Dynamical systems in neuroscience

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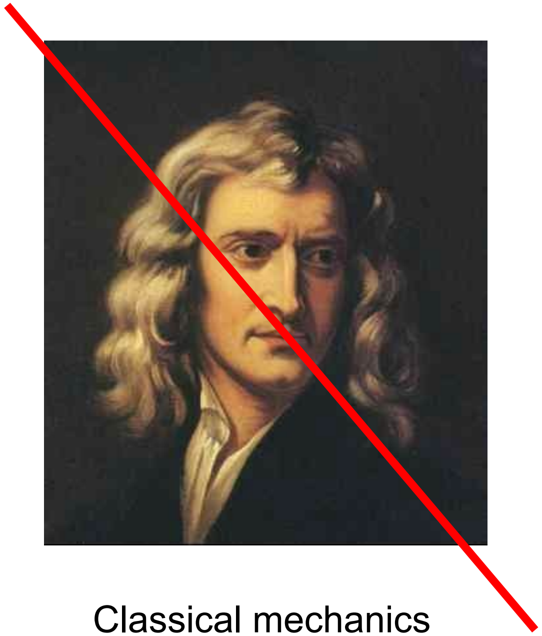
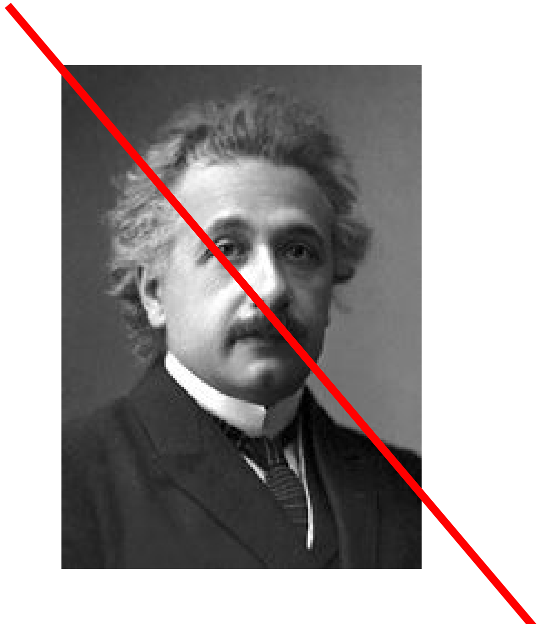
Purpose: impart some basic intuition, introduction to common models and provide terminology

1. Comment on models
2. Introduction to dynamical systems
3. Linear dynamical systems
4. Stochastics
5. Evolution of probability density
6. Leaky integrate and fire neurons

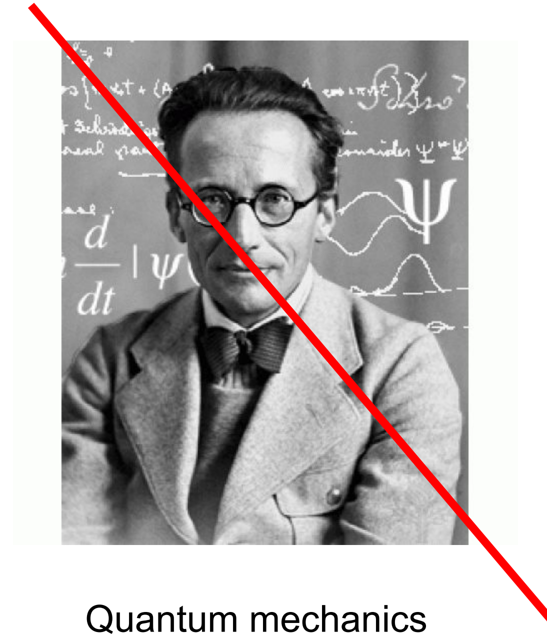
Comment on models



Classical mechanics



Relativistic mechanics

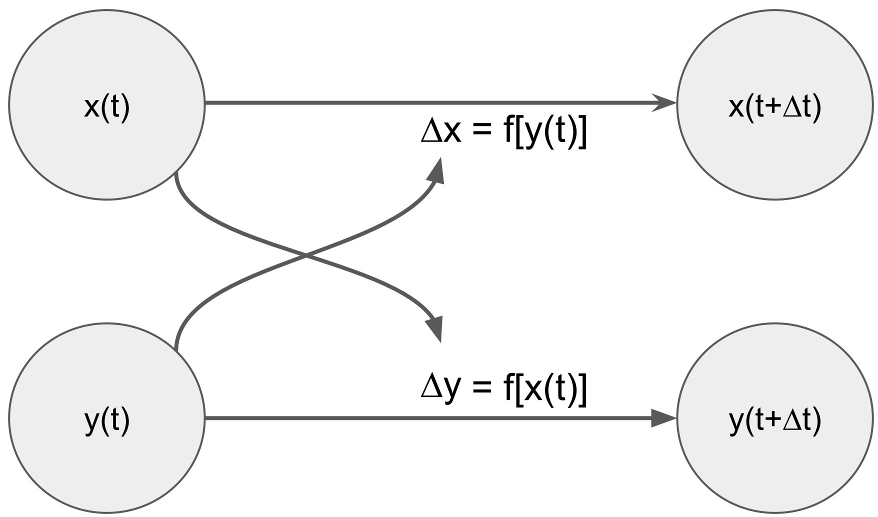


Quantum mechanics

No model is “right”! Models are only *useful* or *not useful* in making predictions and providing insight

What is meant by “dynamical system”?

Quite simply, a set of variables with time-dependence:



Dynamic systems and differential equations

$$x(t + \Delta t) = x(t) + \Delta x$$

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{\Delta x}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}$$

$$\frac{dx}{dt} = f\left[x(t), y(t), t\right]$$

Simple, linear system

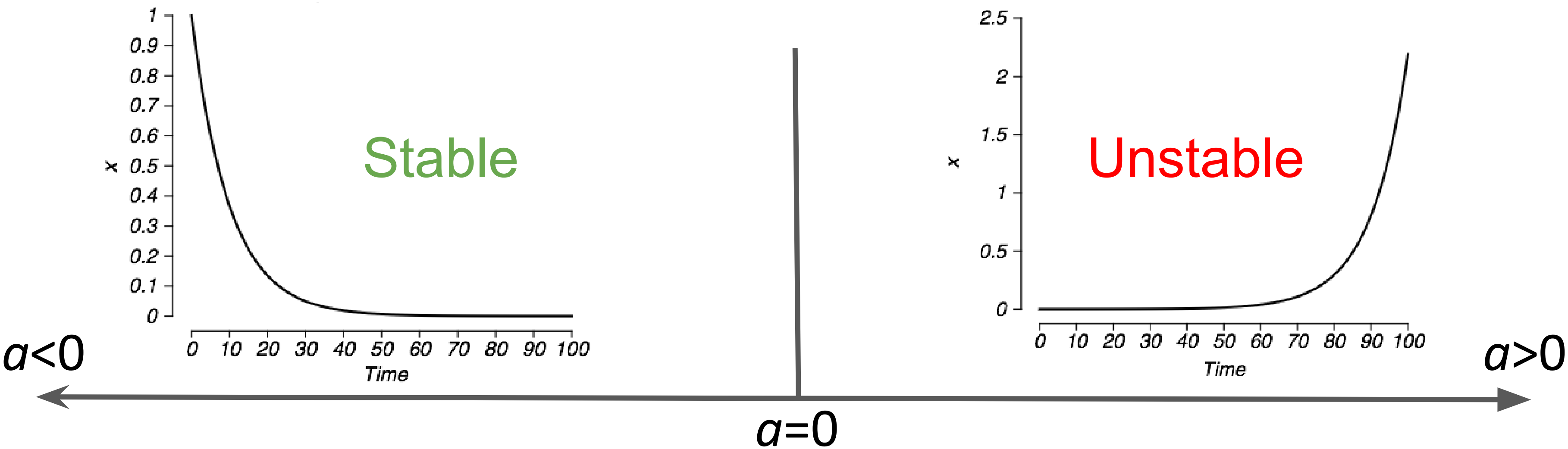
$$\frac{dx}{dt} = ax(t)$$

$$\frac{dx}{ax} = dt$$

$$\int \frac{dx}{ax} = \int dt$$

$$\ln x(t) - \ln x(0) = at$$

$$x(t) = x(0)e^{at}$$



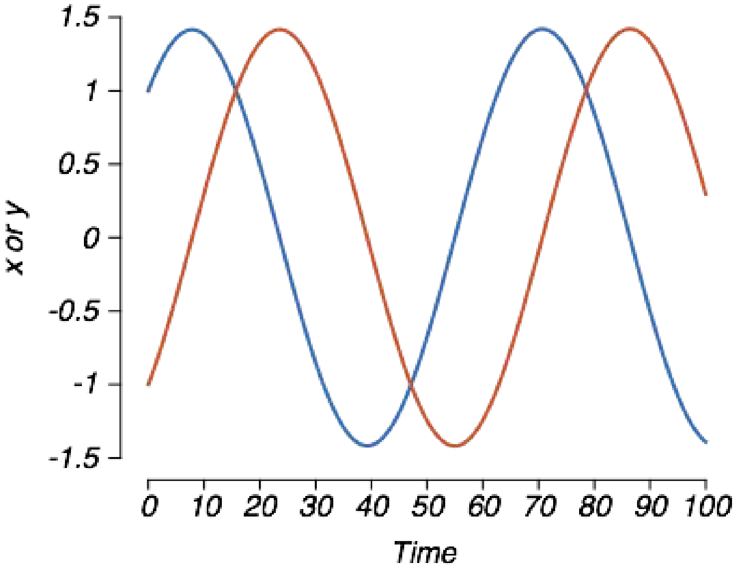
System of two variables

$$\frac{dx}{dt} = ax(t) + by(t)$$

$$\frac{dy}{dt} = cx(t) + dy(t)$$

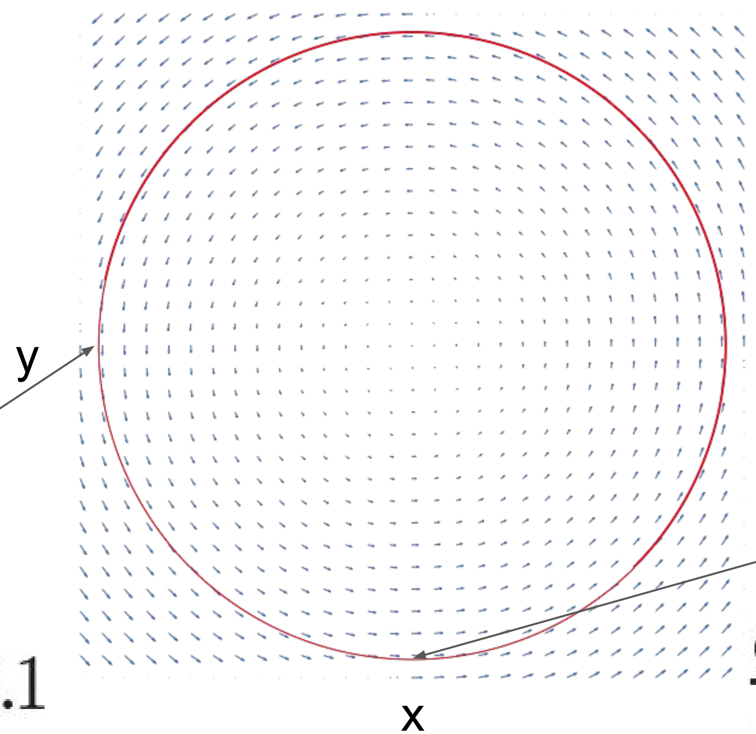
$a = 0 \quad b = -0.1$

$c = 0.1 \quad d = 0$



Phase plane of the system

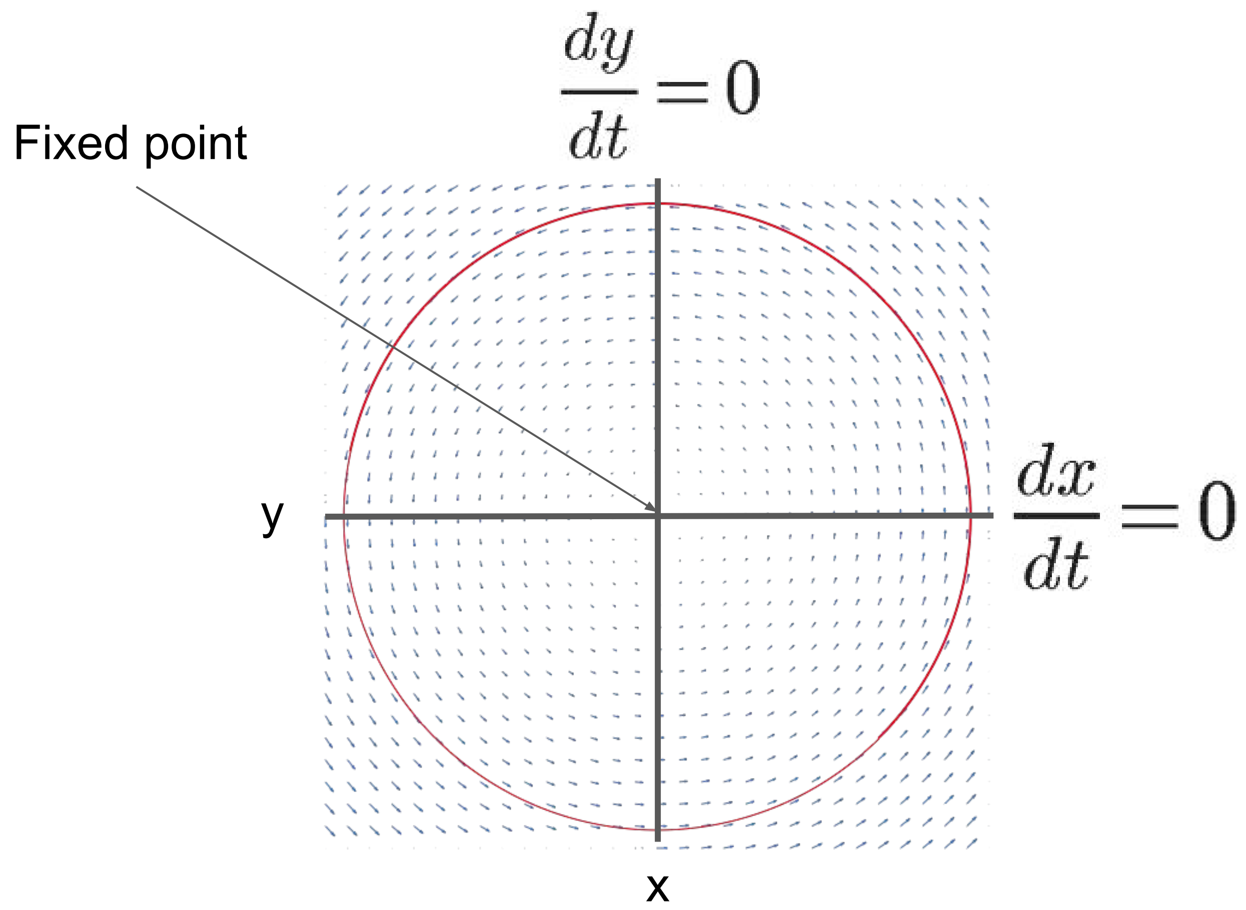
$$\frac{dx}{dt} = ax(t) + by(t) \qquad \frac{dy}{dt} = cx(t) + dy(t)$$



$$x = -1$$
$$y = 0$$
$$\frac{dx}{dt} = 0 \quad \frac{dy}{dt} = -0.1$$

$$x = 0$$
$$y = -1$$
$$\frac{dx}{dt} = 0.1 \quad \frac{dy}{dt} = 0$$

Nullcline: points in space where the value of a variable does *not* change with time



Solutions of linear differential equations

$$\frac{dx}{dt} = ax(t) + by(t)$$

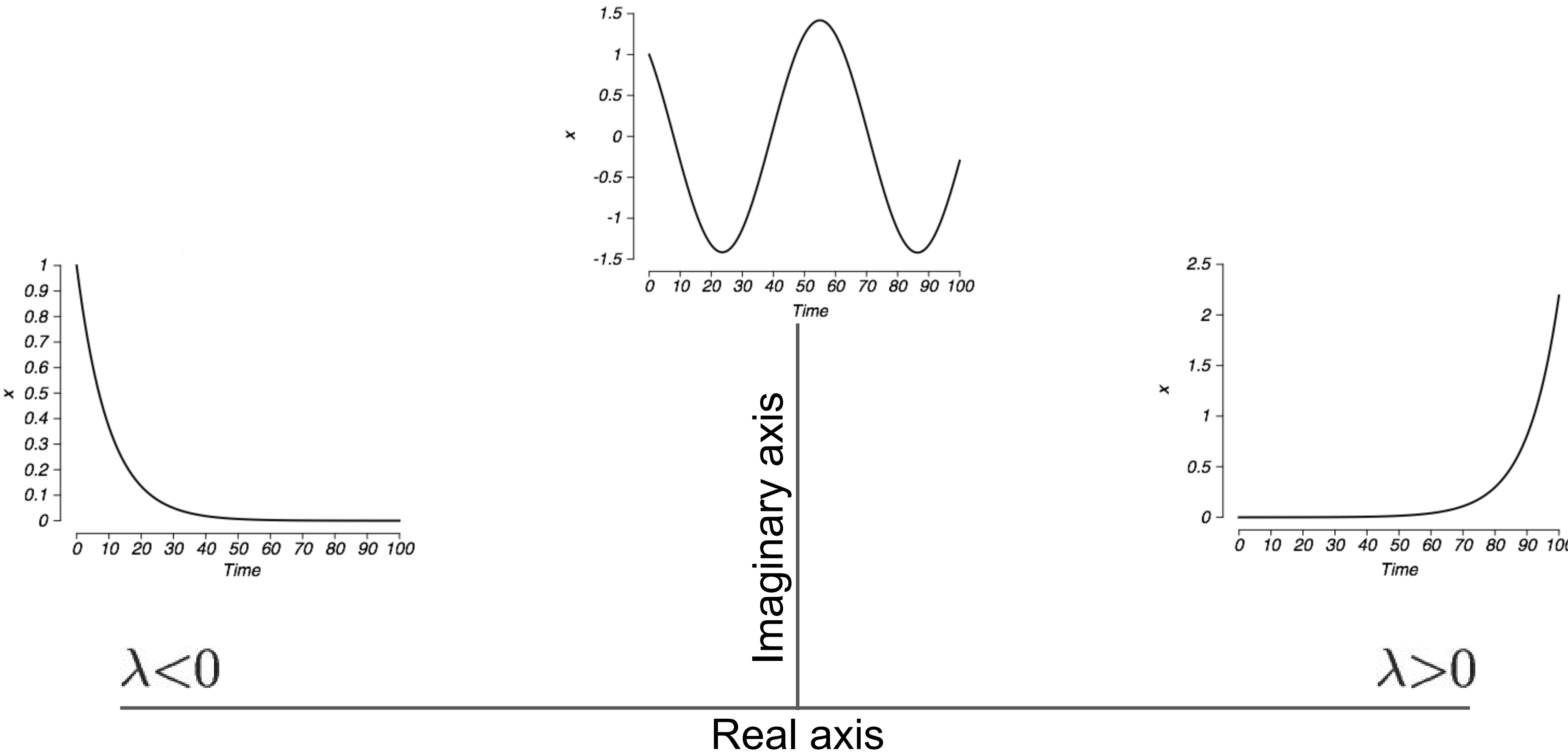
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{dy}{dt} = cx(t) + dy(t)$$

$$\boldsymbol{v}' = \boldsymbol{M}\boldsymbol{v}$$

Generalizes differential equations to arbitrary number of dimensions

Solutions of linear dynamical systems: Eigenvalues of the matrix, $\boldsymbol{M}$



Caution!

- Biological systems are not generally linear!
- Linear models often provide a good *approximation* to complex systems
- Studying linear models provides intuition about dynamical systems

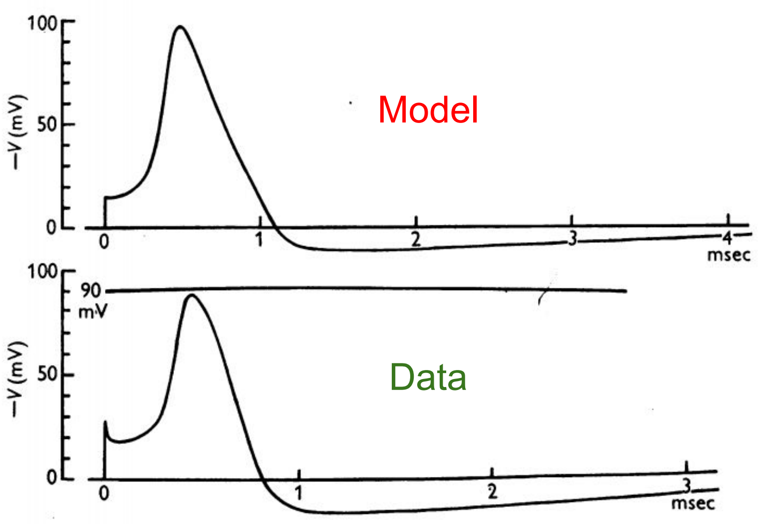
Example from neuroscience

$$\frac{dV}{dt} = -\frac{1}{C_M}\{\bar{g}_K n^4 (V - V_K) + \bar{g}_{Na} m^3 h (V - V_{Na}) + \bar{g}_l (V - V_l)\} + \frac{z}{K}$$

$$dn/dt = \alpha_n(1-n) - \beta_n n,$$

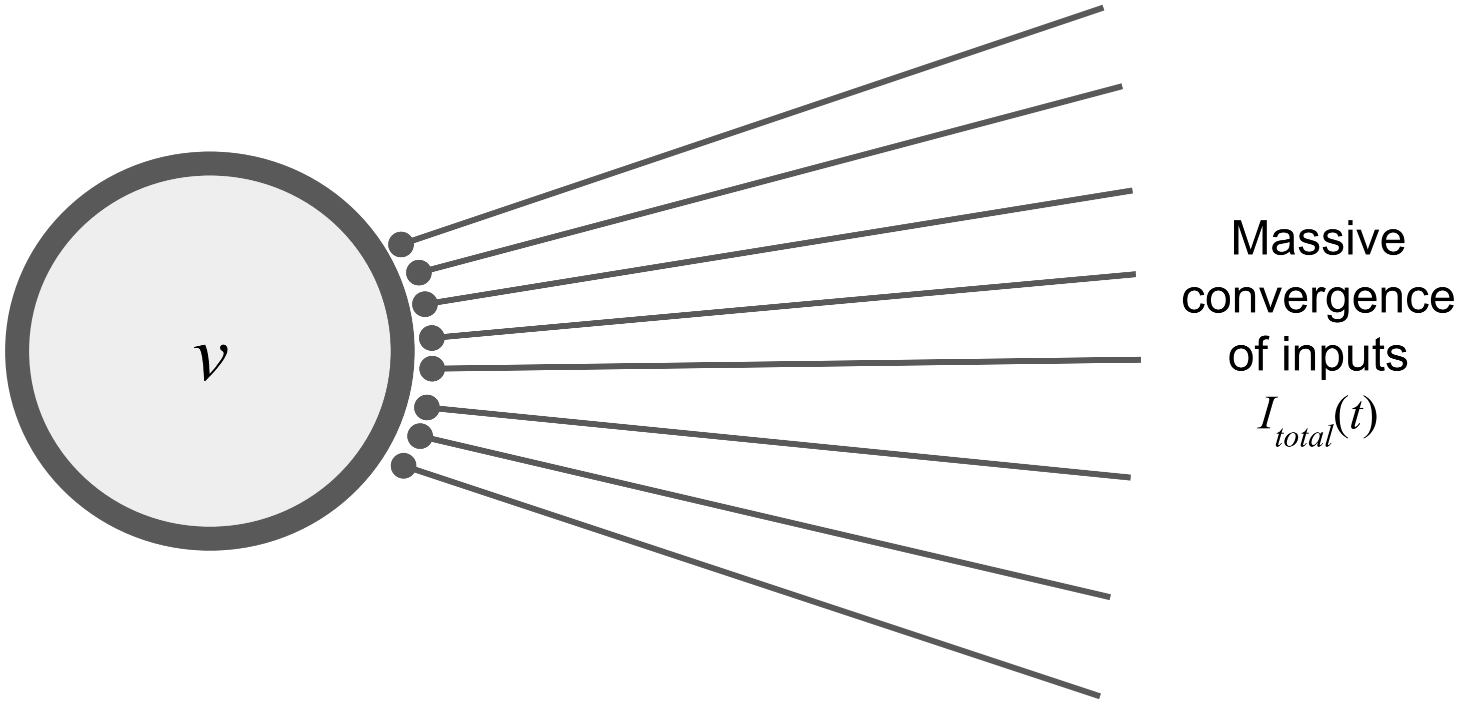
$$dm/dt = \alpha_m(1-m) - \beta_m m,$$

$$dh/dt = \alpha_h(1-h) - \beta_h h,$$

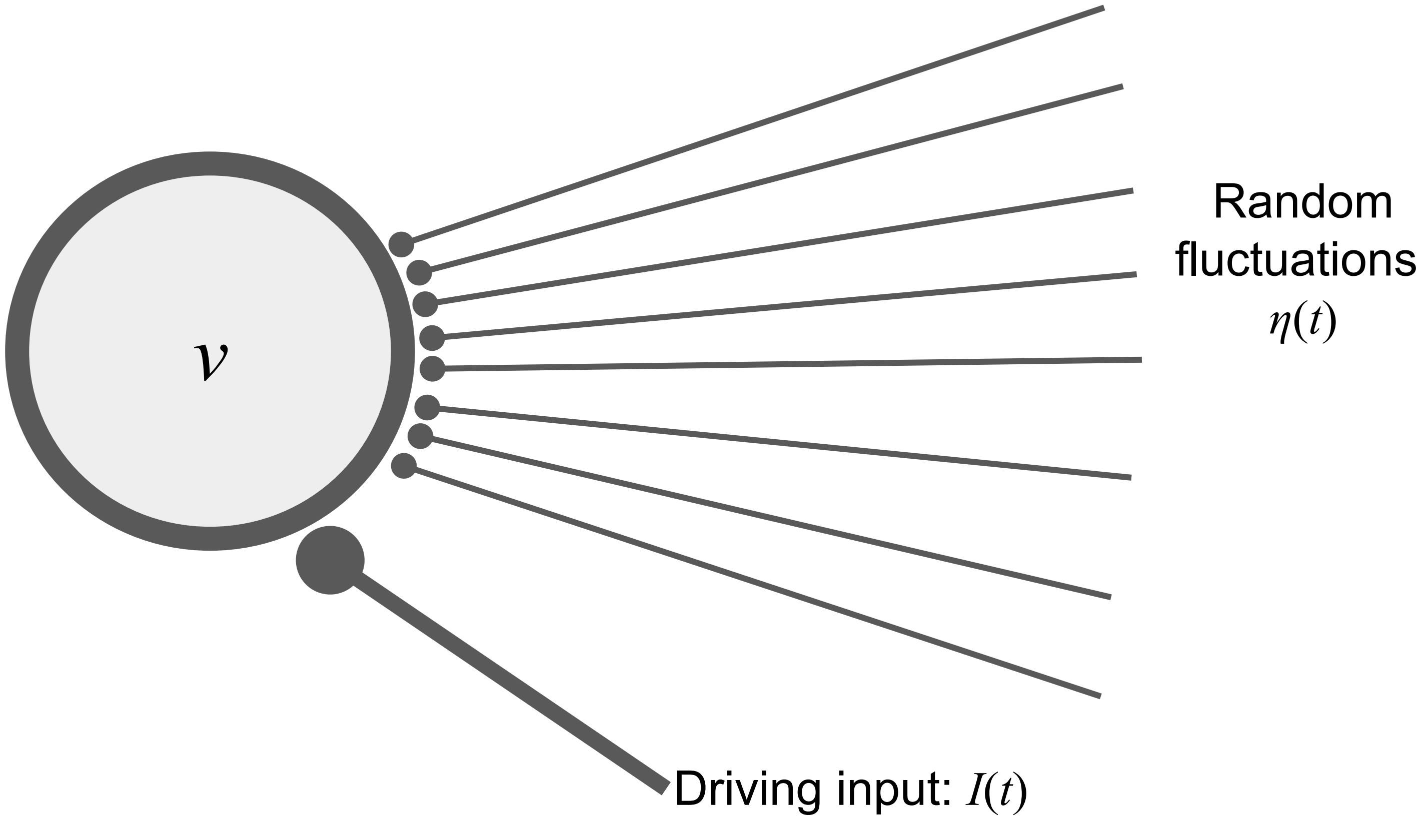


Hodgkin and Huxley (1952)

Stochasticity and dynamic systems



Stochasticity and dynamic systems



$$I_{total}(t) = I(t) + \eta(t)$$

Stochastic differential equations

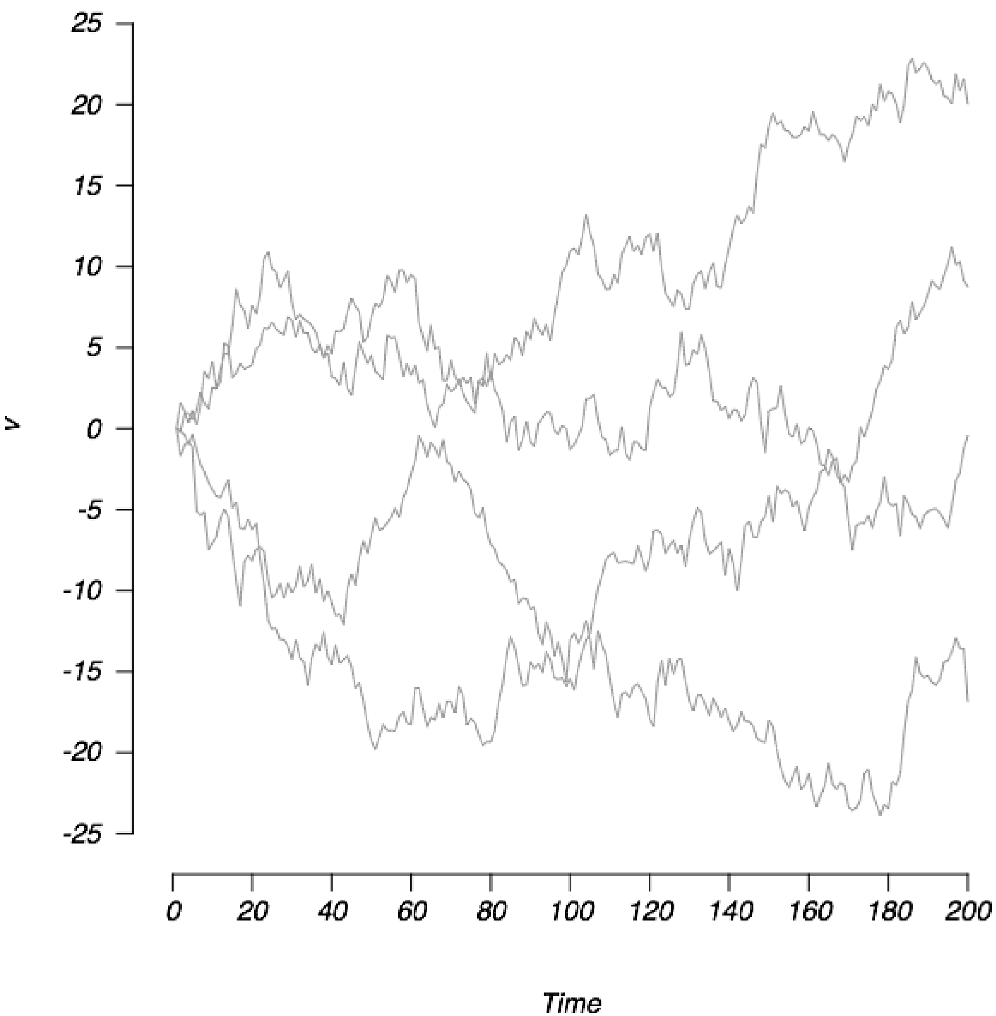
$$\frac{dv}{dt} = f(v,t) + g(v,t)\eta(t)$$

$$\eta(t) \sim N(0,\Delta t)$$

Ornstein-Uhlenbeck process

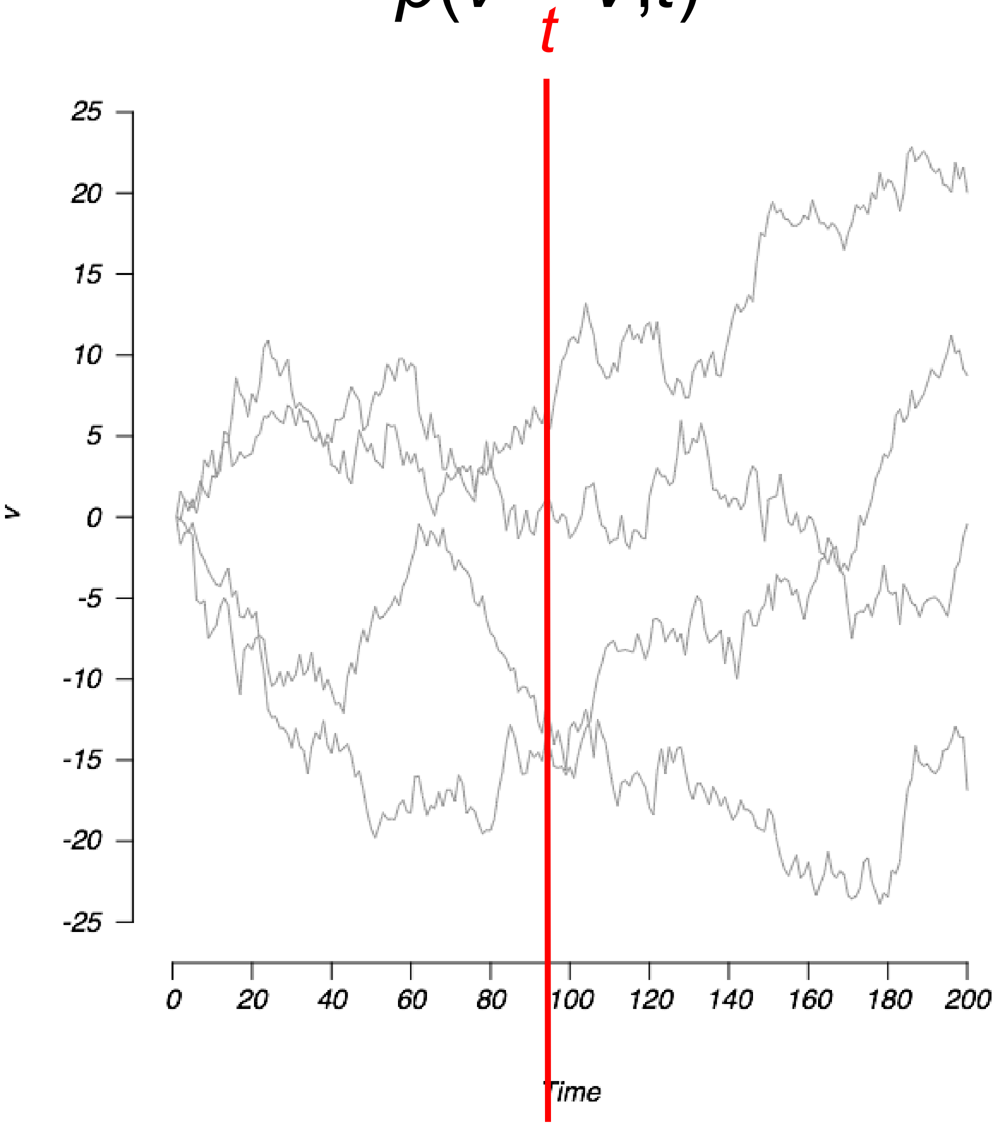
$$\frac{dv}{dt} = -av(t) + b\eta(t)$$

Ornstein-Uhlenbeck process: just noise

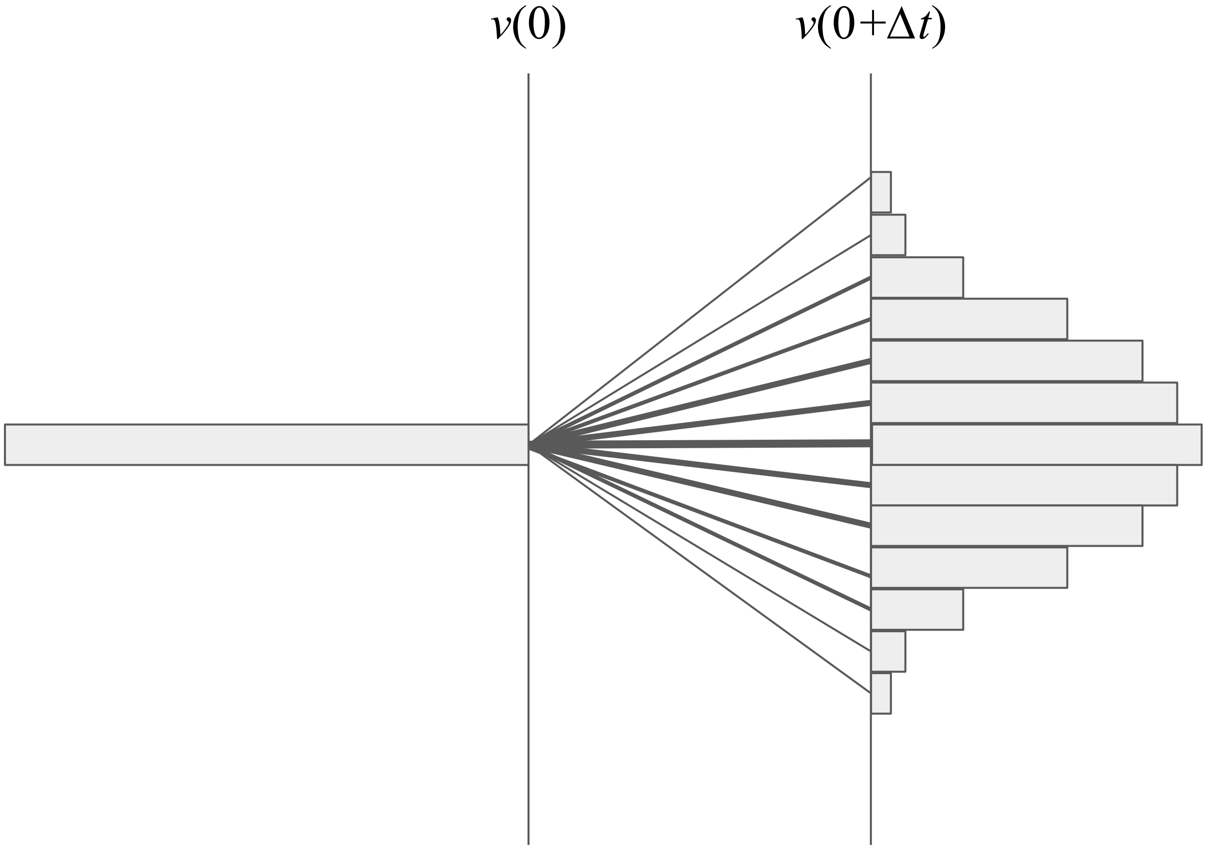


What is the probability of $v = V$ at some time, t?

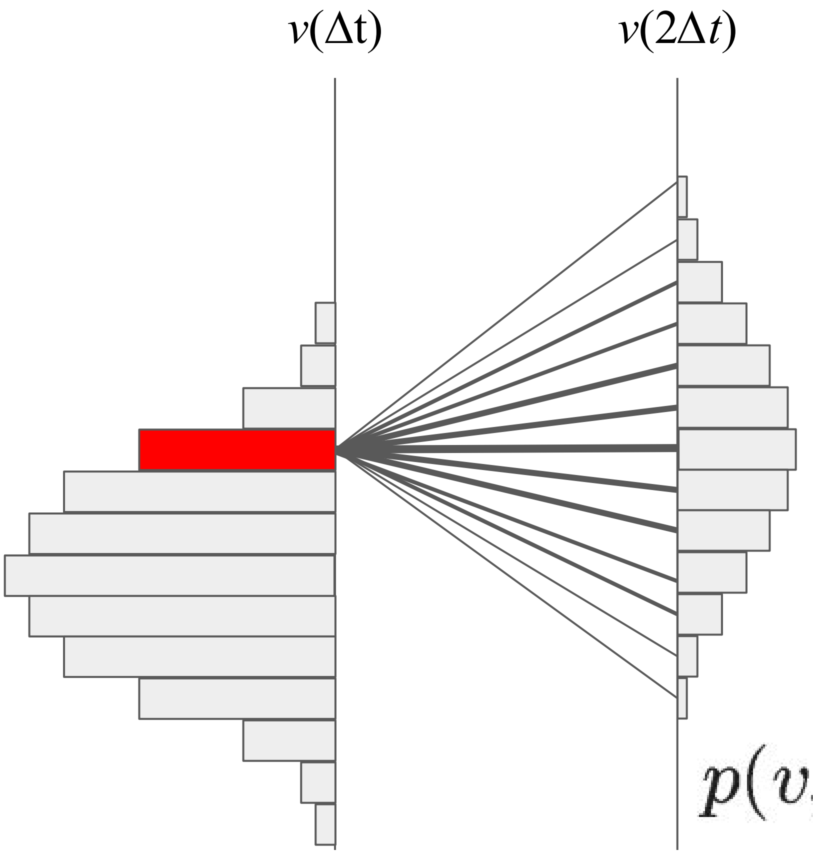
$p(v = V,t)$



Distribution of *v* a short time after initiation of the process: p(v,Δt)

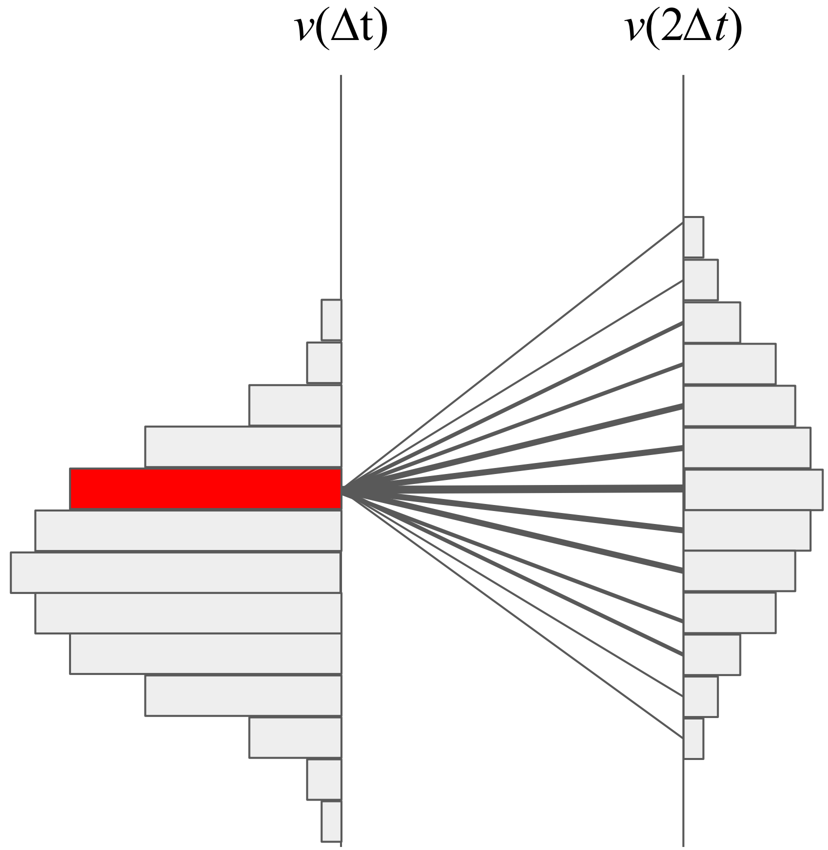


What is the distribution of *v* at the next time step?



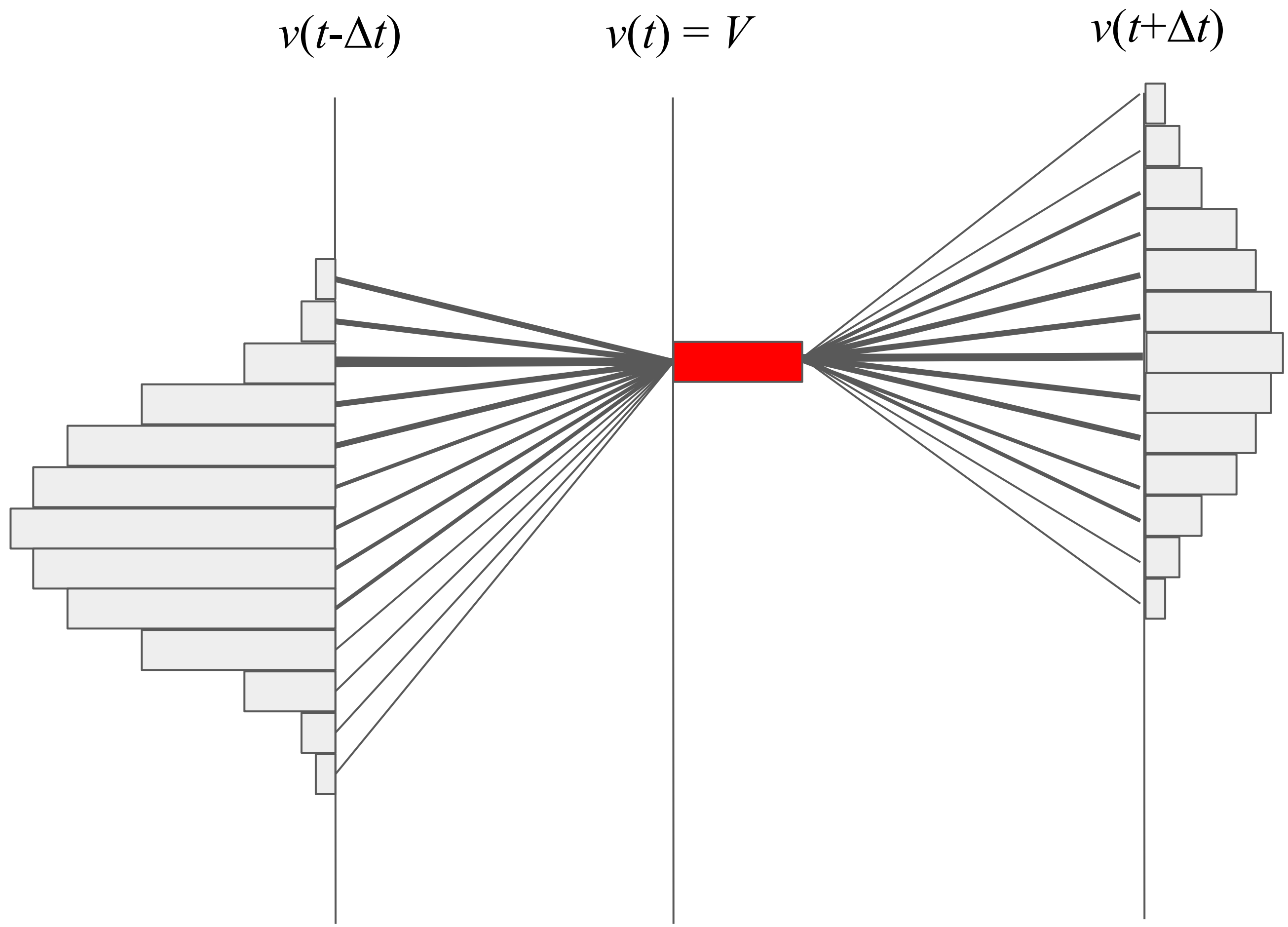
$$p(v, 2\Delta t \mid v = V, \Delta t)$$

What is the distribution of *v*?

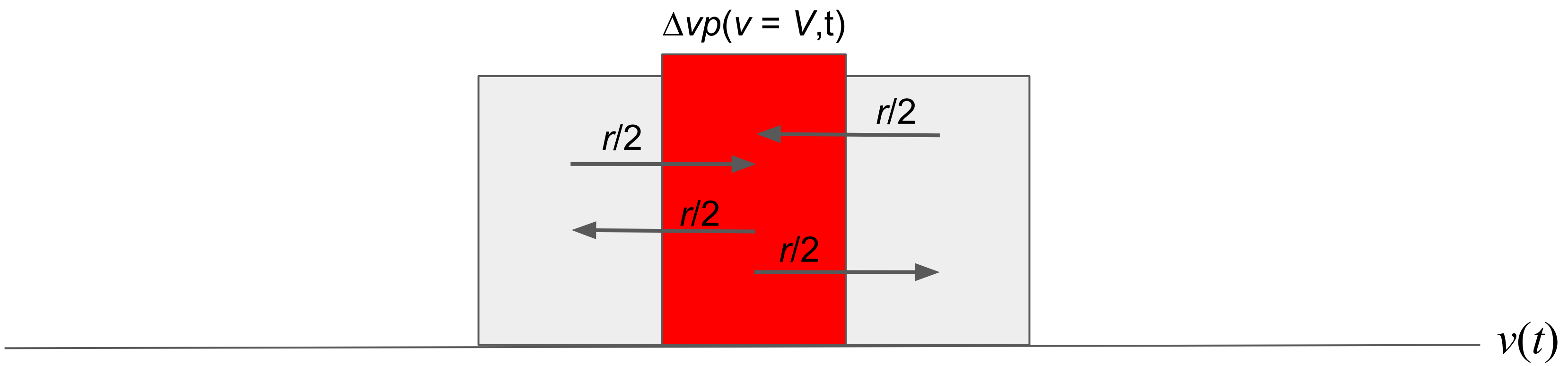


$$p(v,2\Delta t) = \sum_i p(v,2\Delta t \,|\, v = V_i, \Delta t)$$

Alternative approach: track the changes in probability for a given bin



Temporal evolution of the probability distribution



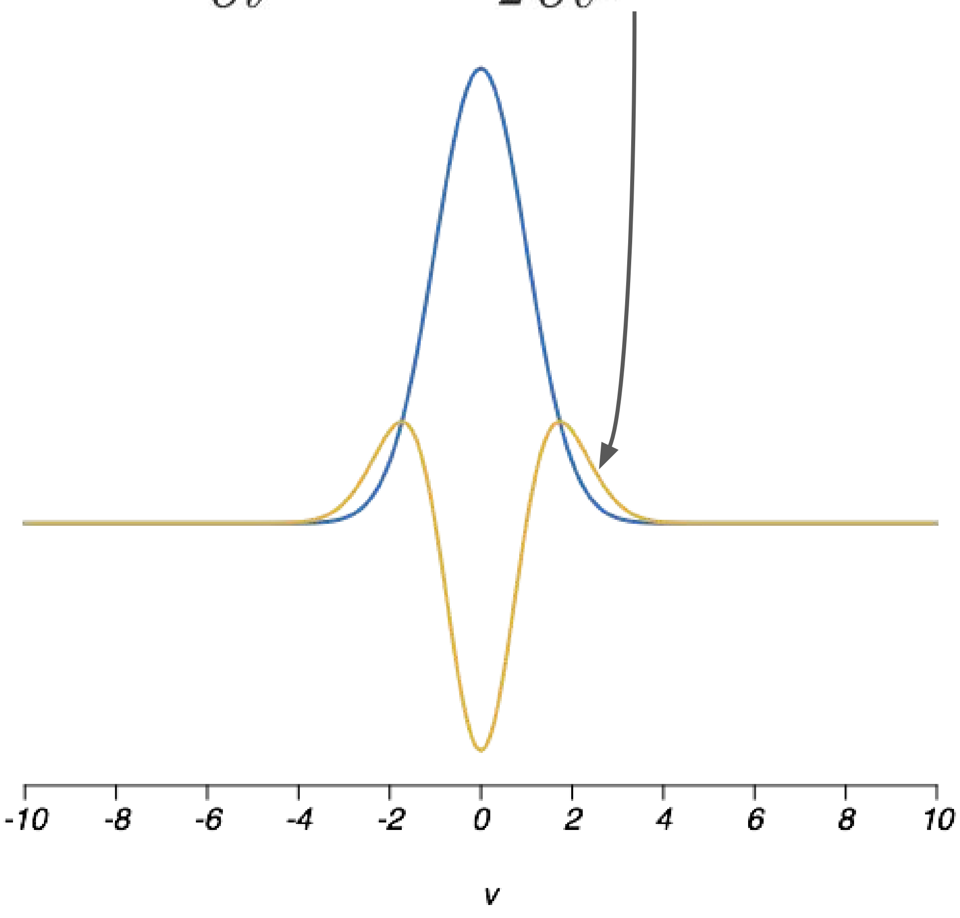
$$\Delta v[p(v,t+\Delta t)-p(v,t)]=\frac{r}{2}\Delta t[p(v-\Delta v,t)-2p(v,t)+p(v+\Delta v,t)]$$

$$r=\frac{b^2}{\Delta v}$$

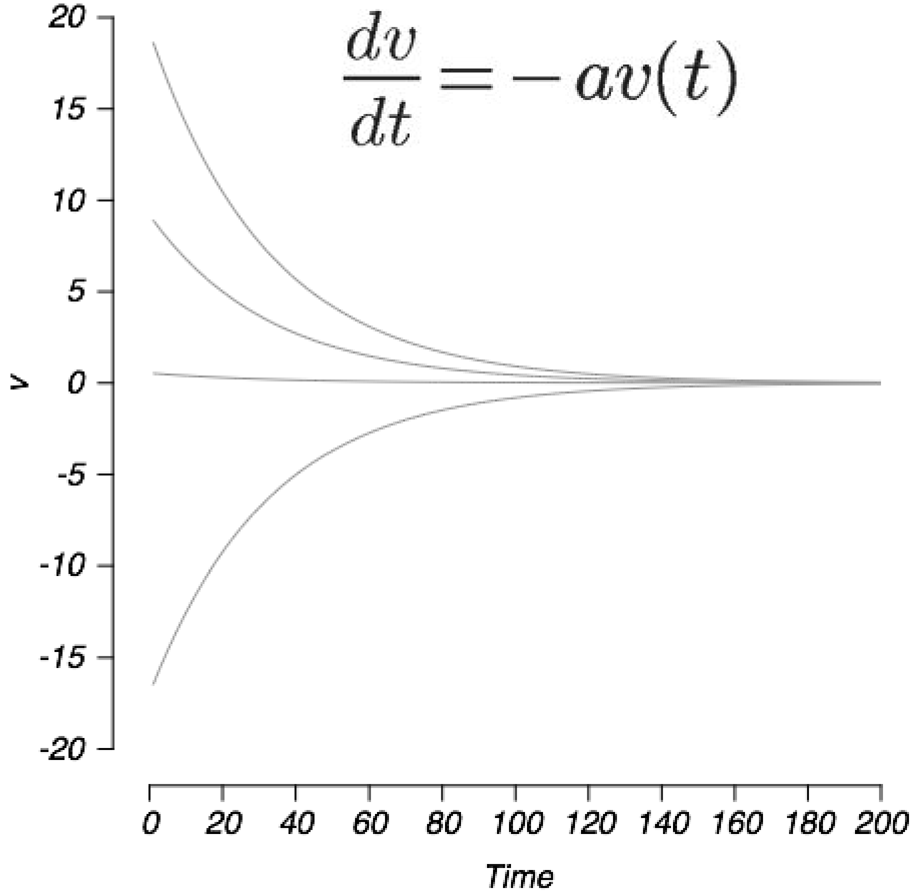
$$\frac{p(v,t+\Delta t)-p(v,t)}{\Delta t}=\frac{b^2}{2}\frac{p(v-\Delta v,t)-2p(v,t)+p(v+\Delta v,t)}{\Delta v^2}$$

Fokker-Planck: diffusion

$$\frac{\partial}{\partial t}p(v,t)=\frac{b^2}{2}\frac{\partial^2}{\partial v^2}p(v,t)$$

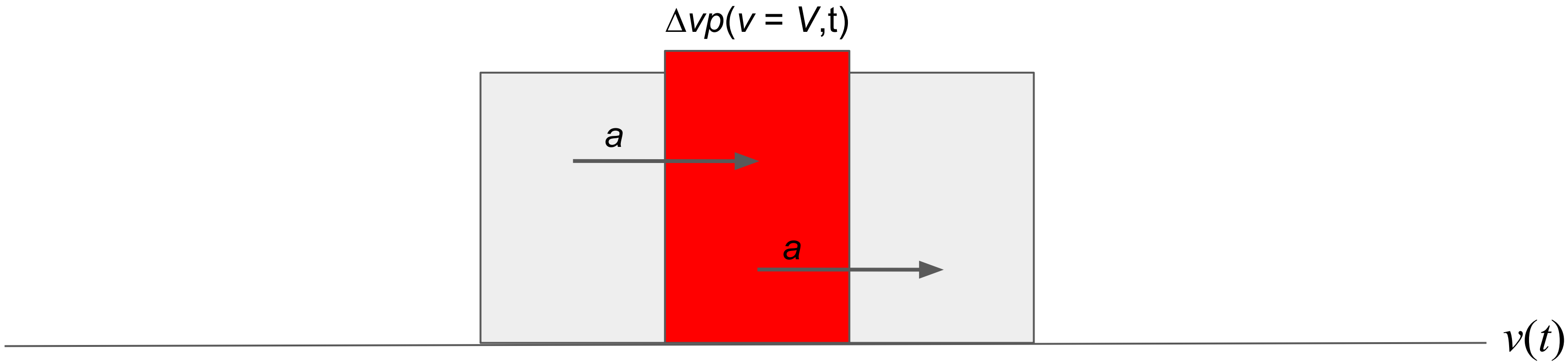


Ornstein-Uhlenbeck: just drift



$$\frac{dv}{dt} = -av(t)$$

Fokker-Planck: drift



$$\Delta v[p(v,t+\Delta t)-p(v,t)]=a\Delta t[p(v-\Delta v,t)-p(v,t)]$$

$$\frac{\partial}{\partial t}p(v,t)=-\frac{\partial}{\partial v}ap(v,t)$$

The full Fokker-Planck equation

For the Ornstein-Uhlenbeck process considered:

$$\frac{dv}{dt} = av(t) + b\eta(t)$$

$$\frac{\partial}{\partial t}p(v,t) = -\frac{\partial}{\partial v}ap(v,t) + \frac{b^2}{2}\frac{\partial^2}{\partial v^2}p(v,t)$$

General solution:

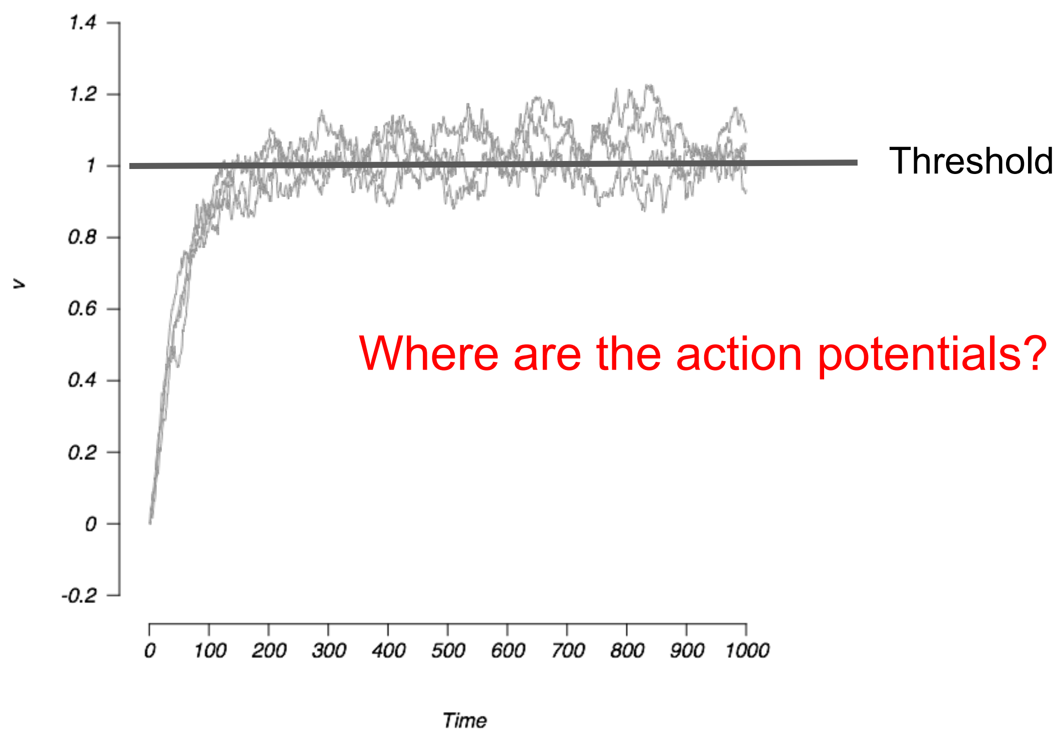
$$\frac{dv}{dt} = -f(v,t) + g(v,t)\eta(t)$$

$$\frac{\partial}{\partial t}p(v,t) = -\frac{\partial}{\partial v}f(v,t)p(v,t) + \frac{1}{2}\frac{\partial^2}{\partial v^2}g^2(v,t)p(v,t)$$

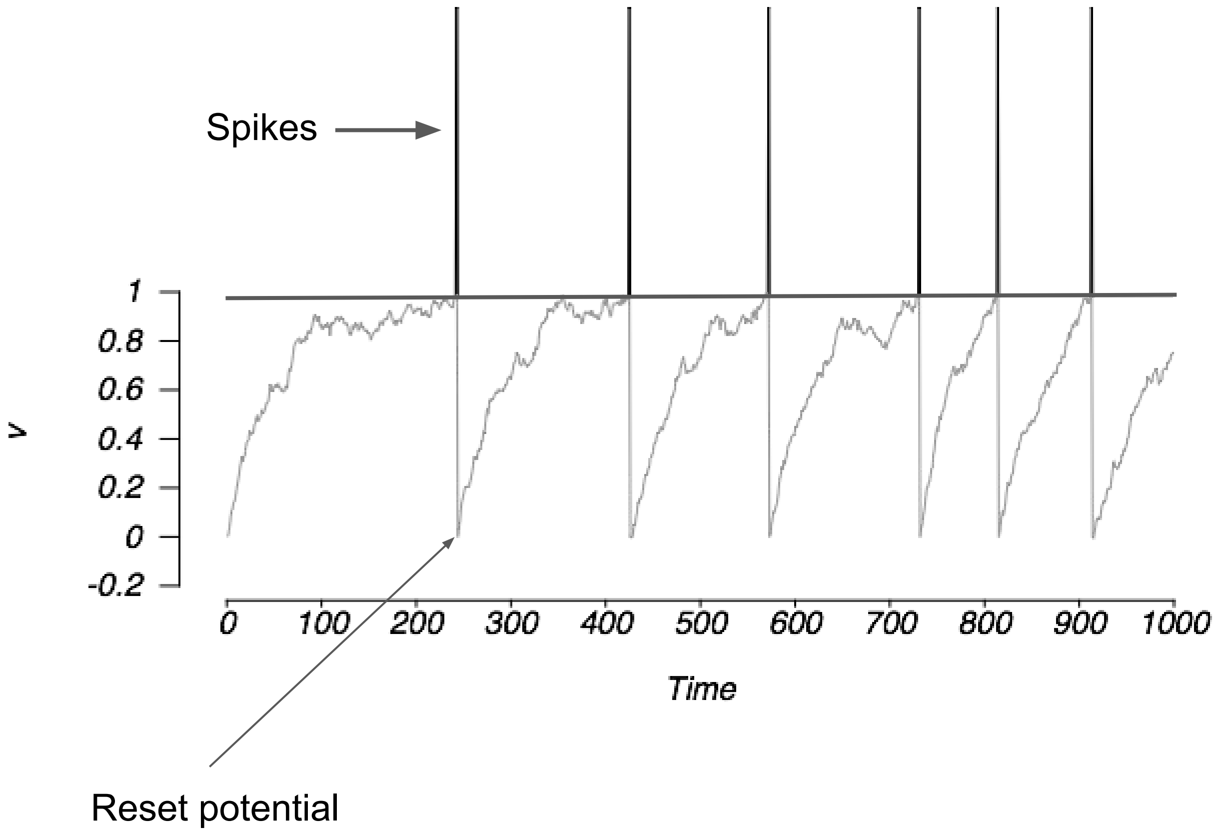
Leaky integrate-and-fire (LIF) model neuron

$$\frac{\partial v}{\partial t} = -a(v(t) - v_{rest}) + I(t) + b\eta(t)$$

Leak conductance — Voltage of the neuron — Resting potential — Input — Noise

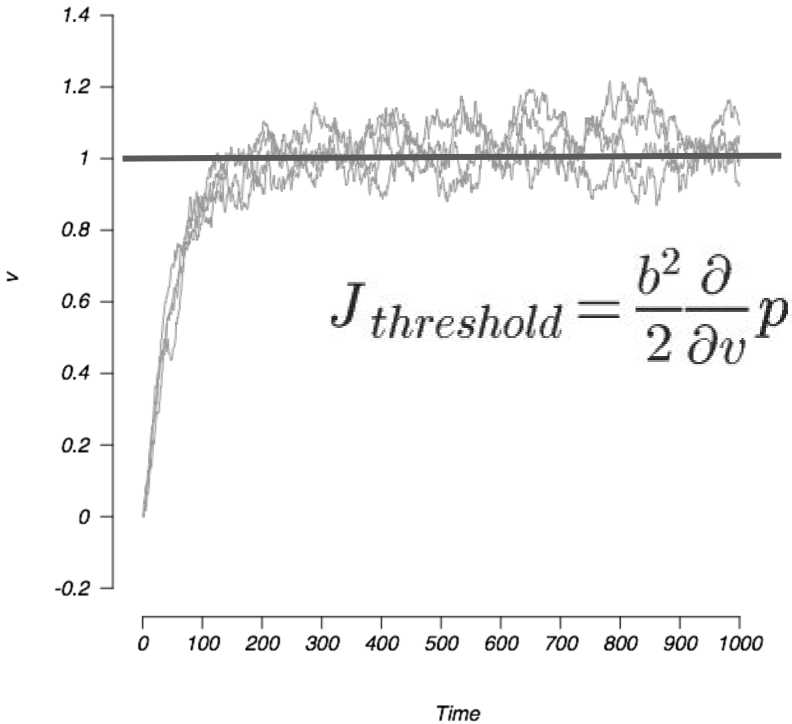


Boundary condition: spike and reset



Boundary condition: augmenting Fokker-Planck

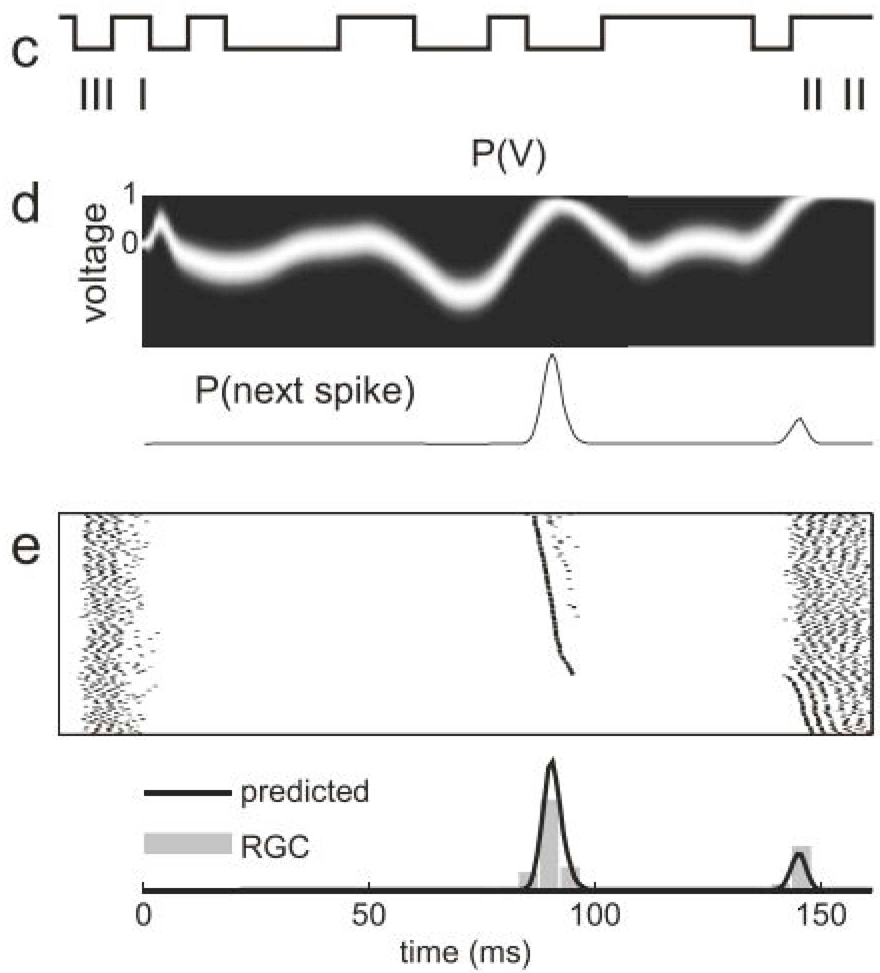
$$\frac{\partial}{\partial t}J(v,t) = -\frac{\partial}{\partial v}\Big[-a(v(t)-v_{rest})+I(t)\Big]+\frac{b^2}{2}\frac{\partial^2}{\partial v^2}p(v,t)$$



$$J_{threshold} = \frac{b^2}{2}\frac{\partial}{\partial v}p(v,t)\Big|_{v=v_{threshold}}$$

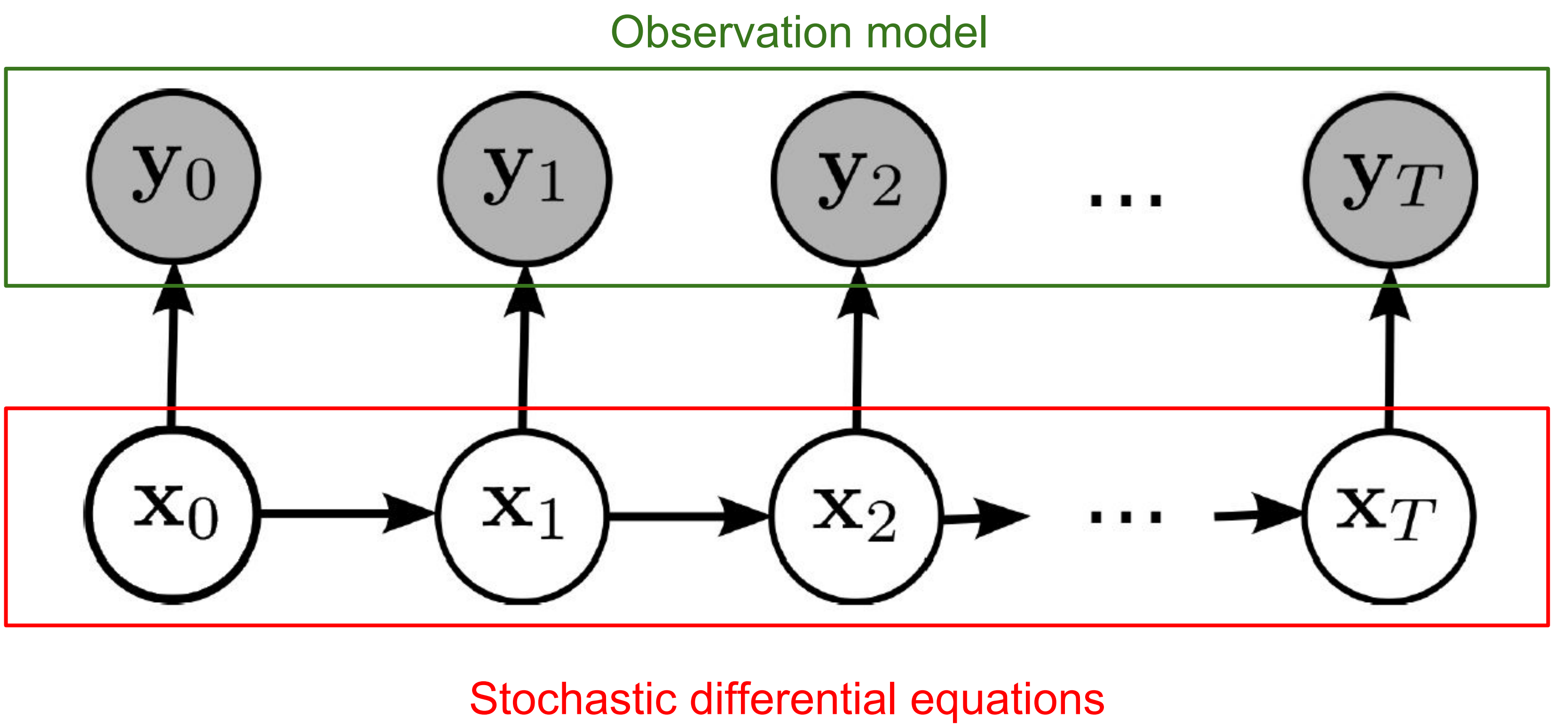
$$\frac{\partial}{\partial t}p(v,t) = \frac{\partial}{\partial v}J(v,t) - J_{threshold}\delta(v-v_{threshold}) + J_{threshold}\delta(v-v_{reset})$$

Example: predicting the time of the next spike



Pillow et. al. 2005

Other applications: inference over time



Suggested reading

Digestible reference for introduction to dynamical systems in neuroscience (no stochastics!)

Izhikevich, E. M. (2007). *Dynamical systems in neuroscience*. MIT press.

General reference for dynamical systems in neuroscience

Ermentrout, G. B., and Terman, D. H. (2010). *Mathematical foundations of neuroscience*. Springer Science & Business Media.

Gerstner, W., Kistler, W. M., Naud, R., and Paninski, L. (2014). *Neuronal Dynamics: From Single Neurons to Networks and Models of Cognition*. Cambridge University Press.

Specific examples of the use of SDEs in neuroscience

Laing, C., and Lord, G. J. (2009). *Stochastic Methods in Neuroscience*. OUP Oxford.