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ABSTRACT

A laboratory experiment, based on a simple electric circuit that can be used to demonstrate the existence of real-world "related rates" problems, is outlined and an equation for voltage across the capacitor terminals during discharge is derived. The necessary materials, setup methods, and experimental problems are described. A student laboratory worksheet is provided. (YP)

A Laboratory Exercise with Related Rates

by

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Abstract

A laboratory experiment based on a simple electric circuit is presented that can be used to demonstrate the existence of real world "related rates" problems. This topic is discussed in the first semester calculus course and it is hoped that exposure to such physical examples will reinforce the student's appreciation of this material.

Discussion

A simple physical process that involves differentiation at the first semester calculus level is the flow, through a resistor, of charge stored in a capacitor. If a battery with voltage V is connected to the terminals of a capacitor (see figure 1) a surplus of negative charge (i.e. electrons) of quantity q, measured in Coulombs, will collect at one terminal.

The storage capacity of the capacitor is called capacitance, C, and is measured in Farads. Physicists tell us that the amount of surplus charge collected at one

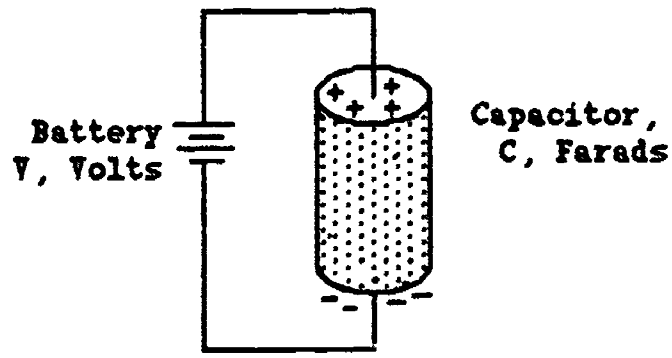

Figure 1
Charging Capacitor

terminal is directly proportional to the voltage, v, across the capacitor with C being the constant of proportionality. Consequently,

(1) $$q = Cv.$$

If the battery is disconnected, the capacitor will hold indefinitely (theoretically) the charge that has been placed on it as depicted in figure 2.

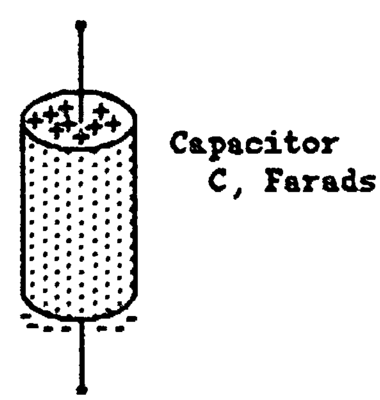

Figure 2
Charged Capacitor

If, at some later time, a resistor (such as a light bulb) is connected across the capacitor terminals, the excess charge will flow from one terminal to the other one as shown in figure 3.

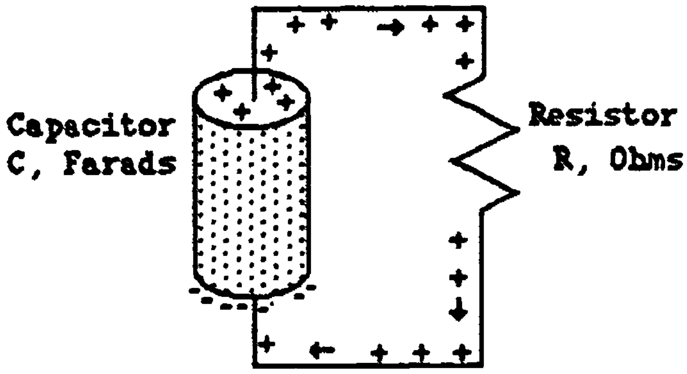

Figure 3
Discharging Capacitor

This continues until the charge is again uniform on the two terminals. While charge is flowing out of the capacitor through the resistor, the capacitor is said to be discharging. It can be shown that during discharge the voltage across the capacitor terminals varies, as a function of time, t, according to equation 2.

$$(2) \qquad v(t) = V_o e^{-\frac{t}{RC}}$$

In this equation V_o is the voltage on the capacitor before discharge begins, R is the resistance, the quantitative measure of the resistor and expressed in Ohms, and C is the capacitance. A reasonable polynomial approximation for this voltage for small values of time is

$$(3) \qquad v(t) = V_o\left(1 - \frac{t}{RC} + \frac{1}{2!}\left(\frac{t}{RC}\right)^2 - \frac{1}{3!}\left(\frac{t}{RC}\right)^3 \right)$$

The discharging process of a capacitor lends itself to a simple experiment involving related rates. Of interest to know is the rate of change of charge on the capacitor, dq/dt, during discharge. This rate is also called current (measured in Amperes) and usually represented with the symbol i(t). Current is an important quantity in electrical circuit design and analysis. From equations 2 or 3 the rate of change of voltage, dv/dt, can be easily calculated. Because the two rates are related through equation 1, $q(t) = Cv(t)$, the task of finding dq/dt represents a classical related rates problem. Given values for R, C, and the expression for v(t) the student can calculate the current, dq/dt. The current can easily be measured physically using a common piece of electronics equipment, the ammeter. The student can measure the current at a given instant after discharge has begun and compare this value with a theoretical result obtained using the previously described analytical process.

Derivation of Equation 2

Consider the schematic diagram for a discharging capacitor given in figure 4. From Kirchhoff's voltage law it is known that the algebraic sum of the voltage drops around a

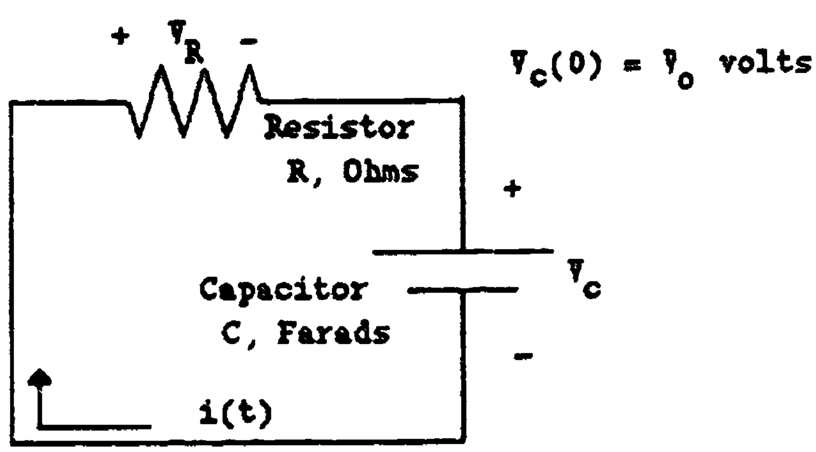

Figure 4
Capacitor Discharging Through a Resistor

closed loop is zero. Using the loop shown in figure 4 and Kirchhoff's voltage law results in the following equation.

$$(4) \qquad v_R(t) + v_C(t) = 0.$$

From Ohm's Law it is known that the voltage drop across a resistor is equal to the product of the current, i(t), and resistance, R. Consequently, $v_R(t) = Ri(t)$ and equation 4 becomes

$$(5) \qquad Ri(t) + v_C(t) = 0.$$

Differentiating equation 5 with respect to time yields

$$(6) \qquad R\frac{di}{dt} + \frac{dv_C}{dt} = 0$$

Recalling equation 1 yields the fact that $q(t) = Cv_C(t)$ and differentiating this equation with respect to time and solving for the derivative of the voltage across the capacitor yields

$$(7) \qquad \frac{dv_C}{dt} = \frac{1}{C}\frac{dq}{dt}.$$

The rate of change of charge with respect to time, dq/dt, is called current and denoted i(t). Substituting into equation 7

$$(8) \qquad \frac{dv_C}{dt} = \frac{1}{C}\, i(t).$$

Substituting dv_C/dt from equation 8 into equation 6

$$(9) \qquad R\frac{di}{dt} + \frac{1}{C}\, i(t) = 0$$

Solving for i(t) by separating variables and integrating

$$(10) \qquad i(t) = k\, e^{-\frac{t}{RC}}$$

To determine the value of the constant of integration note the physical restriction that the voltage across a capacitor can not change instantaneously. This is due to the fact that charge can neither suddenly appear nor suddenly disappear. Consequently, substituting $t = 0$ into equation 4 and recalling that $v_C(0) = V_O$ from figure 4

$$v_R(0) + V_o = 0$$

$$i(0)R + V_o = 0$$

$$i(0) = -\frac{V_o}{R}$$

From equation 10 it is clear that $i(0) = k$ and so

$$k = -\frac{V_o}{R}$$

Equation 10 becomes, upon substitution for k,

$$(11) \qquad i(t) = -\frac{V_o}{R} e^{-\frac{t}{RC}}$$

From equation 5, $v_c = -i(t)R$ and substituting into equation 11 yields

$$(12) \qquad v_c(t) = V_o e^{-\frac{t}{RC}}$$

Expanding the exponential using a Maclaurin series yields

$$v_c(t) = V_o\left(1 - \frac{t}{RC} + \frac{1}{2!}\left(\frac{t}{RC}\right)^2 - \frac{1}{3!}\left(\frac{t}{RC}\right)^3 + \cdot \cdot \cdot\right)$$

Laboratory Equipment and Setup

A simple electric circuit, such as the one in figure 5, must be constructed for this experiment. It requires two resistors, a capacitor, a single throw switch, and a small length of wire. These components are quite inexpensive. Needed also are traditional laboratory devices that should be available on a short term loan basis from the physics or electronics technology department: a ten volt direct current power supply, an ammeter with 1 milliAmpere maximum scale, a voltmeter with a 0 to 10 volt scale. Only one set up is needed for the class. Students will work in groups of two. While one group is actually taking data, the other groups are calculating the expected outcome for the experiment or comparing this value with the experimentally observed value.

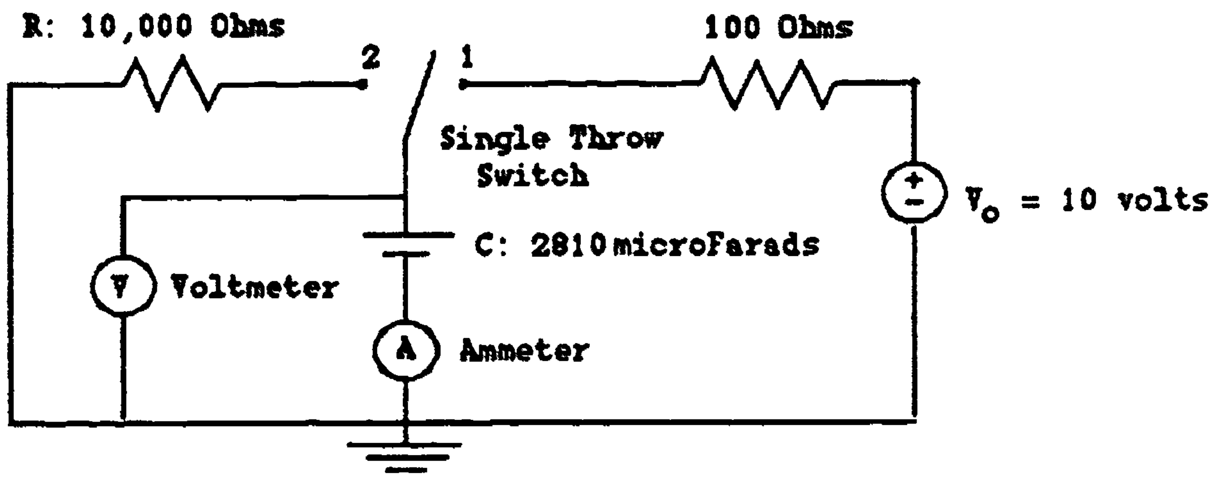

Figure 5
Example of an Experimental Circuit

Experiment Execution

1. Place the switch in position 1. This will allow the capacitor to charge up to its desired initial value of 10 volts. Monitor the voltmeter and when the reading is 10 volts the experiment may proceed with step 3.

2. While waiting for the capacitor to become fully charged, the student is given an observation time, t^*, in the interval (0,15) seconds.

3. The student moves the switch to position 2, and using a

watch with a seconds display, begins counting the seconds. When t^* is reached the value of the current through the capacitor is recorded following observation of the ammeter reading.

4. The switch is moved back to position 1 in preparation for the next student group.

5. The students return to their desks and proceed with the following calculations.

6. Using the fact that q(t)=Cv(t) the student wishes to verify the observed value of current, i(t), and is told that i(t) = dq/dt and given v(t) for the discharge phase along with the values for V_o, R, and C. If exponential functions have been covered in class, the student is given equation 2. Otherwise, the first few terms of the Maclaurin series for v(t) are given, as shown in equation 3. The students calculate

$$i(t) = \frac{dq}{dt} = \frac{d}{dt}(Cv(t)) = C\frac{dv}{dt}.$$

After evaluating this expression at t^*, the students have the theoretical value of $i(t^*)$.

7. The students compare the observed and theoretical values of $i(t^*)$.

Student Laboratory Worksheet

The material below the dotted line represents an example of the actual written materials a student might be given in support of a laboratory exercise involving related rates.

A simple physical process that involves differentiation is the flow, through a resistor, of charge stored in a capacitor. If a battery with voltage v is connected to the terminals of a capacitor (see figure 1) a surplus of negative charge (i.e. electrons) of quantity q will collect at one terminal.

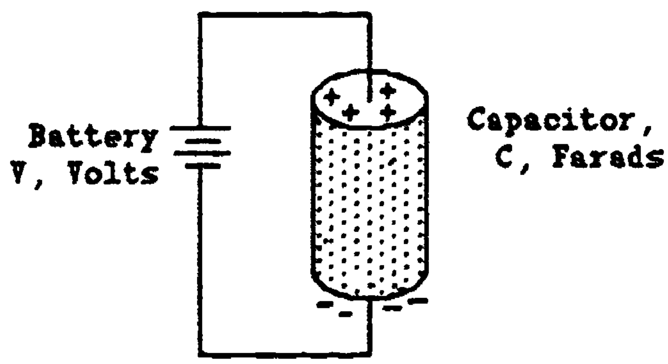

Figure 1
Charging Capacitor

The storage capacity of the capacitor is called capacitance, C, and is measured in Farads. Physicists tell us that the amount of surplus charge collected at one terminal is directly proportional to the voltage, v, across the capacitor with C being the constant of proportionality. Consequently,

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If the battery is disconnected, the capacitor will hold indefinitely (theoretically) the charge that has been placed on it as depicted in figure 2.

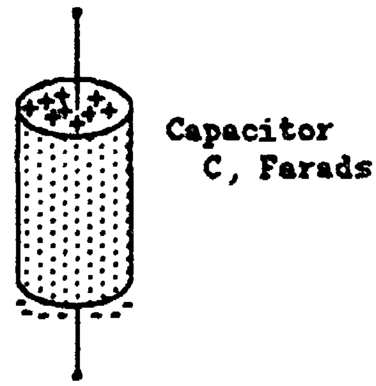

Figure 2
Charged Capacitor

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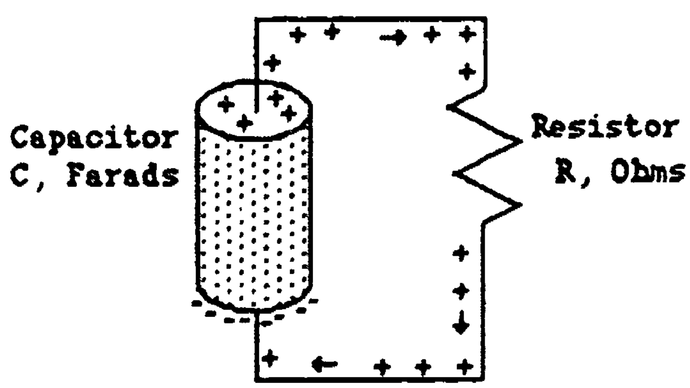

Figure 3
Discharging Capacitor

This continues until the charge is again uniform on the two terminals. While charge is flowing out of the capacitor through the resistor, the capacitor is said to be discharging. It can be shown that during discharge the voltage across the capacitor terminals, v(t), varies as a function of time, t, approximately according to equation 2.

$$(2) \qquad v(t) = V_o\left(1 - \frac{t}{RC} + \frac{1}{2!}\left(\frac{t}{RC}\right)^2 - \frac{1}{3!}\left(\frac{t}{RC}\right)^3\right)$$

In this equation V_o is the voltage on the capacitor before discharge begins, R is the resistance, the quantitative measure of the resistor and expressed in Ohms, and C is the capacitance.

The discharging process of a capacitor lends itself to an experiment involving related rates. We would like to know the rate of change of charge on the capacitor, dq/dt, during discharge. This rate is also called current (measured in Amperes) and usually represented with the symbol i(t). Current is an important quantity in electrical circuit design and analysis. From equation 2 the rate of change of voltage, dv/dt, can be calculated. These two rates are related through equation 1, q(t) = Cv(t), and we thus have a related rates problem. Given values for R, C, and the expression for v(t) you can calculate the current, dq/dt. The current can be measured physically using a common piece of electronics equipment, the ammeter. In this laboratory you will measure the current at a given instant after discharge has begun and compare this value with a theoretical result obtained as suggested above.

Experiment Execution

A schematic diagram of the electrical circuit you will use for this laboratory is shown below

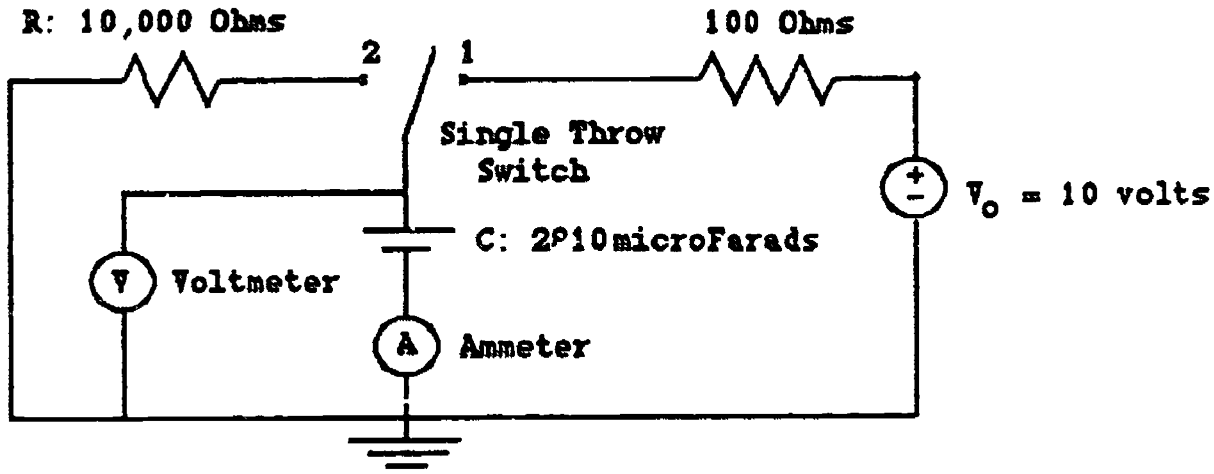

Figure 4
Experimental Circuit

1. Place the switch in position 1 if it is not already there. This will allow the capacitor to charge. Monitor the voltmeter and when the reading is 10 volts proceed to the next step.

2. You will be given an observation time, t^*. Write that value in the space below:

Observation time, t^* is ______________________

3. Move the switch to position 2 and using a watch with a seconds display begin counting the seconds. When the observation time, t^*, is reached read the value of current displayed on the ammeter and record it below:

Current at t^*: $i(t^*)$ = ____________________

4. Move the switch back to positon 1 in preparation for the next student.

5. Return to your desk and calculate the theoretical value of the current at t^* by filling in the blanks below:

5a. Given that V_o = 10 volts, R = 10,000 Ohms, $C = 2810 \times 10^{-6}$ Farads, and recalling equation 2 repeated below find v(t) and dv/dt.

$$v(t) = V_o\left(1 - \frac{t}{RC} + \frac{1}{2!}\left(\frac{t}{RC}\right)^2 - \frac{1}{3!}\left(\frac{t}{RC}\right)^3\right) \qquad (2)$$

v(t) = ______________________________

$\frac{dv}{dt}$ = ______________________________

5b. Find i(t) using the facts that q(t) = C v(t) and that i(t) = dq(t)/dt.

i(t) = ______________________________

5c. Find the current at the observation time, $i(t^*)$.

$i(t^*)$= ______________________________

This is the theoretical value of the current at the observation time.

5d. Calculate the percent error between the observed and predicted value of the current at the observation time.______________________________

5e. Discuss possible reasons for this error below: