Distributed Formation Control of Multi-Agent Systems Using Complex Laplacian

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***Abstract*—The paper concentrates on the fundamental coordination problem that requires a network of agents to achieve a specific but arbitrary formation shape. A new technique based on complex Laplacian is introduced to address the problems of which formation shapes specified by inter-agent relative positions can be formed and how they can be achieved with distributed control ensuring global stability. Concerning the first question, we show that all similar formations subject to only shape constraints are those that lie in the null space of a complex Laplacian satisfying certain rank condition and that a formation shape can be realized almost surely if and only if the graph modeling the inter-agent specification of the formation shape is 2-rooted. Concerning the second question, a distributed and linear control law is developed based on the complex Laplacian specifying the target formation shape, and provable existence conditions of stabilizing gains to assign the eigenvalues of the closed-loop system at desired locations are given. Moreover, we show how the formation shape control law is extended to achieve a rigid formation if a subset of knowledgable agents knowing the desired formation size scales the formation while the rest agents do not need to re-design and change their control laws.**

***Index Terms*—Distributed control, formation, graph Laplacian, multi-agent systems, stability.**

I. Introduction

In recent years, there has been a tremendous surge of interest among researchers from various disciplines of engineering and science in a variety of problems on networked multi-agent systems. Modeling the interaction topology of distributed agents as a graph, a main stream of research ([3], [23], [28], [31], [35]) concentrates on understanding and designing the mechanisms from the structure point of view on how collective behaviors emerge from local interaction in absence of high-level centralized supervision and global information exchange. An interesting example and area of ongoing research is the control of teams of autonomous mobile robots, unmanned aerial vehicles (UAVs), and autonomous underwater vehicles (AUVs), so that they work cooperatively to accomplish a common goal without centralized control and a global coordinate system. As teams of agents working together in formation can be found in various applications such as satellite formation flying, source seeking and exploration, ocean data retrieval, and map construction, much attention has been given to the control of formations. Studies concerning this subject focus primarily on the formation architecture as well as the stability of the formation systems. The former mainly concentrates on defining a formation using graph-theoretic rigidity [4], [16], [17], [22], [33], [40], while the latter concerns stabilization to a formation [7], [8], [10], [24], [32], [38], [39] and control of formation shape in moving [3], [5], [12]–[14], [20].

With regard to rigid formations, there have been several types of control strategies, e.g., affine feedback control laws [1], [2], [10], [18], [25], [29], [34], nonlinear gradient control laws [8], [13], [20], [24], [38], and very recently, angle-based control algorithms [6], [21], [30]. The goal is to achieve a formation with a determined size, which has only freedoms of translations and rotations. On the other hand, [9] studies the formation control problem with the objective of steering a team of agents into a formation of variable size. By allowing the size of the formation to change, the group can dynamically adapt to changes in the environment such as unforseen obstacles, adapt to changes in group objectives, or respond to threats.

In this paper, we concentrate on the fundamental coordination problem that requires the agents to achieve a specific but arbitrary formation shape. By *formation shape*, we are referring to the geometrical information that remains when location, scale, and rotational effects are removed. Thus, formation shape is invariant under the Euclidean similarity transformations of translation, rotation and scaling. The formation shape control problem is of its own interest if the agents do not have a notion of the world coordinate system's origin as well as unit of length or if the goal is to just form a pattern such that the agents can then agree on their respective roles in a subsequent, coordinated action. Moreover, formation shape control also serves as a basis for rigid formation control. As we show in this paper, when formation shape control is possible, a task of rigid formation control can be accomplished with a subset of knowledgable agents knowing the desired formation size, for which the advantage is that the rest agents do not need to re-design and change their control laws in order to achieve the desired formation scaled by the desired size.

In this context, the main research questions are which formation shape specified by inter-agent relative positions can be formed and how they can be achieved with distributed control ensuring global stability. Concerning the first question, we

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introduce the notion of similar formation and show that all similar formations subject to only shape constraints are those that lie in the null space of a complex Laplacian satisfying certain rank condition. Moreover, we prove that an equivalent graphical condition such that a formation shape can be realized is that the graph modeling the inter-agent specification of the formation shape is 2-rooted. This is a kind of new connectivity in graph theory, meaning that there exists a subset of two nodes from which every other node is 2-reachable. Concerning the second question, we develop a distributed and linear control law that is based on the complex Laplacian specifying the target formation shape and can be locally implemented by onboard sensing using relative position measurements. It is shown that for almost all complex Laplacians specifying the target shape, stabilizing gains exist to ensure not only globally asymptotic stability but also other performance specifications such as robustness and fast convergence speed by assigning the eigenvalues of the closed-loop system at desired locations. A procedure is also provided on how to find stabilizing gains. In addition, we show how the formation shape control law is extended to achieve a rigid formation with the formation size controlled by at least a pair of agents when they know the desired formation size.

The contributions of the paper are three-fold. First, the paper presents a systematic approach based on complex Laplacian for the formation shape control problem that is significant in the field. The work is an extension of our conference paper [37], including new developments on systematic construction of complex Laplacian for a given target formation shape, on finding stabilizing gains arbitrarily assigning the eigenvalues of the closed-loop system, and on how a rigid formation can be accomplished by controlling a subset of agents while the remaining agents still implement the same formation shape control law. Second, it provides a new way for rigid formation control by imposing one edge length constraints. Compared with globally rigid formation specified by interagent distances and nonlinear gradient control laws, the approach requires much less relative position measurements. Also, the approach makes possible that a large number of agents achieve a rigid formation almost globally by combining the nonlinear gradient control laws for a small number of agents to attain the edge length constraints, which are well studied with ensured almost global stability properties ([7], [8], [15], [20], [38]), and the simple linear formation shape control laws for the remaining agents. The approach has an advantage that a group of agents can easily change their formation size without a re-design of the control laws for all the agents. This property is more desirable in situations where the environment change is only observed by a minority of agents in the group. Most importantly, due to the use of linear control laws by most agents, it brings the hope by extending the approach to solve those challenging formation control problems in the setup of directed (time-varying) topology and in higher dimensional spaces. Third, the work provides an original analysis for understanding the relationship between complex graph Laplacians and graphical connectivity, which researchers from other disciplines may be interested in. Though the paper mainly focuses on the formation control problem of networked agents in the plane. The methods, however, are general, and they have applicability beyond multi-robot formations, e.g., distributed beamforming of communication systems and power networks where a pattern in the state is an objective.

The organization of the paper is as followers. We review the notations and some knowledge of graph theory in Section II. In Section III necessary and sufficient (algebraic and graphical) conditions are analyzed for similar formations. Global stabilization and stability analysis of multi-agent formations are presented in Section IV. Simulation and experiment results are given in Section V. Section VI concludes our work and points out several open problems along the path introduced in the paper.

II. Notation and Graph Theory

A. Notation

We denote by $\mathbb{C}$ and $\mathbb{R}$ the set of complex and real numbers, respectively. $\iota = \sqrt{-1}$ denotes the imaginary unit. For a complex number $p \in \mathbb{C}$, $|p|$ represents its modulus. For a set $\mathcal{E}$, $|\mathcal{E}|$ represents the cardinality. $\mathbf{1}_n$ represents the n-dimensional vector of ones and I_n denotes the identity matrix of order n. A *block diagonal matrix*, which has main diagonal block matrices $A_1, \ldots, A_n$ and off-diagonal blocks zero matrices, is denoted as $\mathrm{bd}[A_1, A_2, \ldots, A_n]$.

B. Graph Theory

An *undirected graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a non-empty node set $\mathcal{V} = \{1, 2, \ldots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ where an edge of $\mathcal{G}$ is a pair of un-ordered nodes. Undirected graphs can be considered as a special class of directed graphs with the edges consisting of pairs of ordered nodes, called *bidirectional graph*, for which each edge is converted into two directed edges, (i, j) and (j, i). In what follows we use the notion of bidirectional graph (or simply a graph for short) because the graph model we study is topologically equivalent to an undirected graph but different weights are considered on the edges of different order for the same pair of nodes. However, the graphical representation of undirected graphs is still used throughout the paper (i.e., we draw a line rather than two lines with arrows in the graph as the edges). A *walk* in a graph $\mathcal{G}$ is an alternating sequence p: $v_1 e_1 v_2 e_2 \cdots e_{k-1} v_k$ of nodes v_i and edges e_i such that $e_i = (v_i, v_{i+1})$ for every $i = 1, 2, \ldots, k-1$. We say that p is a *walk* from v_1 to v_k. If the nodes of a walk p are distinct, p ia a *path*. v_1 and v_k are called *terminal nodes* and other nodes are called *internal nodes*. A path is called a *Hamiltonian path* if it visits every node in the graph exactly once. Throughout the paper, we let $\mathcal{N}_i$ denote the neighbor set of node i, i.e., $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$. In the paper, we assume that a bidirectional graph does not have self-loops, which means $i \notin \mathcal{N}_i$ for any node i.

Next, we introduce two concepts.

Definition 2.1: For a bidirectional graph $\mathcal{G}$, a node v is said to be *2-reachable* from a non-singleton set $\mathcal{U}$ of nodes if there exists a path from a node in $\mathcal{U}$ to v after removing any one node except node v.

Definition 2.2: A bidirectional graph $\mathcal{G}$ is said to be *2-rooted* if there exists a subset of two nodes, from which every other

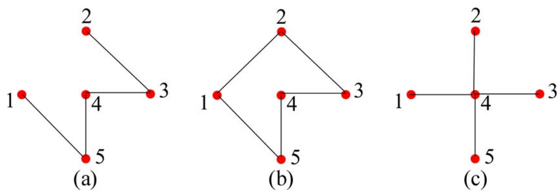

Fig. 1. Graphs that are 2-rooted and not 2-rooted.

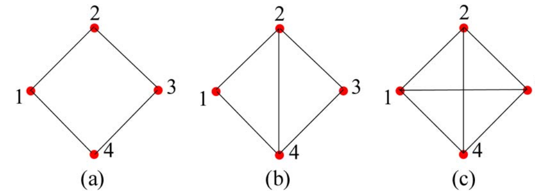

Fig. 2. (a) Not rigid. (b) Rigid but not globally rigid. (c) Globally rigid.

node is 2-reachable. These two nodes are called *roots* in the graph.

Consider for example the graphs in Fig. 1. In Fig. 1(a), let $\mathcal{U} = \{1, 2\}$ and it can be checked that node 3 is 2-reachable from $\mathcal{U}$ as after removing any one other node we are still able to find a path from a node in $\mathcal{U}$ to node 3. Similarly, it is known that node 4 and 5 are also 2-reachable from $\mathcal{U}$ in Fig. 1(a). Thus the graph in Fig. 1(a) is 2-rooted with the two roots being nodes 1 and 2. In Fig. 1(b), the graph is 2-rooted as well and any two nodes can be considered as roots in the graph. In Fig. 1(c), again let $\mathcal{U} = \{1, 2\}$ and it is known that node 3 is not 2-reachable from the set $\mathcal{U}$ as if we remove node 4, there is no path any more from any node in $\mathcal{U}$ to node 3. Furthermore, it can be verified that no matter how we select a subset of two nodes, there always exists another node that is not 2-reachable from the selected subset of nodes. Therefore, the graph in Fig. 1(c) is not 2-rooted.

Finally, we introduce a complex Laplacian for a bidirectional graph. The complex-valued Laplacian L of a bidirectional graph $\mathcal{G}$ is defined as follows: The ijth entry

$$L(i,j) = \begin{cases} -w_{ij} & \text{if } i \neq j \text{ and } j \in \mathcal{N}_i \\ 0 & \text{if } i \neq j \text{ and } j \notin \mathcal{N}_i \\ \sum\limits_{j \in \mathcal{N}_i} w_{ij} & \text{if } i = j \end{cases}$$

where $w_{ij} \in \mathbb{C}$. Note that the graph is a bidirectional graph, so the pattern of zero and nonzero entries of L is symmetric, but L may not be symmetric due to possibly different weights on the edges of the same pair of nodes but with different order.

The definition of complex Laplacian is nothing new from real Laplacian except that the nonzero entries can be complex numbers. Consequently, it is also true that a complex Laplacian has at least one eigenvalue at the origin whose associated eigenvector is $\mathbf{1}_n$ (namely, $L\mathbf{1}_n = 0$).

A *permutation matrix* is a square binary matrix that has exactly one entry 1 in each row and each column and 0's elsewhere. Renumbering the nodes of a graph is equivalent to apply a permutation transformation to the Laplacian. That is, $L' = PLP^T$ where L and L' are the Laplacian before and after renumbering the nodes, and P is the corresponding permutation matrix.

III. Necessary and Sufficient Conditions for Similar Formations

A. Overview of Rigid Frameworks With Distance Specifications

To introduce the notion of similar formation we will embed a graph in the complex plane $\mathbb{C}$ as a framework. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a bidirectional graph with n nodes. We embed $\mathcal{G}$ into $\mathbb{C}$ by assigning to each node i a location (complex number) $\xi_i \in \mathbb{C}$ in a reference frame Σ. Define the n-dimensional composite complex vector $\xi = [\xi_1, \xi_2, \ldots, \xi_n]^T \in \mathbb{C}^n$ a *formation configuration* in the reference frame Σ. A *framework* is a pair $(\mathcal{G}, \xi)$. Throughout the paper, we assume that $\xi_i \neq \xi_j$ if $i \neq j$, meaning that no two nodes are overlapping each other.

In the following, we review a little bit about rigidity of graphs using the distance specifications. The materials below are taken from [24]. Associated with the framework $(\mathcal{G}, \xi)$, define a function $g : \mathbb{C}^n \to \mathbb{R}^{|\mathcal{E}|}$ by

$$g(\xi) := \left[\cdots \ |\xi_i - \xi_j|^2 \ \cdots \right]^T,$$

called a *rigid function*. The kth component of $g(\xi)$, $|\xi_i - \xi_j|^2$, corresponds to the edge $e_k \in \mathcal{E}$, where nodes i and j are connected by e_k, and specifies a desired edge length d_k. Let $d = [\cdots \quad d_k \quad \cdots]^T$ be the composite vector describing the distance specifications on the edges in $\mathcal{G}$. Then the notions of rigidity and global rigidity can be stated as follows.

Definition 3.1: A framework $(\mathcal{G}, \xi)$ specified by $g(\xi) = d$ is *rigid* if there exists a neighborhood $\mathcal{B} \subset \mathbb{C}^n$ of ξ such that

$$g^{-1}(d) \cap \mathcal{B} = \left\{ c_1 \mathbf{1}_n + e^{\iota\theta}\xi : \ c_1 \in \mathbb{C} \text{ and } \theta \in [0, 2\pi) \right\}.$$

Definition 3.2: A framework $(\mathcal{G}, \xi)$ specified by $g(\xi) = d$ is *globally rigid* if

$$g^{-1}(d) = \left\{ c_1 \mathbf{1}_n + e^{\iota\theta}\xi : \ c_1 \in \mathbb{C} \text{ and } \theta \in [0, 2\pi) \right\}.$$

The level set $g^{-1}(d)$ consists of all possible points that have the same edge lengths as the framework $(\mathcal{G}, \xi)$. The set $\{c_1 \mathbf{1}_n + e^{\iota\theta}\xi : c_1 \in \mathbb{C}$ and $\theta \in [0, 2\pi)\}$ consists of points related by rotations θ and translations c_1, i.e., rigid body motions, of the framework $(\mathcal{G}, \xi)$. Therefore, a framework is rigid if the level set $g^{-1}(d)$ in a neighborhood of ξ contains only points corresponding to rotations and translations of the formation configuration ξ. A framework is globally rigid if the level set $g^{-1}(d)$ in $\mathbb{C}^n$ contains only points corresponding to rotations and translations of the formation configuration ξ.

For example, consider the framework in Fig. 2(a). It is possible to translate only nodes 1 and 2, while maintaining the four edge lengths, to a formation that is not attained by rigid body motions, so the framework specified by $g(\xi) = d$ is not rigid. If we add one more edge to obtain a framework as in Fig. 2(b), the only motion to maintain the five edge lengths in the neighborhood is a rigid body motion (rotations and translations). As a result, the framework is rigid. But node 1 can have a flip along the edge connecting 2 and 4, while the edge lengths are preserved, so it is not globally rigid. Fig. 2(c) shows a globally rigid framework.

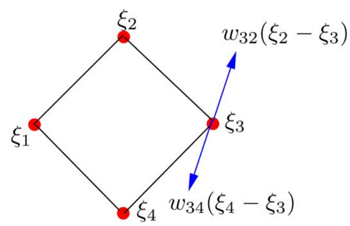

Fig. 3. Illustration of a linear constraint for a framework.

B. Linear Constraints and Similar Formations

From the preceding subsection, it is clear that in order to make a framework rigid (or globally rigid), each node in the graph has to have at least two neighbors as otherwise if a node has only one neighbor, this node can swing around its neighbor. By observing this fact, we will then introduce a new linear constraint for a framework rather than the distance constraints on the edges of the graph. For each node i in the graph, since it has at least two neighbors, we can define a linear constraint for the framework as follows:

$$\sum_{j\in\mathcal{N}_i} w_{ij}(\xi_j - \xi_i) = 0$$

for appropriate complex weights w_{ij}'s defined on the edges linking to node i. The complex weights make the relative state vectors rotated and scaled so that the summation becomes 0 for a given framework, and thus provide a linear constraint. Take Fig. 3 as an example. Node 3 has two neighbors (namely, 2 and 4). So the complex weights w_{32} and w_{34} rotate and scale the relative states $\xi_2 - \xi_3$ and $\xi_4 - \xi_3$ respectively so that the summation is zero as shown in Fig. 3. We should point out that the choice of such complex weights is not unique.

Taking the linear constraint on every node, we derive a composite constraint for the framework as follows:

$$L\xi = 0$$

where L is the complex Laplacian corresponding to the bidirectional graph $\mathcal{G}$ whose nonzero off-diagonal entry is $-w_{ij}$, the negative weight on edge (j,i). Now we are ready to introduce the notion of similar formation.

Definition 3.3: A framework $(\mathcal{G},\xi)$ specified by $L\xi = 0$ is *similar* if

$$\ker(L) = \{c_1\mathbf{1}_n + c_2\xi : \ c_1, c_2 \in \mathbb{C}\}.$$

Remark 3.1: Note that a complex number c_2 can be written in the polar coordinate form (namely, $c_2 = \rho e^{\iota\theta}$). So the solutions to the linear constraint $L\xi = 0$ consist of points related by translations c_1, rotations θ, and scaling ρ (four degrees of freedom). That is, the formations subject to the linear constraint $L\xi = 0$ are scalable from the formation configuration ξ in addition to rigid body motions (translations and rotations). Therefore, one additional distance constraint on an edge will make the framework become globally rigid.

Fig. 4. A path graph of n nodes with its terminal nodes labeled as 1 and 2.

C. Necessary and Sufficient Conditions

In this subsection we are going to explore the necessary and sufficient algebraic and graphical conditions for similar frameworks.

Theorem 3.1: A framework $(\mathcal{G},\xi)$ specified by $L\xi = 0$ is similar if and only if $\operatorname{rank}(L) = n-2$.

Proof: (Sufficiency) L has a zero eigenvalue with an associated eigenvector ξ because $L\xi = 0$. Furthermore, since L is a Laplacian matrix, so $L\mathbf{1}_n = 0$, meaning that $\mathbf{1}_n$ is another eigenvector associated with the zero eigenvalue. The two eigenvectors ξ and $\mathbf{1}_n$ are linearly independent because $\xi_i \neq \xi_j$. Moreover, by the assumption $\operatorname{rank}(L) = n-2$, we know that L has only two zero eigenvalues. Thus the null space of L is $\{c_1\mathbf{1}_n + c_2\xi : c_1, c_2 \in \mathbb{C}\}$ and so the framework $(\mathcal{G},\xi)$ specified by $L\xi = 0$ is similar.

(Necessity) Suppose on the contrary that $\operatorname{rank}(L) \neq n-2$. Then $\operatorname{rank}(L)$ must be less than $n-2$ since we already have $L\xi = 0$ and $L\mathbf{1}_n = 0$. Thus, it follows that the null space of L is of 3-dimension at least and $\ker(L) \neq \{c_1\mathbf{1}_n + c_2\xi : c_1, c_2 \in \mathbb{C}\}$, which contradicts to the condition that the framework $(\mathcal{G},\xi)$ specified by $L\xi = 0$ is similar. ■

Theorem 3.1 presents an algebraic condition requiring to check whether $\operatorname{rank}(L) = n-2$. We give a graphical condition.

Theorem 3.2: A framework $(\mathcal{G},\xi)$ specified by $L\xi = 0$ (for almost all L satisfying $L\xi = 0$) is similar if and only if $\mathcal{G}$ is 2-rooted.

The proof requires a lemma.

Lemma 3.1: Consider a framework $(\mathcal{G},\xi)$ where $\mathcal{G}$ is a path graph of n nodes with its terminal nodes labeled as 1 and 2 (Fig. 4). If $\xi_i \neq \xi_j$ for $i \neq j$, then there exists a complex Laplacian

$$L = \left[\begin{array}{c|c} A_{2\times 2} & B_{2\times(n-2)} \\ \hline C_{(n-2)\times 2} & D_{(n-2)\times(n-2)} \end{array}\right]$$

such that $L\xi = 0$ and D is of rank $n-2$.

Proof: If necessary, relabel the internal nodes of the path graph $\mathcal{G}$ in an order from 3 to n as shown in Fig. 4. Under this labelling scheme, it is then clear that D is tri-diagonal. Denote the first row of D by d_1^T and the remaining rows of D by $\bar{D}$. Moreover, note that node 1 has only one neighbor (namely, node 3), so in the first column of C only the (1,1)-entry is nonzero by the definition of L. Denote the (1,1)-entry of C by c_1. Then we can write C as

$$C = \left[\begin{array}{c|c} c_1 & 0 \\ \hline 0 & \bar{c}_2 \end{array}\right]$$

where $\bar{c}_2 \in \mathbb{C}^{(n-3)}$. From the definition of Laplacian, it follows that

$$c_1 = -d_1^T\mathbf{1}, \quad \bar{c}_2 = -\bar{D}\mathbf{1}. \tag{1}$$

Suppose for an L satisfying $L\xi = 0$ that D is not of rank $n-2$. Moreover, notice that the rows of $\bar{D}$ are linearly independent.

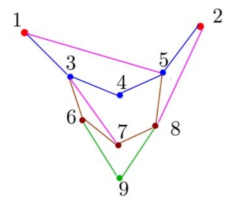

Fig. 5. Example of the relabeling procedure, where $\mathcal{U}_0 = \{1, 2\}$, $\mathcal{U}_1 = \{3, 4, 5\}$, and so on.

So there must exist an $(n-3)$-dimensional vector λ such that $d_1^T = \lambda^T \bar{D}$. Moreover, using (1), we obtain that $c_1 = \lambda^T \bar{c}_2 \neq 0$. From $L\xi = 0$, thus we have

$$c_1\xi_1 + d_1^T \xi' = 0 \tag{2}$$

and

$$\bar{c}_2\xi_2 + \bar{D}\xi' = 0 \tag{3}$$

where ξ' is the sub-vector formed by the last $n-2$ entries of ξ. Pre-multiplying λ^T to (3) and using $c_1 = \lambda^T \bar{c}_2$ and $d_1^T = \lambda^T \bar{D}$ result in

$$c_1\xi_2 + d_1^T \xi' = 0. \tag{4}$$

Comparing (4) and (2) we obtain that $\xi_1 = \xi_2$, a contradiction. Therefore, D is of rank $n-2$. ■

Proof of Theorem 3.2: (Sufficiency) If $\mathcal{G}$ is 2-rooted, then from Definition 2.2, there is a subset of two nodes, from which every other node is 2-reachable. Without loss of generality, denote the subset by $\mathcal{U}_0$ and label the two nodes in $\mathcal{U}_0$ by 1 and 2. Select any node i not in $\mathcal{U}_0$ and then we can find two disjoint paths (no common nodes in these two paths except i) from 1 to i and from 2 to i since node i is 2-reachable from $\mathcal{U}_0$. Denote the set of nodes in these two paths excluding the nodes in $\mathcal{U}_0$ by $\mathcal{U}_1$ and denote n_1 the total number of nodes in $\mathcal{U}_1$. Relabel the nodes in $\mathcal{U}_1$ from 3 to $n_1 + 2$. The next step is then to select another node, say j, not in $\mathcal{U}_0 \cup \mathcal{U}_1$. Also, because node j is 2-reachable from $\mathcal{U}_0$, there must be two disjoint paths from two different nodes in $\mathcal{U}_0 \cup \mathcal{U}_1$ to node j, for which only the two terminal nodes are in $\mathcal{U}_0 \cup \mathcal{U}_1$. Denote n_2 the total number of nodes in these two paths excluding the two terminal nodes in $\mathcal{U}_0 \cup \mathcal{U}_1$ and relabel these nodes from $n_1 + 3$ to $n_1 + n_2 + 2$. Repeat the procedure until all the nodes are included. An illustration is presented in Fig. 5. According to the procedure, it is clear that

$$\sum_i n_i + 2 = n.$$

Take the graph $\mathcal{G}'$ with only edges included in the paths in the procedure. It is a subgraph of $\mathcal{G}$ with the same node set. Notice that if a node i in $\mathcal{U}_{m_1}$ is also a terminal node of some paths composed of nodes in $\mathcal{U}_{m_2}$ for some $m_2 > m_1$, this node has more than two neighbors as it already has two neighbors in $\cup_{k=0,\ldots,m_1}\mathcal{U}_k$. So we can select 0 for the complex weight w_{ij} where $i \in \mathcal{U}_{m_1}$ and $j \in \mathcal{U}_{m_2}$ with $m_2 > m_1$. Thus, the Laplacian L' is of the following form:

$$L' = \begin{bmatrix} L_0 & * & * & * \\ * & L_1 & 0 & 0 \\ * & * & L_2 & 0 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where L_i is the corresponding block to the the subgraph induced by $\mathcal{U}_i$ in $\mathcal{G}'$. By our construction, we know that the subset $\mathcal{U}_i$ of nodes together with its two terminal nodes form a path graph. Thus, by applying Lemma 3.1 it follows that $\text{rank}(L_i) = n_i$. Considering the particular structure of L', we know that

$$\text{rank}(L') \geq \sum_{i=1,\ldots,} \text{rank}(L_i) = \sum_{i=1,\ldots,} n_i = n - 2.$$

Notice that L' can be considered as a Laplacian of the graph $\mathcal{G}$ for a special choice of weights with some being 0. Thus, by using the fact that either a polynomial is zero or it is not zero almost everywhere, it follows that for almost all complex weights satisfying $L\xi = 0$, there exists a non-zero principal minor of $(n-2)$th order. Therefore, $\text{rank}(L) \geq n-2$. On the other hand, since ξ and $\mathbf{1}_n$ are two independent eigenvectors of L corresponding to the zero eigenvalue, we have $\text{rank}(L) = n-2$. As a result of Theorem 3.1, the framework $(\mathcal{G}, \xi)$ specified by $L\xi = 0$ (for almost all L satisfying $L\xi = 0$) is similar.

(Necessity) We prove it in a contrapositive form. Suppose that the graph $\mathcal{G}$ is not 2-rooted. As a result, we can not find two nodes to be roots from which all the nodes are 2-reachable. Since $L\xi = 0$ and $L\mathbf{1} = 0$, there must be two rows of L, say l_q and l_p, which can be transformed to zero vectors by elementary row operations. Choose the two nodes p and q corresponding to the two rows as roots and after removing a node, some nodes are not reachable from the subset of roots. Without loss of generality, suppose after removing a node k there exist a subset $\mathcal{W}$ consisting of $k-1$ nodes which are not reachable from any root and a set $\bar{\mathcal{W}}$ consisting of $n-k$ nodes which are reachable from one of the roots. Relabel the nodes in $\mathcal{W}$ as $1, \ldots, k-1$ and relabel the nodes in $\bar{\mathcal{W}}$ as $k+1, \ldots, n$. Then it is certain that the nodes in $\mathcal{W}$ are not reachable from any node in $\bar{\mathcal{W}}$. Equivalently, $L(i,j) = 0$ for $i \in \mathcal{W}$ and $j \in \bar{\mathcal{W}}$. Thus L is of the following form:

$$\left[\begin{array}{c|c|c} L_w & c_w & 0 \\ \hline * & * & * \end{array}\right]$$

where $L_w \in \mathbb{C}^{(k-1)\times(k-1)}$ and $c_w \in \mathbb{C}^{k-1}$. Denote the formation configuration ξ after relabelling by $[\xi_a^T, \xi_b^T]^T$ where $\xi_a \in \mathbb{C}^k$ and $\xi_b \in \mathbb{C}^{(n-k)}$. According to the definition of L, then we have

$$[L_w \; c_w]\mathbf{1}_k = 0 \text{ and } [L_w \; c_w]\xi_a = 0.$$

As $\mathbf{1}_k$ and ξ_a are linearly independent by assumption, then $\text{rank}([L_w \; c_w]) \leq k-2$. That is, there exists a row which can be turned into the zero vector under elementary row operations. Therefore, $\text{rank}(L) \leq n-3$, or equivalently by Theorem 3.1,

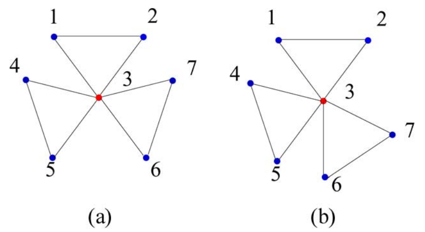

Fig. 6. If a graph $\mathcal{G}$ is not 2-rooted then the framework $(\mathcal{G}, \xi)$ specified by the distance constraint $g(\xi) = d$ is not rigid.

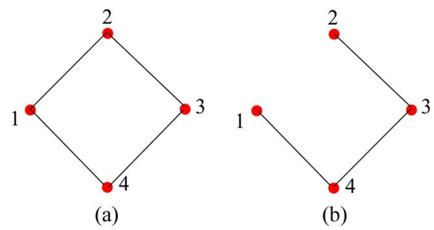

Fig. 7. Frameworks $(\mathcal{G}, \xi)$ specified by the distance constraint $g(\xi) = d$ are not rigid, but they are similar when specified by the linear constraint $L\xi = 0$.

it is not true that the framework $(\mathcal{G}, \xi)$ specified by $L\xi = 0$ is similar. ■

Theorem 3.2 shows that 2-rooted connectivity is a necessary and sufficient condition for a framework $(\mathcal{G}, \xi)$ specified by the linear constraint $L\xi = 0$ to be similar for almost all complex Laplacian $L(\mathcal{G})$.

Remark 3.2: It is worth to point out that a graph $\mathcal{G}$ (of $n \geq 3$ nodes) for a rigid framework $(\mathcal{G}, \xi)$ using the distance constraint $g(\xi) = d$ must also be 2-rooted. This can be seen by the following fact. If $\mathcal{G}$ is not 2-rooted, then for any subset of two nodes, there always exists another node that is not 2-reachable from the subset. That is, after removing a node, the graph can be divided into at least two subgraphs that are not connected to each other. An example is given in Fig. 6, for which after removing node 3, it results in three subgraphs that are not connected. This means, in addition to rigid body motions, another motion exists while preserving the distance constraint $g(\xi) = d$ [see for example Fig. 6(a) and 6(b)].

However, the reverse is not true. In other words, to make a framework $(\mathcal{G}, \xi)$ specified by the distance constraint $g(\xi) = d$ rigid, the graph $\mathcal{G}$ requires more links than just 2-rooted connectivity. From the well-known result by Laman in 1970 [26], the minimal requirement for a framework specified by $g(\xi) = d$ to be rigid is that the graph should have at least $2n - 3$ edges where n is the number of nodes. From our analysis we can know that the minimally 2-rooted graph requires only $n - 1$ edges, which corresponds to the path graph. So it requires much less links when specifying a similar framework in terms of the linear constraint $L\xi = 0$.

In Fig. 7, both (a) and (b) are not rigid if the framework is specified by the distance constraint $g(\xi) = d$, while they are similar if the framework is specified by the linear constraint $L\xi = 0$. Fig. 7(b) is a minimally 2-rooted graph that has only $n - 1$ edges.

D. A Systematic Approach for the Construction of L

In the following, we present a systematic approach for the construction of L from the individual viewpoint. That is, for a

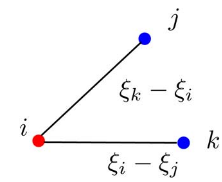

Fig. 8. Example of weight selection for a node having two neighbors.

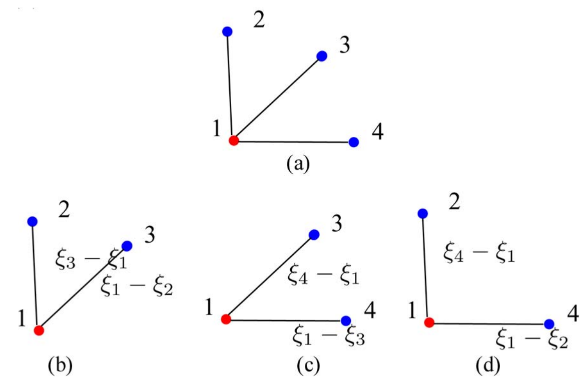

Fig. 9. Example of weight selection for a node having more than two neighbors.

given 2-rooted graph $\mathcal{G}$ and a formation configuration ξ, each agent i finds the weights w_{ij}'s for $j \in \mathcal{N}_i$ such that $L\xi = 0$.

As we discussed above, every node of a 2-rooted graph has at least two neighbors. In the following, we consider two cases.

First, consider the case that node i has exactly two neighbors. Suppose without loss of generality, its two neighbors are j and k. Then the weights w_{ij} and w_{ik} can be parameterized as follows:

$$[\, w_{ij} \quad w_{ik} \,] = p_i^1 [\, \xi_k - \xi_i \quad \xi_i - \xi_j \,]$$

where p_i^1 is a nonzero complex number and can be chosen randomly. That is, $[w_{ij}\ w_{ik}]$ is in the linear span of $[\xi_k - \xi_i\ \xi_i - \xi_j]$ that solely depends on the formation configuration ξ. An example is given in Fig. 8.

Second, consider the case that node i has more than two neighbors. Say without loss of generality that it has totally m $(m > 2)$ neighbors, labeled by $i_1, \ldots, i_m$. Select any two neighbors, denoted by i_j and i_k, from the m neighbors, and define an m-dimensional vector ζ_h with the i_jth entry being $\xi_{i_k} - \xi_i$, the i_kth entry being $\xi_i - \xi_{i_j}$, and the others being zero. Note that there are totally C_m^2 (the binomial coefficient) selections of two neighbors out of m neighbors. Thus, the weights $w_{ii_1}, \ldots, w_{ii_m}$ can be parameterized as follows:

$$[\, w_{ii_1} \quad \cdots \quad w_{ii_m} \,] = \sum_{h=1}^{C_m^2} p_i^h \zeta_h \tag{5}$$

where p_i^h, $h = 1, \ldots, C_m^2$, is a nonzero complex number and can be chosen randomly. An illustrative example is given Fig. 9(a) for which node 1 has three neighbors. So it has three choices of selecting any two neighbors as shown in Fig. 9(b)–(d). Then the weight vector is a linear combination according to (5).

IV. Formation Control of Multi-Agent Systems

A. Stabilization Problem of Multi-Agent Formations

We consider a group of n agents (for example, mobile robots) in the plane. The positions of n agents are denoted by complex numbers $z_1, \ldots, z_n \in \mathbb{C}$. Each agent i is assumed to have an onboard sensor allowing it to measure the relative positions of some of the other agents, that is, $z_j - z_i$ when agent j is a neighbor of agent i. We consider that each agent i has a point kinematic model given by the single integrator

$$\dot{z}_i = u_i \tag{6}$$

where $u_i \in \mathbb{C}$ represents the velocity control input. Define the aggregate state $z = [z_1 \ \cdots \ z_n]^T$, as a complex vector in $\mathbb{C}^n$.

The target formation is described by a framework $(\mathcal{G}, \xi)$ where $\mathcal{G}$ is a bidirectional graph whose nodes represent the agents, and $\xi \in \mathbb{C}^n$ is a formation configuration defined in a reference frame. We refer to $\mathcal{G}$ as the *formation graph*. The agents achieve the formation shape when z is in $\{c_1\mathbf{1}_n + c_2\xi : c_1, c_2 \in \mathbb{C}\}$, i.e., it reaches a formation that is a translated, rotated, and scaled version of ξ.

Associated with the formation shape control problem is also a *sensor graph* that describes the sensor data seen by each agent in the closed-loop system. We assume here the sensor graph is the same as the formation graph. That is, it is also a bidirectional graph with each node i representing an agent, and each edge (j, i) representing a relative state measurement (i.e., $(z_j - z_i)$ available to agent i). For this, no global knowledge such as a common reference frame and a common unit of length is needed, and no communication is required.

The problem is then given as follows. Consider the system (6) and a target formation described by a framework $(\mathcal{G}, \xi)$ specified by $L\xi = 0$, which is similar. Design a distributed control law u_i based on the sensed relative state information such that (i) every point z^* in $\{c_1\mathbf{1}_n + c_2\xi : c_1, c_2 \in \mathbb{C}\}$ is a stable equilibrium of the closed-loop system and (ii) for every initial condition $z(0)$, the closed-loop trajectory approaches to a unique equilibrium in $\{c_1\mathbf{1}_n + c_2\xi : c_1, c_2 \in \mathbb{C}\}$. The problem is referred as a *global stabilization problem of multi-agent formations*.

B. A Local Control Law for Shape Control

In this subsection, we propose a distributed control law to solve the global stabilization problem of multi-agent formations. We consider the following control law

$$u_i = k_i \sum_{j \in \mathcal{N}_i} w_{ij}(z_j - z_i), \quad i = 1, \ldots, n \tag{7}$$

where $k_i \in \mathbb{C}$ is a control parameter to be designed, and w_{ij} is the complex weight on the corresponding edge in the formation graph that defines L such that $L\xi = 0$. The selection of w_{ij} is shown in Section III-D and can be done by agents themselves in a distributed manner.

The control law (7) can be locally implemented by onboard sensors without requiring all the agents to have a common sense of direction and scale unit. However, a common notion of clockwise rotation should be shared by all the agents. Consider for example that agent i has two neighbors j and k. With the onboard sensor (e.g., camera) on agent i, it can measure the relative states $z_j - z_i$ and $z_k - z_i$ in its local frame with the x-axis coincident to the optical axis. Then it rotates the two vectors in its local frame and takes the sum to get the velocity control input as shown in Fig. 10. A more detailed discussion on how to locally implement a control law on a point-mass robot using relative position measurements refers to [27, pp. 141–143].

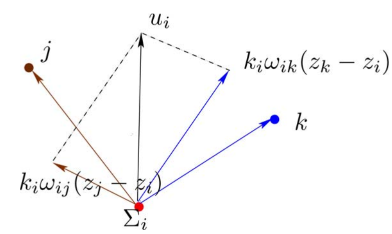

Fig. 10. Illustration of the locally implementable control law.

Under the distributed control law (7), the overall closed-loop dynamics of n agents becomes

$$\dot{z} = -KLz \tag{8}$$

where $K = \operatorname{diag}\{k_1, \ldots, k_n\}$ is an n-by-n diagonal complex matrix.

It is clear that if $k_i \neq 0$ for $i = 1, \ldots, n$ and the target framework $(\mathcal{G}, \xi)$ specified by $L\xi = 0$ is a similar framework, then the equilibrium set of system (8) is $E = \{c_1\mathbf{1}_n + c_2\xi : c_1, c_2 \in \mathbb{C}\}$. Then the next issue is how to ensure that every trajectory asymptotically converges to an equilibrium in E.

C. Existence of a Stabilizing Matrix

Note that the closed-loop system (8) is a linear time-invariant system. So the asymptotic stability can be veryfied by checking the eigenvalues of KL. Unlike real Laplacian matrices that always have all eigenvalues in the right complex plane, a complex Laplacian may have eigenvalues in the left complex plane. Consider for example the following complex Laplacian:

$$L = \begin{bmatrix} -1-\iota & 2 & -1+\iota \\ 1+\iota & -2 & 1-\iota \\ 1+\iota & -2 & 1-\iota \end{bmatrix}.$$

In addition to two eigenvalues at the origin, it has an eigenvalue at $-2 - 2\iota$, that lies in the open left complex plane. Therefore, it is important to design a proper K such that the eigenvalues of KL lie in the right complex plane. We refer to K as a *stabilizing matrix* if it is able to shift the eigenvalues of $-KL$ to the open left complex plane in addition to two fixed eigenvalues at the origin.

Theorem 4.1: Given a 2-rooted graph $\mathcal{G}$ and a formation configuration ξ, for almost all Laplacians L of $\mathcal{G}$ satisfying $L\xi = 0$, a stabilizing matrix K exists and can assign the eigenvalues of $-KL$ at any desired locations in addition to the two fixed zero eigenvalues.

The proof requires a result related to the multiplicative inverse eigenvalue problem by Friedland in 1975.

Theorem 4.2 ([19]): Let A be an $n \times n$ complex-valued matrix. Let $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ be an arbitrary set of n complex numbers. If all principal minors of A are distinct from zero, then there exists a diagonal complex valued matrix M, such that the spectrum of MA is the set σ. Moreover, the number of different matrices M is at most $n!$.

Proof of Theorem 4.1: Given a 2-rooted graph $\mathcal{G}$, there are two nodes called roots, from which every other node is 2-reachable. If necessary, relabel the two roots by $n-1$ and n, and relabel other nodes accordingly. Let L be a complex Laplacian of the graph $\mathcal{G}$ after relabeling and satisfy $L\xi = 0$. Then L has the following form

$$L = \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]$$

in which $B_1 \in \mathbb{C}^{(n-2)\times(n-2)}$, $B_2 \in \mathbb{C}^{(n-2)\times 2}$, $B_3 \in \mathbb{C}^{2\times(n-2)}$ and $B_4 \in \mathbb{C}^{2\times 2}$.

Next we show that all principal minors of B_1 are distinct from zero for almost all choices of L that is a Laplacian of a 2-rooted graph $\mathcal{G}$ and satisfies $L\xi = 0$.

Since the graph $\mathcal{G}$ is 2-rooted, it is clear that each node other than the roots has at least two neighbors. So by the definition of Laplacian matrix, we know that the diagonal entries of B_1 are nonzero for almost all L's, which means all the 1st order principal minors of B_1 are distinct from zero. Suppose now all the principal minors of B_1 up to the $(m-1)$-th order are distinct from zero. We will show that all the m-th $(m \leq n-2)$ order principal minors of B_1 are distinct from zero as well. Consider a subset of any m nodes that are not roots and denote it as $\mathcal{W} = \{i_1, \ldots, i_m\}$. Correspondingly, denote W the m-by-m matrix formed from the entries of B_1 by selecting the rows and columns with indices in $\mathcal{W}$. We discuss two cases. First, if the subgraph induced by the node set $\mathcal{W}$ has a Hamiltonian path, then the Hamiltonian path together with two nodes outside of $\mathcal{W}$ forms a path graph. Thus by Lemma 3.1, it follows that W is of full rank. So the determinant of W is nonzero for a choice of L. Second, if the subgraph induced by the node set $\mathcal{W}$ does not have a Hamiltonian path, then we can find a subset of nodes, denoted as $\mathcal{W}_1$, belonging to a path $\mathcal{S}$ connecting two nodes outside of $\mathcal{W}$ and a subset of remaining nodes, denoted as $\mathcal{W}_2$, that might or might not connect to the path. Note that every node i in $\mathcal{W}_1$ has two neighbors on the path $\mathcal{S}$. So we can select 0 for the complex weight w_{ij} where $i \in \mathcal{W}_1$ and $j \in \mathcal{W}_2$, while satisfying $L\xi = 0$. Thus, via an appropriate permutation transformation Q, W for this special choice of weights is of the following form:

$$QWQ^T = \left[\begin{array}{c|c} W_1 & 0 \\ \hline * & W_2 \end{array}\right].$$

By our assumption that all the principal minors of B_1 up to the $(m-1)$-th order are distinct from zero (namely, the determinants of both W_1 and W_2 are nonzero). Thus the determinant of W is nonzero for this special choice of weights. For both cases, applying the fact that either a polynomial is zero or is not zero almost everywhere, we can conclude that for almost all complex weights satisfying $L\xi = 0$, the determinant of W is nonzero, or equivalently, any m-th order principal minor of B_1 is distinct from zero.

By induction, we just showed that all principal minors of B_1 are distinct from zero for almost all L satisfying $L\xi = 0$. Therefore, by Theorem 4.2, there exists a diagonal complex matrix M_1 arbitrarily assigning the eigenvalues of $M_1 B_1$, which implies, a stabilizing matrix K exists and the eigenvalues of KL can be assigned at any locations in addition to the two fixed zero eigenvalues. ■

Remark 4.1: With the help of Theorem 4.1, we can know that by randomly choosing the parameters p_i^h to construct L as discussed in Section III-D, the obtained L has its principal minors up to the $(n-2)$th order distinct from zero in probability one. Thus, we can select k_i so that the eigenvalues of the closed-loop system (8) lie at any desired locations. For only the purpose of asymptotic stability, it is sufficient to have the eigenvalues of the closed-loop system at the open left half complex plane. For some additional performance requirements such as robustness, it may be desirable to have the eigenvalues of the closed-loop system far away from the imaginary axis.

Remark 4.2: In the formation control literature [36], [39], the design of a stabilizing matrix is also studied. It is proven that choosing stabilizing gains is possible if a certain sub-matrix of the rigidity matrix has all leading principal minors nonzero and is shown that this condition holds for all minimally persistent leader-remote-follower and co-leader formations with generic agent positions. In Theorem 4.1 we show that not only leading principal minors but also all principal minors are nonzero for generic Laplacian L satisfying $L\xi = 0$, which ensures the existence of a stabilizing matrix that can assign the eigenvalues not only in the left complex plane but also at any desired locations in the left complex plane in addition to the fixed zero eigenvalues.

Theorem 4.3: Given a 2-rooted graph $\mathcal{G}$ and a formation configuration ξ, if K assigns the eigenvalues of KL in the open right complex plane in addition to the two fixed zero eigenvalues, then a network of agents under the distributed control law (7) globally asymptotically converges to a formation $c_1\mathbf{1}_n + c_2\xi$ with $c_1, c_2 \in \mathbb{C}$.

Proof: If K assigns the eigenvalues of KL in the open right complex plane in addition to the two fixed zero eigenvalues with associated linearly independent eigenvectors $\mathbf{1}_n$ and ξ, then there is a similarity transformation V with its first two columns being $\mathbf{1}_n$ and ξ such that

$$-V^{-1}KLV = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda \end{bmatrix}$$

where $\Lambda \in \mathbb{C}^{(n-2)\times(n-2)}$ has all its eigenvalues in the open left complex plane. Thus, by the coordinate transformation

$$\begin{bmatrix} y_1 \\ y_2 \\ \bar{y} \end{bmatrix} = V^{-1}z \quad (9)$$

the system (8) is transformed to

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\bar{y}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \bar{y} \end{bmatrix}. \tag{10}$$

Then from (10), it follows that $y_1(t) = y_1(0)$, $y_2(t) = y_2(0)$, and $\bar{y}(t)$ globally asymptotically converges to 0 as $t \to \infty$. Therefore, by the coordinate transformation (9), it is obtained that $z(t)$ globally asymptotically converges to $y_1(0)\mathbf{1}_n + y_2(0)\xi$, which completes the proof. ■

Remark 4.3: Let v_1^T and v_2^T be the first two rows of V^{-1} (that is, they are the left eigenvectors of KL associated to the zero eigenvalues). Then it is clear from (9) that $y_2(0) \neq 0$ if $v_2^T z(0) \neq 0$. In other words, for almost all initial conditions $z(0)$, they are not orthogonal to the vector v_2 and convergence does not occur to a formation where all agents are coincident.

D. Design of Stabilizing Matrix

Theorem 4.1 shows the existence of a stabilizing matrix K such that the closed-loop trajectory globally asymptomatically converges to an equilibrium formation. Next we are going to present an algorithm on how to design a stabilizing matrix K such that the eigenvalues of the closed-loop system (8) lie exactly at $\sigma = \{\lambda_1, \ldots, \lambda_{n-2}, 0, 0\}$. The desired eigenvalues $\lambda_1, \ldots, \lambda_{n-2}$ can be chosen according to additional performance requirements.

Since L has rank $n-2$, it then follows that L can be factorized into $L = UV$ where $U \in \mathbb{C}^{n\times(n-2)}$ and $V \in \mathbb{C}^{(n-2)\times n}$ satisfy $\operatorname{rank}(U) = \operatorname{rank}(V) = n-2$. Notice that

$$\det(sI + KL) = \det(sI + KUV) = s^2 \det(sI + VKU).$$

The problem of designing k_i $(i = 1, \ldots, n)$ such that the spectrum of $-KL$ is the set σ is equivalent to the problem of finding k_i $(i = 1, \ldots, n)$ such that VKU has eigenvalues at

$$\{-\lambda_1, \ldots, -\lambda_{n-2}\}.$$

Theorem 4.1 ensures the existence of K, but in a generic sense there are infinite number of solutions for the above problem. So we could arbitrarily assign two values to two k_i's. Without loss of generality, select k_{n-1} and k_n and set $k_{n-1} = k_n = 1$. Denote $\bar{k} = (k_1, k_2, \ldots, k_{n-2})$ and denote $A(\bar{k}) = VKU$ with $k_{n-1} = k_n = 1$. Generically, there are $(n-2)!$ solutions of $\bar{k}$ to assign the eigenvalues of $A(\bar{k})$ at

$$\{-\lambda_1, \ldots, -\lambda_{n-2}\}.$$

In the following, we consider a Newton iteration method to solve $\bar{k}$. Define

$$F(\bar{k}) = \begin{bmatrix} \det\left(A(\bar{k}) + \lambda_1 I\right) \\ \vdots \\ \det\left(A(\bar{k}) + \lambda_{n-2} I\right) \end{bmatrix} = \begin{bmatrix} F_1(\bar{k}) \\ \vdots \\ F_{n-2}(\bar{k}) \end{bmatrix} \tag{11}$$

where $\det(\cdot)$ represents the determinant of a matrix. Clearly, $\bar{k}$ is a solution of the eigenvalue assignment problem if and only if $F(\bar{k}) = 0$.

To apply the Newton iteration method, we need to calculate the derivative of $F(\bar{k})$ with respect to $\bar{k}$. Denote

$$g_{ij} = \frac{\partial F_i(\bar{k})}{\partial k_j}, \quad i, j = 1, \ldots, n-2 \quad \text{and} \quad G(\bar{k}) = [g_{ij}].$$

For the function $F(\bar{k})$ defined in (11), we know from the Trace Theorem of Dacidenko [11] that

$$\begin{aligned} g_{ij} &= \operatorname{tr}\left(\operatorname{adj}\left(A(\bar{k}) + \lambda_i I\right) \cdot \frac{\partial\left(A(\bar{k}) + \lambda_i I\right)}{\partial k_j}\right) \\ &= \operatorname{tr}\left(\operatorname{adj}\left(A(\bar{k}) + \lambda_i I\right) \cdot V(:, j) \cdot U(j, :)\right) \end{aligned}$$

and if $F_i(\bar{k}) \neq 0$

$$g_{ij} = F_i(\bar{k}) \cdot \operatorname{tr}\left(\left(A(\bar{k}) + \lambda_i I\right)^{-1} \cdot V(:, j) \cdot U(j, :)\right)$$

where $\operatorname{adj}(\cdot)$ and $\operatorname{tr}(\cdot)$ mean adjugate and trace respectively, and $V(:, j)$ and $U(j, :)$ represent the j-th column of V and the j-th row of U, respectively.

The Newton iteration method then provides us a solution for a given initial estimate $\bar{k}(0)$

$$\bar{k}(m+1) = \bar{k}(m) - G\left((\bar{k}(m)\right)^{-1} F\left(\bar{k}(m)\right).$$

For different initial estimate $\bar{k}(0)$, it may reach different solutions for $\bar{k}$ as the problem has $(n-2)!$ solutions generically.

E. Extension to Rigid Formation Control

The control law (7) achieves a scalable formation with the location, orientation and scale dependent on the initial condition. However, with at least one pair of agents attaining the desired distance between them, a rigid formation can also be achieved, which means that the formation scale can be controlled by a minority of knowledgeable agents. So it is more convenient in applications where variation of the formation scale is required in responding to the change of environments, such as passing through a narrow area.

Consider any two agents who are roots of a 2-rooted graph $\mathcal{G}$. Without loss of generality, label them by $n-1$ and n, and suppose that they know the desired distance $\bar{d}$ between them. Then the two agents, called *leaders*, take the following control law.

$$\begin{bmatrix} \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \alpha(z_n - z_{n-1})\left(\|z_n - z_{n-1}\|^2 - \bar{d}^2\right) \\ \alpha(z_{n-1} - z_n)\left(\|z_{n-1} - z_n\|^2 - \bar{d}^2\right) \end{bmatrix} \tag{12}$$

where $\alpha \in \mathbb{R}^+$ is a scalar parameter.

The other agents still take the control law

$$\dot{z}_i = k_i \sum_{j \in \mathcal{N}_i} w_{ij}(z_j - z_i), \quad i = 1, \ldots, n-2 \tag{13}$$

with the selection of w_{ij} discussed in Section III-D and the selection of k_i discussed in the preceding subsection.

Denote by $f(z^l)$ the right-hand side of (12), where $z^l = [z_{n-1}, z_n]^T$ represents the aggregate state of the two leaders.

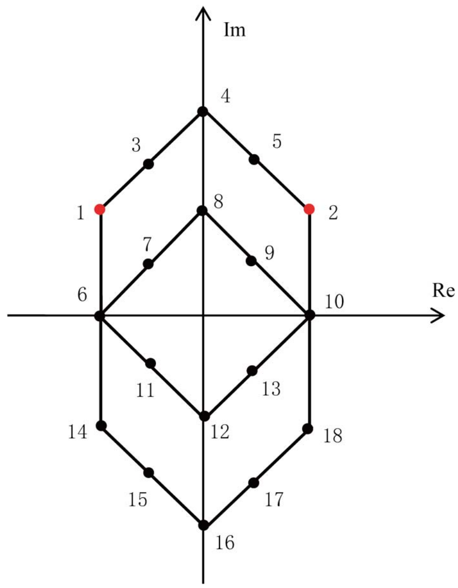

Fig. 11. Target formation is described by the framework $(\mathcal{G}, \xi)$.

Then the overall closed-loop dynamics is given as below

$$\dot{z} = -KLz + \begin{bmatrix} 0 \\ f(z^l) \end{bmatrix} \tag{14}$$

where L is a complex Laplacian of the following form:

$$L = \left[\begin{array}{c|c} L_f & L_l \\ \hline 0_{2\times(n-2)} & 0_{2\times 2} \end{array} \right].$$

Then we present a main result of rigid formation with its scale controlled by two leaders.

Theorem 4.4: Given a 2-rooted graph $\mathcal{G}$ with two roots (namely, $n-1$ and n) and a formation configuration ξ, if K assigns the eigenvalues of KL in the open right complex plane in addition to the two fixed zero eigenvalues, then for any initial state satisfying $z_{n-1}(0) \neq z_n(0)$, the agents under the dynamics (14) asymptotically achieve a rigid formation $c_1\mathbf{1}_n + c_2\xi$, where $c_1 \in \mathbb{C}$ and $c_2 = [\bar{d}/|\xi_{n-1} - \xi_n|]e^{\iota\phi}$ with $\phi \in [0, 2\pi)$.

Proof: Denote by z^f the aggregate state of the agents $1, \ldots, n-2$ and denote by K_f the sub-matrix of K by deleting the rows and columns of indices $n-1$ and n. Then the system (14) can be re-written as

$$\begin{cases} \dot{z}^f = -K_f L_f z^f - K_f L_l z^l, \\ \dot{z}^l = f(z^l). \end{cases}$$

Note that L_f is nonsingular. We make a coordinate transformation $x = z^f + L_f^{-1} L_l z^l$ to the above system and obtain

$$\dot{x} = -K_f L_f x + L_f^{-1} L_l \dot{z}^l, \tag{15}$$

$$\dot{z}^l = f(z^l). \tag{16}$$

Since $x \to 0$ is equivalent to $[z^f, z^l]^T \to c_1\mathbf{1} + c_2\xi$ for some c_1 and c_2 in $\mathbb{C}$, and moreover $|c_2| \to (\bar{d}/|\xi_{n-1} - \xi_n|)$ when $|z_{n-1} - z_n|$ approaches $\bar{d}$, it remains to show that $x(t)$ in (15) asymptotically converges to 0 and $|z_{n-1} - z_n|$ asymptotically converges to $\bar{d}$ under the dynamics (16). By Theorem 1 in [20],

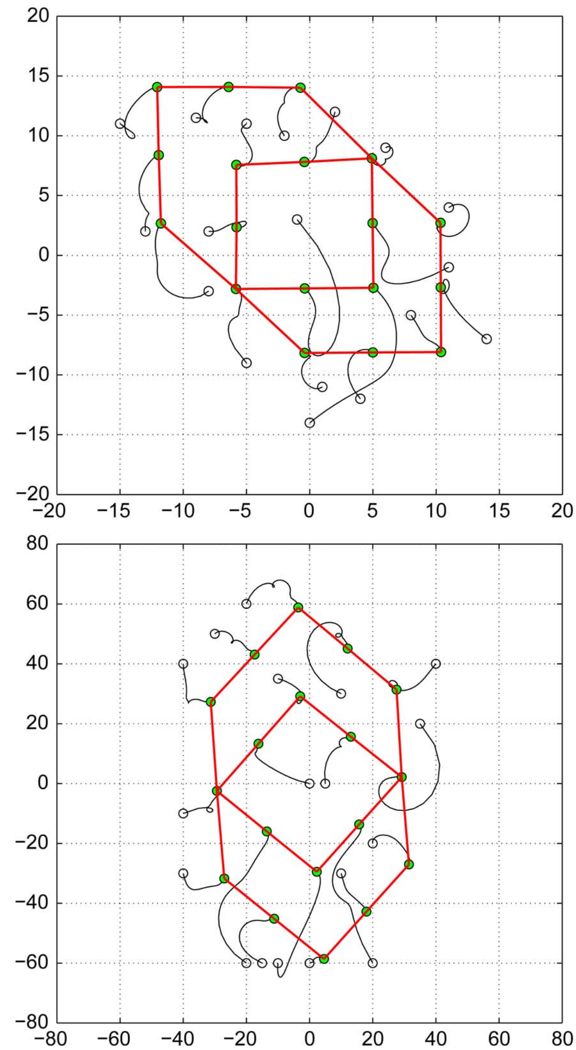

Fig. 12. Eighteen agents converge to a desired formation shape.

it is known that for the dynamic system (16) with any initial state satisfying $z_{n-1}(0) \neq z_n(0)$, we have $|z_{n-1} - z_n| \to \bar{d}$, $\dot{z}_{n-1} \to 0$, and $\dot{z}_n \to 0$ exponentially as $t \to \infty$. Also, note that (15) is a linear system with $-K_f L_f$ Hurwitz since K assigns the eigenvalues of KL in the open right complex plane in addition to the two fixed zero eigenvalues. So the observation that $\dot{z}^l$ exponential converges to zero implies the solution $x(t)$ in (15) asymptotically converges to 0. Thus, the conclusion follows. ■

V. Simulation and Experiment Results

In this section, we present a simulation and an experiment result based on *Rovio* mobile robots.

First, we consider an example of 18 agents. The target formation is described by the framework $(\mathcal{G}, \xi)$ in Fig. 11 where the formation configuration $\xi = [-2 + 2\iota, 2 + 2\iota, -1 + 3\iota, 4\iota, 1 + 3\iota, -2, -1+\iota, 2\iota, 1 + \iota, 2, -1 - \iota, -2\iota, 1-\iota, -2- 2\iota, -1 - 3\iota, -4\iota, 1 - 3\iota, 2 - 2\iota]^T$. In the graph, nodes 1 and 2 can be treated as the two roots and all other nodes are 2-reachable from them. So the graph is 2-rooted. A simulation result under the distributed control law (7) is presented in Fig. 12 with two different initial conditions showing the globally asymptotic

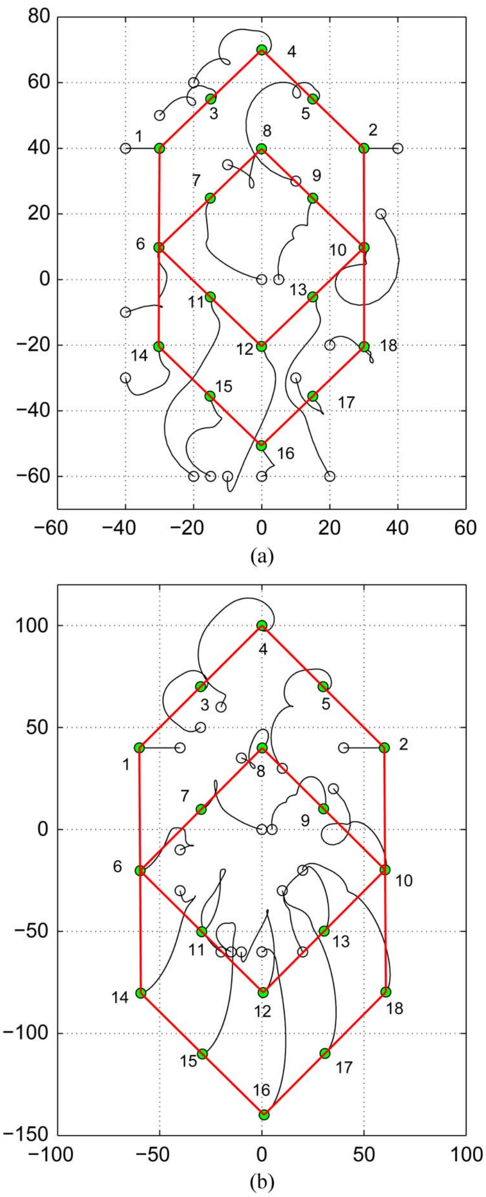

Fig. 13. Eighteen agents converge to a rigid formation with determined scales. (a) $\bar{d} = 60$. (b) $\bar{d} = 120$.

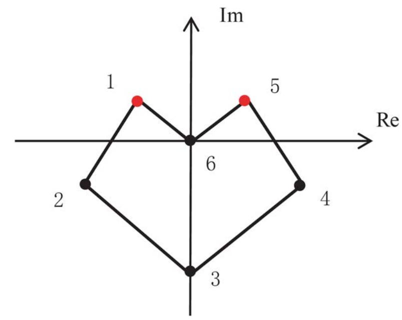

Fig. 14. Target formation is described by the framework $(\mathcal{G}, \xi)$.

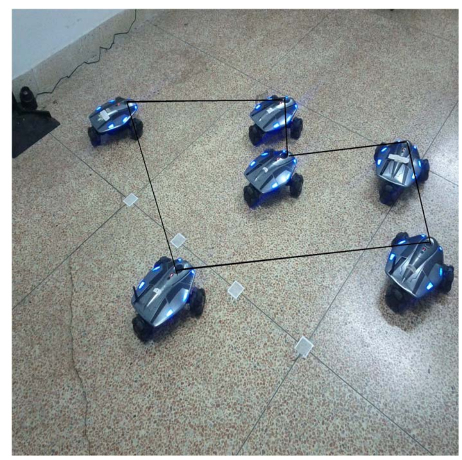

Fig. 15. Snapshot of final formation achieved in the experiment.

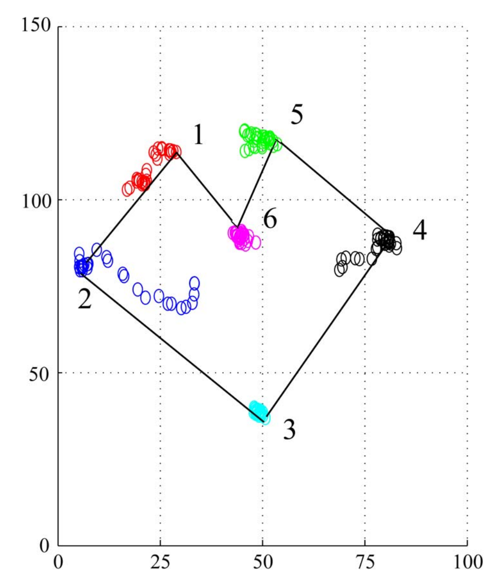

Fig. 16. Closed-loop trajectories recorded in the experiment.

stability. For the same graph $\mathcal{G}$, when agents 1 and 2 know the desired distance between them for the target formation, they are used to control the formation size by taking the control law (12). The simulation result is plotted in Fig. 13 with $\bar{d} = 60$ in Fig. 13(a) and $\bar{d} = 120$ in Fig. 13(b). As expected, the team achieves a rigid formation with the desired scales.

Second, we show an experiment result with our proposed distributed control strategy implemented on six *Rovio* mobile robots. Rovio robots are equipped with three Omni-directional wheels and thus can move freely in the plane like point masses. Moreover, every *Rovio* robot includes a true-track beacon, with which it can localize itself based on the indoor North-Star Localization System. In the experiment, we use a central computer to get all the locations of *Rovio* robots in the plane in real time, but only utilize the relative position information to control the movement of each one for the purpose of mimicking distributed and local implementation of the algorithm. The moving direction and moving speed are quantized from the continuous control signal calculated from our proposed control law. The target formation is described by the framework $(\mathcal{G}, \xi)$ in Fig. 14. A snapshot of the final formation achieved in the experiment under the distributed control law (7) is given in Fig. 15, and the experimental trajectories are recorded in Fig. 16 from the indoor North-Star Localization System. It can be seen from Fig. 15 and 16 that the experimental results also validate our proposed control scheme though there exist localization errors and quantization errors in the experiment.

VI. Conclusion and Future Work

In the paper, we introduce a novel approach for the study of multi-agent formations in the plane. Mainly, a linear constraint is used to describe a similar formation, which relates to the complex Laplacian of the formation graph. A necessary and sufficient algebraic and graphical condition is obtained, showing that a framework specified by the linear constraint is similar if and only if the graph is 2-rooted, a new type of connectivity requiring much less edges compared with the rigid framework specified by a distance constraint. According to the new idea for the representation of a similar formation, a distributed control law is also provided using relative state measurements described by a sensor graph which is the same as the formation graph. It is shown that a linear stabilizer exists almost surely to ensure the globally asymptotic stability provided that the graph is 2-rooted. A procedure is developed as well for the design of control parameters to assign the eigenvalues of the closed-loop system at any desired locations and to meet other performance specifications. Besides, we also show that a rigid formation can be achieved if at least a pair of nodes can apply a control law to control their distance.

In the paper, we focus on the formation control problem of networked mobile robots in the plane. The methods, however, are general, and they have applicability beyond multi-robot formations, e.g., distributed beamforming of communication systems and power networks where consensus is not an objective but achieving a pattern is a goal. The work in the paper is mainly limited to the setup of bidirectional topology, but this work serves as a starting point for many problems in this framework. It can be explored from many directions. For example, the topology can be directed, and/or even time-varying, or stochastic; The dynamics of agents can be more complicated and more realistic such as double integrators, unicycles, or nonlinear systems; The sensing information allows measurement errors or even measurement loss; And the formation pattern can be in 3-dimensional or higher dimensional spaces. Moreover, it is also desired to develop a distributed approach for the controller design without knowing global information of the whole network.

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