Dynamic Games with General Time Preferences

Ichiro Obara Jaeok Park

UCLA Yonsei University

November 21, 2014

Motivation

- We almost always use the DU ("discounted utility") model with geometric/exponential discounting in dynamic decision problems/dynamic games (Samuelson 1937).
- There are many empirical and experimental evidences that are at odds with the DU model.
- "nonstandard" time preferences have been introduced. One example: $\beta - \delta$ discounting. It is very simple and tractable.

Motivation

- Our goal is to provide a framework in which we can study dynamic decision problems/dynamic games with as many time preferences as possible.

Motivation

- Our goal is to provide a framework in which we can study dynamic decision problems/dynamic games with as many time preferences as possible.
- Why is this interesting?
  - We can examine the robustness of results based on a particular "nonstandard" time preference.
  - We may be able to find a new useful class of time preferences.
  - Given a time preference in continuous time, we like to study discrete time dynamic games with different period length by changing a frequency of plays.

- For today's talk, I show that a recursive characterization of equilibrium payoffs is possible for some class of time preferences for dynamic games (and dynamic decision problems).

- For today's talk, I show that a recursive characterization of equilibrium payoffs is possible for some class of time preferences for dynamic games (and dynamic decision problems).
- Which class? Time preferences that are eventually recursive.

Some Examples

- **General Discounting**

$$W_i(g_i^\infty) = g_i^1 + \beta_1 \delta g_i^2 + ... + \beta_{K-1} \delta^{K-1} g_i^K + \beta_K \delta^K \left[\sum_{t=1}^{\infty} \delta^{t-1} g_i^{K+t} \right]$$

  - geometric discounting when $\beta_1 = \beta_2 = ... = \beta_K = 1$.
  - "quasi-hyperbolic" $(\beta - \delta)$ discounting when $\beta_1 = ... = \beta_K = \beta \in (0, 1)$.
  - Hyperbolic discounting.

Special Case: 1 player

- We first consider a special case with 1 player. We consider usual environments for infinite horizon dynamic programming problem with deterministic transition.
  - State $s^t \in S$. A player takes an action $a^t \in A$ in period t, which generates payoff $g(s^t, a^t)$ and determines the next state $s^{t+1} = f(s^t, a^t)$.
  - S and A are compact and g and f are continuous. Let $g(s, a) \in [0, \overline{g}]$.

Assumption on Behavior/Equilibrium

- What would a player with general time preference ($W(g^1, g^2, ...)$) do? What is optimal for you today may not be optimal for your tomorrow-self (who has the same W)
- This problem is not a simple optimization problem, it is like a game among multiple selves.
- We assume that each player at each history takes a strategy of future selves as given. Essentially we treat a player at different histories as different players.

Standard Case with Geometric Discounting: Bellman Equation

$$V(s^t) = \max_{a^t \in A} g(s^t, a^t) + \delta V(f(s^t, a^t))$$

Special Case: 1 player

- To be concrete, assume $W(g^{\infty}) = g^1 + \beta_1 \delta g^2 + \beta_2 \delta^2 (g^3 + \delta g^4 + ...)$.
- Two things to note:
  - Typically no unique optimal plan and multiple equilibria. Thus no value function. Maybe we need a value correspondence $V(s) \subset \left[0, \frac{\overline{g}}{1-\delta}\right]$?
  - We would like a value correspondence to be recursive/self-generating. How to do it?

Special Case: 1 player

- **Main Idea:** We consider a slightly more complicated correspondence $Z(s) \subset \left(A \times \left[0, \frac{\overline{g}}{1-\delta}\right]\right)$, which is a set of all possible pairs of a current action and a continuation payoff evaluated by geometric discounting (which we call **continuation score**) given current state s.
- Now consider the following problem for each s given some Z:

$$\max_{a^1 \in A} g(s, a^1) + \beta_1 \delta^1 g(f(s, a^1), a^2(s, a^1)) + \beta_2 \delta^2 V(s, a^1)$$

, where $\left(a^2(s, a^1), V(s, a^1)\right) \in Z(f(s, a^1))$.

- The solution of this provides a new $\widetilde{Z}(s)$: the set of all possible pairs of a^1 and a continuation score $V = g(f(s, a^1), a^2(s, a^1)) + \delta V(s, a^1)$.

Special Case: 1 player

- Let $Z^*(s)$ be the set of all possible pairs of a current action and a continuation score given state s that can arise in equilibrium.
- "**Theorem**": Z^* is the largest fixed point of $Z \to \widetilde{Z}$.
- Once we have Z^*, then we can characterize all the equilibrium payoffs given s by solving:

$$\max_{a^1 \in A} g(s, a^1) + \beta_1 \delta^1 g(f(s, a^1), a^2(s, a^1)) + \beta_2 \delta^2 V(s, a^1)$$

where $\left(a^2(s, a^1), V(s, a^1)\right) \in Z^*(f(s, a^1))$.

Dynamic Games

- We can extend this to dynamic games.
  - n players with action sets $A_i, i = 1, .., n$.
  - **Payoff:** $g_i(s, a)$, **Transition:** $f(s, a)$

K-Recursive Preference

- Our main assumption about time preferences is **K-Recursivitiy**.

K-Recursivity

W_i is K-**recursive** if there exists a function G_i and a recursive function $\widehat{W}_i$ such that $W_i(g^\infty) = G_i(g^{1,K}, \widehat{W}_i(g^{K+1,\infty}))$.

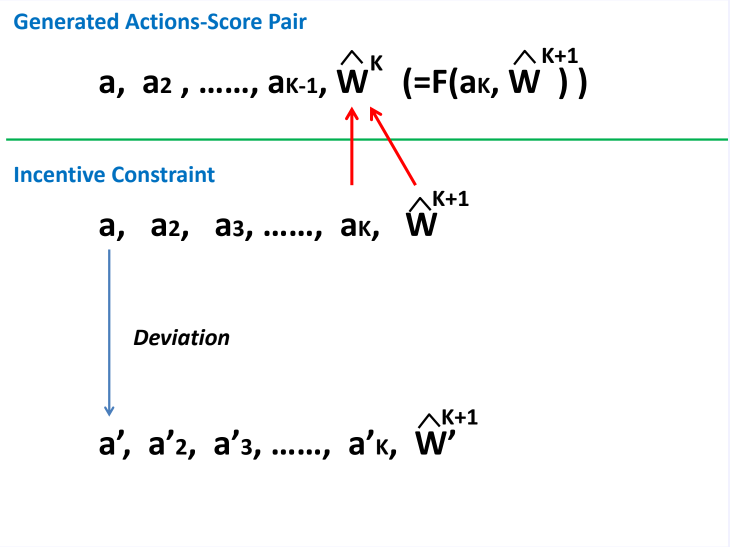
- $\widehat{W}_i$ is **recursive** if there exists a function F_i such that $\widehat{W}_i(g^\infty) = F_i(g^1, \widehat{W}_i(g^{2,\infty}))$.
- In addition, continuity, monotonicity etc. are assumed.
- Let $V^\dagger \subset \mathbb{R}^n$ be a bounded set that contains the range of $\widehat{W} = (\widehat{W}_1, ..., \widehat{W}_n)$.

- **Idea:** W_i depends on the first K period payoffs and a recursive summary statistic (**score**) of all future payoffs from period $K+1$.

Rough Summary

- Assume K-recursive time preferences.
- What can be supported today depends on the set of all possible pairs of the $K-1$ action profiles from the 2nd period and continuation scores from the $K+1$st period given each state s in the next period. Denote this set by $Z(s) \subset \left(A^{K-1} \times V^{\dagger}\right)$
- Given $Z(s)$, we can derive the set of all possible pairs of the $K-1$ action profiles from the current period and continuation scores from the Kth period given current s. Thus we have a mapping from Z to $\widetilde{Z}$.
- Let $Z^{*}(s)$ be the set of all possible pairs of the $K-1$ action profiles from the current period and continuation scores from the Kth period given state s that can arise in equilibrium.
- Z^{*} is the largest fixed point of the mapping $Z \to \widetilde{Z}$.

Some special case with no state (i.e. repeated game).



Future Research

- There are many things we can do.
  - Use different equilibrium concepts.
  - Characterization of Markov equilibrium payoffs.
  - Existence.
  - Apply our method to well-known problems (ex. intergenerational altruism).