Divide and conquer

A hands-on exploration of divisibility

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In this article, John West explores student misconceptions relating to division. A range of activities that promote the understanding of divisibility rules are provided.

Introduction

According to Posamentier (2003), "divisibility rules provide an interesting 'window' into the nature of numbers and their properties" (p. 52). He encourages teachers to present the topic so that "students will not see this as something that they must memorize. Rather, they should try to understand the underpinnings of the rules" (p. 53). Research suggests that understanding division is critical for work with rational numbers, and that "a deep understanding of division concepts themselves is necessary to work flexibly with division strategies" (Young-Loveridge & Mills, 2012, p. 17).

An examination of the research literature reveals that students hold a number of common misconceptions about division. These include misconceptions about division by numbers less than one and that division always 'makes smaller'. It is also not uncommon for students to read division statements in the order that they appear, thus confusing the divisor and the dividend. Students may also resist dividing a bigger number into a smaller number (Willis et al., 2004). Many of these misconceptions appear to be the result of an incomplete or biased understanding of division (Roche & Clarke, 2013). Students' early experiences are crucial since it appears that many students build their understanding of operations on their experiences with small numbers (Swan, 1983).

Students' earliest experiences with number theory arise from the study of odd and even numbers (Jorgensen & Dole, 2011). Beckmann (2005) suggests that the study of odd and even numbers provides a "fertile ground for investigating and exploring math" (p. 562). The use of physical manipulatives (such as counters) or pairing up groups of children can be used to demonstrate divisibility by two (Mooney, Briggs, Fletcher, Hansen & McCulloch, 2009). The concept can be readily extended to dividing objects into three or more groups, with the ultimate goal of developing an understanding of divisibility that no longer requires the use of physical manipulatives.

While students should be provided with opportunities to develop their own theories about odd and even numbers (Jorgensen & Dole, 2011), rarely do students appear to be given the opportunity to develop their own conjectures in practice. The student-centred approach described here was developed in response to teacher concerns that, despite having taught the divisibility 'rules', the concept did not appear to be well-understood by students. It is argued that such an approach is in line with the Australian Curriculum and will enable students to develop a deeper understanding of the concept of divisibility than is typically achieved by teaching students a set of 'rules'. According to the *Australian Curriculum: Mathematics* (Australian Curriculum and Reporting Authority, 2014), students should:

- "Investigate the conditions required for a number to be odd or even and identify odd and even numbers" (ACMNA051) in Year 3;
- "Investigate and use the properties of odd and even numbers" (ACMNA071) in Year 4; and
- "Identify and describe factors and multiples of prime numbers" (ACMNA098) in Year 5.

The exploration of divisibility suggested here may serve as a possible enrichment activity for middle to upper primary students who are ready to move beyond the mere mechanics of multiplication and division. An understanding of divisibility provides the foundation for the subsequent study of prime numbers. The teacher plays a key role in managing this inquiry process. Rather than presenting divisibility as a set of 'rules', which implies that understanding is not the focus, students should be encouraged to formulate, test and refine their own conjectures, with scaffolding provided as necessary. Providing students with the opportunity to wrestle with the problem within a supportive environment enables them to build resilience and stamina in problem solving (Stacey & Groves, 1985). Careful monitoring is needed to ensure that students do not become frustrated by the investigative process, and the teacher may assist by providing encouragement and by guiding students towards salient information where necessary.

Investigating divisibility

Divisibility by two, five and ten

For most students, the exploration of odd and even numbers is their first encounter with the concept of divisibility. While it may appear that determining whether a number is odd or even addresses the concept of divisibility at only the most superficial level, Beckmann (2005) noted that there are many equivalent ways of demonstrating that a number is even. For example, a number is even if it is exactly divisible by two, or it can be factored into two times another number (or another number times two), or if you can divide that number of things into two equal groups (or groups of two) while leaving no remainder.

The use of physical objects (such as counters) provides a tangible means for students to determine if the number of objects in a collection is odd or even. The number of objects is even if the counters can be arranged into pairs or two equal (i.e., even) rows or columns, and odd if the division leaves a remainder. Thus an unpaired counter serves as an obvious visual cue that enables students to recognise an odd number (as shown in Figure 1). Older students may also reason that combining two odd numbers (and therefore two unpaired counters) will always produce an even total, thus beginning to develop their own conjectures about the properties of odd and even numbers.

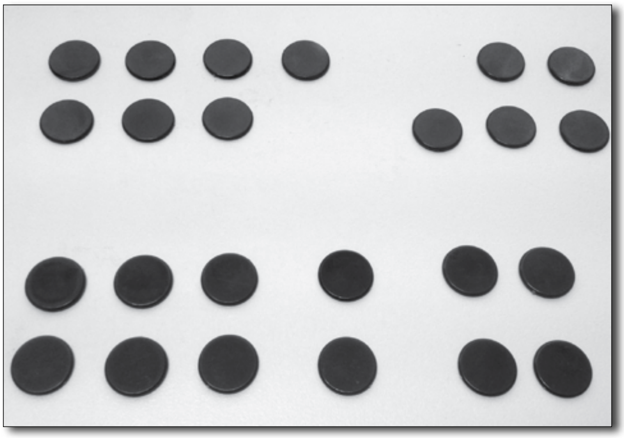

Figure 1. A student uses visual clues to identify two odd numbers and shows their sum is even by arranging the counters into two equal rows.

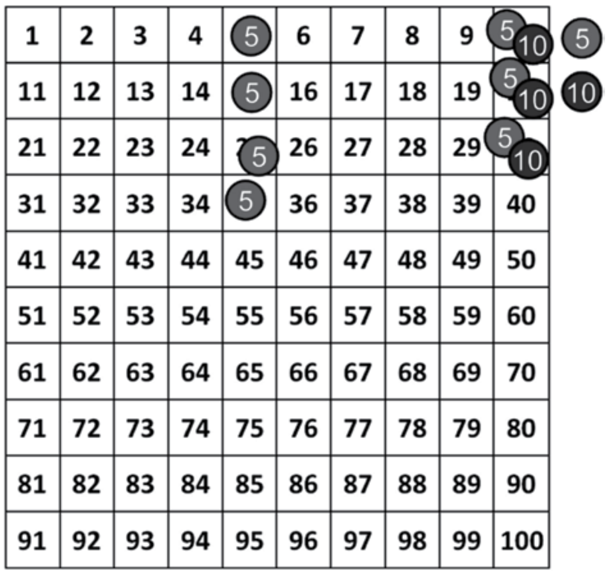

Figure 2. Students can use an interactive hundred square to explore patterns in the multiples of five and ten.

An examination of a one hundred square quickly reveals the pattern of multiples of five and ten. Using an interactive whiteboard enables a one hundred square to be set up so that the multiples of a number change colour when touched (Mooney et al., 2009, p. 56). Teachers and students can also use an interactive whiteboard to explore conjectures such as "all multiples of five are also multiples of ten" and "all multiples of ten are also multiples of five" (as shown in Figure 2). Such activities also enable the teacher to emphasise that subtle variations in language are used to represent ideas that are mathematically quite distinct; e.g., all factors of four are factors of eight while not all factors of eight are factors of four.

Divisibility by three and nine

Teachers can assist students to develop their own conjectures about the rules for divisibility by three in the following manner. Divide the class into groups of approximately three or four students, and provide each group with a unique set of three single-digit cards (e.g., 1, 3 and 4 or 2, 3 and 4). The cards provided to each group should be purposefully selected so that each set has a different digit sum (e.g., 1 + 3 + 4 = 8, 2 + 3 + 4 = 9), although students should be allowed to discover this for themselves in due course.

Have each group list all of the three-digit number combinations that can be formed using their three digits (i.e., 134, 143, 314, 341, 413 and 431). Students should then determine which, if any, of these three-digit numbers they have created are divisible by three, using a calculator if necessary. Since the digit sum determines whether a number is divisible by three, students will discover that precisely all or none of the three-digit numbers they have created are exactly divisible by three. To ensure this result it may be necessary to correct any calculation errors at this point; this can be done by asking, "Are you sure?" or having students double-check their results.

The teacher should then review the digits provided to each group, the possible three-digit numbers combinations, and which (if any) of them are divisible by three. On showing that all or none of the three-digit numbers provided to each group are divisible by three, students can be prompted with: "Do you think it is a coincidence that all of the numbers you (i.e., Group A) could make with the digits I gave you were divisible by three and all of your numbers (i.e., Group B) were not?" Providing each group with a different set of digits (either with the same or different digit sum) allows students to be guided towards the discovery that numbers which are divisible by three have a digit sum that is a multiple of three (i.e., 1 + 2 + 3 = 6).

While this process can be modelled effectively using an interactive whiteboard, the use of physical manipulatives is highly recommended for the discovery phase. The use of the number cards allows students to physically reorder the digits (e.g., 123, 213 or 312) without affecting divisibility (since 1 + 2 + 3 = 6).

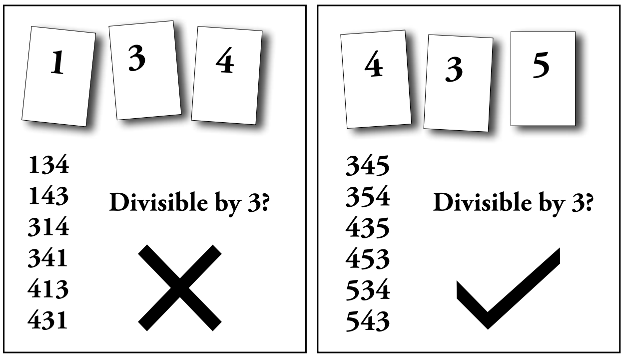

Figure 3. Teachers can assist students to discover that the rule for divisibility by three is related to the digit sum.

Extending the idea

Students can also be encouraged to explore whether their conjecture applies to numbers with more or less than three digits (e.g., 1234) or numbers which contain repeated digits (e.g., 223). Exploring whether patterns in the digit sum exist for other numbers provides students with the opportunity to test conjectures and, ultimately, to discover that numbers which are divisible by nine have a digit sum that is divisible by nine.

Divisibility by four and eight

An understanding of the approach to determining whether a number is divisible by four or eight can be derived from the patterns evident in the multiples of four and eight. A hundred square (or a flat from a set of Base 10 blocks) and four Base 5 flats provide a tangible representation of 100 that can readily be divided into four equal parts. The teacher may encourage students to think about different ways in which the hundred

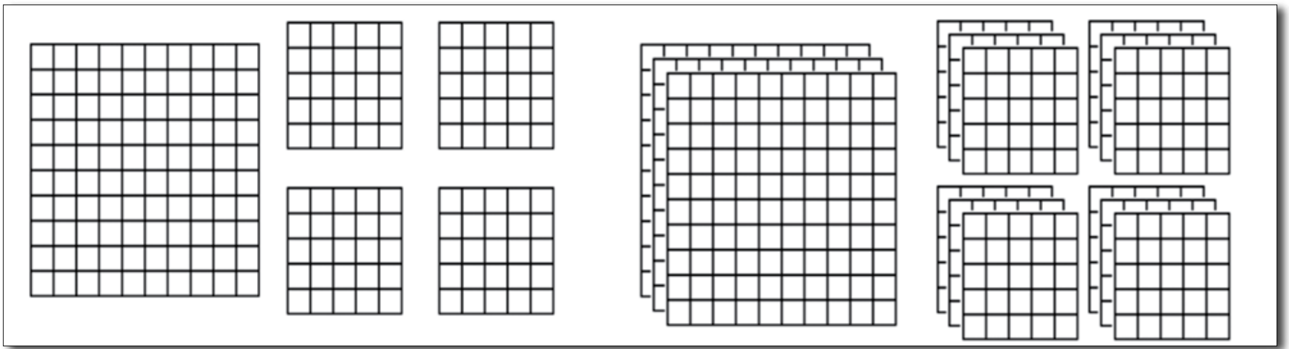

Figure 4. Using Base 10 flats to establish that 100 (and multiples thereof) are divisible by four.

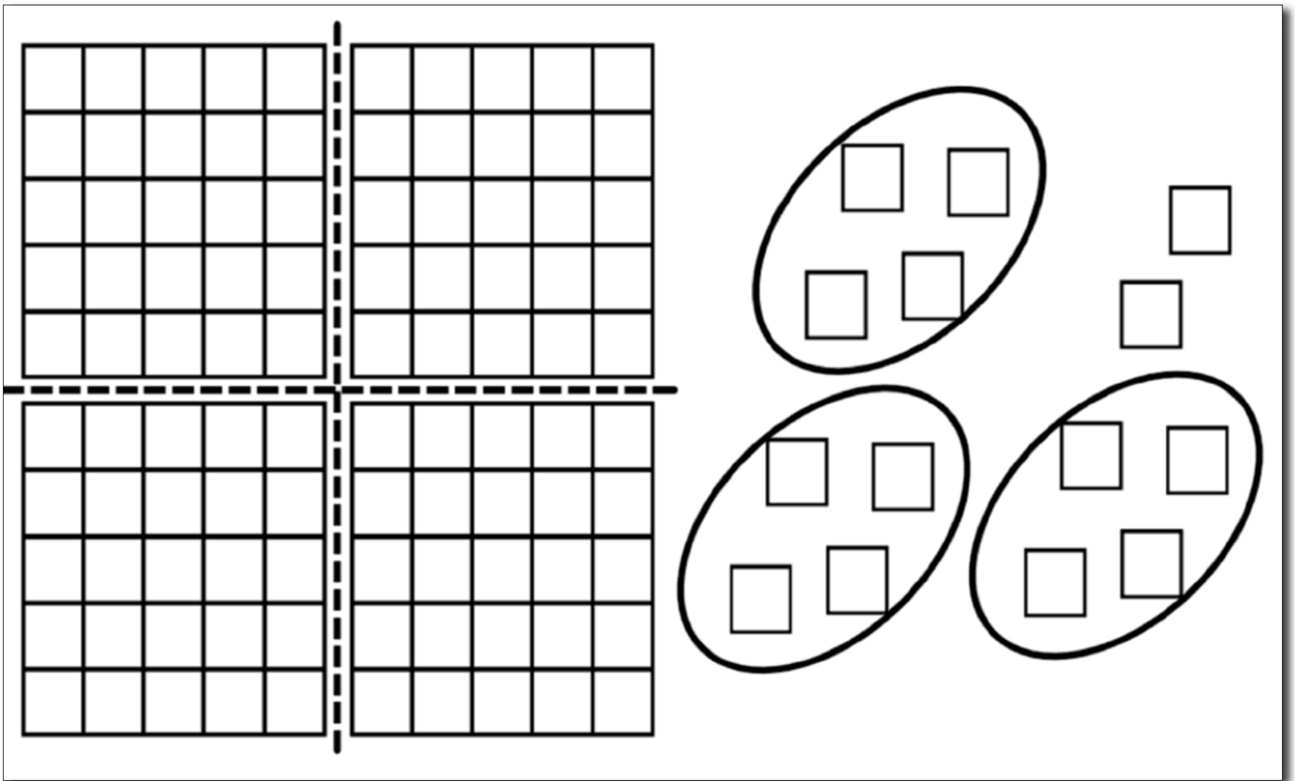

Figure 5. As any multiple of 100 is divisible by four, divisibility by four depends only on the number formed by the tens and units digits.

square could be partitioned into four equal parts. This process should then be extended to multiples of 100, so that students can establish that any multiple of 100 are also divisible by four. Since all multiples of 100 are divisible by four, we can add one or more hundreds to established multiples of four (i.e., 4, 8, 12, 16, … 96) to produce still other multiples of four (e.g., 216, 308 and 1296). Conversely, any number larger than 100 is divisible by four if the number formed by its tens and unit digits is a multiple of four i.e., 04, 08, 12, 16, … 96. The teacher can reinforce that a number is a multiple of four if a whole number results when the number formed by the tens and units digits is halved and then halved again.

In a similar manner, a thousands cube from a set of Base 10 blocks can be seen to readily divide into eight equal octants by halving its length, width and height. The Base 10 thousands cube can be shown to be equivalent to eight 5 x 5 x 5 cubes. Hence the use of Base 5 blocks provides a useful illustration that 1000 (and any multiple thereof) is divisible by eight.

Since all multiples of 1000 are divisible by eight, adding multiples of 1000 to the established multiples of eight (i.e., 8, 16, 24, … 992) produce still other multiples of eight (e.g., 1120, 2320 and 10064). Conversely, any number larger than 1000 will be divisible by eight if the number formed by the hundreds, tens and units digits is a multiple of eight; i.e., 008, 016, 024, … 992. A number is a multiple of eight

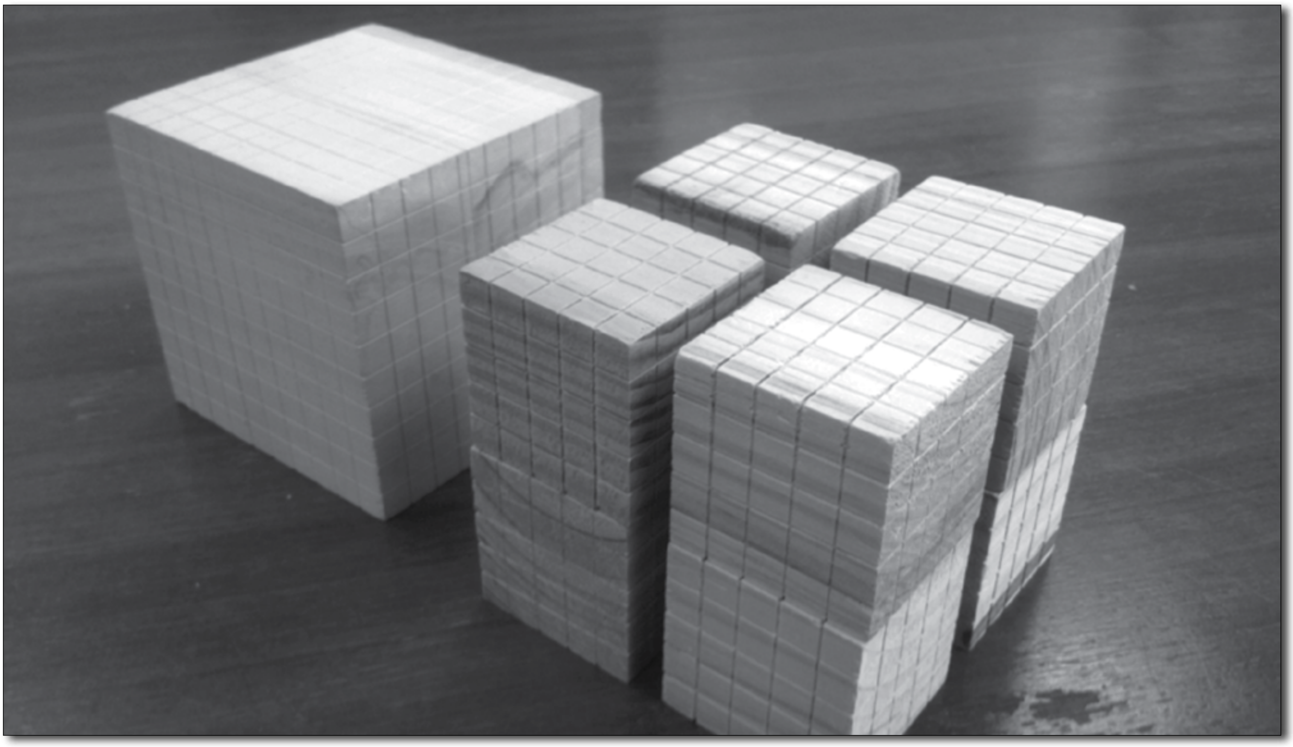

Figure 6. Using Base 5 blocks to establish that 1000 is divisible by eight.

if a whole number results when it is halved, halved and halved again. Students should also come to recognise that any multiple of eight must necessarily be even and a multiple of four.

Other divisibility rules

We have explored ways in which students can develop their own understanding of divisibility rules for two, three, four, five, eight, nine and ten. These rules can be combined to yield divisibility rules for other composite numbers. For example, a number is divisible by six if it is divisible by two and three, and a number is divisible by 15 if it is divisible by three and five. In addition, Posamentier (2003) describes systematic procedures for exploring divisibility by prime numbers from seven through 47 (and beyond).

Conclusion

The concept of divisibility can be explored at various levels and various degrees of abstraction. The hands-on approach described here ensures that students can access the concept at the concrete level, while building a deeper understanding of the division operation and greater flexibility in mental computation. Throughout the investigation, the teacher should encourage students to develop and refine their own divisibility rules by providing appropriate scaffolding. Allowing students to formulate and test their own conjectures allows them to take ownership of the rules they discover and provides an introduction to the notion of mathematical proof.

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