle with the current number of disks. If one disk is added to the pile, the number of moves needed to transfer the new pile is equal to twice the number of moves needed to transfer the previous pile plus 1. One more disk on a pile means you have to do the moves for the former pile twice to be ready to put the new disk on top. With help they can discover and use the recursive pattern shown at the right of the table above to predict required moves for other numbers of disks.

As students develop an understanding of the recursive pattern, encourage them to look at the table they made to find another pattern, an inductive process. Using their past experience with finding powers of 2, help them see the relationship between their chart and the "Powers of 2" chart. (Adding a new column which lists Powers of 2 to the table the students made above may help.)

No. of disks	Powers of 2	No. of moves	
1	$2^1 = 2$	1	$2^1 - 1 = 1$
2	$2^2 = 4$	3	$2^2 - 1 = 3$
3	$2^3 = 8$	7	$2^3 - 1 = 7$
4	$2^4 = 16$	15	$2^4 - 1 = 15$
5	$2^5 = 32$	31	$2^5 - 1 = 31$

These activities reinforce many of the key ideas in the K-12 Components at the middle school level, especially:

- create, communicate, and defend a strategy for winning a game
- explore patterns in more complex geometric and numerical problems (e.g., Fibonacci sequence, Pascal's triangle)
- employ recursion in counting contexts (e.g., moves in Tower of Hanoi puzzle)

About this Activity

- The Tower of Hanoi problem is believed to originate in the great temple at Benares, India. The story claims there are three diamond needles and sixty-four golden disks, graduated in size, near the temple. The temple monks are to move the disks with the same rules as in the game. When all disks have been transferred, the world will come to an end. If the monks start at the beginning of time and make one move per second, how close are we to having the world end? This provides an opportunity to estimate. (There are 2^{64}-1 = 18 446 744 073 709 551 615 moves which is about 5.8 X 10^{13} years.)

Where do we go from here?

- With graphing calculators, students can plot the number of disks against the number of moves and find a curve of best fit. Use calculators to do such things as predict the number of moves required to move 10 disks or to compute how many disks there are if 1023 moves are required.
- If you color code disks from smallest to largest as red, green, blue, yellow, orange, you can highlight the relationship between the moves in this problem to the binary code. The chart below shows the first 7 moves. Encourage students to use the code to work the puzzle with increasing number of pegs.

largest ◀———— smallest

Move #	orange	yellow	blue	green	red	description of move
0	0	0	0	0	0	start
1	0	0	0	0	1	move red
2	0	0	0	1	0	move green
3	0	0	0	1	1	move red onto green
4	0	0	1	0	0	move blue
5	0	0	1	0	1	move red **not** on blue
6	0	0	1	1	0	move green to blue
7	0	0	1	1	1	red to green on blue

Recommended Teaching Resources

Chavey, D. (1987). *Drawing pictures with one line, module #21*. Lexington, MA: COMAP.

Jacobs, H.R. (1995). *Mathematics: A human endeavor.* New York, NY: W.H. Freeman & Co.

Maletsky, E. (Ed.). (1987). *Teaching with* Student math notes (pp. 1-6, 13-18). Reston, VA: NCTM.

Phillips, E., Gardella, T., Kelly, C., & Stewart, J. (1991*). Patterns and functions: Curriculum and evaluation standards for school mathematics addenda series, grades 5-8.* Reston, VA: NCTM.

Seymour, D. (1986). *Visual patterns in Pascal's triangle.* Palo Alto, CA: Dale Seymour Publications.

Sample Activity
High School

Wait for Me!

Adapted with permission from: Sloyer, C., Copes, W., Sacco, W., & Stark, R. (1987). *Queues, will this wait never end!* Chicago, IL: Janson Publications, Inc.

The bank in a small midwestern town has the capacity of providing four tellers to work simultaneously. Customers enter a queue and are served by the first available teller. Data was collected on a Friday that was a pay day for many of the companies and businesses in town. A certain number of one-minute intervals were sampled. In each interval, the number of customers who entered the line during that interval was counted. The following table shows the proportion of sampled intervals for which a given number, *k*, of customers entered the line.

Number of arrivals								
k	0	1	2	3	4	5	6	7
F_k	0.10	0.22	0.16	0.31	0.08	0.03	0.06	0.04
Relative frequency of k in the sample of one-minute intervals.								

This table is interpreted in the following way: For 10% of the one-minute intervals sampled, no customers arrived; for 22% of the one-minute intervals sampled, exactly one customer arrived, and so on.

1. Determine the average number of arrivals for the sampled intervals. (2.58 arrivals per minute)

2. Based on the data collected, estimate the average number of arrivals per minute during the day.

The following table contains data on customer service times. The first row gives service times j (rounded up to the nearest 30 seconds) and the second row is the relative frequency of j.

Service times (seconds)								
j	30	60	90	120	150	180	210	240
F_j	0.24	0.30	0.17	0.12	0.03	0.09	0.01	0.04
Relative frequency of j								

3. Determine the average service time. (87.3 seconds)

Use a random number generator to simulate the number of arrivals for each minute of a 30 minute period. Use a random number generator to simulate the amount of service time required for each arrival. We have now defined the "workload" which the bank will face during one possible 30 minute interval.

4. Examine the capability of the bank to process the workload if two tellers are available.

5. Examine the capability of the bank to process the workload if three tellers are available.

6. Examine the capability of the bank to process the workload if four tellers are available.

This activity reinforces many of the key ideas in the K-12 Components at the High School Graduation Standard Level:

- use concepts of order and repetition to differentiate situations involving permutations and combinations or other techniques of counting
- investigate algorithmic thinking to solve problems involving social choice and decision making (e.g., weighted voting, fair division, apportion-ment)

About this Activity

- Note that the first table is in minutes, the second in seconds.
- When simulation is used, many replications are made. Mathematicians are more comfortable with the results if they involve many replications. The variability of the results can be analyzed and certain quantities can be estimated, such as the probability that the average waiting time exceeds a certain value. The computer is a perfect tool for doing these repetitive calculations.
- A table of random numbers or the random number function on a graphing calculator can be used to simulate the number of arrivals and the service times.
- A discussion of what is a reasonable amount of time for customers to wait can be centered around exercise 3. No customer should wait over _______ minutes.
- What is the average wait time in exercises 4 - 6?
- Since the addition of more tellers reduces customer wait time, why wouldn't the bank just always keep the maximum number of tellers on duty?

Where do we go from here?

We have modeled a very realistic situation by making assumptions to simplify the problem. Generally, banks have busy periods. How could we adjust the simulation to reflect both busy and slack periods? How could the simulation be adjusted to simulate an entire day at the bank and then how could it be adjusted to simulate several days, taking into account some days may be busier than others? How would knowing about slack and busy periods influence staffing schedules?

Examine two queuing methods. In the first, the line forms at each window. In the second method, there is a single queue and tellers serve customers from the head of the line. Given identical service and arrival patterns, is one system better than another? What other factors should be taken into account?

Recommended Teaching Resources

Consortium for Mathematics and Its Applications (COMAP). (1997). *For all practical purposes: Introduction to contemporary mathematics* (4th edition). New York, NY: W.H. Freeman & Company.

Gnanadesikan, M., Schaeffer, R.L., & Swift, J. (1987). *The art and technique of simulation: Quantitative literacy series.* Palo Alto, CA: Dale Seymour Publications.

Maletsky, E. (Ed.). (1987). *Teaching with* Student math notes (pp. 1-6, 13-18). Reston, VA: NCTM.

Meiring, S.P., Rubenstein, R.N., Schultz, J.E., deLange, J., & Chambers, D.L. (1992). *A core curriculum: Making mathematics count for everyone: Curriculum and evaluation standards for school mathematics addenda series, grades 9-12:.* Reston, VA: NCTM.

Sloyer, C., Copes, W., Sacco, W., & Stark, R. (1987). *Queues, will this wait never end!* Chicago, IL: Janson Publications, Inc.

High School Mathematics and Its Applications Project (HiMAP) has developed modules which can be used to integrate discrete topics into an existing curriculum. This is a project of the Consortium for Mathematics and Its Applications (COMAP): Lexington, MA 1-800-772-6627.

An excellent Internet resource is the CHANCE database: http://www.geom.umn.edu/docs/education/chance/

Sample Activity: Beyond High School

Encoding Personal Data

Adapted with permission from: Garfunkel, S. (1997). *For all practical purposes: Introduction to contemporary mathematics, fourth edition.* New York, NY: W.H. Freeman & Company. pp. 365-378.

Consider this social security number: 189-31-9431. What information about the holder can be deduced from the number? Only that the holder obtained it in Pennsylvania. An Illinois driver's license number is 1225-1637-2133. What information about the holder can be deduced from this number? This time we can determine the date of birth, gender, and much about the person's name.

These two examples illustrate the extremes in coding personal data. The social security number has no personal data encoded in the number. It is entirely determined by the place and time it is issued, not the individual to whom it is assigned. In contrast, in some states the driver's license numbers are entirely determined by personal information about the holders. It is no coincidence that the unsophisticated social security numbering scheme predates computers. Agencies that have large data bases that include personal information such as names, gender, and dates of birth find it convenient to encode these data into identification numbers. Coding license numbers solely from personal data enables automobile insurers, government entities, and law enforcement agencies to determine the personal data from the number.

Many states encode the surname, first name, middle initial, date of birth, and gender by quite sophisticated schemes. In one scheme the first four characters of the license number are obtained by applying the **Soundex Coding System** to the surname as follows:

1. Delete all occurrences of h and w. (For example, Schworer becomes Scorer and Hughgill becomes uggill.)
2. Assign numbers to the remaining letters as follows:

a,e,i,o,u,y → 0	b,f,p,v → 1	c,g,j,k,q,s,x,z → 2
d,t → 3	l → 4	m,n → 5
r → 6		

3. If two or more letters with the same numerical value are adjacent, omit all but the first. (For example, Scorer becomes Sorer and uggill becomes ugil.)
4. Delete the first character of the original name if still present. (Sorer becomes orer.)
5. Delete all occurrences of a,e,i,o,u, and y.
6. Retain only the first three digits corresponding to the remaining letters: append trailing zeros if fewer than three letters remain; precede the three digits by the first letter of the surname.

What are possible advantages of this system?

There are many schemes for encoding the date of birth and the gender in driver's license numbers. For example, the last five digits of Illinois and Florida driver's license numbers capture the year and date of birth as well as the gender. In Illinois, each day of the year is assigned a three-digit number in sequence beginning with 001 for January 1. However, each month is assumed to have 31 days. Thus, March 1 is given the number 063 since both January and February are assumed to have 31 days. These numbers are then used to identify the month and day of birth of male drivers. For females, the scheme is identical except 600 is added to the number. The last two digits of the year of birth, separated by a dash, are listed in the fifth and fourth positions from the end of the driver's license number. In Florida, the scheme is the same except each month is assumed to have 40 days and 500 is added for women.

This activity reinforces the following key idea in the K-12 Components at the High School Beyond Graduation Standard Level:

- explore and analyze a variety of algorithms (e.g., simple sorting algorithms, data coding, backtracking).

- What would be the Illinois driver's license of a man born October 13, 1940? of a woman born that day? What would the licenses be in Florida?
- Using variables m for month and b for birth date, have students write formulas to determine the last five digits of Florida male and female driver's licenses.
- Have students work backwards to determine the dates of birth of people whose Florida Driver's licenses end in 42218 and 53953.
- Have students create a computer program to "issue" driver's licenses according to a devised algorithm.

About this Activity

- The information given is from the "Coding Information " part of the *For All Practical Purposes* text. The part adapted here is selected to give an introductory look at an area of discrete mathematics.
- Social security numbers issued in Minnesota begin with the first three digits lying between 468 and 477. For a table showing which digits correspond with which states, see p. 367 of *For All Practical Purposes*.
- **The Soundex Coding System** is an error-correcting scheme. It is designed so that alternative spellings and likely misspellings of a name result in the correct coding of the name. For example, have the students use the Soundex Coding System to encode Erickson, Ericksen, Eriksen, Ericson, and Ericsen. Under this type of encoding the computer searches the data bank for records encoded E-625 despite which variation of spelling is entered.
- Advanced students should complete one or more of the explorations below.

Where do we go from here?

Examine the New York Driver's License numbering system described on page 368 of *For All Practical Purposes.* Determine driver's license numbers for the members of your study group (family, etc.)

Examine the state of Utah method and the Canadian province of Quebec method of issuing driver's license numbers. Evaluate and compare both methods described on page 376 of *For All Practical Purposes.*

Prepare a report on coded information in your location. Possibilities include student ID numbers or bar codes used by the school and city library.

Imagine you are employed by a small company that doesn't have identification numbers for employees. Prepare a report for your boss discussing various methods and make a recommendation.

Recommended Teaching Resources

Consortium for Mathematics and Its Applications (COMAP). (1997). *For all practical purposes: Introduction to contemporary mathematics* (4th edition). New York, NY: W.H. Freeman & Company.

Gallian, J. (1992). "Assigning driver's license numbers." *Mathematics Magazine*, 64, 13-22.

Malkevitch, J., Froehlich, G., & Froehlich, D. (1991). *Codes galore, module #18.* Lexington, MA: COMAP.

Roughton, K. & Tyckosen, D.A. (1985, June). "Browsing with sound: Sound-based codes and automated authority control." *Information Technology and Libraries,* 130-136.

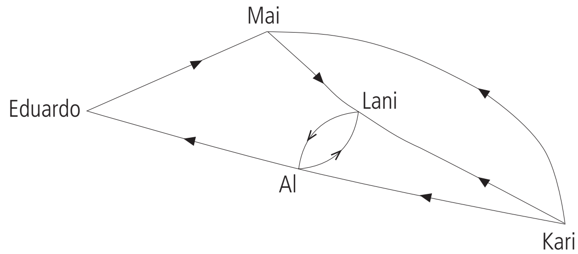
Glossary

A Brief Glossary for Discrete Mathematics

This glossary is an attempt to clarify the meaning of terms used in this document. A combination of formal definitions, contextual situations, and sample problems has been provided with that goal in mind. Terminology in the field of discrete mathematics is not standardized. Some terms used in this document may be referred to by different names in other documents. For example, "vertex-edge graphs" may be called "connected graphs" or "finite graphs" elsewhere.

adjacency matrix

A symbolic notation for the connectedness of the vertices of a graph. A one is placed in row i, column j provided that there is an edge connecting vertex i with vertex j. For example, some group-behavior studies show who influences who in a social setting. The directed graph shown shows a set of such influences for a given set of individuals.



The adjacency matrix for this graph is represented below.

	E	M	L	K	A
Eduardo	0	1	0	0	0
Mai	0	0	1	0	0
Lani	0	0	0	0	1
Kari	0	1	1	0	1
Al	1	0	1	0	0

algorithm

An explicit list of directions for carrying out a procedure. Designing and applying algorithms is an important method for solving problems.

backtracking

A systematic method involving thinking backward through an enumeration process to list out all possible cases in a complicated graph analysis problem.

Consider the example of a maze where at each node you can continue by going down the left, right, or center paths. When you reach a dead end, the backtracking method can be used. By always choosing left until you reach a dead end, you can retreat one step. If you went left, try the center path; if you had followed center, try the right path. But if you went right, retreat another step. Continue in this manner until the path through the maze is found.

circuit

A path in a vertex-edge graph that begins and ends with the same vertex.

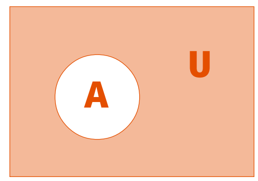
combination

Let S be a set. A combination of S of size *k* is an *unordered* selection of *k* elements of S (a subset of S of size *k*). For example, there are four combinations of size 3 taken from the set {a,b,c,d}: {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}.

Note: {a,b,c} = {b,a,c}

complement (of a set)

Let A be a subset of U. The complement of A (denoted $\bar{A}$) is the set of all elements x in U that are not in A. In the diagram below the complement of A is shaded:



conditional

If p and q are statements, the statement "if p then q" is called a conditional statement. The statement p is called the antecedent (hypothesis), and the statement q is called the conclusion. The conditional "if p then q" is symbolized "$p \rightarrow q$". For example, "If it is Tuesday, then I have Math Club."

conjunction

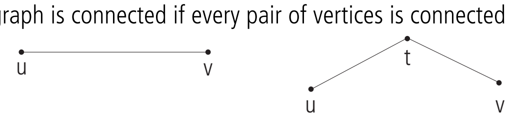
If p and q are statements, the conjunction of p and q is the statement "p and q." The conjunction of p and q is symbolized as "$p \wedge q$."

For example, p = "It is Tuesday." and q = "I have Math Club today."

$p \wedge q$ is " It is Tuesday and I have Math Club today."

connected graph

Two vertices u and v in a graph are connected if there is a path in the graph from one to the other. The whole graph is connected if every pair of vertices is connected.



contrapositive

The contrapositive of the conditional statement "if p then q" is the statement "if not q, then not p." Symbolically, the contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$. The conditional and the contrapositive are equivalent statements (one is true if and only if the other is true). For example, the statement is "If it is Tuesday, then I have Math Club today." The contrapositive is "If I do not have Math Club today, then it is not Tuesday."

converse

The converse of the conditional statement "if p then q" is the statement "if q then p."
For example, the statement is "If it is Tuesday, then I have Math Club." The converse is "If I have Math Club, then it is Tuesday."

critical path

The path in a weighted graph that is the longest in terms of the variable under investigation (e.g. distance, time, cost).

difference (of two sets)

Let A and B be subsets of a set U. The difference of B minus A (denoted B – A) is the set of all elements that are in B and not in A.
For example, let U be the set of all fruit, A ={apple, grape, pear}, B ={apple, banana, pear, kiwi}; then B – A = {banana, kiwi}

directed graph

A vertex-edge graph in which each edge has been given a direction. Directed graphs are also known as **digraphs**. These directions may be used to depict routes, one-way streets, power relationships, etc.

disjunction

If p and q are statements, the disjunction of p and q is the statement "p or q" denoted "$p \vee q$". Unless otherwise stated, it is inclusive which means "either p or q or both." For example, p = "It is Tuesday." and q = "I have Math Club today." $p \vee q$ is "Either it is Tuesday or I have Math Club today or it is both Tuesday and Math Club day."

Euler path

A path which contains each and every edge of a connected graph exactly once. When an Euler path starts and ends at the same vertex, it is called an **Euler circuit**.

Fibonacci sequence

The sequence 1, 1, 2, 3, 5, 8, 13…in which each successive term is the sum of the preceding two terms. This sequence may be seen in the grouping of black and white keys on the piano and in many naturally occurring objects. The golden pincushion of a daisy consists of tiny little flowers arranged in two sets of curved lines which spiral out from the center, 21 spiraling clockwise and 34 counterclockwise. The spiraling of the knobbles of pineapples, scales of pine cones, and plant leaves around a stem are among other natural objects demonstrating parts of the Fibonacci sequence. For more examples of the Fibonacci sequence in nature, see the following website: www.ee.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html

flow chart

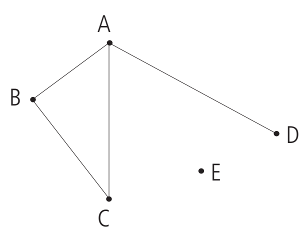
A diagram that illustrates graphically the sequence of operations to be performed in executing an algorithm or to assist in algorithmic thinking.

fundamental counting principle

(See *multiplication principle*.)

graph (vertex-edge graph)

A mathematical structure consisting of vertices and edges in which some pairs of vertices are connected by edges. For example, let the set of vertices be {A, B, C, D, E} and the set of edges be {{A,B}, {A,C}, {A,D}, {B,C}}; then the graph is:



graph theory

The theoretical study of vertex-edge graphs.

Hamiltonian path

In a vertex-edge graph, a path that contains each vertex exactly once. If the path returns to the starting vertex it is called a **Hamiltonian circuit**.

highway inspector problem

A classic problem in discrete mathematics in which a thrifty highway inspector has the problem of finding a route that takes her/him over each section of the highway exactly once.

inclusion/exclusion principle

If S is a finite set, let n(S) denote the number of elements in S. If A and B are finite sets, the inclusion/exclusion principle states that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

For three finite sets, A, B, and C, the principle states that $n(A \cup B \cup C) =$

$[n(A) + n(B) + n(C)] - [n(A \cap B) + n(A \cap C) + n(B \cap C)] + n(A \cap B \cap C)$.
[# in all sets combined] - [# in all pairwise intersections] + [# in intersection of all 3]

The general formula for four or more sets follows a similar pattern of overcounting and undercounting.

inductive reasoning

The process of making generalizations from several known cases. An important problem-solving skill is the "discovery of a pattern," a skill which requires inductive reasoning.

intersection (of two sets)

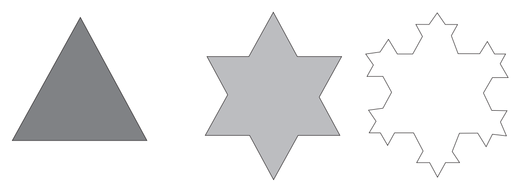
Let A and B be subsets of a set G. The intersection of A and B, denoted $A \cap B$ is the set of elements x in G such that *x* is in A and *x* is in B. For example, let G be the set of all fruit, A = {apple, grape, pear}, B = {apple, banana, kiwi, pear}; then $A \cap B$ = {apple, pear}

iteration

Repetition. Iteration involves performing an algorithmic process over and over again, each time using the result of the previous step as the starting point of the new step. The Koch snowflake is an iterative process, as is the compounding of interest on a bank account.

Koch snowflake

Let S_1 be an equilateral triangle, and define S_{n+1} in terms of S_n as follows. Replace the middle third of each edge of S_n by the two outward-directed edges of an equilateral triangle erected on the deleted middle. The first three figures in this sequence are shown below:



The Koch snowflake is the figure toward which the sequence converges.

logical connectives

A statement is a sentence that is either true or false. Statements can be combined by logical connectives to form other statements. The most common logical connectives are *not, and, or, if…then,* and *if and only if*.

mathematical induction

A type of mathematical proof that uses the following reasoning: suppose P(n) is a statement that depends on positive integer *n,* and let *m* be a positive integer.

If (i) (Basis step) P(*m*) is true; and

(ii) (Inductive step) for all positive integers $k \geq m$, P(*k*+1) is true if P(*k*) is true;

Then P(*n*) is true for all positive integers $n \geq m$.

For a proof by induction, we must show the two steps to be true. First, we verify that a statement holds for the least positive integer m. Then for the inductive step, we assume *k* is a positive integer greater than or equal to *m* and that the statement holds for k. Using this assumption, we show the statement is true for the next positive integer *k*+1. With this completed, we can conclude that the statement holds for all positive integers $n \geq m$.

multiplication principle

If X can be done in n_1 different ways, and independently, Y can be done in n_2 different ways, then X and Y together can be done in $n_1 n_2$ ("n_1 multiplied by n_2") different ways.

To find the number of ways of making several decisions in succession, multiply the number of choices that can be made in each decision. In a series of choices, if the first step can be done in n_1 ways, the second in n_2 ways, the third in n_3 ways, and the last decision in n_k ways, then the number of possible ways of making the choices is

$n_1 \bullet n_2 \bullet n_3 \bullet \ldots \bullet n_k$

(The multiplication principle is sometimes referred to as the *Fundamental Counting Principle*.)

negation

If p is a statement, the negation of p is the statement "not p." The negation of p is symbolized by "~p." For example, p = "Math Club meets today." The negation of p is "Math Club does not meet today."

network

(See *graph*.)

optimization problem

The problem of finding a solution, among all solutions, that optimizes some feature. Sometimes one seeks a "largest," sometimes a "smallest," or sometimes a "best" solution based on a variable in the problem.

outcome

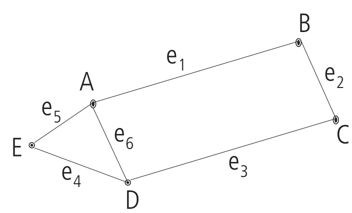
One of the possible results of a probability experiment. For instance, if we toss a coin, there are two outcomes: heads or tails. If we toss a die, there are six outcomes: 1, 2, 3, 4, 5, or 6.

partition of a set

Let S be a set. A collection of nonempty subsets $P_1, P_2, \ldots, P_k$ of S is called a partition of S if the subsets are pairwise disjoint (for each possible pair, their intersection is the empty set) and when you join them all together, you get the set S ($P_1 \cup \ldots \cup P_k = S$). For example, {1}, {2,3}, {4,5,6} is a partition of {1, 2, 3, 4, 5, 6}.

path

A walk from one vertex to another in which no edge is repeated.



A path from A to C could be $e_1 e_2$ or $e_5 e_4 e_6 e_1 e_2$, but not $e_6 e_4 e_5 e_6 e_3$ which repeats one edge.

permutation

Let S be a set. A permutation of S is an *ordered* arrangement of the elements of S. For example, there are six permutations of the set of letters {a, b, c}: {a,b,c}, {a,c,b}, {b,a,c}, {b,c,a}, {c,a,b}, and {c,b,a}.

pigeonhole principle

If *n* + 1 objects are placed in *n* pigeonholes, at least one of the pigeonholes must contain at least two objects.

The pigeonhole principle is an important way of showing that there is at least one solution. The pigeonhole principle can be used to solve problems like the following: A drawer contains unsorted black, blue, brown, and red socks. If I pull socks from the drawer in the dark, how many socks must I pull to be certain I have a matching pair?

predicate calculus

The logic of ordinary compound statements (statements built from other statements by means of logical connectives) plus statements that include the quantifiers ∀ (for all) and ∃ (there exists). For example, in the predicate calculus the negation of "All grass is green" is "Some grass is not green."

recursion

The determination of a succession of elements by an operation on one or more preceding elements, according to some prescribed rule involving only a finite number of steps. For example, if the initial condition of a recurrence relation specifies the first term is one and each succeeding term is twice the previous term, then the sequence is 1, 2, 4, 8, 16 and so on. Expressed recursively, if $x_1 = 1$, then $x_n = 2(x_{n-1})$ for $n \geq 2$.

routing problems

Problems that involve finding paths or circuits in a vertex-edge graph. Examples are the traveling salesperson problem and the highway inspector problem.

sequence

2, 4, 6, 8, 10…is a sequence. The numbers are the terms of the sequence and often referred to as the first term, second term, third term, and so on. In a sequence a function (rule or formula) maps each integer greater than or equal to a fixed integer to a term in the sequence. In the example of even numbers, the function "2 times the term number" gives the value of the term. Sometimes sequences are finite and sometimes they are infinite. We speak of a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ and mean that a_1 is the first term of the sequence, a_2 is the second term of the sequence, and so forth.

series

The sum of the consecutive terms of a sequence. The finite sequence 2,4,6,8 is related to the series 2 + 4 + 6 + 8 which has a sum of 20.

set theory

The notion of a set is fundamental throughout mathematics: sets of blocks, sets of numbers, sets of functions, and so forth. Set theory addresses such topics as how sets can be combined to make other sets (union, intersection, complement) and relationships that exist between these operations.

shortest path problem

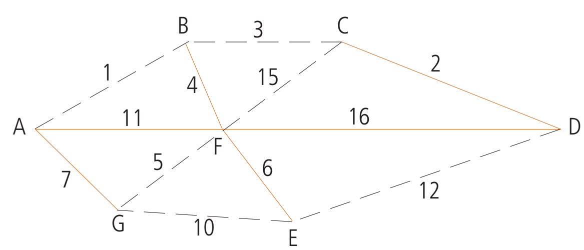
The problem of finding a minimal weight path from any vertex to any other vertex in a vertex-edge graph.

sorting algorithm

An algorithm that sorts a list of elements such as numbers, letters, words, or objects, into a predetermined order (for example, alphabetizing a list of names or arranging a list of integers in descending order.)

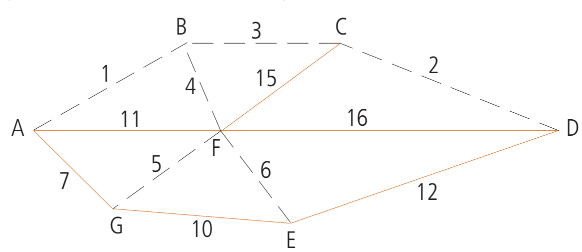
spanning tree

A spanning tree of a vertex-edge graph is a tree formed by using edges and all the vertices of the graph.



Graph M
spanning tree – – – – – – –

A spanning tree of least or minimal weight (or value) is called a minimum spanning tree



Graph M
minimum spanning tree – – – – – – –

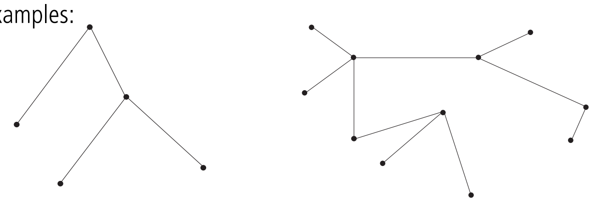
traveling salesperson problem

The classic discrete mathematics problem of finding a route that will allow a salesperson to visit each town in a territory exactly once. The salesperson would like to find the circuit which minimizes his/her total mileage.

tree

A connected vertex-edge graph which contains no circuits (i.e. no path begins and ends with the same vertex).

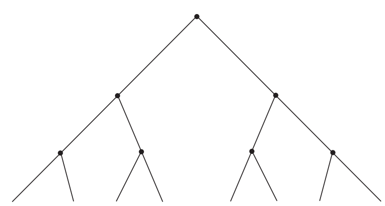
Examples:



tree diagram

A tree diagram is used to represent the outcomes of a sequence of events whose outcomes at each stage may influence future outcomes.

For example, the Elmdale Soccer Club has boys and girls from about 500 families. When a game is canceled, the club president calls two families who in turn call two families, each of whom calls two more families and so on. This is called a telephone calling tree and can be represented by a tree diagram that starts like this:



Each vertex is a person called, who in turn calls two more people. Each edge represents a call. To find how many people have been called, we could count the vertices.

union (of two sets)

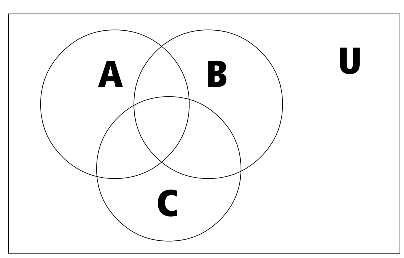
Let A and B be subsets of a set G. The union of A and B, denoted by A U B, is the set of all elements x in G such that x is in A or x is in B (or both).

For example, let G be the set of all fruit, A ={apple, grape, pear}, B = {apple, banana, kiwi, pear} so A U B = {apple, banana, grape, kiwi, pear}

Venn diagram

A pictorial method for visualizing abstract sets and their relationships. The figure below is a Venn diagram for three sets A, B, and C.

If U is the set of all students who are sophomores in high school this year and A is the set of students who are have studied French, B is the set of students who have studied Spanish, and C is the set of students who have studied German, then the students that are in U outside of any circles can be described as students who have not studied French, Spanish, or German.

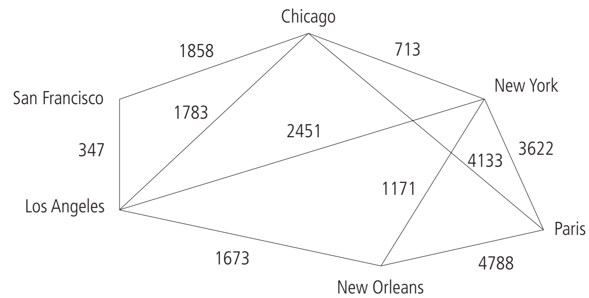


vertex-edge graph

(See *graph*)

weighted graph

A vertex-edge graph that has a number called a weight assigned to each of its edges. The total weight of the graph is the sum of the weights of all of its edges. The numbers in the graph below show the distances between cities.



winning coalition

A coalition of voters in favor of a measure is a winning coalition if the sum of its weights equals or exceeds the quota q. For example: There are 24 votes at a stockholders' meeting and a quota of 13 or more votes is needed to pass a resolution. Ms. A controls 7 votes, Mr. B controls 3 votes, Mrs. C controls 4 votes. Together they control 7+3+4 = 14 votes, so they can pass the resolution by all voting in favor. Ms. A, Mr. B, and Mrs. C form a winning coalition in this case.

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Internet Resources

Chance Database
http://www.geom.umn.edu/locate/chance
Abstracts and full text articles in the news, a handbook for teaching a chance course and other teaching resources.

DIMACS
Center for Discrete Mathematics and Theoretical Computer Science - A National Science Foundation Science and Technology Center
http://dimacs.rutgers.edu
Publications and resources for discrete mathematics and computer science, links to other sites, New Jersey State Frameworks, public events, education and research programs, scientific achievements

Eisenhower National Clearinghouse (ENC)
http://www.enc.org
Resources for mathematics and science education, reform ideas, news items, search functions, links to other sites

Geometry Center
http://www.geom.umn.edu
Current projects, Web and Java applications, multimedia documents, software, video productions, geometry reference archives, course materials, workshops, awards gallery

Math Archives
http://archives.math.utk.edu/topics/discreteMath.html
Constructive Theory of Discrete Mathematics, counting problems, Discrete Applied Mathematics (journal), Discrete Mathematics (journal), links to other sites