Duality in Linear Programming

- The dual of a linear program: motivation
- Formal definition, the Karush-Kuhn-Tucker (KKT) conditions
- Primal–dual relationships, the Weak and the Strong Duality Theorems
- The Farkas Lemma

Linear Programming Duality: Motivation

- Consider the below linear program

$$
\begin{array}{rrrrrrrrrr}
z & = & \max & x_1 & + & 2x_2 & - & x_3 & & \\
& & \text{s.t.} & 3x_1 & + & 2x_2 & + & x_3 & \leq & 12 \\
& & & -x_1 & & & - & x_3 & \leq & -3 \\
& & & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}
$$

- We give upper bounds for the objective function
- Since variables are nonnegative, the first constraint is immediately an upper bound

$$z = x_1 + 2x_2 - x_3 \leq 3x_1 + 2x_2 + x_3 \leq 12$$

- Since component-wise $x_1 \leq 3x_1$, $2x_2 \leq 2x_2$, and $-x_3 \leq x_3$
- Is there any tighter upper bound?

Linear Programming Duality: Motivation

- Summing the two constraints

$$
\begin{array}{rrrrrrl}
z = & x_1 & + & 2x_2 & - & x_3 & & \\
\hline
 & 3x_1 & + & 2x_2 & + & x_3 & \leq & 12 \\
\oplus & -x_1 & & & - & x_3 & \leq & -3 \\
\hline
 & 2x_1 & + & 2x_2 & + & 0x_3 & \leq & 9
\end{array}
$$

- Yields the tighter bound $z = x_1 + 2x_2 - x_3 \leq 2x_1 + 2x_2 \leq 9$
- Even tighter bound is obtained if we add two times the second constraint to the first one:

$$z = x_1 + 2x_2 - x_3 \leq (3-2*1)x_1 + (2-2*0)x_2 + (1-2*1)x_3 \leq 6$$

- This is the tightest possible bound, since the optimal objective function value is $z = 6$

Linear Programming Duality: Motivation

- In fact, for any $w_1 \geq 0$ and $w_2 \geq 0$ for which the expression

$$w_1 \left(3x_1 + 2x_2 + x_3\right) + w_2 \left(-x_1 - x_3\right)$$

component-wise upper bounds the objective function $z = x_1 + 2x_2 - x_3$, that is, for which

$$3w_1 - w_2 \geq 1, \quad 2w_1 \geq 2, \quad \text{and} \quad w_1 - w_2 \geq -1$$

holds, we get a new upper bound:

$$z = x_1 + 2x_2 - x_3 \leq w_1 \left(3x_1 + 2x_2 + x_3\right) + w_2 \left(-x_1 - x_3\right)$$

- $w_1 \geq 0$ and $w_2 \geq 0$ needed, otherwise the sign would change
- The tightest bound is the one for which $12w_1 + (-3)w_2$ is minimal

Linear Programming Duality: Motivation

- Yields another linear program: the **dual linear program**:

$$
\begin{array}{lrcrcr}
\min & 12w_1 & - & 3w_2 & & \\
\text{s.t.} & 3w_1 & - & w_2 & \geq & 1 \\
 & 2w_1 & & & \geq & 2 \\
 & w_1 & - & w_2 & \geq & -1 \\
 & w_1, & & w_2 & \geq & 0
\end{array}
$$

- To distinguish, the original linear program will be called the **primal**
- Interestingly, the dual optimal solution is also 6
- In fact this is guaranteed to hold and, what is more, there are very deep relationships between the primal and the dual

The Dual Linear Program

- **Theorem:** given a linear program as a maximization problem in the **standard form**

$$\max \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t. } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

  the dual is the standard form minimization problem

$$\min \boldsymbol{w}^T \boldsymbol{b}$$
$$\text{s.t. } \boldsymbol{w}^T \boldsymbol{A} - \boldsymbol{v}^T = \boldsymbol{c}^T$$
$$\boldsymbol{v}^T \geq \boldsymbol{0},\ \boldsymbol{w}^T \text{ arbitrary}$$

- One dual variable for each constraint of the primal and one dual constraint for each variable of the primal

The Dual Linear Program

- The variables $\boldsymbol{v}^T$ and $\boldsymbol{w}^T$ are called **dual variables** (or Lagrangean multipliers)
- The dual variables $\boldsymbol{w}^T = [w_1 \quad w_2 \; \ldots \quad w_m]$ correspond to the primal constraints $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$: for every constraint $\boldsymbol{a}^{\boldsymbol{i}}\boldsymbol{x} = b_i$ there is a dual variable w_i, precisely m
- The dual variables $\boldsymbol{v}^T = [v_1 \quad v_2 \; \ldots \quad v_n]$ correspond to the nonnegativity constraints for the primal variables $\boldsymbol{x}$: for every constraint $x_j \geq 0$ there is a dual variable v_j, exactly n
- In fact, $\boldsymbol{v}^T$ act as slack-variables that we can as well omit

$$\begin{aligned} \min\ & \boldsymbol{w}^T\boldsymbol{b} \\ \text{s.t. } & \boldsymbol{w}^T\boldsymbol{A} \geq \boldsymbol{c}^T \\ & \boldsymbol{w}^T \text{ arbitrary} \end{aligned}$$

The Dual Linear Program: Example

- Obtain the dual of the canonical form linear program:

$$
\begin{array}{rlrcrcl}
P: & \max & 6x_1 & + & 8x_2 & & \\
 & \text{s.t.} & 3x_1 & + & x_2 & \leq & 4 \\
 & & 5x_1 & + & 2x_2 & \leq & 7 \\
 & & x_1, & & x_2 & \geq & 0
\end{array}
$$

- Converting to standard form by introducing slack variables

$$
\begin{array}{rlrcrcrcrcl}
P: & \max & 6x_1 & + & 8x_2 & & & & & & \\
 & \text{s.t.} & 3x_1 & + & x_2 & + & x_3 & & & = & 4 \\
 & & 5x_1 & + & 2x_2 & & & + & x_4 & = & 7 \\
 & & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

The Dual Linear Program: Example

- Two primal constraints, so in the dual there will be two dual variables: $\boldsymbol{w}^T = \begin{bmatrix} w_1 & w_2 \end{bmatrix}$
- Dual variables $\boldsymbol{v}^T$ will be handled as slack-variables
- The dual objective function is $\min \boldsymbol{w}^T \boldsymbol{b} = \min \boldsymbol{b}^T \boldsymbol{w}$, where $\boldsymbol{b}^T = \begin{bmatrix} 4 & 7 \end{bmatrix}$

$$\min 4w_1 + 7w_2$$

- One dual condition for each primal variable
- The dual constraint for the primal variable x_1: $\boldsymbol{w}^T \boldsymbol{a_1} \geq c_1$, where $\boldsymbol{a_1}$ is the column of $\boldsymbol{A}$ corresponding to x_1 (the first column) and c_1 is the objective coefficient for x_1

$$\begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 3w_1 + 5w_2 \geq 6$$

The Dual Linear Program: Example

- Similarly, the dual constraint for x_2: $\boldsymbol{w}^T \boldsymbol{a_2} \geq c_2$

$$\begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = w_1 + 2w_2 \geq 8$$

- For the slack variables we obtain the dual constraints in a single step:

$$\begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \geq 0 \equiv w_1 \geq 0,\ w_2 \geq 0$$

- The dual linear program:

$$\begin{array}{llcccl} D: & \min & 4w_1 & + & 7w_2 & \\ & \text{s.t.} & 3w_1 & + & 5w_2 & \geq & 6 \\ & & w_1 & + & 2w_2 & \geq & 8 \\ & & w_1, & & w_2 & \geq & 0 \end{array}$$

The Dual Linear Program: Example

- The primal and the dual in canonical form:

$$\begin{aligned} \max\ & 6x_1 + 8x_2 \\ \text{s.t.}\ & 3x_1 + x_2 \le 4 \\ & 5x_1 + 2x_2 \le 7 \\ & x_1, \quad x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \min\ & 4w_1 + 7w_2 \\ \text{s.t.}\ & 3w_1 + 5w_2 \ge 6 \\ & w_1 + 2w_2 \ge 8 \\ & w_1, \quad w_2 \ge 0 \end{aligned}$$

- In general, the primal in dual in canonical form:

$$\begin{aligned} P:\ \max\ & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.}\ & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

$$\begin{aligned} D:\ \min\ & \boldsymbol{w}^T \boldsymbol{b} \\ \text{s.t.}\ & \boldsymbol{w}^T \boldsymbol{A} \ge \boldsymbol{c}^T \\ & \boldsymbol{w}^T \ge \boldsymbol{0} \end{aligned}$$

The Dual Linear Program

- If there are constraints of the type $(\leq)$, $(\geq)$ and $(=)$ in the linear program

$$\begin{aligned} \max\ & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t. } & \boldsymbol{A}_1\boldsymbol{x} \leq \boldsymbol{b}_1 \\ & \boldsymbol{A}_2\boldsymbol{x} = \boldsymbol{b}_2 \\ & \boldsymbol{A}_3\boldsymbol{x} \geq \boldsymbol{b}_3 \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- In standard form:

$$\begin{array}{llcccrcl} \max & \boldsymbol{c}^T\boldsymbol{x} & & & & & & \\ \text{s.t.} & \boldsymbol{A}_1\boldsymbol{x} & + & \boldsymbol{I}\boldsymbol{x_s} & & & = & \boldsymbol{b}_1 \\ & \boldsymbol{A}_2\boldsymbol{x} & & & & & = & \boldsymbol{b}_2 \\ & \boldsymbol{A}_3\boldsymbol{x} & & & - & \boldsymbol{I}\boldsymbol{x_t} & = & \boldsymbol{b}_3 \\ & \boldsymbol{x}, & & \boldsymbol{x_s}, & & \boldsymbol{x_t} & \geq & \boldsymbol{0} \end{array}$$

The Dual Linear Program

- Let ${\boldsymbol{w_1}}^T$ be the dual variables corresponding to the primal constraints $\boldsymbol{A}_1\boldsymbol{x} \leq \boldsymbol{b}_1$, ${\boldsymbol{w_2}}^T$ to constraints $\boldsymbol{A}_2\boldsymbol{x} = \boldsymbol{b}_2$, and ${\boldsymbol{w_3}}^T$ to $\boldsymbol{A}_3\boldsymbol{x} \geq \boldsymbol{b}_3$

$$\begin{array}{rccccccl}
\min & {\boldsymbol{w_1}}^T\boldsymbol{b} & + & {\boldsymbol{w_2}}^T\boldsymbol{b}_2 & + & {\boldsymbol{w_3}}^T\boldsymbol{b}_3 & & \\
\text{s.t.} & {\boldsymbol{w_1}}^T\boldsymbol{A}_1 & + & {\boldsymbol{w_2}}^T\boldsymbol{A}_2 & + & {\boldsymbol{w_3}}^T\boldsymbol{A}_3 & \geq & \boldsymbol{c}^T \\
& {\boldsymbol{w_1}}^T\boldsymbol{I} & & & & & \geq & \boldsymbol{0} \\
& & & & & -{\boldsymbol{w_3}}^T\boldsymbol{I} & \geq & \boldsymbol{0} \\
& {\boldsymbol{w_1}}^T, & & {\boldsymbol{w_2}}^T, & & {\boldsymbol{w_3}}^T & & \text{tetszőleges}
\end{array}$$

- Consequently, ${\boldsymbol{w_1}}^T \geq \boldsymbol{0}$ and ${\boldsymbol{w_3}}^T \leq \boldsymbol{0}$
- From the constraints of the primal problem
  - of the type "$\leq$" yield dual variables of the type "≥ 0",
  - of the type "$\geq$" yield "≤ 0" variables, and
  - of "$=$" type yield free dual variables (no sign restriction)

The Dual Linear Program

	Maximization problem		Minimization problem	
Constraint	$\geq$	$\longleftrightarrow$	≤ 0	Variable
	$\leq$	$\longleftrightarrow$	≥ 0	
	$=$	$\longleftrightarrow$	arbitrary	
Variable	≥ 0	$\longleftrightarrow$	$\geq$	Constraint
	≤ 0	$\longleftrightarrow$	$\leq$	
	arbitrary	$\longleftrightarrow$	$=$	

The Dual Linear Program: Example

- Write the dual for the below linear program

$$\begin{array}{rrcrcr} \max & 8x_1 & + & 3x_2 & & \\ \text{s.t.} & x_1 & - & 6x_2 & \geq & 2 \\ & 5x_1 & + & 7x_2 & = & -4 \\ & x_1 & & & \leq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

- The dual linear program

$$\begin{array}{rrcrcl} \min & 2w_1 & - & 4w_2 & & \\ \text{s.t.} & w_1 & + & 5w_2 & \leq & 8 \\ & -6w_1 & + & 7w_2 & \geq & 3 \\ & w_1 & & & \leq & 0 \\ & & & w_2 & & \text{arbitrary} \end{array}$$

The Dual Linear Program

	Primal	Dual
Standard form	$\max \boldsymbol{c}^T\boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ $\boldsymbol{x} \geq \boldsymbol{0}$	$\min \boldsymbol{w}^T\boldsymbol{b}$ s.t.$\boldsymbol{w}^T\boldsymbol{A} \geq \boldsymbol{c}^T$ $\boldsymbol{w}^T$ arbitrary
Canonical form	$\max \boldsymbol{c}^T\boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ $\boldsymbol{x} \geq \boldsymbol{0}$	$\min \boldsymbol{w}^T\boldsymbol{b}$ s.t.$\boldsymbol{w}^T\boldsymbol{A} \geq \boldsymbol{c}^T$ $\boldsymbol{w}^T \geq \boldsymbol{0}$

The KKT Conditions

- Consider the primal–dual pair of linear programs:

$$P: \max \boldsymbol{c}^T \boldsymbol{x} \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}$$

$$D: \min \boldsymbol{w}^T \boldsymbol{b} \quad \text{s.t. } \boldsymbol{w}^T \boldsymbol{A} - \boldsymbol{v}^T = \boldsymbol{c}^T, \quad \boldsymbol{v}^T \geq \boldsymbol{0},\ \boldsymbol{w}^T \text{ arbitrary}$$

- **Theorem: The Karush-Kuhn-Tucker (KKT) Optimality Conditions:** some $\boldsymbol{x}$ is an optimal solution to the primal and some $(\boldsymbol{w}^T, \boldsymbol{v}^T)$ is an optimal solution to the dual, if and only if all the following conditions hold:

  P: $\boldsymbol{x}$ is primal feasible: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$

  D: $(\boldsymbol{w}^T, \boldsymbol{v}^T)$ is dual-feasible: $\boldsymbol{w}^T \boldsymbol{A} - \boldsymbol{v}^T = \boldsymbol{c}^T, \boldsymbol{v}^T \geq \boldsymbol{0}$

  CS: complementary slackness conditions hold: $\boldsymbol{v}^T \boldsymbol{x} = 0$

The KKT Conditions

- The (P) and (D) conditions are straight forward: these require the primal and the dual solutions to be feasible
- Complementary slackness (CS) may need more explanation
- Factoring the (CS) conditions: $\boldsymbol{v}^T \boldsymbol{x} = \sum_{j=1}^{n} v_j x_j = 0$
- Since $v_j \geq 0$ and $x_j \geq 0$, this can only hold if for each $j \in \{1, \ldots, n\} : v_j x_j = 0$
- This gives a deep complementarity relation between the optimal and primal and dual solutions:

$$v_j > 0 \Rightarrow x_j = 0$$
$$x_j > 0 \Rightarrow v_j = 0$$

- For instance, if v_j is strictly positive in the optimal dual solution, then the corresponding primal x_j must be zero

The KKT Conditions: Proof

- **Proof:** We prove only the following simpler claim: given a primal linear program $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$, if $\boldsymbol{x}$ is an optimal basic feasible solution in the primal then there is $(\boldsymbol{v}^T, \boldsymbol{w}^T)$ that satisfies (P), (D) and (CS)
- So let $\boldsymbol{x}$ be an optimal basic feasible solution and let $\boldsymbol{B}$ be the corresponding basis, and consider the simplex tableau

	z	$\boldsymbol{x_B}$	$\boldsymbol{x_N}$	RHS	
z	1	$\boldsymbol{0}$	$\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T$	$\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{b}$	row 0
$\boldsymbol{x_B}$	$\boldsymbol{0}$	$\boldsymbol{I}_m$	$\boldsymbol{B}^{-1}\boldsymbol{N}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$	rows 1...m

- Note that $\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T \geq \boldsymbol{0}$ since the tableau is optimal
- We use the optimal tableau to obtain the dual solution

The KKT Conditions: Proof

- Choose the dual variable $\boldsymbol{v}^T$ to the objective row of the optimal simplex tableau and set $\boldsymbol{w}^T$ as follows:

$$\boldsymbol{w}^T = \boldsymbol{c_B}^T \boldsymbol{B}^{-1} \qquad \boldsymbol{v}^T = [\underbrace{\mathbf{0}}_{\text{basic}} \quad \underbrace{\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T}_{\text{nonbasic}}] \geq \mathbf{0}$$

- (P) holds since x is primal optimal by assumption
- (D) holds, since $\boldsymbol{v}^T \geq \mathbf{0}$ due to the optimality condition for the tableau and $\boldsymbol{c}^T - \boldsymbol{w}^T \boldsymbol{A} + \boldsymbol{v}^T = \mathbf{0}$ because it holds separately for both the basic and the nonbasic components:

$$\boldsymbol{c_B}^T - \boldsymbol{w}^T \boldsymbol{B} + 0 = \boldsymbol{c_B}^T - \boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{B} = \mathbf{0} \qquad \text{(basic)}$$

$$\boldsymbol{c_N}^T - \boldsymbol{w}^T \boldsymbol{N} + (\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T) = -\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} + \boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} = \mathbf{0} \qquad \text{(nonbasic)}$$

The KKT Conditions: Proof

- What remained to be done is to show that the complementary slackness (CS) conditions also hold
- In fact, (CS) also holds, i.e., $\boldsymbol{v}^T\boldsymbol{x} = 0$, since

$$\begin{bmatrix}\mathbf{0} & (\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T)\end{bmatrix}\begin{bmatrix}\boldsymbol{x_B} \\ \mathbf{0}\end{bmatrix} = 0$$

- Consequently, if $\boldsymbol{x}$ is a primal optimal basic feasible solution then we can easily read the dual variables $\boldsymbol{v}^T$ and $\boldsymbol{w}^T$ from the optimal tableau that satisfy the KKT conditions □
- This sheds new light on the simplex method itself
- In fact, the simplex is an iterative algorithm to find a point that satisfies the KKT conditions: (P) and (CS) hold in each iteration and (D) is also satisfied at optimality

The KKT Conditions: Example

- Solve the below linear program using the KKT conditions

$$\begin{aligned} \max \quad & x_1 + 3x_2 && (1) \\ \text{s.t.} \quad & -x_1 + 2x_2 \leq 4 && (2) \\ & x_1 + x_2 \leq 4 && (3) \\ & x_1, x_2 \geq 0 && (4) \end{aligned}$$

- Introduce x_3, x_4 slack variables to convert to standard form
- Find $\boldsymbol{x}$ primal and $\boldsymbol{w}^T = \begin{bmatrix} w_1 & w_2 \end{bmatrix}$, $\boldsymbol{v}^T = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$ dual variables so that the KKT conditions hold

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \qquad \boldsymbol{x} \geq \boldsymbol{0} \qquad \text{(P)}$$

$$\boldsymbol{c}^T - \boldsymbol{w}^T \boldsymbol{A} + \boldsymbol{v}^T = \boldsymbol{0}, \qquad \boldsymbol{v}^T \geq \boldsymbol{0} \qquad \text{(D)}$$

$$\boldsymbol{v}^T \boldsymbol{x} = 0 \qquad \text{(CS)}$$

The KKT Conditions: Example

- Consider the point $\boldsymbol{x} = \begin{bmatrix} 0 & 0 & 4 & 4 \end{bmatrix}^T$
  - using (CS): $x_j > 0 \Rightarrow v_j = 0$, so $v_3 = v_4 = 0$
  - writing (D) for the slack variables: $\boldsymbol{c}^T - \boldsymbol{w}^T \boldsymbol{A} + \boldsymbol{v}^T = \boldsymbol{0}$

$$0 - w_1 + 0 = 0$$
$$0 - w_2 + 0 = 0$$

  - from this we get $w_1 = w_2 = 0$
  - writing (D) for x_1 and x_2 and using that $\boldsymbol{v}^T \geq \boldsymbol{0}$

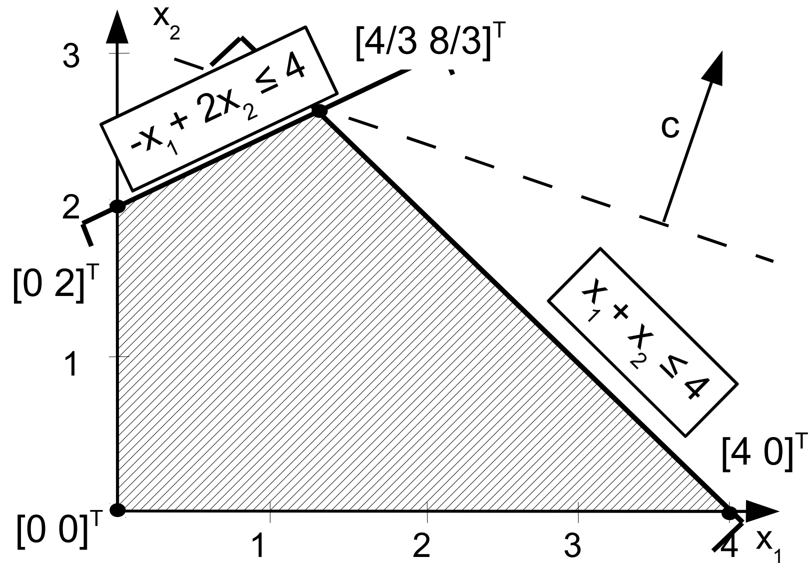
$$1 + w_1 - w_2 \leq 0$$
$$3 - 2w_1 - w_2 \leq 0$$

  - contradiction since $w_1 = w_2 = 0$, so $\boldsymbol{x}$ is not optimal

The KKT Conditions: Example

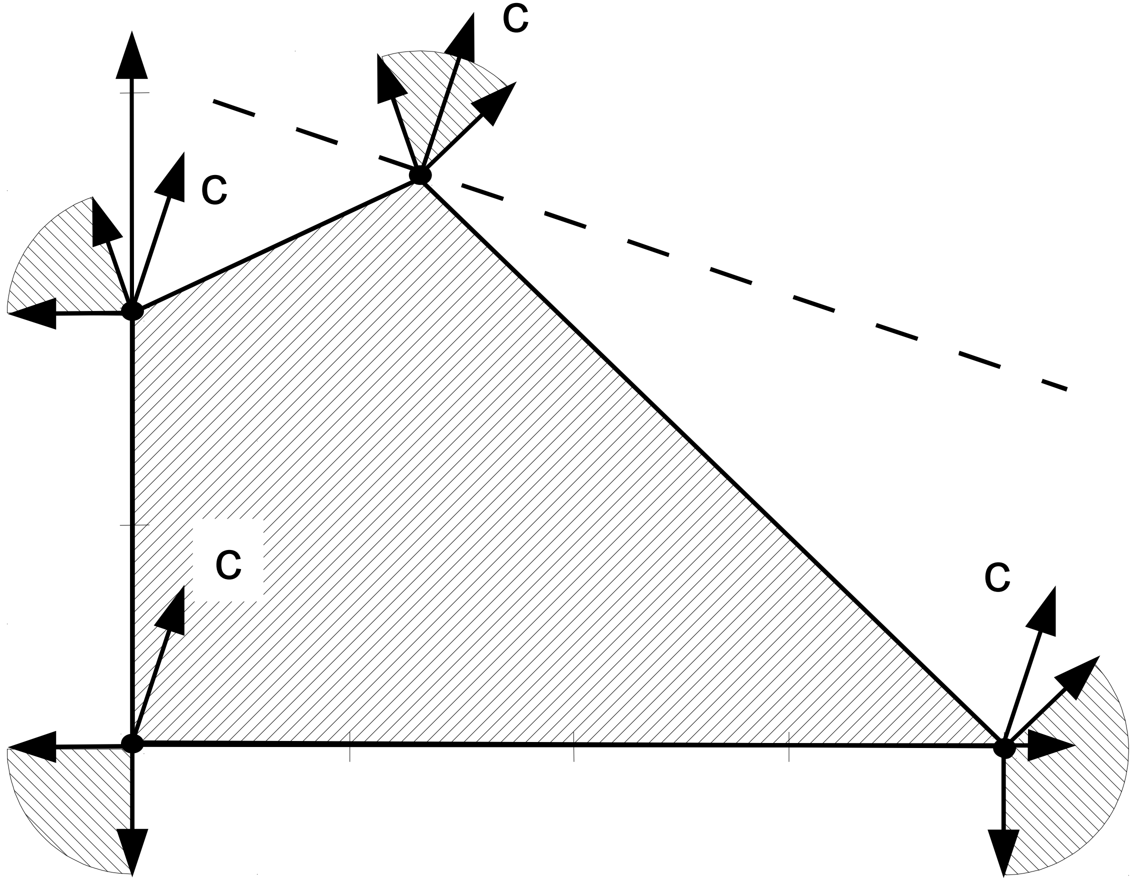
- Now choose $\boldsymbol{x} = [\frac{4}{3} \quad \frac{8}{3}]^T$
  - $x_1 = \frac{4}{3} > 0 \Rightarrow v_1 = 0$, and $x_2 = \frac{8}{3} > 0 \Rightarrow v_2 = 0$
  - the first to rows of (D) (that correspond to x_1 and x_2)

$$\boldsymbol{w}^T \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} = [1 \quad 3]$$



  - from this: $\boldsymbol{w}^T = [\frac{2}{3} \quad \frac{5}{3}]$
  - using the rest of (D): $v_3 = w_1 = \frac{2}{3}$, $v_4 = w_2 = \frac{5}{3}$
  - the KKT conditions hold, so $\boldsymbol{x} = [\frac{4}{3} \quad \frac{8}{3}]^T$ is optimal

The Geometry of the KKT Conditions



- Geometrically, $\boldsymbol{x} = [\frac{4}{3} \quad \frac{8}{3}]^T$ is the only point where $\boldsymbol{c}^{\boldsymbol{T}} = [1 \quad 3]$ can be written as the nonnegative combination of the gradients (normal vectors) of the tight constraints

Primal–dual Relationships

- **Theorem:** the dual of the dual linear program is the primal
- **Proof:** the dual for the canonical form:

$$P: \max \boldsymbol{c}^T\boldsymbol{x} \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b},\ \boldsymbol{x} \ge \boldsymbol{0}$$

$$D: \min \boldsymbol{w}^T\boldsymbol{b} \quad \text{s.t.} \boldsymbol{w}^T\boldsymbol{A} \ge \boldsymbol{c}^T,\ \boldsymbol{w}^T \ge \boldsymbol{0} \quad \equiv \quad -\max\ -\boldsymbol{b}^T\boldsymbol{w} \quad \text{s.t. } -\boldsymbol{A}^T\boldsymbol{w} \le -\boldsymbol{c},\ \boldsymbol{w} \ge \boldsymbol{0}$$

- Taking the dual D^2 of D:

$$D^2: \quad -\min\ -\boldsymbol{x}^T\boldsymbol{c} \quad \text{s.t. } -\boldsymbol{x}^T\boldsymbol{A}^T \ge -\boldsymbol{b}^T,\ \boldsymbol{x}^T \ge \boldsymbol{0} \quad \equiv \quad \max \boldsymbol{c}^T\boldsymbol{x} \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b},\ \boldsymbol{x} \ge \boldsymbol{0}$$

$\square$

Primal–dual Relationships

- Consider the primal–dual pair of linear programs in canonical form:

$$P: \quad \max \boldsymbol{c}^T \boldsymbol{x} \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}$$

$$D: \quad \min \boldsymbol{w}^T \boldsymbol{b} \quad \text{s.t.} \boldsymbol{w}^T \boldsymbol{A} \geq \boldsymbol{c}^T, \quad \boldsymbol{w}^T \geq \boldsymbol{0}$$

- Let $\boldsymbol{x}$ be primal-feasible and let $\boldsymbol{w}^T$ be dual-feasible
  - multiply the primal constraint $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ from the left by $\boldsymbol{w}^T \geq \boldsymbol{0}$: $\boldsymbol{w}^T \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{w}^T \boldsymbol{b}$
  - multiply the dual constraint $\boldsymbol{w}^T \boldsymbol{A} \geq \boldsymbol{c}^T$ from the right by $\boldsymbol{x} \geq \boldsymbol{0}$: $\boldsymbol{w}^T \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{c}^T \boldsymbol{x}$
- Then,

$$\boldsymbol{c}^T \boldsymbol{x} \leq \boldsymbol{w}^T \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{w}^T \boldsymbol{b}$$

The Weak Duality Theorem

- **Theorem:** the objective function value for *any* feasible solution for the primal maximization problem is less than, or equal to the objective function value for *any* feasible solution for the dual minimization problem
- **Proof:** using the above: $\boldsymbol{c}^T\boldsymbol{x} \leq \boldsymbol{w}^T\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{w}^T\boldsymbol{b}$ $\square$
- Note the importance of the *any* quantification: *any* primal-feasible $\boldsymbol{x}$ gives a lower bound $\boldsymbol{c}^T\boldsymbol{x}$ for the dual, and of course *any* dual-feasible $\boldsymbol{w}^T$ gives an upper bound $\boldsymbol{w}^T\boldsymbol{b}$ for the primal
- **Corollaries:**
  - if $\boldsymbol{x}$ is primal-feasible, $\boldsymbol{w}^T$ is dual-feasible, and $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{w}^T\boldsymbol{b}$, then $\boldsymbol{x}$ is optimal in the primal and $\boldsymbol{w}^T$ is optimal in the dual
  - if the primal is unbounded then the dual is infeasible and *vice versa*

Weak Duality: Example

- Consider the previous example:

$$
\begin{array}{llrl}
P: & \max & 6x_1 + 8x_2 & \\
 & \text{s.t.} & 3x_1 + x_2 & \leq 4 \\
 & & 5x_1 + 2x_2 & \leq 7 \\
 & & x_1, \quad x_2 & \geq 0
\end{array}
\qquad
\begin{array}{llrl}
D: & \min & 4w_1 + 7w_2 & \\
 & \text{s.t.} & 3w_1 + 5w_2 & \geq 6 \\
 & & w_1 + 2w_2 & \geq 8 \\
 & & w_1, \quad w_2 & \geq 0
\end{array}
$$

- Choose some primal and dual solution
  - let $\boldsymbol{x} = [\frac{1}{6} \quad 3]^T$ and $\boldsymbol{w}^T = [2 \quad 3]$
  - then, $\boldsymbol{c}^T\boldsymbol{x} = 25$ and $\boldsymbol{w}^T\boldsymbol{b} = 29$, and so for the optimal solution $\bar{\boldsymbol{x}} = \begin{bmatrix}\bar{x}_1 & \bar{x}_2\end{bmatrix}$ of the primal we have the bounds:

$$25 \leq 6\bar{x}_1 + 8\bar{x}_2 \leq 29$$

  - same applies to the dual

A Note on Weak Duality

- If the primal is unbounded then the dual is infeasible
- Similarly, if the dual is unbounded than the primal is infeasible
- This does not hold in the reverse direction: from the infeasibility of the primal it *does not* follow that the dual is unbounded (nor the other way around)
- For instance, the below primal–dual pair of linear programs are both infeasible

$$
\begin{array}{llrcrcr}
P: & \max & 8x_1 & + & 3x_2 & & \\
 & \text{s.t.} & x_1 & - & 6x_2 & \geq & 2 \\
 & & 5x_1 & + & 7x_2 & = & -4 \\
 & & x_1 & & & \leq & 0 \\
 & & & & x_2 & \geq & 0
\end{array}
\qquad
\begin{array}{llrcrcl}
D: & \min & 2w_1 & - & 4w_2 & & \\
 & \text{s.t.} & w_1 & + & 5w_2 & \leq & 8 \\
 & & -6w_1 & + & 7w_2 & \geq & 3 \\
 & & w_1 & & & \leq & 0 \\
 & & & & w_2 & & \text{arb.}
\end{array}
$$

The Strong Duality Theorem

- **Theorem:** for the primal–dual pair of linear programs exactly one of the below claims holds true
  - the primal has an optimal solution $\bar{\boldsymbol{x}}$ and the dual also has an optimal solution $\bar{\boldsymbol{w}}^T$, and $\boldsymbol{c}^T\bar{\boldsymbol{x}} = \bar{\boldsymbol{w}}^T\boldsymbol{b}$
  - one of the problems is unbounded and therefore the other is infeasible
  - neither problem is feasible

P optimal	$\Longleftrightarrow$	D optimal
P unbounded	$\Longrightarrow$	D infeasible
D unbounded	$\Longrightarrow$	P infeasible
P infeasible	$\Longrightarrow$	D unbounded or infeasible
D infeasible	$\Longrightarrow$	P unbounded or infeasible

Duality: Example

- We can use the dual to solve the primal

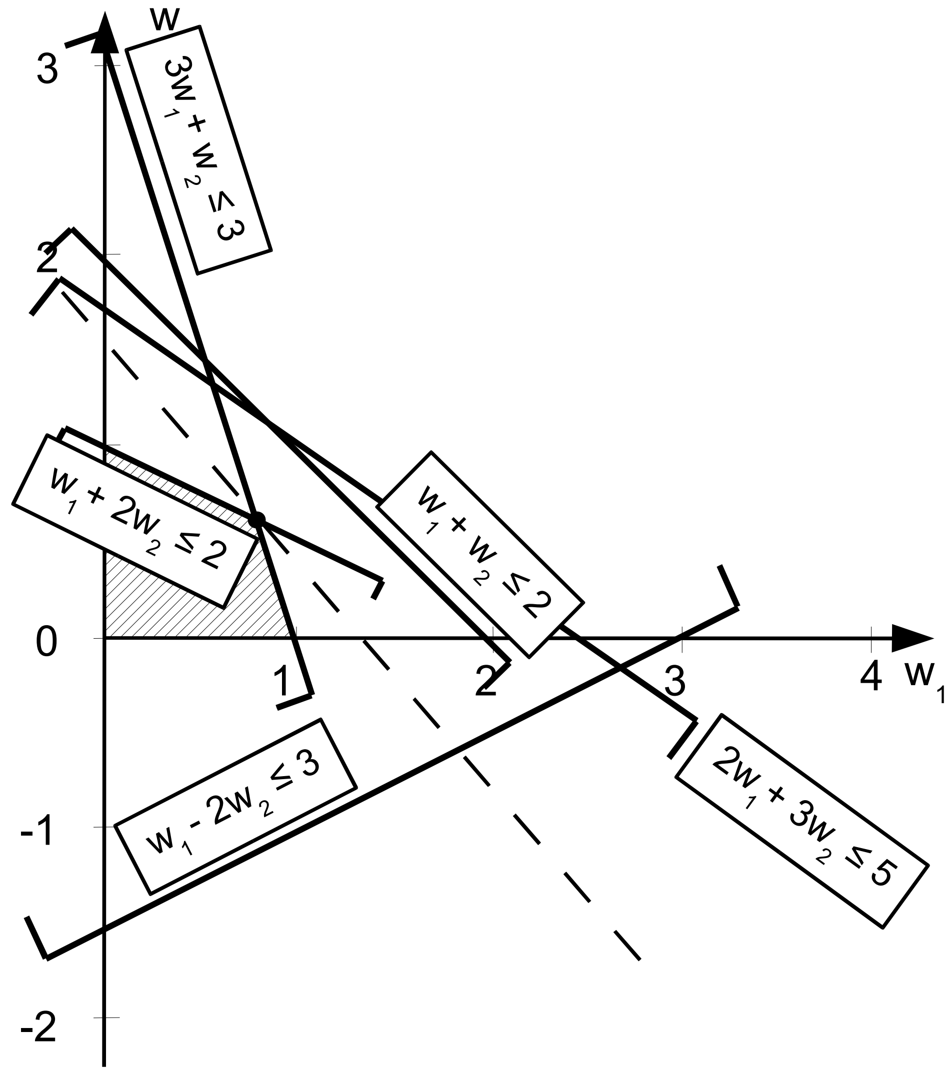
$$
\begin{array}{lrcrcrcrcrcl}
\min & 2x_1 & + & 3x_2 & + & 5x_3 & + & 2x_4 & + & 3x_5 & & \\
\text{s.t.} & x_1 & + & x_2 & + & 2x_3 & + & x_4 & + & 3x_5 & \geq & 4 \\
 & 2x_1 & - & 2x_2 & + & 3x_3 & + & x_4 & + & x_5 & \geq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}
$$

- Only two constraints: the dual has only two variables:

$$
\begin{array}{lrcrcl}
\max & 4w_1 & + & 3w_2 & & \\
\text{s.t.} & w_1 & + & 2w_2 & \leq & 2 \\
 & w_1 & - & 2w_2 & \leq & 3 \\
 & 2w_1 & + & 3w_2 & \leq & 5 \\
 & w_1 & + & w_2 & \leq & 2 \\
 & 3w_1 & + & w_2 & \leq & 3 \\
 & w_1, & & w_2, & \geq & 0
\end{array}
$$

Duality: Example

- Solve the dual graphically
- The optimal solution: $\bar{\boldsymbol{w}}^T = [\frac{4}{5} \quad \frac{3}{5}]$ and $z_0 = 5$
- We immediately know that the primal optimum is 5 by the Strong Theorem
- We could also obtain the primal solution itself
- We do not discuss that here



Duality: Example

- Solve the below linear program

$$
\begin{array}{lrcrcrcr}
\max & -5x_1 & - & 2x_2 & - & x_3 & & \\
\text{s.t.} & -x_1 & - & 2x_2 & & & \leq & 1 \\
 & -2x_1 & - & 2x_2 & & & \leq & 3 \\
 & -5x_1 & + & x_2 & - & x_3 & \leq & -5 \\
 & 5x_1 & + & 3x_2 & - & x_3 & \leq & -2 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}
$$

- In standard form:

$$
\begin{array}{lccccccccc}
\max & -5x_1 & -2x_2 & -x_3 & & & & & & \\
\text{s.t.} & -x_1 & -2x_2 & & +x_4 & & & & = & 1 \\
 & -2x_1 & -2x_2 & & & +x_5 & & & = & 3 \\
 & -5x_1 & +x_2 & -x_3 & & & +x_6 & & = & -5 \\
 & 5x_1 & +3x_2 & -x_3 & & & & +x_7 & = & -2 \\
 & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq & 0
\end{array}
$$

Duality: Example

- Find an initial feasible basis
- The trivial choice would be to choose the columns of the slack variables into the initial basis, in particular if $B = \{x_4, x_5, x_6, x_7\}$ then $\boldsymbol{B} = \boldsymbol{B}^{-1} = \boldsymbol{I}_4$
- Unfortunately, this trivial basis is not (primal) feasible, since $\bar{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{b} \ngeq \mathbf{0}$
- Let us write the dual, in the hope that it will be easier to find an initial basis for that

$$
\begin{array}{lrcrcrcrcr}
\min & w_1 & + & 3w_2 & - & 5w_3 & - & 2w_4 & & \\
\text{s.t.} & -w_1 & - & 2w_2 & - & 5w_3 & + & 5w_4 & \geq & -5 \\
 & -2w_1 & - & 2w_2 & + & w_3 & + & 3w_4 & \geq & -2 \\
 & & & & - & w_3 & - & w_4 & \geq & -1 \\
 & w_1, & & w_2, & & w_3, & & w_4 & \geq & 0
\end{array}
$$

Duality: Example

- Converting to standard form and rewriting the objective as a maximization problem (note to ourselves: we'll need to invert the resultant objective function due to the $\min \Rightarrow \max$ conversion!)

$$
\begin{array}{llllllllll}
\max & -w_1 & -3w_2 & +5w_3 & +2w_4 & & & & & \\
\text{s.t.} & -w_1 & -2w_2 & -5w_3 & +5w_4 & -w_5 & & & = & -5 \\
 & -2w_1 & -2w_2 & +w_3 & +3w_4 & & -w_6 & & = & -2 \\
 & & & -w_3 & -w_4 & & & -w_7 & = & -1 \\
 & w_1, & w_2, & w_3 & w_4, & w_5, & w_6, & w_7 & \geq & 0
\end{array}
$$

- The slack variables form an initial feasible basis, as

$$\boldsymbol{B} = \boldsymbol{B}^{-1} = -\boldsymbol{I}_3 \text{ and } \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \geq \boldsymbol{0}$$

- We can use the (primal) simplex from here

Duality: Example

- The initial simplex tableau:

	z	w_1	w_2	w_3	w_4	w_5	w_6	w_7	RHS
z	1	1	3	−5	−2	0	0	0	0
w_5	0	1	2	$\boxed{5}$	−5	1	0	0	5
w_6	0	2	2	−1	−3	0	1	0	2
w_7	0	0	0	1	1	0	0	1	1

- Recall the pivot rules
  - optimality condition: $z_k = \min_{j \in N} z_j \geq 0$
  - k enters the basis, if $k = \operatorname{argmin}_{j \in N} z_j$
  - r leaves the basis, if $r = \underset{i \in \{1,\ldots,m\}}{\operatorname{argmin}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$
- So w_3 enters and w_5 (or w_7) leaves the basis

Duality: Example

- After the first pivot

	z	w_1	w_2	w_3	w_4	w_5	w_6	w_7	RHS
z	1	2	5	0	-7	1	0	0	5
w_3	0	$\frac{1}{5}$	$\frac{2}{5}$	1	-1	$\frac{1}{5}$	0	0	1
w_6	0	$\frac{11}{5}$	$\frac{12}{5}$	0	-4	$\frac{1}{5}$	1	0	3
w_7	0	$-\frac{1}{5}$	$-\frac{2}{5}$	0	$\boxed{2}$	$-\frac{1}{5}$	0	1	0

- w_4 enters and w_7 leaves the basis: degenerate pivot

	z	w_1	w_2	w_3	w_4	w_5	w_6	w_7	RHS
z	1	$\frac{13}{10}$	$\frac{18}{5}$	0	0	$\frac{3}{10}$	0	$\frac{7}{2}$	5
w_3	0	$\frac{1}{10}$	$\frac{1}{5}$	1	0	$\frac{1}{10}$	0	$\frac{1}{2}$	1
w_6	0	$\frac{9}{5}$	$\frac{8}{5}$	0	0	$-\frac{1}{5}$	1	2	3
w_4	0	$-\frac{1}{10}$	$-\frac{1}{5}$	0	1	$-\frac{1}{10}$	0	$\frac{1}{2}$	0

Duality: Example

- The optimal dual solution: $\boldsymbol{w}^T = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 3 \quad 0]$
- The objective function value is -5, since we must invert the result due to the $\min \Rightarrow \max$ objective function conversion
- This is the optimum of the primal as well (Strong Theorem)
- For the optimal primal solution we need to work a bit, in that we must calculate $\mathbf{x} = \boldsymbol{c_B}^T \boldsymbol{B}^{-1}$
- Since $B = \{w_3, w_4, w_6\}$, so $\boldsymbol{B} = \begin{bmatrix} -5 & 5 & 0 \\ 1 & 3 & -1 \\ -1 & -1 & 0 \end{bmatrix}$
- From this: $\boldsymbol{B}^{-1} = \begin{bmatrix} -\frac{1}{10} & 0 & -\frac{1}{2} \\ \frac{1}{10} & 0 & -\frac{1}{2} \\ \frac{1}{5} & -1 & -2 \end{bmatrix}$
- Finally: $\mathbf{x} = \boldsymbol{c_B}^T \boldsymbol{B}^{-1} = [\frac{3}{10} \quad 0 \quad \frac{7}{2}]$

The Farkas Lemma

- **Theorem:** given matrix $\boldsymbol{A}$ $(m \times n)$ and vector $\boldsymbol{b}$ (column m-vector), precisely one of the below claims hold:
  1.) exists $\boldsymbol{x}$ so that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$, or
  2.) exists $\boldsymbol{w}^T$ so that $\boldsymbol{w}^T\boldsymbol{A} \geq \boldsymbol{0}$ and $\boldsymbol{w}^T\boldsymbol{b} < 0$
- **Proof:** consider the primal–dual pair of linear programs

$$P: \quad \max \boldsymbol{0}\boldsymbol{x} \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}$$

$$D: \quad \min \boldsymbol{w}^T\boldsymbol{b} \quad \text{s.t.} \boldsymbol{w}^T\boldsymbol{A} \geq \boldsymbol{0}, \quad \boldsymbol{w}^T \text{ arbitrary}$$

- If (1) holds, i.e., when $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ is feasible, then the primal optimum is 0
- The primal optimum 0 is a lower bound for the dual objective for *any* dual solution: $0 \leq \boldsymbol{w}^T\boldsymbol{b}$ (Weak Theorem)
- This contradicts $\boldsymbol{w}^T\boldsymbol{b} < 0$, thus (2) cannot hold

The Farkas Lemma

- The reverse direction: if (1) does not hold, i.e., when $\boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}$ is infeasible, then the primal (P) is infeasible
- Due to the Strong Theorem, the dual is either unbounded or infeasible
- Observe that the dual is trivially feasible, since at least $\boldsymbol{w}^T = \mathbf{0}$ is a solution
- Thus, the dual in unbounded, so it is feasible and (2) holds $\square$
- The Farkas lemma is a seemingly innocuous result, yet it underlies basically the entire field of mathematical programming
- This time we have proved the Farkas lemma using linear programming duality
- We could have gone the other way around: in fact, the Farkas lemma predates linear programming theory