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ABSTRACT

This book is a sequel to MATHEMATICAL CHALLENGES, which was published in 1965. In this sequel are 100 problems, together with their printed solutions. The problems range from those that are quite simple to those that will challenge even the most ardent problem solver, and they include examples from algebra, geometry, number theory, probability, and trigonometry. They are directed to students at the junior and senior high school levels, and with few exceptions they are taken from the pages of the MATHEMATICS STUDENT JOURNAL (recently renamed the MATHEMATICS STUDENT), a periodical publication of the National Council of Teachers of Mathematics. (JP)

SELECTED PROBLEMS FROM THE MATHEMATICS STUDENT JOURNAL

MATHEMATICAL CHALLENGES II plus Six

compiled and annotated by
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SUMMARY: One hundred problems in algebra, geometry, number theory, probability, and trigonometry, varying in difficulty. Solutions are included as well as six articles on mathematics.

"Sequel to Mathematical challenges [compiled by M. Charosh] which was published in 1965."

Includes 6 articles originally published in the Mathematics student journal.

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Contents

Preface

Mathematical Challenges II—plus Six is a sequel to *Mathematical Challenges*, which was published in 1965. In this sequel are 100 problems, together with their printed solutions. The problems range from those that are quite simple to those that will challenge even the most ardent problem solver, and they include examples from algebra, geometry, number theory, probability, and trigonometry. All of them are dedicated to the proposition that problem solving is one of the most fascinating and productive facets of mathematics.

The problems are directed to students at the junior and senior high school levels, and with few exceptions they are taken from the pages of the *Mathematics Student Journal* (recently renamed the *Mathematics Student*), a periodical publication of the National Council of Teachers of Mathematics. Thus many of the problems and all the solutions were submitted by students. And except for minor editorial revisions, these are reproduced here as originally printed so that this publication (like the *Journal* itself) may be one that is for the most part not only *for* students but also *by* students.

The "plus six" of the title refers to the fact that in addition to the problems and solutions, six articles originally published in the *Journal* are included here. Three of these were written by high school students, and they show convincingly that students are capable not only of solving problems but also of thinking and writing clearly about mathematics.

Finally, this publication would not exist were it not for the problems and solutions submitted by students from all over the world to the *Mathematics Student Journal*. No individual acknowledgments are included here; however, we acknowledge a debt of gratitude to all those students whose spirit of adventure and inquiry have helped to demonstrate the excitement of problem solving. Acknowledgment is also made to Stephen Willoughby and Elroy Bolduc, who, like myself, served as editor of the *Mathematics Student Journal* for a part of the period (1965–1973) in which the problems were first printed, and especially to Mannis Charosh, Joan Levine, Elroy Bolduc, and Stephen Conrad, who have been editors of the Problem Department of the *Journal* during this same period.

PROBLEMS

Introduction

The problems are numbered consecutively from 1 through 100, and they are grouped by content area. The first twelve are miscellaneous ones that fall in the general category of "puzzle problems"; the groupings after that are of problems from algebra, geometry, number theory, probability, and trigonometry.

No real attempt has been made to determine an ordering of the problems by degree of difficulty (for one thing, it would be virtually impossible to agree on which of two problems is the more difficult), but in general the earlier problems in any one content grouping are easier than the later ones.

Complete solutions to the problems are presented beginning on page 28, and in many cases alternate solutions are discussed, thus emphasizing the fact that there are usually many different ways to solve a problem.

1

Mostly for Fun

Puzzle problems have fascinated people since the beginning of mathematics, and they continue to do so. One popular kind of puzzle problem is that which seems to involve a paradox—things seem right and wrong at the same time. These are often stated as riddles, but regardless of the form in which they are expressed, they are always fun and sometimes challenging. Problems 1 and 2 are of the one-line riddle variety; problems 3 and 4 are somewhat more challenging.

1. Two men divided an amount 50-50; yet one got 100 times as much as the other. How can this be?

2. Two coins add up to fifty-five cents, and yet one of them is not a nickel. What are the two coins?

3. A father left 17 cows to his three sons, with instructions that the first son get one-half, the second son one-third, and the youngest son one-ninth of the cows. This, however, would have resulted in fractions of a cow, which would be of no use to anybody. How could they follow the father's instructions?

   A friend then offered to give them his one cow, making a total of 18. So it was possible to give the first son one-half (9) of the cows, the second son one-third (6), and the youngest son one-ninth (2). Since $9 + 6 + 2 = 17$, it was possible to return the loaned cow.

   This seems to be somewhat analogous to an enzyme that promotes a chemical reaction without itself being used up. Can you explain the "mystery" in the case of the cows?

4. Once three traveling men stopped overnight at a hotel and decided to share one room. The room cost them \$10 each. Later that night, the manager returned from a trip and said that the rate for the room should have been a total of \$25; so he gave the bellboy \$5 to return to the men. On his way to the room, the bellboy decided that three

men couldn't divide \$5 evenly among themselves; so he kept \$2 himself and returned \$3 to the men. Now this made the cost of the room amount to \$9 each. In addition, the bellboy had \$2. But $9 + 9 + 9 + 2 = 29$. What happened to the "missing dollar"?

A cryptarithmetic problem is a puzzle in which some or all of the numerals in the problem have been replaced by letters (or some other symbols). The object is to determine the missing numerals by studying the structure of the problem. Some, such as problems 5 and 6, are relatively easy; others, such as problem 7, may require both patience and ingenuity.

5. Find the digit represented by each letter in the following division problem where each letter represents a different digit.

$$\begin{array}{r}
X\,T\,X\,T \\
T\,D\,X\,\overline{)\,E\,D\,E\,I\,T\,X} \\
\underline{E\,S\,X} \\
S\,S\,I \\
\underline{T\,D\,X} \\
E\,X\,T \\
\underline{E\,S\,X} \\
S\,N\,X \\
\underline{T\,D\,X} \\
T\,S\,I
\end{array}$$

6. Find the digit represented by each letter in the following multiplication problem where each letter represents a different digit in *base five*.

$$\begin{array}{r}
T\,X\,X \\
\underline{N\,T\,S\,N} \\
X\,D\,N\,N \\
D\,X\,T\,T\,S \\
\underline{X\,D\,N\,N} \\
N\,X\,T\,D\,N\,N\,N
\end{array}$$

7. In the following division problem all the digits except an 8 are represented by x's. Reconstruct the division by determining a proper value for each x.

$$\begin{array}{r}
x\,x\,8\,x\,x \\
x\,x\,x\,\overline{)\,x\,x\,x\,x\,x\,x\,x\,x} \\
\underline{x\,x\,x} \\
x\,x\,x\,x \\
\underline{x\,x\,x} \\
x\,x\,x\,x \\
\underline{x\,x\,x\,x}
\end{array}$$

The next three problems are also puzzle problems from the pages of the Mathematics Student Journal. *They don't fall into any particular class of problems as the earlier ones do, but, like the others, they involve some mathematical thinking and are fun to solve.*

8. In a certain classroom there are 25 seats—all occupied—that are arranged in 5 rows with 5 seats in each row. Suppose that the teacher announces that every student is to change his seat in one of the following ways: by going to the seat in front of his present one, by going to the seat behind his present one, by going to the seat to the right of his present one, or by going to the seat to the left of his present one. Of course at the end of the changes, each seat is to be occupied. Is it possible for the class to carry out these instructions? Why or why not?

9. Three teams—A, B, and C—compete in a track meet with ten events. Three points and a gold medal are awarded for each first place, two points and a silver medal for each second place, and one point and a bronze medal for each third place. Team C wins more gold medals than either A or B. Also the total number of medals won by team C is one more than the total number won by B and is two more than the total won by A. Nevertheless, team A comes in first with one point more than B and two points more than C. Determine the number of medals of each type won by each team.

10. A, B, C, and D are four different weights. When they are placed on a balance scale, the following observations are made: A and B exactly balance C and D. A and C together outweigh B and D together. C is lighter than D. Problem: Arrange the weights in order from heaviest to lightest.

We close this introductory section with two problems that are somewhat more advanced. Problem 11 is one of those classic puzzle problems calling for deductive analysis; problem 12 requires an understanding of mathematical induction in order to detect the fallacy in the purported proof.

11. Andy, Bob, and Carl are three boys, only *one* of whom is a student of mathematics. Each of the boys in turn makes the following truthful statements.

    Andy says:

    1. If I'm not a math student, I won't pass physics.
    2. If I am a math student, I will pass chemistry.

    Bob says:

    3. If I'm not a math student, I won't pass chemistry.
    4. If I am a math student, I will pass physics.

Carl says:

5. If I'm not a math student, I won't pass physics.
6. If I am a math student, I will pass physics.

If, in addition, it is known that the mathematics student is the only one who will pass one of the two subjects and also the only one who will fail the other, which of the boys is a student of mathematics?

12. The following argument purports to prove that *all horses are the same color*. The method of proof used is that of mathematical induction. What is wrong?

The statement is true for $n = 1$.

If there is only one horse in a set, it certainly has the same color as itself.

If the statement is true for $n = k$, then it is true for $n = k + 1$.

Consider a set of $k + 1$ horses. Remove one horse, say A, from the set. This leaves a set of k horses who, by assumption, have the same color. Now place A back in the set and remove another horse, say B. Again this leaves a set of k horses, all of whom have the same color. Now by this argument, both A and B have the same color as the other $k - 1$ horses. Therefore all $k + 1$ horses have the same color.

2

Algebra

13. *ABCD* is a square whose side measures ten units. It is divided into five noncongruent rectangular regions with equal areas. Find the dimensions of each rectangle.

14. A man is walking on a railroad bridge that goes from *A* to *B*. When he is $\frac{3}{8}$ of the way across the bridge he hears a train approaching, and it is known that the speed of the train is 60 mph. The man's rate of running is such that if he runs toward *A* he will meet the train at *A*, and if he runs toward *B* the train will overtake him at that point. How fast can he run?

15. The *freshman sum* is defined as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

Under what conditions will the freshman sum of two fractions be equal to the arithmetic mean of the fractions?

16. The set $\{1, 2, 3, \ldots, 15\}$ of the first fifteen consecutive positive integers is to be partitioned into five subsets. Each subset is to have three elements, and the sum of these elements is to be the same in each subset. Prove that $\{7, 8, 9\}$ cannot be one of these subsets.

17. The Up and Down Airline has a policy that allows each passenger to carry k pounds of baggage, with an additional charge for baggage over this weight. The combined weight of the baggage of two passengers on one particular flight was 105 pounds. One passenger had to pay \$1.00 extra, and the second passenger had to pay \$1.50 extra. On that same flight a third passenger had 105 pounds of baggage and was charged \$6.50 for excess baggage over the k-pound limit. How many pounds of baggage is allowed per person without charge?

18. How many different integral Pythagorean triples exist in which 60 is one of the two smaller integers?

19. If a, b, c form an arithmetic progression of three terms, and

$$a = x^2 + xy + y^2,$$
$$b = x^2 + xz + z^2,$$
$$c = y^2 + yx + z^2,$$

and the sum $x + y + z$ is not zero, prove that x, y, z is also an arithmetic progression.

20. In a geometric progression in which the terms are positive integers (no two the same), prove that the sum of the first, second, and sixth terms cannot be prime. Why is the condition of unequal terms necessary to the solution?

21. Prove: If a, b, c are odd integers, then the roots of $ax^2 + bx + c = 0$ are not rational.

22. If $f(x) = \dfrac{x - 1}{x + 1}$, express $f(2x)$ in terms of $f(x)$.

23. If the ratio of the difference of the two roots of one quadratic equation to the difference of the two roots of a second quadratic equation is rational, prove that the ratio of the discriminants of the two equations is a rational square. (Assume the coefficients of the equations to be rational.)

24. A mathematics teacher was getting his exercise by rowing a boat upstream. As he passed under a low bridge, his hat was knocked off. It was not, however, until five minutes later that he noticed his hat was missing. At this time he turned around and rowed with the same effort downstream. He retrieved his hat at a point precisely one mile downstream from the offending bridge. How fast was the current flowing?

25. When Mr. Smith cashed a check (for less than $100), the bank clerk accidentally mistook the number of dollars for the number of cents, and conversely. After Mr. Smith had spent 68 cents, he discovered that he had twice as much money as the check had been written for. What was the amount for which the check had been written? Is there a *unique* solution?

26. Two positive numbers, r_1 and r_2, have arithmetic mean A and geometric mean G. If $r_1 > r_2$, find a formula for r_1 and a formula for r_2 in terms of A and G. What order relation exists between A and G?

27. In the multiplicative magic square shown in figure 1, the products for all rows, all columns, and all diagonals are equal. Thus, $abc = def = ghi = aei = \ldots = k$, some constant. If all entries in the magic square are taken as positive integers, find all possible values of k between 1 and 100.

a	b	c
d	e	f
g	h	i

Fig. 1

28. Find positive integers x and y such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{12}.$$

29. In the coordinate plane, a line passing through the point $P(3, 3)$ intersects the x-axis at $(u, 0)$ and the y-axis at $(0, v)$.

a. Find v if $u = 4$; if $u = 5$; if $u = 6$.

b. Find the sum of the reciprocals of u and v for each of the cases in part *a*.

c. Prove a generalization suggested by part *b*.

d. Describe all points of the plane for which this generalization is valid.

30. **a.** Create a function rule $f(x, y)$, x and y both real numbers, such that $f(x, y)$ is always the greater of the two numbers x, y. (What happens if $x = y$?)

b. Extend part *a* to a function of three real numbers x, y, z so that for all x, y, z, $f(x, y, z)$ is the greatest of the three. (What happens if two of the three numbers are equal?)

31. The midpoints of each side of an equilateral triangle are joined, forming a smaller triangle. The midpoints of this smaller triangle are then joined, forming a third triangle, and so on. What is the total perimeter of all triangles that can be formed in this way? Take the length of

each side of the original triangle as 1. (Fig. 2 shows the first three triangles.) Note that this is an "infinite" sum.

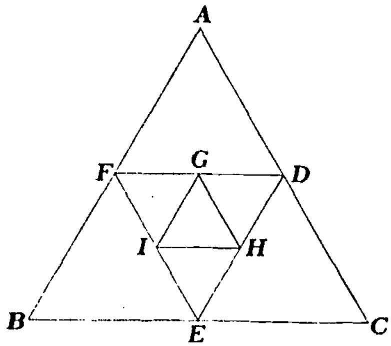


Fig. 2

32. In a certain class there are more than 20 and fewer than 40 students. On a recent test the average passing mark was 75, the average failing mark was 48, and the class average was 66. The teacher then raised every grade 5 points (the highest score had been 95). As a result the average passing mark became $77\frac{1}{2}$, and the average failing mark became 45. If 65 is the established minimum for passing, how many students had their grades changed from failing to passing?

33. An army column 25 miles long marches 25 miles on a particular day. At the start of the day a messenger goes from the rear of the column, carrying a message to the front of the column, after which he immediately turns and starts back to his original position at the rear. As it turns out, the messenger reaches the rear just as the day's march comes to an end. How many miles did the messenger travel?

34. If the fraction $\frac{1}{x^3 - 13x + 12}$ is expressed as the sum of fractions of the form $\frac{a}{bx + c}$ where a, b, and c are all integers with $b > 0$, what will be the sum of the denominators of these fractions?

35. Find all possible positive integral values for x in the following equation, where $a \neq b, a \neq 0, b \neq 0$:

$$a^{(14x+6)}b^2 - 2ba^{(14x+7)} + a^{(14x+8)} = a^{(2x^2+32)} - 2ba^{(2x^2+31)} + b^2a^{(2x^2+30)}$$

36. What is the least positive integral base (of numeration) in which the two alphametics

$$ONE + ONE = TWO$$

and

$$TWO + TWO = FOUR$$

both hold? (Each letter in the alphametic represents a digit. Any letter represents the same digit each time it occurs, and two different letters do not represent the same digit.)

37. Find the least integer that is one-half of a perfect square, one-third of a perfect cube, one-fifth of a fifth power, and one-seventh of a seventh power. How many digits are there in the solution?

38. If $f(n)$ is defined as $f(n - 1) + 2n - 1$, for $n > 1$, and $f(1) = 1$, express $f(2n)$ as a polynomial in n. Prove your answer.

39. Find all positive integral values of a between 100 and 200 for which

$$ax^2 + x - 6$$

can be expressed as the product of two linear factors, each with integral coefficients.

40. Can an arithmetic progression consisting of at least three integers have a sum that is prime? Consider two cases—(**a**) one in which all terms of the progression are positive and (**b**) one in which not all terms are positive.

41. An enclosure is to be built in the form of a stair-patterned set of squares. (See fig. 3.) If n is the number of squares on a "side" of such a set and T is the total number of "walls" (i.e., segments), then from the figure it is apparent that when $n = 1$, $T = 4$; when $n = 2$, $T = 10$; when $n = 3$, $T = 18$; and so on. In the particular problem at hand, 317 walls are available, but they are not enough to complete a stair-shaped pattern. What is the least number of additional walls needed to complete the pattern?

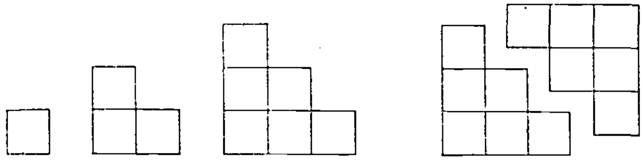


Fig. 3

42. Find all real numbers x such that

$$\sqrt{5 - \sqrt{5 - x}} = x.$$

3

Geometry

43. $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{CD}$, and E and F are points on $\overline{AD}$ and $\overline{BC}$ respectively such that

$$\frac{AE}{ED} = \frac{BF}{FC} = \frac{2}{1}.$$

If $AB = 7$ and $DC = 10$, find the length of $\overline{EF}$. (*Hint*: Through B draw a segment that is parallel to $\overline{AD}$ and that intersects $\overline{EF}$ and $\overline{DC}$ at points H and G respectively, as shown in fig. 4).

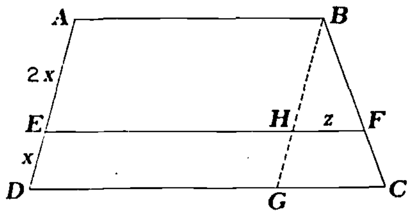


Fig. 4

44. $\overrightarrow{OC}$ is the bisector of $\angle AOB$ and D is a given point on side $\overrightarrow{OB}$. Determine a segment $\overline{DF}$ (intersecting $\overrightarrow{OC}$ at E) such that $\frac{DE}{EF} = \frac{1}{2}$. (See fig. 5.)

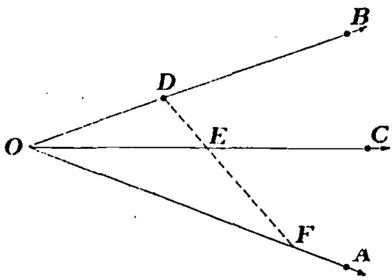


Fig. 5

45. A circle with radius 5 is escribed to triangle ABC, being tangent to side $\overline{BC}$ and also tangent to sides $\overline{AB}$ and $\overline{AC}$ extended. The center of the circle is 13 units from vertex A of the triangle. What is the perimeter of the triangle?

46. $ABCD$ is a square; E is the midpoint of $\overline{CD}$, and $\overleftrightarrow{AE} \perp \overleftrightarrow{EF}$, as shown in figure 6. If x, y, z are the measures of $\angle EAD$, $\angle FAE$, and $\angle BAF$, respectively, prove the following:

a. $x = y$

b. $z > x$.

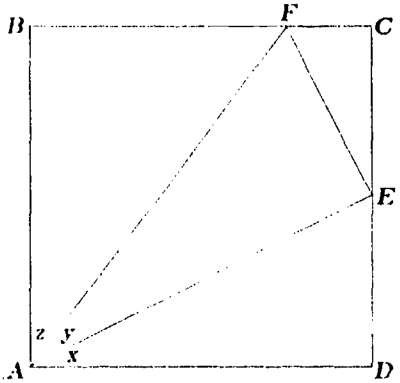


Fig. 6

47. Using the angle measures shown in figure 7, determine the shortest segment in the figure. Then decide which of the segments is the longest.

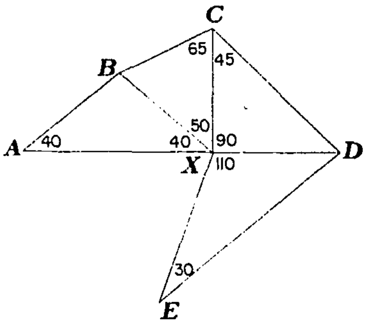


Fig. 7

48. Every triangle has six "parts"—three sides and three angles. Show that it is possible for five of the parts of one triangle to be congruent

to five of the parts of a second triangle without the triangles themselves being congruent to each other.

49. $ABCD$ is a rectangle with E an interior point. $\overline{AE}$, $\overline{DE}$, and $\overline{BE}$ have measures a, b, and c, respectively. Find the measure of $\overline{CE}$ expressed in terms of a, b, and c. (See fig. 8.)

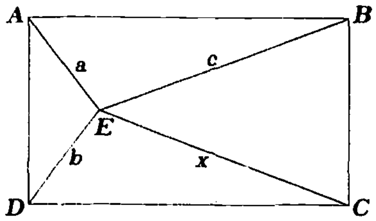


Fig. 8

50. Hypotenuse $\overline{AB}$ of right triangle ABC is divided into three equal parts. If the segment between one of these points and the vertex C measures 7 units and the segment between the other point and C measures 9 units, what is the length of hypotenuse $\overline{AB}$?

51. In a 3-4-5 right triangle, find the length of the one of the two trisectors of the right angle that is nearer the shorter leg of the triangle.

52. In a plane, three regular polygons of m, n, and p sides respectively have a vertex in common, and at this vertex they exactly "fill up" the plane. Prove that for all sets of numbers m, n, p satisfying this condition, the sum

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p}$$

is the same, and find this sum.

53. In figure 9, $ABCD$ is a parallelogram, and K is the point on segment $\overline{BD}$ such that $\frac{BK}{KD} = \frac{3}{7}$. Line AK intersects $\overline{BC}$ at X. Find the ratio of the area of quadrilateral $DKXC$ to the area of parallelogram $ABCD$.

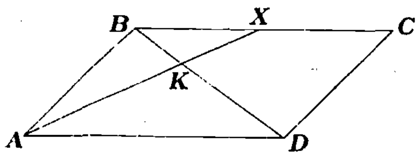


Fig. 9

54. A rectangular parallelepiped has edges with (integral) lengths x, y, and z. The sum of the lengths of all twelve edges is 48. The sum of the areas of all 6 faces is 94, and the volume of the solid is 60. Find the dimensions x, y, and z.

55. In a quadrilateral $ABCD$, $AB = BC = CD$, and also $AD = AC = BD$ ($\overline{AC}$ and $\overline{BD}$ being diagonals of the quadrilateral). Find the measure of $\angle ABC$.

56. If $\frac{2}{v} = \frac{1}{u} + \frac{1}{w}$, then v is called the *harmonic mean* between u and w. In figure 10, $\overline{BG} \perp \overline{AC}$, $\overline{FE} \perp \overline{AC}$, and $FC = CE$. Prove that AD is the harmonic mean between AB and AC.

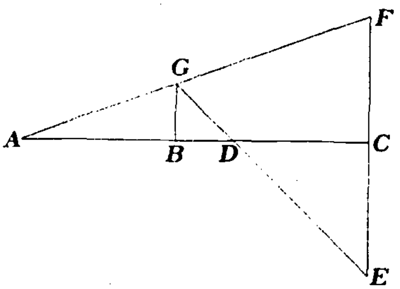


Fig. 10

57. In the coordinate plane, equilateral triangle ABC has vertex A at the origin and all other points in the first quadrant. (See fig. 11.) Side $\overline{BC}$ is then extended to intersect the coordinate axes, forming triangles AEC and ABD. If the area of one of these two triangles is twice that of the other, find the ratio of the area of the smaller triangle to the area of $\triangle ABC$.

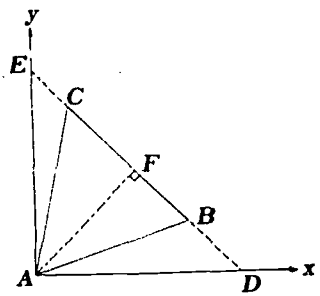


Fig. 11

58. In a square, the midpoint of each side is joined to the two opposite vertices. If the length of each side of the square is 2 units, what is the area of the octagonal region formed within the square?

59. In a certain scalene triangle the inscribed circle has a radius of 4 units. The points of tangency are 1 unit from the midpoint of one side, $\frac{1}{2}$ unit from the midpoint of a second side, and also $\frac{1}{2}$ unit from the midpoint of the third side. Find the length of each side of the triangle.

60. Circles are constructed on each of the three sides of right triangle ABC. In each case the center of the circle is the midpoint of the side, with the side being a diameter of the circle. If the area of $\triangle ABC$ is 12 square units, what is the total area of that part of the two smaller circles that lies outside the largest circle? (The area in question is that of the shaded region shown in fig. 12.)

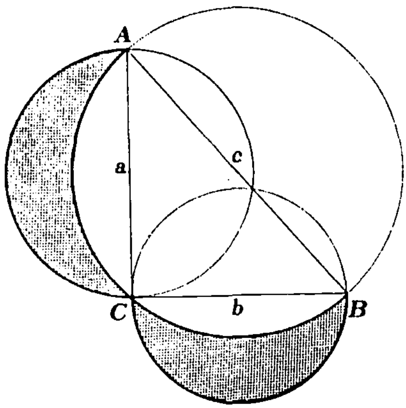


Fig. 12

61. If the measures of the sides of a triangle are consecutive integers and it is known that the measure of the greatest angle is twice that of the smallest angle, find the measures of the sides of the triangle.

62. Given a regular tetrahedron, find the ratio of the volume of the inscribed sphere to that of the circumscribed sphere.

63. $\overline{AB}$ is a common side of right triangles ABC and ABD. Hypotenuse $\overline{AC}$ of triangle ABC and hypotenuse $\overline{BD}$ of triangle ABD intersect at point E, determining a scalene triangle ABE. If hypotenuse $\overline{AC}$ measures 12 units and hypotenuse $\overline{BD}$ measures 10 and if the altitude of triangle ABE drawn from E to $\overline{AB}$ measures 4, approximate (to the nearest tenth) the length of $\overline{AB}$.

64. In figure 13, D is the midpoint of $\overline{BC}$, $m\angle BAD = 2(m\angle DAC)$, and $\overline{CA} \perp \overline{CE}$. Prove that $AE = 2(AB)$.

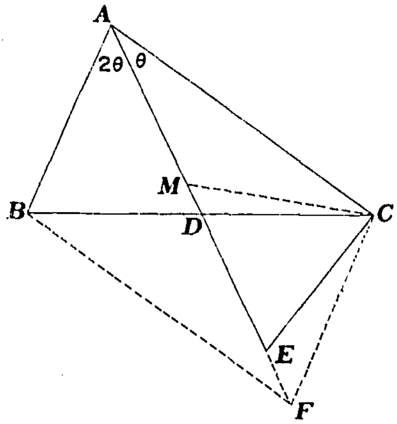


Fig. 13

65. Two circles with centers M and N (see fig. 14) are each internally tangent to a third circle (center at O) at points A and B. Also the two circles are externally tangent to each other at C. $\overrightarrow{AC}$ and $\overrightarrow{BC}$ intersect the circle O at points D and E. Prove that $\overline{DE}$ is a diameter of circle O.

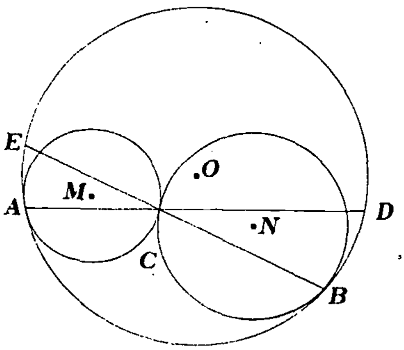


Fig. 14

66. Squares $AGFC$ and $BDEC$ are constructed on sides $\overline{AC}$ and $\overline{BC}$ of (any) triangle ABC, as shown in figure 15. $\overline{CH}$ is the altitude from C to side AB. Prove that when extended, this altitude bisects the segment joining E and F.

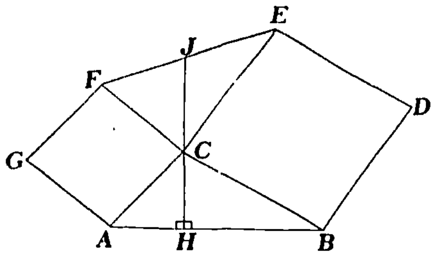


Fig. 15

67. In figure 16, ABC is an isosceles right triangle with hypotenuse $\overline{AB}$. Through C, a line is drawn parallel to $\overline{AB}$, and on this line, point D is taken so that $BD = BA$. Thus there are two possible positions for D; for each of them find $m\angle DBC$.

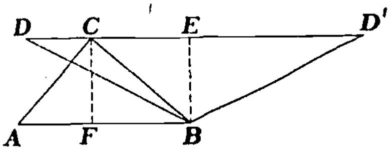


Fig. 16

68. ABC is a triangle in which $BC = 2(AB)$. D is the midpoint of $\overline{BC}$, and E is the midpoint of $\overline{BD}$. Prove that $\overrightarrow{AD}$ bisects $\angle CAE$. (See fig. 17.) (*Hint*: Draw $\overline{DF}$ parallel to $\overline{AC}$, where F is a point of $\overline{AB}$.)

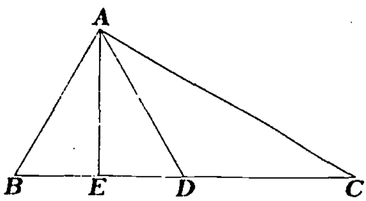


Fig. 17

69. Prove that in any triangle the sum of the medians is *less* than the perimeter.

4

Number Theory

70. Show that the sum of the squares of three consecutive integers cannot end in either 3 or 8.

71. What is the remainder when $5^{999,999}$ is divided by 7?

72. In how many different ways can the number 288 be represented as the product of three different positive integers (other than the number 1) if the order of the factors is not considered?

73. Write the integers 1, 2, 3, 4, 5, 6, 7 in a row; then beneath that row write the same integers but in a different order. For example, one such display would be this:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 6 & 3 & 5 & 7 & 1 & 4 \end{array}$$

If you now subtract by columns, taking the absolute value in each case, the result is this:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 6 & 3 & 5 & 7 & 1 & 4 \\ \hline 1 & 4 & 0 & 1 & 2 & 5 & 3 \end{array}$$

Note that the number 1 occurs twice among these differences. Now prove that no matter what the order in the second row, these differences cannot all be different.

74. Find the number of terminal zeros in the standard numeral for 100!

75. Find three positive integers m, n, and p that satisfy the equation

$$5^m + 9^n = 7p^2,$$

or prove that no such solutions exist.

76. Two numbers arc said to be congruent to each other with respect to a third nonzero integer if and only if they have the same positive remainder when divided by that third integer. For example, 5 and 11 are congruent (mod 3), and 8 and 1 are congruent (mod 7). Using this definition, find the smallest integer that is congruent to 2^{100} (mod 7).

77. If n is an *odd* integer, prove that 8 is a factor of $n^2 - 1$.

78. If n is a *positive* integer, prove that 6 is a factor of $n(n - 1)(2n - 1)$.

79. Find all the integers between 200 and 300 that have exactly six positive factors.

80. Prove that an infinite number of integral squares exist that begin with 7 and do not end with 0.

81. Prove that if a, b, and c are three integers such that

$$a^2 + b^2 = c^2,$$

then one of the following numbers is divisible by 7:

$$a, b, a + b, a - b.$$

5

Probability

82. A jar contains four balls—two red, one white, and one blue. They are drawn in succession and at random from the jar until a red ball is obtained; then no more are drawn. If it is known that a blue ball was drawn at some point in the process, what is the probability that the white ball was also drawn?

83. A certain football team has the reputation of being a very good team when the weather is bad. In fact, the probability of their winning when the weather is bad is .8, whereas the probability of their winning on a clear day is only .4. During the month of November, weather statistics indicate the probability of bad weather to be .6. If it is known that the team won a given game in November, what is the probability that the weather was bad on that day?

84. Three "fair" coins are tossed. Those coins that come up heads are removed, the remaining ones are tossed again, and so forth. What is the probability that after exactly three tosses, all the coins will have been removed?

85. Fifty tickets, numbered consecutively from 1 to 50, are placed in a jar, and two of them are drawn at random (without replacement). What is the probability that the difference of the two numbers drawn is 10 or less?

86. An integer between (but not including) 0 and 100,000 is to be chosen at random. What is the probability that the integer will be one whose digits can be reversed in order without changing the number?

87. A jar contains ten red balls and five white balls. One ball is drawn from the jar and is replaced by one of the opposite color. If a ball is now drawn from the jar, what is the probability that it will be red?

88. A publishing company is preparing a manuscript for publication and, of course, wants it to be as free of errors as possible. The company

hires two proofreaders—call them A and B—who read the manuscript independently. Proofreader A finds 36 errors, and proofreader B finds 32 errors. The publisher finds that included in these totals are 24 errors that were found by both proofreaders. What estimate can he make of the number of errors still unfound? (The answer to this question will help him decide if it is worthwhile to hire a third proofreader.)

89. There are three cards in a hat. One card is white on both faces, one is red on both faces, and the third has one red face and one white face. The cards are mixed, and one is drawn from the hat and then placed on a table *without* showing the underside. If the face up is red, what are the odds that the other face is also red?

90. Three jars are arranged in a row. The left jar contains two red balls and one white. The center jar contains one red ball and one white. The right jar contains one red ball and two white. A ball is chosen randomly from the appropriate jar and then discarded. If the chosen ball is white, the jar to the *left* is used next; if it is red, the jar to the *right* is used next. The game continues until either an empty jar is reached *or* a move is made past an end jar. Assuming that the game starts with the center jar, what is the probability that the game ends because an empty jar is reached?

6

Trigonometry

91. If in triangle ABC, with sides of length a, b, and c,

$$\frac{a^3 + b^3 + c^3}{a + b + c} = c^2,$$

determine the measure of $\angle C$, which is opposite the side of length c.

92. $ACJD$, $CBGH$, and $BAEF$ are squares constructed on the sides of triangle ABC, as in figure 18. If the sum of the measures of the areas of squares I and II is equal to the measure of the area of the rest of the figure, determine the measure of $\angle ABC$.

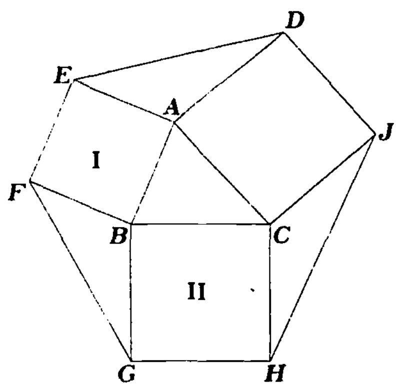


Fig. 18

93. Suppose it is given that the measures of the angles of a triangle form an arithmetic progression. Then consider the two questions below:

a. If the measures of the sides also form an arithmetic progression, is the triangle equilateral?

b. If the sides form a geometric progression, is the triangle equilateral?

94. A ladder is placed against the side of a building, forming with the ground an angle of measure u. The foot of the ladder is then moved a distance a farther away from the house (to D in fig. 19), and as a

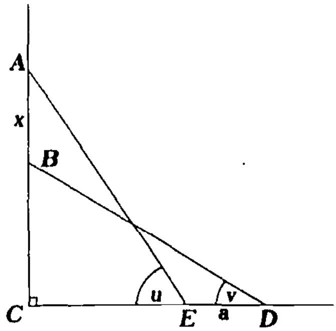


Fig. 19

result the ladder now forms with the ground an angle of measure v. If x is the distance the top of the ladder moves down the house as the foot is being moved out, prove that

$$x = a\left(\cot \frac{u+v}{2}\right).$$

95. Two coplanar circles are externally tangent, with the radius of one circle twice that of the other. From the center of the smaller circle two rays are drawn tangent to the larger circle. Determine the area of the shaded region as shown in figure 20.

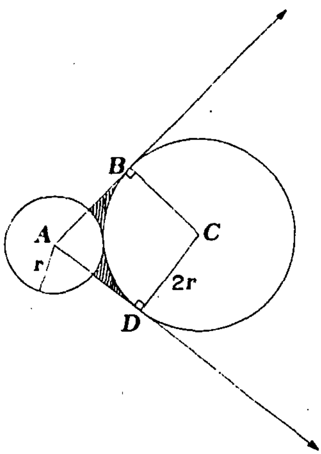


Fig. 20

96. Find the angle measure θ for which the following is true:

$$\left(\frac{16}{81}\right)^{\sin^2\theta} + \left(\frac{16}{81}\right)^{\cos^2\theta} = \frac{26}{27}.$$

97. Given the equation

$$\sin x = \frac{x}{100},$$

determine the *number* of solutions.

98. If in triangle ABC,

$$\sin B + \sin C = 2(\sin A),$$

find the numerical value of

$$\left(\tan \frac{B}{2}\right)\left(\tan \frac{C}{2}\right).$$

99.

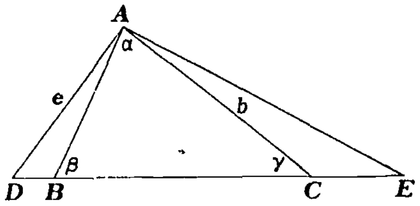


Fig. 21

If in figure 21 $\overline{DA} \perp \overline{AC}$ and $\overline{BA} \perp \overline{AE}$, prove that

$$DE = (BC)(\tan \angle ABC)(\tan \angle ACB).$$

100. In figure 22, $\overline{BD} \cong \overline{DC}$, $m\angle BAD = 2\,(m\angle DAC)$, and $\overline{CA} \perp \overline{CE}$. Prove that $AE = 2AB$.

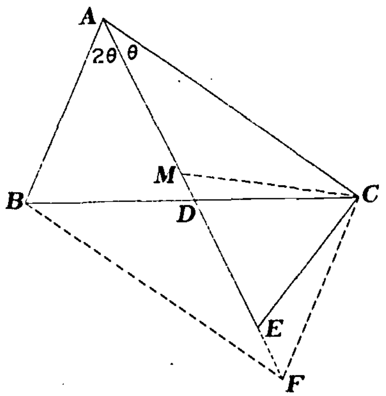


Fig. 22

SOLUTIONS

1

SOLUTIONS

Mostly for Fun

1. One got \$50; the other, \$.50.
2. *One* of them isn't a nickel, but the *other* one is.
3. The explanation is that the three specified fractions—$\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{9}$—do not have a sum of 1; instead their sum is only $\frac{17}{18}$.
4. The cost established by the manager was $\$30 - \$5 = \$25$. The cost to the customers was 3(\$9), or \$27. The difference of \$2 was the bellboy's theft. That is, $27 + (-2) = 25$, not 29.
5. The digits, each represented by a letter in the division problem, follow: $I = 0$; $T = 1$; $S = 2$; $D = 4$; $X = 5$; $N = 6$; and $E = 7$.

```
       5151
145)747015
    725
     220
     145
      751
      725
       265
       145
       120
```

6. 1. If $N \cdot X = N$ and $T \cdot X = T$, X must be 1.
   2. Since the multiplication developed by $TXX \cdot S = S$, S must be 0.
   3. If $D + D = a\,1$, where a is the number carried over to X in the next column and 1 is the value of X, D must be 3 or 4.

4. When $N \cdot T = 1D$, the only two possible answers are 11_{five} and 13_{five}; therefore, D must be 3 and T must be 4.
5. By the process of elimination, N must be 2.

$$S = 0;\ X = 1;\ N = 2;\ D = 3;\ \text{and } T = 4.$$

$$\begin{array}{r} 411_{five} \\ \underline{2402_{five}} \\ 1322\phantom{_{five}} \\ 31440\phantom{0_{five}} \\ \underline{1322\phantom{000_{five}}} \\ 2143222_{five} \end{array}$$

7. The sixth and eighth digits of the dividend are brought down with the fifth and seventh. Hence, the second and fourth digits of the quotient are zeros.

Since 8 times the divisor is a three-digit number (thereby making the divisor no more than 124) and since the last digit of the quotient times the divisor is a four-digit number, the last digit of the quotient must be a 9.

If the divisor were less than 124—for example, 123—the first digit of the quotient would have to be a 9 in order for the dividend to be an eight-digit number. But 9×123 is a four-digit number, and the product of the divisor and the first digit of the quotient is supposed to be a three-digit number. Hence, the divisor is exactly 124, and the first digit of the quotient is less than 9.

Finally, in order for the dividend to be an eight-digit number, the first digit of the quotient must be an 8 or a 9. Since it has been established that this digit cannot be a 9, it must be an 8. The division is thus reconstructed as follows:

$$\begin{array}{r} 80809 \\ 124\overline{)10020316} \\ \underline{992} \\ 1003 \\ \underline{992} \\ 1116 \\ \underline{1116} \end{array}$$

8. **First Solution**

Think of the seats as squares on a 5×5 checkerboard, colored alternately black and white. Then each student must move from a seat of one color to a seat of the other color. But the colors are divided into twelve of one and thirteen of the other. Therefore one of the thirteen will have nowhere to go. Thus the class cannot obey the teacher's order.

Second Solution

Number the rows and columns each from 1 to 5, and assign to each student a number corresponding to the sum of his row and column numbers. For example, the student in the third seat of the second row is assigned number 5. Let x be the sum of all 25 numbers. After the students have moved, each student will have a new number; but the sum for the class will still be x. Each student's number will have changed by 1, since he has changed his row or his column, but not both. Thus there will be 25 changes of 1 each. Thus if x is even (or odd), the odd number of changes will make the sum odd (or even), contrary to the fact that the sum must remain the same. Therefore the teacher's order cannot be obeyed.

9. If c is the number of medals won by team C, then $c + c - 1 + c - 2 = 30$. Therefore C, B, and A won 11, 10, and 9 medals, respectively. If a is the number of points obtained by A, then $a + a - 1 + a - 2 = 60$. Therefore A, B, and C earned 21, 20, and 19 points, respectively.

Team C must have at least 4 gold medals to have more than either A or B, but it cannot have more than 6 because it has a total of only 19 points. But if C had 5 or 6 gold medals, there would be no possible way for the team to have a total of 11 medals. Therefore C must have 4 gold medals. With 7 points and 7 medals more to account for, C must also have 7 bronze medals.

Because A and B must have fewer gold medals than C, they must each have 3 gold medals. For A, we must still account for 6 medals and 12 points. This is possible only if A has 6 silver medals.

There remain 3 bronze and 4 silver medals that must belong to B.

Summary:

Team A. 3 gold and 6 silver medals;
Team B. 3 gold, 4 silver, and 3 bronze medals;
Team C. 4 gold and 7 bronze medals.

10. We are given that

(1) $A + B = C + D,$

(2) $A + C > B + D,$ and

(3) $D > C.$

Subtracting (1) from (2) and simplifying, we get

(4) $C > B.$

Adding (1) and (4) and simplifying, we get

(5) $A > D.$

From (5), (3), and (4), we see that $A > D > C > B$.

11. If Andy is a mathematics student, then—from statement 2—he will pass chemistry. This implies that he would then fail physics. (Remember that the mathematics student will pass one subject, fail the other.) If, on the other hand, Andy is not a mathematics student, then—from 1—he will fail physics, which means he will pass chemistry.

If Bob is the mathematics student, then—by statement 4—we know he will pass physics and hence will fail chemistry. If he is not the mathematics student, then—from 3—we know he will not pass chemistry, and so he will pass physics.

If Carl is the student of mathematics, then—from 6—we know he will pass physics and therefore fail chemistry. If he is not the student of mathematics, then—from 5—he will not pass physics and therefore will pass chemistry.

From all of this we deduce that Andy cannot be the one boy who is a mathematics student. If he were, then both he and Carl would pass chemistry. (Remember that the mathematics student is the *only* one who will pass one of the two subjects and the only one who will fail the other.)

Also we know that Carl cannot be the mathematics student, since this would imply that both he and Bob pass physics.

Bob *is* the mathematics student. The six statements show that in this case he is the only one to pass physics and the only one to fail chemistry, thus satisfying the conditions of the problem.

12. The fallacy of the purported proof lies in the fact that it attempts to apply the method of mathematical situation to an argument about sets. Mathematical induction applies to theorems about the positive integers, where every integer k has a unique successor $k + 1$. However, sets are not ordered in this way, since given two sets of, say, ten elements, no one can say which is the successor of the other.

2

SOLUTIONS

Algebra

13. Let $AF = x$, $AE = 10 - x$, and $JK = 10 - 2x$. Then $x(10 - x) = (10 - 2x)^2$ and $x = 5 \pm \sqrt{5}$. If $x < 10 - x$, then $x = 5 - \sqrt{5}$ and $10 - x = 5 + \sqrt{5}$. These are the dimensions of each of the four outside rectangles. The measure of a side of the inner square is $10 - 2x$, or $2\sqrt{5}$. (See fig. 23.)

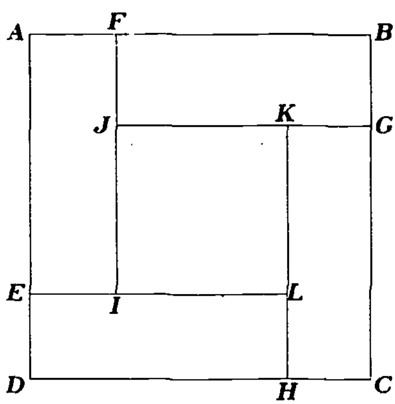


Fig. 23

14.

First Solution

First, assume that the train and the man meet at A (see fig. 24). Let y equal the man's speed in mph. Since time equals distance divided

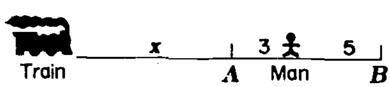


Fig. 24

by rate, it takes him $3k/y$ hours to get to A, where $8k$ is the length of the bridge in miles. Since he meets the train at A it also takes the train

$3k/y$ hours to get to A. For the train, distance equals rate times time, and so

$$x = (60)\left(\frac{3k}{y}\right),$$

or

$$x = \frac{180k}{y}.$$

Next, assume that they meet at B. It takes the man $\frac{5k}{y}$ hours to get to B. Since the train overtakes him at B, it also takes the train $\frac{5k}{y}$ hours to get to B, and

$$8k + x = (60)\left(\frac{5k}{y}\right),$$

or

$$8k + x = \frac{300k}{y}.$$

Substitute the value for x obtained above. Then

$$8 + \frac{180}{y} = \frac{300}{y}$$

$$8 = \frac{120}{y}$$

$$y = 15 \text{ mph}.$$

Second Solution

The man has two alternatives: he can run to A and meet the train there, or he can run to B and be overtaken by the train at that point. First, let us say that the man runs to A. When the man has run $\frac{3}{8}$ of the length of the bridge, the train will be at A. Therefore, if he ran the other way, toward B, he would have covered

$$\frac{3}{8} + \frac{3}{8} = \frac{3}{4}$$

of the length of the bridge by the time the train reaches A. Since the train and the man meet at point B, the man must travel $\frac{1}{4}$ of the length of the bridge while the train travels the entire length. Since the time is the same, the man must be traveling $\frac{1}{4}$ of the speed of the train, or 15 mph.

15. If the freshman sum is equal to the arithmetic mean, then

$$\frac{a+c}{b+d} = \frac{\frac{a}{b}+\frac{c}{d}}{2},$$

with $b \neq 0$, $d \neq 0$, and $(b+d) \neq 0$.

Then

$$abd + cbd = cb^2 + ad^2,$$
$$abd - ad^2 = cb^2 - cbd,$$
$$ad(b - d) = cb(b - d),$$

and

$$(ad - cb)(b - d) = 0.$$

So

$$(b - d) = 0 \rightarrow b = d,$$

and

$$(ab - cb) = 0 \rightarrow ad = cb,$$

or

$$\frac{a}{b} = \frac{c}{d}.$$

Thus, in order for the freshman sum to be equal to the mean, we must have either $b = d$ or $\frac{a}{b} = \frac{c}{d}$.

16.

First Solution

Suppose $\{7, 8, 9\}$ is one of the sets. Then the sum of the elements in each set is 24. The only possible sets containing 15 are $\{3, 6, 15\}$ and $\{4, 5, 15\}$. The only possible set containing 14 is $\{4, 6, 14\}$, which duplicates one of the elements in each of the sets containing 15. Hence $\{7, 8, 9\}$ is not one of the sets.

Second Solution

If $\{7, 8, 9\}$ is one of the five sets, then the sets containing 1 and 2 must each contain two elements $\geqq 10$. Since each set of three has as its sum 24, the other two elements in the set containing 1 total 23, and those in the set containing 2 total 22. Thus, the sum of the 4 elements from both sets totals 45. However, the lowest this sum can be is $10 + 11 + 12 + 13 = 46$. This is impossible; hence $\{7, 8, 9\}$ is not one of the sets.

17. The single passenger with 105 pounds of baggage would have to pay for $(105 - k)$ pounds. The two passengers with a total of 105 pounds of baggage would have to pay for $(105 - 2k)$ pounds, since each one would receive k pounds free. The two passengers had to pay a total of \$2.50 on their overweight baggage, and the single passenger had to pay \$6.50 on his overweight baggage. There is therefore a ratio between the two situations, as follows:

$$\frac{105 - k}{6.50} = \frac{105 - 2k}{2.50}$$

Solving this equation for k, we obtain

$$k = 40.$$

Therefore, each passenger was allowed to carry 40 pounds of baggage without charge.

18. Since 60 represents one of the two lesser integers of a Pythagorean triple and since $x^2 + y^2 = z^2$, we know that

$$(60)^2 + y^2 = z^2,$$
$$z^2 - y^2 = 3600,$$

and

$$(z - y)(z + y) = 3600.$$

Possible integral values for $(z - y)$ and $(z + y)$ are shown in the following chart. However, many of these values are eliminated if $(z - y) + (z + y)$ is an odd integer, since then $2z$ would be an odd integer and z would be half an odd integer. In this case z would not be an integer.

$z - y$	$z + y$	z	y	x
1	3600	—	—	—
2	1800	901	899	60
3	1200	—	—	—
4	900	452	448	60
5	720	—	—	—
6	600	303	297	60
8	450	229	221	60
9	400	—	—	—
10	360	185	175	60
12	300	156	144	60
15	240	—	—	—
16	225	—	—	—
18	200	109	91	60
20	180	100	80	60
24	150	87	63	60
25	144	—	—	—
30	120	75	45	60

$z - y$	$z + y$	z	y	x
36	100	68	32	60
40	90	65	25	60
45	80	—	—	—
48	75	—	—	—
50	72	61	11	60
60	60	—	—	—

Checking the last three columns, we find exactly 13 different integral Pythagorean triples that satisfy the given conditions.

19. Since a, b, and c are in arithmetic progression,

$$b - a = c - b.$$

Thus,

$$(x^2 + xz + z^2) - (x^2 + xy + y^2) = (y^2 + yz + z^2) - (x^2 + xz + z^2).$$

Simplifying, we obtain

$$(x + y + z)(z - y) = (y - x)(x + y + z).$$

Since $x + y + z \neq 0$,

$$z - y = y - x.$$

Therefore, x, y, and z are in arithmetic progression.

20. Let a be the first term and r the common ratio. Thus ar is the second term, and ar^5 the sixth term. Then,

$$a + ar + ar^5 = a(1 + r + r^5).$$

If $a \neq 1$, then $a(1 + r + r^5)$ is not a prime number. If $a = 1$, then

$$1 + r + r^5 = (1 - r^2 + r^3)(1 + r + r^2),$$

and this is not a prime number. If $a = 1$ and $r = 1$, then each term is 1, and $1 + 1 + 1 = 3$ is a prime number.

21.

First Solution

If p/q is a rational root of the given equation, p and q must each be odd, since they are integral factors of c and a, respectively. Substitute $x = p/q$ and clear fractions:

$$ap^2 + bpq + cq^2 = 0$$

This states that the sum of three odd integers is zero—an impossibility. Therefore the given equation cannot have rational roots.

Second Solution

For the given equation to have rational roots, its discriminant must be a perfect square. That is, $b^2 - 4ac = k^2$, or $b^2 - k^2 = 4ac$. Since

b and k are odd, let $b = 2r + 1$ and $k = 2s + 1$. Substituting, we obtain

$$4r^2 + 4r + 1 - (4s^2 + 4s + 1) = 4ac.$$

Therefore $(r^2 + r) - (s^2 + s) = ac$. Now $r(r + 1)$ is the product of consecutive integers and is therefore even. For the same reason $s^2 + s$ is even. But ac is odd; therefore, the last equation states that the difference between two even numbers is an odd number—an impossibility.

22. If

$$f(x) = \frac{x - 1}{x + 1},$$

then

$$x = \frac{f(x) + 1}{1 - f(x)}.$$

Also

$$f(2x) = \frac{2x - 1}{2x + 1}.$$

If we now substitute for x, we obtain

$$f(2x) = \frac{2f(x) + 2}{1 - f(x)} - 1 \Big/ \frac{2f(x) + 2}{1 - f(x)} + 1,$$

which reduces to

$$\frac{3f(x) + 1}{f(x) + 3}.$$

In general

$$f(nx) = \frac{(n + 1)f(x) + (n - 1)}{(n - 1)f(x) + (n + 1)}.$$

23. The difference of the roots of $a_1x^2 + b_1x + c_1 = 0$ is

$$\frac{-b_1 + \sqrt{D_1}}{2a_1} - \frac{-b_1 - \sqrt{D_1}}{2a_1} = \frac{\sqrt{D_1}}{a_1},$$

where $D_1 = b_1^2 - 4a_1c_1$.

The difference may also be taken as the negative of this, but the final result is not affected. Similarly, the difference of the roots of $a_2x^2 + b_2x + c_2 = 0$ is $\frac{\sqrt{D_2}}{a_2}$, where $D_2 = b_2^2 - 4a_2c_2$. Since the ratio of the differences $\frac{a_2\sqrt{D_1}}{a_1\sqrt{D_2}}$ is rational, $\sqrt{\frac{D_1}{D_2}}$ is also rational. This is possible only if $\frac{D_1}{D_2}$ is a rational square.

24. **First Solution**

Let x equal the time (in minutes) between the teacher's turning downstream and his meeting up with his hat, a equal the rowing speed of the teacher in feet/minute, and c equal the speed of the current in feet/minute. Using these variables, we can set up an equation that shows that the distance covered by the hat is equal to the distance traveled by the rowing teacher after he passes under the bridge for the second time:

$$(5 + x)c = (a + c)x - 5(a - c)$$

Solving this equation for x, we obtain

$$x = 5 \text{ minutes.}$$

Since we also know that $(5 + x)c$ is equal to one mile, the fact that x is equal to 5 minutes shows us that the current travels one mile in ten minutes. Therefore

$$c = 6 \text{ miles per hour.}$$

Second Solution

Since both the hat and the boat are affected the same way by the current, the time taken to row to the hat would be equal to the time spent rowing away from the hat. The teacher spent 5 minutes rowing away from the hat; thus it would take 5 minutes for him to row back to his hat. In these 10 minutes the hat was carried 1 mile from the bridge. In 1 hour the hat would be carried 6 times as far, or 6 miles. Therefore, the current was 6 miles per hour.

25. **First Solution**

Let x be the number of dollars and y the number of cents that the check originally called for. Then, $100x + y$ is the real value of the check, and $100y + x$ is what Mr. Smith received. Thus

$$100y + x - 68 = 2(100x + y),$$

and

$$98y = 68 + 199x.$$

The last equation can be expressed as

$$98y \equiv 68 \pmod{199}.$$

Also, since we know that

$$199y \equiv 199 \pmod{199},$$

we have

$$199y - (2)(98y) \equiv [199 - (2)(68)] \pmod{199},$$

and thus

$$3y \equiv 63 \pmod{199}$$
$$y \equiv 21 \pmod{199}.$$

Thus y is equal to 21 plus some integral multiple of 199 and x will correspondingly equal 10 plus some integral multiple of 98. Since the original check was for less than \$100.00, the only correct answer is \$10.21.

Second Solution

The problem can also be solved by using the FORTRAN program shown in figure 25. This program simply takes all values from \$00.01 to \$99.99 and tries them separately to see if they work. The word STOP at the end signifies that the computer was finished trying all possibilities and that \$10.21 was the only answer found.

```
     REST
09 NOV 70, 12:31
       LINE 10
FORTRAN II READY
      C            MSJ PROBLEM 337
OK.   C            STEVE DENENBERG
OK.          100   FORMAT($THE SOLUTION(S) TO THE PROBLEM$/)
OK.                TYPE 100
OK.                I = O
OK.            1  FORMAT (1H$, I2, $.$, I2)
OK.            2   FORMAT(I2, $,←.$, I2)
OK.                DO 20 J = 0, 99
OK.                DO 20 JA = 0, 99
OK.                JB = 2*(J*100 + JA) - (JA*100 + J - 68)
OK.                IF(JB)20, 21, 20
OK.           21   JC = JA + 100
OK.                I = I + 1
OK.                IF(I - 1)22, 22, 23
OK.           22   TYPE 1, J, JC
OK.                GO TO 20
OK.           23   TYPE 2, J, JC
OK.           20   CONTINUE
OK.                END
OK.  RUN
THE SOLUTION(S) TO THE PROBLEM
$10.21
STOP
```

Fig. 25

26. By definition, $A = \frac{1}{2}(r_1 + r_2)$ and $G = \sqrt{r_1 r_2}$; therefore, $2A = r_1 + r_2$ and $G^2 = r_1 r_2$. Solving these equations for r_2 and combining, we obtain

$$G^2 = r_1(2A - r_1);$$

thus,

$$r_1^2 - 2Ar_1 + G^2 = 0.$$

Applying the quadratic formula, we obtain

$$r_1 = A \pm \sqrt{A^2 - G^2}.$$

Similarly, if we solve the equations above for r_2 and combine, we obtain a quadratic equation in r_2, and

$$r_2 = A \pm \sqrt{A^2 - G^2}$$

Since we know $r_1 > r_2$, then

$$r_1 = A + \sqrt{A^2 - G^2}$$

and

$$r_2 = A - \sqrt{A^2 - G^2}.$$

The relationship $A > G$ is always true because r_1 and r_2 must represent positive numbers. If $A < G$, then $A^2 - G^2$ would be negative and the roots would be complex. If $A = G$, the roots would be equal.

27. Since the product of any row, column, or diagonal equals k, $aei = k$, $beh = k$, $ceg = k$, and $abc = k$. Dividing, we obtain:

$$a = \frac{k}{ei}; \quad b = \frac{k}{eh}; \quad \text{and } c = \frac{k}{eg}.$$

Therefore,

$$\left(\frac{k}{ei}\right)\left(\frac{k}{eh}\right)\left(\frac{k}{eg}\right) = k,$$

$$\frac{k^3}{eieheg} = k,$$

and

$$k^2 = eieheg.$$

But $ghi = k$;

$$\therefore k = e^3.$$

Since k is a positive integer between 1 and 100, k may assume the values of the cubes of 2, 3, or 4.

$$\therefore k \in \{8, 27, 64\}.$$

28. If

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{12},$$

then

$$xy - 12x - 12y = 0.$$

Therefore

$$xy - 12x - 12y + 144 = 144,$$

or

$$(x - 12)(y - 12) = 144.$$

If we now write 144 as the product of two positive integral factors, we may set $x - 12$ equal to one, and $y - 12$ equal to the other. Thus $[(x - 12), (y - 12)] = (1, 144), (2, 72), (3, 48), (4, 36), (6, 24), (8, 18), (9, 16), (12, 12)$, and the first seven pairs in reverse order. Therefore $(x, y) = (13, 156), (14, 84), (15, 60), (16, 48), (18, 36), (20, 30), (21, 28), (24, 24)$, and the seven reversed pairs—a total of 15 answers.

29. **a.** The intercept form of the equation of a line whose x and y intercepts are u and v respectively is

$$\frac{x}{u} + \frac{y}{v} = 1.$$

With $(x, y) = (3, 3)$ and $u = 4, 5, 6$, v becomes $12, \frac{15}{2}, 6$, respectively.

b. For $u = 4, 5, 6$,

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{4} + \frac{1}{12}, \quad \frac{1}{5} + \frac{2}{15}, \quad \frac{1}{6} + \frac{1}{6}.$$

In each case, this is $\frac{1}{3}$.

c. The generalization is that for any line passing through the point (k, k) and intersecting the x-axis at $(u, 0)$ and the y-axis at $(0, v)$, $(u, v \neq 0$, and hence $k \neq 0)$,

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{k}.$$

Proof. Let $(x, y) = (k, k)$ in

$$\frac{x}{u} + \frac{y}{v} = 1.$$

Then

$$\frac{k}{u} + \frac{k}{v} = 1,$$

or

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{k}.$$

d. All points (k, k) lie on the line $y = x$. But not all points on $y = x$ yield the given result. As noted in *c*, the exception is $(0, 0)$.

30. **First Solution**

a. Let $F_1 = \frac{1}{2}(x + y + |x - y|)$, where the symbol $|a|$ is defined as the absolute-value function.

b. Let $F_2 = \frac{1}{2}(F_1 + z + |F_1 - z|)$, where F_1 is defined as in **a.**

Second Solution

a. Let $F(x)$ be defined so that $F(x) = -1$ for $x < 0$, $F(x) = 0$ for $x = 0$, and $F(x) = +1$ for $x > 0$. Then the desired function will be

$$f(x,y) = \frac{1}{2}\{x[1 + F(x - y)] + y[1 + F(y - x)]\}.$$

b. Although the expression is lengthy, this solution may be extended to solve part **b** also.

31. Since a segment joining the midpoints of the two sides of a triangle has a length equal to one-half the length of the third side, each side of the second triangle is one-half the length of a side of the original triangle. The perimeter of the second triangle is therefore one-half the perimeter of the original triangle. It is also clear that the new triangle is also equilateral.

These statements will also be true as we continue constructing triangles; that is, the third triangle has a perimeter one-half that of the second triangle, the fourth triangle has a perimeter one-half that of the third triangle, and so forth.

In this way, we can form the following infinite geometric progression with a common ratio of $\frac{1}{2}$, where the terms are the perimeters of the triangles:

$$3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \ldots$$

The total perimeter of all the triangles that can be formed in this way will equal the sum of this convergent infinite geometric series. In order to find this sum, we use the formula

$$S = \lim_{n \to \infty} S_n = \frac{a}{1 - r}$$

where n is the number of terms, a is the first term, and r is the common ratio.

Thus, the total perimeter of all the triangles is

$$\frac{3}{1 - \frac{1}{2}} = \frac{3}{\frac{1}{2}} = 6.$$

If the problem were interpreted to include not only the triangles like *ABC*, *DEF*, *GHI*, but also triangles like *BFE*, *AFD*, *CED*, and so on, the series would be a little different:

$$3 + 4\left(\frac{3}{2}\right) + 4\left(\frac{3}{4}\right) + \ldots$$

In this case the total perimeter would be 15.

32. The average failing grade, 48, times the number of failing grades, (F_1), plus the average passing grade, 75, times the number of passing grades, (P_1), must equal the class average, 66, times the total number of students in the class, ($F_1 + P_1 = T$). Thus

$$48F_1 + 75P_1 = 66(F_1 + P_1),$$

or

$$P_1 = 2F_1,$$

and

(1) $$T = P_1 + F_1 = 3F_1.$$

After the grades were raised 5 points, we have

$$45F_2 + 77.5P_2 = 71(F_2 + P_2),$$

or

$$P_2 = 4F_2,$$

and

(2) $$T = P_2 + F_2 = 5F_2.$$

From equations (1) and (2) we see that T must be divisible by 3 and by 5. The only such number between 20 and 40 is 30. Thus, $T = 30$, and we have

$$30 = 3F_1, \text{ or } F_1 = 10$$

and

$$30 = 5F_2, \text{ or } F_2 = 6.$$

Therefore the number of students who had their grade changed is $10 - 6 = 4$.

33. In figure 26, we see that the rear of the line starts at A and travels to B. The front starts at B and travels to C. The messenger starts at A, meets the front of the line D, and returns x miles to B. The ratio of the total distances the messenger and the front of the column traveled is equal to the ratio of the distances each traveled until they met at D:

$$\frac{25 + 2x}{25} = \frac{25 + x}{x}$$

$$25x + 2x^2 = 625 + 25x$$

and

$$x^2 = \frac{625}{2},$$

or

$$x = \frac{25\sqrt{2}}{2}.$$

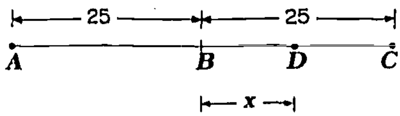


Fig. 26

The total number of miles traveled was $25 + 2x$. Using the value of x obtained above, we have the total number of miles

$$25 + 25\sqrt{2}.$$

Therefore, the messenger traveled approximately 60.35 miles.

34. Since $x^3 - 13x + 12 = (x - 1)(x - 3)(x + 4)$, we know that the required fractions must be of the form

$$\frac{a}{(x-1)},\ \frac{b}{(x-3)},\ \text{and}\ \frac{c}{(x+4)}.$$

To find a, b, and c, we combine the above fractions and obtain

$$\frac{(a + b + c)x^2 + (a + 3b - 4c)x + (-12a - 4b + 3c)}{x^3 - 13x + 12}.$$

Thus, since the numerator is equal to 1, we obtain the following system of equations:

$$\begin{aligned} a + b + c &= 0 \\ a + 3b - 4c &= 0 \\ -12a - 4b + 3c &= 1. \end{aligned}$$

Solving simultaneously gives

$$a = -\frac{1}{10},\ b = \frac{1}{14},\ c = \frac{1}{35}.$$

Substituting and simplifying, we obtain

$$\frac{-1}{10x - 10} + \frac{1}{14x - 42} + \frac{1}{35x + 140}.$$

Therefore, the sum of the denominators is $59x + 88$.

35. Factoring the given expression, we obtain

$$a^{(14x+6)}(b^2 - 2ba + a^2) = a^{(2x^2+30)}(b^2 - 2ba + a^2),$$

and

$$a^{(14x+6)}(b - a)^2 = a^{(2x^2+30)}(b - a)^2.$$

Since we know that $(b - a)$ does not equal zero, division by $(b - a)^2$ is permitted, and

$$a^{(14x+6)} = a^{(2x^2+30)}.$$

For all a not equal to 1 or -1, the exponents must be equal.

$$\begin{aligned} 14x + 6 &= 2x^2 + 30 \\ 2x^2 - 14x + 24 &= 0 \\ x^2 - 7x + 12 &= 0 \\ (x-3)(x-4) &= 0. \end{aligned}$$

and

$$x = 3 \text{ or } x = 4.$$

Since both $14x + 6$ and $2x^2 + 30$ represent even numbers for any positive integer x, the solution set is all positive integers when a is 1 or -1.

36. Since each letter in the alphametic represents a unique and different "digit" and there are eight different letters used (O, N, E, T, W, F, U, R), the positive integral base must be equal to, or greater than, eight.

$O + O = R$, but $O + O = T$; therefore, in $ONE + ONE = TWO$, 1 must be carried over from $N + N$.

$$\therefore 2O = R,\ 2O + 1 = T,\ R + 1 = T.$$

(R is even, T is odd) and $F = 1$.

If $O =$ zero, then $R =$ zero. $\therefore O \neq$ zero and $O \neq 1$. Also, O must be less than one-half of the base, since $1 + O + O = T$ does not exceed the base, and $O + O = R$ does not exceed the base.

Label the base b.

$$1 < O < \frac{b}{2}.$$

If $b = 8$, $O = 2$ or $O = 3$. (Neither works.)

If $b = 9$, $O = 2$, $O = 3$, or $O = 4$. (None of these works.)

If $b = 10$, then $O = 2$, $O = 3$, or $O = 4$. (None of these checks).

If $b = 11$ or $b = 12$, then $O = 2$, $O = 3$, $O = 4$, or $O = 5$. (None of these checks in either base.)

If $b = 13$, $O = 2$, $O = 3$, $O = 4$, $O = 5$, or $O = 6$. (None of these checks in base 13.)

If $b = 14$, $O = 2$, $O = 3$, $O = 4$, $O = 5$, or $O = 6$.

If the digits of base 14 are

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, \text{tn}, \text{el}, \text{tw}, \text{th}\},$$

then when $O = 4$, a solution is found.

$$ONE + ONE = TWO \leftrightarrow 4(\text{tn})2 + 4(\text{tn})2 = 964,$$
$$TWO + TWO = FOUR \leftrightarrow 964 + 964 = 14(\text{tw})8.$$

Therefore, $O = 4$, $N =$ tn, $E = 2$, $T = 9$, $W = 6$, $F = 1$, $U =$ tw, and $R = 8$.

Thus, the smallest base for which these alphametics hold true is 14.

37. If the answer is to be one-half of a perfect square, then twice the answer is a perfect square and contains 2 as a factor. However, when 2 is a factor in a square number, 2 must be a factor more than once. Similar reasoning indicates that the number in question contains 2, 3, 5, and 7 as factors. Call the number $2^w \cdot 3^x \cdot 5^y \cdot 7^z$.

For this number to be one-half of a perfect square, x, y, and z must be multiples of 2, and w must be one less than a multiple of 2. For this number to be one-third of a perfect cube, w, y, and z must be multiples of 3, and x must be one less than a multiple of 3. Similar reasoning holds for 5 and 7.

Thus, w must be a multiple of 3, 5, and 7, and one less than a multiple of 2. 105 is divisible by 3, 5, and 7, and is an odd number. $\therefore w = 105$. 140 is divisible by 2, 5, and 7 and is one less than a multiple of 3. $\therefore x = 140$.

Although 42 is a multiple of 2, 3, and 7, it is not one less than a multiple of 5. Since 84 meets all these requirements, $y = 84$.

90 meets all the requirements for z, since it is a multiple of 2, 3, and 5 and is one less than a multiple of 7. $\therefore z = 90$.

Therefore, the least integer that fits all the stated requirements is $2^{105} \cdot 3^{140} \cdot 5^{84} \cdot 7^{90}$. By using logarithms, the number of digits in $2^{105} \cdot 3^{140} \cdot 5^{84} \cdot 7^{90}$ is 234.

38.

First Solution

We have

$$\begin{aligned} f(1) &= 1 \\ f(2) &= f(1) + 2(2) - 1 \\ f(3) &= f(2) + 2(3) - 1 \\ &\;\;\vdots \\ f(n) &= f(n-1) + 2(n) - 1. \end{aligned}$$

Adding, we obtain

$$\sum_1^n f(i) = 1 + \sum_1^{n-1} f(i) + 2\sum_2^n i - (n-1) \quad .$$

Therefore,

$$f(n) = 1 - n + 1 + 2\left[\frac{n-1}{2}(n+2)\right] = n^2.$$

Replace n by $2n$ to obtain $f(2n) = 4n^2$.

Second Solution

Note that $1^2 = 1$ and $n^2 = (n-1)^2 + 2n - 1$. Thus $f(n)$ as defined here is equal to n^2. Therefore (as in solution 1), $f(2n) = 4n^2$.

39. If the linear factors of $ax^2 + x - 6$ are to have integral coefficients, the discriminant $24a + 1$ must be an integral square. Therefore $1 + 24a = N^2$, and since N^2 is odd, N is odd and may be written $2M + 1$. Therefore

$$a = \frac{(N^2 - 1)}{24} = M\,\frac{(M + 1)}{6},$$

where $600 \leq M(M + 1) \leq 1{,}200$ and 6 divides $M(M + 1)$. $M = 24$, 26, 27, 29, 30, 32, 33 satisfy these conditions, and $a = 100$, 117, 126, 145, 155, 176, 187.

40. **a.** (All terms positive.) Consider the formula

$$S = n\left(\frac{a + l}{2}\right),$$

in which a is the first term, n the number of terms, l the last term, and S the sum of the terms of the arithmetic progression. For S to be prime, there are four possibilities:

1. $n = 1$; $(a + l)/2$ prime
2. $n/2 = 1$; $a + l$ prime
3. $a + l = 1$; $n/2$ prime
4. $(a + l)/2 = 1$; n prime

Cases 1 and 2 may be ruled out because it is given that $n \geq 3$. Since all terms are to be positive, $a \geq 1$ and $l \geq 1$, and $a + l \geq 2$, thereby ruling out case 3. In case 4, $a + l = 2$, which is satisfied only by $a = 1$, $l = 1$. In this case, the common difference is zero. Thus the sum of a prime number of 1's is that prime. For example,

$$1 + 1 + 1 + 1 + 1 + 1 + 1 = 7.$$

b. (Not all terms positive.) If we are not limited to positive integers, cases 3 and 4 yield any number of possibilities. For example, if $a + l = 1$, $n = 2p$ (p is a prime). Let $d = 1$, $p = 3$; then $n = 6$, $a = -2$, and $S = (-2) + (-1) + 0 + 1 + 2 + 3 = 3$. If $a + l = 2$, $n = p$. Let $d = 3$, $p = 5$; then $a = -5$, $l = 7$, and

$$S = (-5) + (-2) + 1 + 4 + 7 = 5.$$

41. The formula $T = n^2 + 3n$ can be determined by a variety of methods, a common one being based on the formula for the sum of an arithmetic series. Thus, the number of vertical walls is $1 + 2 + \ldots + n + n = n\left(\frac{n+1}{2}\right) + n$. The number of horizontal walls is the same. Therefore the total number of walls is

$$T = n(n + 1) + 2n = n^2 + 3n.$$

Now for $n = 16$,

$$n^2 + 3n = 304;$$

and for $n = 17$,

$$n^2 + 3n = 340.$$

Hence the number of additional walls required is $340 - 317 = 23$.

42. Square both sides (this may introduce extraneous solutions) to obtain

$$5 - x^2 = \sqrt{5 - x}.$$

Square again to get

$$x^4 - 10x^2 + x + 20 = 0,$$

or

$$(x^2 + x - 5)(x^2 - x - 4) = 0.$$

Setting $x^2 - x - 4$ equal to zero gives $x = \dfrac{1 \pm \sqrt{17}}{2}$. However, neither of these numbers is a solution of the original equation (they were introduced by squaring). Setting $x^2 + x - 5$ equal to zero yields $x = \dfrac{-1 \pm \sqrt{21}}{2}$. Of these numbers only $\dfrac{-1 + \sqrt{21}}{2}$ is a solution. (Remember that the symbol "$\sqrt{}$" denotes a positive number.)

3

SOLUTIONS

Geometry

43. By drawing $\overline{BG} \parallel \overline{AD}$ we know that $AB = EH = DG = 7$, and

$$\frac{AE}{ED} = \frac{BH}{HG} = \frac{BF}{FC} = \frac{2}{1}.$$

This, of course, makes $\triangle BHF$ similar to $\triangle BGC$, so

$$\frac{BH}{BG} = \frac{HF}{GC}$$

and

$$\frac{2x}{3x} = \frac{z}{3},$$

which means that $z = 2$; therefore, $EF = EH + HF = 7 + 2 = 9$.

44. Mark off point F on OA such that $OF = 2(OD)$. The segment connecting D to F will be the required segment.

Proof. From geometry we know that the bisector of a vertex angle of a triangle divides the base in a proportion equal to that of the other two sides of the triangle. Consider the triangle DOF. Since $OD{:}OF = \frac{1}{2}$ by construction, $\overrightarrow{OE}$ divides the base in the same proportion as the sides of the triangle, that is, $\frac{1}{2}$. So $\frac{DE}{EF} = \frac{1}{2}$.

45. In figure 27, $(AD)^2 + (DO)^2 = (AO)^2$, and $(AD)^2 + 5^2 = 13^2$ (Pythagorean theorem and substitution). Thus, $AD = 12$; likewise, $AE = 12$.

$BD = BF$ and $CE = CF$ (tangents drawn from an external point to a circle are equal).

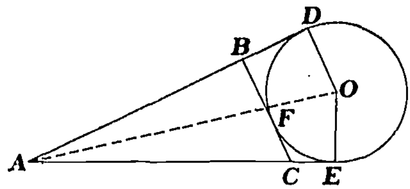


Fig. 27

The perimeter of $\triangle ABC$ is equal to $AB + BC + AC$, or $AB + BF + CF + AC$.
$\therefore$ the perimeter of $\triangle ABC$ is 24.

46. a. *First solution.* Let $\overrightarrow{FE}$ intersect AD at G. $\triangle FCE \cong EDG$ ($\overline{CE} \cong \overline{ED}$, right angles, vertical angles). Therefore $\overline{EG} \cong \overline{EF}$, whence $\triangle AFE \cong \triangle AGE$ (SAS). Thus $x = y$.

Second solution. Through E draw $\overrightarrow{EH} \parallel \overrightarrow{AD}$ intersecting $\overline{FA}$ at J (see fig. 28). Since $CE = ED$, and $\overline{BC} \parallel \overline{HE} \parallel \overline{AD}$, it follows that $\overline{FJ} \cong \overline{JA}$. Thus EJ is the median to the hypotenuse of right triangle AEF. So $EJ = \frac{1}{2} AF = AJ$; whence $m\ \angle JEA = y$. But $m\ \angle JEA = x$. Therefore $x = y$.

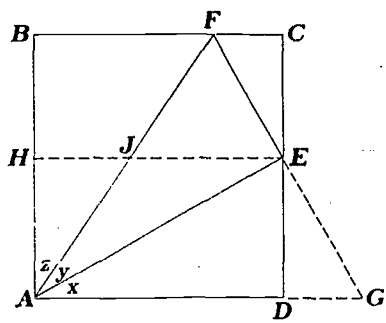


Fig. 28

b. $ED = \frac{1}{2} AD < \frac{1}{2} AE$. Therefore $x < 30°$, and $x + y < 60°$. Thus $z > 30° > x$.

47. In triangle EXD, $\overline{XD}$ is the shortest side, since it is opposite the smallest angle. In triangle DXC, $\overline{XD}$ and $\overline{XC}$ are the same length, each being shorter than $\overline{CD}$. In triangle CXB, $\overline{BC}$ is the shortest side, which means that it is shorter than $\overline{XC}$ and consequently shorter than $\overline{XD}$.

In triangle BXA, $BX = BA$, and since $BX > BC$, we have $BA > BC$. Also in triangle BXA, $\overline{AX}$ is the longest side, and so $AX > BC$. Therefore, by elimination, $\overline{BC}$ is the shortest segment.

Using the fact that if two sides of one triangle are congruent to two sides of another triangle but the included angle is greater in one case than the other then the side opposite the greater angle is longer, one can quickly eliminate all segments except $\overline{AX}$ and $\overline{ED}$ when searching for the longest segment. Which of these two is longer? Locate point E' on segment ED so that $m \angle DXE' = 100$. Note that E' will be between E and D, since $m \angle DXE' < m \angle DXE$. Then triangles ABX and DXE' are congruent by angle-side-angle; so $DE' = AX$. But since $DE' < DE$, this means $AX < DE$, and so $\overline{DE}$ is the longest segment.

48. Let one triangle be triangle ABC with $AB = 12$, $BC = 18$, and $AC = 27$; then let the other be triangle DEF with $DE = 8$, $EF = 12$, and $DF = 18$. Since these triangles are similar, the three angles of one have the same measure as the three angles of the other. Also two of the sides of one agree with two of the sides of the other. Thus the triangles agree in a total of five parts, but they are not congruent!

(*Note*: Of course it is impossible to set up a one-to-one correspondence between the vertices of the triangles so that five *corresponding* parts are congruent. The solution cited above is but one of an infinite number. For a full discussion of this problem, see the article "5-Gon Triangles," by Richard G. Pawley, in the May 1967 issue of the *Mathematics Teacher*.

49. Construct a perpendicular from E to $\overline{DC}$. It is also perpendicular to $\overline{AB}$, since $ABCD$ is a rectangle. This line cuts both $\overline{AB}$ and $\overline{CD}$ into two parts, say q and n. (See fig. 29.)

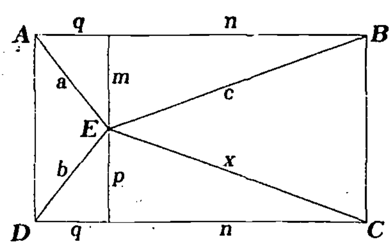


Fig. 29

Using the Pythagorean theorem on the upper triangles yields

$$q^2 + m^2 = a^2 \text{ and } m^2 + n^2 = c^2.$$

Subtracting the latter equation from the former gives us

(1) $$q^2 - n^2 = a^2 - c^2.$$

On the lower triangles, we again use the Pythagorean theorem, and

$$q^2 + p^2 = b^2 \text{ and } n^2 + p^2 = x^2$$

Subtracting as before, we have

(2) $$q^2 - n^2 = b^2 - x^2.$$

Substituting equation (1) into equation (2), we obtain

$$a^2 - c^2 = b^2 - x^2,$$

and finally

$$x = \sqrt{b^2 + c^2 - a^2}.$$

50. As shown in figure 30, E and D are the trisection points of hypotenuse $\overline{AB}$ in right $\triangle ABC$. $CE = 9$ and $CD = 7$. Through E and D draw lines parallel to $\overline{CB}$. Since $\overline{EG} \parallel \overline{DF} \parallel \overline{CB}$, $\overline{EG} \perp \overline{AC}$ and $\overline{DF} \perp \overline{AC}$ $(\overline{BC} \perp \overline{AC})$. Through E and D lines are drawn parallel to $\overline{AC}$ making $\overline{DJ} \parallel \overline{EH} \parallel \overline{AC}$, and $\overline{DJ} \perp \overline{CB}$, $\overline{EH} \perp \overline{CB}$, and $\overline{AC} \perp \overline{CB}$. $\triangle AGE \cong \triangle END \cong \triangle DJB$. Therefore, $AG = GF = FC$, and $GE = FN = ND$.

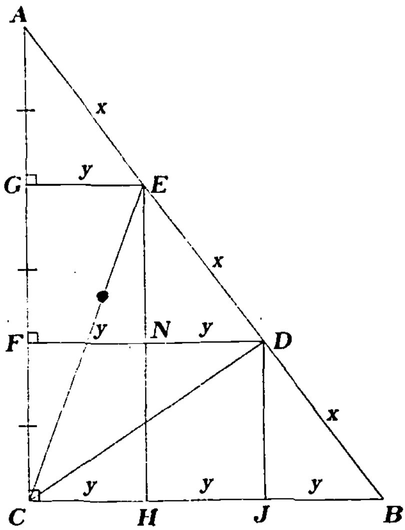


Fig. 30

Using the Pythagorean theorem,

$$AG = GF = FC = \sqrt{x^2 - y^2},$$
$$(GE)^2 + (GC)^2 = 9^2,$$

and

$$(FD)^2 + (FC)^2 = 7^2.$$

Solving these equations and substituting, we obtain $AB = 3\sqrt{26}$.

51. As shown in figure 31, $\triangle ABC$ represents the 3-4-5 right triangle and $\overline{CD}$ represents the trisector of the right angle that is adjacent to the shorter side, $\overline{CB}$. A perpendicular is dropped from D to $\overline{AC}$ forming the 30-60 right triangle DEC. Letting $DC = 2x$, we see that $EC = x$, and $DE = x\sqrt{3}$.

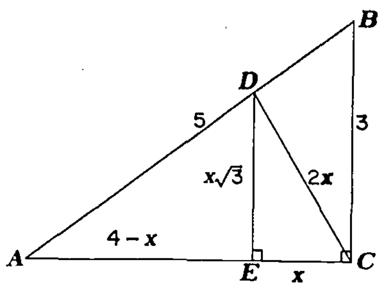


Fig. 31

Since $\triangle AED \sim \triangle ACB$,

$$\frac{4 - x}{x\sqrt{3}} = \frac{4}{3},$$

and

$$x = \frac{12}{3 + 4\sqrt{3}}.$$

By substitution ($DC = 2x$) and rationalizing, we obtain

$$DC = \frac{32\sqrt{3} - 24}{13}.$$

52. If the polygons just fill up the space, the sum of the angles at the junction is 360°, or 2π (radians). Therefore

$$\pi\left(\frac{m - 2}{m} + \frac{n - 2}{n} + \frac{p - 2}{p}\right) = 2\pi,$$

or

$$1 - \frac{2}{m} + 1 - \frac{2}{n} + 1 - \frac{2}{p} = 2,$$

from which it follows that

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = \frac{1}{2}.$$

This can be generalized for k polygons with $n_1, n_2, \ldots, n_k$ sides respectively. As in the proof above,

$$\pi\left(\frac{n_1 - 2}{n_1} + \frac{n_2 - 2}{n_2} + \ldots + \frac{n_k - 2}{n_k}\right) = 2\pi,$$

or

$$\frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k} = \frac{(k - 2)}{2}.$$

53. Since $ABCD$ is a parallelogram, the congruent alternate-interior angles assure us that $\triangle BXK \sim \triangle DAK$. Further, since $\frac{BK}{KD} = \frac{3}{7}$, if $AD = b$, then $BX = \frac{3b}{7}$, and if the altitude from K to $\overline{AD}$ is h, then the altitude from K to $\overline{BX}$ is $\frac{3h}{7}$ and an altitude (to $\overline{AD}$) of the parallelogram is $\frac{10h}{7}$. We can now compute areas.

$$\text{Area of } \triangle BXK = \frac{9bh}{98}.$$

$$\text{Area of } \triangle BCD = \frac{10bh}{14} = \frac{70bh}{98}.$$

$$\text{Area of } DKXC = \frac{70bh}{98} - \frac{9bh}{98} = \frac{61bh}{98}.$$

$$\text{Area of } \square ABCD = \frac{140bh}{98}.$$

Using the above results, we see that the ratio of the areas of $DXKC$ and $ABCD$ is $\frac{61}{140}$.

54.

First Solution

$$4x + 4y + 4z = 48,$$

or

$$x + y + z = 12.$$

$$2xy + 2xz + 2yz = 94,$$

or

$$xy + yz + xz = 47.$$

$$xyz = 60.$$

Solving simultaneously, our solution set is $\{3, 4, 5\}$ in some order.

Second Solution

After setting up the same three basic equations, we note that x, y, and z are roots of some cubic equation

$$(r - x)(r - y)(r - z) = 0.$$

Expanding, we obtain

$$r^3 - (x + y + z)r^2 + (xy + yz + xz)\,r - xyz = 0.$$

Substituting, we obtain

$$r^3 - 12r^2 + 47r - 60 = 0.$$

Let $s = r - \frac{12}{3} = r - 4$; then $r = s + 4$, and substituting and simplifying, we see that $(s^2 - 1)s = 0$. Thus, $s = 0, s = 1$, or $s = -1$, and $r = 3$, $r = 4$, or $r = 5$. Therefore,

$$(x, y, z) = (3, 4, 5).$$

55. From the given information, $\triangle ABC$ and $\triangle BCD$ are both congruent and isosceles. Let x represent the measure of $\angle BAC$, $\angle BCA$, $\angle CBD$, and $\angle CDB$, as illustrated in figure 32. Since $\triangle ACD$ and $\triangle DBA$ are also congruent and isosceles, let $\angle ACD$, $\angle ADC$, $\angle DBA$, and $\angle DAB$ each have measure y.

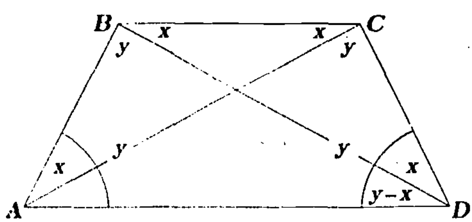


Fig. 32

Since the measure of $\angle BDA$ is $y - x$, we have from $\triangle DBA$

$$3y - x = 180,$$

and from $\triangle ABC$

$$y + 3x = 180.$$

Solving, we find $m\angle ABC = 108$.

56. **First Solution**

$\triangle CDE \sim \triangle BGD$, since $\angle BDG \cong \angle CDE$ (vertical angles) and $\angle GBD \cong \angle ECD$ (right angles).

$$\therefore \frac{EC}{BG} = \frac{DC}{BD}$$

$\triangle FCA \sim \triangle BGA$, since $\angle CAF \cong \angle CAF$ and $\angle FCA \cong \angle GBA$.

$$\therefore \frac{CF}{BG} = \frac{AC}{AB}.$$

Also $FC = CE$, $DC = AC - AD$, and $BD = AD - AB$.

$$\therefore \frac{AC}{AB} = \frac{AC - AD}{AD - AB},$$

and $AD(AC + AB) = 2(AC)(AB)$. Thus,

$$\frac{2}{AD} = \frac{1}{AB} + \frac{1}{AC}.$$

Second Solution

Using coordinate geometry and letting C be the origin of a coordinate system with $\overleftrightarrow{FE}$ as the y-axis and $\overleftrightarrow{AC}$ as the x-axis, we see that the coordinates of C are $(0, 0)$. Let the coordinates of F be $(0, 1)$ and the coordinates of E be $(0, -1)$. Since both A and D are on the x-axis, let their coordinates be $(a, 0)$ and $(b, 0)$, respectively.

An equation for $\overleftrightarrow{AF}$ is

$$y = \frac{-1}{a} x + 1,$$

and an equation for $\overleftrightarrow{EG}$ is

$$y = \frac{1}{b} x - 1.$$

Solving the equations simultaneously, we find that the x-coordinate of G is

$$\frac{2ab}{a + b}.$$

This is also the x-coordinate of B, since $\overleftrightarrow{GB} \perp \overleftrightarrow{AC}$.
Then

$$AD = a - b,$$
$$AB = a - \frac{2ab}{a + b},$$

and

$$AC = a.$$

Therefore, AD is the harmonic mean between AB and AC, since

$$\frac{2}{b - a} = \frac{1}{\frac{2ab}{a + b} - a} + \frac{1}{a}.$$

57. As shown in figure 11, equilateral $\triangle ABC$ is in the first quadrant, vertex A is at the origin, and $\overline{BC}$ is extended through B to intersect the x-axis at D and extended through C to intersect the y-axis at E.

Since $\overline{AF}$ is an altitude of $\triangle ABC$, $\triangle DBA$, and $\triangle EAC$, the ratio of their areas is equal to the ratio of the length of their bases. With respect to $\triangle DBA$ and $\triangle EAC$, the area of one of these triangles is double the area of the other. Let $2(DB) = CE$.

For simplicity, set the length of each side of $\triangle ABC$ equal to 1. Thus,

$AF = \frac{\sqrt{3}}{2}$ and $BF = \frac{1}{2}$ ($\triangle ABF$ is a 30-60 right triangle).

$$\frac{DF}{AF} = \frac{AF}{FE}$$

(The length of the altitude drawn to the hypotenuse of a right triangle is the mean proportional between the lengths of the segments of the hypotenuse). Substituting in this equation, we have

$$\frac{DB + \frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2} + 2(DB)},$$

and solving, we get $DB = \frac{1}{4}$. Therefore the ratio of $\triangle DBA$ to $\triangle ABC$ is

$$\frac{\frac{1}{4}}{1}, \text{ or } \frac{1}{4}.$$

58.

First Solution

As shown in figure 33, $ABCD$ represents the square placed on a coordinate plane with A at the origin, $\overline{AB}$ along the x-axis, and $\overline{AD}$ along the y-axis. Octagon $EFHIJKLME$ is shown at the center of square $ABCD$, G represents the center of the square and the center of the octagon, and the four diagonals of the octagon divide the octagon into eight congruent triangles.

The coordinates of the vertices of $\triangle EFG$ are $\left(\frac{2}{3}, \frac{2}{3}\right)$, $\left(1, \frac{1}{2}\right)$, and $(1, 1)$, respectively. $\overline{EN} \perp \overline{FG}$, and the coordinates of N are $\left(1, \frac{2}{3}\right)$. $FG = \frac{1}{2}$, $EN = \frac{1}{3}$, and the area of triangular region EFG is $\frac{1}{12}$.

The area of the octagonal region at the center of the square is eight times the area of triangular region EGF, or

$$8 \cdot \frac{1}{12} = \frac{2}{3}.$$

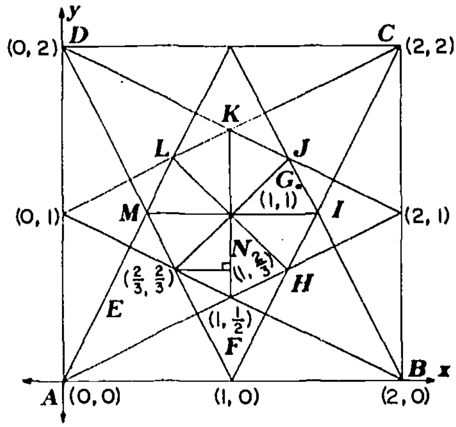


Fig. 33

Second Solution

As shown in figure 34, *HGEL* represents the square with *F*, *M*, *N*, *O* the midpoints of the respective sides. *EFG* is a 1-2- $\sqrt{5}$ right triangle, and $\triangle GFD \sim \triangle EFG$. $\therefore FD = \frac{\sqrt{5}}{5}$ and $DE = \frac{4\sqrt{5}}{5}$. $AD = \frac{1}{2}(DE) = \frac{2\sqrt{5}}{5}$, and the area of square region *ABCD* is $\frac{4}{5}$.

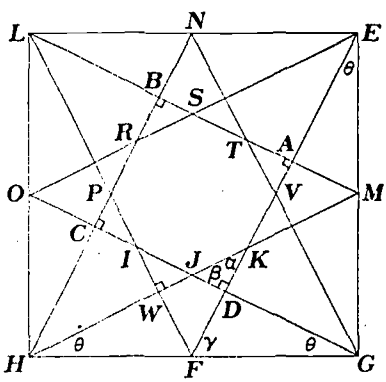


Fig. 34

The area of the octagonal region is equal to the area of square region *ABCD* minus the area of the four triangular regions at the corners of the square.

In $\triangle DKJ$, $\beta = 2\theta$ and $\alpha = \gamma - \theta$ (exterior angle theorem). Tan $\theta = \frac{1}{2}$, tan $\gamma = 2$, tan $\alpha = \tan(\gamma - \theta) = \frac{3}{4}$, and tan $\beta = \frac{4}{3}$. Sin $\alpha = \frac{3}{5}$.

Now

$$\triangle DKJ \cong \triangle CIP,\ \overline{DK} \cong \overline{CI},\ \text{and}\ DK = CI.$$

Also

$$\triangle DKJ \cong \triangle WIJ,\ \overline{KJ} \cong \overline{IJ},\ KJ = IJ.$$

And

$$DK = \tan\beta \cdot JD = \frac{4}{3}(JD).\quad JK = \frac{1}{\sin\alpha} \cdot JD = \frac{5}{3}(JD).$$

So $CI + IJ + JD = \frac{2\sqrt{5}}{5}$ (since $CD = AD$), and (by substitution) $JD = \frac{\sqrt{5}}{10}$, $DK = \frac{2\sqrt{5}}{15}$, and the area of triangular region $DKJ = \frac{5}{150}$.

Furthermore, since $\triangle DKJ \cong \triangle CIP \cong \triangle BRS \cong \triangle ATV$, the area of the four triangular regions at the corners of square $ABCD$ is equal to $4\left(\frac{5}{150}\right) = \frac{2}{15}$.

Thus the area of the octagonal region is equal to $\frac{4}{5} - \frac{2}{15} = \frac{2}{3}$.

59. Let the lengths of the three sides of the triangle be $2x$, $2y$, and $2z$. Let the inscribed circle be tangent $\frac{1}{2}$ unit from the midpoints of the $2x$ and $2y$ sides.

Since tangents to the same circle from the same point are of equal length, one of the following statements must be true (tangents are given in terms of x and y):

1. $x + \frac{1}{2} = y + \frac{1}{2}$; and $x = y$.
2. $x - \frac{1}{2} = y + \frac{1}{2}$; and $x = y + 1$.
3. $x + \frac{1}{2} = y - \frac{1}{2}$; and $x + 1 = y$.
4. $x - \frac{1}{2} = y - \frac{1}{2}$; and $x = y$.

Statements 1 and 4 are false (the triangle is scalene), and statements 2 and 3 are essentially equivalent, since the lengths of the sides $2x$ and $2y$ have not been specified.

As shown in figure 35, the lengths of all the tangents have been expressed in terms of x, and the lengths of the sides of the triangle become $2x$, $2x + 1$, and $2x + 2$. Using the formula stating that the area of a triangular region is equal to one-half the sum of the lengths of the sides times the length of the radius of the inscribed circle, we see that

$$\text{area } \triangle ABC = \frac{1}{2}(2x + 2x + 1 + 2x + 2)(4)$$
$$= 12x + 6.$$

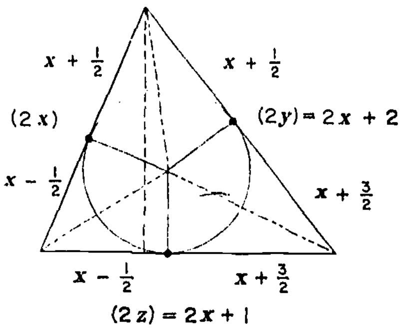


Fig. 35

The area of triangular region ABC is also equal to $\frac{1}{2}(2x + 1)(h_{2x+1})$, where h_{2x+1} represents the length of the altitude to side $2x + 1$. Equating the values for the area of triangular region ABC, we obtain $h_{2x+1} = 12$. Using the fact that the altitude to side $2x + 1$ divides this side into two parts and solving the equations thus obtained, we get the following quadratic equation:

$$4x^2 + 4x - 195 = 0.$$

Solving this equation, we get

$$x = 6.5 \text{ or } x = -7.5.$$

Since x must be positive, the sides of the triangle are 13, 14, and 15, respectively.

60. Since the sides of the triangle form the diameters of the three circles and since the area of a circular region is proportional to the square of its diameter, the Pythagorean relation $a^2 + b^2 = c^2$ implies that the area of the large circular region is equal to the sum of the areas of the small circular regions.

The area of the shaded region is equal to the sum of the areas of the small semicircular regions minus the difference of the area of the large semicircular region and the area of the triangular region.

Therefore, the area of the shaded region is equal to the area of the triangular region, or 12 square units.

61. In figure 36, let $m\angle C = \theta$ and $m\angle A = 2\theta$. Also let $AB = x$, $AC = x + 1$, $BC = x + 2$, and $BD = y$. If $\overrightarrow{AD}$ is the bisector of $\angle A$, then $m\angle BAD = m\angle DAC = \theta$, and $m\angle BDA = 2\theta$. From these we

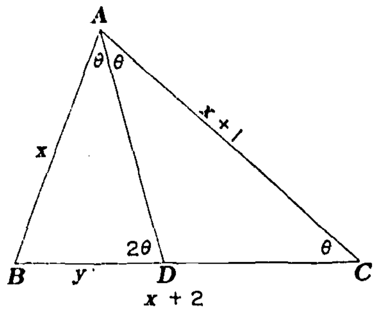


Fig. 36

deduce first that $\overrightarrow{AD}$ divides $\overline{BC}$ into segments with lengths proportional to AB and AC, and thus $AB/BD = AC/DC$, or

(1) $$x/y = (x + 1)/(x + 2 - y);$$

and second that $\triangle ABD \sim \triangle ABC$, so that

(2) $$x/y = (x + 2)/x.$$

From (1) and (2), we obtain

$$(x + 2)/x = (x + 1)/(x + 2 - y).$$

If $\dfrac{x^2}{(x + 2)}$, from (2), replaces y, the result reduces to $x^2 - 2x - 8 = 0$. Of the two roots, 4 and -2, only $x = 4$ is acceptable. Thus $x = 4$, $x + 1 = 5$, and $x + 2 = 6$ are the required lengths.

62. In a regular tetrahedron, the centers of both the inscribed and the circumscribed spheres must coincide with the center of the tetrahedron, which will be called X. As shown in figure 37, let F be the center of equilateral triangle CDB, located by the intersection of medians $\overline{BE}$, $\overline{CG}$, and $\overline{DH}$. Since X lies on $\overline{AF}$, which is the segment from point A perpendicular to the base, X is also in the plane of triangle ABE. Let J be the center of equilateral triangle ACD, the intersection of medians $\overline{AE}$, $\overline{DL}$, and $\overline{CK}$. Since the medians of a triangle meet $\frac{2}{3}$ of the way from any vertex to the midpoint of the opposite side, $AJ = \frac{2}{3}(AE)$ and $EJ = \frac{1}{3}(AE)$. Similarly, $BF = \frac{2}{3}(BE)$ and $EF = \frac{1}{3}(BE)$. For a sphere to be inscribed within the tetrahedron, radii of the sphere must extend from X to the faces of the tetrahedron and meet them perpendicularly at their centers. Therefore, $\overline{XJ}$ and $\overline{XF}$ must be radii of the inscribed sphere. Similarly, since the circumscribed sphere passes through A, B, C, and D, $\overline{XA}$ and $\overline{XB}$ are radii of that sphere.

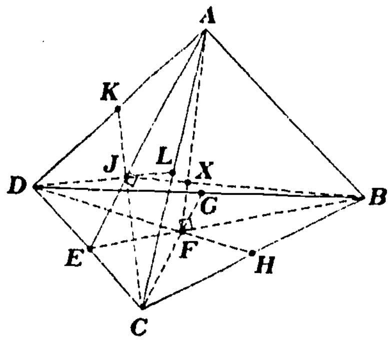


Fig. 37

Triangles BEJ and BXF can be shown to be similar, and so $\frac{EJ}{EB} = \frac{XF}{XB}$. Since medians of congruent triangles are congruent, $\overline{EB} \cong \overline{EA}$, and since $EJ = \frac{1}{3}(EA)$,

$$\frac{EJ}{EB} = \frac{(1/3)\ EA}{EA} = \frac{1}{3} = \frac{XF}{XB}.$$

Finally, since the volume of a sphere is proportional to the cube of its radius ($V = \frac{4}{3}\pi r^3$), the ratio of the volume of the inscribed sphere to the volume of the circumscribed sphere is $\frac{1^3}{3^3}$, or $\frac{1}{27}$.

63. Using triangles ABE, AED, and ABD in figure 38, we see that the area of triangle ABD is equal to the area of triangle ABE plus the area of triangle AED. Thus

(1) $$bx = 4x + bc.$$

Using the Pythagorean theorem on triangle ABD yields

(2) $$100 = b^2 + x^2.$$

The area of triangle ABE equals the area of triangle ABC minus the area of triangle BCE, and so

(3) $$4x = ac.$$

Using the Pythagorean theorem on triangle BCA yields

(4) $$144 = a^2 + x^2.$$

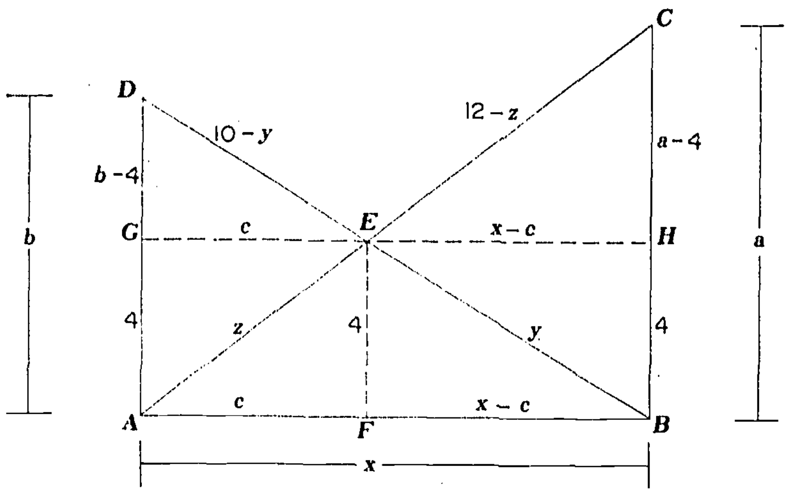


Fig. 38

Substituting equation (3) into equation (1) gives

$$(5) \qquad bx = 4x + \frac{4xb}{a}.$$

Substituting equation (4) into equation (5) gives

$$(6) \qquad bx = \frac{4xb}{\sqrt{144 - x^2}} + 4x.$$

Substituting equation (2) into equation (6) gives

$$(7) \qquad x\sqrt{100 - x^2} = \frac{4x\sqrt{100 - x^2}}{\sqrt{144 - x^2}} + 4x.$$

Simplifying equation (7) by removing the radicals gives the rather formidable equation

$$x^8 - 424x^6 + 64{,}912x^4 - 4{,}200{,}448x^2 + 95{,}420{,}416 = 0.$$

Using synthetic division to solve this equation, we obtain

$$x \approx 7.33.$$

64. **First Solution**

Let $m\angle CAE = \theta$ and $m\angle BAD = 2\theta$. Take F on $\overrightarrow{AD}$ so that $\overline{AD} \cong \overline{DF}$, and let M be the midpoint of $\overline{AE}$. Draw $\overline{BF}$, $\overline{FC}$, and $\overline{MC}$. $ABFC$ is a parallelogram ($\overline{AF}$ and $\overline{BC}$ bisect each other). Therefore $\overline{CF} \cong \overline{AB}$. Also $\overline{CM}$ is the median to the hypotenuse of right triangle ACE. Therefore $CM = \frac{1}{2}AE$, and $m\angle CMF = 2\theta$. Since $\overline{AB}$ is parallel to $\overline{CF}$, $m\angle CFA = 2\theta$. Therefore $\overline{CM} \cong \overline{CF} \cong \overline{AB}$, and $AE = 2AB$.

Second Solution

Since $\overline{BD} \cong \overline{DC}$, the triangular regions ABD and ADC have areas of equal measure. Then $\frac{1}{2}AB \cdot AD \cdot \sin 2\theta = \frac{1}{2}AC \cdot AD \cdot \sin \theta$. Substitute $2\sin\theta\cos\theta$ for $\sin 2\theta$ and AC/AE for $\cos\theta$, and we obtain $2AB = AE$.

65.

First Solution

Let M, N be the centers of the inner circles, as shown in figure 39. Draw $\overrightarrow{AF}$, the common tangent of circles M and O, and $\overrightarrow{BG}$, the common tangent of circles N and O. Since radii $\overline{AM}$ and $\overline{AO}$ are each perpendicular to tangent $\overrightarrow{AF}$ at A, we know that A, M, and O are collinear. Similarly, B, N, and O are collinear. Draw $\overline{OMA}$ and $\overline{ONB}$. Since $\overline{CM}$ and $\overline{CN}$ are each perpendicular to the common tangent of circles M and N at C (not shown), M, C, and N are collinear. Draw $\overline{MCN}$.

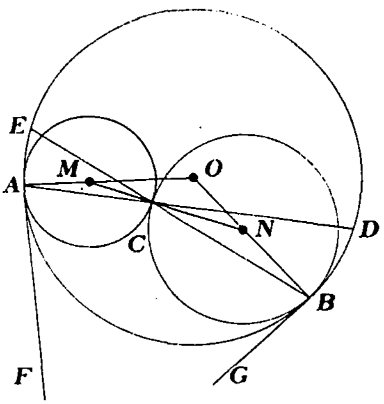


Fig. 39

We have

$m\angle NBC = m\angle NCB$, and $m\angle MAC\ (= m\angle MCA) = m\angle NCD$; therefore

$$m\angle MAC + m\angle NBC = m\angle NCD + m\angle NCB = m\angle BCD.$$

But

$$m\angle BCD \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{AE} + \overset{\frown}{DB});$$

therefore,

(1) $$m\angle MAC + m\angle NBC \overset{\circ}{=} \tfrac{1}{2}(\widehat{AE} + \widehat{DB}).$$

We have also

(2) $$90^\circ - m\angle NBC\ (= m\angle EBG) \overset{\circ}{=} \tfrac{1}{2}(\widehat{AB} + \widehat{AE})$$

and

(3) $$90^\circ - m\angle MAC\ (= m\angle DAF) \overset{\circ}{=} \tfrac{1}{2}(\widehat{AB} + \widehat{DB}).$$

Adding (2) and (3), we obtain

(4) $$180 - (m\angle NBC + m\angle MAC) \overset{\circ}{=} AB + \tfrac{1}{2}(\widehat{AE} + \widehat{DB}).$$

If we now substitute from (1) and combine terms, we have

$$180 \overset{\circ}{=} \widehat{AB} + \widehat{AE} + \widehat{DB}.$$

This means that $\widehat{EBD}$ is a semicircle, and so $\overline{ED}$ is a diameter.

Second Solution

In figure 39, draw $\overline{OE}$ and $\overline{OD}$. Then $m\angle ADO = m\angle DAO = m\angle ACM$, and $m\angle BEO = m\angle EBO = m\angle BCN$. Therefore $\overline{OD}$ and $\overline{OE}$ are parallel to $\overline{MN}$. Thus E, O, D are collinear, making $\overline{ED}$ a diameter.

66. Let $m\angle CAB = x$ and $m\angle ABC = y$. Then, since $\angle FCJ$ and $\angle CAB$ are complements of the same angle ACH, $m\angle FCJ = x$. Likewise, $m\angle JCE = y$. Construct a ray from F parallel to $\overline{CE}$, and let it intersect $\overleftrightarrow{HJ}$ at K. Also construct $\overline{KE}$. Since $\overline{FK} \parallel \overline{CE}$, $m\angle FKC = y$. Therefore $\triangle ABC \cong \triangle CKF$ (side-angle-angle), and $\overline{AB} \cong \overline{CK}$. Now we have $\triangle ABC \cong \triangle KCE$ (side-angle-side), and thus $m\angle CKE = x$. Hence $\overline{KE} \parallel \overline{FC}$, and $FKEC$ is a parallelogram. Its diagonals bisect each other; therefore, $\overline{FJ} \cong \overline{JE}$. (See fig. 40.)

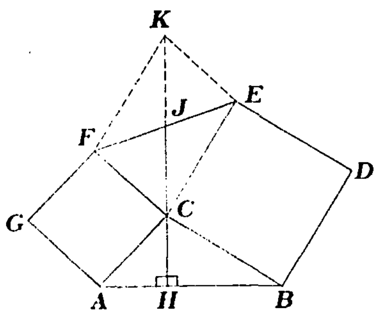


Fig. 40

67. Figure 16 shows both cases at once, with BD and BD' each equal to $\overline{BA}$. Construct $\overline{BE}$ perpendicular to $\overline{DD'}$ and $\overline{CF}$ perpendicular to $\overline{AB}$. Then $BE = CF = \frac{1}{2}AB = \frac{1}{2}BD = \frac{1}{2}BD'$. Hence $\triangle DEB$ and $\triangle D'EB$ are 30-60 right triangles. Thus $m\angle EDB = 30° = m\angle DBA$, and $m\angle CBD = 45° - 30° = 15°$. Also $m\angle ED'B = 30°$, and $m\angle CBD' = 45° + 60° = 105°$.

68. D is the midpoint of $\overline{BC}$ and $\overline{DF} \parallel \overline{AC}$. So F is the midpoint of $\overline{AB}$. Also, $AB = BD = \frac{1}{2}(BC)$. So $BF = BE$.

Therefore triangles ABE and DBF are congruent, and $\angle EAF \cong \angle EDF$. But since $\angle BAD \cong \angle BDA$, this means (by subtraction) that $\angle EAD \cong \angle FDA$. Also, since $\overline{DF} \parallel \overline{AC}$, $\angle FDA \cong \angle DAC$. Therefore, $\angle EAD \cong \angle DAC$.

69. In triangle ABC, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are the medians, which we know are concurrent. In the figure, the point of concurrency is G. Extend each of these medians so that $AD = DR$ (as shown in the figure), $BE = ES$, and $CF = FT$.

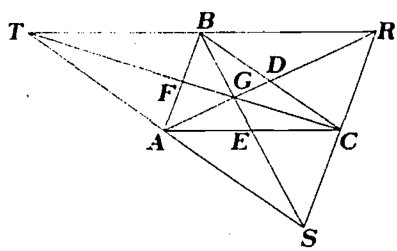


Fig. 41

Now $ABCS$ is a parallelogram, since its diagonals bisect each other ($AE = EC$ and $BE = ES$). In the same way it can be shown that $ABRC$ and $ACBT$ are parallelograms.

Thus, since opposite sides of a parallelogram are congruent, $TB = AC = BR$, $BC = TA = AS$, and $AB = CS = CR$.

However, $BS < AB + AS$, $CT < TB + BC$, and $AR < AC + CR$, by triangle inequality. So $(BS + CT + AR) < (AB + BC + AC) + (AS + TB + CR)$.

Also $AS = BC$, $TB = AC$, and $CR = AB$. Furthermore, $BS = 2(BE)$, $AR = 2(AD)$, and $CT = 2(CF)$.

Therefore, $2(AD + BE + CF) < 2(AB + BC + AC)$, and $AD + BE + CF < AB + BC + AC$, which was to be proved.

4

SOLUTIONS

Number Theory

70. Three consecutive integers can be represented by $A - 1$, A, and $A + 1$. The sum of the squares of these three numbers is

$$3A^2 + 2.$$

Since A is an integer, A^2 must end in 0, 1, 4, 5, 6, or 9. If we now use this information to find the units digit for $3A^2 + 2$, we find that it must be one of the following: 0, 2, 4, 5, 7, or 9. Therefore the sum of the squares of three consecutive integers cannot end in 3 or 8 (nor can it end in 1 or 6).

71.

First Solution

$$5^3 \equiv -1 \pmod 7$$
$$(5^3)^{333,333} \equiv (-1)^{333,333} \pmod 7$$
$$5^{999,999} \equiv -1 \pmod 7$$

or

$$5^{999,999} \equiv 6 \pmod 7.$$

Therefore, the remainder is 6.

Second Solution

The remainders obtained when successive powers of 5 are divided by 7 form a repeating cycle of six integers: $\{5, 4, 6, 2, 3, 1\}$. $5^{999,999}$ goes through the cycle 166,666 times and through three remainders of the 166,667th cycle. Therefore, the remainder is the third integer of the cycle, namely, 6.

72. As a product of prime factors, $288 = 1 \times 2^5 \times 3^2$. The set X such that each element of X is a product of one or more of the prime factors, excluding 1, is

$$X = \{2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 32, 36, 48, 72, 96, 144, 288\}.$$

All possible combinations of a, b, and $c \in X$ such that a, b, and c are unique numbers and $a \times b \times c = 288$ are the following:

$2 \times 4 \times 36$	$3 \times 4 \times 24$
$2 \times 6 \times 24$	$3 \times 6 \times 16$
$2 \times 8 \times 18$	$3 \times 8 \times 12$
$2 \times 3 \times 48$	$4 \times 6 \times 12$
$2 \times 9 \times 16$	$4 \times 8 \times 9$

Thus, there are ten such combinations.

73.

First Solution

The absolute values of the differences in the third row must be 0, 1, 2, 3, 4, 5, and 6, in some order. Also the algebraic differences must add up to zero because the sum of the integers in the second row is the same as the sum in the top row. Thus we want to know if

$$(1) \qquad 0 \pm 1 \pm 2 \pm 3 \pm 4 \pm 5 \pm 6 = 0$$

is possible for some choice of signs. But the left side of (1) contains three odd numbers, which, regardless of signs, will combine to give an odd number; and the even numbers, regardless of signs, will combine to give an even number. Since an odd and an even number cannot combine to give zero, (1) is impossible. Therefore the absolute differences in row 3 cannot all be different.

Second Solution

Let the absolute differences $|1 - x_1|, |2 - x_2|, \ldots, |7 - x_7|$, where the x's represent the integers 1 to 7 in some order, all be different. Then

$$(1) \quad |1 - x_1| + |2 - x_2| + \ldots + |7 - x_7| = 0 + 1 + \ldots + 6,$$

and

$$(2) \quad |1 - x_1|^2 + |2 - x_2|^2 + \ldots + |7 - x_7|^2 = 0 + 1 + 4 + \ldots + 36 = 91.$$

Note that $|1 - x_1|^2 = 1^2 - 2x_1 + x_1^2$, and so forth.

If we expand the left side of (2) and rearrange the terms, we obtain

$$(3) \quad (1^2 + 2^2 + \ldots + 7^2) - 2(x_1 + 2x_2 + \ldots + 7x_7) + (x_1^2 + x_2^2 + \ldots + x_7^2) = 91.$$

But

$$1^2 + 2^2 + \ldots + 7^2 = x_1^2 + x_2^2 + \ldots + x_7^2 = 140.$$

Therefore, we would have

$$140 - 2(x_1 + 2x_2 + \ldots + 7x_7) + 140 = 91.$$

This is impossible because the left member is even and the right is odd. Therefore the absolute differences cannot all be different.

74. 100! can be thought of as the result of multiplying all the prime factors of all the integers less than 101. The only way a terminal zero will arise is to have both a factor of two and a factor of five appear in the set of prime factors. Since there are many even numbers between 0 and 101, the only limiting quantity is the number of fives available. There are twenty numbers that have at least one factor of five, that is, 5, 10, 15, 20, and so on; but when we inspect the numbers 25, 50, 75, and 100, we find two five factors in each. Thus there are 20 + 4, or 24, terminal zeros in 100!

75.

First Solution

5^m ends in 5 (m a positive integer). 9^n ends in either 1 or 9 (n a positive integer). Therefore, $5^m + 9^n$ ends in either 6 or 4.

p^2 ends in 0, 1, 4, 5, 6, or 9. Therefore, $7p^2$ must end in 0, 7, 8, 5, 2, or 3.

It can be seen that under the given conditions no solution exists.

Second Solution

For any natural numbers m and n, 5^m and 9^n must both be odd, and hence their sum will be even. This sum, however, is equal to $7p^2$; therefore, p^2 is even, and so is p. We can write p as $2k$ for some natural number k, and p^2 as $4k^2$. This can be restated as

$$p^2 \equiv 0 \pmod 4$$

and

$$7p^2 \equiv 0 \pmod 4.$$

However, $5 \equiv 1 \pmod 4$, and $9 \equiv 1 \pmod 4$. Therefore, since any power of 1 is 1, $5^m \equiv 1 \pmod 4$, and $9^n \equiv 1 \pmod 4$ and $5^m + 9^n \equiv 2 \pmod 4$. Thus we see that $7p^2$ must be congruent to both 0 and 2—an impossibility—and the equation has no solution.

Third Solution

$5^m + 9^n \equiv 7p^2 \pmod 5$ and $5^m \equiv 0 \pmod 5$, $9 \equiv -1 \pmod 5$, and $7 \equiv 2 \pmod 5$. Therefore, $(-1)^n \equiv 2p^2 \pmod 5$. Since $(-1)^n \equiv 1$, or $4 \pmod 5$, this gives $p^2 \equiv 3$, or $2 \pmod 5$. Neither 2 nor 3 is a quadratic residue of 5; so the original equation is impossible.

76. We know that $8 \equiv 1 \pmod 7$ and also that $8 \equiv 2^3$; therefore,

$$2^3 \equiv 1 \pmod 7.$$

Now, raising this congruence to the 33d power, we obtain

$$2^{99} \equiv (2^3)^{33} \equiv 1 \pmod 7,$$

and thus

$$2^{100} \equiv 2(2^3)^{33} \equiv 2 \pmod 7.$$

77. Since n is odd,

$$n = 2k + 1$$

for some integer k. Then

$$\begin{aligned} n^2 - 1 &= (2k+1)^2 - 1 \\ &= 4k^2 + 4k \\ &= 4(k^2 + k) \\ &= (2)(2)(k)(k+1). \end{aligned}$$

Now either k or $k + 1$ is even and thus has 2 as a factor. Therefore, $n^2 - 1$ is the product of factors in which $(2)(2)(2) = 8$ occurs, and by definition it is divisible by 8.

78. Either n or $n - 1$ must be even; so the factor 2 is present in the product. One of the three consecutive integers $n + 1$, n, $n - 1$ must have the factor 3. If it is n or $n - 1$, we already have 3 present in the given product. If it is $n + 1$, then $2n - 1$ has the factor 3, since $(2n - 1) - (n + 1) = n - 2$, which is a multiple of 3 if $n + 1$ is a multiple of 3. Thus, both 2 and 3—and hence 6—are factors of the given product.

79. If a number $N = p_1^{a_1} \cdot p_2^{a_2} \ldots p_n^{a_n}$, then its positive factors are the terms of the product

$$(1 + p_1 + p_1^2 + \ldots + p_1^{a_1})(1 + p_2 + p_2^2 + \ldots + p_2^{a_2}) \ldots (1 + p_n + p_n^2 + \ldots + p_n^{a_n}).$$

The number of factors is therefore $(a_1 + 1)(a_2 + 1) \ldots (a_n + 1)$. If this product is to equal 6, there are two possibilities: $N = p_1^5$ or $N = p_1^2 \cdot p_2$ where p_1 and p_2 are any primes, subject to the given conditions. The answers are therefore as follows:

$3^5 = 243$	$23 \times 3^2 = 207$	$2^2 \times 59 = 236$
$2 \times 11^2 = 242$	$29 \times 3^2 = 261$	$2^2 \times 61 = 244$
$5 \times 7^2 = 245$	$31 \times 3^2 = 279$	$2^2 \times 67 = 268$
$11 \times 5^2 = 275$	$2^2 \times 53 = 212$	$2^2 \times 71 = 284$
		$2^2 \times 73 = 292$

80. $(28 \times 10^n + 1)^2$ always begins with 7 for all integers n, $n \geq 2$. For $n = 2$, $(2{,}800 + 1)^2 = 7{,}845{,}601$. For $n > 2$, $(28 \times 10^n + 1) = 784 \ldots (n - 2)$ zeros $\ldots 56 \ldots (n - 1)$ zeros $\ldots 1$.
These numbers will always end in 1, and there are an infinite number of them. In general, if K is a number with k digits such that K^2 begins with the numeral 7, then $(K \times 10^n + 1)^2$ always begins with 7 and ends with 1, for all $n \geq k$.

81. Every integer can be put into one of seven mutually exclusive sets: $7n$, $7n + 1$, $7n + 2$, $7n + 3$, . . . , $7n + 6$.

If a is of form	then a^2 is of form
$7n$	$7m$
$7n + 1$	$7m + 1$
$7n + 2$	$7m + 4$
$7n + 3$	$7m + 2$
$7n + 4$	$7m + 2$
$7n + 5$	$7m + 4$
$7n + 6$	$7m + 1$

Thus, as shown in the table, the square of an integer belongs to one of four mutually exclusive sets—$7m$, $7m + 1$, $7m + 2$, $7m + 4$.

If a^2 is of form $7m$, a is also of that form—that is, it is a multiple of 7.

If a^2 is of form $7m + 1$, a is of form $7m + 1$ or $7m + 6$. Since $a^2 + b^2 = c^2$ in the given problem, b^2 must be one of the following forms: $7m$, whereby b is of the same form and hence is a multiple of 7; $7m + 1$, whereby b is of the form $7n + 1$ or $7n + 6$. If a and b are of the same form, then $a - b$ is a multiple of 7. If a and b are of different forms, then $a + b$ is a multiple of 7.

If a^2 is of the form $7m + 2$, then a is of form $7n + 3$ or $7m + 4$. Since $a^2 + b^2 = c^2$, b^2 must be one of the following forms: $7m$, whereby b is of the same form and hence a multiple of 7; $7m + 2$, whereby b is of the form $7n + 3$ or $7n + 4$. If both a and b are of the same form, $a - b$ is a multiple of 7. If a and b are of different forms, then $a + b$ is a multiple of 7.

If a^2 is of the form $7m + 4$, then the same sort of argument holds for a and b of form either $7n + 2$ or $7n + 5$.

5

SOLUTIONS

Probability

82. The sample space and probabilities for this problem are as follows:

$$\text{RED: } \frac{1}{2}$$

$$\text{*BLUE, RED: } \left(\frac{1}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{6}$$

$$\text{*WHITE, RED: } \left(\frac{1}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{6}$$

$$\text{*BLUE, WHITE, RED: } \left(\frac{1}{4}\right)\left(\frac{1}{3}\right)(1) = \frac{1}{12}$$

$$\text{*WHITE, BLUE, RED: } \left(\frac{1}{4}\right)\left(\frac{1}{3}\right)(1) = \frac{1}{12}.$$

Those outcomes preceded by the * are those that include the drawing of a blue ball. Therefore, the probability that a white ball was drawn, given that a blue ball was drawn, is

$$P(W \mid B) = \frac{P(B,W,R) + P(W,B,R)}{P(B,W,R) + P(W,B,R) + P(B,R)}$$

$$= \frac{\frac{1}{12} + \frac{1}{12}}{\frac{1}{12} + \frac{1}{12} + \frac{1}{6}} = \frac{\frac{1}{6}}{\frac{2}{6}} = \frac{1}{2}$$

83. **First Solution**

A sample space showing all the possible outcomes can be set up as in figure 42. From this sample space we see that out of the 25 possible cases, the team will win 16 times. Of these 16 wins, 12 occur on bad days and 4 on clear days. Thus 12 out of 16 wins occur when the weather is bad, and this probability would be $\frac{3}{4}$, or .75.

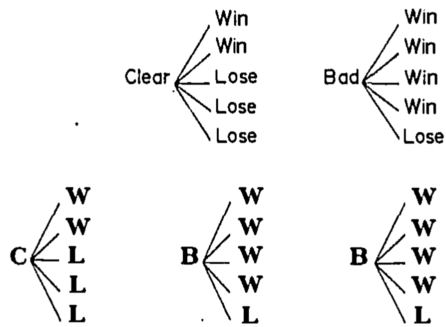


Fig. 42

Second Solution

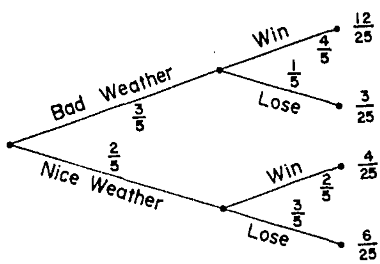


Fig. 43

P(bad weather | the team won)

$$= \frac{P(\text{bad weather} \cap \text{the team won})}{P(\text{the team won})}$$

The "tree" diagram in figure 43 shows the necessary probabilities. The above fraction is therefore

$$\frac{\frac{12}{25}}{\frac{12}{25}+\frac{4}{25}} = \frac{3}{4},$$

and the probability that the weather was bad is .75.

84. A "tree" showing all the possible outcomes is set up as shown in figure 44. If in the course of any one toss a coin comes up heads, it is removed, and the chances of following turns do not exist. The chances of each possibility are calculated by figuring the probability of each toss and multiplying it by the next probability. (Just follow along a branch and

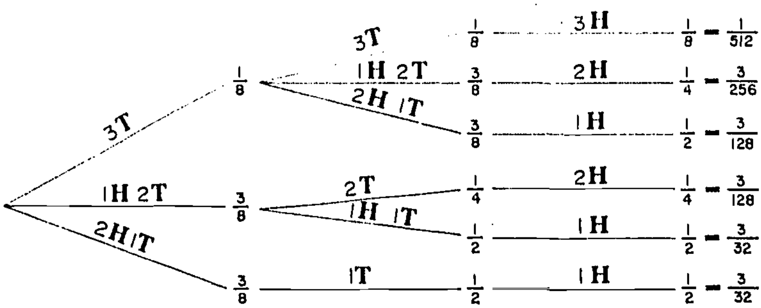


Fig. 44

multiply the fractions to find the various probabilities.) Add the final set of probabilities to determine the chances that none of the coins remain. Thus,

$$\frac{1}{512} + \frac{3}{256} + \frac{3}{128} + \frac{3}{128} + \frac{3}{32} + \frac{3}{32} = \frac{127}{512}.$$

85. The tickets can be drawn in 50×49, or 2,450 ways. Hence, the probability that the difference of the numbers on the two tickets is 10 or less is 890 to 2,450.

Ways of Drawing		
First Ticket	Second Ticket	Total
1 or 50	10	20
2 or 49	11	22
3 or 48	12	24
4 or 47	13	26
5 or 46	14	28
6 or 45	15	30
7 or 44	16	32
8 or 43	17	34
9 or 42	18	36
10 or 41	19	38
11 through 40	20	600
	Total	890

Fig. 45

86.

Case	Possibilities
x	9 (any integer from 1 through 9)
xx	9 (integers from 1 through 9, since the two digits must be the same)
xxx	9·10 (the first and third digits are the same [integers from 1 through 9]. The second digit is an integer from 0 through 9)
$xxxx$	9·10 (the first and fourth digits are the same [from 1 through 9], the second and third digits are the same [from 0 through 9])
$xxxxx$	9·10·10 (the first and fifth digits are the same [from 1 through 9], the second and fourth digits are the same [from 0 through 9], and the third digit is an integer from 0 through 9)

Fig. 46

Adding the possibilities gives

$$9 + 9 + 90 + 90 + 900 = 1{,}098.$$

The total number of integers between 0 and 100,000 is 99,999.

$$\text{Probability} = \frac{1{,}098}{99{,}999} = \frac{122}{11{,}111}.$$

87. We know that the chance of drawing a red ball the first time is $\frac{2}{3}$ and that of drawing a white ball is $\frac{1}{3}$. Replacing each ball drawn by the "same amount" of the other ball, we would now have $9\frac{2}{3}$ red balls and $5\frac{1}{3}$ white balls. From this we see that the probability of drawing a red ball is $\frac{29}{45}$.

88. A Venn diagram (fig. 47) shows the figures in the problem. Note that the number 24 appears in the intersection of *A* and *B* because this is the number of errors found by *both* proofreaders. Then the number 12 appears in that part of *A* not also included in *B* because this brings A's total to 36; similarly, the number 8 appears in that part of B not included in A. This makes it easy to see that the total number of errors found was $12 + 24 + 8 = 44$.

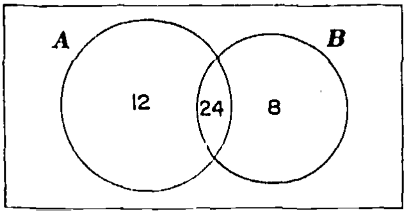


Fig. 47

Next we find an efficiency rating for proofreader A; that is, we estimate the *fraction* of errors he can be expected to find in a given sample or, otherwise stated, the *probability* that he will find an error. To do this, we use as a sample the set of errors B found. B found 32 errors; and of these 32, A found 24. Thus, A found $\frac{24}{32}$ or $\frac{3}{4}$ of the errors in the sample. We therefore estimate that A will find $\frac{3}{4}$ of the errors in any given sample. If n denotes the total number of errors in the manuscript, then we estimate

$$\frac{3}{4}n = 36,$$

which means $n = 48$. Since 44 errors were found, the estimate is that 4 errors remain undetected. (Incidentally, the same result is obtained if you find a proficiency rating for B based on the sample of errors that A found.)

89. We know that the card that was picked cannot have white on both faces. Since one card has red on both faces, it is twice as likely to be the one picked as the card with red on one face. Therefore, the odds that the other face is red are two to one.

90. The desired goal of selecting a ball that forces one to move over to an empty jar can be reached in only two ways, as can be seen in figure 48. In this diagram, the lines run from the present jar to the jar moved to. The letters on the line indicate the color of the ball selected, and the numbers on the lines indicate the probabilities for that branch as opposed to the other branch. The numbers in the boxes represent the current contents of the jars before a ball is selected. The circles represent

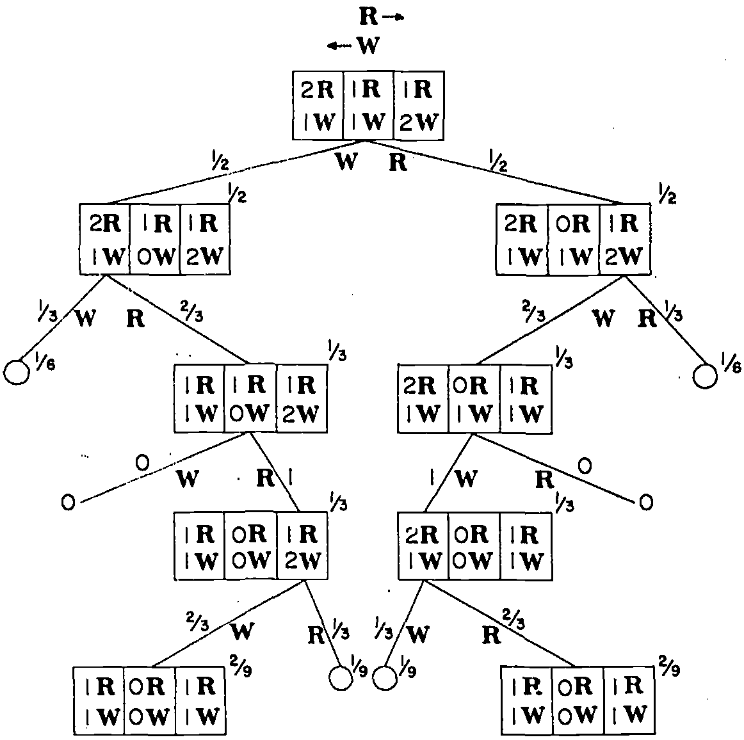


Fig. 48

movement past an end jar. The numbers at the upper right of the boxes and circles and at the ends of the lines indicate the cumulative probabilities for the given situations. The game ends when a circle is drawn or a line leads to a box whose contents are 0R and 0W. As can be seen, the required probability is $\frac{2}{9} + \frac{2}{9} = \frac{4}{9}$.

6

SOLUTIONS

Trigonometry

91.
$$a^3 + b^3 + c^3 = c^2a + c^2b + c^3.$$
Therefore
$$c^2 = (a^3 + b^3)/(a + b) = a^2 - ab + b^2.$$
By the law of cosines
$$c^2 = a^2 + b^2 - 2ab \cos C.$$
Hence
$$\cos C = 1/2, \text{ and } m\angle C = 60°.$$

92. Because $\sin(180 - m\angle ABC) = \sin \angle ABC$, $AB = BF$, and $BC = BG$, the measures of the areas of $\triangle BFG$ and $\triangle ABC$ are equal. Similarly, the measures of the areas of $\triangle ADE$ and $\triangle CHJ$ equal that of $\triangle ABC$. The measure of the area of region $ACJD = (AC)^2$, of I $= (AB)^2$, of II $= (BC)^2$, and of $\triangle ABC = \frac{1}{2} AB \cdot BC \cdot \sin \angle ABC$. The sum of the measures of the areas of the four triangles is thus $2 \cdot AB \cdot BC \cdot \sin \angle ABC$. From the given data
$$(AB)^2 + (BC)^2 = (AC)^2 + 2 \cdot AB \cdot BC \cdot \sin \angle ABC.$$
Then by the law of cosines
$$\sin \angle ABC = \frac{[(AB)^2 + (BC)^2 - (AC)^2]}{2 \cdot AB \cdot BC} = \cos \angle ABC.$$
Hence $m\angle ABC = 45°$.

93. Let the measures of the angles of this triangle be $\theta - x$, θ, and $\theta + x$. Since their sum $3\theta = 180°$, we have $\theta = 60°$.

a. Let the measures of the sides of the triangle be $a - b$, a, and $a + b$, so that the side of measure a is opposite the $60°$ angle. Then by the law of cosines
$$a^2 = (a + b)^2 + (a - b)^2 - 2(a + b)(a - b) \cos 60°.$$

Then since $\cos 60° = \frac{1}{2}$, this becomes

$$3b^2 = 0, \text{ or } b = 0.$$

Therefore the three sides of the triangle are equal, and the triangle is equilateral.

b. Let the measures of the sides of the triangle be a/r, a, and ar; let opposite angles of measure be $60° - x$, $60°$, and $60° + x$, respectively. Then by the cosine law

$$a^2 = a^2r^2 + a^2/r^2 - 2a^2 \cos 60°.$$

This leads to $r^4 - 2r^2 + 1 = 0$, from which $r^2 = 1$ and $r = 1$. Thus all three sides of the triangle are equal, and the triangle is equilateral.

94. Let $AE = BD = y$. $CE = y \cos u$ and $CB = y \sin v$. Then,

$$\cos v = \frac{a + y \cos u}{y}$$

or

$$y = \frac{a}{\cos v - \cos u}$$

so that

$$\sin u = \frac{x + y \sin v}{y}$$

or

$$x = \frac{a \sin u - a \sin v}{\cos v - \cos u}$$

Thus

$$x = a\left(\frac{2 \cos \frac{u+v}{2} \sin \frac{u-v}{2}}{2 \sin \frac{u+v}{2} \sin \frac{u-v}{2}}\right)$$

and

$$x = a \cot \frac{u+v}{2}.$$

95. Figure 49 is the right triangle formed by the line segment connecting the centers of the circles, a tangent of the larger circle, and a radius. By the Pythagorean theorem, the "unknown" side of the triangle is $r\sqrt{5}$, and the area of quadrilateral $ABCD$ is $2r^2\sqrt{5}$. The area of the shaded portion in figure 21 is the area of the quadrilateral minus the area of the pie-shaped pieces. The area of such a piece is $\frac{1}{2}r^2\theta$, where θ

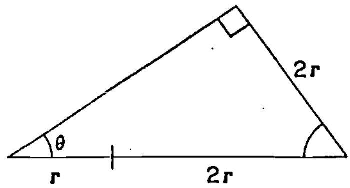


Fig. 49

is the central angle measured in radians. The angle marked θ in figure 49 is Arcsin $\frac{2}{3}$, and thus the area of the sector of the smaller circle is

$$\frac{1}{2}r^2\left(2 \text{ Arcsin } \frac{2}{3}\right),$$

and the area of the sector of the larger circle is

$$\frac{1}{2}(2r)^2\left[2\left(\frac{\pi}{2} - \text{Arcsin } \frac{2}{3}\right)\right].$$

Their sum, subtracted from the area of the quadrilateral, is

$$r^2\left(2\sqrt{5} + 3\text{Arcsin } \frac{2}{3} - 2\pi\right),$$

or approximately $.39r^2$.

96. Since $\cos^2\theta = 1 - \sin^2\theta$, the substitution $y = (16/81)^{\sin^2\theta}$ transforms the given equation into

$$y + \frac{(16/81)}{y} = \frac{26}{27},$$

or

$$81y^2 - 78y + 16 = 0,$$

from which we find that $y = 2/3$ or $y = 8/27$. If $y = 2/3$, then $(16/81)^{\sin^2\theta} = 2/3$, or $\sin^2\theta = 1/4$, from which $\theta = 30°$ is found to be the acute solution. The value $y = 8/27$ gives us the solution $\theta = 60°$. Thus $\theta = 30°$ or $\theta = 60°$.

97. Consider the graphs of the two functions $y = x/100$ and $y = \sin x$. Let $y = x/100$ be a line through the origin with slope $1/100$. Every cycle (interval 2π) of the sine curve will intersect this line twice. Furthermore, these intersections will occur when x is in intervals of form $(2n\pi, 2(n + 1)\pi)$, since for positive x the sine curve is above the x-axis (where the line is) and for negative x the sine curve is below the x-axis (again where the line is). However, it can be computed that $32\pi/100 > 1$ and $30\pi/100 < 1$. Thus the last two intersections occur in the interval $(30\pi, 31\pi)$. This means that there are 31 intersections of the graphs for positive values of x, and by symmetry 31 intersections for negative x. Including the intersection at the origin, this produces a total of 63 intersections; hence the given equation has 63 distinct solutions.

98. **First Solution**

Since $B + C$ is the supplement of $\angle A$,

$$\sin A = \sin (B + C) = 2 \sin \left(\frac{B + C}{2}\right) \cos \left(\frac{B + C}{2}\right).$$

Also,

$$\sin B + \sin C = 2 \sin \left(\frac{B + C}{2}\right) \cos \left(\frac{B - C}{2}\right).$$

Substituting in the given expression yields

$$4 \sin \left(\frac{B + C}{2}\right) \cos \left(\frac{B + C}{2}\right) = 2 \sin \left(\frac{B + C}{2}\right) \cos \left(\frac{B - C}{2}\right).$$

Therefore

$$2 \cos \frac{B + C}{2} = \cos \frac{B - C}{2}.$$

Expanding each side gives

$$2 \cos B/2 \cos C/2 - 2 \sin B/2 \sin C/2 = \cos B/2 \cos C/2 + \sin B/2 \sin C/2.$$

Therefore,

$$\cos B/2 \cdot \cos C/2 = 3 \sin B/2 \sin C/2.$$

Divide both sides by $\cos B/2 \cos C/2$ to obtain

$$\text{Tan } (B/2) \text{ Tan } (C/2) = \frac{1}{3}.$$

Second Solution

By the law of sines,

$$\text{Sin } A/a = \sin B/b = \sin C/c.$$

Let each fraction $= t$, so that $\sin A = at$, $\sin B = bt$, $\sin C = ct$. Substitute in the given statement: $\sin B + \sin C = 2 \sin A$, whence $b + c = 2a$. Now

$$(\text{Tan } B/2)\ (\text{Tan } C/2) = \left(\sqrt{\frac{(s - a)(s - c)}{s(s - b)}}\right)\left(\sqrt{\frac{(s - a)(s - b)}{s(s - c)}}\right)$$

$$= \frac{s - a}{s} = \frac{a/2}{3a/2} = \frac{1}{3}.$$

Note: $s = \dfrac{(a + b + c)}{2}$.

Third Solution

For an equilateral triangle,

$$\sin B + \sin C = 2 \sin A.$$

Therefore

$$(\text{Tan } B/2)(\text{Tan } C/2) = (\text{Tan } 30°)(\text{Tan } 30°) = \frac{1}{3}.$$

99. Let $m\angle BAC = \alpha$, $m\angle ABC = \beta$, $m\angle ACB = \gamma$, $AC = b$, and $AD = e$. Then $m\angle D = 90 - \gamma$, and $m\angle E = 90 - \beta$. Applying the law of sines to $\triangle ABC$ yields

$$BC = \frac{b \sin \alpha}{\sin \beta} = \frac{b \sin(\beta + \gamma)}{\sin \beta},$$

since $\beta + \gamma$ is the supplement of α.

In $\triangle ADE$, by the law of sines,

$$DE = e \sin DAE/\sin E = \frac{e \sin[180 - (90 - \beta) - (90 - \gamma)]}{\sin(90 - \beta)}$$
$$= e \sin(\beta + \gamma)/\cos \beta.$$

Now

$$\frac{e \sin(\beta + \gamma)}{\cos \beta} = \frac{b \sin(\beta + \gamma)}{\sin \beta} \cdot \frac{\sin \beta}{\cos \beta} \cdot \frac{e}{b}.$$

Therefore $DE = BC \tan \beta \tan \gamma$.

First Solution

100. Let $m\angle CAE = \theta$, and $m\angle BAD = 2\theta$. Take F on $\overrightarrow{AD}$ so that $\overline{AD} \cong \overline{DF}$, and let M be the midpoint of $\overline{AE}$. Draw $\overline{BF}$, $\overline{FC}$, and $\overline{MC}$. $ABFC$ is a parallelogram ($\overline{AF}$ and $\overline{BC}$ bisect each other). Therefore $\overline{CF} \cong \overline{AB}$. Also $\overline{CM}$ is the median to the hypotenuse of right triangle ACE. Therefore $CM = \frac{1}{2}AE$, and $m\angle CMF = 2\theta$. Since $\overline{AB}$ is parallel to $\overline{CF}$, $m\angle CFA = 2\theta$. Therefore $\overline{CM} \cong \overline{CF} \cong \overline{AB}$, and $AE = 2AB$.

Second Solution

Since $\overline{BD} \cong \overline{DC}$, the triangular regions ABD and ADC have areas of equal measure. Then $\frac{1}{2}AB \cdot AD \cdot \sin 2\theta = \frac{1}{2}AC \cdot AD \cdot \sin \theta$. Substitute $2 \sin \theta \cos \theta$ for $\sin 2\theta$, and AC/AE for $\cos \theta$, and we obtain $2AB = AE$.

ARTICLES

Introduction

Through the years many outstanding articles have been printed in the *Mathematics Student Journal*, and six of them are included here. Like the problems preceding them, they have been chosen to represent different areas of interest in mathematics.

The first three articles were written by high school students and were the prizewinners in an essay contest sponsored by the *Journal* in conjunction with the fiftieth anniversary of the National Council of Teachers of Mathematics. The first, "Consequences of Russell's Paradox," deals with the logical foundations of mathematics. The second, "The King and **i**," is a bit of whimsy in the form of a play and deals with the attack on the real-number line by "imaginary" forces from the complex plane. The third, "Perfect Numbers," discusses one of the many fascinating topics from number theory.

The final three articles were written, not by students, but by members of the mathematics community writing for high school students. "The Underwater World of Harvey Swartz" is concerned with the rapidly expanding field of computer mathematics and with how computers can help solve problems. "Probability, Geometry, and Witches," although couched as a fanciful tale about two witches, suggests a serious method by which some important problems can be solved. The last article, "Transformation Geometry," discusses, mostly in an intuitive way, some of the basic ideas involved in the study of geometry by means of transformations.

Consequences of Russell's Paradox

ROBERT ROSEN

THE SECOND HALF of the nineteenth century ushered in a new era of mathematical philosophy. The works of Boole, Frege, Pierce, and Cantor tied together the hitherto separate fields of mathematics and logic. Set theory and mathematical "laws of thought" were developed—developed toward the accomplishment of a common metaphysical goal, namely, expressing all of mathematics in what Frege termed "analytic propositions," that is, propositions that follow only from recognized laws of logic and definitions developed therefrom. In this half-century, "Cantor's paradise," as Hilbert called Cantor's intuitionistic set theory, ruled the regal realm of mathematics. Alas, as Adam was exiled from the Garden of Eden, so were these men forced to leave behind them the elaborate abstract paradise that they had created.

The Paradox

In 1901, Bertrand Russell shattered intuitionistic set theory by showing how a paradox, a logical contradiction, could be derived from one of Cantor's basic assumptions. This one step irrevocably changed the course of mathematical logic and set theory. As a direct result, not one but many different systems were built from the pieces of the now shattered paradise, each trying to eradicate the flaw of the original in the best way possible.

Our intuition tells us that it is logical to assume that if we take any property, there exists a set that consists of only those elements having that property; this is known as the *axiom of abstraction*. From it Russell showed how a contradiction could be derived.

Obviously, Russell stated, there are two types of classes: those that have the property of being members of themselves (for example, the class of all classes, which is itself a class), and secondly those classes that have the property of not being members of themselves (for example, the class of all houses, which is itself not a house.) This second type of class—that which is not a member of itself—is called an *impredicable* class. The property of impredicability is formally defined as

$$F \text{ isa imp} \Longleftrightarrow \sim (F \text{ isa } F).$$

Substituting "imp" for "F" in the above yields

$$\text{imp isa imp} \Longleftrightarrow \sim (\text{imp isa imp}).$$

With "P" substituted for "imp isa imp," this formula is of the form $P \Longleftrightarrow \sim P$, which is equivalent to the contradiction $P \wedge \sim P$, and ergo the paradox. The following statement illustrates this point:

$$\underbrace{(\text{imp isa imp})}_{P} \Longleftrightarrow \sim \underbrace{(\text{imp isa imp})}_{P}$$

Not only is the contradiction derivable from the definition of impredicability, but it is also derivable from the axiom of abstraction. Formally presented, the axiom is

$$(\exists y)(\forall x)(x \in y \Longleftrightarrow F(x)).$$

(Read: There exists y such that for all x, x is an element of y if and only if F of x.") To derive the paradox from the axiom, substitute "$x \notin x$" for "$F(x)$":

$$(\exists y)(\forall x)(x \in y \Longleftrightarrow x \notin x).$$

The substitution of "y" for "x" yields

$$y \in y \Longleftrightarrow y \notin y,$$

again the contradiction.

Russell's paradox clearly showed that the axiom of abstraction must be dispensed with; yet its use is inherently tied to the needs of set theory. How might the paradox be lost?

Three main types of set theory have emerged in answer to Russell's argument: (1) Russell's own theory of types, (2) Zermelo-Fraenkel's axiomatic set theory, and (3) Quine's "new foundations."

It is general practice in mathematical circles to have an answer ready when you ask an unanswerable question. Accordingly, Russell proposed his answer, the theory of types, in 1908.

The Theory of Types

The theory of types is a formal device that states the existence of a logical hierarchy of types into which all logical expressions can be classified. Individuals, according to Russell, are all of type 0; properties of individuals are of type 1; properties of properties of individuals are of type 2 and so on. The essential point is that any argument of type n, under the theory of types, can only be predicated with a predicate of type $n + 1$. Intuitively this concept seems fairly obvious. Although it is meaningful to say that a *rose* (type 0) is *fragrant*, it clearly does not make any sense to say that a *property* (type 1 or greater) is fragrant.

Formally, any expression

$$(\exists x)(\forall y)(x \in y)$$

is meaningful in Russell's system if and only if x is of type n and y is of type $n + 1$. The above expression is then logically analogous under the theory of types to

$$(\exists x^n)(\forall y^{n+1})(x^n \in y^{n+1}).$$

How this method avoids the contradiction is also obvious. All expressions involving self-reference—that is, $x^n \in x^n$—are declared meaningless because they violate the formation rule of the theory; their predicate is of the same type as their argument. With respect to the substitution transformation of "y" for "x" in the derivation of the paradox, we cannot substitute "y^{n+1}" for "x^n" under the theory, since they are of different types.

On reviewing the theory of types, we find that although it does indeed prevent the derivation of the paradox, it has certain drawbacks. First, it is difficult to work with, and second, the question whether it is true that each variable, in all cases, is restricted to one type of level appears. Another major debit is that every logical entity—be it the null set, the complement of a set, the universal set, and so forth—needs an infinite chain (one for each type) of statements for its complete definition. Cardinal numbers, for example, cannot be simply and uniquely defined as classes of equivalent classes; there must be an infinity of these definitions to define cardinal numbers. This multiplicity of overlapping definitions not only is undesirable for intuitive simplicity but also creates undesired notational restrictions.

Zermelo-Fraenkel

The second of the three systems was developed by Ernst Zermelo and expanded by Abraham Fraenkel in the 1920s. Its structure is more closely related to Cantor's set theory than the theory of types is, and it is more satisfying in that it does not need to use a new formal device to prevent the derivation of the paradox.

In this system Russell's contradiction is avoided by redefining the axiom of abstraction. In the way we have already defined this axiom, it reads

$$(\exists y)(\forall x)(x \in y \iff F(x)).$$

Zermelo converted this to the following form:

$$(\forall z)(\exists y)(\forall x)(x \in y \iff x \in z \wedge F(x)).$$

Zermelo names the above formula the *Aussonderung Axiom* (axiom of separation); it is so called because it permits us, given a set z, to separate from it all members that satisfy a given property and thus to form a set of only those elements.

Under this system, therefore, if we know that the set of houses is defined, we may use this axiom to show the existence of the set of houses that satisfy the property of being red—we can *separate* the red houses from all other houses.

In the Zermelo-Fraenkel system, then, the existence of subsets is conditional to there being an original set z. Only with this proviso is a definition of sets allowed. This is in contrast to the axiom of abstraction, under which any and all sets can be defined.

Let us now show how the derivation of the paradox is prevented. If we begin with the axiom of separation instead of the axiom of abstraction,

$$(\forall z)(\exists y)(\forall x)(x \in y \iff x \in z \wedge F(x)).$$

Taking $F(x) = x \not\in x$, we have

$$(\forall z)(\exists y)(\forall x)(x \in y \iff x \in z \wedge x \not\in x).$$

Again, using $x = y$, we end with

$$y \in y \iff y \in z \wedge y \not\in y.$$

This last statement is not contradictory; all it does is specify a property—$x \not\in x$—to which no set corresponds. The simple proviso, $x \in z$, therefore succeeds in inhibiting the successful derivation of the contradiction.

As with theory of types, Zermelo-Fraenkel's axiomatic set theory is not fully satisfying. Showing the existence of classes tends to be difficult and uncertain. There is another important difficulty to which we have already alluded. This difficulty is the existence of conditions (for example, $x \not\in x$) to which no class corresponds. Russell's system, of course, is not subject to this difficulty, for it rejects such formulas as meaningless.

The Zermelo-Fraenkel system, though, does have some advantages over the theory of types. First, Zermelo is free of the objections that we raised in analyzing Russell's system, most notably the redundancy of definitions. Second, a ramified version of the axiom of separation, developed by Fraenkel and known as the *replacement axiom*, allows the derivation of transfinite induction and ordinal arithmetic in this system. Furthermore, as a general guideline, if one system of logic can be proved to be consistent in another, then the second system is considered to be the stronger of the two. Using this track of reasoning, J. G. Kemeny showed that Zermelo's system is stronger than the theory of types.

New Foundations

The last system that will be analyzed was developed by Willard Van Orman Quine in 1937 and is known as *new foundations*, after the paper in which it was first described. "New foundations" is an attempt to strike a middle course between Russell and Zermelo.

In attempting to strike this compromise, Quine returned to the axiom of abstraction, adding to it a new requirement, namely, that the defining property, $F(x)$, be a "stratified" formula. A stratified formula is one in which it is possible to associate indices with each distinct variable in that formula in such a fashion that the formula obeys the ordering of the theory of types, namely, n, $n + 1$, around every "$\in$" sign. The process of stratification, then, is merely the replacement of each variable with a numeral, starting from the left with zero; for example,

$$(\exists x)(\forall y)(x \in y)$$

is analogous to

$$(0)(1)(0 \in 1)$$

and is thus stratified because "x" and "y" can be replaced with 0 and 1, respectively. Note that the second formula is not be to given any meaning. It is only an illustrative instrument that may be compared to a truth table. This is contrasted to Russell's system where a formula, for example,

$$(\exists x^n)(\forall y^{n+1})(x^n \in y^{n+1})$$

is an intrinsic part of the system and is meaningful.

It is important to realize in looking at "new foundations" that the indices used in showing whether a formula is stratified are not representations of any hierarchy of types. There are no types as such in Quine's system. Another major premise is that it is not necessary for a formula to be stratified to be meaningful. Stratification, then, is only the regulation to which a formula must conform if it is to qualify as an $F(x)$ in the axiom of abstraction. Even so, it is enough to eliminate the existence of the contradiction; it disallows the use of "$x \in x$" as an "$F(x)$" because it is not stratified.

"New foundations" is akin to Zermelo's system in that it uses all of Zermelo's axioms except those concerning basic class existence, of which the axiom of separation is the most prominent.

Quine's "new foundations" shares the good points and some—but not all—of the debits of the two systems already discussed; it is a very suitable compromise (if one can be satisfied with compromises). Quine's system is also the only axiomatic set theory to contain a universal set and the condition that the complement of any set is itself a set. On the negative side, the axiom of choice has been disproved for this system.

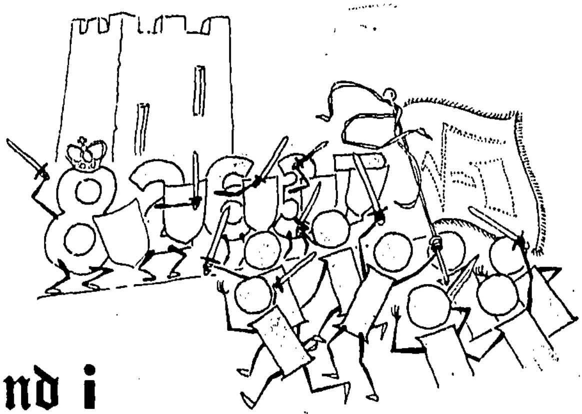
Conclusion

Any logical system, in the final analysis, must be graded with respect to two qualifications: (1) Can we be sure of its consistency? (2) How much naive set theory and mathematics in general can be developed within the confines of the system?

All three of these systems have been shown to be consistent; so no differentiation among them comes from this source. As to the second qualification, however, the theory of types is the least meritorious; both Quine's and Zermelo's systems encompass more set theory than the theory of types.

In comparison to each other, the difference is not so apparent. "New foundations" contains the universal set and the unconditional complement, whereas Zermelo's theory does not. The reverse is true, however, with respect to the axiom of choice. To confuse the matter even further, the theory of types has some advantages over the other two because of its distinctive membership conditions.

As the situation now stands, it would appear that the relative merits of each system must be judged according to one's particular needs. If none is indeed a "paradise," each is at least a part of one.



The King and i

ROBERT BLUESTEIN

CHARACTERS

EPSILONIUS, *King of the Real Numbers*
EXPONENTIO *and* DIFFERENTIO, *two noblemen*
A MESSENGER

Scene I

A room in the royal castle of real numbers

[*Enter* EXPONENTIO *and* DIFFERENTIO.]

EXPONENTIO. O sire! O sire! Our number line has been attacked through a dreadful crime!
The imaginaries invade with beams of destructive art.
Our one-dimensional system is falling apart!
DIFFERENTIO. Our line is cracking all around.
And we'll soon be falling through the ground,
Out of our system to another dimension
And down through infinity's endless extension!
KING. This can't be true, this villainous offense!
My kingdom attacked? It makes no sense.
What coordinates are they through which we drop?
We must cover these holes! This madness must stop!
DIFFERENTIO. Our mathematicians have studied, and near zero they think
Lie the terrible cracks through which we sink.
KING. The center of our number line!
O diabolical deed! O devilish crime!
Who dares try this assault of the strong on the weak?
Who leads the attack? I bid you, speak!

EXPONENTIO. 'Tis i, my lord, who with fiendish crew
  Has made the holes we are now falling through.
KING. 'Tis you! I've no time to joke and laugh!
EXPONENTIO. Not *me*, sire, I meant "negative one to the one-half."
KING. Oh.

[*Enter* MESSENGER.]

MESSENGER. Sire! O sire! The holes are being plugged up tight!
KING. Excellent! Now we are ready to fight!
  Exponentio, lead my forces; go out and be a hero.
  I expect news of our victory at the Battle of Zero.
EXPONENTIO. But surely, sire, how do you know
  That Reals can fight an imaginary foe?
KING. As king of our empire, I cannot turn my back;
  For the honor of our realm, we must counterattack.
  So onward to battle; let loose our might;
  We shall triumph in glory or be crushed in the fight!

[*Exit* EXPONENTIO, DIFFERENTIO, *and* MESSENGER.]

Scene II

[*Reenter* MESSENGER.]

KING. What news from the battle?
MESSENGER. We are now in retreat!
  These imaginary numbers are too powerful to defeat.
  Imaginary beams from the complex plane
  Are tearing apart our one-dimensional domain.
KING. I can feel the vibrations of these devastating beams.
MESSENGER. Our empire is falling apart at the seams!
  They puncture our line on every blast,
  And down these holes we're sinking fast!
KING. If you were king, what would you do?
MESSENGER. I have no idea. It's up to you.
KING. Beware the beam!

[*A hole suddenly appears, and* MESSENGER *falls through into darkness.*]

Farewell, poor friend.
  It looks not long before the end.

[*Looks out window.*]

  Yonder lie our forces in chaotic withdrawal!
  Yet we'll not give up hope!
  All for one! One for all!

[*Exit.*]

Scene III

[*Enter* DIFFERENTIO.]

DIFFERENTIO. My Lord! I've just come from the battlefield!
Gallantly we fight, but we'll soon have to yield.
KING. This enemy's not as powerful as it seems;
Bid the army turn 'round and repulse all the beams!

[*Exit* DIFFERENTIO. *A beam flies by.*]

These beams and blasts may fill the air,
And they may come from who knows where,
And though our line cracks beneath our feet
We won't give up and we won't retreat!

[*A deafening crash of thunder and a sudden illumination of the skies.*]

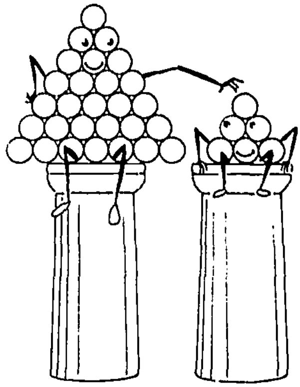
What is this explosion of light and sound
That stops the beams and shakes the ground?

[*Enter* EXPONENTIO *out of breath.*]

EXPONENTIO. O sire! The fight's over!
By an incredible phenomenon
We've won the battle! The enemy's gone!
KING. If t'was not the valor of our defense,
Then what caused this happy turn of events?
EXPONENTIO. It's hard to explain, but from out of the skies
Came a loud peal of thunder and a beam of such size
That all who observed it stood frozen in awe
At the ocean of dazzling light that they saw.
It was aimed at the source of all the other beams.
And made them shrink to the size of mere streams.
KING. What happened then?
EXPONENTIO. While everyone waited,
The enemy's beams disintegrated.
Then the colossal beam. with a noise like a drum,
Left just as quickly as it had come.
KING. The imaginaries are gone, but is victory ours?
Who sent forth this beam of amazing powers?
EXPONENTIO. It didn't come from our number line's domain!
KING. It wasn't from the complex plane.
EXPONENTIO. And it certainly was not our mind's invention.
KING. So it must have come from a third dimension.
EXPONENTIO. Or fourth, or fifth, or any extension.
KING. But this is beyond our comprehension!

No, victory's not ours as it may seem,
But belongs to the owner of the powerful beam.
There's someone watching from the sky,
And I doubt we'll ever know who or why;
Yet let the trumpets blare away
To celebrate this happy day.

[*Flourish. Exeunt.*]



Perfect Numbers

JOE MOLLING

IF YOU WERE ASKED to find a set of whole numbers that have a fascinating history, that possess elegant properties, that are surrounded by great depths of mystery, and yet that are more useless than any other set, your answer might very well be the perfect numbers and their close relatives, the amicable (or friendly) numbers.

A perfect number is simply a whole number that is equal to the sum of its divisors (not including the number itself). The smallest such number is 6, since 6 is equal to the sum $(1 + 2 + 3)$ of its divisors. The next perfect number is 28, since $28 = 1 + 2 + 4 + 7 + 14$. Many Jewish people in Old Testament times were impressed by the "perfection" of 6 and 28. They thought that God could easily have created the world in an instant, but He preferred to take six days because the perfection of 6 signifies the perfection of the created world. Twenty-eight was also thought of as a perfect number; after all, the moon circles the earth in 28 days.

The first great step ahead in the theory of perfect numbers was Euclid's ingenious proof that the expression $2^{n-1}(2^n - 1)$ always yields an even perfect number if the expression in parentheses is prime. (It is never prime unless the exponent n is also prime. If n is prime, 2^{n-1} need not be, and rarely is, prime.) It was not until two thousand years later that Leonhard Euler proved that this formula yields all even perfect numbers. From this point on, "perfect number" will be taken to mean "even perfect number," since no odd perfects are known; in fact, odd perfect numbers may not even exist, although no one has been able to prove (or disprove) this.

To see how closely Euclid's formula ties the perfect numbers to the familiar "doubling" sequence, 1, 2, 4, 8, 16, . . . , the following story comes to mind. A Persian king was so delighted with the game of chess that he told the inventor that he could have anything he wanted. The man made what seemed a reasonable request. He wanted a single grain of wheat on the first square of the chessboard, two grains on the second square, four on the third, eight on the fourth, and so on, proceeding with the powers of 2 up to the sixty-fourth square of the board. (See fig. 1.) It turns out that the last square would require 9,223,372,036,854,775,808 grains! The total of all the grains is twice that number minus one—or a few thousand times the world's annual wheat crop!

1 2^0	2 2^1			16 2^4		64 2^6	
256 2^8	512 2^9	1024 2^{10}	2048 2^{11}	4096 2^{12}		16384 2^{14}	32768 2^{15}
65536 2^{16}		2^{18}		2^{20}	2^{21}	2^{22}	2^{23}
2^{24}	2^{25}	2^{26}	2^{27}	2^{28}	2^{29}	2^{30}	
2^{32}	2^{33}	2^{34}	2^{35}	2^{36}	2^{37}	2^{38}	2^{39}
2^{40}	2^{41}	2^{42}	2^{43}	2^{44}	2^{45}	2^{46}	2^{47}
2^{48}	2^{49}	2^{50}	2^{51}	2^{52}	2^{53}	2^{54}	2^{55}
2^{56}	2^{57}	2^{58}	2^{59}	2^{60}		2^{62}	2^{63}

Fig. 1

If each square of the chessboard is marked according to the number of grains it is to hold (as in fig. 1) and if one grain is removed from any one of the squares, then the number left in that square will be a number of the kind—$2^n - 1$—enclosed in parentheses in Euclid's formula. If that number is prime, multiply it by the number of grains on the preceding square—2^{n-1}—and the product $2^{n-1}(2^n - 1)$ is a perfect number. Primes of the form 2^{n-1} are now called Mersenne primes, after the French mathematician who studied them. The shaded squares of figure 1, after losing one grain, become Mersenne primes, which in turn produce—by the process described above—the first nine perfect numbers.

Properties of Perfect Numbers

From Euclid's formula it is not hard to prove all kinds of beautiful (and even weird) properties of perfect numbers. For example, all perfects are triangular. This means that a perfect number of grains can be arranged to form an equilateral triangle, such as 10 bowling pins or 15 billiard balls. Another property of the perfect numbers (except 6) is that each is a partial sum of the series formed by consecutive odd cubes: $1^3 + 3^3 + 5^3 + 7^3 + \ldots$. Still more surprising is the fact that the sum of the reciprocals of the divisors of a perfect number (this time including the number itself) is always 2. For instance, for the perfect number 28, we have

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{14} + \frac{1}{28} = 2.$$

Also, the digital root of every perfect number (again excepting 6) is 1; that is, if the sum of the digits is divided by 9, the remainder is 1. In the case of 28, the sum of the digits is $2 + 8 = 10$; dividing 10 by 9 results in a remainder of 1.

There is a great air of mystery surrounding the perfect numbers. The end digits of all the twenty-three known perfect numbers is either 6 or 8. When the number ends with 8, the preceding digit is 2. When it ends in 6, the preceding digit is 1 or 3 or 5 or 7. (The numbers 6 and 496, however, are exceptions.) The ancients knew the first four perfects—6, 28, 496, and 8,128—and thought that the 6s and 8s alternated as the sequence of perfect numbers continued. However, this was proved wrong when in the fifteenth century the next two perfect numbers were discovered. The sequence of last digits of the twenty-three known perfect numbers is as follows: 6, 8, 6, 8, 6, 6, 8, 8, 6, 6, 8, 8, 6, 8, 8, 8, 6, 6, 6, 8, 6, 6, 6. The sequence contains some infuriating hints of regularity. The first four digits alternate between 6 and 8, then 6, 6 and 8, 8 alternate for a total of eight entries, followed by a meaningless 6, then 8, 8, 8 and then the climax 6, 6, 6, 8, 6, 6, 6. So far, no one has been able to find a reliable rule for predicting the last digit of the next undiscovered perfect. Is this sequence of last digits trying to tell us something, or is it all an accident?

Unanswered Questions

The two greatest unanswered questions about perfect numbers are these:

1. Is there an odd perfect?
2. Is there a largest even perfect?

No odd perfect has yet been found; nor, although it might look easy, has anybody proved that such a number cannot exist.

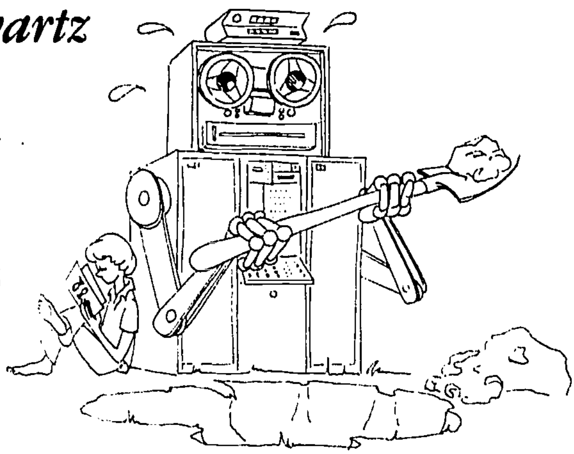
Now for the second question. The early mathematicians (Euclid and, later, Mersenne and Pierre de Fermat) discovered that using the Mersenne primes for n in the expression $2^n - 1$ resulted in higher Mersenne primes. However, their method worked only for the first four such primes: 3, 7, 31, and 127. A computer proved that their method would not work for all primes. In 1876 the French mathematician Edouard Lucas, who wrote a classic, four-volume work on recreational mathematics, announced the twelfth perfect number, $(2^{126})(2^{127} - 1)$. The twelfth Mersenne prime, on which it is based, is one less than the number of grains on the last square of the second chessboard. It was the largest Mersenne prime to be found without the aid of modern computers. The list in figure 2 contains the formulas for the twenty-three known perfect numbers, the number of digits in each, and (until they get too lengthy) numerals for the numbers themselves. The last perfect, which has 22,425 divisors, was discovered at the University of Illinois in 1963 when a computer determined the 23d Mersenne prime.

	Formula	Number	Number of Digits
1	$2^{1}(2^{2}-1)$	6	1
2	$2^{2}(2^{3}-1)$	28	2
3	$2^{4}(2^{5}-1)$	496	3
4	$2^{6}(2^{7}-1)$	8128	4
5	$2^{12}(2^{13}-1)$	33,550,336	8
6	$2^{16}(2^{17}-1)$	8,589,869,056	10
7	$2^{18}(2^{19}-1)$	137,438,691,328	12
8	$2^{30}(2^{31}-1)$		19
9	$2^{60}(2^{61}-1)$		37
10	$2^{88}(2^{87}-1)$		54
11	$2^{106}(2^{107}-1)$		65
12	$2^{126}(2^{127}-1)$		77
13	$2^{520}(2^{521}-1)$		314
14	$2^{606}(2^{607}-1)$		366
15	$2^{1278}(2^{1279}-1)$		770
16	$2^{2202}(2^{2203}-1)$		1327
17	$2^{2280}(2^{2281}-1)$		1373
18	$2^{3216}(2^{3217}-1)$		1937
19	$2^{4252}(2^{4353}-1)$		2561
20	$2^{4422}(2^{4423}-1)$		2663
21	$2^{9688}(2^{9689}-1)$		5834
22	$2^{9940}(2^{9941}-1)$		5985
23	$2^{11212}(2^{11213}-1)$		6751

Fig. 2

So if you are really ambitious—and would like to have your name in every scientific journal—all you have to do is spend a little spare time finding the 24th perfect number.

The Underwater World of Harvey Swartz



RICHARD T. MORGAN

HARVEY LIKED FISH; he had always liked fish. More than anything else he wanted to build an aquarium. Having recently purchased a piece of land adjacent to the stockcar raceway, he was about to make his life-long dream come true.

The first job was to figure out how deep a hole to dig. For the size and number of fish Harvey had in mind, he knew he needed approximately 350,000 gallons of water. The problem looked difficult, to be sure; however, because his son, Harvey, Jr., was using computers in his study of mathematics, he felt confident the problem could be solved.

That evening Harvey and Harvey, Jr., spread a blueprint of the property on the kitchen table. (See fig. 1.) The aquarium was to be located on the property bounded by the raceway, Main Street, and 4th Avenue. Harvey, Jr., drew in a coordinate system, using a scale of 100 to 1, with the center of the raceway as the origin. (See fig. 2.) He had been studying conic sections in his mathematics class, and knew that the equation of an ellipse with center at the origin has an equation of form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with x-intercepts $(\pm\ a, 0)$ and y-intercepts $(0,\ \pm b)$. Using his scale marked in hundreds of feet, he wrote the equation of the curve:

$$\frac{x^2}{4} + \frac{y^2}{1} = 1.$$

Harvey asked his son what formula he was going to use to solve the problem and was disappointed when Harvey, Jr., told him he didn't know a simple

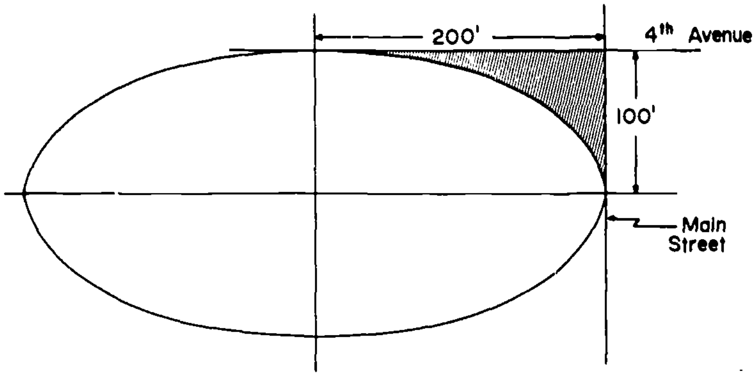


Fig. 1

formula for solving the problem. Harvey thought this strange, since all the problems he had done in school could be solved by using a formula; however, Harvey, Jr. assured him that this problem could be solved at school with the aid of a computer, formula or no formula.

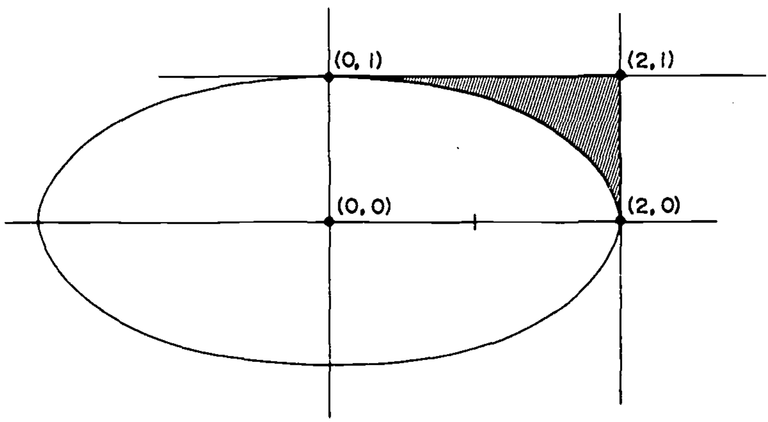


Fig. 2

A Solution Appears

The next evening Harvey, Jr., explained his solution to his father. Since 350,000 gallons of water were needed and since there are approximately 7.481 gallons of water in each cubic foot, the desired volume V must be approximately $350{,}000 \div 7.481$, or about 46,800 cubic feet.

The aquarium was to be of the shape shown in figure 3. Since $V = h \cdot A$, where h is the depth of the hole and A the area of the land purchased, the depth of the hole could be found once the area was known.

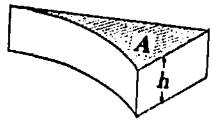


Fig. 3

An approximation of the area was determined in the following way. The areas of eight rectangles, shown in figure 4, were used to approximate the area of ABD. The desired area BCD was found by subtracting the area of ABD from the area of $ABCD$. To find the area of each rectangle, the measure of

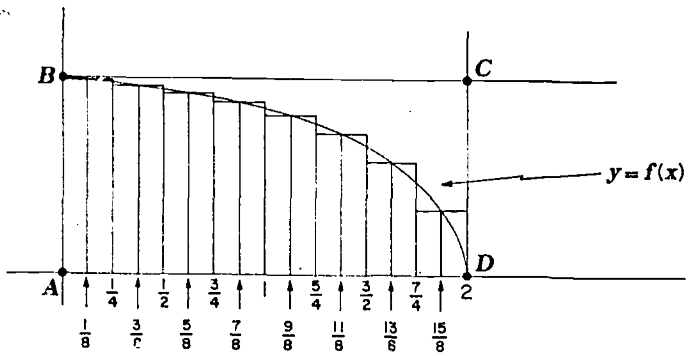


Fig. 4

each base was chosen to be $\frac{1}{4}$ unit. The height was the value of $y = f(x)$ calculated at the midpoint of each rectangle.

$$A \approx \frac{1}{4}f\left(\frac{1}{8}\right) + \frac{1}{4}f\left(\frac{3}{8}\right) + \ldots + \frac{1}{4}f\left(\frac{15}{8}\right)$$

or

$$A \approx \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + \ldots + f\left(\frac{15}{8}\right)\right].$$

Since the equation of the ellipse is

$$\frac{x^2}{4} + \frac{y^2}{1} = 1,$$

we have

$$y^2 = 1 - \frac{x^2}{4}$$

$$y = \frac{\sqrt{4 - x^2}}{2}$$

for values in the first quadrant. To evaluate

$$y = f(x) = \frac{\sqrt{4 - x^2}}{2}$$

for $x = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8}$ is a tedious job; however, it is a trivial task for the computer.

To be able to solve a problem on the computer requires a knowledge of a language the computer can "understand." One such language is known as BASIC. In BASIC, a sequence of steps called a *program* can be written to instruct the computer in solving the problem. The following is a computer program written in BASIC that will solve the problem in the manner discussed above. Read through the program and determine as well as you can the purpose of each step. Then, as you read the explanation below, refer to the program as needed.

```
10  LET S = 0
20  LET X = .125
30    LET Y = SQR(4 − X↑2)/2
40    LET S = S + Y
50    LET X = X + .25
60    IF X ≤ 1.875 THEN 30
70  LET A = 2 − S/4
80  PRINT "AREA ="A
90  END
```

Notice that the nine lines of the program are numbered in ascending order. Many of the words and symbols are familiar to you. In line 30, however, notice the arrow ↑ and letters SQR. The arrow denotes exponentiation, and SQR represents square root. Thus, the function described in line 30 is

$$y = \frac{\sqrt{4 - x^2}}{2}.$$

A special word of caution is given concerning the equality sign in the program. It should be interpreted as "is replaced by" instead of "is equal to." For example, in ordinary algebra, line 50 is false for any real number X, if the usual meaning of "=" is used.

Line 20 assigns X its initial value, $X = .125$. Lines 30 through 60 form what is known as a *loop*. That is, that block of four statements is repeated over and over again. Y is calculated in line 30 and then S in line 40. Line 50 increases the current value of X by .25. Notice that line 60 is different from all the lines considered thus far. It is a decision step. Since the current value of X (the first time through the loop) is $X = .125 + .25 = .375$, which is less than or equal to 1.875, control is transferred back to line 30. Thus, Y and S are calculated again. (S is just the sum of the Y values. S was set equal to zero in line 10, and each time through the loop it is increased by the then current value of Y.)

Each time through the loop, X is increased by .25. Y and S are calculated for the last time when $X = 1.875$. When X is incremented once again, $X = 1.875 + .25 = 2.125$. Thus X is no longer less than or equal to 1.875, and control is transferred to the next statement, line 70.

The desired area, BCD, is found by subtracting $S/4$ (area of ABD) from 2 (area of $ABCD$) in line 70. Line 90 instructs the computer to print AREA = and then the value of A. Line 90 signals the end of the program. The output of this program was as follows:

AREA = .4216565

Another Method

Needless to say, Harvey, Jr., impressed his father a great deal by the way he solved the problem. But as if one solution were not enough, he proceeded to explain an alternative method.

An important use of computers is in the process known as simulation. Simulation is the imitation of some process from the physical world effected by using the mathematics of random numbers and probability. To choose a random number from a set means that all members are equally likely to be chosen. BASIC provides the user with random numbers between 0 and 1 to six decimal places. The following BASIC statement would assign the variable X a random number r such that $0 < r < 1$:

LET X = RND(Z)

The variable Z is said to be a "dummy" variable because it is used for each use of the random number statement, but it does not affect the answer.

For example, to simulate the flip of a coin, X could represent a head when $0 < X < \frac{1}{2}$, and represent a tail when $\frac{1}{2} < X < 1$. The BASIC program below shows one hundred "flips of a coin" on a computer:

```
 5 LET I = 1
10 LET X = RND(Z)
15 IF X ≤ .5 THEN 30
20 PRINT "TAILS"
25 GO TO 35
30 PRINT "HEADS"
35 LET I = I + 1
40 IF I ≤ 100 THEN 10
45 END
```

Line 5 sets I equal to 1. I is going to be used as a "counter"; after each flip of the coin, I is increased by 1 (line 35) and then tested in line 40. Line 10 generates a random number X such that $0 < X < 1$. Line 15 is a decision statement. If X is less than or equal to .5, control is transferred to line 30, the computer prints HEADS, and line 35 is next. If X is greater than .5, control is transferred to line 20, the computer prints TAILS, and line 25 is next. Line 25 simply instructs the computer to go to line 35. Do you see why this statement is necessary? Thus, whether the random number represents a head or a tail, control always ends up at line 35. After I is increased by 1, the new value for I is tested against 100 in line 40. When I is less than or equal to 100, control is transferred back to line 10 for the next "flip of the coin." When I becomes 101, control is transferred to line 45, and the problem is completed.

Just before Harvey could interrupt his son to ask what this all had to do with fish, Harvey, Jr., drew another picture of the aquarium site. Reassured that there was a direct link between the area of the property and random numbers, Harvey, Sr., settled back for another dose of "modern mathematics."

The method involved choosing two random numbers, X and Y, to represent the coordinates of any point P in the rectangular region $ABCD$. (See fig. 5.)

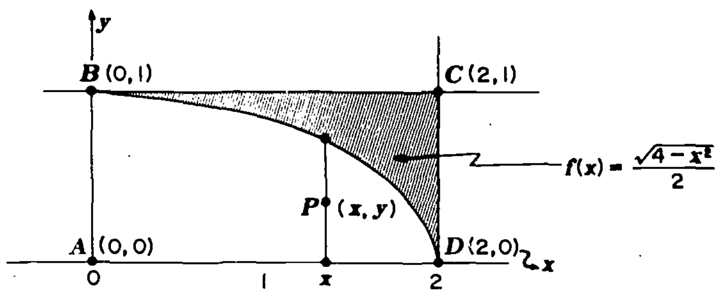


Fig. 5

Once the coordinates of a point P are selected, it is necessary to determine if the point is on or above or below the curve whose equation is

$$f(x) = \frac{\sqrt{4 - x^2}}{2}.$$

For a given value of x, if the ordinate for point P is less than the value of $f(x)$—that is, $y - f(x) < 0$—then the point is below the curve. If $y - f(x) \geq 0$, the point is on or above the curve. Thus, if 25 out of 100 points were above the curve, it would seem reasonable to say that 25/100, or 25 percent, of the total area is above the curve. Since the total area is 2, the area within the rectangle and above the curve would be 25 percent of 2, or .5, square units. Read the following BASIC program carefully. See if you can follow each of the steps already outlined. (Recall that the random numbers are between 0 and 1. Therefore, in this problem, to represent X, it is necessary to multiply the random number by 2.)

```
 10 LET S = 0
 20 LET I = 1
 30 LET X = 2*RND(Z)
 40 LET Y = RND(Z)
 50 IF Y - SQR(4 - N ↑ 2)/2 < 0
    THEN 70
 60 LET S = S + 1
 70 LET I = I + 1
 80 IF I ≤ 1000 THEN 30
 90 LET A = (S/1000)*2
100 PRINT "AREA" = A
110 END
```

(The * denotes multiplication.) In this program two counters are used. I is used to count the 1,000 trials of the experiment, and S is used to count the number of times the point fell on or above the curve. Note that when the point is below the curve, line 60 is skipped. When control is finally transferred to line 90, the area is calculated by multiplying the relative frequency of successes $S/1{,}000$ (the ratio of the number of points on or above the line to the number of trials) by the area of the rectangle, which was 2. Ten runs of the experiment of 1,000 trials each yielded the following output for A:

.462,	.400,	.382,	.442,	.384,
.442,	.436,	.406,	.446,	.402

The arithmetic mean of these numbers is .420. Thus, to two decimal places, Harvey, Jr., found the answer to be .42, both by areas of rectangles and by simulation.

At this point, Harvey, Sr., said he thought he could finish the job. The actual area of the rectangular piece of ground was 100 × 200, or 20,000 square feet (10,000 times the area of the rectangular model). The actual area of the piece of land purchased was therefore 10,000 × .42, or 4,200 square feet. Since $V = A \cdot h$, and the necessary volume was 46,800 cubic feet, the depth of the hole, h, must be 46,800/4,200 or 11.14 feet.

Harvey shook his son's hand vigorously and exclaimed, "We did it, didn't we, Harvey, my boy?" And with visions of dolphins and barracudas bobbing in his head, Harvey, Sr., strolled off into the next room puzzling over his next problem—"If John can dig a hole in 10 weeks and Bill can dig a hole in 7 weeks, how many weeks will it take . . .?"

Probability, Geometry, and Witches

J. PHILIP SMITH

Two WITCHES enjoy meeting each evening over a cauldron of tea. Both witches have two serious shortcomings, however. First, each witch is poorly organized and arrives at the meeting place randomly between midnight and 1:00 A.M. Second, each is notoriously evil-tempered and becomes outraged on having to wait fifteen minutes or longer for her companion. Thus, the following temper-saving arrangement has been agreed on: when either witch has waited fifteen minutes—or when one o'clock arrives and she is still alone—she disappears at once, not returning until the next night. Here is our problem: On a given night, what is the probability that the two witches meet?

One excellent method of attacking an unfamiliar and apparently difficult problem is to ask, "Can I solve a simpler but related question?" For us, the answer is yes. Let's try to see why. A more familiar situation results if we assume that each witch can arrive only at ten-minute intervals between midnight and 1:00 A.M.; that is, at 12:10, 12:20, 12:30, 12:40, or 12:50. To couch the problem in more familiar and more useful language, let us choose one witch and let x represent that fraction of the hour that has elapsed at the moment she appears. Let y be the result of a similar measure applied to the other witch. We can now represent the outcomes of our experiment by ordered pairs (x, y). For example, the first witch's arrival at 12:30 and the second's appearance at 12:10 will be considered as a single outcome and will be denoted by $\left(\frac{3}{6}, \frac{1}{6}\right)$. How many such outcomes are possible? We can not only count them—we can graph them as shown in figure 1.

Of the 25 possible outcomes, how many will result in a witches' meeting? Clearly, the witches will meet if their arrival times do not differ

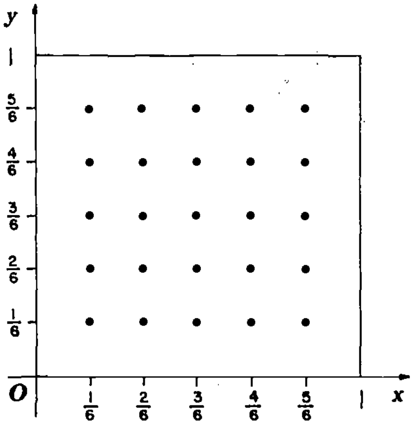


Fig. 1

by more than $\frac{1}{4}$ of an hour. The ordered pairs $\left(\frac{1}{6}, \frac{1}{6}\right)$ and $\left(\frac{1}{6}, \frac{2}{6}\right)$ represent two such outcomes. Let us call such outcomes *favorable outcomes* to indicate our interest in them. The ordered pair $\left(\frac{1}{6}, \frac{3}{6}\right)$ does not represent a favorable outcome. How many favorable outcomes exist? If you look carefully at figure 1, you will find exactly 13 represented. The 13 ordered pairs are those shown between the dotted lines of figure 2. For such pairs (x, y), either $x = y$ or $|x - y| = \frac{1}{6}$. All other pairs represent outcomes where the witches' arrival times differ by $\frac{1}{3}$ of an hour or more. Thus in 13 of 25 possible outcomes the

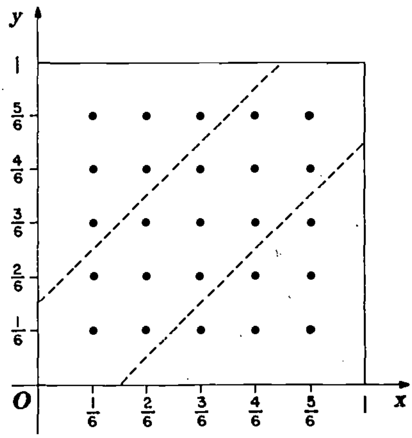


Fig. 2

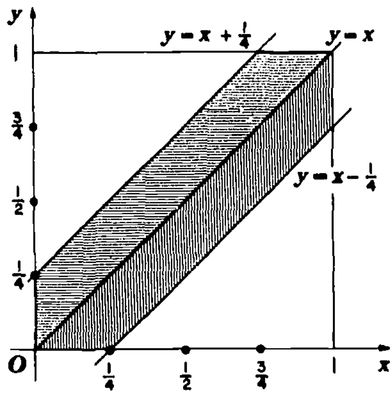
witches will arrive within fifteen minutes of one another and will get together for a chat. Since we are assuming that any one outcome is as likely to occur as another, we can now give the probability of the witches' meeting as $\frac{13}{25}$, or .52.

Let us return to the original problem: What is the probability that the witches meet when arrival times are not restricted to ten-minute intervals? Here, then, x and y are allowed to be any numbers in the interval (0, 1). We no longer have 25 outcomes—in fact, we no longer have a finite number of outcomes! How would you approach this problem? The method we used for the simpler problem gives us a clue. In that case, 25 outcomes were possible. Of the 25 equally likely outcomes, 13 resulted in the witches' meeting. We then took the number $\frac{13}{25}$ as the probability that the witches meet on a given night. Theoretically, this is what we are saying: Let all outcomes be considered equally likely. Let S be the set of all outcomes and let A be some subset of S. (We call S the outcome set and A, an event.) If we denote the probability of A by $P(A)$, then we have $P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } S}$, or, more concisely, $P(A) = \frac{n(A)}{n(S)}$. In our case, A was the set of outcomes whose entries differed by less than $\frac{1}{4}$, and S was the entire set of 25 ordered pairs. We obtained $P(A) = \frac{n(A)}{n(S)} = \frac{13}{25}$. The numbers 13 and 25 were obtained by the use of a graph. Can we, perhaps, also graph the outcomes of our more general problem and then distinguish the favorable outcomes from the others?

What outcomes (x, y), $0 < x < 1, 0 < y < 1$, lead to the witches' meeting? Clearly, outcomes for which x and y differ by less than $\frac{1}{4}$, since a meeting occurs only if neither witch has to wait as long as $\frac{1}{4}$ of an hour. Thus the outcome (x, y) represents a meeting if and only if either (1) $y \geq x$ and $y - x < \frac{1}{4}$ or (2) $y < x$ and $x - y < \frac{1}{4}$. We restate this result: An ordered pair (x, y) represents a favorable outcome if and only if either (1) $y \geq x$ and $y < x + \frac{1}{4}$ or (2) $y < x$ and $y > x - \frac{1}{4}$. Figure 3 shows those outcomes for which conditions (1) and (2) hold. The favorable outcomes are those falling between the two lines given by

$$y = x - \frac{1}{4} \text{ and } y = x + \frac{1}{4}.$$

What is the probability that a favorable outcome occurs? In the finite problem we measured the set A of favorable outcomes and the set S of all outcomes and then took a ratio of the two results as our probability. Our measure in such a case was a very simple one: We just counted the number of elements in A and S. Can we now, in the more general problem, think of a way to measure the set of favorable outcomes and the set of all outcomes with the intention of comparing the two?



(i) ▒: $\{(x,y): y > x \text{ and } y < x + \frac{1}{4}\}$
(ii) |||: $\{(x,y): y < x \text{ and } y > x + \frac{1}{4}\}$

Fig. 3

Look again at figure 3. How "large" is the shaded region—let's call it A—in comparison to the set—call it S—representing all possible outcomes? (Since we require $0 < x < 1$, $0 < y < 1$, it follows that S is the set of all ordered pairs (x, y) lying within the unit square of figure 3. What method would you use to compare the size of A to the size of S? You have probably already thought of one very natural method—the use of an area measure. Let us denote the area of a region T by $a(T)$. Clearly, $a(S) = 1$. You can apply your knowledge of geometry to figure 3 and discover that $a(A) = \frac{7}{16}$. We can now let

$$P(A) = \frac{a(A)}{a(S)} = \frac{\frac{7}{16}}{1} = \frac{7}{16},$$

or approximately 0.44.

Notice that although we have enlarged our view of probability theory here, our result resembles that of the finite case discussed earlier. We take as the probability of A the ratio of A to the total area involved. More precisely, if we assume all outcomes are equally likely, S is an outcome set, and A is a subset of S. If we can assign an area measure to both A and S, we can then obtain the probability of A by the equation $P(A) = \frac{a(A)}{a(S)}$. In the earlier finite case we wrote $P(A) = \frac{n(A)}{n(S)}$.

As a further example, let us apply our method to answer this question: Suppose each witch agrees to arrive between 12:30 and 1:00 A.M. How has the probability of a meeting changed? Here,

$$S = \left\{(x, y)\colon \frac{1}{2} < x < 1 \text{ and } \frac{1}{2} < y < 1\right\}$$

and

$$A = \left\{(x, y)\colon |y - x| < \frac{1}{4}, \frac{1}{2} < x < 1, \text{ and } \frac{1}{2} < y < 1\right\}.$$

As you can see from figure 4, $P(A) = \frac{a(A)}{a(S)} = \frac{3}{4}$, or 0.75—a much more favorable result from the witches' viewpoint.

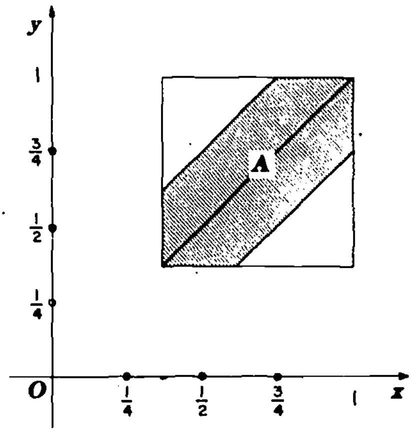


Fig. 4

What does it mean to say that the probability of the witches' meeting for tea on a given night is $\frac{7}{16}$? One interpretation is the following. If we sat by the cauldron and each night computed the ratio $\frac{\text{total number of meetings}}{\text{total number of nights}}$, we would expect after many nights that the ratio would be close to $\frac{7}{16}$. Thus, the probability of our event can be viewed as a prediction of the relative frequency of its occurrence in a large number of trials. Is the number $\frac{7}{16}$ a good prediction? Does our method provide reasonable results? Perhaps you would like to answer these questions yourself by finding two witches who behave in the appropriate manner. You then must watch them for many nights and see what happens. Have they met about $\frac{7}{16}$ of the total number of nights? If you do not enjoy witch-huntings, you can test the reasonableness of the number $\frac{7}{16}$ in other ways. We shall mention one method recently used by a student.

The student found a table of random numbers, whose first few entries looked like this:

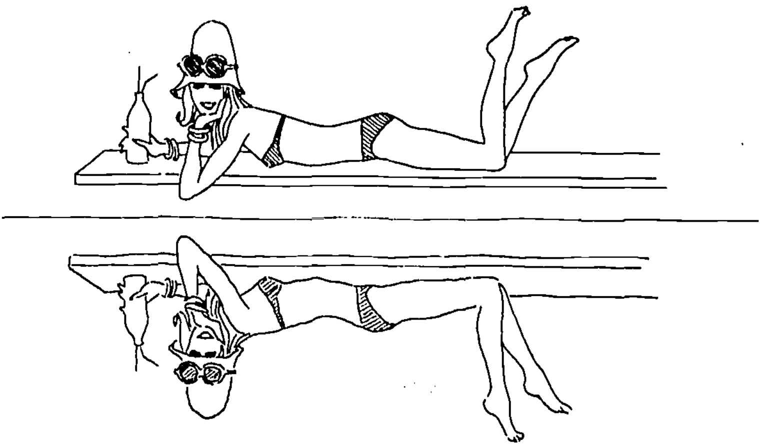
23018	70826
76576	38158
61272	23923
.	.
.	.
.	.

He placed an imaginary decimal point before each five-digit number and regarded the result as a fraction of an hour between midnight and 1:00 A.M. The numbers in the nth row can be regarded as the arrival times of the two witches on the nth night. Thus, the first night's outcome is $(x, y) = (.23018, .70826)$. Here $|y - x| < \frac{1}{4}$, and so the witches fail to meet. In fact, the outcome (.23018, .70826) indicates that witch x has arrived at about 12:14 and that witch y has arrived around 12:42 (because $60 \times .23018 \approx 14$ and $60 \times .70826 \approx 42$). Since witch x disappears in a huff at about 12:29, it is clear that no meeting occurs.

Proceeding this way through the first 100 pairs of random numbers, the student counted exactly 46 pairs that differed by less than $\frac{1}{4}$, indicating that out of the first 100 nights, the relative frequency of witches' meetings was $\frac{46}{100}$. This compares favorably with our theoretical prediction of $\frac{7}{16}$, or .4375. Our new method (in this case, anyway) appears to give sensible results.

For students interested in trying a further problem or two that may be solved by methods of geometrical probability, the following are suggested:

1. A one-meter rod breaks accidently into three pieces. What is the probability that a triangle can be formed from the pieces?
2. At a carnival game, contestants toss a disk of diameter 4 cm onto a grid composed of squares 6 cm on a side. If the tossed disk lies entirely within a square, the contestant wins a stuffed dog. Assume that the disk's center lands at random and find the probability of winning a prize.



TRANSFORMATION GEOMETRY

HOWARD F. FEHR

GEOMETRY IS A STUDY of space. The elements of space are called *points*. A physical representation could be the room in which you are now reading, with the points being specific locations in the room. The principal subsets of space that are studied in geometry are planes and line figures. In this article we limit our study to a plane and line figures in the plane. This page, when flattened out on a table, is a representation of a plane; however, a plane, unlike this page, is endless in extent.

Mappings

There are many ways to study the relations of points, lines, and figures in a plane; but a modern method, gaining in popularity, is that of transformations. Since transformations are described by mappings, we shall consider mappings first. A mapping of a plane into itself is merely assigning to each point in the plane, by some rule, exactly one other point of the plane. A very simple way to do this is to assign each point to itself; this mapping is called the *identity mapping*. Now we shall consider other types of mappings.

Line reflection. Let m be any line in the plane (fig. 1). Let P be any point of the plane. We assign to point P the point P' so that line m is the perpendicular bisector of PP'. Another way to construct this image P' is to draw a

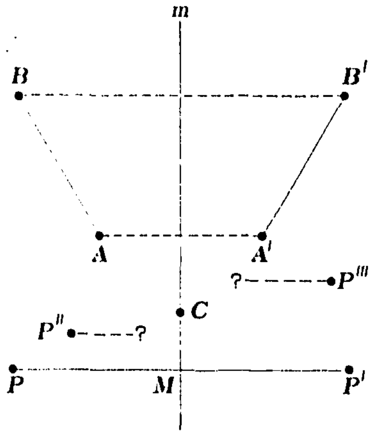


Fig. 1

line through P that is perpendicular to m at point M. Select P' on this line so that $PM = MP'$. If we consider a flat piece of paper as a plane, we can crease the paper along line m by folding. At point P we can put a pinhole through the double thickness of paper. The two pinpoints thus formed are images of each other with respect to the fold.

You can readily convince yourself that no matter where you select a point in the plane—at C, P'', P''', or elsewhere—there is one and only one point of the plane that is assigned to it by a line reflection. A point on line m, such as C, is assigned to itself. Not only does each point of the plane have an assignment (or image point), but each point is itself an image point. This kind of mapping is called *one-to-one onto.* A photograph of a tree at the edge of a lake, together with its reflection in the water, represents a line reflection, the line being the edge of the water.

In contrast to a one-to-one onto mapping, let us consider a different type. In a plane having perpendicular coordinate axes, any point has an ordered pair of coordinates (x, y). We shall simply call (x, y) a point. We now assign to each point (x, y) the image point (x^2, y), as in figure 2. For example, $(2, 3) \rightarrow (4, 3)$, and $(-2, 3) \rightarrow (4, 3)$. Here two different points, P_1 and P_2, have

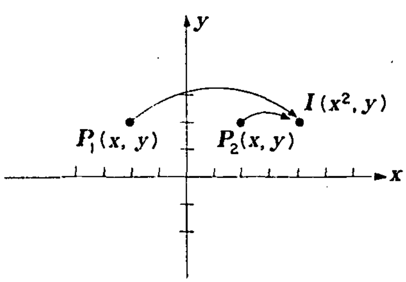


Fig. 2

the same image point I; that is, the same point I is assigned to both P_1 and P_2. In this mapping no point to the left of the y-axis is an image point. All the points of the plane are mapped into the right half of the plane only. This is not a one-to-one onto mapping. It is called a mapping of the plane *into* the plane.

Transformations

Here we shall not be concerned with into mappings. We shall instead deal only with one-to-one onto mappings. *A one-to-one onto mapping of the plane is called a transformation of the plane.* A line reflection, for instance, is a transformation of the plane. Here are some other transformations.

Point reflection. Let P be a fixed point of the plane (fig. 3). For any point A of the plane, the reflection of A through the point P is the point A' such that P is the bisector of AA'. For the point B, we can construct the image B' by first drawing line BP, then choosing B' so that $PB' = BP$. By making drawings of point reflections of various figures, we can discover a number of properties of this transformation. The following properties are important ones.

1. A segment AB is reflected into a congruent segment $A'B'$.
2. The reflection of a line AB is a parallel line $A'B'$.
3. Angle BAD reflects into a congruent angle $B'A'D'$.
4. If C is between A and B, its image C' is between A' and B'.
5. Perpendicular lines reflect into perpendicular lines.
6. Parallel lines reflect into parallel lines.

You may check that all these properties except 2 also hold for a line reflection.

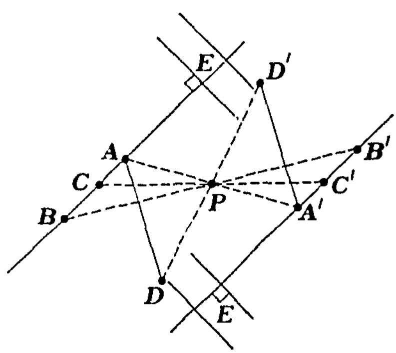


Fig. 3

Translation. A vector may be described as an arrow, or directed segment, in which the length of the segment is the magnitude, and the arrow head indicates the direction. Let $\vec{a}$ represent a vector (fig. 4). Let A, B, and C be three points. Assign to these points the images A', B', and C' respectively so that AA', BB', and CC' are equal to $\overrightarrow{a}$ (that is, have the same magnitude and direction). In this way we can establish a one-to-one onto mapping of the plane. This transformation is called a *translation*. You can check, by making appropriate drawings, that all six properties listed for a point reflection also hold for a translation.

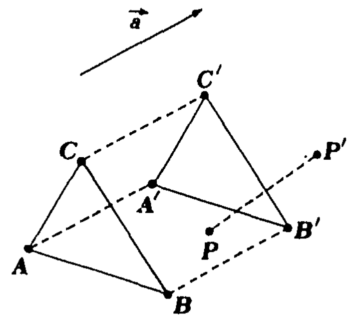


Fig. 4

Rotation. By a rotation of the plane about a fixed point P, through an angle θ, we assign to a point A the point A', so that A' is on the circle with radius PA and angle $APA' = \theta$ (fig. 5). The point P is its own image. The image of B is B', where angle $BPB' = \theta$. The rotation can be either clockwise or counterclockwise. Again you can show, by making constructions with compass, ruler, and protractor, that most of the properties listed earlier also hold for a rotation. A rotation is another kind of one-to-one onto mapping of the plane. You might note also that a rotation of 180° (or half-turn) is identical to a point reflection.

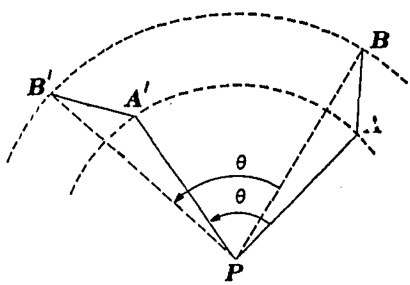


Fig. 5

The four transformations above are of a special kind called *isometries*. They are so called because they *preserve distance*. That is, if K and L are any two distinct points, with images K' and L', then the distance $K'L'$ is the same as the distance KL. Not all transformations are isometries. Let P be a fixed point of the plane (fig. 6). We now assign to any point A the point A' so that $PA' = k \cdot PA$, where k is a given number (constant). Similarly, to B we assign B' such that $PB' = k \cdot PB$, and to C is assigned C' such that $PC' = k \cdot PC$. This mapping may be thought of as a *stretching* (if $k > 1$) or as a shrinking ($k < 1$). It is easy to see that in this kind of transformation every point has exactly one image, and every point is the image of exactly one point. However, distance is not preserved. In fact, if $k = 2$, the distances AB and $A'B'$ are such that $A'B' = 2 \cdot AB$. Such a transformation is called a *similitude.* It is not in general an isometry. (Question: For what value of k is a similitude an isometry?)

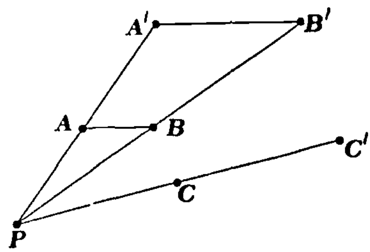


Fig. 6

Symmetry

A point A is symmetric to a point A' with respect to point O if O is the midpoint of AA'. A point P is symmetric to a point P' with respect to a line m if the line m is the perpendicular bisector of PP' (fig. 7a). A figure has line symmetry if the reflection in this line maps the figure into itself. The line is then called an axis of symmetry. Figure 7b shows a line m, a rectangle $ABCD$, and its reflection in m, $A'B'C'D'$. Since $ABCD$ and $A'B'C'D'$ do not coincide, m is not an axis of symmetry. Is there an axis of symmetry for a rectangle? Consider figure 7c, where line r is perpendicular to BC at its midpoint M. Then A maps into D, D into A, B into C, and C into B; in fact, the rectangle maps into itself. The line r is therefore an axis of symmetry. Every rectangle has two axes of symmetry; find the other axis of symmetry for $ABCD$ in figure 7c.

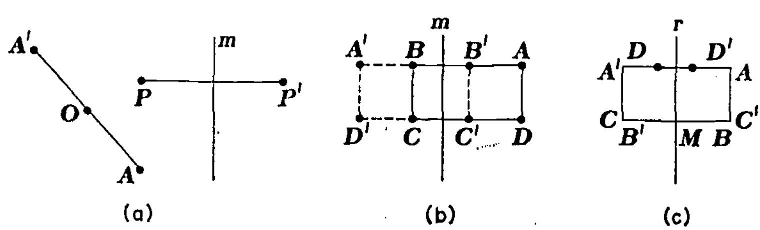


Fig. 7

In figure 8 there are two rectangles, R and R'. The one rectangle is the reflection of the other in the line m. Thus line m is an axis of symmetry for the pair of rectangles, but it is not an axis of symmetry for either of the rectangles alone. Can you find another axis of symmetry for the pair of rectangles? Figure 9 shows that a circle has an infinite number of axes of symmetry, and figure 10 shows that an oblique parallelogram has no axis of symmetry.

A figure has central symmetry if the figure reflects into itself through a fixed point. The point is then called a center of symmetry. The center of a circle is its center of symmetry. In figure 10, there is a center of symmetry. This becomes evident when we consider that a point reflection maps a segment into a congruent parallel segment. It is easy to find this center of symmetry and to prove that it is the center.

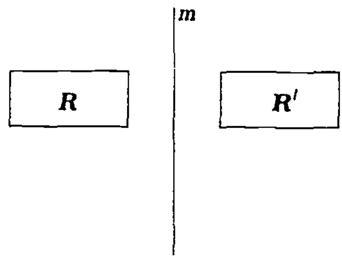


Fig. 8

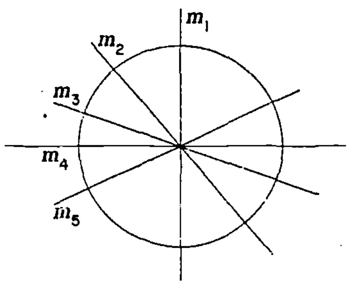


Fig. 9

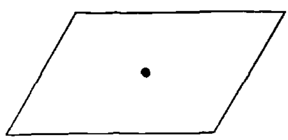


Fig. 10

Some figures have only one axis of symmetry, as shown in figure 11. A rectangle has two axes of symmetry and one center of symmetry. An equilateral triangle (fig. 12) has three axes of symmetry, but it has no center of

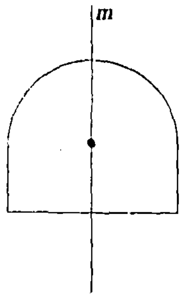


Fig. 11

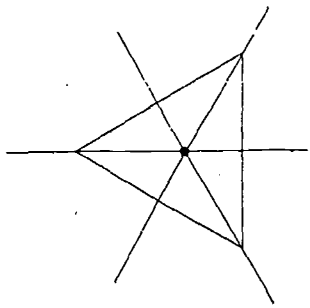


Fig. 12

symmetry. See how many axes of symmetry and centers of symmetry you can find for a square. Then do the same for a regular pentagon, a regular hexagon, a regular heptagon (7 sides), and so on, and see if you can find a general rule for the number of axes and centers of symmetry of a regular polygon. Also show by using figure 13 that if a figure has two perpendicular axes of symmetry it also has a center of symmetry.

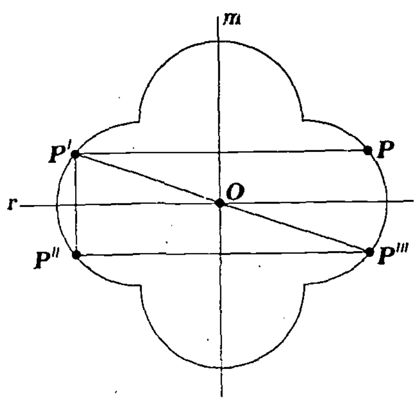


Fig. 13

There is another kind of symmetry called rotational symmetry. A figure having this type of symmetry maps into itself by rotation through an angle. In figure 14, the equilateral triangle has the centroid G. A rotation of the triangle about G, through 120°, maps the triangle into itself. Similarly, a rotation through 240° is a symmetry, and a rotation of 0° is the identity mapping. Try to show the rotational symmetries for a square and for a rectangle, Can you find a center of symmetry and the rotational symmetries for a pentagon?

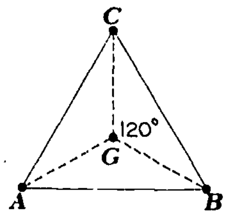


Fig. 14

All the figures mentioned so far have had more than one axis of symmetry; for instance, a regular hexagon has six axes of symmetry. Can a figure have more than one center of symmetry? To answer this question we first consider repeating designs and tessellations. A tessellation is a collection of tiles, all regular polygons, which fit together with no overlapping to cover the plane. Hexagonal tiles provide a simple example, as shown in figure 15. Note that there are many axes of symmetry—vertical ones, horizontal ones, and oblique ones. There are also many centers of symmetry; in fact, the center of

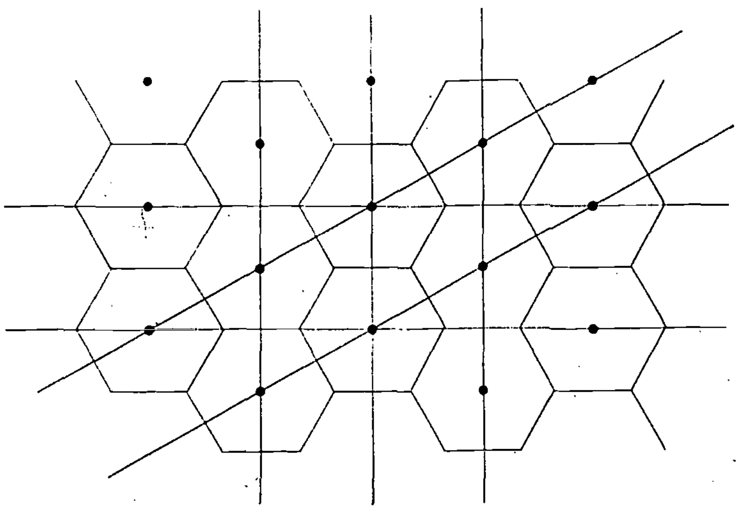


Fig. 15

each hexagon becomes a center of symmetry for the endlessly repeating design. By examining designs on gift-wrapping paper and wallpaper, you can discover many instances of mappings, transformations, symmetries, and axes and centers of symmetry.

You are now in a position to prove an interesting and important theorem on symmetry. We shall assume understanding of bounded figures. In general terms, a bounded figure is one about which you can draw a circle so that all parts of the figure are within the circle. The bounded figure cannot extend indefinitely in any direction.

THEOREM. *A bounded figure has at most one center of symmetry.* How shall we prove this? We shall do it in an indirect way, by proving that if a figure has two or more centers of symmetry, then the figure is unbounded.

In figure 16, let A and B be two centers of symmetry of a figure F. Take AB as a unit of length. Let P be a point of the figure F. Then P reflected in A yields P', which is also a point of F because of central symmetry. Now reflect P' in B to obtain Q. Then Q is also a point of figure F, since B was given as a center of symmetry. Since A is the midpoint of $P'P$ and B is the midpoint of $P'Q$, we know that $PQ = 2 \cdot AB$ (see the stretching transformation discussed above).

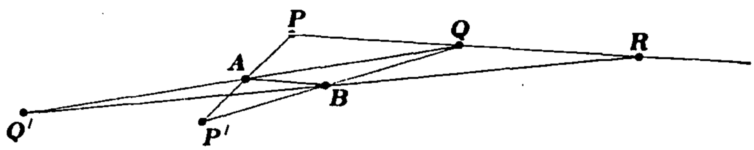


Fig. 16

Next, let Q' be the symmetric point of Q with respect to center A; then find R, the symmetric point of Q' with respect to center B. Again $QR = 2 \cdot AB$, and thus $PR = 4 \cdot AB$. Continuing indefinitely in this manner, we obtain the sequence $2 \cdot AB, 4 \cdot AB, 6 \cdot AB, \ldots, n \cdot AB, \ldots$, which increases beyond all bounds. Hence, figure F is unbounded.

In the preceding two paragraphs we have proved that if a figure has two (or more) centers of symmetry, then it is unbounded. The contrapositive of this statement, which is necessarily true, is: *If a figure is bounded, then it does not have two (or more) centers of symmetry.* Hence a bounded figure has at most one center of symmetry.

Uses of Transformations

This introduction to transformation geometry has been very informal and elementary. There are entire books written on just line reflections alone. Transformation geometry finds application wherever there is motion in space. The physicist speaks of displacements of physical bodies, such as the earth, in space. As a body moves, it may retain its physical shape and size. Thus, at two different times the body is the same, but the positions in space are different. Transformation geometry does not consider the physical body, but instead makes a mapping of the points it occupied at one instant onto the points it occupies at a later instant. If the body were rigid, then the mapping would be an isometry. If, however, the body were elastic and changed its size (and shape), then the mapping could be a similitude, or even some other type. The study of the various transformations, and compositions of them (that is, one transformation followed by another) enables the mathematician to make predictions about what may be happening in a physical situation.