DYNAMICS AND CONTROL
OF
FLEXIBLE MULTIBODY STRUCTURES

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(ASTRACT)

The goal of this study is to present a method for deriving equations of motion capable of modeling the controlled motion of an open loop multibody structure comprised of an arbitrary number of rigid bodies and slender beams. The procedure presented here for deriving equations of motion for flexible multibody systems is carried out by means of the Principle of Virtual Work (often referred to in the dynamics literature as d'Alembert's Principle).

We first consider the motion of a general flexible body relative to the inertial space, and then derive specific formulas for both rigid bodies and slender beams. Next, we make a small motions assumption, with the end result being equations for a Rayleigh beam, which include terms which account for the axial motion, due to bending, of points on the beam central axis. This process includes a novel application of the exponential form of an orthogonal matrix, which is ideally suited for truncation. Then, the generalized coordinates and quasi-velocities used in the mathematical model, including those needed in the spatial discretization process of the beam equations are discussed. Furthermore, we develop a new set of recursive relations used to compute the inertial motion of a body in terms of the generalized coordinates and quasi-velocities.

This research was motivated by the desire to model the controlled motion of a flexible space robot, and consequently, we use the multibody dynamics equations to simulate the motion of such a structure, providing a demonstration of the computer program. For this particular example we make use of a new sequence of shape functions, first used by Meirovitch and Stemple to model a two dimensional building frame subjected to earthquake excitations.

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Chapter 1 Introduction

1.1 Overview

The goal of this study is to present a method for deriving equations of motion capable of modeling the controlled motion of an open loop multibody structure comprised of an arbitrary number of rigid bodies and slender beams. The equations of motion for flexible multibody structures consist of simultaneous ordinary differential equations for the rigid-body motions and boundary-value problems, composed of partial differential equations and boundary conditions, for the elastic deformations. Such coupled sets of differential equations are referred to as hybrid [33]. The solution of hybrid sets of equations is difficult, with the problem further complicated by the fact that the equations are generally nonlinear, even when the elastic deformations are small.

The procedure presented here for deriving equations of motion for flexible multibody systems is carried out by means of the Principle of Virtual Work [31], often referred to in the dynamics literature as d'Alembert's Principle [32]. The Principle of Virtual Work allows for a fairly streamlined and systematic approach to the derivation of multibody dynamics equations of motion. This is particularly true when the ultimate goal is to develop a general purpose computer program which can be applied to any number of different structures.

There are five basic steps involved in the derivation. In the first place we consider the motion of a general flexible body relative to the inertial space, and then derive specific formulas for both rigid bodies and slender beams. The slender beam equations, however, are not in a form suitable for numerical analysis, so that the second step involves making a small motions assumption. This process includes a novel application of the exponential form of an orthogonal matrix, which is ideally suited for truncation, with the end result being equations for a Rayleigh beam [52]. In fact, the equations include terms which account for the axial motion, due to bending, of points on the beam central axis. The third step is to introduce the generalized coordinates and quasi-velocities used in the mathematical model, including those needed in the spatial discretization process of the beam equations. In the fourth step we discuss the relative motion of one body with respect to another and develop a new set of recursive relations used to compute the inertial motion of a body in terms of the generalized coordinates and quasi-velocities. In the fifth and final step, we combine the results of the previous four steps to derive the system ordinary differential equations of motion used to model the multibody structure.

This research was motivated by the desire to model the controlled motion of a flexible space robot, such as the one used on the space shuttle. Consequently, we use the multibody dynamics equations to simulate the motion of such a structure, providing a demonstration of the multibody dynamics code. For this particular example we make use of a sequence of shape functions first used by Meirovitch and Stemple [46] to model a two dimensional building frame subjected to earthquake excitations. The results are presented in the form of plots of the various joint displacements, the displacements of points on the beams due to the elastic motion, as well as the control torques of the various actuators.

1.2 Literature Survey

Many common engineering structures, including various types of spacecraft, land vehicles, industrial machinery and robots, can be modeled as multibody systems. The technical literature on the subject is vast, indeed, with publications going back several decades. We include here a literature survey which represents a fair cross-section of all the various aspects of multibody dynamic analysis.

In some applications multibody structures can be modeled by assuming that all bodies in the structure are rigid, with the derivation of equations of motion carried out by a variety of techniques such as Newton-Euler equations, d'Alembert's principle, Lagrange's equations, or the method popularized by Kane. The literature devoted to rigid multibody structures is well established, and here we simply single out the books by Haug [17], Huston [20], Roberson and Schwertassek [51], Shabana [53] and Wittenburg [64], as well as papers by Kane and Levinson [22, 23].

The modeling of flexible multibody systems relies heavily on principles and techniques for modeling single flexible bodies. In an early paper, Meirovitch and Nelson [40] considered the problem of stability of spinning spacecraft modeled as a rigid core with two flexible beams simulating antennas. They derived Lagrange's ordinary differential equations in terms of quasi-coordinates for the rigid-body rotations of the core coupled with Lagrange's partial differential equations for the elastic deformations relative to a reference frame embedded in the rigid core. Using the concept of "floating reference frame," de Veubeke [13] derived equations of motion for a mean rigid body motion in terms of quasi-coordinates and modal equations for elastic motions measured relative to the mean rigid-body motion. Cavin and Dusto [8] developed a variational formulation yielding finite element equations of motion for a single unconstrained elastic body. Using a Lagrangian approach, Meirovitch and Quinn [41] derived equations of motion for maneuvering flexible spacecraft. Then, they derived a set of perturbation equations with the rigid-body maneuvers as the unperturbed motion. Shabana [54] derived generalized Newton-Euler equations for deformable bodies undergoing large translational and rotational displacements. The equations were formulated in terms of time-invariant scalars and matrices depending on both spatial coordinates as well as the assumed displacement field. General hybrid equations of motion for an arbitrarily shaped body in space were derived by Meirovitch [33], who extended the concept of quasi-coordinates to translations in terms of body axes components. An example involving a flexible beam attached to a translating and rotating rigid disk was provided. Using Lagrange's form of d'Alembert's principle, Weng and Greenwood [63] derived equations of motion for a body undergoing large elastic deformations, and then applied the formulation to a beam attached to a rotating base. Zhang, Liu and Huston [68] used Kane's equations to derive equations for overall large motions of an arbitrarily shaped body. Two examples were provided, a rotating beam and a rotating plate.

Mathematical models for flexible multibody systems tend to be considerably more involved than those for single flexible bodies, or rigid bodies with flexible parts. Ho [18] used the "direct path method" to derive equations of motion for multibody spacecraft with topological tree configuration, with the terminal bodies being flexible and the interconnecting bodies assumed to be rigid. Three methods for deriving the equations were considered, Lagrangian-Newtonian, all Newtonian and all Lagrangian. Vu-Quoc and Simo [61] considered the dynamics of flexible multibody spacecraft by referring the motion to an inertial frame. To this end, they introduced a floating reference frame translating relative to the inertial space. Kim and Haug [25] used a recursive formulation to model the dynamics of flexible multibody systems, whereby the elastic deformation of each body is represented by deformation modes. Two models for flexible multibody systems were considered by Yamada, Tsuchiya and Ohkami [65], one based on a Newton equation for each body in conjunction with constraint equations and one based on Kane's equations. In the context of flexible multibody systems, Géradin, Cardona and Granville [15] discussed a nonlinear beam model, a substructuring technique for incorporating the flexibility of individual members into the overall equations of motion and the incorporation of kinematic constraints. Chang and Shabana [10] addressed the problem of modeling flexible multibody structures subjected to changes in topology due to changes in the connectivity between bodies. They discretized the system by the finite element method. Extending the approach of Ref. [33], Meirovitch and Kwak [37] considered the problem of pointing flexible antennas mounted on a spacecraft by stabilizing the spacecraft relative to the inertial space and maneuvering the antennas relative to the spacecraft at the same time. Keat [24] developed an nth order algorithm capable of modeling systems of rigid or flexible bodies. The formulation can accomodate open-chain, tree and closed-loop topologies, and the joints connecting adjacent bodies can have 0 to 6 degrees of freedom. A formulation of the dynamics of rotorcraft consisting of flexible and rigid components was presented by Agrawal [1]. The result of using the finite element method in conjunction with a multibody approach is a set of differential-algebraic equations of motion. Avello, de Jalon and Bayo [3] used Lagrange's equations to derive equations of motion for flexible slender bodies modeled as nonlinear Timoshenko beams. Constraints are introduced with a penalty formulation and the resulting differential equations were integrated by Newmark's methods. Cyril, Angeles and Misra [12] developed a model consisting of both rigid and flexible links and used it to simulate a typical maneuver of the Space Shuttle remote manipulator. Euler-Lagrange equations are derived for each body separately and then assembled to obtain the constrained dynamical equations for the multibody system. A new recursive dynamics analysis of flexible multibody systems was formulated by Lai, Haug, Kim and Bae [28] using a kinematic graph concept and a variational vector calculus approach. Assuming small deformations, the flexibility was modeled by means of modal coordinates. To illustrate the approach, a flexible closed-loop spatial robot was analyzed. Nikravesh and Ambrosio [48] formulated the equations of motion for multibody systems containing both rigid and flexible bodies. They used joint coordinates to derive the minimum number of equations for the rigid bodies and the finite element method to discretize the flexible bodies. A systematic method for deriving the minimum number of equations of motion of spatial flexible multibody systems was presented by Pereira and Proenca [49]. Relative kinematics in terms of relative joint coordinates and velocities was used to formulate the equations of motion. Shabana [55] considered issues related to the dynamics modeling of constrained deformable bodies undergoing large rigid-body displacement. He discretized the bodies by the finite element method. A formulation capable of treating the problem of maneuvering and vibration control of a flexible multibody system was developed by Kwak and Meirovitch [27]. Equations of motion in terms of quasi-coordinates were derived for each body separately and then combined by means of a consistent kinematical synthesis. Vukasovic, Celigueta and de Jalon [60] used Cartesian coordinates to model flexible multibody structures, with the elastic deformations represented by linear combinations of Ritz vectors with respect to a local reference frame. An example involving a satellite deployment was provided. Various issues in the structural modeling of flexible multibody systems were discussed by Suleman, Modi and Venkayya [57]. Comparative analyses between component and system modal discretization techniques were presented. When the system undergoes large three-dimensional rigid-body motions, in addition to elastic motions, Lagrange's equations in terms of quasi-coordinates [32, 40, 50] provide an alternative to ordinary Lagrange's equations [41]. Hybrid equations of motion in terms of quasi-coordinates were derived by Meirovitch [34] directly from Hamilton's principle and by Meirovitch and Stemple [43] by transforming ordinary Lagrange's equations. The developments of Refs. [34] and [43] are carried out using symbolic vector operations in conjunction with recursive kinematical relations to eliminate redundant coordinates. The hybrid equations have been discretized by Meirovitch and Stemple [42] by the approach of Ref. [38] and the resulting ordinary differential equations cast in state form for control.

The development of multibody dynamics formulations especially for modeling mechanisms and land vehicles has been made necessary to a large extent by the fact that the kinematics for such systems is substantially more complicated than for aerospace structures. In fact, the modeling of mechanisms and land vehicles often requires multibodies forming closed loops, and in some cases even involves nonholonomic constraints. Because the interest in this paper lies mainly in aerospace structures and flexible robots, we concentrate the review on a few papers in which the flexibility is included in the model.

Yoo and Haug [66, 67] developed a flexible multibody model in which the individual bodies can undergo large rigid-body motions but the elastic displacements remain small. The deformation modes for flexible bodies are generated by a lumped mass finite element approach and the equations of motion are derived by using a Lagrange multipliers formulation. Koppens et. al. [26] considered the dynamics of a deformable body allowed to undergo large displacements, with the deformation modeled independently from the rigid-body motions by means of a linear combination of assumed displacement fields. The system equations of motion were derived by d'Alembert's principle and illustrated by means of a uniform beam and a crank-slider mechanism. Lieh [29] presented a separated-form formulation for the dynamics and control of multibody systems with elastic members treated as Euler-Bernoulli beams, where "separated" is in the sense that the inertia matrix, nonlinear coupling vector, generalized force vector and base motion-induced terms are determined individually. Examples include an elastic vehicle with active suspension and an elastic crank-slider mechanism. The problem of including the rotational, or dynamic stiffening, effect in the analysis of flexible bodies has been considered by Wallrapp and Schwertassek [62].

Industrial robots have been modeled traditionally as chains of rigid bodies moving relative to an inertial space, with adjacent bodies connected by motors providing internal control torques. With the advent of lightweight industrial robots and space robots, such as the Space Shuttle manipulator arm, it has become necessary to include the flexibility in the system model.

Book et. al. [6] considered the control of a flexible robot consisting of two pinned beams using two flexibility models. They investigated and compared three types of linear feedback control. Later, Book [7] developed nonlinear equations of motion for flexible manipulator arms with adjacent arms connected by rotary joints. Assuming small deformations, the displacement of the flexible links was expressed as linear combinations of shape functions. The efficiency of the formulation was compared to that of a rigid-link model. Hughes [19] used the "direct path method" to derive equations of motion for a chain of bodies, with the two end bodies being rigid and the intermediate bodies capable of small flexible motions. A computer program based on these dynamical equations was written and used to model the Space Shuttle robot arm. Low [30] used Hamilton's principle to derive hybrid equations of motion for flexible robots including inertial, Coriolis, centrifugal, gravitational and external force effects. Two examples were studied, a three-link flexible manipulator with revolute joints and a flexible manipulator consisting of a prismatic bar and a discrete mass. Naganathan and Soni [47] used a Newton-Euler formulation in conjunction with Timoshenko beam theory discretized by the finite element method to develop a nonlinear model capable of predicting the response of spatial manipulators with flexible links. From studies of both planar and spatial manipulators, they concluded that nonlinear kinematic coupling has significant effect on the positioning errors of the end-effector. Baruh and Tadikonda [4] used an approach similar to substructure synthesis to model a robot with elastic arms. They concluded that the centrifugal stiffening effect plays a large role in the overall system behavior. Bayo et. al. [5] considered the inverse dynamics and kinematics of multilink flexible robots, with the links modeled by the Timoshenko beam theory discretized by the finite element method. A method was developed for determining the joint torques required to produce a specified end-effector motion, with the performance tested both by simulation and experiment. Equations of motion for a planar model of the proposed Space Station-based Mobile Servicing System were derived by means of Lagrange's equations by Chan and Modi [9], who designed controls using linear quadratic theory. A parallel processing algorithm to simulate the dynamical equations for constrained flexible multibody systems undergoing large rotations was developed by Ider [21], who tested its performance by simulating a spatial robotic manipulator. Amirouche and Xie [2] used Kane's equations to develop a model for the dynamic simulation of rigid/flexible multibody systems. Using finite element discretization, a recursive formulation was developed and applied to a two-link robot manipulator. Meirovitch and Lim [39] considered the controlled response of a flexible space robot. A perturbation approach applied to the original nonlinear equations of motion permitted the control law to be divided into two parts, one for rigid-body maneuvering carried out open-loop and another for the elastic motions controlled by a closed-loop discrete-time linear quadratic regulator with prescribed degree of stability. Meirovitch and Chen [36] considered the problem of designing controls for a flexible space robot required to ferry a payload in space and to dock with an orbiting target. The robot trajectory is determined by an optimization procedure, with controls based on a perturbation approach. The rigid-body maneuvering control is determined using inverse dynamics and the elastic vibration is controlled by a linear quadratic regulator for time-varying systems. Chen and Meirovitch [11] developed a control scheme for a docking maneuver of a flexible space robot with a moving target. The problem is complicated by the assumption that the target's motion is not known a priori. Lagrange's equations are used to derive equations of motion, which are separated into two sets suitable for rigid-body maneuver and vibration suppression control. Van Woerkom and de Boer [59] developed an "order-n" algorithm used to simulate the motion of a flexible space manipulator consisting of a base (the spacecraft) and an end-effector connected by a chain of flexible arms. A discussion of the parametric Lagrange form of d'Alembert's principle is included. Van Woerkom [58] discussed the problem of including the axial deformation of beams, as applied to models of space robots. Four methods of including the axial deformation, i.e., nonlinear deformation field modeling, perturbed dynamics modeling, fictitious joints modeling and bracket joints modeling, are discussed. The hybrid equations of Refs. [34] and [43] were first discretized by means of substructure synthesis [38] and then used by Meirovitch and Stemple [42, 44, 45] to carry out rest-to-rest maneuvers of flexible robots. Controls were designed by the Liapunov direct method and direct feedback control was used to suppress the elastic vibration of the links.

Chapter 2 Flexibility Models

In this chapter we develop the equations required to describe the inertial kinematics and dynamics of the bodies making up the structure. Vectors and tensors in three-dimensional Euclidian space are denoted with bold type ($\mathbf{R}$, $\mathbf{u}$, $\mathbf{F}$, $\boldsymbol{\Omega}$), points are denoted with italic type (O_o, P_1, P_2) and, as usual, the notation $\overrightarrow{P_1P_2}$ represents a vector with initial point P_1 and terminal point P_2. Scalars and matrices are denoted by plane type (R, u, F, Ω) with the usual basis in R^3 given by $\mathsf{e}_1 = [1\ \ 0\ \ 0]^{\mathsf{T}}$, $\mathsf{e}_2 = [0\ \ 1\ \ 0]^{\mathsf{T}}$ and $\mathsf{e}_3 = [0\ \ 0\ \ 1]^{\mathsf{T}}$. The terminology *component matrix* refers to either a 3-by-1 matrix, r for example, containing the components of a vector $\mathbf{r}$ along specified orthogonal axes, or a 3-by-3 matrix, F for example, containing the components of a tensor $\mathbf{F}$ with respect to specified orthogonal axes. Furthermore, we let $\{O; \mathsf{xyz}\}$ stand for an orthogonal coordinate system with axes xyz and origin O. Some basic matrix definitions and identities are developed in the Appendix.

2.1 Principle of Virtual Work

The starting point for deriving equations of motion is the *Principle of Virtual Work* [31], which, for a structure comprised of N bodies labeled $\mathsf{i} = 1, 2, \ldots, \mathsf{N}$, takes the form

$$\sum_{\mathsf{i}=1}^{\mathsf{N}} \mathcal{R}_{\mathsf{i}} \;=\; 0, \tag{2.1}$$

where a general expression for $\mathcal{R}_{\mathsf{i}}$ valid for any model of a solid body is given by

$$\begin{aligned} \mathcal{R}_{\mathsf{i}} \;=\; & \int_{\mathcal{B}_{\mathsf{oi}}} \rho_{\mathsf{oi}}(\mathbf{r}_{\mathsf{i}})\ddot{\mathbf{z}}_{\mathsf{i}} \cdot \delta\mathbf{z}_{\mathsf{i}}(\mathbf{r}_{\mathsf{i}}, \mathsf{t})\, \mathsf{d}\mathcal{V} + \int_{\mathcal{B}_{\mathsf{oi}}} \mathrm{Tr}\Big[\mathbf{T}_{\mathsf{oi}}(\mathbf{r}_{\mathsf{i}}, \mathsf{t})\delta\mathbf{F}_{\mathsf{i}}(\mathbf{r}_{\mathsf{i}}, \mathsf{t})\Big]\, \mathsf{d}\mathcal{V} \\ & - \int_{\mathcal{B}_{\mathsf{oi}}} \mathbf{b}_{\mathsf{oi}}(\mathbf{r}_{\mathsf{i}}, \mathsf{t}) \cdot \delta\mathbf{z}_{\mathsf{i}}(\mathbf{r}_{\mathsf{i}}, \mathsf{t})\, \mathsf{d}\mathcal{V} - \int_{\partial\mathcal{B}_{\mathsf{oi}}} \boldsymbol{\tau}_{\mathsf{oi}}(\mathbf{r}_{\mathsf{i}}, \mathsf{t}) \cdot \delta\mathbf{z}_{\mathsf{i}}(\mathbf{r}_{\mathsf{i}}, \mathsf{t})\, \mathsf{d}\mathcal{A}. \end{aligned} \tag{2.2}$$

This is where the region $\mathcal{B}_{oi}$, which serves as a reference configuration for body i, is occupied by the body at time $t = 0$, and $\partial\mathcal{B}_{oi}$ is the boundary of $\mathcal{B}_{oi}$ (Fig. 1). Furthermore, $\mathbf{r}_i = \overrightarrow{O_{oi}P_{oi}}$ is the position vector of a typical point P_{oi} in $\mathcal{B}_{oi}$, $\rho_{ci}(\mathbf{r}_i)$ is the mass density of the body, when in the reference configuration, $\mathbf{T}_{oi}(\mathbf{r}_i, t)$ is the (first) Piola-Kirchoff stress tensor for body i, and $\mathbf{b}_{oi}(\mathbf{r}_i, t)$ and $\boldsymbol{\tau}_{oi}(\mathbf{r}_i, t)$ are, respectively, the body force density (force per unit reference volume) and surface traction (force per unit reference surface area). As the bodies in the structure translate, rotate and deform, the points O_{oi} and P_{oi} of body i will occupy new points in space, which we denote by O_i and P_i, respectively. To keep track of the position of points O_i and P_i, we let $\mathbf{R}_i(t) = \overrightarrow{O_oO_i}$ and $\mathbf{z}_i(\mathbf{r}_i, t) = \overrightarrow{O_oP_i}$, where O_o is a point fixed in the inertial space. The gradient of $\mathbf{z}_i(\mathbf{r}_i, t)$ with respect to $\mathbf{r}_i$, denoted by $\mathbf{F}_i(\mathbf{r}_i, t) = \nabla\mathbf{z}_i(\mathbf{r}_i, t)$, is a second order tensor referred to as the deformation gradient [16] of body i.

Although we could proceed with the derivation directly from Eq. 2.2, it is more efficient to first rewrite the formula for $\mathcal{R}_i$ in terms of vector and tensor components. To this end, we introduce the inertial coordinate system $\{O_o; x_oy_oz_o\}$ (Fig. 1) and the moving coordinate system $\{O_i; x_iy_iz_i\}$, fixed on body i, for $i = 1, 2, \ldots, N$; body axes $x_iy_iz_i$, for time $t = 0$, are denoted by $x_{oi}y_{oi}z_{oi}$. Then, $r_i = [x\ \ y\ \ z]^T$ denotes the component matrix of vector $\mathbf{r}_i$, along axes $x_{oi}y_{oi}z_{oi}$, $R_i(t)$ and $z_i(r_i, t) = z_i(x, y, z, t)$ are component matrices for vectors $\mathbf{R}_i(t)$ and $\mathbf{z}_i(\mathbf{r}_i, t)$, respectively, both along inertial axes $x_oy_oz_o$, and $r_{ei}(r_i, t)$ is the component matrix, along body axes $x_iy_iz_i$, for vector $\mathbf{r}_{ei}(\mathbf{r}_i, t) = \overrightarrow{O_iP_i}$, implying that $r_{ei}(r_i, 0) = r_i$. Note that the components of r_i are Lagrangian (or material) coordinates and the components of z_i are Eulerian (or spatial) coordinates for body i. We also let T_{oi}, F_i, b_{oi} and τ_{oi} be the component matrices of $\mathbf{T}_{oi}$, $\mathbf{F}_i$, $\mathbf{b}_{oi}$ and $\boldsymbol{\tau}_{oi}$, respectively, all with components along inertial axes $x_oy_oz_o$. The component matrix T_{oi} of the Piola-Kirchoff stress tensor, which measures force per unit area in the reference configuration, is related to T_{ci}, the component matrix (also with respect to inertial axes $x_oy_oz_o$) of the Cauchy stress tensor, by the formula [16]

$$T_{oi} = (\det F_i)T_{ci}F_i^{-T}. \tag{2.3}$$

Recall that the conservation of angular momentum principle implies that the Cauchy stress tensor is symmetric, that is,

$$T_{ci}^T = T_{ci}. \tag{2.4}$$

Then, since z_i, T_{oi}, F_i, b_{oi} and τ_{oi} are all in terms of components along the same set of axes, we can rewrite Eq. 2.2 in the form

$$\begin{aligned} \mathcal{R}_i = & \int_{\mathcal{B}_{oi}} \rho_{oi}(r_i)\delta z_i^T(r_i, t)\ddot{z}_i(r_i, t)\, d\mathcal{V} + \int_{\mathcal{B}_{oi}} \mathrm{Tr}\Big[T_{oi}^T(r_i, t)\delta F_i(r_i, t)\Big]\, d\mathcal{V} \\ & - \int_{\mathcal{B}_{oi}} \delta z_i^T(r_i, t)b_{oi}(r_i, t)\, d\mathcal{V} - \int_{\partial\mathcal{B}_{oi}} \delta z_i^T(r_i, t)\tau_{oi}(r_i, t)\, d\mathcal{A}. \end{aligned} \tag{2.5}$$

The four terms making up $\mathcal{R}_i$ will be referred to, in order, as the inertia term, the internal force term, the body force term and the surface traction term. However, since we are con-

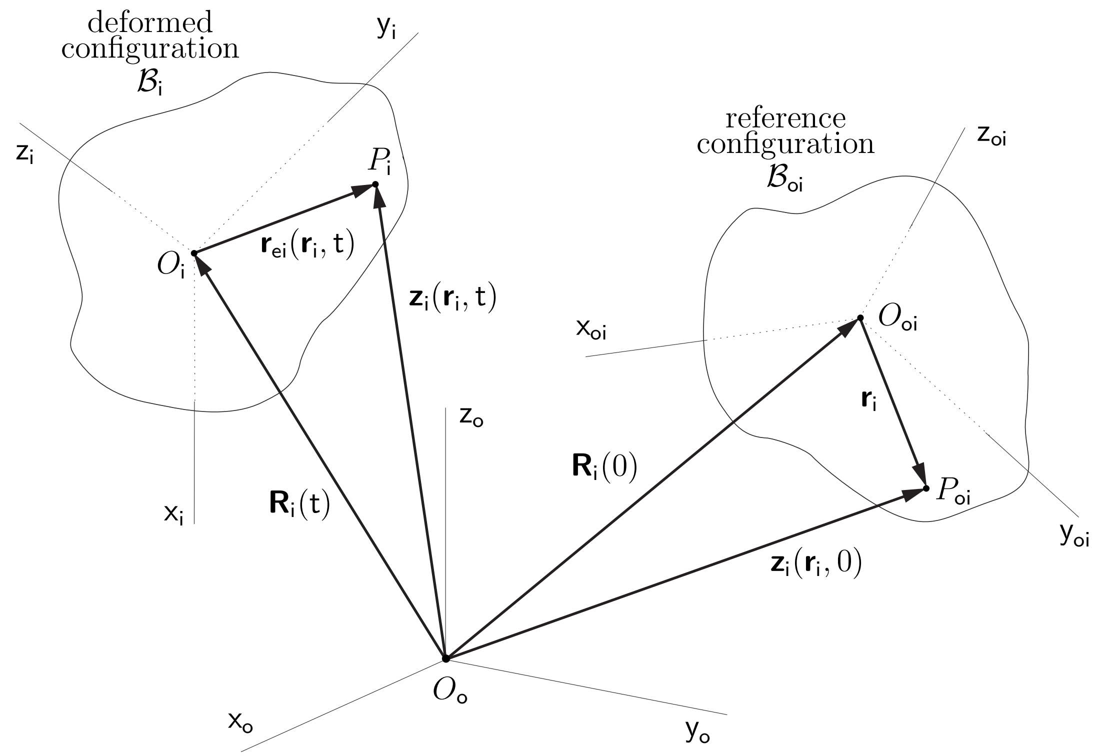

Figure 1: Reference and Deformed Configuration of a Body

cerned primarily with space structures, we assume that the effect of body forces is negligible for the duration of the simulations to be performed, so that $b_{oi} \equiv 0$.

It is worth pointing out that the Principle of Virtual Work is simply a more primitive version of the extended Hamilton's Principle [31, 32]. In fact, the internal force term, which gives the virtual work done by internal stresses, is often written as the variation of a strain energy plus terms used to model internal damping. Then, in combination with the body force term and surface traction term, the last three terms of $\mathcal{R}_i$ are written as $\delta V_{pe,i} - \overline{\delta W}_{nc,i}$, where $V_{pe,i}$ is the potential energy and $\overline{\delta W}_{nc,i}$ the nonconservative virtual work for body i. Note first that this allows writing the Principle of Virtual Work in the form

$$\sum_{i=1}^{N} \left(\int_{\mathcal{B}_{oi}} \rho_{oi} \delta z_i^T \ddot{z}_i d\mathcal{V} + \delta V_{pe,i} - \overline{\delta W}_{nc,i} \right) = 0. \tag{2.6}$$

Then, integrating with respect to time from $t = t_1$ to $t = t_2$ and applying the integration by parts formula yields the extended Hamilton's Principle,

$$\int_{t_1}^{t_2} (\delta L + \overline{\delta W}_{nc})\, dt \;=\; 0, \tag{2.7}$$

$$\delta z_i(r_i, t_1) = \delta z_i(r_i, t_2) = 0, \quad r_i \in \mathcal{B}_{oi}, \quad i = 1, 2, \ldots, N, \tag{2.8}$$

where

$$L \;=\; T - V \tag{2.9}$$

is the system Lagrangian, in which

$$T \;=\; \frac{1}{2}\sum_{i=1}^{N}\int_{\mathcal{B}_{oi}} \rho_{oi}\dot{r}_i^T \dot{r}_i\, d\mathcal{V} \tag{2.10}$$

is the kinetic energy of the system,

$$V_{pe} \;=\; \sum_{i=1}^{N} V_{pe,i} \tag{2.11}$$

is the potential energy and

$$\overline{\delta W}_{nc} \;=\; \sum_{i=1}^{N} \overline{\delta W}_{nc,i} \tag{2.12}$$

is the system nonconservative virtual work.

2.2 Rigid Body Motion

The motion of coordinate system $\{O_i; x_i y_i z_i\}$ with respect to coordinate system $\{O_o; x_o y_o z_o\}$ involves both the translation of point O_i as well as the rotation of axes $x_i y_i z_i$. We have earlier introduced the component matrices $R_i(t)$ and $z_i(r_i, t)$, which involve the position of points O_i and P_i, respectively, with respect to the inertial space. We also require the matrix of direction cosines of body axes $x_i y_i z_i$ relative to the inertial axes $x_o y_o z_o$, which is denoted by $P_i(t)$. Note that since $P_i^T(t)$ transforms components of a vector, along axes $x_i y_i z_i$, to components of the same vector, but along axes $x_o y_o z_o$, the matrix form of the vector equation (Fig. 1)

$$\mathbf{z}_i(\mathbf{r}_i, t) \;=\; \mathbf{R}_i(t) + \mathbf{r}_{ei}(\mathbf{r}_i, t) \tag{2.13}$$

is given by

$$z_i(r_i, t) \;=\; R_i(t) + P_i^T r_{ei}(r_i, t). \tag{2.14}$$

We also make use of the quasi-velocities $V_i(t)$ and $\Omega_i(t)$, which are, respectively, component matrices, along axes $x_i y_i z_i$, for the velocity vector $\dot{\mathbf{R}}_i(t)$, and angular velocity vector $\boldsymbol{\Omega}_i(t)$ of axes $x_iy_iz_i$ with respect to axes $x_oy_oz_o$. Referring to Eqs. A.24 and A.25 in the appendix, this implies that

$$V_i = P_i\dot{R}_i, \tag{2.15}$$
$$\widetilde{\Omega}_i = P_i\dot{P}_i^T. \tag{2.16}$$

Analogous to V_i and Ω_i, the quasi-virtual displacements δR_i^* and $\delta\Theta_i^*$ are defined by

$$\delta R_i^* = P_i\,\delta R_i, \tag{2.17}$$
$$\widetilde{\delta\Theta_i^*} = P_i\,\delta P_i^T, \tag{2.18}$$

and, due to the fact that $P_i(t)$ is an orthogonal matrix, Eqs. 2.16 and 2.18 can be rewritten in the forms

$$\dot{P}_i = -\widetilde{\Omega}_iP_i, \tag{2.19}$$
$$\dot{P}_i^T = P_i^T\widetilde{\Omega}_i, \tag{2.20}$$
$$\delta P_i = -\widetilde{\delta\Theta_i^*}P_i, \tag{2.21}$$
$$\delta P_i^T = P_i^T\,\widetilde{\delta\Theta_i^*}. \tag{2.22}$$

At this point we derive a formula for $F_i(r_i, t)$, which was introduced earlier as the component matrix, with respect to inertial axes $x_oy_oz_o$, of the deformation gradient $\nabla\mathbf{z}_i$ of body i. Note first that the constant matrix $P_i(0)$ is the matrix of direction cosines of axes $x_{oi}y_{oi}z_{oi}$ with respect to axes $x_oy_oz_o$. Then, by definition [16], $F_i = \partial z_i/\partial\bar{r}_i$, in which $\bar{r}_i = P_i^T(0)r_i$ is the component matrix, along axes $x_oy_oz_o$, of position vector $\mathbf{r}_i$. Referring now to Eq. A.39 and the comment immediately preceding it, we see that

$$F_i(r_i, t) = \frac{\partial z_i(r_i, t)}{\partial r_i^T}P_i(0). \tag{2.23}$$

The equations developed so far in this section are valid for any of the bodies in the structure. However, for the time being at least, we proceed with the assumption that body i is rigid. This means that the distance between any two points in the body remains constant, which, when applied to points O_i and P_i (Fig. 1), implies that

$$r_{ei}(r_i, t) = r_i. \tag{2.24}$$

Substituting this into Eq. 2.14, the component matrix, along inertial axes $x_oy_oz_o$, for the position vector of an arbitrary point in a rigid body is given by

$$z_i = R_i + P_i^Tr_i. \tag{2.25}$$

Recalling that r_i is independent of time, so that $\dot{r}_i = 0$, we take time derivatives of Eq. 2.25 and make use of Eqs. 2.15 and 2.20 to get that

$$\begin{aligned}\dot{z}_i &= \dot{R}_i + P_i^T\widetilde{\Omega}_ir_i \\ &= P_i^T(V_i - \tilde{r}_i\Omega_i), \qquad (2.26)\\ \ddot{z}_i &= P_i^T(\dot{V}_i - \tilde{r}_i\dot{\Omega}_i) + P_i^T\widetilde{\Omega}_i(V_i - \tilde{r}_i\Omega_i) \\ &= P_i^T(\dot{V}_i - \tilde{r}_i\dot{\Omega}_i + \widetilde{\Omega}_iV_i - \widetilde{\Omega}_i\tilde{r}_i\Omega_i). \qquad (2.27)\end{aligned}$$

Then, analogous to Eq. 2.26, we take the variation of Eq. 2.25 and use Eqs. 2.17 and 2.22 to get that

$$\begin{aligned} \delta z_i &= \delta R_i + P_i^T \widetilde{\delta\Theta_i^*} r_i \\ &= P_i^T(\delta R_i^* - \tilde{r}_i \delta\Theta_i^*). \end{aligned} \tag{2.28}$$

Considering now the deformation gradient for the body, Eqs. 2.23, 2.25 and A.5 imply that

$$F_i = \frac{\partial z_i}{\partial r_i^T} P_i(0) = \frac{\partial(R_i + P_i^T r_i)}{\partial r_i^T} P_i(0) = P_i^T P_i(0). \tag{2.29}$$

Furthermore, we can use Eq. 2.22 to get the variation of the deformation gradient as

$$\delta F_i = P_i^T \widetilde{\delta\Theta_i^*} P_i(0), \tag{2.30}$$

and also, since P_i and $P_i(0)$ are both proper orthogonal matrices, F_i is also proper orthogonal, so that

$$\det F_i = 1. \tag{2.31}$$

We now have enough information to construct the formula for $\mathcal{R}_i$. Substituting Eqs. 2.27 and 2.28 into Eq. 2.5 and making use of Eq. A.11, the inertia term for a rigid body takes the form

$$\begin{aligned} \int_{\mathcal{B}_{oi}} \rho_{oi} \delta z_i^T \ddot{z}_i \, d\mathcal{V} &= \int_{\mathcal{B}_{oi}} \rho_{oi} \left(\delta R_i^{*T} + \delta\Theta_i^{*T} \tilde{r}_i\right) P_i P_i^T \left(\dot{V}_i - \tilde{r}_i \dot{\Omega}_i + \tilde{\Omega}_i V_i - \tilde{\Omega}_i \tilde{r}_i \Omega_i\right) d\mathcal{V} \\ &= \delta R_i^{*T} \left(m_i \dot{V}_i - \tilde{S}_{ei} \dot{\Omega}_i + \tilde{\Omega}_i V_i - \tilde{\Omega}_i \tilde{S}_{ei} \Omega_i\right) \\ &\quad + \delta\Theta_i^{*T} \left(\tilde{S}_{ei} \dot{V}_i + J_{ei} \dot{\Omega}_i + \tilde{S}_{ei} \tilde{\Omega}_i V_i + \tilde{\Omega}_i J_{ei} \Omega_i\right), \end{aligned} \tag{2.32}$$

in which

$$m_i = \int_{\mathcal{B}_{oi}} \rho_{oi}(r_i) \, d\mathcal{V}, \tag{2.33-a}$$

$$S_{ei} = \int_{\mathcal{B}_{oi}} \rho_{oi}(r_i) r_i \, d\mathcal{V}, \tag{2.33-b}$$

$$J_{ei} = \int_{\mathcal{B}_{oi}} \rho_{oi}(r_i) \tilde{r}_i \tilde{r}_i^T \, d\mathcal{V}, \tag{2.33-c}$$

$$\tag{2.33-d}$$

are the mass, first moments of inertia, and mass moments of inertia, respectively, for the body. Next, substituting Eqs. 2.3, 2.29 and 2.30 into Eq. 2.5, and making use of Eqs. A.37, 2.4 and 2.31, the internal force term for a rigid body takes the form

$$\begin{aligned}\int_{\mathcal{B}_{oi}} \mathrm{Tr}\Big[\mathsf{T}_{oi}^{\mathsf{T}}\delta\mathsf{F}_i\Big]\,d\mathcal{V} &= \int_{\mathcal{B}_{oi}} \mathrm{Tr}\Big[(\det \mathsf{F}_i)\mathsf{F}_i^{-1}\mathsf{T}_{ci}\mathsf{P}_i^{\mathsf{T}}\,\widetilde{\delta\Theta}_i^{*}\mathsf{P}_i(0)\Big]\,d\mathcal{V} \\ &= \int_{\mathcal{B}_{oi}} \mathrm{Tr}\Big[\mathsf{P}_i^{\mathsf{T}}(0)\mathsf{P}_i\,\mathsf{T}_{ci}\mathsf{P}_i^{\mathsf{T}}\,\widetilde{\delta\Theta}_i^{*}\mathsf{P}_i(0)\Big]\,d\mathcal{V} \\ &= \int_{\mathcal{B}_{oi}} \mathrm{Tr}\Big[\widetilde{\delta\Theta}_i^{*}\mathsf{P}_i(0)\mathsf{P}_i^{\mathsf{T}}(0)(\mathsf{P}_i\,\mathsf{T}_{ci}\mathsf{P}_i^{\mathsf{T}})\Big]\,d\mathcal{V} \\ &= \int_{\mathcal{B}_{oi}} \mathrm{Tr}\Big[\widetilde{\delta\Theta}_i^{*}(\mathsf{P}_i\,\mathsf{T}_{ci}\mathsf{P}_i^{\mathsf{T}})\Big]\,d\mathcal{V} \;=\; 0, \end{aligned} \tag{2.34}$$

where the last equality follows from Eq. A.38 and the fact that $\widetilde{\delta\Theta}_i^{*}$ is skew-symmetric and $\mathsf{P}_i\,\mathsf{T}_{ci}\mathsf{P}_i^{\mathsf{T}}$ is symmetric.

We consider now the surface traction term. In this regard, first note that the surface traction τ_{oi} can be written as

$$\tau_{oi} = \tau_{oi}^{*} + \mathsf{P}_i^{\mathsf{T}}\tau_{si}, \tag{2.35}$$

where τ_{oi}^{*} is nonzero only on that part of the surface of body i which contacts another body in the structure, and τ_{si} is nonzero only on that part of the surface of body i which does not contact any other body in the structure. Note that including $\mathsf{P}_i^{\mathsf{T}}$ in this formula implies that τ_{si} is in terms of components along body axes $x_iy_iz_i$. Then, substituting into Eq. 2.5 and making use of Eq. 2.28, we get that

$$\begin{aligned}\int_{\partial\mathcal{B}_{oi}} \delta\mathsf{z}_i^{\mathsf{T}}\tau_{oi}\,d\mathcal{A} &= \int_{\partial\mathcal{B}_{oi}} \delta\mathsf{z}_i^{\mathsf{T}}\tau_{oi}^{*}\,d\mathcal{A} + \int_{\partial\mathcal{B}_{oi}} \Big(\delta\mathsf{R}_i^{*\mathsf{T}} + \delta\Theta_i^{*\mathsf{T}}\tilde{\mathsf{r}}_i\Big)\mathsf{P}_i\,\mathsf{P}_i^{\mathsf{T}}\tau_{si}\,d\mathcal{A} \\ &= \int_{\partial\mathcal{B}_{oi}} \delta\mathsf{z}_i^{\mathsf{T}}\tau_{oi}^{*}\,d\mathcal{A} + \delta\mathsf{R}_i^{*\mathsf{T}}\mathsf{F}_{si} + \delta\Theta_i^{*\mathsf{T}}\mathsf{M}_{si}, \end{aligned} \tag{2.36}$$

where

$$\mathsf{F}_{si}(t) = \int_{\partial\mathcal{B}_{oi}} \tau_{si}(\mathsf{r}_i, t)\,d\mathcal{A}, \tag{2.37-a}$$

$$\mathsf{M}_{si}(t) = \int_{\partial\mathcal{B}_{oi}} \tilde{\mathsf{r}}_i\tau_{si}(\mathsf{r}_i, t)\,d\mathcal{A}, \tag{2.37-b}$$

are external force and moment component matrices, both along body axes $x_iy_iz_i$. Now combine

Eqs. 2.32, 2.34 and 2.36, plus the fact that the body force is assumed to be zero, to get that

$$\mathcal{R}_i = \begin{bmatrix} \delta R_i^{*T} & \delta\Theta_i^{*T} \end{bmatrix} \left(M_{rri} \begin{bmatrix} \dot{V}_i \\ \dot{\Omega}_i \end{bmatrix} - G_{ri} \right) - \int_{\partial\mathcal{B}_{oi}} \delta z_i^T \tau_{oi}^* \, d\mathcal{A}, \tag{2.38}$$

where

$$M_{rri} = \begin{bmatrix} m_i I & -\widetilde{S}_{ei} \\ \widetilde{S}_{ei} & J_{ei} \end{bmatrix}, \tag{2.39-a}$$

$$G_{ri} = \begin{bmatrix} -m_i\widetilde{\Omega}_i & \widetilde{\Omega}_i\widetilde{S}_{ei} \\ -\widetilde{S}_{ei}\widetilde{\Omega}_i & -\widetilde{\Omega}_i J_{ei} \end{bmatrix} \begin{bmatrix} V_i \\ \Omega_i \end{bmatrix} + \begin{bmatrix} F_{si} \\ M_{si} \end{bmatrix}. \tag{2.39-b}$$

2.3 Slender Beams

The reference configuration $\mathcal{B}_{oi}$ for a slender beam can be written in the form [52]

$$\mathcal{B}_{oi} = \{(x, y, z) | (y, z) \in \mathcal{A}_{ci}(x) \text{ for } 0 \le x \le \ell_i\}, \tag{2.40}$$

where ℓ_i is the length of the beam and $\mathcal{A}_{ci}(x) \subseteq \mathrm{R}^2$ defines a cross-section. We make the common simplifying assumption that cross-sections remain planar, which allows the introduction of an orthogonal coordinate system $\{O_{ci}; \xi_i\eta_i\zeta_i\}$ (Fig. 2) fixed on cross-section $\mathcal{A}_{ci}(x)$. We also assume that the ξ_i-axis coincides with the central axis of the beam in undeformed state, and that body axes $x_iy_iz_i$ coincide with cross-section axes $\xi_i\eta_i\zeta_i$ at $x = 0$.

We start by considering the motion of points in cross-section $\mathcal{A}_{ci}(x)$ relative to the inertial space. In fact, due to the assumption that cross-sections remain planar, the analysis is essentially the same as for a rigid body. To this end, $R_{ci}(x, t)$ denotes the component matrix, along inertial axes $x_oy_oz_o$, of position vector $\mathbf{R}_{ci}(x, t) = \overrightarrow{O_oO_{ci}}$ (Fig. 2) and $P_{ci}(x, t)$ is the matrix of direction cosines of cross-section axes $\xi_i\eta_i\zeta_i$ with respect to inertial axes $x_oy_oz_o$. Furthermore, $V_{ci}(x, t)$ is the component matrix of velocity vector $\dot{\mathbf{R}}_{ci}(x, t)$ and $\Omega_{ci}(x, t)$ is the component matrix of the angular velocity vector, $\boldsymbol{\Omega}_{ci}(x, t)$, of axes $\xi_i\eta_i\zeta_i$ with respect to axes $x_oy_oz_o$, both with components along cross-section axes $\xi_i\eta_i\zeta_i$. We also make use of quasi-virtual displacements analogous to $V_{ci}(x, t)$ and $\Omega_{ci}(x, t)$, denoted by $\delta R_{ci}^*(x, t)$ and $\delta\Theta_{ci}^*(x, t)$, respectively, and let

$$r_p = [0 \;\; y \;\; z]^T \tag{2.41}$$

be the component matrix, along cross-section axes $\xi_i\eta_i\zeta_i$, of vector $\overrightarrow{O_{ci}P_i}$, where P_i is a point in the beam with Lagrangian coordinates (x, y, z).

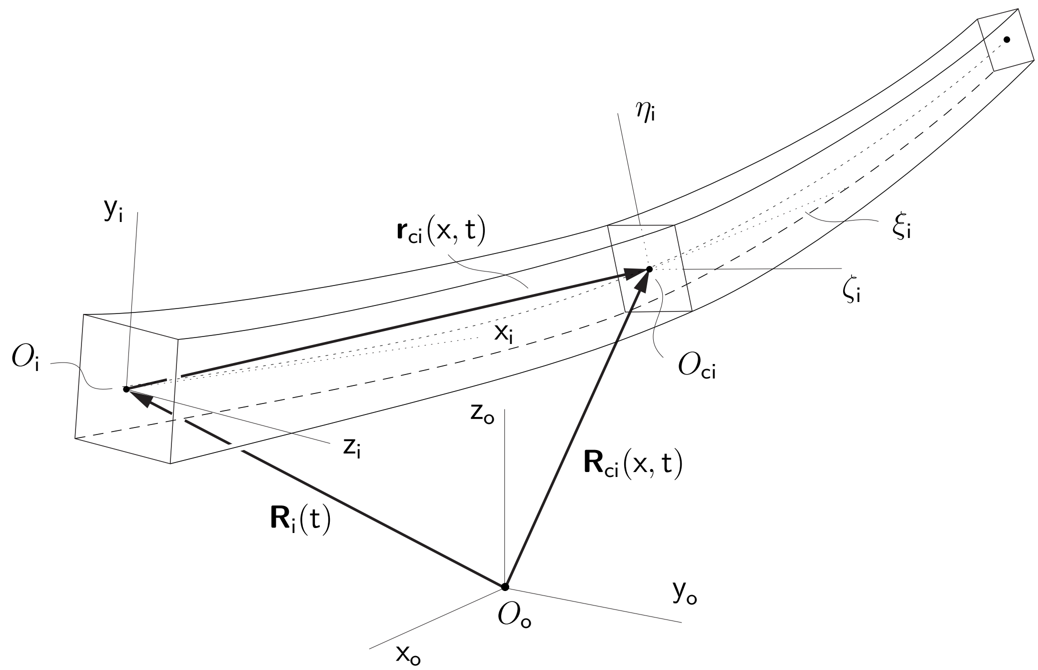

Figure 2: A Slender Beam in Deformed Configuration

Based on the preceding definitions we can invoke Eqs. A.24, A.25, A.26 and A.27 to get that

$$V_{ci} = P_{ci}\,\dot{R}_{ci}, \tag{2.42}$$
$$\widetilde{\Omega}_{ci} = P_{ci}\,\dot{P}^T_{ci}, \tag{2.43}$$
$$\delta R^*_{ci} = P_{ci}\,\delta R_{ci}, \tag{2.44}$$
$$\widetilde{\delta\Theta}^*_{ci} = P_{ci}\,\delta P^T_{ci}. \tag{2.45}$$

Furthermore, since $R_{ci}(x,t)$ is the component matrix of a position vector, and $P_{ci}(x,t)$ is a matrix of direction cosines, both functions of the spatial variable x, we can introduce two 3-by-1 matrices, $\beta_{ci}(x,t)$ and $\phi_{ci}(x,t)$, defined by [52] (refer to the comment immediately preceding Eqs. A.26 and A.27)

$$\beta_{ci}(x,t) = P_{ci}(x,t)R'_{ci}(x,t), \tag{2.46}$$
$$\widetilde{\phi}_{ci}(x,t) = P_{ci}(x,t)P'^T_{ci}(x,t), \tag{2.47}$$

where the “prime” represents partial differentiation with respect to x. As will be seen shortly, the components of β_{ci} and ϕ_{ci} are related to the strain in the beam.

Before proceeding, we point out that for a slender beam, integration over the body and integration over the surface of the body, excluding the two ends, can be decomposed as follows:

$$\int_{\mathcal{B}_{oi}} d\mathcal{V} = \int_0^{\ell_i} \iint_{\mathcal{A}_{ci}(x)} dydz\, dx, \tag{2.48-a}$$

$$\int_{\partial\mathcal{B}_{oi}} d\mathcal{A} = \int_0^{\ell_i} \oint_{\partial\mathcal{A}_{ci}(x)} ds\, dx, \tag{2.48-b}$$

where s measures arc length around the boundary of cross-section $\mathcal{A}_{ci}(x)$.

Because of the assumption that a beam cross-section translates and rotates as a thin rigid plate, we can make use of the rigid body derivations and substitute r_p and R_{ci} for r_i and R_i, respectively, in Eq. 2.25, and then substitute V_{ci}, Ω_{ci}, δR^*_{ci} and $\delta\Theta^*_{ci}$ for V_i, Ω_i, δR^*_i and $\delta\Theta^*_i$, respectively, in Eqs. 2.25, 2.26, 2.27 and 2.28 to arrive at

$$z_i = R_{ci} + P^T_{ci} r_p, \tag{2.49}$$

$$\dot{z}_i = P^T_{ci}\Big(V_{ci} - \tilde{r}_p\Omega_{ci}\Big), \tag{2.50}$$

$$\ddot{z}_i = P^T_{ci}\Big(\dot{V}_{ci} - \tilde{r}_p\dot{\Omega}_{ci} + \widetilde{\Omega}_{ci}V_{ci} - \widetilde{\Omega}_{ci}\tilde{r}_p\Omega_{ci}\Big), \tag{2.51}$$

$$\delta z_i = P^T_{ci}\Big(\delta R^*_{ci} - \tilde{r}_p\delta\Theta^*_{ci}\Big). \tag{2.52}$$

To derive a formula for the inertia term, combine Eqs. 2.51 and 2.52, and then make use of Eq. 2.48-a to get that

$$\begin{aligned}
\int_{\mathcal{B}_{oi}} \rho_{oi}\delta z_i^T \ddot{z}_i\, d\mathcal{V} &= \int_0^{\ell_i} \left\{ \iint_{\mathcal{A}_{ci}(x)} \rho_{oi}\Big(\delta R^{*T}_{ci} + \delta\Theta^{*T}_{ci}\tilde{r}_p\Big) P_{ci} P^T_{ci}\Big(\dot{V}_{ci} - \tilde{r}_p\dot{\Omega}_{ci} + \widetilde{\Omega}_{ci}V_{ci} - \widetilde{\Omega}_{ci}\tilde{r}_p\Omega_{ci}\Big)\, dydz \right\} dx \\
& \int_0^{\ell_i} \left\{ \iint_{\mathcal{A}_{ci}(x)} \rho_{oi}\Big[\delta R^{*T}_{ci}\Big(\dot{V}_{ci} - \tilde{r}_p\dot{\Omega}_{ci} + \widetilde{\Omega}_{ci}V_{ci} - \widetilde{\Omega}_{ci}\tilde{r}_p\Omega_{ci}\Big)\right. \\
& \qquad \left. +\delta\Theta^{*T}_{ci}\Big(\tilde{r}_p\dot{V}_{ci} - \tilde{r}^2_p\dot{\Omega}_{ci} + \tilde{r}_p\widetilde{\Omega}_{ci}V_{ci} - \widetilde{\Omega}_{ci}\tilde{r}^2_p\Omega_{ci}\Big)\Big]\, dydz \right\} dx \\
&= \int_0^{\ell_i} \Big[\delta R^{*T}_{ci}\Big(\rho_{ci}\dot{V}_{ci} + \rho_{ci}\widetilde{\Omega}_{ci}V_{ci}\Big) + \delta\Theta^*_{ci}\Big(\hat{J}_{ci}\dot{\Omega}_{ci} + \widetilde{\Omega}_{ci}\hat{J}_{ci}\Omega_{ci}\Big)\Big]\, dx,
\end{aligned} \tag{2.53}$$

where

$$\rho_{ci}(x) = \iint_{\mathcal{A}_{ci}(x)} \rho_{oi}(x,y,z)\, dydz \tag{2.54}$$

is the mass per unit length,

$$\begin{aligned}
\hat{J}_{ci}(x) &= \iint_{\mathcal{A}_{ci}(x)} \rho_{oi}(x,y,z)\tilde{r}_p\tilde{r}_p^T\, dydz \\
&= \text{diag}\Big[\hat{J}_{xxi}(x),\ \hat{J}_{yyi}(x),\ \hat{J}_{zzi}(x)\Big]
\end{aligned} \tag{2.55}$$

is the mass moment of inertia density, with

$$\hat{J}_{yyi}(x) = \iint\limits_{\mathcal{A}_{ci}(x)} \rho_{oi}(x,y,z)z^2\, dydz, \tag{2.56-a}$$

$$\hat{J}_{zzi}(x) = \iint\limits_{\mathcal{A}_{ci}(x)} \rho_{oi}(x,y,z)y^2\, dydz, \tag{2.56-b}$$

$$\hat{J}_{xxi}(x) = \hat{J}_{yyi}(x) + \hat{J}_{zzi}(x), \tag{2.56-c}$$

and we have assumed that each cross-section is symmetric, implying that

$$\iint\limits_{\mathcal{A}_{ci}(x)} \rho_{oi}(x,y,z) r_p\, dydz = 0. \tag{2.57}$$

For future use, we also define

$$J^*_{ci}(x) = \mathrm{diag}\Big[0,\ \hat{J}_{zzi}(x),\ \hat{J}_{yyi}(x)\Big], \tag{2.58}$$

and note that

$$\hat{J}_{ci} + J^*_{ci} = \hat{J}_{xxi} I, \tag{2.59}$$

in which I is the 3-by-3 identity matrix.

To derive a formula for the deformation gradient we make use of Eqs. 2.23, 2.41, 2.46, 2.47 and 2.49 to get that

$$\begin{aligned} F_i &= \frac{\partial z_i}{\partial r_i^T} P_i(0) = \Big[\frac{\partial z_i}{\partial x} \vdots \frac{\partial z_i}{\partial y} \vdots \frac{\partial z_i}{\partial z}\Big] P_i(0) = \Big[R'_{ci} + P^T_{ci}\widetilde{\phi}_{ci} r_p \vdots P^T_{ci} e_2 \vdots P^T_{ci} e_3\Big] P_i(0) \\ &= P^T_{ci}\Big[\beta_{ci} - \tilde{r}_p \phi_{ci} \vdots e_2 \vdots e_3\Big] P_i(0) = P^T_{ci}\Big[\epsilon_{ci} + e_1 \vdots e_2 \vdots e_3\Big] P_i(0) \\ &= P^T_{ci} \begin{bmatrix} \epsilon_{xi}+1 & 0 & 0 \\ \epsilon_{yi} & 1 & 0 \\ \epsilon_{zi} & 0 & 1 \end{bmatrix} P_i(0) \qquad (2.60\text{-a}) \\ &= P^T_{ci}(\epsilon_{ci} e_1^T + I) P_i(0), \qquad (2.60\text{-b}) \end{aligned}$$

in which the component matrix of a *strain vector* for the body is defined by

$$\epsilon_{ci}(r_i,t) = \begin{bmatrix} \epsilon_{xi}(r_i,t) \\ \epsilon_{yi}(r_i,t) \\ \epsilon_{zi}(r_i,t) \end{bmatrix} = \beta_{ci}(x,t) - e_1 - \tilde{r}_p \phi_{ci}(x,t). \tag{2.61}$$

Now take the variation of Eq. 2.60-b, and then use Eqs. A.29 and 2.45 to get that

$$\begin{aligned} \delta F_i &= P^T_{ci}\widetilde{\delta\Theta^*_{ci}}\left(\epsilon_{ci} e_1^T + I\right) P_i(0) + P^T_{ci}\Big(\delta\epsilon_{ci} e_1^T\Big) P_i(0) \\ &= \mathrm{Tilde}\Big(P^T_{ci}\delta\Theta^*_{ci}\Big) F_i + P^T_{ci}\Big(\delta\epsilon_{ci} e_1^T\Big) P_i(0). \end{aligned} \tag{2.62}$$

Furthermore, we can use Eq. 2.60-a to arrive at the formulas

$$\det \mathsf{F_i} = \epsilon_{xi} + 1, \tag{2.63}$$

$$(\det \mathsf{F_i})\mathsf{F_i}^{-1} = \mathsf{P_i^T}(0)\begin{bmatrix} 1 & 0 & 0 \\ -\epsilon_{yi} & 1+\epsilon_{xi} & 0 \\ -\epsilon_{zi} & 0 & 1+\epsilon_{xi} \end{bmatrix}\mathsf{P_{ci}}\,. \tag{2.64}$$

Consequently, making use of Eqs. A.37, A.38, 2.3, 2.4, 2.62 and 2.64, the internal force term takes the form

$$\begin{aligned}
\int_{\mathcal{B}_{oi}} \mathrm{Tr}[\mathsf{T_{oi}^T}\delta \mathsf{F_i}]\,\mathsf{d}\mathcal{V} &= \int_{\mathcal{B}_{oi}} \mathrm{Tr}\Big[(\det \mathsf{F_i})\mathsf{F_i}^{-1}\mathsf{T_{ci}}\delta\mathsf{F_i}\Big]\,\mathsf{d}\mathcal{V} \\
&= \int_{\mathcal{B}_{oi}} \mathrm{Tr}\Big[(\det \mathsf{F_i})\mathsf{F_i}^{-1}\mathsf{T_{ci}}\Big\{\mathrm{Tilde}\Big(\mathsf{P_{ci}^T}\delta\Theta_{ci}^*\Big)\mathsf{F_i} + \mathsf{P_{ci}^T}\Big(\delta\epsilon_{ci}\mathsf{e_1^T}\Big)\mathsf{P_i}\,(0)\Big\}\Big]\,\mathsf{d}\mathcal{V} \\
&= \int_{\mathcal{B}_{oi}} \mathrm{Tr}\Big[(\det \mathsf{F_i})\mathrm{Tilde}\Big(\mathsf{P_{ci}^T}\delta\Theta_{ci}^*\Big)\mathsf{F_i}\,\mathsf{F_i}^{-1}\mathsf{T_{ci}}\Big]\,\mathsf{d}\mathcal{V} \\
&\quad + \int_{\mathcal{B}_{oi}} \mathrm{Tr}\left\{\mathsf{P_i^T}(0)\begin{bmatrix} 1 & 0 & 0 \\ -\epsilon_{yi} & 1+\epsilon_{xi} & 0 \\ -\epsilon_{zi} & 0 & 1+\epsilon_{xi} \end{bmatrix}\mathsf{P_{ci}}\,\mathsf{T_{ci}}\mathsf{P_{ci}^T}\Big(\delta\epsilon_{ci}\mathsf{e_1^T}\Big)\mathsf{P_i}\,(0)\right\}\mathsf{d}\mathcal{V} \\
&= \int_{\mathcal{B}_{oi}} \mathrm{Tr}\left\{\Big(\mathsf{P_{ci}}\,\mathsf{T_{ci}}\mathsf{P_{ci}^T}\Big)\begin{bmatrix} \delta\epsilon_{xi} & 0 & 0 \\ \delta\epsilon_{yi} & 0 & 0 \\ \delta\epsilon_{zi} & 0 & 0 \end{bmatrix}\mathsf{P_i}\,(0)\mathsf{P_i^T}(0)\begin{bmatrix} 1 & 0 & 0 \\ -\epsilon_{yi} & 1+\epsilon_{xi} & 0 \\ -\epsilon_{zi} & 0 & 1+\epsilon_{xi} \end{bmatrix}\right\}\mathsf{d}\mathcal{V}.
\end{aligned} \tag{2.65}$$

Now take note that $\mathsf{P_{ci}}\,\mathsf{T_{ci}}\mathsf{P_{ci}^T}$ is the component matrix, with respect to cross-section axes $\xi_i\eta_i\zeta_i$, of the Cauchy stress tensor, and let

$$\tau_{ci}(\mathsf{r_i},\mathsf{t}) = \begin{bmatrix} \tau_{xi}(\mathsf{r_i},\mathsf{t}) \\ \tau_{yi}(\mathsf{r_i},\mathsf{t}) \\ \tau_{zi}(\mathsf{r_i},\mathsf{t}) \end{bmatrix} \tag{2.66}$$

be the component matrix, along cross-section axes $\xi_i\eta_i\zeta_i$, for the stress vector acting on the cross-section $\mathcal{A}_{ci}(\mathsf{x})$, as shown in Fig. 3. Recall from basic continuum mechanics [16] that since the ξ_i-axis is perpendicular to the cross-section, τ_{ci} is both the first row and first column of component matrix $\mathsf{P_{ci}T_{ci}P_{ci}^T}$. This allows writing

$$\mathsf{P_{ci}}\,\mathsf{T_{ci}}\mathsf{P_{ci}^T} = \begin{bmatrix} \tau_{xi} & \tau_{yi} & \tau_{zi} \\ \tau_{yi} & \mathsf{T}_{22} & \mathsf{T}_{32} \\ \tau_{zi} & \mathsf{T}_{32} & \mathsf{T}_{33} \end{bmatrix}, \tag{2.67}$$

in which T_{22}, T_{32} and T_{33} are stress tensor components which enforce the "cross-sections remain planar" assumption. Continuing from Eq. 2.65, and making use of Eqs. 2.48-a and 2.61,

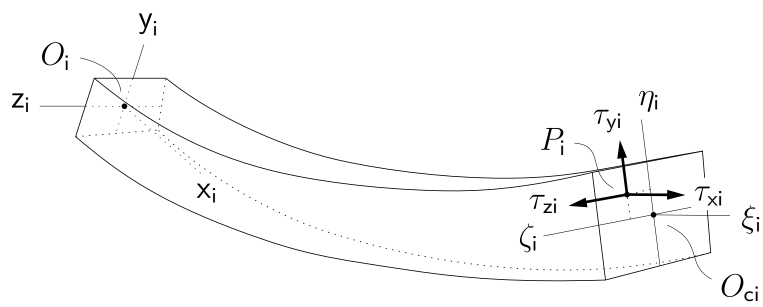

Figure 3: Components of Stress Vector Acting on a Cross-Section

we see then that the internal force term can be written as

$$\begin{aligned}
\int_{\mathcal{B}_{oi}} \mathrm{Tr}\Big[\mathsf{T}_{oi}^{\mathsf{T}}\delta\mathsf{F}_i\Big]\, d\mathcal{V} &= \int_{\mathcal{B}_{oi}} \mathrm{Tr}\left\{\begin{bmatrix} \tau_{xi} & \tau_{yi} & \tau_{zi} \\ \tau_{yi} & \mathsf{T}_{22} & \mathsf{T}_{32} \\ \tau_{zi} & \mathsf{T}_{32} & \mathsf{T}_{33}\end{bmatrix}\begin{bmatrix} \delta\epsilon_{xi} & 0 & 0 \\ \delta\epsilon_{yi} & 0 & 0 \\ \delta\epsilon_{zi} & 0 & 0\end{bmatrix}\right\} d\mathcal{V} \\
&= \int_{\mathcal{B}_{oi}} \delta\epsilon_{ci}^{\mathsf{T}}\tau_{ci}\, d\mathcal{V} \qquad (2.68\text{-a}) \\
&= \int_0^{\ell_i}\iint_{\mathcal{A}_{ci}(x)} \Big(\delta\beta_{ci}^{\mathsf{T}} + \delta\phi_{ci}^{\mathsf{T}}\tilde{\mathsf{r}}_p\Big)\tau_{ci}\ dy dz\, dx \\
&= \int_0^{\ell_i}\Big(\delta\beta_{ci}^{\mathsf{T}}\mathsf{f}_{ci} + \delta\phi_{ci}^{\mathsf{T}}\mathsf{m}_{ci}\Big)\, dx, \qquad (2.68\text{-b})
\end{aligned}$$

in which

$$\mathsf{f}_{ci}(x) = \begin{bmatrix} \mathsf{f}_{xi}(x) \\ \mathsf{f}_{yi}(x) \\ \mathsf{f}_{zi}(x)\end{bmatrix} = \iint_{\mathcal{A}_{ci}(x)} \tau_{ci}(x,y,z,t)\, dydz, \qquad (2.69\text{-a})$$

$$\mathsf{m}_{ci}(x) = \begin{bmatrix} \mathsf{m}_{xi}(x) \\ \mathsf{m}_{yi}(x) \\ \mathsf{m}_{zi}(x)\end{bmatrix} = \iint_{\mathcal{A}_{ci}(x)} \tilde{\mathsf{r}}_p\tau_{ci}(x,y,z,t)\, dydz. \qquad (2.69\text{-b})$$

Note that f_{xi} is the axial force, f_{yi} and f_{zi} are shear forces, m_{xi} is the twisting moment, and m_{yi} and m_{zi} are bending moments for the beam.

Considering the fact that β_{ci} and ϕ_{ci} depend on R_{ci} and P_{ci}, it is possible to derive a relationship between $\delta\beta_{ci}$ and $\delta\phi_{ci}$ on the one hand, and $\delta\mathsf{R}_{ci}^*$ and $\delta\Theta_{ci}^*$ on the other hand. To this end, we take the variation of both sides of Eqs. 2.46 and 2.47, and then, making use of Eqs. 2.44, 2.45, 2.46 and 2.47, plus the fact that the variational operator and partial differentiation with respect to x commute, we get that

$$\begin{aligned}\delta\beta_{ci} &= (\delta P_{ci})R'_{ci} + P_{ci}\,\delta R'_{ci} \\ &= -\widetilde{\delta\Theta}^*_{ci}\,P_{ci}\,R'_{ci} + (P_{ci}\,\delta R_{ci})' - P'_{ci}\delta R_{ci} \\ &= -\widetilde{\delta\Theta}^*_{ci}\,\beta_{ci} + \delta R^{*\prime}_{ci} + \widetilde{\phi}_{ci}P_{ci}\,\delta R_{ci} \\ &= \delta R^{*\prime}_{ci} + \widetilde{\phi}_{ci}\delta R^*_{ci} + \widetilde{\beta}_{ci}\delta\Theta^*_{ci}\,, \end{aligned} \tag{2.70}$$

$$\begin{aligned}\widetilde{\delta\phi}_{ci} &= (\delta P_{ci})P'^{T}_{ci} + P_{ci}\,\delta P'^{T}_{ci} \\ &= -\widetilde{\delta\Theta}^*_{ci}\,P_{ci}\,P'^{T}_{ci} + (P_{ci}\,\delta P^{T}_{ci})' - P'_{ci}\delta P^{T}_{ci} \\ &= -\widetilde{\delta\Theta}^*_{ci}\,\widetilde{\phi}_{ci} + \widetilde{\delta\Theta}^{*\prime}_{ci} + \widetilde{\phi}_{ci}P_{ci}\,\delta P^{T}_{ci} \\ &= \widetilde{\delta\Theta}^{*\prime}_{ci} + \widetilde{\phi}_{ci}\widetilde{\delta\Theta}^*_{ci} - \widetilde{\delta\Theta}^*_{ci}\,\widetilde{\phi}_{ci} \\ &= \widetilde{\delta\Theta}^{*\prime}_{ci} + (\widetilde{\widetilde{\phi}_{ci}\delta\Theta^*_{ci}}), \end{aligned} \tag{2.71}$$

where the last equality follows from Eq. A.10. Equation 2.71 then implies that

$$\delta\phi_{ci} = \delta\Theta^{*\prime}_{ci} + \widetilde{\phi}_{ci}\delta\Theta^*_{ci}\,. \tag{2.72}$$

We turn now to the surface traction term. As with the rigid body case, we can split τ_{oi} into two parts,

$$\tau_{oi} = \tau^*_{oi} + P^{T}_{ci}\tau_{si}, \tag{2.73}$$

where τ^*_{oi} is nonzero only on that part of the surface of body i which contacts another body in the structure, which in this case means one or both ends of the beam, and τ_{si} is nonzero only on that part of the surface of body i which does not contact any other body in the structure. Note that including P^{T}_{ci} in this formula implies that τ_{si} is in terms of components along cross-section axes $\xi_i\eta_i\zeta_i$. Then, making use of Eqs. 2.48-b and 2.52, we get that

$$\begin{aligned}\int_{\partial\mathcal{B}_{oi}} \delta z_i^{T}\tau_{oi}\,d\mathcal{A} &= \int_{\partial\mathcal{B}_{oi}} \delta z_i^{T}\tau^*_{oi}\,d\mathcal{A} + \int_0^{\ell_i}\oint_{\partial\mathcal{A}_{ci}(x)} \left(\delta R^{*T}_{ci} + \delta\Theta^{*T}_{ci}\tilde{r}_p\right)P_{ci}\,P^{T}_{ci}\,\tau_{si}\,ds\,dx \\ &= \int_{\partial\mathcal{B}_{oi}} \delta z_i^{T}\tau^*_{oi}\,d\mathcal{A} + \int_0^{\ell_i}\left(\delta R^{*T}_{ci}\hat{f}_{si} + \delta\Theta^{*T}_{ci}\hat{m}_{si}\right)dx, \end{aligned} \tag{2.74}$$

where

$$\hat{f}_{si}(x,t) = \begin{bmatrix}\hat{f}_{xi}(x,t)\\ \hat{f}_{yi}(x,t)\\ \hat{f}_{zi}(x,t)\end{bmatrix} = \oint_{\partial\mathcal{A}_{ci}(x)} \tau_{si}(x,y,z,t)\,ds, \tag{2.75-a}$$

$$\hat{m}_{si}(x,t) = \begin{bmatrix}\hat{m}_{xi}(x,t)\\ \hat{m}_{yi}(x,t)\\ \hat{m}_{zi}(x,t)\end{bmatrix} = \oint_{\partial\mathcal{A}_{ci}(x)} \tilde{r}_p\tau_{si}(x,y,z,t)\,ds, \tag{2.75-b}$$

are surface force and moment component matrices, respectively, which can be used to account both for forces acting along the length of the beam, as well as forces and moments external to the structure as a whole acting at the ends of the beam.

Now combine Eqs. 2.53, 2.68-b, 2.70, 2.72 and 2.74, plus the assumption that body forces are zero, with Eq. 2.5, to get that

$$\mathcal{R}_i = \int_0^{\ell_i} \Big[\delta R_{ci}^{*T}\Big(\rho_{ci}\dot{V}_{ci} + \rho_{ci}\widetilde{\Omega}_{ci}V_{ci} - \hat{f}_{si}\Big) + \delta\Theta_{ci}^{*T}\Big(\hat{J}_{ci}\dot{\Omega}_{ci} + \widetilde{\Omega}_{ci}\hat{J}_{ci}\Omega_{ci} - \hat{m}_{si}\Big)$$

$$+\delta\beta_{ci}^{T}f_{ci} + \delta\phi_{ci}^{T}m_{ci}\Big]\,dx - \int_{\partial\mathcal{B}_{oi}} \delta z_i^{T}\tau_{oi}^{*}\,d\mathcal{A} \qquad (2.76\text{-a})$$

$$= \int_0^{\ell_i} \Big[\delta R_{ci}^{*T}\Big(\rho_{ci}\dot{V}_{ci} + \rho_{ci}\widetilde{\Omega}_{ci}V_{ci} - \hat{f}_{si}\Big) + \delta\Theta_{ci}^{*T}\Big(\hat{J}_{ci}\dot{\Omega}_{ci} + \widetilde{\Omega}_{ci}\hat{J}_{ci}\Omega_{ci} - \hat{m}_{si}\Big)$$

$$+\Big(\delta R_{ci}^{*\prime T} - \delta R_{ci}^{*T}\widetilde{\phi}_{ci} - \delta\Theta_{ci}^{*T}\widetilde{\beta}_{ci}\Big)f_{ci} + \Big(\delta\Theta_{ci}^{*\prime T} - \delta\Theta_{ci}^{*T}\widetilde{\phi}_{ci}\Big)m_{ci}\Big]dx - \int_{\partial\mathcal{B}_{oi}} \delta z_i^{T}\tau_{oi}^{*}\,d\mathcal{A}$$

$$= \int_0^{\ell_i} \Big[\delta R_{ci}^{*T}\Big(\rho_{ci}\dot{V}_{ci} + \rho_{ci}\widetilde{\Omega}_{ci}V_{ci} - \widetilde{\phi}_{ci}f_{ci} - \hat{f}_{si}\Big) + \delta\Theta_{ci}^{*T}\Big(\hat{J}_{ci}\dot{\Omega}_{ci} + \widetilde{\Omega}_{ci}\hat{J}_{ci}\Omega_{ci} - \widetilde{\beta}_{ci}f_{ci} - \widetilde{\phi}_{ci}m_{ci} - \hat{m}_{si}\Big)$$

$$+\delta R_{ci}^{*\prime T}f_{ci} + \delta\Theta_{ci}^{*\prime T}m_{ci}\Big]\,dx - \int_{\partial\mathcal{B}_{oi}} \delta z_i^{T}\tau_{oi}^{*}\,d\mathcal{A} \qquad (2.76\text{-b})$$

$$= \int_0^{\ell_i} \Big[\delta R_{ci}^{*T}\Big(\rho_{ci}\dot{V}_{ci} + \rho_{ci}\widetilde{\Omega}_{ci}V_{ci} - f_{ci}' - \widetilde{\phi}_{ci}f_{ci} - \hat{f}_{si}\Big) + \delta\Theta_{ci}^{*T}\Big(\hat{J}_{ci}\dot{\Omega}_{ci} + \widetilde{\Omega}_{ci}\hat{J}_{ci}\Omega_{ci} - m_{ci}' - \widetilde{\beta}_{ci}f_{ci}$$

$$-\widetilde{\phi}_{ci}m_{ci} - \hat{m}_{si}\Big)\Big]\,dx + \Big(\delta R_{ci}^{*T}f_{ci} + \delta\Theta_{ci}^{*T}m_{ci}\Big)\Big|_0^{\ell_i} - \int_{\partial\mathcal{B}_{oi}} \delta z_i^{T}\tau_{oi}^{*}\,d\mathcal{A}, \qquad (2.76\text{-c})$$

where the last equality follows from integration by parts. Note that Eqs. 2.76-a and 2.76-b are in variational form, suitable for producing discretized equations of motion.

Chapter 3 Spatial Discretization

The discretization process involves three steps. To begin with, we introduce local kinematic variables which describe the motion of points on the beam relative to the moving coordinate system $\{O_i; x_i y_i z_i\}$. This, in itself, does not introduce any approximations. Next, we specialize to a Rayleigh beam and make a small motions assumption, and then finally introduce shape functions and generalized coordinates to model the elastic displacement of the beam.

3.1 Non-Inertial Beam Equations

The goal of this section is to rewrite Eq. 2.76-a in a form suitable for discretization. To this end, we let $r_{ci}(x,t)$ be the component matrix, along body axes $x_i y_i z_i$, of the vector $\mathbf{r}_{ci}(x,t) = \overrightarrow{O_i O_{ci}}$ (Fig. 2) and $E_{ci}(x,t)$ the matrix of direction cosines of the cross-section axes $\xi_i \eta_i \zeta_i$ with respect to body axes $x_i y_i z_i$. Furthermore,

$$u_{ci}(x,t) = \left[u_{xi}(x,t) \;\; u_{yi}(x,t) \;\; u_{zi}(x,t) \right]^T \tag{3.1}$$

is the component matrix of the displacement vector of point O_{ci} on the central axis of the beam, along axes $x_i y_i z_i$, and

$$\psi_{ci}(x,t) = \left[\psi_{xi}(x,t) \;\; \psi_{yi}(x,t) \;\; \psi_{zi}(x,t) \right]^T \tag{3.2}$$

is the component matrix for the vector which defines the axis of rotation of E_{ci}. That is, the magnitude of ψ_{ci} equals the angle of rotation associated with E_{ci}, and the direction indicates the axis of rotation. As discussed in the Appendix, the formula for E_{ci} in terms of the components of ψ_{ci} can be written as a power series (Eq. A.30), and consequently is well suited for truncation.

Based on the above statements, we have that

$$\mathsf{r}_{ci} = x\mathsf{e}_1 + \mathsf{u}_{ci}, \tag{3.3}$$

$$\mathsf{E}_{ci} = \exp(-\widetilde{\psi}_{ci}) = \sum_{k=0}^{\infty} \frac{(-1)^k \widetilde{\psi}_{ci}^{\,k}}{k!}, \tag{3.4}$$

$$\mathsf{R}_{ci} = \mathsf{R}_i + \mathsf{P}_i^T \mathsf{r}_{ci}, \tag{3.5}$$

$$\mathsf{P}_{ci} = \mathsf{E}_{ci}\mathsf{P}_i\,. \tag{3.6}$$

As discussed in the Appendix, E_{ci} is, indeed, a proper orthogonal matrix, and furthermore,

$$\mathrm{Tilde}\left(\mathsf{D}_{ci}\dot{\psi}_{ci}\right) = \mathsf{E}_{ci}\dot{\mathsf{E}}_{ci}^T, \tag{3.7}$$

$$\mathrm{Tilde}\left(\mathsf{D}_{ci}\psi'_{ci}\right) = \mathsf{E}_{ci}\mathsf{E}_{ci}'^T, \tag{3.8}$$

$$\mathrm{Tilde}\left(\mathsf{D}_{ci}\delta\psi_{ci}\right) = \mathsf{E}_{ci}\delta\mathsf{E}_{ci}^T, \tag{3.9}$$

in which

$$\mathsf{D}_{ci} = \sum_{k=0}^{\infty} \frac{(-1)^k \widetilde{\psi}_{ci}^{\,k}}{(k+1)!}. \tag{3.10}$$

We note, however, that the specific formulas given here for E_{ci} and D_{ci} are not used until the next section. Consequently, the equations developed in this section, particularly Eqs. 3.27, 3.28-c, 3.28-d and 3.28-e, are valid for other characterizations of the matrix of direction cosines E_{ci}.

Now substitute Eqs. 3.5 and 3.6 into Eq. 2.42, and use Eqs. 2.15, 2.20 and 3.3 to get that

$$\begin{aligned} \mathsf{V}_{ci} = \mathsf{P}_{ci}\dot{\mathsf{R}}_{ci} &= \mathsf{E}_{ci}\mathsf{P}_i\left(\dot{\mathsf{R}}_i + \mathsf{P}_i^T\widetilde{\Omega}_i\mathsf{r}_{ci} + \mathsf{P}_i^T\dot{\mathsf{u}}_{ci}\right) \\ &= \mathsf{E}_{ci}\left(\mathsf{V}_i - \tilde{\mathsf{r}}_{ci}\Omega_i + \dot{\mathsf{u}}_{ci}\right). \end{aligned} \tag{3.11}$$

Furthermore, substitute Eq. 3.6 into Eq. 2.43, and use Eqs. 2.20, 3.6 and 3.7 to get

$$\begin{aligned} \widetilde{\Omega}_{ci} = \mathsf{P}_{ci}\dot{\mathsf{P}}_{ci}^T &= \mathsf{E}_{ci}\mathsf{P}_i\frac{\mathsf{d}\left(\mathsf{P}_i^T\mathsf{E}_{ci}^T\right)}{\mathsf{dt}} \\ &= \mathsf{E}_{ci}\mathsf{P}_i\left[\mathsf{P}_i^T\widetilde{\Omega}_i\mathsf{E}_{ci}^T + \mathsf{P}_i^T\mathsf{E}_{ci}^T\left(\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}}\right)\right] \\ &= \mathrm{Tilde}\left(\mathsf{E}_{ci}\Omega_i + \mathsf{D}_{ci}\dot{\psi}_{ci}\right), \end{aligned} \tag{3.12}$$

so that

$$\Omega_{ci} = \mathsf{E}_{ci}\Omega_i + \mathsf{D}_{ci}\dot{\psi}_{ci}. \tag{3.13}$$

Analogous to Eqs. 3.11 and 3.13, we also have

$$\delta\mathsf{R}_{ci}^* = \mathsf{E}_{ci}\left(\delta\mathsf{R}_i^* - \tilde{\mathsf{r}}_{ci}\delta\Theta_i^* + \delta\mathsf{u}_{ci}\right), \tag{3.14}$$

$$\delta\Theta_{ci}^* = \mathsf{E}_{ci}\delta\Theta_i^* + \mathsf{D}_{ci}\delta\psi_{ci}. \tag{3.15}$$

Next, substitute Eqs. 3.5 and 3.6 into Eqs. 2.46 and 2.47 and use Eqs. 3.3 and 3.8 to get

$$\beta_{ci} = P_{ci}R'_{ci} = E_{ci}P_iP_i^T r'_{ci} = E_{ci}\left(e_1 + u'_{ci}\right), \tag{3.16}$$

$$\widetilde{\phi}_{ci} = P_{ci}P'^T_{ci} = E_{ci}P_iP_i^T E'^T_{ci} = E_{ci}E'^T_{ci} = \left(\widetilde{D_{ci}\psi'_{ci}}\right), \tag{3.17}$$

so that

$$\phi_{ci} = D_{ci}\psi'_{ci}. \tag{3.18}$$

Equations 3.11 and 3.13, together with Eqs. 3.3 and 3.7 imply that

$$\begin{aligned}
\dot{V}_{ci} + \widetilde{\Omega}_{ci}V_{ci} &= E_{ci}\left(\dot{V}_i - \tilde{r}_{ci}\dot{\Omega}_i - \tilde{\dot{u}}_{ci}\Omega_i + \ddot{u}_{ci}\right) - \left(\widetilde{D_{ci}\dot{\psi}_{ci}}\right)V_{ci} + \left(\widetilde{E_{ci}\Omega_i} + \widetilde{D_{ci}\dot{\psi}_{ci}}\right)V_{ci} \\
&= E_{ci}\left(\dot{V}_i - \tilde{r}_{ci}\dot{\Omega}_i + \ddot{u}_{ci} - \tilde{\dot{u}}_{ci}\Omega_i\right) + \left(\widetilde{E_{ci}\Omega_i}\right)E_{ci}\left(V_i - \tilde{r}_{ci}\Omega_i + \dot{u}_{ci}\right) \\
&= E_{ci}\left(\dot{V}_i - \tilde{r}_{ci}\dot{\Omega}_i + \ddot{u}_{ci} + \widetilde{\Omega}_iV_i - \widetilde{\Omega}_i\tilde{r}_{ci}\Omega_i - 2\tilde{\dot{u}}_{ci}\Omega_i\right),
\end{aligned} \tag{3.19}$$

$$\dot{\Omega}_{ci} = E_{ci}\dot{\Omega}_i + D_{ci}\ddot{\psi}_{ci} - \left(\widetilde{D_{ci}\dot{\psi}_{ci}}\right)E_{ci}\Omega_i + \dot{D}_{ci}\dot{\psi}_{ci}. \tag{3.20}$$

Combining Eqs. 3.11, 3.13, 3.14, 3.15, 3.19 and 3.20, we get that

$$\begin{aligned}
&\int_0^{\ell_i} \left\{\delta R^{*T}_{ci}\left(\rho_{ci}\dot{V}_{ci} + \rho_{ci}\widetilde{\Omega}_{ci}V_{ci}\right) + \delta\Theta^{*T}_{ci}\left(\hat{J}_{ci}\dot{\Omega}_{ci} + \widetilde{\Omega}_{ci}\hat{J}_{ci}\Omega_{ci}\right)\right\}dx \\
&= \int_0^{\ell_i} \Big\{\left(\delta R^{*T}_i + \delta\Theta^{*T}_i\tilde{r}_{ci} + \delta u^T_{ci}\right)\left[\rho_{ci}\dot{V}_i - \rho_{ci}\tilde{r}_{ci}\dot{\Omega}_i + \rho_{ci}\ddot{u}_{ci} + \rho_{ci}\widetilde{\Omega}_iV_i - \rho_{ci}\widetilde{\Omega}_i\tilde{r}_{ci}\Omega_i - 2\rho_{ci}\tilde{\dot{u}}_{ci}\Omega_i\right] \\
&\quad + \left(\delta\Theta^{*T}_iE^T_{ci} + \delta\psi^T_{ci}D^T_{ci}\right)\Big[\hat{J}_{ci}E_{ci}\dot{\Omega}_i + \hat{J}_{ci}D_{ci}\ddot{\psi}_{ci} - \hat{J}_{ci}(\widetilde{D_{ci}\dot{\psi}_{ci}})E_{ci}\Omega_i + \hat{J}_{ci}\dot{D}_{ci}\dot{\psi}_{ci} \\
&\quad + \left(\widetilde{E_{ci}\Omega_i} + \widetilde{D_{ci}\dot{\psi}_{ci}}\right)\hat{J}_{ci}\left(E_{ci}\Omega_i + D_{ci}\dot{\psi}_{ci}\right)\Big]\Big\}dx \\
&= \int_0^{\ell_i} \Big\{\delta R^{*T}_i\left[\rho_{ci}\dot{V}_i - \rho_{ci}\tilde{r}_{ci}\dot{\Omega}_i + \rho_{ci}\ddot{u}_{ci} + \rho_{ci}\widetilde{\Omega}_iV_i - \rho_{ci}\widetilde{\Omega}_i\tilde{r}_{ci}\Omega_i - 2\rho_{ci}\tilde{\dot{u}}_{ci}\Omega_i\right] \\
&\quad + \delta\Theta^{*T}_i\Big[\rho_{ci}\tilde{r}_{ci}\dot{V}_i - \rho_{ci}\tilde{r}^2_{ci}\dot{\Omega}_i + \rho_{ci}\tilde{r}_{ci}\ddot{u}_{ci} + \rho_{ci}\tilde{r}_{ci}\widetilde{\Omega}_iV_i - \rho_{ci}\widetilde{\Omega}_i\tilde{r}^2_{ci}\Omega_i - 2\rho_{ci}\tilde{r}_{ci}\tilde{\dot{u}}_{ci}\Omega_i + E^T_{ci}\hat{J}_{ci}E_{ci}\dot{\Omega}_i \\
&\quad + E^T_{ci}\hat{J}_{ci}D_{ci}\ddot{\psi}_{ci} - E^T_{ci}\hat{J}_{ci}(\widetilde{D_{ci}\dot{\psi}_{ci}})E_{ci}\Omega_i + E^T_{ci}\hat{J}_{ci}\dot{D}_{ci}\dot{\psi}_{ci} + \widetilde{\Omega}_iE^T_{ci}\hat{J}_{ci}E_{ci}\Omega_i + \widetilde{\Omega}_iE^T_{ci}\hat{J}_{ci}D_{ci}\dot{\psi}_{ci} \\
&\quad + E^T_{ci}(\widetilde{D_{ci}\dot{\psi}_{ci}})\hat{J}_{ci}E_{ci}\Omega_i + E^T_{ci}(\widetilde{D_{ci}\dot{\psi}_{ci}})\hat{J}_{ci}D_{ci}\dot{\psi}_{ci}\Big] \\
&\quad + \delta u^T_{ci}\left[\rho_{ci}\dot{V}_i - \rho_{ci}\tilde{r}_{ci}\dot{\Omega}_i + \rho_{ci}\ddot{u}_{ci} + \rho_{ci}\widetilde{\Omega}_iV_i - \rho_{ci}\widetilde{\Omega}_i\tilde{r}_{ci}\Omega_i - 2\rho_{ci}\tilde{\dot{u}}_{ci}\Omega_i\right] \\
&\quad + \delta\psi^T_{ci}\Big[D^T_{ci}\hat{J}_{ci}E_{ci}\dot{\Omega}_i + D^T_{ci}\hat{J}_{ci}D_{ci}\ddot{\psi}_{ci} - D^T_{ci}\hat{J}_{ci}(\widetilde{D_{ci}\dot{\psi}_{ci}})E_{ci}\Omega_i + D^T_{ci}\hat{J}_{ci}\dot{D}_{ci}\dot{\psi}_{ci} + D^T_{ci}(\widetilde{E_{ci}\Omega_i})\hat{J}_{ci}E_{ci}\Omega_i \\
&\quad + D^T_{ci}(\widetilde{E_{ci}\Omega_i})\hat{J}_{ci}D_{ci}\dot{\psi}_{ci} + D^T_{ci}(\widetilde{D_{ci}\dot{\psi}_{ci}})\hat{J}_{ci}E_{ci}\Omega_i + D^T_{ci}(\widetilde{D_{ci}\dot{\psi}_{ci}})\hat{J}_{ci}D_{ci}\dot{\psi}_{ci}\Big]\Big\}dx \\
&= \int_0^{\ell_i} \Big\{\delta R^{*T}_i\left[\rho_{ci}\dot{V}_i - \rho_{ci}\tilde{r}_{ci}\dot{\Omega}_i + \rho_{ci}\ddot{u}_{ci} + \rho_{ci}\widetilde{\Omega}_iV_i - \rho_{ci}\widetilde{\Omega}_i\tilde{r}_{ci}\Omega_i - 2\rho_{ci}\tilde{\dot{u}}_{ci}\Omega_i\right] \\
&\quad + \delta\Theta^{*T}_i\Big[\rho_{ci}\tilde{r}_{ci}\dot{V}_i + \left(E^T_{ci}\hat{J}_{ci}E_{ci} - \rho_{ci}\tilde{r}^2_{ci}\right)\dot{\Omega}_i + \rho_{ci}\tilde{r}_{ci}\ddot{u}_{ci} + E^T_{ci}\hat{J}_{ci}D_{ci}\ddot{\psi}_{ci} \\
&\quad + \rho_{ci}\tilde{r}_{ci}\widetilde{\Omega}_iV_i + \widetilde{\Omega}_i\left(E^T_{ci}\hat{J}_{ci}E_{ci} - \rho_{ci}\tilde{r}^2_{ci}\right)\Omega_i + E^T_{ci}\left(\hat{J}_{ci}\dot{D}_{ci} + (\widetilde{D_{ci}\dot{\psi}_{ci}})\hat{J}_{ci}D_{ci}\right)\dot{\psi}_{ci}\Big] \\
&\quad + \Big(-2\rho_{ci}\tilde{r}_{ci}\tilde{\dot{u}}_{ci} - E^T_{ci}\hat{J}_{ci}(\widetilde{D_{ci}\dot{\psi}_{ci}})E_{ci} + E^T_{ci}(\widetilde{D_{ci}\dot{\psi}_{ci}})\hat{J}_{ci}E_{ci} - \mathrm{Tilde}\left(E^T_{ci}\hat{J}_{ci}D_{ci}\dot{\psi}_{ci}\right)\Big)\Omega_i
\end{aligned}$$

$$+\delta \mathsf{u}_{ci}^{\mathsf{T}}\Big[\rho_{ci}\dot{\mathsf{V}}_i - \rho_{ci}\tilde{\mathsf{r}}_{ci}\dot{\Omega}_i + \rho_{ci}\ddot{\mathsf{u}}_{ci} + \rho_{ci}\tilde{\Omega}_i\mathsf{V}_i - \rho_{ci}\tilde{\Omega}_i\tilde{\mathsf{r}}_{ci}\Omega_i - 2\rho_{ci}\tilde{\dot{\mathsf{u}}}_{ci}\Omega_i\Big]$$
$$+\delta\psi_{ci}^{\mathsf{T}}\Big[\mathsf{D}_{ci}^{\mathsf{T}}\hat{\mathsf{J}}_{ci}\mathsf{E}_{ci}\dot{\Omega}_i + \mathsf{D}_{ci}^{\mathsf{T}}\hat{\mathsf{J}}_{ci}\mathsf{D}_{ci}\ddot{\psi}_{ci} + \mathsf{D}_{ci}^{\mathsf{T}}(\widetilde{\mathsf{E}_{ci}\Omega_i})\hat{\mathsf{J}}_{ci}\mathsf{E}_{ci}\Omega_i + \mathsf{D}_{ci}^{\mathsf{T}}\Big(\hat{\mathsf{J}}_{ci}\dot{\mathsf{D}}_{ci} + (\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}})\hat{\mathsf{J}}_{ci}\mathsf{D}_{ci}\Big)\dot{\psi}_{ci}$$
$$+\mathsf{D}_{ci}^{\mathsf{T}}\Big(-\hat{\mathsf{J}}_{ci}(\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}}) - (\widetilde{\hat{\mathsf{J}}_{ci}\mathsf{D}_{ci}\dot{\psi}_{ci}}) + (\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}})\hat{\mathsf{J}}_{ci}\Big)\mathsf{E}_{ci}\Omega_i\Big]\Big\}\mathsf{dx}. \tag{3.21}$$

Before proceeding with the derivation, we note that Eq. 2.55 implies that $\mathrm{Tr}(\hat{\mathsf{J}}_{ci}) = 2\hat{\mathsf{J}}_{xxi}$, and then, making use of Eqs. A.9 and 2.59 we see that

$$\begin{aligned} -\hat{\mathsf{J}}_{ci}(\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}}) - (\widetilde{\hat{\mathsf{J}}_{ci}\mathsf{D}_{ci}\dot{\psi}_{ci}}) + (\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}})\hat{\mathsf{J}}_{ci} &= -\hat{\mathsf{J}}_{ci}(\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}}) - \mathrm{Tr}(\hat{\mathsf{J}}_{ci})(\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}}) + \hat{\mathsf{J}}_{ci}(\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}}) \\ &\quad +2(\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}})\hat{\mathsf{J}}_{ci} = (\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}})\Big[2\hat{\mathsf{J}}_{ci} - (\mathrm{Tr}\hat{\mathsf{J}}_{ci})\mathrm{I}\Big] \\ &= -2(\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}})\mathsf{J}_{ci}^{*}. \end{aligned} \tag{3.22}$$

Furthermore, premultiplying by $\mathsf{E}_{ci}^{\mathsf{T}}$ and post-multiplying by E_{ci} implies that

$$-\mathsf{E}_{ci}^{\mathsf{T}}\hat{\mathsf{J}}_{ci}(\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}})\mathsf{E}_{ci} - (\mathsf{E}_{ci}^{\mathsf{T}}\widetilde{\hat{\mathsf{J}}_{ci}\mathsf{D}_{ci}\dot{\psi}_{ci}}) + \mathsf{E}_{ci}^{\mathsf{T}}(\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}})\hat{\mathsf{J}}_{ci}\mathsf{E}_{ci} = -2\mathsf{E}_{ci}^{\mathsf{T}}(\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}})\mathsf{J}_{ci}^{*}\mathsf{E}_{ci}. \tag{3.23}$$

Substituting Eqs. 3.22 and 3.23 into Eq. 3.21 we have that

$$\begin{aligned} &\int_0^{\ell_i}\Big\{\delta\mathsf{R}_{ci}^{*\mathsf{T}}\Big(\rho_{ci}\dot{\mathsf{V}}_{ci} + \rho_{ci}\tilde{\Omega}_{ci}\mathsf{V}_{ci}\Big) + \delta\Theta_{ci}^{*\mathsf{T}}\Big(\hat{\mathsf{J}}_{ci}\dot{\Omega}_{ci} + \tilde{\Omega}_{ci}\hat{\mathsf{J}}_{ci}\Omega_{ci}\Big)\Big\}\,\mathsf{dx} \\ &\quad= \delta\mathsf{R}_i^{*\mathsf{T}}\int_0^{\ell_i}\Big\{\rho_{ci}\dot{\mathsf{V}}_i - \rho_{ci}\tilde{\mathsf{r}}_{ci}\dot{\Omega}_i + \rho_{ci}\ddot{\mathsf{u}}_{ci} + \rho_{ci}\tilde{\Omega}_i\mathsf{V}_i - \rho_{ci}\tilde{\Omega}_i\tilde{\mathsf{r}}_{ci}\Omega_i - 2\rho_{ci}\tilde{\dot{\mathsf{u}}}_{ci}\Omega_i\Big\} \\ &\quad+ \delta\Theta_i^{*\mathsf{T}}\int_0^{\ell_i}\Big\{\rho_{ci}\tilde{\mathsf{r}}_{ci}\dot{\mathsf{V}}_i + \Big(\mathsf{E}_{ci}^{\mathsf{T}}\hat{\mathsf{J}}_{ci}\mathsf{E}_{ci} - \rho_{ci}\tilde{\mathsf{r}}_{ci}^2\Big)\dot{\Omega}_i + \rho_{ci}\tilde{\mathsf{r}}_{ci}\ddot{\mathsf{u}}_{ci} + \mathsf{E}_{ci}^{\mathsf{T}}\hat{\mathsf{J}}_{ci}\mathsf{D}_{ci}\ddot{\psi}_{ci} \\ &\qquad+\rho_{ci}\tilde{\mathsf{r}}_{ci}\tilde{\Omega}_i\mathsf{V}_i + \tilde{\Omega}_i\Big(\mathsf{E}_{ci}^{\mathsf{T}}\hat{\mathsf{J}}_{ci}\mathsf{E}_{ci} - \rho_{ci}\tilde{\mathsf{r}}_{ci}^2\Big)\Omega_i + \mathsf{E}_{ci}^{\mathsf{T}}\Big(\hat{\mathsf{J}}_{ci}\dot{\mathsf{D}}_{ci} + (\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}})\hat{\mathsf{J}}_{ci}\mathsf{D}_{ci}\Big)\dot{\psi}_{ci} \\ &\qquad+\Big(-2\rho_{ci}\tilde{\mathsf{r}}_{ci}\tilde{\dot{\mathsf{u}}}_{ci} + \mathsf{E}_{ci}^{\mathsf{T}}(\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}})\mathsf{J}_{ci}^{*}\mathsf{E}_{ci}\Big)\Omega_i\Big\}\,\mathsf{dx} \\ &\quad+\int_0^{\ell_i}\Big(\delta\mathsf{u}_{ci}^{\mathsf{T}}\Big\{\rho_{ci}\dot{\mathsf{V}}_i - \rho_{ci}\tilde{\mathsf{r}}_{ci}\dot{\Omega}_i + \rho_{ci}\ddot{\mathsf{u}}_{ci} + \rho_{ci}\tilde{\Omega}_i\mathsf{V}_i - \rho_{ci}\tilde{\Omega}_i\tilde{\mathsf{r}}_{ci}\Omega_i - 2\rho_{ci}\tilde{\dot{\mathsf{u}}}_{ci}\Omega_i\Big\} \\ &\qquad+ \delta\psi_{ci}^{\mathsf{T}}\Big\{\mathsf{D}_{ci}^{\mathsf{T}}\hat{\mathsf{J}}_{ci}\mathsf{E}_{ci}\dot{\Omega}_i + \mathsf{D}_{ci}^{\mathsf{T}}\hat{\mathsf{J}}_{ci}\mathsf{D}_{ci}\ddot{\psi}_{ci} + \mathsf{D}_{ci}^{\mathsf{T}}(\widetilde{\mathsf{E}_{ci}\Omega_i})\hat{\mathsf{J}}_{ci}\mathsf{E}_{ci}\Omega_i \\ &\qquad+\mathsf{D}_{ci}^{\mathsf{T}}\Big(\hat{\mathsf{J}}_{ci}\dot{\mathsf{D}}_{ci} + (\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}})\hat{\mathsf{J}}_{ci}\mathsf{D}_{ci}\Big)\dot{\psi}_{ci} - 2\mathsf{D}_{ci}^{\mathsf{T}}(\widetilde{\mathsf{D}_{ci}\dot{\psi}_{ci}})\mathsf{J}_{ci}^{*}\mathsf{E}_{ci}\Omega_i\Big\}\Big)\mathsf{dx}. \end{aligned} \tag{3.24}$$

To arrive at a formula for the surface traction term, we make use of Eqs. 3.14 and 3.15 to get that

$$\begin{aligned}
\int_0^{\ell_i} \left(\delta R_{ci}^{*T} f_{si} + \delta\Theta_{ci}^{*T} m_{si}\right) dx &= \int_0^{\ell_i} \left[\left(\delta R_i^{*T} + \delta\Theta_i^{*T}\tilde{r}_{ci} + \delta u_{ci}^T\right)E_{ci}^T f_{si} + \left(\delta\Theta_i^{*T} E_{ci}^T + \delta\psi_{ci}^T D_{ci}^T\right) m_{si}\right] dx \\
&= \int_0^{\ell_i} \left[\delta R_i^{*T}\left(E_{ci}^T f_{si}\right) + \delta\Theta_i^{*T}\left(\tilde{r}_{ci}E_{ci}^T f_{si} + E_{ci}^T m_{si}\right)\right. \\
&\qquad \left. + \delta u_{ci}^T\left(E_{ci}^T f_{si}\right) + \delta\psi_{ci}^T\left(D_{ci}^T m_{si}\right)\right] dx \\
&= \delta R_i^{*T} F_{si} + \delta\Theta_i^{*T} M_{si} + \int_0^{\ell_i} \left[\delta u_{ci}^T\left(E_{ci}^T f_{si}\right) + \delta\psi_{ci}^T\left(D_{ci}^T m_{si}\right)\right] dx, \qquad (3.25)
\end{aligned}$$

where

$$F_{si} = \begin{bmatrix} F_{xi} \\ F_{yi} \\ F_{zi} \end{bmatrix} = \int_0^{\ell_i} E_{ci}^T f_{si}\, dx, \qquad (3.26\text{-a})$$

$$M_{si} = \begin{bmatrix} M_{xi} \\ M_{yi} \\ M_{zi} \end{bmatrix} = \int_0^{\ell_i} \left(\tilde{r}_{ci}E_{ci}^T f_{si} + D_{ci}^T m_{si}\right) dx. \qquad (3.26\text{-b})$$

Now combine Eqs. 3.24 and 3.25 with Eq. 2.76-a to get that

$$\begin{aligned}
\mathcal{R}_i = \; & \delta R_i^{*T}\Big\{m_i\dot{V}_i - \widetilde{S}_{ei}\dot{\Omega}_i + \int_0^{\ell_i} \rho_{ci}\ddot{u}_{ci}\, dx + m_i\widetilde{\Omega}_i V_i - \widetilde{\Omega}_i\widetilde{S}_{ei}\Omega_i - 2\dot{\widetilde{S}}_{ei}\Omega_i - F_{si}\Big\} \\
& + \delta\Theta_i^{*T}\Big\{\widetilde{S}_{ei}\dot{V}_i + J_{ei}\dot{\Omega}_i + \int_0^{\ell_i}\left(\rho_{ci}\tilde{r}_{ci}\ddot{u}_{ci} + E_{ci}^T\hat{J}_{ci}D_{ci}\ddot{\psi}_{ci}\right) dx + \widetilde{S}_{ei}\widetilde{\Omega}_i V_i + \widetilde{\Omega}_i J_{ei}\Omega_i \\
& \qquad -2\Pi_{ei}\Omega_i + \sigma_{ei} - M_{si}\Big\} \\
& + \int_0^{\ell_i}\Bigg(\delta u_{ci}^T\Big\{\rho_{ci}\dot{V}_i - \rho_{ci}\tilde{r}_{ci}\dot{\Omega}_i + \rho_{ci}\ddot{u}_{ci} + \rho_{ci}\widetilde{\Omega}_i V_i - \rho_{ci}\widetilde{\Omega}_i\tilde{r}_{ci}\Omega_i - 2\rho_{ci}\dot{\tilde{u}}_{ci}\Omega_i - E_{ci}^T f_{si}\Big\} \\
& \quad + \delta\psi_{ci}^T\Big\{D_{ci}^T\hat{J}_{ci}E_{ci}\dot{\Omega}_i + D_{ci}^T\hat{J}_{ci}D_{ci}\ddot{\psi}_{ci} - D_{ci}^T\left(\widetilde{E_{ci}\Omega_i} + 2\widetilde{D_{ci}\dot{\psi}_{ci}}\right)J_{ci}^* E_{ci}\Omega_i \\
& \qquad + D_{ci}^T\left[\hat{J}_{ci}\dot{D}_{ci} + (\widetilde{D_{ci}\dot{\psi}_{ci}})\hat{J}_{ci}D_{ci}\right]\dot{\psi}_{ci} - D_{ci}^T m_{si}\Big\} + \delta\beta_{ci}^T f_{ci} + \delta\phi_{ci}^T m_{ci}\Bigg) dx \\
& \qquad - \int_{\partial\mathcal{B}_{oi}} \delta z_i^T \tau_{oi}^*\, d\mathcal{A}, \qquad (3.27)
\end{aligned}$$

in which

$$m_i = \int_0^{\ell_i} \rho_{ci}\, dx, \qquad (3.28\text{-a})$$

$$S_{ei} = \int_0^{\ell_i} \rho_{ci} r_{ci}\, dx, \qquad (3.28\text{-b})$$

$$J_{ei} = \int_0^{\ell_i} \left(E_{ci}^T \hat{J}_{ci} E_{ci} - \rho_{ci} \tilde{r}_{ci}^2 \right) dx, \tag{3.28-c}$$

$$\Pi_{ei} = \int_0^{\ell_i} \left[\rho_{ci} \tilde{r}_{ci} \tilde{\dot{u}}_{ci} + E_{ci}^T \widetilde{(D_{ci} \dot{\psi}_{ci})} J_{ci}^* E_{ci} \right] dx, \tag{3.28-d}$$

$$\sigma_{ei} = \int_0^{\ell_i} E_{ci}^T \left[\hat{J}_{ci} \dot{D}_{ci} + \widetilde{(D_{ci} \dot{\psi}_{ci})} \hat{J}_{ci} D_{ci} \right] \dot{\psi}_{ci}\, dx. \tag{3.28-e}$$

3.2 Second-Order Rayleigh Beam

At this point we specialize to a Rayleigh beam, which is simply a beam model which includes rotatory inertia and which satisfies the same kinematical constraints as an Euler-Bernoulli beam. These constraints can be stated as:

1. The distance, along the central axis of the beam, between any two points on the central axis remains constant.
2. The tangent line to the central axis remains perpendicular to the central axis.

In order to derive usable consequences of these two assumptions, note first that R'_{ci} is the component matrix, along inertial axes $x_o y_o z_o$, of the tangent vector to the central axis. Consequently, $\beta_{ci}(x) = P_{ci}(x) R'_{ci}(x)$ is the component matrix of the tangent vector, but this time along cross-section axes $\xi_i \eta_i \zeta_i$. Then, since x measures arc length along the central axis when the beam is in undeformed state, postulate 1 requires that

$$x = \int_a^{a+x} \sqrt{\beta_{ci}^T(s,t) \beta_{ci}(s,t)}\; ds. \tag{3.29}$$

Now set a to zero, take the derivative with respect to x and then square the result to get that

$$\beta_{ci}^T(x,t) \beta_{ci}(x,t) = 1. \tag{3.30}$$

Postulate 2 requires that

$$\beta_{ci}^T(x,t) e_2 = \beta_{ci}^T(x,t) e_3 = 0, \tag{3.31}$$

which, taken together with Eq. 3.30, shows that $\beta_{ci} = \pm e_1$. However, it is obvious that β_{ci} points away from the same side of the cross-section as e_1 does, so that $\beta_{ci} = e_1$. Combining this with Eq. 3.16, we see that

$$\beta_{ci} = E_{ci}(e_1 + u'_{ci}) = e_1. \tag{3.32}$$

Before proceeding, first rewrite this as

$$u'_{ci} = (E_{ci}^T - I) e_1, \tag{3.33}$$

which represents three scalar equations involving the six components of u_{ci} and ψ_{ci}. Based on this equation alone, it would seem reasonable to take the components of ψ_{ci} as independent; however, the standard approach, and the one we adopt here, is to take ψ_{xi}, u_{yi} and u_{zi} as independent.

Our interest is in deriving equations of motion which retain only up to second-order terms involving the components of u_{ci}, ψ_{ci}, $\dot{\mathsf{u}}_{ci}$, $\dot{\psi}_{ci}$, $\ddot{\mathsf{u}}_{ci}$, $\ddot{\psi}_{ci}$, u'_{ci}, ψ'_{ci}, u''_{ci}, ψ''_{ci}, $\dot{\mathsf{u}}'_{ci}$, $\dot{\psi}'_{ci}$, $\delta\mathsf{u}_{ci}$, $\delta\psi_{ci}$, f_{si} and m_{si}. To this end, we retain the first three terms in the power series in Eq. 3.4, and substitute into Eq. 3.33 to arrive at

$$\begin{bmatrix} \mathsf{u}'_{xi} \\ \mathsf{u}'_{yi} \\ \mathsf{u}'_{zi} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(\psi_{yi}^2+\psi_{zi}^2) & -\psi_{zi}+\frac{1}{2}\psi_{xi}\psi_{yi} & \psi_{yi}+\frac{1}{2}\psi_{xi}\psi_{zi} \\ \psi_{zi}+\frac{1}{2}\psi_{xi}\psi_{yi} & -\frac{1}{2}(\psi_{xi}^2+\psi_{zi}^2) & -\psi_{xi}+\frac{1}{2}\psi_{yi}\psi_{zi} \\ -\psi_{yi}+\frac{1}{2}\psi_{xi}\psi_{zi} & \psi_{xi}+\frac{1}{2}\psi_{yi}\psi_{zi} & -\frac{1}{2}(\psi_{xi}^2+\psi_{yi}^2) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{3.34}$$

which then implies that

$$\mathsf{u}'_{xi} = -\tfrac{1}{2}(\psi_{yi}^2+\psi_{zi}^2), \tag{3.35-a}$$
$$\mathsf{u}'_{yi} = \psi_{zi}+\tfrac{1}{2}\psi_{xi}\psi_{yi}, \tag{3.35-b}$$
$$\mathsf{u}'_{zi} = -\psi_{yi}+\tfrac{1}{2}\psi_{xi}\psi_{zi}. \tag{3.35-c}$$

As can be checked by direct substitution, the second and third of these equations can be solved for ψ_{yi} and ψ_{zi}, yielding

$$\psi_{yi} = (-\mathsf{u}'_{zi}+\tfrac{1}{2}\psi_{xi}\mathsf{u}'_{yi})/(1+\tfrac{1}{4}\psi_{xi}^2), \tag{3.36-a}$$
$$\psi_{zi} = (\ \mathsf{u}'_{yi}+\tfrac{1}{2}\psi_{xi}\mathsf{u}'_{zi})/(1+\tfrac{1}{4}\psi_{xi}^2). \tag{3.36-b}$$

Retaining up to second-order terms only, and then substituting the results into Eq. 3.35-a and integrating yields

$$\mathsf{u}_{xi}(\mathsf{x}) = -\tfrac{1}{2}\int_0^{\mathsf{x}} \left\{[\mathsf{u}'_{yi}(\mathsf{s})]^2+[\mathsf{u}'_{zi}(\mathsf{s})]^2\right\} d\mathsf{s}, \tag{3.37-a}$$
$$\psi_{yi}(\mathsf{x}) = -\mathsf{u}'_{zi}(\mathsf{x})+\tfrac{1}{2}\psi_{xi}(\mathsf{x})\mathsf{u}'_{yi}(\mathsf{x}), \tag{3.37-b}$$
$$\psi_{zi}(\mathsf{x}) = \mathsf{u}'_{yi}(\mathsf{x})+\tfrac{1}{2}\psi_{xi}(\mathsf{x})\mathsf{u}'_{zi}(\mathsf{x}). \tag{3.37-c}$$

Referring to Eqs. 3.4 and 3.10, second-order expressions for E_{ci} and D_{ci} are given by

$$\mathsf{E}_{ci} = \begin{bmatrix} 1-\frac{1}{2}\{(\mathsf{u}'_{yi})^2+(\mathsf{u}'_{zi})^2\} & \mathsf{u}'_{yi} & \mathsf{u}'_{zi} \\ -\mathsf{u}'_{yi}-\psi_{xi}\mathsf{u}'_{zi} & 1-\frac{1}{2}\{\psi_{xi}^2+(\mathsf{u}'_{yi})^2\} & \psi_{xi}-\frac{1}{2}\mathsf{u}'_{yi}\mathsf{u}'_{zi} \\ -\mathsf{u}'_{zi}+\psi_{xi}\mathsf{u}'_{yi} & -\psi_{xi}-\frac{1}{2}\mathsf{u}'_{yi}\mathsf{u}'_{zi} & 1-\frac{1}{2}\{\psi_{xi}^2+(\mathsf{u}'_{zi})^2\} \end{bmatrix}, \tag{3.38-a}$$

$$\mathsf{D}_{ci} = \begin{bmatrix} 1-\frac{1}{6}\{(\mathsf{u}'_{yi})^2+(\mathsf{u}'_{zi})^2\} & \frac{1}{2}\mathsf{u}'_{yi}+\frac{1}{12}\psi_{xi}\mathsf{u}'_{zi} & \frac{1}{2}\mathsf{u}'_{zi}-\frac{1}{12}\psi_{xi}\mathsf{u}'_{yi} \\ -\frac{1}{2}\mathsf{u}'_{yi}-\frac{5}{12}\psi_{xi}\mathsf{u}'_{zi} & 1-\frac{1}{6}\{\psi_{xi}^2+(\mathsf{u}'_{yi})^2\} & \frac{1}{2}\psi_{xi}-\frac{1}{6}\mathsf{u}'_{yi}\mathsf{u}'_{zi} \\ -\frac{1}{2}\mathsf{u}'_{zi}+\frac{5}{12}\psi_{xi}\mathsf{u}'_{yi} & -\frac{1}{2}\psi_{xi}-\frac{1}{6}\mathsf{u}'_{yi}\mathsf{u}'_{zi} & 1-\frac{1}{6}\{\psi_{xi}^2+(\mathsf{u}'_{zi})^2\} \end{bmatrix}. \tag{3.38-b}$$

Substituting Eqs. 3.37-b, 3.37-c and 3.38-b in Eq. 3.18 yields, to second-order

$$\begin{aligned}
\phi_{xi} &= \psi'_{xi} - \tfrac{1}{2}u'_{yi}u''_{zi} + \tfrac{1}{2}u''_{yi}u'_{zi}, && (3.39\text{-a})\\
\phi_{yi} &= -u''_{zi} + \psi_{xi}u''_{yi}, && (3.39\text{-b})\\
\phi_{zi} &= u''_{yi} + \psi_{xi}u''_{zi}. && (3.39\text{-c})
\end{aligned}$$

Recall now that for an Euler-Bernoulli beam the bending moments and twisting moment are given by

$$\begin{aligned}
m_{xi} &= \hat{k}_{xi}\hat{I}_{xxi}G_i(\psi'_{xi} + \zeta_i\dot{\psi}'_{xi}), && (3.40\text{-a})\\
m_{yi} &= -E_i\hat{I}_{yyi}(u''_{zi} + \zeta_i\dot{u}''_{zi}), && (3.40\text{-b})\\
m_{zi} &= E_i\hat{I}_{zzi}(u''_{yi} + \zeta_i\dot{u}''_{yi}), && (3.40\text{-c})
\end{aligned}$$

in which E_i is the elastic modulus, G_i is the shear modulus, ζ_i is a damping factor, $\hat{k}_{xi}$ is a fudge factor and $\hat{I}_{yyi}$, $\hat{I}_{zzi}$ and $\hat{I}_{xxi} = \hat{I}_{yyi} + \hat{I}_{zzi}$ are the area moments of inertia for the beam. Then, since Eq. 3.32 implies that $\delta\beta_{ci} = 0$, we have that

$$\begin{aligned}
\delta\beta_{ci}^T f_{ci} + \delta\phi_{ci}^T m_{ci} &= \delta\psi'_{xi}\Big[\hat{k}_{xi}\hat{I}_{xxi}G_i(\psi'_{xi} + \zeta_i\dot{\psi}'_{xi})\Big] + \delta u''_{yi}\Big[E_i\hat{I}_{zzi}(u''_{yi} + \zeta_i\dot{u}''_{yi})\Big]\\
&\quad + \delta u''_{zi}\Big[E_i\hat{I}_{yyi}(u''_{zi} + \zeta_i\dot{u}''_{zi})\Big]. && (3.41)
\end{aligned}$$

At this point, the best way to proceed is to use a computer program, such as Mathematica, to perform the algebraic computations required after substituting second-order approximations into Eqs. 3.26-a, 3.26-b, 3.28-b, 3.28-c, 3.28-d, 3.28-e and 3.27. Note also that, with reference to Eq. 3.37-a, a simple change of order of integration allows writing

$$\int_0^{\ell_i} f(x)u_{xi}(x)\,dx = -\tfrac{1}{2}\int_0^{\ell_i}\left[\int_x^{\ell_i} f(s)\,ds\right]\Big\{[u'_{yi}(x)]^2 + [u'_{zi}(x)]^2\Big\}\,dx. \qquad (3.42)$$

The result of these calculations is:

$$S_{ei} = \begin{bmatrix} \tfrac{1}{2}m_i\ell_i \\ 0 \\ 0 \end{bmatrix} + \int_0^{\ell_i}\rho_{ci}\begin{bmatrix} -\tfrac{1}{2}(\ell_i - x)\{(u'_{yi})^2 + (u'_{zi})^2\} \\ u_{yi} \\ u_{zi} \end{bmatrix} dx, \qquad (3.43\text{-a})$$

$$\dot{S}_{ei} = \int_0^{\ell_i}\rho_{ci}\begin{bmatrix} -(\ell_i - x)\{u'_{yi}\dot{u}'_{yi} + u'_{zi}\dot{u}'_{zi}\} \\ \dot{u}_{yi} \\ \dot{u}_{zi} \end{bmatrix} dx, \qquad (3.43\text{-b})$$

$$J_{ei} = \begin{bmatrix} \hat{J}_{xxi}\ell_i & 0 & 0 \\ 0 & \hat{J}_{yyi}\ell_i + \tfrac{1}{3}m_i\ell_i^2 & 0 \\ 0 & 0 & \hat{J}_{zzi}\ell_i + \tfrac{1}{3}m_i\ell_i^2 \end{bmatrix}$$

$$+\int_0^{\ell_i}\left[\begin{matrix} \rho_{ci}u_{yi}^2 + \rho_{ci}u_{zi}^2 - \hat{J}_{zzi}(u'_{yi})^2 - \hat{J}_{yyi}(u'_{zi})^2 \\ \hat{J}_{zzi}u'_{yi} - \rho_{ci}xu_{yi} + (\hat{J}_{zzi} - \hat{J}_{yyi})\psi_{xi}u'_{zi} \\ \hat{J}_{yyi}u'_{zi} - \rho_{ci}xu_{zi} + (\hat{J}_{zzi} - \hat{J}_{yyi})\psi_{xi}u'_{yi} \end{matrix}\quad \cdots\cdots\right.$$

$$
\begin{array}{c}
\hat{J}_{zzi}u'_{yi} - \rho_{ci}xu_{yi} + (\hat{J}_{zzi} - \hat{J}_{yyi})\psi_{xi}u'_{zi} \\
\rho_{ci}u_{zi}^2 - \frac{1}{2}\rho_{ci}(\ell_i^2 - x^2)\{(u'_{yi})^2 + (u'_{zi})^2\} + \hat{J}_{zzi}(u'_{yi})^2 + (\hat{J}_{zzi} - \hat{J}_{yyi})\psi_{xi}^2 \\
(\hat{J}_{yyi} - \hat{J}_{zzi})\psi_{xi} - \rho_{ci}u_{yi}u_{zi} + \frac{1}{2}\hat{J}_{xxi}u'_{yi}u'_{zi}
\end{array}
$$

$$
\left.\begin{array}{c}
\hat{J}_{yyi}u'_{zi} - \rho_{ci}xu_{zi} + (\hat{J}_{zzi} - \hat{J}_{yyi})\psi_{xi}u'_{yi} \\
(\hat{J}_{yyi} - \hat{J}_{zzi})\psi_{xi} - \rho_{ci}u_{yi}u_{zi} + \frac{1}{2}\hat{J}_{xxi}u'_{yi}u'_{zi} \\
\rho_{ci}u_{yi}^2 - \frac{1}{2}\rho_{ci}(\ell_i^2 - x^2)\{(u'_{yi})^2 + (u'_{zi})^2\} + \hat{J}_{yyi}(u'_{zi})^2 + (\hat{J}_{yyi} - \hat{J}_{zzi})\psi_{xi}^2
\end{array}\right] dx, \tag{3.43-c}
$$

$$
\Pi_{ei} = \int_0^{\ell_i} \left[\begin{array}{c}
-\rho_{ci}u_{yi}\dot{u}_{yi} - \rho_{ci}u_{zi}\dot{u}_{zi} + \hat{J}_{zzi}u'_{yi}\dot{u}'_{yi} + \hat{J}_{yyi}u'_{zi}\dot{u}'_{zi} \\
\rho_{ci}x\dot{u}_{yi} + \hat{J}_{yyi}u'_{zi}\dot{\psi}_{xi} \\
\rho_{ci}x\dot{u}_{zi} - \hat{J}_{zzi}u'_{yi}\dot{\psi}_{xi}
\end{array}\right.
$$

$$
\begin{array}{c}
-\hat{J}_{zzi}\dot{u}'_{yi} - \hat{J}_{zzi}u'_{zi}\dot{\psi}_{xi} + (\hat{J}_{yyi} - \hat{J}_{zzi})\psi_{xi}\dot{u}'_{zi} \\
(\hat{J}_{yyi} - \hat{J}_{zzi})\psi_{xi}\dot{\psi}_{xi} - \hat{J}_{zzi}u'_{yi}\dot{u}'_{yi} - \rho_{ci}u_{zi}\dot{u}_{zi} + \frac{1}{2}\rho_{ci}(\ell_i^2 - x^2)(u'_{yi}\dot{u}'_{yi} + u'_{zi}\dot{u}'_{zi}) \\
\hat{J}_{zzi}\dot{\psi}_{xi} + \rho_{ci}u_{yi}\dot{u}_{zi} - \frac{1}{2}\hat{J}_{zzi}u'_{zi}\dot{u}'_{yi} - \frac{1}{2}\hat{J}_{zzi}u'_{yi}\dot{u}'_{zi}
\end{array}
$$

$$
\left.\begin{array}{c}
\hat{J}_{yyi}u'_{yi}\dot{\psi}_{xi} + (\hat{J}_{yyi} - \hat{J}_{zzi})\psi_{xi}\dot{u}'_{yi} - \hat{J}_{yyi}\dot{u}'_{zi} \\
-\hat{J}_{yyi}\dot{\psi}_{xi} + \rho_{ci}u_{zi}\dot{u}_{yi} - \frac{1}{2}\hat{J}_{yyi}u'_{zi}\dot{u}'_{yi} - \frac{1}{2}\hat{J}_{yyi}u'_{yi}\dot{u}'_{zi} \\
(\hat{J}_{zzi} - \hat{J}_{yyi})\psi_{xi}\dot{\psi}_{xi} - \rho_{ci}u_{yi}\dot{u}_{yi} - \hat{J}_{yyi}u'_{zi}\dot{u}'_{zi} + \frac{1}{2}\rho_{ci}(\ell_i^2 - x^2)(u'_{yi}\dot{u}'_{yi} + \dot{u}'_{zi}u'_{zi})
\end{array}\right] dx. \tag{3.43-d}
$$

Furthermore, substituting Eqs. 2.75-a and 2.75-b into Eqs. 3.26-a and 3.26-b, we have that

$$
F_{si} = \int_0^{\ell_i} \left[\begin{array}{l}
\hat{f}_{xi} - u'_{yi}\hat{f}_{yi} - u'_{zi}\hat{f}_{zi} \\
u'_{yi}\hat{f}_{xi} + \hat{f}_{yi} - \psi_{xi}\hat{f}_{zi} \\
u'_{zi}\hat{f}_{xi} + \psi_{xi}\hat{f}_{yi} + \hat{f}_{zi}
\end{array}\right] dx, \tag{3.43-e}
$$

$$
M_{si} = \int_0^{\ell_i} \left[\begin{array}{c}
-u_{zi}\hat{f}_{yi} + u_{yi}\hat{f}_{zi} + \hat{m}_{xi} - \frac{1}{2}u'_{yi}\hat{m}_{yi} - \frac{1}{2}u'_{zi}\hat{m}_{zi} \\
(u_{zi} - xu'_{zi})\hat{f}_{xi} - x\psi_{xi}\hat{f}_{yi} - x\hat{f}_{zi} + \frac{1}{2}u'_{yi}\hat{m}_{xi} + \hat{m}_{yi} - \frac{1}{2}\psi_{xi}\hat{m}_{zi} \\
(xu'_{yi} - u_{yi})\hat{f}_{xi} + x\hat{f}_{yi} - x\psi_{xi}\hat{f}_{zi} + \frac{1}{2}u'_{zi}\hat{m}_{xi} + \frac{1}{2}\psi_{xi}\hat{m}_{yi} + \hat{m}_{zi}
\end{array}\right] dx. \tag{3.43-f}
$$

Now, making the appropriate substitutions into Eq. 3.27, we have that

$$
\begin{aligned}
\mathcal{R}_i = \delta R_i^{*T} \Bigg\{ & m_i \dot{V}_i - \widetilde{S}_{ei}\dot{\Omega}_i + \int_0^{\ell_i} \rho_{ci} \begin{bmatrix} -(\ell_i - x)(u'_{yi}\ddot{u}'_{yi} + u'_{zi}\ddot{u}'_{zi}) \\ \ddot{u}_{yi} \\ \ddot{u}_{zi} \end{bmatrix} dx + m_i \widetilde{\Omega}_i V_i \\
& - \widetilde{\Omega}_i \widetilde{S}_{ei}\Omega_i - 2\dot{\widetilde{S}}_{ei}\Omega_i \quad + \int_0^{\ell_i} \rho_{ci} \begin{bmatrix} -(\ell_i - x)\{(\dot{u}'_{yi})^2 + (\dot{u}'_{zi})^2\} \\ 0 \\ 0 \end{bmatrix} dx - F_{si} \Bigg\} \\
+ \delta\Theta_i^{*T} \Bigg\{ & \widetilde{S}_{ei}\dot{V}_i + J_{ei}\dot{\Omega}_i + \int_0^{\ell_i} \begin{bmatrix} \hat{J}_{xxi}\ddot{\psi}_{xi} - \rho_{ci}u_{zi}\ddot{u}_{yi} + \rho_{ci}u_{yi}\ddot{u}_{zi} + \frac{1}{2}(\hat{J}_{yyi} - \hat{J}_{zzi})(u'_{zi}\ddot{u}'_{yi} + u'_{yi}\ddot{u}'_{zi}) \\ \hat{J}_{xxi}u'_{yi}\ddot{\psi}_{xi} - \rho_{ci}x\ddot{u}_{zi} - \hat{J}_{yyi}\ddot{u}'_{zi} + (\hat{J}_{yyi} - \hat{J}_{zzi})\psi_{xi}\ddot{u}'_{yi} \\ \hat{J}_{xxi}u'_{zi}\ddot{\psi}_{xi} + \rho_{ci}x\ddot{u}_{yi} + \hat{J}_{zzi}\ddot{u}'_{yi} + (\hat{J}_{zzi} - \hat{J}_{yyi})\psi_{xi}\ddot{u}'_{zi} \end{bmatrix} dx \\
& + \widetilde{S}_{ei}\widetilde{\Omega}_i V_i + \widetilde{\Omega}_i J_{ei}\Omega_i - 2\Pi_{ei}\Omega_i + \int_0^{\ell_i} \begin{bmatrix} (\hat{J}_{yyi} - \hat{J}_{zzi})\dot{u}'_{yi}\dot{u}'_{zi} \\ 2\hat{J}_{yyi}\dot{\psi}_{xi}\dot{u}'_{yi} \\ 2\hat{J}_{zzi}\dot{\psi}_{xi}\dot{u}'_{zi} \end{bmatrix} dx - M_{si} \Bigg\} \\
+ \int_0^{\ell_i} \Bigg(& \delta\psi_{xi}\Big\{ \hat{J}_{xxi}\dot{\Omega}_{xi} + \hat{J}_{xxi}u'_{yi}\dot{\Omega}_{yi} + \hat{J}_{xxi}u'_{zi}\dot{\Omega}_{zi} + \hat{J}_{xxi}\ddot{\psi}_{xi} + 2\hat{J}_{zzi}\dot{u}'_{yi}\Omega_{yi} + 2\hat{J}_{yyi}\dot{u}'_{zi}\Omega_{zi} \\
& \quad + (\hat{J}_{zzi} - \hat{J}_{yyi})\Big[\psi_{xi}(\Omega_{zi}^2 - \Omega_{yi}^2) - u'_{zi}\Omega_{xi}\Omega_{yi} - u'_{yi}\Omega_{xi}\Omega_{zi} + \Omega_{yi}\Omega_{zi}\Big] - \hat{m}_{xi}\Big\} \\
& \quad + \delta\psi'_{xi}\Big\{\hat{k}_{xi}\hat{I}_{xxi}G_i(\psi'_{xi} + \zeta_i\dot{\psi}'_{xi})\Big\} \\
& + \delta u_{yi}\Big\{\rho_{ci}\dot{V}_{yi} - \rho_{ci}u_{zi}\dot{\Omega}_{xi} + \rho_{ci}x\dot{\Omega}_{zi} + \rho_{ci}\ddot{u}_{yi} + \rho_{ci}(V_{xi}\Omega_{zi} - V_{zi}\Omega_{xi}) - 2\rho_{ci}\dot{u}_{zi}\Omega_{xi} \\
& \quad - \rho_{ci}u_{yi}(\Omega_{xi}^2 + \Omega_{zi}^2) + \rho_{ci}x\Omega_{xi}\Omega_{yi} + \rho_{ci}u_{zi}\Omega_{yi}\Omega_{zi} - \hat{f}_{yi}\Big\} \\
& \quad + \delta u'_{yi}\Big\{ - \rho_{ci}(\ell_i - x)u'_{yi}\dot{V}_{xi} + (\hat{J}_{yyi} - \hat{J}_{zzi})(\tfrac{1}{2}u'_{zi}\dot{\Omega}_{xi} + \psi_{xi}\dot{\Omega}_{yi}) + \hat{J}_{zzi}(\dot{\Omega}_{zi} + \ddot{u}'_{yi}) \\
& \quad - \rho_{ci}(\ell_i - x)u'_{yi}(V_{zi}\Omega_{yi} - V_{yi}\Omega_{zi}) - 2\hat{J}_{zzi}\dot{\psi}_{xi}\Omega_{yi} + \hat{J}_{zzi}u'_{yi}(\Omega_{xi}^2 - \Omega_{yi}^2) - \hat{J}_{zzi}\Omega_{xi}\Omega_{yi} \\
& \quad + \tfrac{1}{2}\rho_{ci}(\ell_i^2 - x^2)u'_{yi}(\Omega_{yi}^2 + \Omega_{zi}^2) + (\hat{J}_{yyi} - \hat{J}_{zzi})\psi_{xi}\Omega_{xi}\Omega_{zi} - \tfrac{1}{2}\hat{J}_{xxi}u'_{zi}\Omega_{yi}\Omega_{zi} - \hat{m}_{zi}\Big\} \\
& \quad + \delta u''_{yi}\Big\{E_i\hat{I}_{zzi}(u''_{yi} + \zeta_i\dot{u}''_{yi})\Big\} \\
& + \delta u_{zi}\Big\{\rho_{ci}\dot{V}_{zi} + \rho_{ci}u_{yi}\dot{\Omega}_{xi} - \rho_{ci}x\dot{\Omega}_{yi} + \rho_{ci}\ddot{u}_{zi} + \rho_{ci}(V_{yi}\Omega_{xi} - V_{xi}\Omega_{yi}) + 2\rho_{ci}\dot{u}_{yi}\Omega_{xi} \\
& \quad - \rho_{ci}u_{zi}(\Omega_{xi}^2 + \Omega_{yi}^2) + \rho_{ci}x\Omega_{xi}\Omega_{zi} + \rho_{ci}u_{yi}\Omega_{yi}\Omega_{zi} - \hat{f}_{zi}\Big\} \\
& \quad + \delta u'_{zi}\Big\{ - \rho_{ci}(\ell_i - x)u'_{zi}\dot{V}_{xi} + (\hat{J}_{yyi} - \hat{J}_{zzi})(\tfrac{1}{2}u'_{yi}\dot{\Omega}_{xi} - \psi_{xi}\dot{\Omega}_{zi}) - \hat{J}_{yyi}(\dot{\Omega}_{yi} - \ddot{u}'_{zi}) \\
& \quad - \rho_{ci}(\ell_i - x)u'_{zi}(V_{zi}\Omega_{yi} - V_{yi}\Omega_{zi}) - 2\hat{J}_{yyi}\dot{\psi}_{xi}\Omega_{zi} + \hat{J}_{yyi}u'_{zi}(\Omega_{xi}^2 - \Omega_{zi}^2) - \hat{J}_{yyi}\Omega_{xi}\Omega_{zi} \\
& \quad + \tfrac{1}{2}\rho_{ci}(\ell_i^2 - x^2)u'_{zi}(\Omega_{yi}^2 + \Omega_{zi}^2) + (\hat{J}_{yyi} - \hat{J}_{zzi})\psi_{xi}\Omega_{xi}\Omega_{yi} - \tfrac{1}{2}\hat{J}_{xxi}u'_{yi}\Omega_{yi}\Omega_{zi} + \hat{m}_{yi}\Big\} \\
& \quad + \delta u''_{zi}\Big\{E_i\hat{I}_{yyi}(u''_{zi} + \zeta_i\dot{u}''_{zi})\Big\}\Bigg) dx - \int_{\partial\mathcal{B}_{oi}} \delta z_i^T \tau_{oi}^* \, d\mathcal{A}. \qquad (3.44)
\end{aligned}
$$

3.3 Shape Functions

In order to derive equations of motion which can be integrated numerically, we follow the standard procedure [35] of approximating the functions $\psi_{xi}(x,t)$, $u_{yi}(x,t)$ and $u_{zi}(x,t)$ as finite linear combinations of shape functions which depend only on the spatial variable x. That is, we assume

$$\begin{aligned} \psi_{xi}(x,t) &= \varphi_{xi}^T(x) q_{xi}(t), && (3.45\text{-a}) \\ u_{yi}(x,t) &= \varphi_{yi}^T(x) q_{yi}(t), && (3.45\text{-b}) \\ u_{zi}(x,t) &= \varphi_{zi}^T(x) q_{zi}(t), && (3.45\text{-c}) \end{aligned}$$

where

$$\begin{aligned} \varphi_{xi}(x) &= \left[\begin{array}{cccc} \varphi_{xi}^1(x) & \varphi_{xi}^2(x) & \cdots & \varphi_{xi}^{N_{xi}}(x) \end{array} \right]^T, && (3.46\text{-a}) \\ \varphi_{yi}(x) &= \left[\begin{array}{cccc} \varphi_{yi}^1(x) & \varphi_{yi}^2(x) & \cdots & \varphi_{yi}^{N_{yi}}(x) \end{array} \right]^T, && (3.46\text{-b}) \\ \varphi_{zi}(x) &= \left[\begin{array}{cccc} \varphi_{zi}^1(x) & \varphi_{zi}^2(x) & \cdots & \varphi_{zi}^{N_{zi}}(x) \end{array} \right]^T, && (3.46\text{-c}) \end{aligned}$$

are arrays of shape functions, and

$$\begin{aligned} q_{xi}(t) &= \left[\begin{array}{cccc} q_{xi}^1(t) & q_{xi}^2(t) & \cdots & q_{xi}^{N_{xi}}(t) \end{array} \right]^T, && (3.47\text{-a}) \\ q_{yi}(t) &= \left[\begin{array}{cccc} q_{yi}^1(t) & q_{yi}^2(t) & \cdots & q_{yi}^{N_{yi}}(t) \end{array} \right]^T, && (3.47\text{-b}) \\ q_{zi}(t) &= \left[\begin{array}{cccc} q_{zi}^1(t) & q_{zi}^2(t) & \cdots & q_{zi}^{N_{zi}}(t) \end{array} \right]^T, && (3.47\text{-c}) \end{aligned}$$

are arrays of generalized coordinates, and define

$$q_{ei}(t) = \left[\begin{array}{ccc} q_{xi}^T(t) & q_{yi}^T(t) & q_{zi}^T(t) \end{array} \right]^T = \left[\begin{array}{cccc} q_{ei}^1(t) & q_{ei}^2(t) & \cdots & q_{ei}^{N_{ei}}(t) \end{array} \right]^T, \qquad (3.48)$$

where

$$N_{ei} = N_{xi} + N_{yi} + N_{zi} \qquad (3.49)$$

is the total number of degrees of freedom used in the discretization of the beam equations. Substituting Eqs. 3.45-a, 3.45-b and 3.45-c into Eqs. 3.43-a, 3.43-b, 3.43-c, 3.43-d, 3.43-e, 3.43-f and 3.44, we have that

$$\begin{aligned} \mathcal{R}_i = & \left[\begin{array}{cc} \delta R_i^{*T} & \delta \Theta_i^{*T} \end{array} \right] \left\{ M_{rri} \left[\begin{array}{c} \dot{V}_i \\ \dot{\Omega}_i \end{array} \right] + M_{rei} \ddot{q}_{ei} - G_{ri} \right\} \\ & + \delta q_{ei}^T \left\{ M_{rei}^T \left[\begin{array}{c} \dot{V}_i \\ \dot{\Omega}_i \end{array} \right] + M_{eei} \ddot{q}_{ei} - G_{ei} \right\} - \int_{\partial \mathcal{B}_{oi}} \delta z_i^T \tau_{oi}^* d\mathcal{A}, \qquad (3.50) \end{aligned}$$

in which

$$M_{rri} = \begin{bmatrix} m_i I & -\widetilde{S}_{ei} \\ \widetilde{S}_{ei} & J_{ei} \end{bmatrix}, \tag{3.51-a}$$

$$G_{ri} = \begin{bmatrix} -m_i\widetilde{\Omega}_i & \widetilde{\Omega}_i\widetilde{S}_{ei} + 2\widetilde{\dot{S}}_{ei} \\ -\widetilde{S}_{ei}\widetilde{\Omega}_i & 2\Pi_{ei} - \widetilde{\Omega}_i J_{ei} \end{bmatrix} \begin{bmatrix} V_i \\ \Omega_i \end{bmatrix} + \begin{bmatrix} \rho_{ci}\dot{q}_{yi}^T\widetilde{M}_{yyi}\dot{q}_{yi} + \rho_{ci}\dot{q}_{zi}^T\widetilde{M}_{zzi}\dot{q}_{zi} \\ 0 \\ 0 \\ (\hat{J}_{zzi} - \hat{J}_{yyi})\dot{q}_{yi}^T\bar{M}_{y'z'i}\dot{q}_{zi} \\ -2\hat{J}_{yyi}\dot{q}_{xi}^T\bar{M}_{xy'i}\dot{q}_{yi} \\ -2\hat{J}_{zzi}\dot{q}_{xi}^T\bar{M}_{xz'i}\dot{q}_{zi} \end{bmatrix} + \begin{bmatrix} F_{xi} \\ F_{yi} \\ F_{zi} \\ M_{xi} \\ M_{yi} \\ M_{zi} \end{bmatrix}, \tag{3.51-b}$$

$$S_{ei} = \begin{bmatrix} \frac{1}{2}m_i\ell_i \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}\rho_{ci}q_{yi}^T\widetilde{M}_{yyi}q_{yi} - \frac{1}{2}\rho_{ci}q_{zi}^T\widetilde{M}_{zzi}q_{zi} \\ \rho_{ci}q_{yi}^T\bar{\varphi}_{yi} \\ \rho_{ci}q_{zi}^T\bar{\varphi}_{zi} \end{bmatrix}, \tag{3.51-c}$$

$$J_{ei} = \begin{bmatrix} \hat{J}_{xxi}\ell_i & 0 & 0 \\ 0 & \hat{J}_{yyi}\ell_i + \frac{1}{3}m_i\ell_i^2 & 0 \\ 0 & 0 & \hat{J}_{zzi}\ell_i + \frac{1}{3}m_i\ell_i^2 \end{bmatrix}$$

$$+ \left[\begin{array}{c} q_{yi}^T(\rho_{ci}\bar{M}_{yyi} - \hat{J}_{zzi}\bar{M}_{y'y'i})q_{yi} + q_{zi}^T(\rho_{ci}\bar{M}_{zzi} - \hat{J}_{yyi}\bar{M}_{z'z'i})q_{zi} \\ q_{yi}^T(\hat{J}_{zzi}\bar{\varphi}_{y'i} - \rho_{ci}\widetilde{\varphi}_{yi}) + (\hat{J}_{zzi} - \hat{J}_{yyi})q_{xi}^T\bar{M}_{xz'i}q_{zi} \\ q_{zi}^T(\hat{J}_{yyi}\bar{\varphi}_{z'i} - \rho_{ci}\widetilde{\varphi}_{zi}) + (\hat{J}_{zzi} - \hat{J}_{yyi})q_{xi}^T\bar{M}_{xy'i}q_{yi} \end{array} \right. \cdots$$

$$\cdots \begin{array}{c} q_{yi}^T(\hat{J}_{zzi}\bar{\varphi}_{y'i} - \rho_{ci}\widetilde{\varphi}_{yi}) + (\hat{J}_{zzi} - \hat{J}_{yyi})q_{xi}^T\bar{M}_{xz'i}q_{zi} \\ (\hat{J}_{zzi} - \hat{J}_{yyi})q_{xi}^T\bar{M}_{xxi}q_{xi} + q_{yi}^T(\hat{J}_{zzi}\bar{M}_{y'y'i} - \frac{1}{2}\rho_{ci}\widetilde{\widetilde{M}}_{yyi})q_{yi} + q_{zi}^T(\rho_{ci}\bar{M}_{zzi} - \frac{1}{2}\rho_{ci}\widetilde{\widetilde{M}}_{zzi})q_{zi} \\ q_{xi}^T(\hat{J}_{yyi} - \hat{J}_{zzi})\bar{\varphi}_{xi} + q_{yi}^T(\frac{1}{2}\hat{J}_{xxi}\bar{M}_{y'z'i} - \rho_{ci}\bar{M}_{yzi})q_{zi} \end{array} \cdots$$

$$\cdots \left. \begin{array}{c} q_{zi}^T(\hat{J}_{yyi}\bar{\varphi}_{z'i} - \rho_{ci}\widetilde{\varphi}_{zi}) + (\hat{J}_{zzi} - \hat{J}_{yyi})q_{xi}^T\bar{M}_{xy'i}q_{yi} \\ q_{xi}^T(\hat{J}_{yyi} - \hat{J}_{zzi})\bar{\varphi}_{xi} + q_{yi}^T(\frac{1}{2}\hat{J}_{xxi}\bar{M}_{y'z'i} - \rho_{ci}\bar{M}_{yzi})q_{zi} \\ (\hat{J}_{yyi} - \hat{J}_{zzi})q_{xi}^T\bar{M}_{xxi}q_{xi} + q_{zi}^T(\hat{J}_{yyi}\bar{M}_{z'z'i} - \frac{1}{2}\rho_{ci}\widetilde{\widetilde{M}}_{zzi})q_{zi} + q_{yi}^T(\rho_{ci}\bar{M}_{yyi} - \frac{1}{2}\rho_{ci}\widetilde{\widetilde{M}}_{yyi})q_{yi} \end{array} \right], \tag{3.51-d}$$

$$\dot{S}_{ei} = \begin{bmatrix} -\rho_{ci}q_{yi}^T\widetilde{M}_{yyi}\dot{q}_{yi} - \rho_{ci}q_{zi}^T\widetilde{M}_{zzi}\dot{q}_{zi} \\ \rho_{ci}\bar{\varphi}_{yi}^T\dot{q}_{yi} \\ \rho_{ci}\bar{\varphi}_{zi}^T\dot{q}_{zi} \end{bmatrix}, \tag{3.51-e}$$

$$
\Pi_{ei} = \left[\begin{array}{c}
q_{yi}^T(\hat{J}_{zzi}\bar{M}_{y'y'i} - \rho_{ci}\bar{M}_{yyi})\dot{q}_{yi} + q_{zi}^T(\hat{J}_{yyi}\bar{M}_{z'z'i} - \rho_{ci}\bar{M}_{zzi})\dot{q}_{zi} \\
\rho_{ci}\widetilde{\varphi}_{yi}^T\dot{q}_{yi} + \hat{J}_{yyi}\dot{q}_{xi}^T\bar{M}_{xz'i}q_{zi} \\
\rho_{ci}\widetilde{\varphi}_{zi}^T\dot{q}_{zi} - \hat{J}_{zzi}\dot{q}_{xi}^T\bar{M}_{xy'i}q_{yi} \\
\hdashline
-\hat{J}_{zzi}\bar{\varphi}_{y'i}^T\dot{q}_{yi} - \hat{J}_{zzi}\dot{q}_{xi}^T\bar{M}_{xz'i}q_{zi} + (\hat{J}_{yyi} - \hat{J}_{zzi})q_{xi}^T\bar{M}_{xz'i}\dot{q}_{zi} \\
(\hat{J}_{yyi} - \hat{J}_{zzi})q_{xi}^T\bar{M}_{xxi}\dot{q}_{xi} + q_{yi}^T(\frac{1}{2}\rho_{ci}\widetilde{\widetilde{\bar{M}}}_{yyi} - \hat{J}_{zzi}\bar{M}_{y'y'i})\dot{q}_{yi} + q_{zi}^T(\frac{1}{2}\rho_{ci}\widetilde{\widetilde{\bar{M}}}_{zzi} - \rho_{ci}\bar{M}_{zzi})\dot{q}_{zi} \\
\hat{J}_{zzi}\bar{\varphi}_{xi}^T\dot{q}_{xi} + q_{yi}^T(\rho_{ci}\bar{M}_{yzi} - \frac{1}{2}\hat{J}_{zzi}\bar{M}_{y'z'i})\dot{q}_{zi} - \frac{1}{2}\hat{J}_{zzi}\dot{q}_{yi}^T\bar{M}_{y'z'i}q_{zi} \\
\hdashline
-\hat{J}_{yyi}\bar{\varphi}_{z'i}^T\dot{q}_{zi} + (\hat{J}_{yyi} - \hat{J}_{zzi})q_{xi}^T\bar{M}_{xy'i}\dot{q}_{yi} + \hat{J}_{yyi}\dot{q}_{xi}^T\bar{M}_{xy'i}q_{yi} \\
-\hat{J}_{yyi}\bar{\varphi}_{xi}^T\dot{q}_{xi} + \dot{q}_{yi}^T(\rho_{ci}\bar{M}_{yzi} - \frac{1}{2}\hat{J}_{yyi}\bar{M}_{y'z'i})q_{zi} - \frac{1}{2}\hat{J}_{yyi}q_{yi}^T\bar{M}_{y'z'i}\dot{q}_{zi} \\
(\hat{J}_{zzi} - \hat{J}_{yyi})q_{xi}^T\bar{M}_{xxi}\dot{q}_{xi} + q_{yi}^T(\frac{1}{2}\rho_{ci}\widetilde{\widetilde{\bar{M}}}_{yyi} - \rho_{ci}\bar{M}_{yyi})\dot{q}_{yi} + q_{zi}^T(\frac{1}{2}\rho_{ci}\widetilde{\widetilde{\bar{M}}}_{zzi} - \hat{J}_{yyi}\bar{M}_{z'z'i})\dot{q}_{zi}
\end{array}\right], \tag{3.51-f}
$$

$$
M_{rei} = \left[\begin{array}{c:c:c}
0\cdots0 & -\rho_{ci}q_{yi}^T\widetilde{\bar{M}}_{yyi} & -\rho_{ci}q_{zi}^T\widetilde{\bar{M}}_{zzi} \\
0\cdots0 & \rho_{ci}\bar{\varphi}_{yi}^T & 0\cdots0 \\
0\cdots0 & 0\cdots0 & \rho_{ci}\bar{\varphi}_{zi}^T \\
\hat{J}_{xxi}\bar{\varphi}_{xi}^T & \frac{1}{2}(\hat{J}_{yyi} - \hat{J}_{zzi})q_{zi}^T\bar{M}_{y'z'i}^T - \rho_{ci}q_{zi}^T\bar{M}_{yzi}^T & \frac{1}{2}(\hat{J}_{yyi} - \hat{J}_{zzi})q_{yi}^T\bar{M}_{y'z'i} + \rho_{ci}q_{yi}^T\bar{M}_{yzi} \\
\hat{J}_{xxi}q_{yi}^T\bar{M}_{xy'i}^T & (\hat{J}_{yyi} - \hat{J}_{zzi})q_{xi}^T\bar{M}_{xy'i} & -\rho_{ci}\widetilde{\varphi}_{zi}^T - \hat{J}_{yyi}\bar{\varphi}_{z'i}^T \\
\hat{J}_{xxi}q_{zi}^T\bar{M}_{xz'i}^T & \rho_{ci}\widetilde{\varphi}_{yi}^T + \hat{J}_{zzi}\bar{\varphi}_{y'i}^T & (\hat{J}_{zzi} - \hat{J}_{yyi})q_{xi}^T\bar{M}_{xz'i}
\end{array}\right], \tag{3.51-g}
$$

$$
M_{eei} = \begin{bmatrix}
\hat{J}_{xxi}\bar{M}_{xxi} & 0 & 0 \\
0 & \rho_{ci}\bar{M}_{yyi} + \hat{J}_{zzi}\bar{M}_{y'y'i} & 0 \\
0 & 0 & \rho_{ci}\bar{M}_{zzi} + \hat{J}_{yyi}\bar{M}_{z'z'i}
\end{bmatrix}, \tag{3.51-h}
$$

$$
G_{ei} = -D_{eei}\dot{q}_{ei} - K_{eei}q_{ei} + d_{ei}, \tag{3.51-i}
$$

$$
K_{eei} = K_{eei}^* + \left[\begin{array}{c:c:c}
(\hat{J}_{zzi} - \hat{J}_{yyi})\bar{M}_{xxi}(\Omega_{zi}^2 - \Omega_{yi}^2) & (\hat{J}_{yyi} - \hat{J}_{zzi})\bar{M}_{xy'i}\Omega_{xi}\Omega_{zi} & (\hat{J}_{yyi} - \hat{J}_{zzi})\bar{M}_{xz'i}\Omega_{xi}\Omega_{yi} \\
\hdashline
(\hat{J}_{yyi} - \hat{J}_{zzi})\bar{M}_{xy'i}^T\Omega_{xi}\Omega_{zi} & \begin{array}{c} -\rho_{ci}\bar{M}_{yyi}(\Omega_{xi}^2 + \Omega_{zi}^2) + \rho_{ci}\widetilde{\bar{M}}_{yyi}(V_{yi}\Omega_{zi} - V_{zi}\Omega_{yi}) \\ +\hat{J}_{zzi}\bar{M}_{y'y'i}(\Omega_{xi}^2 - \Omega_{yi}^2) + \frac{1}{2}\rho_{ci}\widetilde{\widetilde{\bar{M}}}_{yyi}(\Omega_{yi}^2 + \Omega_{zi}^2) \end{array} & (\rho_{ci}\bar{M}_{yzi} - \frac{1}{2}\hat{J}_{xxi}\bar{M}_{y'z'i})\Omega_{yi}\Omega_{zi} \\
\hdashline
(\hat{J}_{yyi} - \hat{J}_{zzi})\bar{M}_{xz'i}^T\Omega_{xi}\Omega_{yi} & (\rho_{ci}\bar{M}_{yzi}^T - \frac{1}{2}\hat{J}_{xxi}\bar{M}_{y'z'i}^T)\Omega_{yi}\Omega_{zi} & \begin{array}{c} -\rho_{ci}\bar{M}_{zzi}(\Omega_{xi}^2 + \Omega_{yi}^2) + \rho_{ci}\widetilde{\bar{M}}_{zzi}(V_{yi}\Omega_{zi} - V_{zi}\Omega_{yi}) \\ +\hat{J}_{yyi}\bar{M}_{z'z'i}(\Omega_{xi}^2 - \Omega_{zi}^2) + \frac{1}{2}\rho_{ci}\widetilde{\widetilde{\bar{M}}}_{zzi}(\Omega_{yi}^2 + \Omega_{zi}^2) \end{array}
\end{array}\right], \tag{3.51-j}
$$

$$\mathsf{D}_{eei} = \mathsf{D}^*_{eei} + \begin{bmatrix} 0 & 2\Omega_{yi}\hat{\mathsf{J}}_{zzi}\bar{\mathsf{M}}_{xy'i} & 2\Omega_{zi}\hat{\mathsf{J}}_{yyi}\bar{\mathsf{M}}_{xz'i} \\ -2\Omega_{yi}\hat{\mathsf{J}}_{zzi}\bar{\mathsf{M}}^T_{xy'i} & 0 & -2\Omega_{xi}\rho_{ci}\bar{\mathsf{M}}_{yzi} \\ -2\Omega_{zi}\hat{\mathsf{J}}_{yyi}\bar{\mathsf{M}}^T_{xz'i} & 2\Omega_{xi}\rho_{ci}\bar{\mathsf{M}}^T_{yzi} & 0 \end{bmatrix}, \tag{3.51-k}$$

$$\mathsf{K}^*_{eei} = \begin{bmatrix} \hat{k}_{xi}\hat{\mathsf{I}}_{xxi}\mathsf{G}_i\bar{\mathsf{K}}_{xxi} & 0 & 0 \\ 0 & \mathsf{E}_i\hat{\mathsf{I}}_{zzi}\bar{\mathsf{K}}_{yyi} & 0 \\ 0 & 0 & \mathsf{E}_i\hat{\mathsf{I}}_{yyi}\bar{\mathsf{K}}_{zzi} \end{bmatrix}, \tag{3.51-l}$$

$$\mathsf{D}^*_{eei} = \zeta_i \mathsf{K}^*_{eei}. \tag{3.51-m}$$

$$\mathsf{d}_{ei} = \begin{bmatrix} (\hat{\mathsf{J}}_{yyi} - \hat{\mathsf{J}}_{zzi})\bar{\varphi}_{xi}\Omega_{yi}\Omega_{zi} \\ \rho_{ci}\bar{\varphi}_{yi}(\mathsf{V}_{zi}\Omega_{xi} - \mathsf{V}_{xi}\Omega_{zi}) + (\hat{\mathsf{J}}_{zzi}\bar{\varphi}_{y'i} - \rho_{ci}\widetilde{\varphi}_{yi})\Omega_{xi}\Omega_{yi} \\ \rho_{ci}\bar{\varphi}_{zi}(\mathsf{V}_{xi}\Omega_{yi} - \mathsf{V}_{yi}\Omega_{xi}) + (\hat{\mathsf{J}}_{yyi}\bar{\varphi}_{z'i} - \rho_{ci}\widetilde{\varphi}_{zi})\Omega_{xi}\Omega_{zi} \end{bmatrix} + \mathsf{F}_{ci}, \tag{3.51-n}$$

$$\mathsf{F}_{ci} = \int_0^{\ell_i} \begin{bmatrix} \varphi_{xi}(x)\hat{m}_{xi}(x) \\ \varphi_{yi}(x)\hat{f}_{yi}(x) + \varphi'_{yi}(x)\hat{m}_{zi}(x) \\ \varphi_{zi}(x)\hat{f}_{zi}(x) - \varphi'_{zi}(x)\hat{m}_{yi}(x) \end{bmatrix} dx, \tag{3.51-o}$$

Furthermore,

$$\bar{\varphi}_{xi} = \int_0^{\ell_i} \varphi_{xi}(x)\, dx, \tag{3.52-a}$$

$$\bar{\varphi}_{yi} = \int_0^{\ell_i} \varphi_{yi}(x)\, dx, \tag{3.52-b}$$

$$\bar{\varphi}_{zi} = \int_0^{\ell_i} \varphi_{zi}(x)\, dx, \tag{3.52-c}$$

$$\widetilde{\varphi}_{yi} = \int_0^{\ell_i} x\varphi_{yi}(x)\, dx, \tag{3.52-d}$$

$$\widetilde{\varphi}_{zi} = \int_0^{\ell_i} x\varphi_{zi}(x)\, dx, \tag{3.52-e}$$

$$\bar{\varphi}_{y'i} = \int_0^{\ell_i} \varphi'_{yi}(x)\, dx, \tag{3.52-f}$$

$$\bar{\varphi}_{z'i} = \int_0^{\ell_i} \varphi'_{zi}(x)\, dx, \tag{3.52-g}$$

$$\bar{\mathsf{M}}_{xxi} = \int_0^{\ell_i} \varphi_{xi}(x)\varphi^T_{xi}(x) dx, \tag{3.52-h}$$

$$\bar{\mathsf{M}}_{yyi} = \int_0^{\ell_i} \varphi_{yi}(x)\varphi^T_{yi}(x) dx, \tag{3.52-i}$$

$$\bar{M}_{zzi} = \int_0^{\ell_i} \varphi_{zi}(x)\varphi_{zi}^T(x)dx, \quad (3.52\text{-j})$$

$$\bar{M}_{yzi} = \int_0^{\ell_i} \varphi_{yi}(x)\varphi_{zi}^T(x)dx, \quad (3.52\text{-k})$$

$$\bar{M}_{xy'i} = \int_0^{\ell_i} \varphi_{xi}(x)\varphi_{yi}'^T(x)dx, \quad (3.52\text{-l})$$

$$\bar{M}_{xz'i} = \int_0^{\ell_i} \varphi_{xi}(x)\varphi_{zi}'^T(x)dx, \quad (3.52\text{-m})$$

$$\bar{M}_{y'y'i} = \int_0^{\ell_i} \varphi_{yi}'(x)\varphi_{yi}'^T(x)dx, \quad (3.52\text{-n})$$

$$\bar{M}_{z'z'i} = \int_0^{\ell_i} \varphi_{zi}'(x)\varphi_{zi}'^T(x)dx, \quad (3.52\text{-o})$$

$$\bar{M}_{y'z'i} = \int_0^{\ell_i} \varphi_{yi}'(x)\varphi_{zi}'^T(x)dx, \quad (3.52\text{-p})$$

$$\widetilde{M}_{yyi} = \int_0^{\ell_i} (\ell_i - x)\varphi_{yi}'(x)\varphi_{yi}'^T(x)dx, \quad (3.52\text{-q})$$

$$\widetilde{M}_{zzi} = \int_0^{\ell_i} (\ell_i - x)\varphi_{zi}'(x)\varphi_{zi}'^T(x)dx, \quad (3.52\text{-r})$$

$$\widetilde{\widetilde{M}}_{yyi} = \int_0^{\ell_i} (\ell_i^2 - x^2)\varphi_{yi}'(x)\varphi_{yi}'^T(x)dx, \quad (3.52\text{-s})$$

$$\widetilde{\widetilde{M}}_{zzi} = \int_0^{\ell_i} (\ell_i^2 - x^2)\varphi_{zi}'(x)\varphi_{zi}'^T(x)dx. \quad (3.52\text{-t})$$

In addition,

$$\bar{K}_{xxi} = \int_0^{\ell_i} \varphi_{xi}'(x)\varphi_{xi}'^T(x)\,dx, \quad (3.53\text{-a})$$

$$\bar{K}_{yyi} = \int_0^{\ell_i} \varphi_{yi}''(x)\varphi_{yi}''^T(x)\,dx, \quad (3.53\text{-b})$$

$$\bar{K}_{zzi} = \int_0^{\ell_i} \varphi_{zi}''(x)\varphi_{zi}''^T(x)\,dx. \quad (3.53\text{-c})$$

Finally, referring to Eqs. 3.43-e and 3.43-f, we also have that

$$\mathsf{F}_{xi} = \int_0^{\ell_i} \hat{\mathsf{f}}_{xi}(x)\,dx - \mathsf{q}_{yi}^T \int_0^{\ell_i} \varphi'_{yi}(x)\hat{\mathsf{f}}_{yi}(x)\,dx - \mathsf{q}_{zi}^T \int_0^{\ell_i} \varphi'_{zi}(x)\hat{\mathsf{f}}_{zi}(x)\,dx, \tag{3.54-a}$$

$$\mathsf{F}_{yi} = \mathsf{q}_{yi}^T \int_0^{\ell_i} \varphi'_{yi}(x)\hat{\mathsf{f}}_{xi}(x)\,dx + \int_0^{\ell_i} \hat{\mathsf{f}}_{yi}(x)\,dx - \mathsf{q}_{xi}^T \int_0^{\ell_i} \varphi_{xi}(x)\hat{\mathsf{f}}_{zi}(x)\,dx, \tag{3.54-b}$$

$$\mathsf{F}_{zi} = \mathsf{q}_{zi}^T \int_0^{\ell_i} \varphi'_{zi}(x)\hat{\mathsf{f}}_{xi}(x)\,dx + \mathsf{q}_{xi}^T \int_0^{\ell_i} \varphi_{xi}(x)\hat{\mathsf{f}}_{yi}(x)\,dx + \int_0^{\ell_i} \hat{\mathsf{f}}_{zi}(x)\,dx, \tag{3.54-c}$$

$$\begin{aligned}\mathsf{M}_{xi} = & -\mathsf{q}_{zi}^T \int_0^{\ell_i} \varphi_{zi}(x)\hat{\mathsf{f}}_{yi}(x)\,dx + \mathsf{q}_{yi}^T \int_0^{\ell_i} \varphi_{yi}(x)\hat{\mathsf{f}}_{zi}(x)\,dx + \int_0^{\ell_i} \hat{\mathsf{m}}_{xi}(x)\,dx \\ & -\tfrac{1}{2}\mathsf{q}_{yi}^T \int_0^{\ell_i} \varphi'_{yi}(x)\hat{\mathsf{m}}_{yi}(x)\,dx - \tfrac{1}{2}\mathsf{q}_{zi}^T \int_0^{\ell_i} \varphi'_{zi}(x)\hat{\mathsf{m}}_{zi}(x)\,dx,\end{aligned} \tag{3.54-d}$$

$$\begin{aligned}\mathsf{M}_{yi} = & \;\mathsf{q}_{zi}^T \int_0^{\ell_i} \left[\varphi_{zi}(x) - x\varphi'_{zi}(x)\right]\hat{\mathsf{f}}_{xi}(x)\,dx - \mathsf{q}_{xi}^T \int_0^{\ell_i} x\varphi_{xi}(x)\hat{\mathsf{f}}_{yi}(x)\,dx \\ & - \int_0^{\ell_i} x\hat{\mathsf{f}}_{zi}(x)\,dx + \tfrac{1}{2}\mathsf{q}_{yi}^T \int_0^{\ell_i} \varphi'_{yi}(x)\hat{\mathsf{m}}_{xi}(x)\,dx \\ & + \int_0^{\ell_i} \hat{\mathsf{m}}_{yi}(x)\,dx - \tfrac{1}{2}\mathsf{q}_{xi}^T \int_0^{\ell_i} \varphi_{xi}(x)\hat{\mathsf{m}}_{zi}(x)\,dx,\end{aligned} \tag{3.54-e}$$

$$\begin{aligned}\mathsf{M}_{zi} = & \;\mathsf{q}_{yi}^T \int_0^{\ell_i} \left[x\varphi'_{yi}(x) - \varphi_{yi}(x)\right]\hat{\mathsf{f}}_{xi}(x)\,dx + \int_0^{\ell_i} x\hat{\mathsf{f}}_{yi}(x)\,dx \\ & -\mathsf{q}_{xi}^T \int_0^{\ell_i} x\varphi_{xi}(x)\hat{\mathsf{f}}_{zi}(x)\,dx + \tfrac{1}{2}\mathsf{q}_{zi}^T \int_0^{\ell_i} \varphi'_{zi}(x)\hat{\mathsf{m}}_{xi}(x)\,dx \\ & +\tfrac{1}{2}\mathsf{q}_{xi}^T \int_0^{\ell_i} \varphi_{xi}(x)\hat{\mathsf{m}}_{yi}(x)\,dx + \int_0^{\ell_i} \hat{\mathsf{m}}_{zi}(x)\,dx.\end{aligned} \tag{3.54-f}$$

Chapter 4 Equations of Motion

To each body i in the multibody system we associate another body j (Fig. 4), referred to as the body *inboard* to body i [20], and chosen so that the rigid body motion of body i relative to body j can be analyzed in an efficient manner. The inertial motion of each body in the structure can be determined recursively, by working backwards from the body, to its inboard body, and then finally to the base body, referred to as body 1. Furthermore, the body inboard to body 1 is taken to be the inertial space, referred to as body 0 (zero).

4.1 Generalized Coordinates

We introduce a coordinate system $\{O_i'; x_i' y_i' z_i'\}$, which is fixed on body j (Fig. 4), and assume that the motion of coordinate system $\{O_i; x_i y_i z_i\}$ relative to coordinate system $\{O_i'; x_i' y_i' z_i'\}$ depends just on generalized coordinates

$$q_{ri} = \begin{bmatrix} q_{ri}^1 & q_{ri}^2 & \cdots & q_{ri}^{N_{qri}} \end{bmatrix}^T, \tag{4.1}$$

and quasi-velocities

$$w_{ri} = \begin{bmatrix} w_{ri}^1 & w_{ri}^2 & \cdots & w_{ri}^{N_{wri}} \end{bmatrix}^T, \tag{4.2}$$

in which $N_{wri} \leq N_{qri}$. Recall from the meaning of quasi-velocity [32] that this implies the existence of two N_{qri}-by-N_{wri} matrices A_{ri} and B_{ri}, both functions of q_{ri}, which satisfy the relations

$$\dot{q}_{ri} = B_{ri} w_{ri}, \tag{4.3}$$

$$w_{ri} = A_{ri}^T \dot{q}_{ri}. \tag{4.4}$$

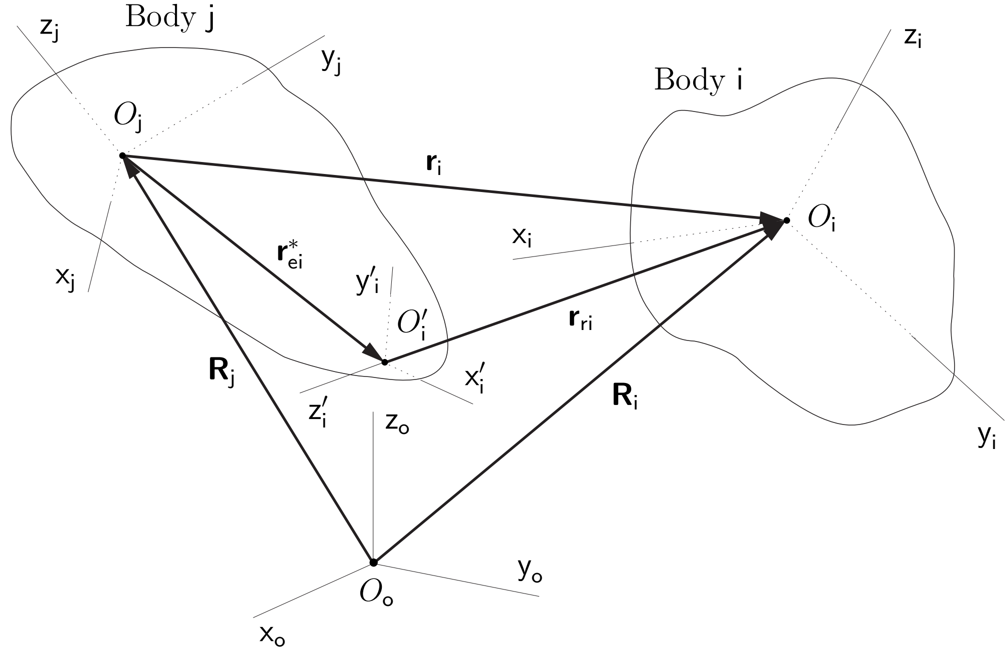

Figure 4: Kinematical Relationship Between Body i and its Inboard Body j

Note that this implies that the matrix product $\mathsf{A}_{ri}^{\mathsf{T}}\mathsf{B}_{ri}$ equals the N_{wri}-by-N_{wri} identity matrix. We also make use of the quasi-virtual displacement

$$\delta\mathsf{q}^*_{ri} \;=\; \left[\begin{array}{cccc} \delta\mathsf{q}^{*1}_{ri} & \delta\mathsf{q}^{*2}_{ri} & \cdots & \delta\mathsf{q}^{*\mathsf{N}_{wri}}_{ri} \end{array}\right]^{\mathsf{T}}, \tag{4.5}$$

which, analogous to Eq. 4.4 is related to $\delta\mathsf{q}_{ri}$ by

$$\delta\mathsf{q}^*_{ri} \;=\; \mathsf{A}_{ri}^{\mathsf{T}}\delta\mathsf{q}_{ri}. \tag{4.6}$$

In many cases, the motion of body i relative to body j will consist solely of rotation about a single axis with direction fixed in the two bodies, in which case $\mathsf{N}_{wri} = \mathsf{N}_{qri} = 1$ and $\mathsf{w}_{ri} = \dot{\mathsf{q}}_{ri}$. However, more general cases of relative rigid body motion are easily included in the multibody system model by allowing $\mathsf{w}_{ri} \neq \dot{\mathsf{q}}_{ri}$ and $\mathsf{N}_{wri} < \mathsf{N}_{qri}$. For example, in some cases we may wish to use the components of the angular velocity of coordinate system $\{O_i; x_i y_i z_i\}$ relative to coordinate system $\{O'_i; x'_i y'_i z'_i\}$ as components of w_{ri}, in which case $\mathsf{w}_{ri} \neq \dot{\mathsf{q}}_{ri}$. Furthermore, in other cases we may want to use Euler parameters as components of q_{ri}, so that in this case $\mathsf{N}_{wri} < \mathsf{N}_{qri}$.

We turn now to a consideration of the structure as a whole, and let

$$\begin{aligned}
\mathsf{q}_\mathsf{r} &= \begin{bmatrix} \mathsf{q}_\mathsf{r}^1 & \mathsf{q}_\mathsf{r}^2 & \cdots & \mathsf{q}_\mathsf{r}^{\mathsf{N}_\mathsf{qr}} \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \mathsf{q}_{\mathsf{r}1}^\mathsf{T} & \mathsf{q}_{\mathsf{r}2}^\mathsf{T} & \cdots & \mathsf{q}_{\mathsf{rN}}^\mathsf{T} \end{bmatrix}^\mathsf{T}, && (4.7)\\
\mathsf{q}_\mathsf{e} &= \begin{bmatrix} \mathsf{q}_\mathsf{e}^1 & \mathsf{q}_\mathsf{e}^2 & \cdots & \mathsf{q}_\mathsf{e}^{\mathsf{N}_\mathsf{e}} \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \mathsf{q}_{\mathsf{e}1}^\mathsf{T} & \mathsf{q}_{\mathsf{e}2}^\mathsf{T} & \cdots & \mathsf{q}_{\mathsf{eN}}^\mathsf{T} \end{bmatrix}^\mathsf{T}, && (4.8)\\
\mathsf{q} &= \begin{bmatrix} \mathsf{q}^1 & \mathsf{q}^2 & \cdots & \mathsf{q}^{\mathsf{N}_\mathsf{q}} \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \mathsf{q}_\mathsf{r}^\mathsf{T} & \mathsf{q}_\mathsf{e}^\mathsf{T} \end{bmatrix}^\mathsf{T}, && (4.9)\\
\mathsf{w}_\mathsf{r} &= \begin{bmatrix} \mathsf{w}_\mathsf{r}^1 & \mathsf{w}_\mathsf{r}^2 & \cdots & \mathsf{w}_\mathsf{r}^{\mathsf{N}_\mathsf{wr}} \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \mathsf{w}_{\mathsf{r}1}^\mathsf{T} & \mathsf{w}_{\mathsf{r}2}^\mathsf{T} & \cdots & \mathsf{w}_{\mathsf{rN}}^\mathsf{T} \end{bmatrix}^\mathsf{T}, && (4.10)\\
\mathsf{w} &= \begin{bmatrix} \mathsf{w}^1 & \mathsf{w}^2 & \cdots & \mathsf{w}^{\mathsf{N}_\mathsf{wr}} \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \mathsf{w}_\mathsf{r}^\mathsf{T} & \dot{\mathsf{q}}_\mathsf{e}^\mathsf{T} \end{bmatrix}^\mathsf{T}, && (4.11)\\
\delta\mathsf{q}_\mathsf{r}^* &= \begin{bmatrix} \delta\mathsf{q}_\mathsf{r}^{*1} & \delta\mathsf{q}_\mathsf{r}^{*2} & \cdots & \delta\mathsf{q}_\mathsf{r}^{*\mathsf{N}_\mathsf{wr}} \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \delta\mathsf{q}_{\mathsf{r}1}^{*\mathsf{T}} & \delta\mathsf{q}_{\mathsf{r}2}^{*\mathsf{T}} & \cdots & \delta\mathsf{q}_{\mathsf{rN}}^{*\mathsf{T}} \end{bmatrix}^\mathsf{T}, && (4.12)\\
\delta\mathsf{q}^* &= \begin{bmatrix} \delta\mathsf{q}^{*1} & \delta\mathsf{q}^{*2} & \cdots & \delta\mathsf{q}^{*\mathsf{N}_\mathsf{w}} \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \delta\mathsf{q}_\mathsf{r}^{*\mathsf{T}} & \delta\mathsf{q}_\mathsf{e}^\mathsf{T} \end{bmatrix}^\mathsf{T}, && (4.13)
\end{aligned}$$

in which $\mathsf{q}_\mathsf{r}^\mathsf{j}$, $\mathsf{q}_\mathsf{e}^\mathsf{j}$, q^j, $\mathsf{w}_\mathsf{r}^\mathsf{j}$, w^j, $\delta\mathsf{q}_\mathsf{r}^{*\mathsf{j}}$ and $\delta\mathsf{q}^{*\mathsf{j}}$ are scalars, and

$$\begin{aligned}
\mathsf{N}_\mathsf{qr} &= \mathsf{N}_{\mathsf{qr}1} + \mathsf{N}_{\mathsf{qr}2} + \cdots + \mathsf{N}_\mathsf{qrN}, && (4.14)\\
\mathsf{N}_\mathsf{wr} &= \mathsf{N}_{\mathsf{wr}1} + \mathsf{N}_{\mathsf{wr}2} + \cdots + \mathsf{N}_\mathsf{wrN}, && (4.15)\\
\mathsf{N}_\mathsf{e} &= \mathsf{N}_{\mathsf{e}1} + \mathsf{N}_{\mathsf{e}2} + \cdots + \mathsf{N}_\mathsf{eN}, && (4.16)\\
\mathsf{N}_\mathsf{q} &= \mathsf{N}_\mathsf{qr} + \mathsf{N}_\mathsf{e}, && (4.17)\\
\mathsf{N}_\mathsf{w} &= \mathsf{N}_\mathsf{wr} + \mathsf{N}_\mathsf{e}. && (4.18)
\end{aligned}$$

Furthermore, we define the N_q-by-N_w matrix

$$\mathsf{B} = \text{block-diag}\Big[\mathsf{B}_{\mathsf{r}1}, \mathsf{B}_{\mathsf{r}2}, \ldots, \mathsf{B}_\mathsf{rN}, \mathrm{I}\Big], \tag{4.19}$$

where in this case I equals the N_e-by-N_e identity matrix, and point out that Eq. 4.3 and the definitions of q and w imply that

$$\dot{\mathsf{q}} = \mathsf{B}\mathsf{w}. \tag{4.20}$$

Finally, we let

$$\mathsf{T}_\mathsf{ri}^* = \begin{bmatrix} \mathsf{T}_\mathsf{ri}^{*1} & \mathsf{T}_\mathsf{ri}^{*2} & \cdots & \mathsf{T}_\mathsf{ri}^{*\mathsf{N}_\mathsf{wri}} \end{bmatrix}^\mathsf{T} \tag{4.21}$$

be the generalized force associated with the quasi-velocity w_ri, in which $\mathsf{T}_\mathsf{ri}^{*\mathsf{j}}$ is a scalar, and define

$$\begin{aligned}
\mathsf{T}_\mathsf{r}^* &= \begin{bmatrix} \mathsf{T}_\mathsf{r}^{*1} & \mathsf{T}_\mathsf{r}^{*2} & \cdots & \mathsf{T}_\mathsf{r}^{*\mathsf{N}_\mathsf{wr}} \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \mathsf{T}_{\mathsf{r}1}^{*\mathsf{T}} & \mathsf{T}_{\mathsf{r}2}^{*\mathsf{T}} & \cdots & \mathsf{T}_\mathsf{rN}^{*\mathsf{T}} \end{bmatrix}^\mathsf{T}, && (4.22)\\
\mathsf{T}^* &= \begin{bmatrix} \mathsf{T}^{*1} & \mathsf{T}^{*2} & \cdots & \mathsf{T}^{*\mathsf{N}_\mathsf{w}} \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \mathsf{T}_\mathsf{r}^{*\mathsf{T}} & 0 \end{bmatrix}^\mathsf{T}, && (4.23)
\end{aligned}$$

in which 0 represents the N_e-by-1 zero matrix, and $\mathsf{T}_\mathsf{r}^{*\mathsf{j}}$ and $\mathsf{T}^{*\mathsf{j}}$ are both scalars. Consequently, the virtual work performed by the generalized force T_ri^* is given by

$$\overline{\delta W}_\mathsf{ri} = \delta\mathsf{q}_\mathsf{ri}^{*\mathsf{T}}\mathsf{T}_\mathsf{ri}^*. \tag{4.24}$$

4.2 Recursive Relations

In this section we develop the transformation from q, w and $\dot{\mathsf{w}}$ to R_i, P_i, V_i, Ω_i, $\dot{\mathsf{V}}_i$ and $\dot{\Omega}_i$, required in formulating the equations of motion. To begin with, r_i is the component matrix of position vector $\mathbf{r}_i = \overrightarrow{O_j O_i}$ (Fig. 4), C_i denotes the matrix of direction cosines of axes $x_i y_i z_i$ with respect to $x_j y_j z_j$ and quasi-velocities v_i and ω_i are defined by the relations

$$\begin{aligned} \mathsf{v}_i &= \mathsf{C}_i \dot{\mathsf{r}}_i, && (4.25)\\ \widetilde{\omega}_i &= \mathsf{C}_i \dot{\mathsf{C}}_i^T, && (4.26) \end{aligned}$$

with associated quasi-virtual displacements given by

$$\begin{aligned} \delta \mathsf{r}_i^* &= \mathsf{C}_i \delta \mathsf{r}_i, && (4.27)\\ \widetilde{\delta\theta}_i^* &= \mathsf{C}_i \delta \mathsf{C}_i^T. && (4.28) \end{aligned}$$

(No confusion should arise from the fact that the symbols $\mathbf{r}_i$ and r_i used here have a different meaning than in Chapters 2 and 3.)

Based on the discussion of the preceding section, r_i and C_i depend only on q_{ej}, q_{ri}, and possibly t. An application of the chain rule implies that

$$\begin{aligned} \mathsf{v}_i &= \mathsf{C}_i \dot{\mathsf{r}}_i,\\ &= \mathsf{C}_i \left(\frac{\partial \mathsf{r}_i}{\partial \mathsf{q}_{ri}^T} \dot{\mathsf{q}}_{ri} + \frac{\partial \mathsf{r}_i}{\partial \mathsf{q}_{ej}} \dot{\mathsf{q}}_{ej} + \frac{\partial \mathsf{r}_i}{\partial \mathsf{t}} \right)\\ &= \left(\mathsf{C}_i \frac{\partial \mathsf{r}_i}{\partial \mathsf{q}_{ri}^T} \mathsf{B}_{ri} \right) \mathsf{w}_{ri} + \left(\mathsf{C}_i \frac{\partial \mathsf{r}_i}{\partial \mathsf{q}_{ej}} \right) \dot{\mathsf{q}}_{ej} + \mathsf{C}_i \frac{\partial \mathsf{r}_i}{\partial \mathsf{t}}, && (4.29) \end{aligned}$$

where the last equality follows from Eq. 4.3; note, too, that a similar expression holds for ω_i. Next, if we were to continue in this manner and calculate $\dot{\mathsf{v}}_i$ and $\dot{\omega}_i$, we would find that the 6-by-1 matrix $[\dot{\mathsf{v}}_i^T \; \dot{\omega}_i^T]^T$ can be written in the form

$$\begin{bmatrix} \dot{\mathsf{v}}_i \\ \dot{\omega}_i \end{bmatrix} = \mathsf{h}_{ri} \dot{\mathsf{w}}_r + \mathsf{h}_{ei} \ddot{\mathsf{q}}_e - \mathsf{d}_i, \quad (4.30)$$

with local quasi-virtual displacements satisfying the relation

$$\begin{bmatrix} \delta \mathsf{r}_i^* \\ \delta \theta_i^* \end{bmatrix} = \mathsf{h}_{ri} \delta \mathsf{q}_r^* + \mathsf{h}_{ei} \delta \mathsf{q}_e, \quad (4.31)$$

with h_{ri} a 6-by-N_{wr} matrix and h_{ei} a 6-by-N_e matrix, with both dependent only on q_{ri} and t, and d_i is a 6-by-1 matrix which depends only on q_{ri}, w_{ri} and t. The derivation of specific formulas for h_{ri}, h_{ei} and d_i must be carried out case by case, depending on the type of joint model involved, and is postponed to a later chapter. In a similar manner, since R_i and P_i are functions of q_r, q_e and t, we conclude that

$$\begin{bmatrix} \dot{V}_i \\ \dot{\Omega}_i \end{bmatrix} = H_{ri}\dot{w}_r + H_{ei}\ddot{q}_e - D_i, \tag{4.32}$$

$$\begin{bmatrix} \delta R_i^* \\ \delta \Theta_i^* \end{bmatrix} = H_{ri}\delta q_r^* + H_{ei}\delta q_e, \tag{4.33}$$

where H_{ri} is a 6-by-N_{wr} matrix and H_{ei} is 6-by-N_e, with both dependent only on q_r, q_e and t, and D_i is a 6-by-1 matrix which depends only on q_r, w_r, q_e, $\dot{q}_e$ and t.

Recall now that the body inboard to body 1 is the inertial space (body 0), which implies that

$$R_1 = r_1, \tag{4.34}$$

$$P_1 = C_1, \tag{4.35}$$

$$\begin{bmatrix} V_1 \\ \Omega_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ \omega_1 \end{bmatrix}, \tag{4.36}$$

$$H_{r1} = h_{r1}, \tag{4.37}$$

$$H_{e1} = h_{e1}, \tag{4.38}$$

$$D_1 = d_1. \tag{4.39}$$

These equations initiate the recursion process described next.

With reference to Fig. 4, we see that the vector equation

$$\mathbf{R}_i = \mathbf{R}_j + \mathbf{r}_i, \tag{4.40}$$

written in terms of component matrices, takes the form

$$R_i = R_j + P_j^T r_i. \tag{4.41}$$

Furthermore, based on the definitions of the matrices of direction cosines, P_i, P_j and C_i, we also have that

$$P_i = C_i P_j. \tag{4.42}$$

Now take derivatives with respect to time, yielding

$$\begin{aligned} V_i &= P_i \dot{R}_i \\ &= C_i P_j \left(\dot{R}_j + P_j^T \widetilde{\Omega}_j r_i + P_j^T \dot{r}_i\right) \\ &= C_i\left(V_j - \tilde{r}_i \Omega_j\right) + v_i, \end{aligned} \tag{4.43}$$

$$\begin{aligned} \widetilde{\Omega}_i &= P_i \dot{P}_i^T \\ &= C_i P_j \left(P_j^T \widetilde{\Omega}_j C_i^T + P_j^T C_i^T \widetilde{\omega}_i\right) \\ &= C_i \widetilde{\Omega}_j C_i^T + \widetilde{\omega}_i \\ &= \left(\widetilde{C_i \Omega_j}\right) + \widetilde{\omega}_i, \end{aligned}$$

which implies that

$$\Omega_i \;=\; C_i\Omega_j + \omega_i. \tag{4.44}$$

Combining Eqs. 4.43 and 4.44, we can write

$$\begin{bmatrix} V_i \\ \Omega_i \end{bmatrix} \;=\; B_i^* \begin{bmatrix} V_j \\ \Omega_j \end{bmatrix} + \begin{bmatrix} v_i \\ \omega_i \end{bmatrix}, \tag{4.45}$$

where

$$B_i^* \;=\; \begin{bmatrix} C_i & -C_i\tilde{r}_i \\ 0 & C_i \end{bmatrix}. \tag{4.46}$$

Analogous to Eq 4.45 we also have that

$$\begin{bmatrix} \delta R_i^* \\ \delta\Theta_i^* \end{bmatrix} \;=\; B_i^* \begin{bmatrix} \delta R_j^* \\ \delta\Theta_j^* \end{bmatrix} + \begin{bmatrix} \delta r_i^* \\ \delta\theta_i^* \end{bmatrix}. \tag{4.47}$$

Now take the time derivative of Eq. 4.45, to get that

$$\begin{bmatrix} \dot{V}_i \\ \dot{\Omega}_i \end{bmatrix} \;=\; B_i^* \begin{bmatrix} \dot{V}_j \\ \dot{\Omega}_j \end{bmatrix} + \begin{bmatrix} \dot{v}_i \\ \dot{\omega}_i \end{bmatrix} + \dot{B}_i^* \begin{bmatrix} V_j \\ \Omega_j \end{bmatrix}, \tag{4.48}$$

where, making use of Eqs. 4.43 and 4.44, we see that

$$\begin{aligned}
\dot{B}_i^* \begin{bmatrix} V_j \\ \Omega_j \end{bmatrix} &= \begin{bmatrix} -\widetilde{\omega}_i C_i & \widetilde{\omega}_i C_i \tilde{r}_i - C_i \dot{\tilde{r}}_i \\ 0 & -\widetilde{\omega}_i C_i \end{bmatrix} \begin{bmatrix} V_j \\ \Omega_j \end{bmatrix} \\
&= \begin{bmatrix} -\omega_i C_i V_j + \widetilde{\omega}_i C_i \tilde{r}_i \Omega_j - C_i \dot{\tilde{r}}_i \Omega_j \\ -\widetilde{\omega}_i C_i \Omega_j \end{bmatrix} \\
&= \begin{bmatrix} -\omega_i (V_i + C_i \tilde{r}_i \Omega_j - v_i) + \widetilde{\omega}_i C_i \tilde{r}_i \Omega_j - \tilde{v}_i C_i \Omega_j \\ -\widetilde{\omega}_i (\Omega_i - \omega_i) \end{bmatrix} \\
&= \begin{bmatrix} -\widetilde{\omega}_i V_i + \widetilde{\omega}_i v_i - \tilde{v}_i (\Omega_i - \omega_i) \\ -\widetilde{\omega}_i \Omega_i \end{bmatrix} \\
&= -\begin{bmatrix} \widetilde{\omega}_i & \tilde{v}_i \\ 0 & \widetilde{\omega}_i \end{bmatrix} \begin{bmatrix} V_i \\ \Omega_i \end{bmatrix},
\end{aligned} \tag{4.49}$$

so that

$$\begin{bmatrix} \dot{V}_i \\ \dot{\Omega}_i \end{bmatrix} \;=\; B_i^* \begin{bmatrix} \dot{V}_j \\ \dot{\Omega}_j \end{bmatrix} + \begin{bmatrix} \dot{v}_i \\ \dot{\omega}_i \end{bmatrix} - X_i^* \begin{bmatrix} V_i \\ \Omega_i \end{bmatrix}, \tag{4.50}$$

in which

$$X_i^* \;=\; \begin{bmatrix} \widetilde{\omega}_i & \tilde{v}_i \\ 0 & \widetilde{\omega}_i \end{bmatrix}. \tag{4.51}$$

Next, we develop recursive relations which can be used to compute H_{ri}, H_{ei} and D_i. To this end, combine Eqs. 4.30, 4.32 and 4.50 to get that

$$\begin{aligned}
H_{ri}\dot{w}_r + H_{ei}\ddot{q}_e - D_i &= \begin{bmatrix} \dot{V}_i \\ \dot{\Omega}_i \end{bmatrix} \\
&= B_i^* \begin{bmatrix} \dot{V}_j \\ \dot{\Omega}_j \end{bmatrix} + \begin{bmatrix} \dot{v}_i \\ \dot{\omega}_i \end{bmatrix} - X_i^* \begin{bmatrix} V_i \\ \Omega_i \end{bmatrix} \\
&= B_i^*(H_{rj}\dot{w}_r + H_{ej}\ddot{q}_e - D_j) + (h_{ri}\dot{w}_r + h_{ei}\ddot{q}_e - d_i) - X_i^* \begin{bmatrix} V_i \\ \Omega_i \end{bmatrix} \\
&= (B_i^* H_{rj} + h_{ri})\dot{w}_r + (B_i^* H_{ej} + h_{ei})\ddot{q}_e - \left(B_i^* D_j + d_i + X_i^* \begin{bmatrix} V_i \\ \Omega_i \end{bmatrix} \right). \qquad (4.52)
\end{aligned}$$

The obvious solution, and the one we use, is to take

$$H_{ri} = B_i^* H_{rj} + h_{ri}, \qquad (4.53)$$

$$H_{ei} = B_i^* H_{ej} + h_{ei}, \qquad (4.54)$$

$$D_i = B_i^* D_j + d_i + X_i^* \begin{bmatrix} V_i \\ \Omega_i \end{bmatrix}. \qquad (4.55)$$

The recursive process is easily carried out by first requiring $j < i$ (this places absolutely no restrictions on the structure). Then, use Eqs. 4.34, 4.35, 4.36, 4.37, 4.38 and 4.39 to compute the motion of body 1, and subsequently use Eqs. 4.41, 4.42, 4.45, 4.53, 4.54 and 4.55 to compute the motion of bodies 2, 3, . . . , N, in that order.

4.3 Discretized Equations of Motion

At this point we have enough information to develop the final form of the discretized equations of motion. To begin with, we note that the formula for $\mathcal{R}_i$ given in Eq. 2.38 for a rigid body has precisely the same form as the formula for $\mathcal{R}_i$ given in Eq. 3.50 for a second-order Rayleigh beam, provided that for the rigid body case we set $N_{ei} = 0$. In fact, this form of $\mathcal{R}_i$ is quite general, in that any other discretized beam model, or for that matter, any discretized plate or shell model, would also result in the same form for $\mathcal{R}_i$. Consequently, we can proceed by substituting Eq. 3.50 into Eq. 2.1, to get that

$$\begin{aligned}
0 = \sum_{i=1}^{N} \Bigg\{ \begin{bmatrix} \delta R_i^{*T} & \delta\Theta_i^{*T} \end{bmatrix} \left(M_{rri} \begin{bmatrix} \dot{V}_i \\ \dot{\Omega}_i \end{bmatrix} + M_{rei}\ddot{q}_{ei} - G_{ri} \right) \\
+ \delta q_{ei}^T \left(M_{rei}^T \begin{bmatrix} \dot{V}_i \\ \dot{\Omega}_i \end{bmatrix} + M_{eei}\ddot{q}_{ei} - G_{ei} \right) \Bigg\} - \sum_{i=1}^{N} \int_{\partial\mathcal{B}_{oi}} \delta z_i^T \tau_{oi}^* d\mathcal{A}. \qquad (4.56)
\end{aligned}$$

Recall that τ_{oi}^*, which is the component matrix, along inertial axes $x_o y_o z_o$, of part of the surface traction, is nonzero only on that part of body i which contacts another body in the structure. However, we include here the forces acting between the base (body 1) and the inertial space (body 0) even in the case when there is no contact between the two bodies. With this in mind, and referring back to Eq. 4.24, we conclude that

$$\sum_{i=1}^{N} \int_{\partial \mathcal{B}_{oi}} \delta z_i^T \tau_{oi}^* d\mathcal{S} = \sum_{i=1}^{N} \delta q_{ri}^{*T} T_{ri}^* \tag{4.57-a}$$

$$= \delta q_r^{*T} T_r^*. \tag{4.57-b}$$

Now substitute Eqs. 4.32, 4.33, and 4.57-b into Eq. 4.56 to get that

$$\begin{aligned}
0 &= \sum_{i=1}^{N} \Big\{ \left(\delta q_r^{*T} H_{ri}^T + \delta q_e^T H_{ei}^T \right) \left[M_{rri} \left(H_{ri} \dot{w}_r + H_{ei} \ddot{q}_e - D_i \right) + M_{rei} \ddot{q}_{ei} - G_{ri} \right] \\
&\quad + \delta q_{ei}^T \left[M_{rei}^T \left(H_{ri} \dot{w}_r + H_{ei} \ddot{q}_e - D_i \right) + M_{eei} \ddot{q}_{ei} - G_{ei} \right] \Big\} - \delta q_r^{*T} T_r^* \\
&= \delta q_r^{*T} \left\{ \left(\sum_{i=1}^{N} H_{ri}^T M_{rri} H_{ri} \right) \dot{w}_r + \left(\sum_{i=1}^{N} H_{ri}^T M_{rri} H_{ei} \right) \ddot{q}_e + \sum_{i=1}^{N} H_{ri}^T M_{rei} \ddot{q}_{ei} \right. \\
&\quad \left. - \sum_{i=1}^{N} H_{ri}^T \left(M_{rri} D_i + G_{ri} \right) - T_r^* \right\} + \delta q_e^T \left\{ \left(\sum_{i=1}^{N} H_{ei}^T M_{rri} H_{ri} \right) \dot{w}_r + \left(\sum_{i=1}^{N} H_{ei}^T M_{rri} H_{ei} \right) \ddot{q}_e \right. \\
&\quad \left. + \sum_{i=1}^{N} H_{ei}^T M_{rei} \ddot{q}_{ei} - \sum_{i=1}^{N} H_{ei}^T \left(M_{rri} D_i + G_{ri} \right) \right\} + \left(\sum_{i=1}^{N} \delta q_{ei}^T M_{rei}^T H_{ri} \right) \dot{w}_r + \left(\sum_{i=1}^{N} \delta q_{ei}^T M_{rei}^T H_{ei} \right) \ddot{q}_e \\
&\quad + \sum_{i=1}^{N} \delta q_{ei}^T M_{eei} \ddot{q}_{ei} - \sum_{i=1}^{N} \delta q_{ei}^T \left(M_{rei}^T D_i + G_{ei} \right) \\
&= \delta q_r^{*T} \left\{ \left(\sum_{i=1}^{N} H_{ri}^T M_{rri} H_{ri} \right) \dot{w}_r + \left(\sum_{i=1}^{N} H_{ri}^T M_{rri} H_{ei} \right. \right. \\
&\quad \left. \left. + \left[H_{r1}^T M_{re1} \,\vdots\, H_{r2}^T M_{re2} \,\vdots\, \cdots \,\vdots\, H_{rN}^T M_{reN} \right] \right) \ddot{q}_e - \sum_{i=1}^{N} H_{ri}^T \left(M_{rri} D_i + G_{ri} \right) - T_r^* \right\} \\
&\quad + \delta q_e^T \left\{ \left(\sum_{i=1}^{N} H_{ei}^T M_{rri} H_{ri} + \left[H_{r1}^T M_{re1} \,\vdots\, H_{r2}^T M_{re2} \,\vdots\, \cdots \,\vdots\, H_{rN}^T M_{reN} \right]^T \right) \dot{w}_r + \left(\sum_{i=1}^{N} H_{ei}^T M_{rri} H_{ei} \right. \right. \\
&\quad + \left[H_{e1}^T M_{re1} \,\vdots\, H_{e2}^T M_{re2} \,\vdots\, \cdots \,\vdots\, H_{eN}^T M_{reN} \right] + \left[H_{e1}^T M_{re1} \,\vdots\, H_{e2}^T M_{re2} \,\vdots\, \cdots \,\vdots\, H_{eN}^T M_{reN} \right]^T \\
&\quad \left. \left. + \begin{bmatrix} M_{ee1} & 0 & \cdots & 0 \\ 0 & M_{ee2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{eeN} \end{bmatrix} \right) \ddot{q}_e - \sum_{i=1}^{N} H_{ei}^T \left(M_{rri} D_i + G_{ri} \right) - \begin{bmatrix} M_{re1}^T D_1 + G_{e1} \\ M_{re2}^T D_2 + G_{e2} \\ \vdots \\ M_{reN}^T D_N + G_{eN} \end{bmatrix} \right\} \\
&= \begin{bmatrix} \delta q_r^{*T} & \delta q_e^T \end{bmatrix} \left\{ \begin{bmatrix} M_{rr} & M_{re} \\ M_{re}^T & M_{ee} \end{bmatrix} \begin{bmatrix} \dot{w}_r \\ \ddot{q}_e \end{bmatrix} - \begin{bmatrix} G_r \\ G_e \end{bmatrix} - \begin{bmatrix} T_r^* \\ 0 \end{bmatrix} \right\} \\
&= \delta q^{*T} \left(M \dot{w} - G - T^* \right),
\end{aligned} \tag{4.58}$$

in which

$$\mathsf{M} = \begin{bmatrix} \mathsf{M}_{rr} & \mathsf{M}_{re} \\ \mathsf{M}_{re}^T & \mathsf{M}_{ee} \end{bmatrix}, \tag{4.59-a}$$

$$\mathsf{G} = \begin{bmatrix} \mathsf{G}_r \\ \mathsf{G}_e \end{bmatrix}, \tag{4.59-b}$$

$$\mathsf{M}_{rr} = \sum_{i=1}^{N} \mathsf{H}_{ri}^T \mathsf{M}_{rr} \mathsf{H}_{ri}, \tag{4.59-c}$$

$$\mathsf{M}_{re} = \sum_{i=1}^{N} \mathsf{H}_{ri}^T \mathsf{M}_{rr} \mathsf{H}_{ei} + \left[\mathsf{H}_{r1}^T \mathsf{M}_{re1} \vdots \mathsf{H}_{r2}^T \mathsf{M}_{re2} \vdots \cdots \vdots \mathsf{H}_{rN}^T \mathsf{M}_{reN} \right], \tag{4.59-d}$$

$$\begin{aligned} \mathsf{M}_{ee} = & \sum_{i=1}^{N} \mathsf{H}_{ei}^T \mathsf{M}_{rr} \mathsf{H}_{ei} + \text{block-diag}\left(\mathsf{M}_{ee1}, \mathsf{M}_{ee2}, \ldots, \mathsf{M}_{eeN}\right) \\ & + \left[\mathsf{H}_{e1}^T \mathsf{M}_{re1} \vdots \mathsf{H}_{e2}^T \mathsf{M}_{re2} \vdots \cdots \vdots \mathsf{H}_{eN}^T \mathsf{M}_{reN} \right] \\ & + \left[\mathsf{H}_{e1}^T \mathsf{M}_{re1} \vdots \mathsf{H}_{e2}^T \mathsf{M}_{re2} \vdots \cdots \vdots \mathsf{H}_{eN}^T \mathsf{M}_{reN} \right]^T, \end{aligned} \tag{4.59-e}$$

$$\mathsf{G}_r = \sum_{i=1}^{N} \mathsf{H}_{ri}^T \left(\mathsf{M}_{rri} \mathsf{D}_i + \mathsf{G}_{ri} \right), \tag{4.59-f}$$

$$\mathsf{G}_e = \sum_{i=1}^{N} \mathsf{H}_{ei}^T \left(\mathsf{M}_{rri} \mathsf{D}_i + \mathsf{G}_{ri} \right) + \begin{bmatrix} \mathsf{M}_{re1}^T \mathsf{D}_1 + \mathsf{G}_{e1} \\ \mathsf{M}_{re2}^T \mathsf{D}_2 + \mathsf{G}_{e2} \\ \vdots \\ \mathsf{M}_{reN}^T \mathsf{D}_N + \mathsf{G}_{eN} \end{bmatrix}. \tag{4.59-g}$$

As it stands, this equation is true for any structure, including those with nonholonomic constraints. However, we restrict our attention to open-loop structures, in which case we can claim that there are no constraint equations involving the components of $\delta \mathbf{q}^*$. Then, combined with Eq. 4.20, the equations of motion for the flexible multibody structure are given as

$$\mathsf{M}\dot{\mathsf{w}} = \mathsf{G} + \mathsf{T}^*, \tag{4.60-a}$$

$$\dot{\mathsf{q}} = \mathsf{B}\mathsf{w}. \tag{4.60-b}$$

Chapter 5 Joint Models

In this chapter we discuss the motion of body i with respect to body j in more detail. There are three cases of interest:

1. Body i does not come in contact with body j.
2. Body i is connected to body j by a revolute joint, with body j rigid.
3. Body i is connected to body j by a revolute joint, with body j a slender beam.

For future reference, we define

$$\mathsf{C}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \tag{5.1}$$

$$\mathsf{C}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{5.2}$$

$$\mathsf{C}_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & & 1 \end{bmatrix}, \tag{5.3}$$

$$\mathsf{D} = \begin{bmatrix} \cos\theta_2\cos\theta_3 & \sin\theta_3 & 0 \\ -\cos\theta_2\sin\theta_3 & \cos\theta_3 & 0 \\ \sin\theta_2 & 0 & 1 \end{bmatrix}, \tag{5.4}$$

$$\mathsf{D}^{-1} = \begin{bmatrix} \cos\theta_3/\cos\theta_2 & -\sin\theta_3/\cos\theta_2 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ -\tan\theta_2\cos\theta_3 & \tan\theta_2\sin\theta_3 & 1 \end{bmatrix}, \tag{5.5}$$

$$\mathsf{E} = \begin{bmatrix} 1-2(\eta_2^2+\eta_3^2) & 2(\eta_1\eta_2+\eta_3\eta_4) & 2(\eta_1\eta_3-\eta_2\eta_4) \\ 2(\eta_1\eta_2-\eta_3\eta_4) & 1-2(\eta_1^2+\eta_3^2) & 2(\eta_2\eta_3+\eta_1\eta_4) \\ 2(\eta_1\eta_3+\eta_2\eta_4) & 2(\eta_2\eta_3-\eta_1\eta_4) & 1-2(\eta_1^2+\eta_2^2) \end{bmatrix}, \tag{5.6}$$

$$\mathsf{D_E} = \begin{bmatrix} \eta_4 & \eta_3 & -\eta_2 & -\eta_1 \\ -\eta_3 & \eta_4 & \eta_1 & -\eta_2 \\ \eta_2 & -\eta_1 & \eta_4 & -\eta_3 \end{bmatrix}. \tag{5.7}$$

Note that if $\mathsf{C} = \mathsf{C}_3(\theta_3)\mathsf{C}_2(\theta_2)\mathsf{C}_1(\theta_1)$, and $\omega = [\,\omega_1\,\omega_2\,\omega_3\,]^\mathsf{T}$ is defined by $\tilde{\omega} = \mathsf{C}\dot{\mathsf{C}}^\mathsf{T}$, then D has the property that $\omega = \mathsf{D}\dot{\theta}$, with $\theta = [\,\theta_1\,\theta_2\,\theta_3\,]^\mathsf{T}$. Furthermore, E is a matrix of direction cosines in terms of the Euler parameters $\eta = [\,\eta_1\,\eta_2\,\eta_3\,\eta_4\,]^\mathsf{T}$, provided $\eta_1^2+\eta_2^2+\eta_3^2+\eta_4^2 = 1$, and if $\Omega = [\,\Omega_\mathsf{x}\,\Omega_\mathsf{y}\,\Omega_\mathsf{z}\,]^\mathsf{T}$ is defined by $\widetilde{\Omega} = \mathsf{E}\dot{\mathsf{E}}^\mathsf{T}$, then $\mathsf{D_E}$ has the property that $\Omega = 2\mathsf{D_E}\dot{\eta}$.

5.1 Body i not connected with body j

If we know that the rotational position of axes $\mathsf{x_iy_iz_i}$ relative to axes $\mathsf{x_jy_jz_j}$ is fairly small, then we can make use of 1-2-3 Euler angles θ_1, θ_2 and θ_3. On the other hand, if we expect the relative rotational motion to be arbitrarily large, then, in order to avoid singularities, it is advisable to use Euler parameters $\eta = [\,\eta_1\,\eta_2\,\eta_3\,\eta_4\,]^\mathsf{T}$. In this case, the motion of coordinate system $\{O_\mathsf{i};\mathsf{x_iy_iz_i}\}$ (Fig. 4) relative to coordinate system $\{O_\mathsf{j};\mathsf{x_jy_jz_j}\}$ involves six degrees of freedom. Furthermore, it makes sense to use quasi-velocities, which in this case means components of $\dot{\mathbf{r}}_\mathsf{i}$ and $\boldsymbol{\omega}_\mathsf{i}$, both along body axes $\mathsf{x_iy_iz_i}$, in which $\mathbf{r}_\mathsf{i} = \overrightarrow{O_\mathsf{j}O_\mathsf{i}}$ (Fig. 4) and $\boldsymbol{\omega}_\mathsf{i}$ is the angular velocity vector of axes $\mathsf{x_iy_iz_i}$ relative to axes $\mathsf{x_jy_jz_j}$. These comments are summarized as follows. (Recall that $\mathsf{r_i}$ is the component matrix, along axes $\mathsf{x_jy_jz_j}$, of vector $\mathbf{r}_\mathsf{i}$, and ω_i is the component matrix, along axes $\mathsf{x_iy_iz_i}$, of vector ω_i.)

$$\mathsf{q_{ri}} = \begin{bmatrix} \mathsf{r_x} & \mathsf{r_y} & \mathsf{r_z} & \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^\mathsf{T}, \quad \text{or}$$

$$= \begin{bmatrix} \mathsf{r_x} & \mathsf{r_y} & \mathsf{r_z} & \eta_1 & \eta_2 & \eta_3 & \eta_4 \end{bmatrix}^\mathsf{T}, \tag{5.8-a}$$

$$\mathsf{w_{ri}} = \begin{bmatrix} \mathsf{v_x} & \mathsf{v_y} & \mathsf{v_z} & \omega_\mathsf{x} & \omega_\mathsf{y} & \omega_\mathsf{z} \end{bmatrix}^\mathsf{T}, \tag{5.8-b}$$

$$\delta\mathsf{q}^*_\mathsf{ri} = \begin{bmatrix} \delta\mathsf{r}^*_\mathsf{x} & \delta\mathsf{r}^*_\mathsf{y} & \delta\mathsf{r}^*_\mathsf{z} & \delta\theta^*_\mathsf{x} & \delta\theta^*_\mathsf{y} & \delta\theta^*_\mathsf{z} \end{bmatrix}^\mathsf{T}, \tag{5.8-c}$$

$$\mathsf{T}^*_\mathsf{ri} = \begin{bmatrix} \mathsf{f_x} & \mathsf{f_y} & \mathsf{f_z} & \mathsf{m_x} & \mathsf{m_y} & \mathsf{m_z} \end{bmatrix}^\mathsf{T}, \tag{5.8-d}$$

$$\mathsf{r_i} = \begin{bmatrix} \mathsf{R_x} & \mathsf{R_y} & \mathsf{R_z} \end{bmatrix}^\mathsf{T}, \tag{5.8-e}$$

$$\mathsf{C_i} = \mathsf{C}_3(\theta_3)\mathsf{C}_2(\theta_2)\mathsf{C}_3(\theta_1), \quad \text{or}$$

$$= \mathsf{E}, \tag{5.8-f}$$

$$\begin{bmatrix} \mathsf{v_i} \\ \omega_\mathsf{i} \end{bmatrix} = \mathsf{w_{ri}}, \tag{5.8-g}$$

$$\mathsf{h_{ri}} = \begin{bmatrix} 0 & \vdots & \mathrm{I}_{6\times6} & \vdots & 0 \end{bmatrix}, \tag{5.8-h}$$

$$
\begin{aligned}
\mathsf{h}_{\mathsf{ei}} &= 0, && (5.8\text{-i})\\
\mathsf{d}_{\mathsf{i}} &= 0, && (5.8\text{-j})\\
\mathsf{B}_{\mathsf{ri}} &= \begin{bmatrix} \mathsf{C}_{\mathsf{i}}^{\mathsf{T}} & 0 \\ 0 & \mathsf{D}^{-1} \end{bmatrix}, \quad \text{or} \\
&= \begin{bmatrix} \mathsf{C}_{\mathsf{i}}^{\mathsf{T}} & 0 \\ 0 & \frac{1}{2}\mathsf{D}_{\mathsf{E}}^{\mathsf{T}} \end{bmatrix}. && (5.8\text{-k})
\end{aligned}
$$

5.2 Body i connected to body j by a revolute joint

In this case, the motion of body i with respect to body j involves both the elastic motion of body j, as well as the rigid body motion of one body with respect to the other. We let r_{ri} be the component matrix, along axes $\mathsf{x}_{\mathsf{i}}'\mathsf{y}'_{\mathsf{i}}\mathsf{z}_{\mathsf{i}}'$ of position vector $\mathbf{r}_{\mathsf{ri}} = \overrightarrow{O_{\mathsf{i}}'O_{\mathsf{i}}}$ and C_{ri} the matrix of direction cosines of axes $\mathsf{x}_{\mathsf{i}}\mathsf{y}_{\mathsf{i}}\mathsf{z}_{\mathsf{i}}$ with respect to axes $\mathsf{x}_{\mathsf{i}}'\mathsf{y}'_{\mathsf{i}}\mathsf{z}_{\mathsf{i}}'$ (Fig. 4). Likewise, $\mathsf{r}_{\mathsf{ei}}^*$ is the component matrix, along axes $\mathsf{x}_{\mathsf{j}}\mathsf{y}_{\mathsf{j}}\mathsf{z}_{\mathsf{j}}$ of vector $\mathbf{r}_{\mathsf{ei}}^* = \overrightarrow{O_{\mathsf{j}}O_{\mathsf{i}}'}$, and $\mathsf{C}_{\mathsf{ei}}^*$ is the matrix of direction cosines of axes $\mathsf{x}_{\mathsf{i}}'\mathsf{y}'_{\mathsf{i}}\mathsf{z}_{\mathsf{i}}'$ with respect to axes $\mathsf{x}_{\mathsf{j}}\mathsf{y}_{\mathsf{j}}\mathsf{z}_{\mathsf{j}}$. These definitions imply that

$$
\begin{aligned}
\mathsf{r}_{\mathsf{i}} &= \mathsf{r}_{\mathsf{ei}}^* + \mathsf{C}_{\mathsf{ei}}^{*\mathsf{T}}\mathsf{r}_{\mathsf{ri}}, && (5.9)\\
\mathsf{C}_{\mathsf{i}} &= \mathsf{C}_{\mathsf{ri}}\mathsf{C}_{\mathsf{ei}}^*. && (5.10)
\end{aligned}
$$

We also let ω_{ri} be the component matrix, along axes $\mathsf{x}_{\mathsf{i}}\mathsf{y}_{\mathsf{i}}\mathsf{z}_{\mathsf{i}}$, of the angular velocity vector of axes $\mathsf{x}_{\mathsf{i}}\mathsf{y}_{\mathsf{i}}\mathsf{z}_{\mathsf{i}}$ with respect to axes $\mathsf{x}_{\mathsf{i}}'\mathsf{y}'_{\mathsf{i}}\mathsf{z}_{\mathsf{i}}'$ and $\delta\theta_{\mathsf{ri}}^*$ the associated quasi-virtual displacement, so that

$$
\begin{aligned}
\widetilde{\omega}_{\mathsf{ri}} &= \mathsf{C}_{\mathsf{ri}}\dot{\mathsf{C}}_{\mathsf{ri}}^{\mathsf{T}}, && (5.11)\\
\widetilde{\delta\Theta}_{\mathsf{ri}}^* &= \mathsf{C}_{\mathsf{ri}}\delta\mathsf{C}_{\mathsf{ri}}^{\mathsf{T}}, && (5.12)
\end{aligned}
$$

and let the 3-by-N_{wr} matrix $\mathsf{h}_{\omega\mathsf{ri}}$ and 3-by-1 matrix $\mathsf{d}_{\omega\mathsf{ri}}$ satisfy the relations

$$
\begin{aligned}
\dot{\omega}_{\mathsf{ri}} &= \mathsf{h}_{\omega\mathsf{ri}}\dot{\mathsf{w}}_{\mathsf{r}} - \mathsf{d}_{\omega\mathsf{ri}}, && (5.13)\\
\delta\theta_{\mathsf{ri}}^* &= \mathsf{h}_{\omega\mathsf{ri}}\delta\mathsf{q}_{\mathsf{r}}^*. && (5.14)
\end{aligned}
$$

For revolute joints, point O_{i}' on body j, and point O_{i} on body i occupy the same location in space, and consequently, $\mathsf{r}_{\mathsf{ri}} = 0$ and $\mathsf{r}_{\mathsf{i}} = \mathsf{r}_{\mathsf{ei}}^*$.

There are several choices available for q_{ri} and w_{ri}, depending on the number of rotational degrees of freedom at the joint. We display next those cases of immediate interest. The particular example we consider in the next chapter has joints with one, two or three rotational degrees of freedom. Furthermore, each degree of freedom is directly controlled by a separate actuator, in the sense that if θ is the rotational degree of freedom and T^* is the torque effected by the actuator, then $\mathsf{T}^*\delta\theta$ is the associated virtual work. We can actually cover all pertinent cases simultaneously by considereing a joint with three degrees of rotational freedom. To be precise, there are three rotations in series, with the first a 1-rotation, the second a 2-rotation and the third a 3-rotation. Equations which correspond to joints with one or two degrees of freedom can be obtained by setting two, or one, of the angles to zero. Letting θ_1, θ_2 and θ_3 be the rotational degrees of freedom and M_1, M_2 and M_3 the corresponding control torques, the pertinent equations take the form

$$\mathsf{q}_{ri} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^{\mathsf{T}}, \tag{5.15-a}$$

$$\mathsf{w}_{ri} = \begin{bmatrix} \dot\theta_1 & \dot\theta_2 & \dot\theta_3 \end{bmatrix}^{\mathsf{T}} = \dot{\mathsf{q}}_{ri}, \tag{5.15-b}$$

$$\delta\mathsf{q}^*_{ri} = \begin{bmatrix} \delta\theta_1 & \delta\theta_2 & \delta\theta_2 \end{bmatrix}^{\mathsf{T}} = \delta\mathsf{q}_{ri}, \tag{5.15-c}$$

$$\mathsf{T}^*_{ri} = \begin{bmatrix} \mathsf{M}_1 & \mathsf{M}_2 & \mathsf{M}_3 \end{bmatrix}^{\mathsf{T}}, \tag{5.15-d}$$

$$\mathsf{C}_{ri} = \mathsf{C}_3(\theta_3)\mathsf{C}_2(\theta_2)\mathsf{C}_1(\theta_1), \tag{5.15-e}$$

$$\omega_{ri} = \mathsf{D}\mathsf{w}_{ri}, \tag{5.15-f}$$

$$\mathsf{h}_{\omega ri} = \begin{bmatrix} 0 & \vdots & \mathsf{D} & \vdots & 0 \end{bmatrix}, \tag{5.15-g}$$

$$\mathsf{d}_{\omega ri} = \dot\theta_1\dot\theta_2 \begin{bmatrix} \sin\theta_2\cos\theta_3 \\ -\sin\theta_2\sin\theta_3 \\ -\cos\theta_2 \end{bmatrix} + \dot\theta_1\dot\theta_3 \begin{bmatrix} \cos\theta_2\sin\theta_3 \\ \cos\theta_2\cos\theta_3 \\ 0 \end{bmatrix} + \dot\theta_2\dot\theta_3 \begin{bmatrix} -\cos\theta_3 \\ \sin\theta_3 \\ 0 \end{bmatrix}, \tag{5.15-h}$$

$$\mathsf{A}_{ri} = \mathrm{I}_{3\times3}, \tag{5.15-i}$$

$$\mathsf{B}_{ri} = \mathrm{I}_{3\times3}. \tag{5.15-j}$$

5.2.1 Body j rigid

In this case, since body j is rigid, r^*_{ei} and C^*_{ei} are both constant, so that $\widetilde{\omega}_i = \mathsf{C}_i\dot{\mathsf{C}}_i^{\mathsf{T}} = \mathsf{C}_{ri}\mathsf{C}^*_{ei}(\mathsf{C}^{*\mathsf{T}}_{ei}\mathsf{C}^{\mathsf{T}}_{ri}\widetilde{\omega}_{ri}) = \widetilde{\omega}_{ri}$. Consequently,

$$\mathsf{r}_i = \mathsf{r}^*_{ei}, \qquad (\mathsf{r}^*_{ei} \text{ a constant}) \tag{5.16}$$

$$\mathsf{C}_i = \mathsf{C}_{ri}\mathsf{C}^*_{ei}, \qquad (\mathsf{C}^*_{ei} \text{ a constant}) \tag{5.17}$$

$$\begin{bmatrix} \mathsf{v}_i \\ \omega_i \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_{ri} \end{bmatrix}, \tag{5.18}$$

$$\mathsf{h}_{ri} = \begin{bmatrix} 0 \\ \mathsf{h}_{\omega ri} \end{bmatrix}, \tag{5.19}$$

$$\mathsf{h}_{ei} = 0, \tag{5.20}$$

$$\mathsf{d}_i = \begin{bmatrix} 0 \\ \mathsf{d}_{\omega ri} \end{bmatrix}. \tag{5.21}$$

5.2.2 Body j a slender beam

In this case, points O_i, O_i' and $O_{cj}|_{x=\ell_j}$ (Figs. 2 and 4) coincide, so that $r^*_{ei} = r_{cj}(\ell_j)$ and $C^*_{ei} = E_{cj}(\ell_j)$. Referring to Eqs. 3.3, 5.9 and 5.10, we see that

$$\begin{aligned} r_i &= r_{cj}(\ell_j) = \ell_j e_1 + u_{cj}(\ell_j), && (5.22)\\ C_i &= C_{ri} E_{cj}(\ell_j). && (5.23) \end{aligned}$$

(To simplify matters, we drop the "(ℓ_j)" for most of the following derivations.) Consequently,

$$\begin{aligned} v_i &= C_i \dot{r}_i \\ &= C_{ri} E_{cj} \dot{u}_{cj}, && (5.24)\\ \widetilde{\omega}_i &= C_i \dot{C}_i^T \\ &= C_{ri} E_{cj} \Big[E_{cj}^T (\widetilde{D_{cj}\dot{\psi}_{cj}}) C_{ri}^T + E_{cj}^T C_{ri}^T \widetilde{\omega}_{ri} \Big] \\ &= \text{Tilde}\Big(C_{ri} D_{cj} \dot{\psi}_{cj} + \omega_{ri} \Big), && (5.25) \end{aligned}$$

where we have used Eqs. 3.7 and 5.23, so that

$$\omega_i = C_{ri} D_{cj} \dot{\psi}_{cj} + \omega_{ri}. \quad (5.26)$$

Next, computing time derivatives of Eqs. 5.24 and 5.26 results in

$$\begin{aligned} \dot{v}_i &= C_{ri} E_{cj} \ddot{u}_{cj} - \widetilde{\omega}_{ri} v_i - C_{ri} (\widetilde{D_{cj}\dot{\psi}_{cj}}) E_{cj} \dot{u}_{cj} \\ &= C_{ri} E_{cj} \ddot{u}_{cj} - \widetilde{\omega}_{ri} v_i - \text{Tilde}\Big(C_{ri} D_{cj} \dot{\psi}_{cj} \Big) C_{ri} E_{cj} \dot{u}_{cj} \\ &= C_{ri} E_{cj} \ddot{u}_{cj} - \widetilde{\omega}_{ri} v_i - (\widetilde{\omega}_i - \widetilde{\omega}_{ri}) v_i \\ &= C_{ri} E_{cj} \ddot{u}_{cj} - \widetilde{\omega}_i v_i, && (5.27)\\ \dot{\omega}_i &= C_{ri} D_{cj} \ddot{\psi}_{cj} + \dot{\omega}_{ri} - \widetilde{\omega}_{ri} (\omega_i - \omega_{ri}) + C_{ri} \dot{D}_{cj} \dot{\psi}_{cj} \\ &= C_{ri} D_{cj} \ddot{\psi}_{cj} + h_{\omega ri} \dot{w}_r - (d_{\omega ri} + \widetilde{\omega}_{ri} \omega_i - C_{ri} \dot{D}_{cj} \dot{\psi}_{cj}). && (5.28) \end{aligned}$$

Combining Eqs. 5.24 and 5.26, and then Eqs. 5.27 and 5.28, we can write

$$\begin{bmatrix} v_i \\ \omega_i \end{bmatrix} = \begin{bmatrix} C_{ri} & 0 \\ 0 & C_{ri} \end{bmatrix} \begin{bmatrix} E_{cj} \dot{u}_{cj} \\ D_{cj} \dot{\psi}_{cj} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_{ri} \end{bmatrix}, \quad (5.29)$$

$$\begin{bmatrix} \dot{v}_i \\ \dot{\omega}_i \end{bmatrix} = \begin{bmatrix} C_{ri} & 0 \\ 0 & C_{ri} \end{bmatrix} \begin{bmatrix} E_{cj} \ddot{u}_{cj} \\ D_{cj} \ddot{\psi}_{cj} \end{bmatrix} + \begin{bmatrix} 0 \\ h_{\omega ri} \end{bmatrix} \dot{w}_r - \begin{bmatrix} \widetilde{\omega}_i v_i \\ d_{\omega ri} + \widetilde{\omega}_{ri} \omega_i - C_{ri} \dot{D}_{cj} \dot{\psi}_{cj} \end{bmatrix}. \quad (5.30)$$

Regardless of the discretization method in use, we can identify a 6-by-N_e matrix Φ_j^* and a 6-by-1 matrix d^*_{ej}, so that

$$\begin{bmatrix} E_{cj} \dot{u}_{cj} \\ D_{cj} \dot{\psi}_{cj} \end{bmatrix} = \Phi_j^* \dot{q}_e, \quad (5.31)$$

$$\begin{bmatrix} E_{cj} \ddot{u}_{cj} \\ D_{cj} \ddot{\psi}_{cj} \end{bmatrix} = \Phi_j^* \ddot{q}_e - d^*_{ej}. \quad (5.32)$$

Combining the previous four equations, and comparing the result with Eq. 4.30, we conclude that

$$\begin{bmatrix} \mathsf{v_i} \\ \omega_\mathsf{i} \end{bmatrix} = \begin{bmatrix} \mathsf{C_{ri}} & 0 \\ 0 & \mathsf{C_{ri}} \end{bmatrix} \Phi_\mathsf{j}^* \dot{\mathsf{q}}_\mathsf{ej} + \begin{bmatrix} 0 \\ \omega_\mathsf{ri} \end{bmatrix}, \tag{5.33}$$

$$\mathsf{h_{ri}} = \begin{bmatrix} 0 \\ \mathsf{h}_{\omega\mathsf{ri}} \end{bmatrix}, \tag{5.34}$$

$$\mathsf{h_{ei}} = \begin{bmatrix} \mathsf{C_{ri}} & 0 \\ 0 & \mathsf{C_{ri}} \end{bmatrix} \Phi_\mathsf{j}^*, \tag{5.35}$$

$$\mathsf{d_i} = \begin{bmatrix} \widetilde{\omega}_\mathsf{i} \mathsf{v_i} \\ \mathsf{d}_{\omega\mathsf{ri}} + \widetilde{\omega}_\mathsf{ri}\omega_\mathsf{i} - \mathsf{C_{ri}}\dot{\mathsf{D}}_\mathsf{cj}\dot{\psi}_\mathsf{cj} \end{bmatrix} + \begin{bmatrix} \mathsf{C_{ri}} & 0 \\ 0 & \mathsf{C_{ri}} \end{bmatrix} \mathsf{d}_\mathsf{ej}^*. \tag{5.36}$$

At this point we derive specific formulas for $\mathsf{u_{cj}}(\ell_\mathsf{j})$, $\mathsf{E_{cj}}(\ell_\mathsf{j})$, Φ_j^* and d_ej^*, for the discretized, second-order Rayleigh beam. Making use of Eqs. 3.37-a, 3.45-a, 3.45-b, 3.45-c, 3.52-n and 3.52-o we see that

$$\begin{aligned} \mathsf{u_{xj}}(\ell_\mathsf{j}) &= -\tfrac{1}{2}\int_0^{\ell_\mathsf{j}} \left[\mathsf{q}_\mathsf{yj}^\mathsf{T}\varphi'_\mathsf{yj}(\mathsf{s})\varphi'^\mathsf{T}_\mathsf{yj}(\mathsf{s})\mathsf{q_{yj}} + \mathsf{q}_\mathsf{zj}^\mathsf{T}\varphi'_\mathsf{zj}(\mathsf{s})\varphi'^\mathsf{T}_\mathsf{zj}(\mathsf{s})\mathsf{q_{zj}}\right] \mathsf{ds} \\ &= -\tfrac{1}{2}\left(\mathsf{q}_\mathsf{yj}^\mathsf{T}\bar{\mathsf{M}}_{\mathsf{y'y'j}}\mathsf{q_{yj}} + \mathsf{q}_\mathsf{zj}^\mathsf{T}\bar{\mathsf{M}}_{\mathsf{z'z'j}}\mathsf{q_{zj}}\right), \end{aligned} \tag{5.37}$$

so that

$$\mathsf{u_{cj}}(\ell_\mathsf{j}) = \begin{bmatrix} -\tfrac{1}{2}(\mathsf{q}_\mathsf{yj}^\mathsf{T}\bar{\mathsf{M}}_{\mathsf{y'y'j}}\mathsf{q_{yj}} + \mathsf{q}_\mathsf{zj}^\mathsf{T}\bar{\mathsf{M}}_{\mathsf{z'z'j}}\mathsf{q_{zj}}) \\ \varphi_\mathsf{yj}^\mathsf{T}(\ell_\mathsf{j})\mathsf{q_{yj}} \\ \varphi_\mathsf{zj}^\mathsf{T}(\ell_\mathsf{j})\mathsf{q_{zj}} \end{bmatrix}, \tag{5.38}$$

$$\dot{\mathsf{u}}_\mathsf{cj}(\ell_\mathsf{j}) = \begin{bmatrix} -\mathsf{q}_\mathsf{yj}^\mathsf{T}\bar{\mathsf{M}}_{\mathsf{y'y'j}}\dot{\mathsf{q}}_\mathsf{yj} - \mathsf{q}_\mathsf{zj}^\mathsf{T}\bar{\mathsf{M}}_{\mathsf{z'z'j}}\dot{\mathsf{q}}_\mathsf{zj} \\ \varphi_\mathsf{yj}^\mathsf{T}(\ell_\mathsf{j})\dot{\mathsf{q}}_\mathsf{yj} \\ \varphi_\mathsf{zj}^\mathsf{T}(\ell_\mathsf{j})\dot{\mathsf{q}}_\mathsf{zj} \end{bmatrix}, \tag{5.39}$$

$$\ddot{\mathsf{u}}_\mathsf{cj}(\ell_\mathsf{j}) = \begin{bmatrix} -\mathsf{q}_\mathsf{yj}^\mathsf{T}\bar{\mathsf{M}}_{\mathsf{y'y'j}}\ddot{\mathsf{q}}_\mathsf{yj} - \mathsf{q}_\mathsf{zj}^\mathsf{T}\bar{\mathsf{M}}_{\mathsf{z'z'j}}\ddot{\mathsf{q}}_\mathsf{zj} - \mathsf{q}_\mathsf{yj}^\mathsf{T}\bar{\mathsf{M}}_{\mathsf{y'y'j}}\dot{\mathsf{q}}_\mathsf{yj} - \dot{\mathsf{q}}_\mathsf{zj}^\mathsf{T}\bar{\mathsf{M}}_{\mathsf{z'z'j}}\dot{\mathsf{q}}_\mathsf{zj} \\ \varphi_\mathsf{yj}^\mathsf{T}(\ell_\mathsf{j})\ddot{\mathsf{q}}_\mathsf{yj} \\ \varphi_\mathsf{zj}^\mathsf{T}(\ell_\mathsf{j})\ddot{\mathsf{q}}_\mathsf{zj} \end{bmatrix}. \tag{5.40}$$

Likewise, referring to Eqs. 3.37-b and 3.37-c, we see that

$$\psi_\mathsf{cj}(\ell_\mathsf{j}) = \begin{bmatrix} \varphi_\mathsf{xj}^\mathsf{T}(\ell_\mathsf{j})\mathsf{q_{xj}} \\ -\varphi'^\mathsf{T}_\mathsf{zj}(\ell_\mathsf{j})\mathsf{q_{zj}} + \tfrac{1}{2}\mathsf{q}_\mathsf{xj}^\mathsf{T}\varphi_\mathsf{xj}(\ell_\mathsf{j})\varphi'^\mathsf{T}_\mathsf{yj}(\ell_\mathsf{j})\mathsf{q_{yj}} \\ \varphi'^\mathsf{T}_\mathsf{yj}(\ell_\mathsf{j})\mathsf{q_{yj}} + \tfrac{1}{2}\mathsf{q}_\mathsf{xj}^\mathsf{T}\varphi_\mathsf{xj}(\ell_\mathsf{j})\varphi'^\mathsf{T}_\mathsf{zj}(\ell_\mathsf{j})\mathsf{q_{zj}} \end{bmatrix}, \tag{5.41}$$

$$\dot{\psi}_\mathsf{cj}(\ell_\mathsf{j}) = \begin{bmatrix} \varphi_\mathsf{xj}^\mathsf{T}(\ell_\mathsf{j})\dot{\mathsf{q}}_\mathsf{xj} \\ -\varphi'^\mathsf{T}_\mathsf{zj}(\ell_\mathsf{j})\dot{\mathsf{q}}_\mathsf{zj} + \tfrac{1}{2}\mathsf{q}_\mathsf{xj}^\mathsf{T}\varphi_\mathsf{xj}(\ell_\mathsf{j})\varphi'^\mathsf{T}_\mathsf{yj}(\ell_\mathsf{j})\dot{\mathsf{q}}_\mathsf{yj} + \tfrac{1}{2}\mathsf{q}_\mathsf{yj}^\mathsf{T}\varphi'_\mathsf{yj}(\ell_\mathsf{j})\varphi_\mathsf{xj}^\mathsf{T}(\ell_\mathsf{j})\dot{\mathsf{q}}_\mathsf{xj} \\ \varphi'^\mathsf{T}_\mathsf{yj}(\ell_\mathsf{j})\dot{\mathsf{q}}_\mathsf{yj} + \tfrac{1}{2}\mathsf{q}_\mathsf{xj}^\mathsf{T}\varphi_\mathsf{xj}(\ell_\mathsf{j})\varphi'^\mathsf{T}_\mathsf{zj}(\ell_\mathsf{j})\dot{\mathsf{q}}_\mathsf{zj} + \tfrac{1}{2}\mathsf{q}_\mathsf{zj}^\mathsf{T}\varphi'_\mathsf{zj}(\ell_\mathsf{j})\varphi_\mathsf{xj}^\mathsf{T}(\ell_\mathsf{j})\dot{\mathsf{q}}_\mathsf{xj} \end{bmatrix}, \tag{5.42}$$

$$\ddot{\psi}_{cj}(\ell_j) = \begin{bmatrix} \varphi_{xj}^T(\ell_j)\ddot{q}_{xj} \\ -\varphi_{zj}'^T(\ell_j)\ddot{q}_{zj} + \frac{1}{2}q_{xj}^T\varphi_{xj}(\ell_j)\varphi_{yj}'^T(\ell_j)\ddot{q}_{yj} + \frac{1}{2}q_{yj}^T\varphi_{yj}'(\ell_j)\varphi_{xj}^T(\ell_j)\ddot{q}_{xj} + \frac{1}{2}\dot{q}_{xj}^T\varphi_{xj}(\ell_j)\varphi_{yj}'(\ell_j)\dot{q}_{yj} \\ \varphi_{yj}'^T(\ell_j)\ddot{q}_{yj} + \frac{1}{2}q_{xj}^T\varphi_{xj}(\ell_j)\varphi_{zj}'^T(\ell_j)\ddot{q}_{zj} + \frac{1}{2}q_{zj}^T\varphi_{zj}'(\ell_j)\varphi_{xj}^T(\ell_j)\ddot{q}_{xj} + \frac{1}{2}\dot{q}_{xj}^T\varphi_{xj}(\ell_j)\varphi_{zj}'(\ell_j)\dot{q}_{zj} \end{bmatrix}. \tag{5.43}$$

Now, referring to Eqs. 3.38-a and 3.38-b, we compute $E_{cj}\dot{u}_{cj}$, $E_{cj}\ddot{u}_{cj}$, $D_{cj}\dot{\psi}_{cj}$ and $E_{cj}\dot{\psi}_{cj}$, and then compare the results with Eqs. 5.31 and 5.32 to obtain

$$\Phi_j^* = \begin{bmatrix} 0\cdots 0 & q_{yj}^T(\varphi_{yj}'\varphi_{yj}^T - \bar{M}_{y'y'j}) & q_{zj}^T(\varphi_{zj}'\varphi_{zj}^T - \bar{M}_{z'z'j}) \\ 0\cdots 0 & \varphi_{yj}^T & q_{xj}^T\varphi_{xj}\varphi_{zj}^T \\ 0\cdots 0 & -q_{xj}^T\varphi_{xj}\varphi_{yj}^T & \varphi_{zj}^T \\ \varphi_{xj}^T & \frac{1}{2}q_{zj}^T\varphi_{zj}'\varphi_{yj}'^T & -\frac{1}{2}q_{yj}^T\varphi_{yj}'\varphi_{zj}'^T \\ 0\cdots 0 & q_{xj}^T\varphi_{xj}\varphi_{yj}'^T & -\varphi_{zj}'^T \\ 0\cdots 0 & \varphi_{yj}'^T & q_{xj}^T\varphi_{xj}\varphi_{zj}'^T \end{bmatrix}, \tag{5.44-a}$$

$$d_{ej}^* = \begin{bmatrix} q_{yj}^T\bar{M}_{y'y'j}\dot{q}_{yj} + \dot{q}_{zj}^T\bar{M}_{z'z'j}\dot{q}_{zj} \\ 0 \\ 0 \\ 0 \\ \dot{q}_{xj}^T\varphi_{xj}\varphi_{yj}'^T\dot{q}_{yj} \\ \dot{q}_{xj}^T\varphi_{xj}\varphi_{zj}'^T\dot{q}_{zj} \end{bmatrix}. \tag{5.44-b}$$

Finally, we note that the required formula for E_{cj} is obtained by substituting Eqs. 3.45-a, 3.45-b and 3.45-c into Eq. 3.38-a.

Chapter 6 Numerical Example

The equations developed in this paper were used to create a general purpose computer program which was applied to a robot consisting of two flexible links — bodies 1 and 2 — with a rigid end effector — body 3 — as shown in Fig. 5. To simplify the mathematical model somewhat, we assume that the joints have no mass and are dimensionless, which implies that the three axes of joint 1 intersect in a common point and that the three axes of joint 3 intersect in a common point. We have also assumed that the robot base is fixed in the inertial space. Although this is certainly not the case for a space-based robot, keep in mind that for the space shuttle for example, the ratio of robot arm mass to shuttle mass is approximately .0035, so that for the duration of the maneuvers we consider here (30 s), the motion of the base should have a minimal effect.

6.1 Shape Functions

In order to determine an appropriate choice of shape functions we first note that, relative to coordinate system $\{O_i; x_i y_i z_i\}$ (Fig. 2), beam i should behave similar to a cantilever beam with a mass attached at the far end, where $x = \ell_i$, $i = 1, 2$. This dictates that the geometric boundary conditions are given by

$$u_{yi}(0) = u_{zi}(0) = \psi_{xi}(0) = u'_{yi}(0) = u'_{zi}(0) = 0. \tag{6.1}$$

The dynamic boundary conditions are not time-invariant, as would be the case if coordinate system $\{O_i; x_i y_i z_i\}$ was fixed in the inertial space and there was simply a rigid-body fixed on the far end of the beam. Specific, albeit extremely complicated, expressions for the dynamic boundary conditions could be obtained from substituting Eq. 3.44, for $i = 1, 2$, and Eq. 2.38, for $i = 3$, into Eq. 2.1. The equations are complicated by the fact that $[\delta R_2^{*T}\ \delta\Theta_2^{*T}]^T$ involves $\delta u_{y1}(\ell_1)$, $\delta u_{z1}(\ell_1)$, $\delta\psi_{x1}(\ell_1)$, $\delta u'_{y1}(\ell_1)$ and $\delta u'_{z1}(\ell_1)$, and $[\delta R_3^{*T}\ \delta\Theta_3^{*T}]^T$ involves $\delta u_{yi}(\ell_i)$, $\delta u_{zi}(\ell_i)$,

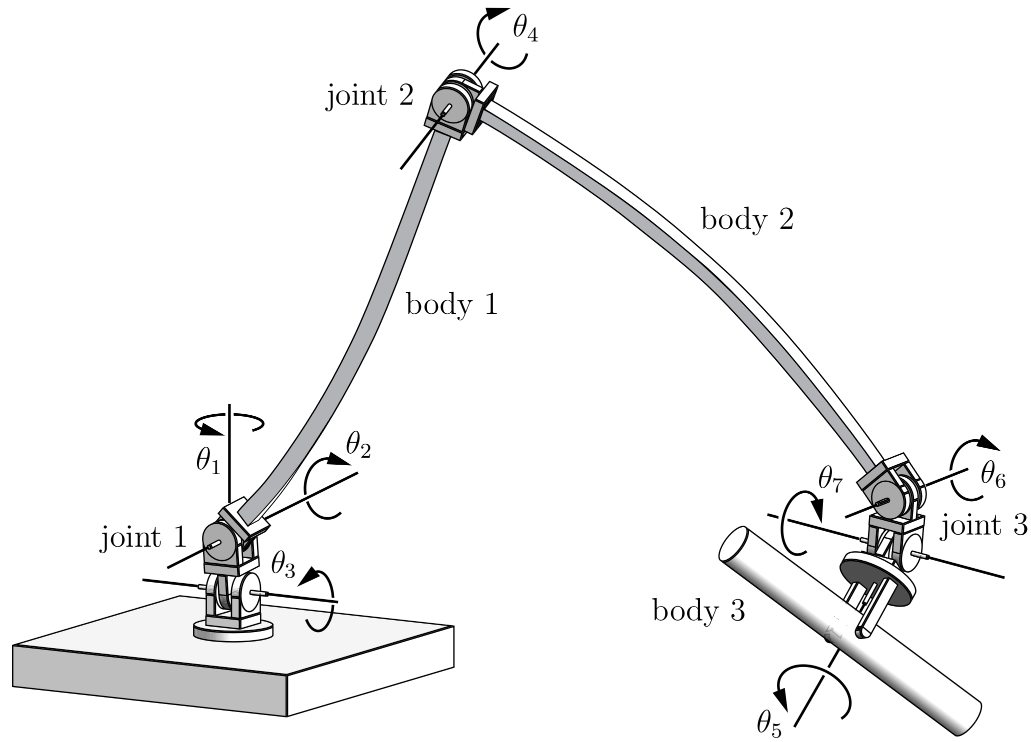

Figure 5: Model of a Flexible Robot

$\delta\psi_{xi}(\ell_i)$, $\delta u'_{yi}(\ell_i)$ and $\delta u'_{zi}(\ell_i)$, for $i = 1, 2$. The choice of shape functions, however, does not require explicit knowledge of the dynamic boundary conditions.

The discretization process is carried out by the hierarchical finite element method, of which a special case is the p-version of the finite element method [35]. According to the p-version of the finite element method, for bending vibration the first four interpolation functions over a finite element are Hermite cubics (Fig. 6-a), with accuracy improved by adding higher-degree polynomials (the p-version) to the same element, rather than refining the mesh (the h-version).

In the hierarchical finite element method, the higher-degree interpolation functions $f_i(\xi)$ do not affect the nodal displacements, i.e., $f_i(a) = f_i(b) = f'_i(a) = f'_i(b) = 0$, where a and b denote the endpoints of the element. Hence, the nodal displacements are defined by the Hermite cubics alone. A standard set of hierarchical functions consists of polynomials [35]

$$f_i(\xi) = \xi^2(1-\xi)\prod_{j=2}^{i}(j-1-i\,\xi), \quad 0 \le \xi \le 1, \quad i = 1, 2, \dots, \tag{6.2}$$

in which $\xi = (x-a)/(b-a)$, and chosen so that $f_i(\xi)$ has $i-1$ equally spaced zeros in the

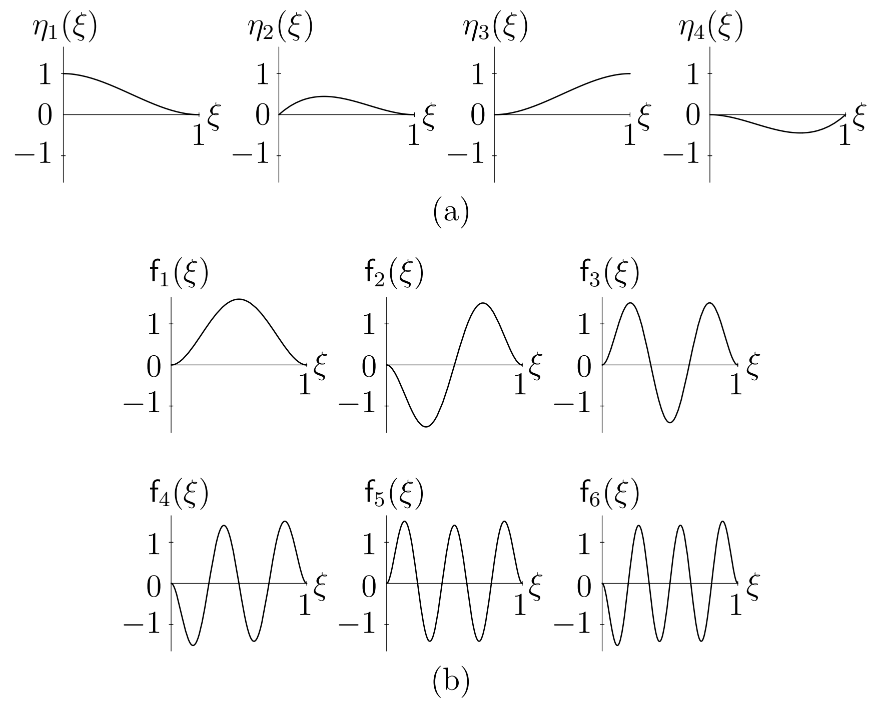

Figure 6: Shape Functions for Bending, (a) Hermite Cubics, (b) Fixed-Fixed Euler-Bernoulli Beam Eigenfunctions

interval $0 \leq \xi \leq 1$.

Another set of hierarchical functions, first used by Meirovitch and Stemple [46] to model framed structures, consists of the sequence of *fixed-fixed* Euler-Bernoulli beam eigenfunctions (Fig. 6-b). We point out that a useful computational formula for these shape functions is

$$\begin{aligned} \mathsf{f}_\mathsf{i}(\xi) &= (\cos\lambda_\mathsf{i} + \sin\lambda_\mathsf{i} - e^{-\lambda_\mathsf{i}})\sin\lambda_\mathsf{i}\xi + (\cos\lambda_\mathsf{i} - \sin\lambda_\mathsf{i} - e^{-\lambda_\mathsf{i}})\cos\lambda_\mathsf{i}\xi \\ &\quad + (e^{-\lambda_\mathsf{i}}\cos\lambda_\mathsf{i} - 1)e^{-\lambda_\mathsf{i}(1-\xi)} + (\sin\lambda_\mathsf{i})e^{-\lambda_\mathsf{i}\xi}, \quad 0 \leq \xi \leq 1, \quad \mathsf{i} = 1, 2, \ldots, \end{aligned} \tag{6.3}$$

where λ_i satisfies the characteristic equation $\cos\lambda_\mathsf{i}\cosh\lambda_\mathsf{i} = 1$. That is, the functions $\mathsf{f}_\mathsf{i}(\xi)$ satisfy the differential equation $\mathsf{f}_\mathsf{i}''''(\xi) - \mu^4\mathsf{f}_\mathsf{i}(\xi) = 0$, with boundary conditions $\mathsf{f}_\mathsf{i}(0) = \mathsf{f}_\mathsf{i}'(0) = \mathsf{f}_\mathsf{i}(1) = \mathsf{f}_\mathsf{i}'(1) = 0$. The quantity μ^2 is the nondimensional natural frequency of the beam.

(As an aside, we note that the eigenfunctions for the *fixed-free* Euler-Bernoulli beam have a similar form, given by

$$\begin{aligned} \mathsf{f}_\mathsf{i}(\xi) &= (\cos\lambda_\mathsf{i} + \sin\lambda_\mathsf{i} + e^{-\lambda_\mathsf{i}})\sin\lambda_\mathsf{i}\xi + (\cos\lambda_\mathsf{i} - \sin\lambda_\mathsf{i} + e^{-\lambda_\mathsf{i}})\cos\lambda_\mathsf{i}\xi \\ &\quad + (e^{-\lambda_\mathsf{i}}\cos\lambda_\mathsf{i} + 1)e^{-\lambda_\mathsf{i}(1-\xi)} + (\sin\lambda_\mathsf{i})e^{-\lambda_\mathsf{i}\xi}, \quad 0 \leq \xi \leq 1, \quad \mathsf{i} = 1, 2, \ldots, \end{aligned} \tag{6.4}$$

with λ_i satisfying the characteristic equation $\cos\lambda_i \cosh\lambda_i = -1$, and with boundary conditions $f_i(0) = f_i'(0) = f_i''(1) = f_i'''(1) = 0$.)

Unlike the standard set of hierarchical functions, the fixed-fixed shape functions are orthogonal, in the sense that

$$\int_0^1 f_i(\xi)f_j(\xi)d\xi = \int_0^1 f_i''(\xi)f_j''(\xi)d\xi = 0, \quad i,j = 1,2,\ldots, \quad i \neq j. \tag{6.5}$$

In addition, if $\eta_i(\xi)$, $i = 1,2,3,4$, are the four Hermite cubics, then it can be shown that

$$\int_0^1 f_i''(\xi)\eta_j''(\xi)d\xi = 0, \quad i = 1,2,\ldots, \quad j = 1,2,3,4. \tag{6.6}$$

These facts imply that the mass and stiffness matrices are substantially sparser than would be the case if polynomials were used for the hierarchical functions.

We also need to consider the discretization process for the torsional motion of the beam. Recall that the torsional equation of motion for the linear case is simply the 1-dimensional wave equation accompanied by the appropriate boundary conditions. Consequently, the first two interpolation functions over a finite element are the linear interpolation functions (Fig. 7-a). Taking a cue from the previous discussion concerning the lateral motion of the beam, we choose the sequence of eigenfunctions for the fixed-fixed torsional beam as heirarchical functions, with formula given by

$$h_i(\xi) = \sin(\pi i\,\xi), \quad 0 \leq \xi \leq 1, \quad i = 1,2,\ldots, \tag{6.7}$$

and shown in Fig. 7-b.

6.2 Control Forces

From an analytical viewpoint, the actuators which control rigid-body motions are relatively simple. Each of the angles θ_i, $i = 1,2,\ldots,7$, (Fig. 5) is controlled by a separate control torque T_i^*. In order to demonstrate the simulation of the robotic structure, we assume that a particular desired trajectory, $\theta_i^*(t)$, $0 \leq t \leq t_F$, which satisfies $\dot{\theta}_i^*(0) = \dot{\theta}_i^*(t_F) = 0$, is associated with each of the angles θ_i. The open-loop control torques $T_{io}^*(t)$, $0 \leq t \leq t_F$, are the joint torques required to force the structure, assumed rigid, to follow the trajectories $\theta_i^*(t)$. It is sufficient to assume that the feedback control torque T_i^* responds only to the error in the angular position θ_i and angular velocity $\dot{\theta}_i$, so that the actual control torques are given by

$$T_i^*(t) = \left\{ \begin{array}{ll} T_{io}^*(t) - \alpha_i[\theta_i(t) - \theta_i^*(t)] - \beta_i[\dot{\theta}_i(t) - \dot{\theta}_i^*(t)], & t \leq 0 \\ \quad - \alpha_i[\theta_i(t) - \theta_i^*(t_F)] - \beta_i[\dot{\theta}_i(t) - 0], & t \geq t_F \end{array} \right\}, \quad i = 1,2,\ldots,7, \tag{6.8}$$

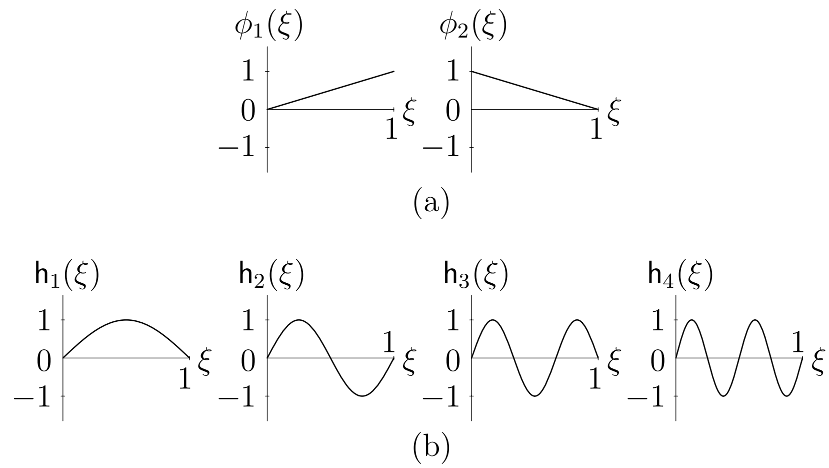

Figure 7: Shape Functions for Torsion, (a) Linear Interpolation Functions, (b) Fixed-Fixed Rod Eigenfunctions

in which α_i and β_i are constant feedback gains.

The elastic motion of the beams is controlled with actuators capable of producing torques about the y_i and z_i axes. We assume that the actuators come in pairs, with the individual actuators of a given pair separated by a short distance, and producing torques of equal magnitude but opposite sign (Fig. 8). This type of setup can be used to model the effect of piezoelectric patches [14], although in this case, it is probably more practical to assume that control-moment-gyros are used to produce the torques. In this case it makes sense to feed back the curvature and its time derivative of the point midway between the two actuators, so that

$$\hat{m}^k_{yi}(t) = \alpha_{cyi} u''_{zi}(x^k_{cyi}, t) + \beta_{cyi} \dot{u}''_{zi}(x^k_{cyi}, t), \quad k = 1, 2, \ldots, N_{cyi}, \tag{6.9}$$

$$\hat{m}^k_{zi}(t) = \alpha_{czi} u''_{yi}(x^k_{czi}, t) + \beta_{czi} \dot{u}''_{yi}(x^k_{czi}, t), \quad k = 1, 2, \ldots, N_{czi}, \quad i = 1, 2. \tag{6.10}$$

This implies that the distributed control forces and torques defined in Eqs. 2.75-a and 2.75-b are given by

$$\hat{f}_{xi}(x, t) = \hat{f}_{yi}(x, t) = \hat{f}_{zi}(x, t) = \hat{m}_{xi}(x, t) = 0, \tag{6.11}$$

$$\hat{m}_{yi}(x, t) = \sum_{k=1}^{N_{cyi}} \hat{m}^k_{yi}(t)[\delta(x^k_{cyi} + \tfrac{1}{2}h_{cyi} - x) - \delta(x^k_{cyi} - \tfrac{1}{2}h_{cyi} - x)], \tag{6.12}$$

$$\hat{m}_{zi}(x, t) = -\sum_{k=1}^{N_{czi}} \hat{m}^k_{zi}(t)[\delta(x^k_{czi} + \tfrac{1}{2}h_{czi} - x) - \delta(x^k_{czi} - \tfrac{1}{2}h_{czi} - x)], \tag{6.13}$$

in which $\delta(x)$ is the Dirac delta function. Note that the sign difference in the formulas for $\hat{m}_{yi}(x, t)$ and $\hat{m}_{zi}(x, t)$ arises from the fact that in Fig. 8-a the y_i-axis points into the paper,

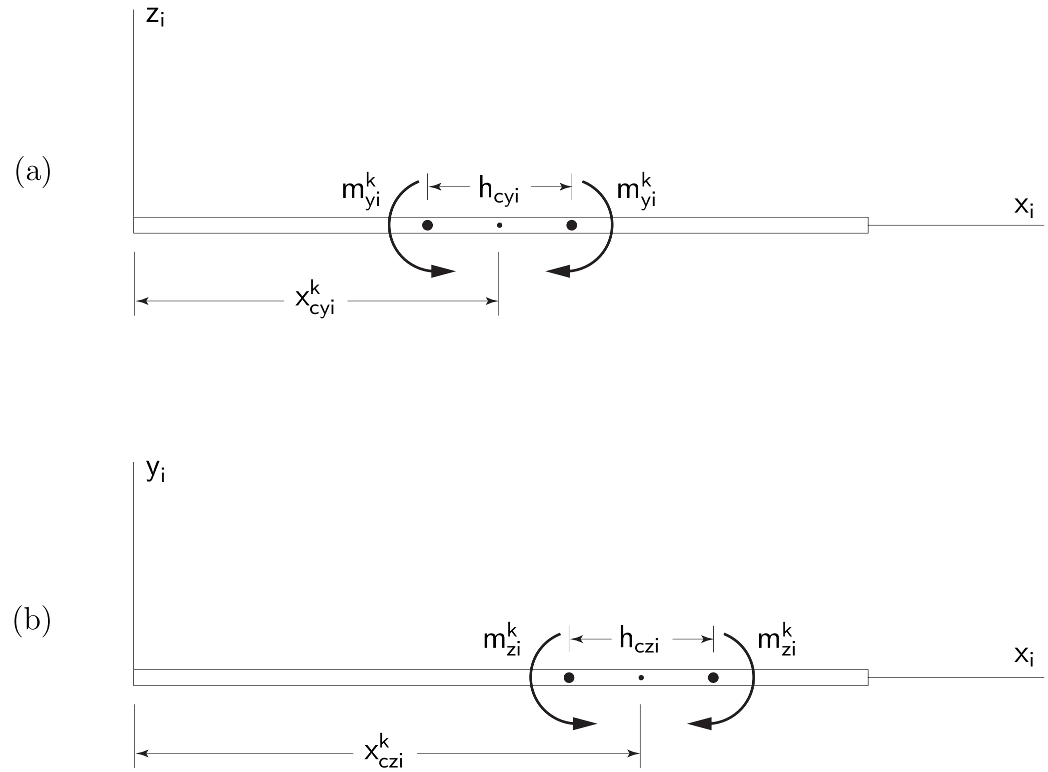

Figure 8: Actuator Pairs for Controlling Beam Vibrations, (a) About the y_i-axis, (b) About the z_i-axis

and in Fig. 8-b the z_i-axis points out of the paper. Now substitute Eqs. 6.11, 6.12 and 6.13 into Eqs. 3.51-o, 3.54-a, 3.54-b, 3.54-c, 3.54-d, 3.54-e and 3.54-f to obtain

$$\mathsf{F}_{ci}(t) = \begin{bmatrix} 0 \\ \sum_{k=1}^{N_{czi}} \hat{m}_{zi}^k(t)[\varphi'_{yi}(x_{czi}^k - \frac{1}{2}h_{czi}) - \varphi'_{yi}(x_{czi}^k + \frac{1}{2}h_{czi})]^T \\ \sum_{k=1}^{N_{cyi}} \hat{m}_{yi}^k(t)[\varphi'_{zi}(x_{cyi}^k - \frac{1}{2}h_{cyi}) - \varphi'_{zi}(x_{cyi}^k + \frac{1}{2}h_{cyi})]^T \end{bmatrix}, \tag{6.14}$$

$$\mathsf{F}_{xi}(t) = \mathsf{F}_{yi}(x,t) = \mathsf{F}_{zi}(x,t) = 0, \tag{6.15}$$

$$\begin{aligned} \mathsf{M}_{xi}(t) = & \frac{1}{2}q_{yi}^T \sum_{k=1}^{N_{cyi}} \hat{m}_{yi}^k(t)[\varphi'_{yi}(x_{cyi}^k - \tfrac{1}{2}h_{cyi}) - \varphi'_{yi}(x_{cyi}^k + \tfrac{1}{2}h_{cyi})] \\ & -\frac{1}{2}q_{zi}^T \sum_{k=1}^{N_{czi}} \hat{m}_{zi}^k(t)[\varphi'_{zi}(x_{czi}^k - \tfrac{1}{2}h_{czi}) - \varphi'_{zi}(x_{czi}^k + \tfrac{1}{2}h_{czi})], \end{aligned} \tag{6.16}$$

$$\mathsf{M}_{yi}(t) = \frac{1}{2}q_{xi}^T \sum_{k=1}^{N_{czi}} \hat{m}_{zi}^k(t)[\varphi_{xi}(x_{czi}^k + \tfrac{1}{2}h_{czi}) - \varphi_{xi}(x_{czi}^k - \tfrac{1}{2}h_{czi})], \tag{6.17}$$

$$M_{zi}(t) = \frac{1}{2} q_{xi}^T \sum_{k=1}^{N_{cyi}} \hat{m}_{yi}^k(t)[\varphi_{xi}(x_{cyi}^k + \tfrac{1}{2}h_{cyi}) - \varphi_{xi}(x_{cyi}^k - \tfrac{1}{2}h_{cyi})]. \quad (6.18)$$

6.3 Simulation Results

As mentioned earlier, we consider a flexible space robot comprised of two slender beams and a rigid end-effector (Fig. 5). Furthermore, the first and third joints have three degrees of freedom, with the motion described by a 1-rotation, 2-rotation, and 3-rotation, in series, and the second joint, which has one degree of freedom, is described by a 2-rotation, so that $N_{qr} = N_{wr} = 7$. The system parameters for the structure are given as follows:

$$\ell_i = 4 \text{ m}, \quad \rho_{ci} = 21.6 \text{ kg/m}, \quad i = 1,\ 2,$$
$$\hat{I}_{xxi} = 9.84 \times 10^{-6} \text{ m}^4, \quad \hat{I}_{yyi} = \hat{I}_{zzi} = 4.92 \times 10^{-6} \text{ m}^4, \quad i = 1,\ 2,$$
$$m_3 = 62.1 \text{ kg}, \quad J_{e3} = \text{diag}[\,35.6,\ 85.4,\ 115.0] \text{ kg-m}^2.$$

For the particular numerical example under consideration, we use only one finite element per beam, with six hierarchical functions for lateral motion, and four hierarchical functions for torsional motion. Due to the nature of the the geometric boundary conditions, Eq. 6.1, the first two Hermite cubics $\eta_1(\xi)$ and $\eta_2(\xi)$ (Fig. 6-a), and the first linear interpolation function $\phi_1(\xi)$ (Fig. 7-a), are not required. Consequently, $N_{xi} = 5$ and $N_{yi} = N_{zi} = 8$, for $i = 1, 2$. Considering also that the first and third joints have three degrees of freedom, and the second joint one degree of freedom, we conclude that the structural model has $N_q = N_w = 49$ degrees of freedom.

The feedback gains for joint control were chosen to be

$$\alpha_i = 1000, \quad \beta_i = 2000, \qquad i = 1, 2, \ldots, 7, \quad (6.19)$$

with four actuator pairs on each of the two beams to control the elastic motion. That is, $N_{cyi} = N_{czi} = 2$, with locations $x_{cyi}^1 = x_{czi}^1 = .15$ m, $x_{cyi}^2 = x_{czi}^2 = 3.75$ m, separation distances $h_{cyi} = h_{czi} = .1$ m, $i = 1, 2$ and feedback gains given by

$$\alpha_{cyi}^k = \alpha_{czi}^k = 15,000, \quad \beta_{cyi}^k = \beta_{czi}^k = 22,500, \quad k = 1, 2, 3, 4, \quad i = 1, 2. \quad (6.20)$$

The desired joint angle trajectories were chosen to be

$$\theta_i^*(t) = \frac{\theta_{Fi}}{2\pi}\left[\frac{2\pi t}{t_F} - \sin\left(\frac{2\pi t}{t_F}\right)\right], \quad 0 \le t \le t_F, \quad (6.21)$$

in which $\theta_{Fi} = \theta_i^*(t_F)$. Note that this function satisfies the conditions $\theta_i^*(0) = 0$, $\theta_i^*(t_F) = \theta_{Fi}$, and is rather smooth, in the sense that $\dot{\theta}_i^*(0) = \ddot{\theta}_i^*(0) = \dot{\theta}_i^*(t_F) = \ddot{\theta}_i^*(t_F) = 0$. The total simulation time was 30 s, with $t_F = 20$ s and

$$\theta_{F1} = -1 \text{ rad}, \quad \theta_{F2} = -.5 \text{ rad}, \quad \theta_{F3} = .5 \text{ rad}, \quad \theta_{F4} = 1.5 \text{ rad},$$
$$\theta_{F5} = 1.25 \text{ rad}, \quad \theta_{F6} = .5 \text{ rad}, \quad \theta_{F7} = -.75 \text{ rad}. \quad (6.22)$$

Plots of the functions $\theta_i^*(t)$ are shown with open circles in Fig. 9.

The system of ordinary differential equations, Eqs. 4.60-a and 4.60-b, were integrated using the IMSL subroutine DIVPAG, which uses the BDF method, also known as Gear's stiff method, with the results plotted in Figs. 9, 10, 11 and 12. Three different simulations were performed. In the first case, the structure is assumed to be rigid, with all feedback gains set to zero. The resulting angular and flexible displacements are simply the desired trajectories, and are shown with open circles. In the second case, the flexibility of the beams is included, but with all of the feedback gains still equal to zero. These results are shown with a dotted line. And finally, in the third case, the feedback gains are as indicated earlier, with the results shown with a solid line. Figure 9 shows time histories of the joint angles, Fig. 10 shows time histories of the beam tip angular, axial and lateral displacements, $\psi_{xi}(\ell_i)$, $u_{xi}(\ell_i)$, $u_{yi}(\ell_i)$ and $u_{zi}(\ell_i)$, $i = 1, 2$, the joint control torques are shown in Fig. 11, and the actuator pair control torque time histories are shown in Fig. 12.

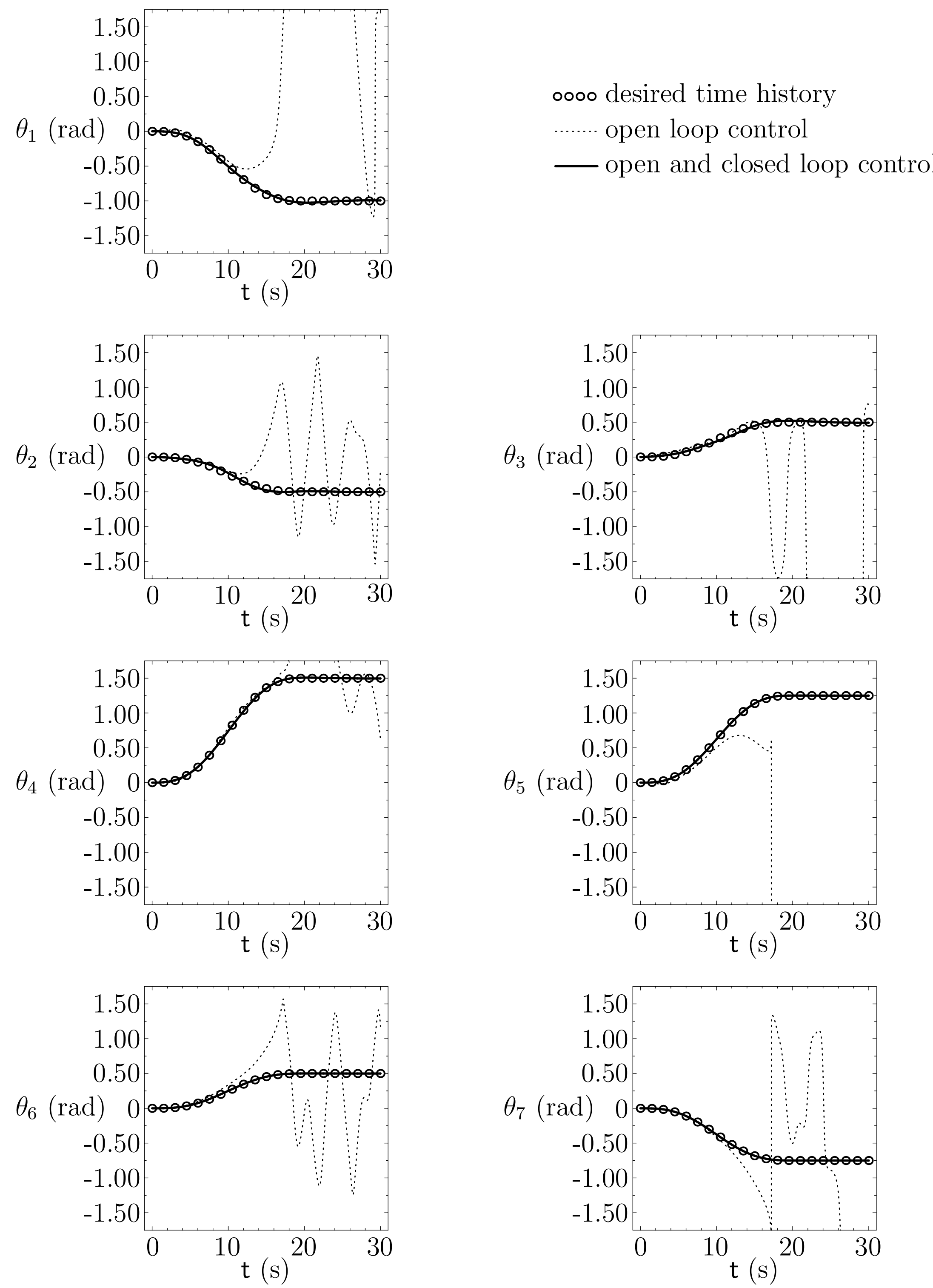

Figure 9: Time Histories of Robot Joint Angles

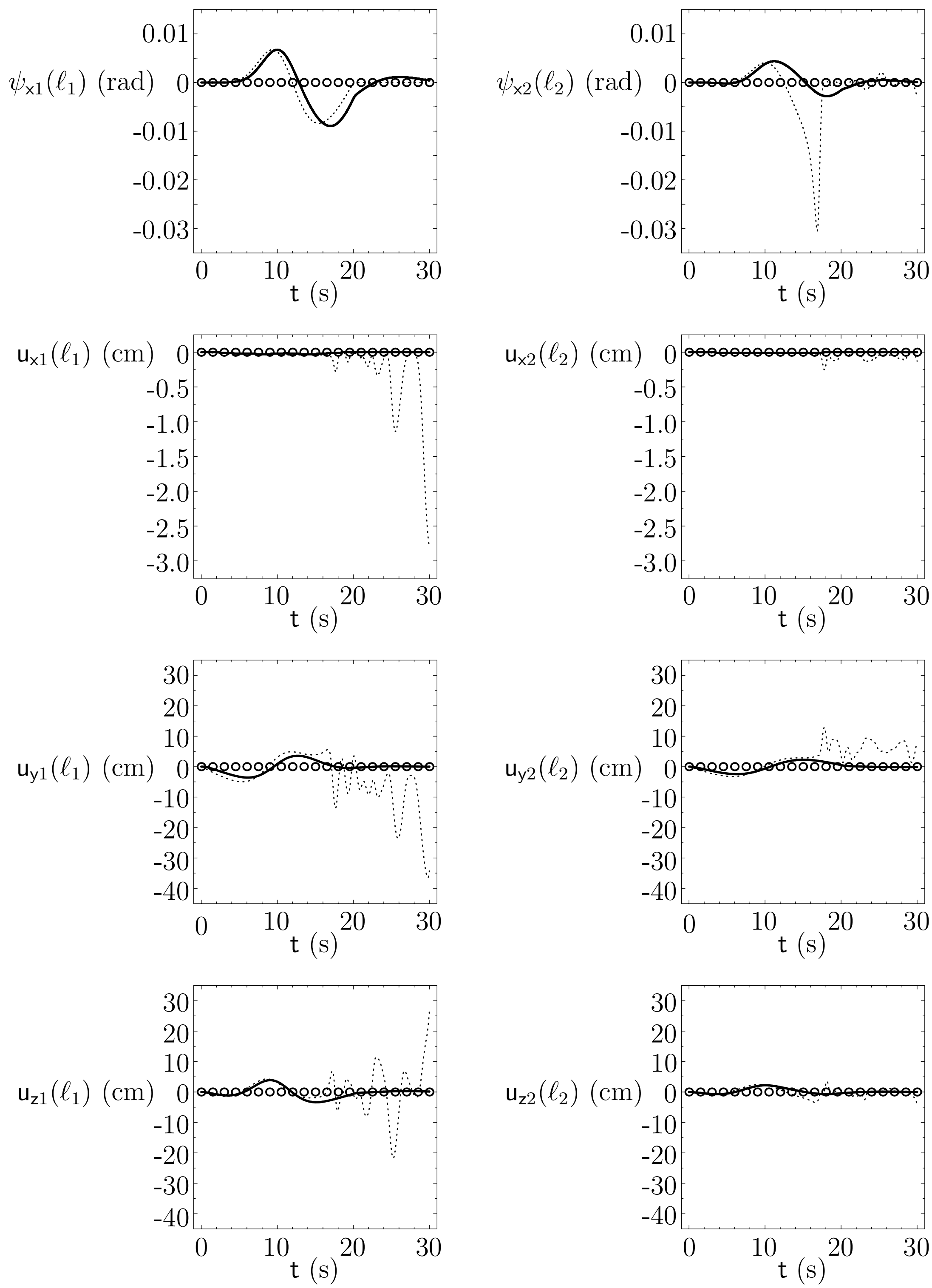

Figure 10: Time Histories of Beam Elastic Motions

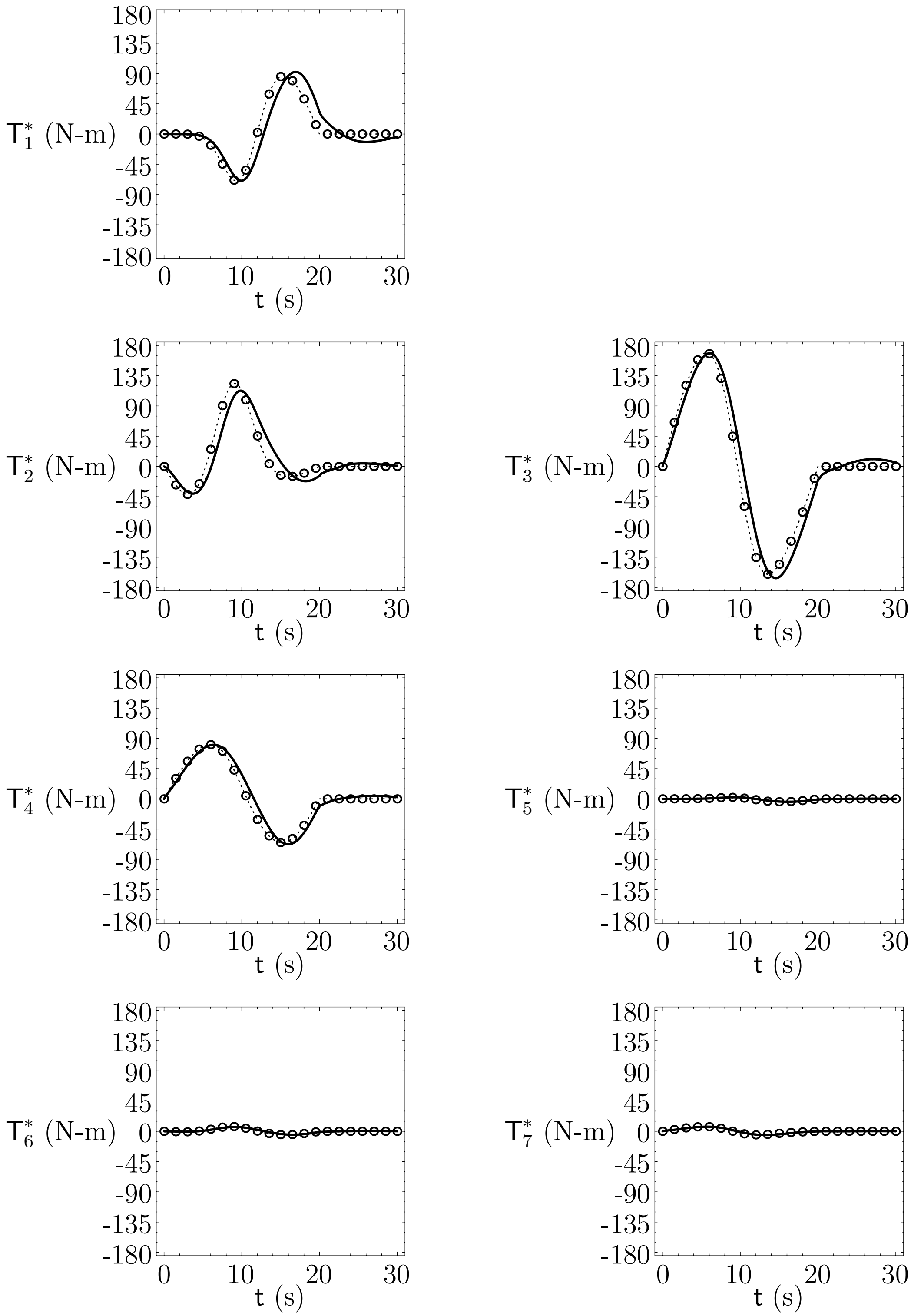

Figure 11: Time Histories of Robot Joint Torques

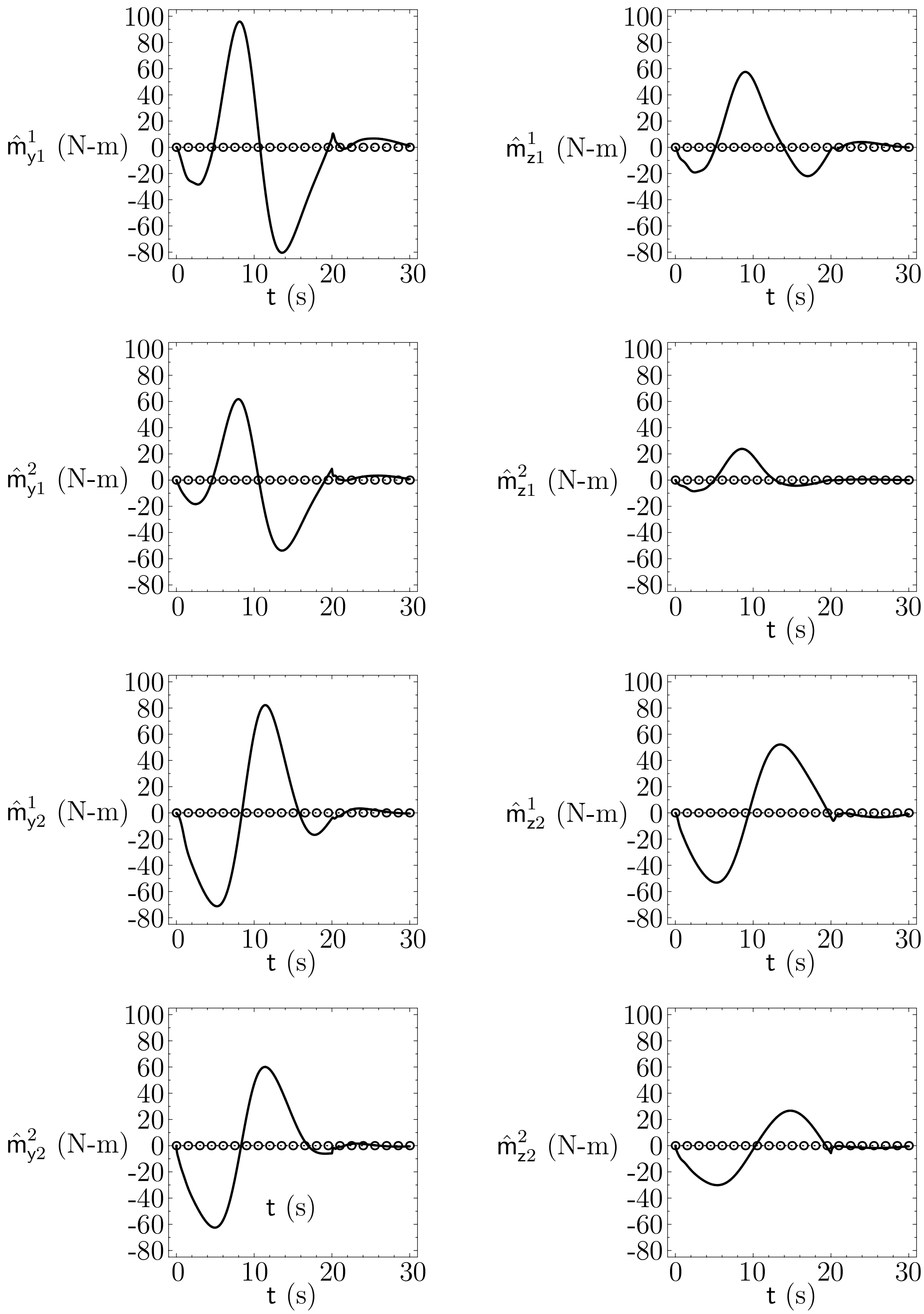

Figure 12: Time Histories of Elastic Torques

Chapter 7 Summary and Conclusions

An efficient technique for generating equations of motion for flexible multibody structures, carried out by means of the Principle of Virtual Work, has been presented. The resulting formulation is quite general, and can be used to model the dynamics of a large class of multibody structures. In particular, open-loop structures comprised of rigid bodies and slender beams connected by any type of revolute joint can be modeled. The structure can have one body fixed in the inertial space, or be freely floating. Furthermore, the joints can be actively controlled, governed by the action of torsional springs and dampers, or torque free. The model also allows for any type of control force or torque to be acting on the surface of the bodies.

Due to the modular nature of the computer program, it would be a simple exercise to extend the setup to include structures with prismatic joints, as well as structures moving in a gravitational field. As far as flexibiltiy models are concerned, the beams are modeled as second-order Rayleigh beams. Not much work would be required, however, to extend the program to include second-order Timoshenko beam models in the class of structures which can be modeled.

The equations of motion are initially presented in hybrid variational form. That is, they include both generalized coordinates and quasi-velocities, which involve the rigid body motions, as well as functions of time and space which model the elastic motions of the flexible bodies in the structure. A form of the hierarchical finite element method, which, for bending, uses the sequence of fixed-fixed Euler-Bernoulli eigenfunctions instead of polynomials for the hierarchical functions, was used to carry out the discretization process. In general, the hierarchical finite element method requires substantially fewer degrees of freedom than the usual, or h-version, of the finite element method. Furthermore, there are also advantages to using eigenfunctions, as opposed to polynomials, as hierarchical functions. For one, the eigenfunctions by themselves are orthogonal, but also, when combined with the Hermite cubics they enjoy a certain amount of additional orthogonality. The end result, of course, is that the mass and stiffness matrices are substantially sparser. Another advantage to using eigenfunctions, as opposed to polynomials, arises from what happens when the number of hierarchical functions is increased. When using polynomials, there is a limit to the number of hierarchical functions which can be used, since increasing this number causes the mass matrix to become more singular, causing the system to become numerically unstable. This phenomenon is postponed consideraby when using eigenfunctions as hierarchical functions. To be more precise, the hierarchical finite element method was applied to a four story framed structure comprised of Euler-Bernoulli beams [46]. Using more than six polynomials as hierarchical functions resulted in numerical instability, whereas when using eigenfunctions, there was no practical limit to the number of hierarchical functions which could be used.

We have also introduced a practical application of the exponential form of a proper orthogonal matrix, which was used in converting general nonlinear slender beam equations to a form involving up through second-order terms in the functions used to measure the displacement of a cross-section relative to body axes. The exponential form of the orthogonal matrix is ideally suited to this operation, since the Taylor series of the exponential function is straightforward and easily truncated.

A numerical example, which involves the motion of a flexible robot arm, was included to demonstrate the effectiveness of the computer program. The structure consists of a rigid base with two flexible arms and a rigid end-effector. However, since the main interest was in the maneuvering of a payload from one location to another, the base was assumed fixed in the inertial space. The joint control torques consist of an open loop plus closed loop torque. The open loop control torques were determined by requiring the structure, assumed rigid, to move in such a way that the joint angles follow specified trajectories. The closed loop portion of the control torque was determined locally. That is, for each of the joint angles, the closed loop control torque associated with that joint angle responded to the error in the desired angular position and angular velocity.

The elastic motion of the beams was controlled with actuators capable of producing torques about the y_i and z_i axes. The actuators are arranged in pairs, with the individual actuators of a given pair separated by a short distance, and producing torques of equal magnitude but opposite sign. The control again is local, with the magnitude of the torque of a given actuator pair responding to the curvature and time rate of change of curvature of the point midway between the two actuators. Four actuator pairs were located on each of the two beams, all situated at the ends of the beams, with two pairs exerting torques about the y_i axis, and two pairs exerting torques about the z_i axis. An examination of the plots of joint angle time histories and elastic displacement time histories shows that the control strategy certainly had the desired effect.

Appendix A Matrix Operations

If r is a 3-by-1 matrix and F is a 3-by-3 matrix, then it is convenient to use the notation

$$\mathsf{r} \;=\; [\mathsf{r}_k] \;=\; \begin{bmatrix} \mathsf{r}_1 \\ \mathsf{r}_2 \\ \mathsf{r}_3 \end{bmatrix} \tag{A.1}$$

and

$$\mathsf{F} \;=\; [\mathsf{F}_{jk}] \;=\; \begin{bmatrix} \mathsf{F}_{11} & \mathsf{F}_{12} & \mathsf{F}_{13} \\ \mathsf{F}_{21} & \mathsf{F}_{22} & \mathsf{F}_{23} \\ \mathsf{F}_{31} & \mathsf{F}_{32} & \mathsf{F}_{33} \end{bmatrix} \tag{A.2}$$

to indicate the components of r and F. When using $[\mathsf{r}_k]$ or $[\mathsf{F}_{jk}]$, it will be understood that the indices j and k range from 1 to 3. In the event that we require a matrix which is neither 3-by-1 nor 3-by-3, the components will be listed explicitly.

The gradient of $\mathsf{r}(\mathsf{x}) = [\mathsf{r}_1(\mathsf{x}) \;\; \mathsf{r}_2(\mathsf{x}) \; \cdots \; \mathsf{r}_m(\mathsf{x})]^T$ with respect to $\mathsf{x} = [\mathsf{x}_1 \;\; \mathsf{x}_2 \; \cdots \; \mathsf{x}_n]^T$ is denoted by

$$\frac{\partial \mathsf{r}}{\partial \mathsf{x}^T} \;=\; \begin{bmatrix} \partial \mathsf{r}_1/\partial \mathsf{x}_1 & \partial \mathsf{r}_1/\partial \mathsf{x}_2 & \dots & \partial \mathsf{r}_1/\partial \mathsf{x}_n \\ \partial \mathsf{r}_2/\partial \mathsf{x}_1 & \partial \mathsf{r}_2/\partial \mathsf{x}_2 & \dots & \partial \mathsf{r}_2/\partial \mathsf{x}_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial \mathsf{r}_m/\partial \mathsf{x}_1 & \partial \mathsf{r}_m/\partial \mathsf{x}_2 & \dots & \partial \mathsf{r}_m/\partial \mathsf{x}_n \end{bmatrix}. \tag{A.3}$$

Consequently, if $\mathsf{r} = \mathsf{B}\mathsf{x}$, where the m-by-n matrix B does not depend on x, then

$$\frac{\partial \mathsf{r}_i}{\partial \mathsf{x}_j} \;=\; \frac{\partial}{\partial \mathsf{x}_j}\left(\sum_{k=1}^{n} \mathsf{B}_{ik}\mathsf{x}_k\right) \;=\; \mathsf{B}_{ij}, \tag{A.4}$$

so that

$$\frac{\partial \mathsf{r}}{\partial \mathsf{x}^T} \;=\; \frac{\partial(\mathsf{B}\mathsf{x})}{\partial \mathsf{x}^T} \;=\; \mathsf{B}. \tag{A.5}$$

Furthermore, if r depends on x (but not necessarily linearly, as above), and x depends on $\mathsf{y} = [\mathsf{y}_1 \;\; \mathsf{y}_2 \;\ldots\; \mathsf{y}_s]^\mathsf{T}$, then

$$\frac{\partial \mathsf{r}_i}{\partial \mathsf{y}_j} = \sum_{k=1}^{n} \frac{\partial \mathsf{r}_i}{\partial \mathsf{x}_k}\frac{\partial \mathsf{x}_k}{\partial \mathsf{y}_j}, \tag{A.6}$$

so that the chain rule takes the form

$$\frac{\partial \mathsf{r}}{\partial \mathsf{y}^\mathsf{T}} = \frac{\partial \mathsf{r}}{\partial \mathsf{x}^\mathsf{T}}\frac{\partial \mathsf{x}}{\partial \mathsf{y}^\mathsf{T}}. \tag{A.7}$$

If $\mathsf{a} = [\mathsf{a}_k]$, then the 3-by-3 skew-symmetric matrix $\mathrm{Tilde}(\mathsf{a}) = \tilde{\mathsf{a}}$ is defined by

$$\mathrm{Tilde}(\mathsf{a}) = \tilde{\mathsf{a}} = \begin{bmatrix} 0 & -\mathsf{a}_3 & \mathsf{a}_2 \\ \mathsf{a}_3 & 0 & -\mathsf{a}_1 \\ -\mathsf{a}_2 & \mathsf{a}_1 & 0 \end{bmatrix} \tag{A.8-a}$$

$$= [(\tilde{\mathsf{a}})_{jk}] = -\epsilon_{jkm}\mathsf{a}_m, \tag{A.8-b}$$

where ϵ_{jkm} is the permutation symbol, i.e., $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$, $\epsilon_{321} = \epsilon_{132} = \epsilon_{213} = -1$, $\epsilon_{ijk} = 0$ if $i * j * k \neq 6$, and we have made use of the Einstein summation convention. If B is any 3-by-3 matrix, then it is a simple exercise to show that

$$(\widetilde{\mathsf{B}\mathsf{a}}) = (\mathrm{Tr}\mathsf{B})\tilde{\mathsf{a}} - \mathsf{B}^\mathsf{T}\tilde{\mathsf{a}} - \tilde{\mathsf{a}}\mathsf{B}, \tag{A.9}$$

where Tr stands for trace. Furthermore, if c is another 3-by-1 matrix, then letting $\mathsf{B} = \tilde{\mathsf{c}}$ in Eq. A.9 implies that

$$(\widetilde{\tilde{\mathsf{c}}\mathsf{a}}) = \tilde{\mathsf{c}}\tilde{\mathsf{a}} - \tilde{\mathsf{a}}\tilde{\mathsf{c}}. \tag{A.10}$$

Multiplying on the right by $\tilde{\mathsf{c}}\mathsf{a}$ results in the left side equaling the 3-by-3 zero matrix, so that

$$\tilde{\mathsf{c}}\tilde{\mathsf{a}}\tilde{\mathsf{c}}\mathsf{a} = \tilde{\mathsf{a}}\tilde{\mathsf{c}}^2\mathsf{a}. \tag{A.11}$$

Another easily verified fact is

$$\tilde{\mathsf{a}}\tilde{\mathsf{c}}\tilde{\mathsf{a}} = -(\mathsf{a}^\mathsf{T}\mathsf{c})\tilde{\mathsf{a}}, \tag{A.12}$$

which can then be used to prove that

$$\tilde{\mathsf{a}}^{2n+1} = (-1)^n(\mathsf{a}^\mathsf{T}\mathsf{a})^n\tilde{\mathsf{a}}, \tag{A.13-a}$$

$$\tilde{\mathsf{a}}^{2n+2} = (-1)^n(\mathsf{a}^\mathsf{T}\mathsf{a})^n\tilde{\mathsf{a}}^2, \quad n = 0, 1, 2, \ldots\,. \tag{A.13-b}$$

The description of the motion of one coordinate system with respect to another is of central importance in the dynamics of multibody systems. To begin with, let orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be associated with a coordinate system with axes $\mathsf{x}_1\mathsf{x}_2\mathsf{x}_3$ and origin A, and orthonormal basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be associated with a coordinate system with axes $\mathsf{y}_1\mathsf{y}_2\mathsf{y}_3$ and origin B. Next, define the time derivative of vector $\mathbf{r}$ with respect to axes $\mathsf{x}_1\mathsf{x}_2\mathsf{x}_3$ by

$$\mathring{\mathbf{r}} = \frac{d}{dt}(\mathbf{r}\cdot\mathbf{e}_1)\mathbf{e}_1 + \frac{d}{dt}(\mathbf{r}\cdot\mathbf{e}_2)\mathbf{e}_2 + \frac{d}{dt}(\mathbf{r}\cdot\mathbf{e}_3)\mathbf{e}_3. \tag{A.14}$$

And then, let $\mathbf{R} = \overrightarrow{AB}$ be the position vector of point B with respect to point A and $\mathbf{\Omega}$ the angular velocity of axes $y_1y_2y_3$ with respect to axes $x_1x_2x_3$.

Note that $\mathbf{\Omega}$ is given by

$$\begin{aligned} \mathbf{\Omega} &= (\mathring{\mathbf{b}}_2 \cdot \mathbf{b}_3)\mathbf{b}_1 + (\mathring{\mathbf{b}}_3 \cdot \mathbf{b}_1)\mathbf{b}_2 + (\mathring{\mathbf{b}}_1 \cdot \mathbf{b}_2)\mathbf{b}_3 && \text{(A.15-a)} \\ &= \tfrac{1}{2}\epsilon_{ijk}(\mathring{\mathbf{b}}_j \cdot \mathbf{b}_k)\mathbf{b}_i, && \text{(A.15-b)} \end{aligned}$$

which endows it with the usual property,

$$\mathring{\mathbf{b}}_k = \mathbf{\Omega} \times \mathbf{b}_k, \qquad k = 1, 2, 3. \tag{A.16}$$

We also introduce the four component matrices

$$\begin{aligned} R &= [R_k] = [\mathbf{R} \cdot \mathbf{e}_k], && \text{(A.17)} \\ P &= [P_{jk}] = [\mathbf{b}_j \cdot \mathbf{e}_k], && \text{(A.18)} \\ V &= [V_k] = [\mathring{\mathbf{R}} \cdot \mathbf{b}_k], && \text{(A.19)} \\ \Omega &= [\Omega_k] = [\mathbf{\Omega} \cdot \mathbf{b}_k], && \text{(A.20)} \end{aligned}$$

where we recognize that R contains the components of position vector $\mathbf{R}$, along axes $x_1x_2x_3$, P is the matrix of direction cosines of axes $y_1y_2y_3$ with respect to axes $x_1x_2x_3$, and V and Ω contain, respectively, the components of velocity vector $\mathring{\mathbf{R}}$ and angular velocity vector $\mathbf{\Omega}$, both along axes $y_1y_2y_3$. Note that this definition of P implies that the 3-by-1 matrix PR contains the components of vector $\mathbf{R}$, along axes $y_1y_2y_3$, and furthermore that

$$\mathbf{e}_k = (\mathbf{e}_k \cdot \mathbf{b}_j)\mathbf{b}_j = P_{jk}\,\mathbf{b}_j. \tag{A.21}$$

Then, making use of the easily verified fact that for any two vectors $\mathbf{a}$ and $\mathbf{b}$, $d(\mathbf{a} \cdot \mathbf{b})/dt = \mathring{\mathbf{a}} \cdot \mathbf{b} + \mathbf{a} \cdot \mathring{\mathbf{b}}$, we see that

$$\begin{aligned} \dot{R} &= [\dot{R}_k] = [d(\mathbf{R} \cdot \mathbf{e}_k)/dt] = [\mathring{\mathbf{R}} \cdot \mathbf{e}_k] = [\mathring{\mathbf{R}} \cdot (P_{jk}\,\mathbf{b}_j)] = [P_{jk}\,V_j] \\ &= P^T V, && \text{(A.22)} \end{aligned}$$

$$\begin{aligned} \dot{P} &= [\dot{P}_{jk}] = [d(\mathbf{b}_j \cdot \mathbf{e}_k)/dt] = [\mathring{\mathbf{b}}_j \cdot \mathbf{e}_k] = [(\mathbf{\Omega} \times \mathbf{b}_j) \cdot \mathbf{e}_k] \\ &= [(\mathbf{b}_j \times \mathbf{e}_k) \cdot \mathbf{\Omega}] = [(\mathbf{b}_j \times P_{nk}\,\mathbf{b}_n) \cdot \mathbf{\Omega}] = [P_{nk}\,\epsilon_{jnm}\mathbf{b}_m \cdot \mathbf{\Omega}] \\ &= [P_{nk}\,\epsilon_{jnm}\Omega_m] = -[(-\epsilon_{jnm}\Omega_m)P_{nk}] = -[(\widetilde{\Omega})_{jn}P_{nk}] \\ &= -\widetilde{\Omega}P. && \text{(A.23)} \end{aligned}$$

Multiply Eq. A.22 on the left by P, and then rearrange Eq. A.23 by taking the transpose of both sides, and then multiplying on the left by P, to get that

$$\begin{aligned} V &= P\dot{R}, && \text{(A.24)} \\ \widetilde{\Omega} &= P\dot{P}^T. && \text{(A.25)} \end{aligned}$$

Two comments are in order. First, if we are given the matrices R and P, we can use Eqs. A.24 and A.25 to define V and Ω, without regard to the physical meaning as components of $\overset{\circ}{\mathbf{R}}$ and $\boldsymbol{\Omega}$. Secondly, we can replace the "dot" in Eqs. A.24 and A.25 with another operator. For example, we can define the quasi-virtual displacements $\delta\mathsf{R}^*$ and $\delta\Theta^*$ by

$$\delta\mathsf{R}^* = \mathsf{P}\delta\mathsf{R}, \tag{A.26}$$

$$\widetilde{\delta\Theta^*} = \mathsf{P}\delta\mathsf{P}^{\mathsf{T}}. \tag{A.27}$$

We will have occasion to use some facts related specifically to matrices of direction cosines. Recall that the matrix of direction cosines P is a proper orthogonal matrix, i.e., $\mathsf{P}^{\mathsf{T}}\mathsf{P} = \mathsf{P}\mathsf{P}^{\mathsf{T}} = \mathrm{I}$ and $\det\mathsf{P} = 1$. Furthermore, it is well known that

$$\mathsf{P} = \mathrm{I} - (\sin\phi)\tilde{\xi} + (1-\cos\phi)\tilde{\xi}^2, \tag{A.28}$$

in which ϕ is a scalar and $\xi = [\,\xi_1\,\xi_2\,\xi_3\,]^{\mathsf{T}}$ satisfies $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$. In fact, ϕ is the angle of rotation about the axis defined by unit vector $\boldsymbol{\xi} = \xi_{\mathsf{i}}\mathbf{e}_{\mathsf{i}} = \xi_{\mathsf{i}}\mathbf{b}_{\mathsf{i}}$. If $\mathsf{a} = [\,\mathsf{a}_1\,\mathsf{a}_2\,\mathsf{a}_3\,]^{\mathsf{T}}$ is any 3-by-1 matrix, then Eqs. A.12 and A.28 can be used to verify the identity

$$(\widetilde{\mathsf{P}\mathsf{a}}) = \mathsf{P}\tilde{\mathsf{a}}\mathsf{P}^{\mathsf{T}}. \tag{A.29}$$

Introducing the Taylor series for $\sin\phi$ and $1-\cos\phi$ into Eq. A.28, we get that

$$\mathsf{P} = \sum_{\mathsf{k}=0}^{\infty} \frac{(-1)^{\mathsf{k}}\tilde{\theta}^{\mathsf{k}}}{\mathsf{k}!} = \exp(-\tilde{\theta}), \tag{A.30}$$

in which the 3-by-1 matrix $\theta = \phi\xi = [\,\phi\xi_1\ \ \phi\xi_2\ \ \phi\xi_3\,]^{\mathsf{T}}$.

Next we assume that the orthogonal matrix P is a function of $\theta = [\,\theta_1\,\theta_2\,\theta_3\,]^{\mathsf{T}}$. Then, since $\mathsf{P}\mathsf{P}^{\mathsf{T}} = \mathrm{I}$, we have that $\mathsf{P}(\partial\mathsf{P}^{\mathsf{T}}/\partial\theta_{\mathsf{k}}) + (\partial\mathsf{P}/\partial\theta_{\mathsf{k}})\mathsf{P}^{\mathsf{T}} = 0$, $\mathsf{k} = 1, 2, 3$. This implies that $\mathsf{P}(\partial\mathsf{P}^{\mathsf{T}}/\partial\theta_{\mathsf{k}})$ is a 3-by-3 skew-symmetric matrix, which allows the introduction of a 3-by-1 matrix $\beta_{\mathsf{k}} = [\,\beta_{1\mathsf{k}}\,\beta_{2\mathsf{k}}\,\beta_{3\mathsf{k}}\,]^{\mathsf{T}}$ defined by

$$\tilde{\beta}_{\mathsf{k}} = \mathsf{P}\frac{\partial\mathsf{P}^{\mathsf{T}}}{\partial\theta_{\mathsf{k}}}, \quad \mathsf{k} = 1, 2, 3. \tag{A.31}$$

Furthermore, the chain rule implies that

$$\begin{aligned} \mathsf{P}\dot{\mathsf{P}}^{\mathsf{T}} &= \sum_{\mathsf{k}=1}^{3} \mathsf{P}\frac{\partial\mathsf{P}^{\mathsf{T}}}{\partial\theta_{\mathsf{k}}}\dot{\theta}_{\mathsf{k}} = \sum_{\mathsf{k}=1}^{3} \tilde{\beta}_{\mathsf{k}}\dot{\theta}_{\mathsf{k}} = \mathrm{Tilde}\left(\sum_{\mathsf{k}=1}^{3} \beta_{\mathsf{k}}\dot{\theta}_{\mathsf{k}}\right) \\ &= (\widetilde{\mathsf{D}\dot{\theta}}), \end{aligned} \tag{A.32}$$

in which

$$\mathsf{D} = \left[\ \beta_1 \ \vdots\ \beta_2 \ \vdots\ \beta_3\ \right] = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix}. \tag{A.33}$$

Applying this procedure to the characterization of P given in Eq. A.28 and taking into account that $\xi = [\theta_1/\phi \;\; \theta_2/\phi \;\; \theta_3/\phi]^{\mathsf{T}}$ where $\phi = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^3}$, it can be shown that

$$\beta_{\mathsf{k}} \;=\; \frac{\sin\phi}{\phi}\mathsf{e}_{\mathsf{k}} + \theta_{\mathsf{k}}\frac{\phi - \sin\phi}{\phi^3}\theta + \frac{1-\cos\phi}{\phi^2}\tilde{\mathsf{e}}_{\mathsf{k}}\theta, \tag{A.34}$$

which, when substituted into Eq. A.33 can be shown to yield the formula

$$\mathsf{D} \;=\; \mathrm{I} - \frac{1-\cos\phi}{\phi^2}\tilde{\theta} + \frac{\phi - \sin\phi}{\phi^3}\tilde{\theta}^2. \tag{A.35}$$

This closed-form expression for D is akin to that given for P in Eq. A.28. Introducing Taylor series expansions for $(1-\cos\phi)/\phi^2$ and $(\phi - \sin\phi)/\phi^3$ into Eq. A.35 and making use also of Eqs. A.13-a and A.13-b, we get that the Taylor series expansion for D is given by

$$\mathsf{D} \;=\; \sum_{\mathsf{k}=0}^{\infty}\frac{(-1)^{\mathsf{k}}\tilde{\theta}^{\mathsf{k}}}{(k+1)!}, \tag{A.36}$$

which, except for the $\mathsf{k}+1$ in the denominator, is identical to the Taylor series formula for P. It is also possible to verify the identity of Eq. A.32 directly from the Taylor series formulas for P (Eq. A.30) and D (Eq. A.36), although the proof is not particularly straightforward.

There are two facts concerning the trace of a square matrix which will be required. If A and B are both n-by-n matrices, then it is easy to verify that

$$\mathrm{Tr}[\mathsf{AB}] \;=\; \mathrm{Tr}[\mathsf{BA}], \tag{A.37}$$

and if S and W are n-by-n matrices, with S symmetric and W skew-symmetric, then

$$\mathrm{Tr}[\mathsf{SW}] \;=\; 0. \tag{A.38}$$

Finally, we will also need a simple formula related to the component matrix of the gradient of a vector valued function $\mathbf{z}(\mathbf{r})$. Let $\mathsf{z} = [\mathsf{z}_{\mathsf{k}}]$ be the component matrix of $\mathbf{z}$, along axes $\mathsf{x}_1\mathsf{x}_2\mathsf{x}_3$, and $\mathsf{r} = [\mathsf{r}_{\mathsf{k}}]$ the component matrix of $\mathbf{r}$, along axes $\mathsf{y}_1\mathsf{y}_2\mathsf{y}_3$. Suppose we are given $\mathsf{z} = [\mathsf{z}_{\mathsf{k}}]$ as a function of $\mathsf{r} = [\mathsf{r}_{\mathsf{k}}]$, so that it is straightforward to compute $\partial\mathsf{z}/\partial\mathsf{r}^{\mathsf{T}}$, but require the component matrix F of the gradient $\nabla\mathbf{z}$ with respect to axes $\mathsf{x}_1\mathsf{x}_2\mathsf{x}_3$. Recall that by definition [16], $\mathsf{F} = \partial\mathsf{z}/\partial\bar{\mathsf{r}}^{\mathsf{T}}$, in which $\bar{\mathsf{r}} = \mathsf{P}^{\mathsf{T}}\mathsf{r}$ is the component matrix of $\mathbf{r}$, along axes $\mathsf{x}_1\mathsf{x}_2\mathsf{x}_3$ and P is the matrix of direction cosines of axes $\mathsf{y}_1\mathsf{y}_2\mathsf{y}_3$ with respect to axes $\mathsf{x}_1\mathsf{x}_2\mathsf{x}_3$. This implies that $\mathsf{r} = \mathsf{P}\bar{\mathsf{r}}$, and consequently,

$$\mathsf{F} \;=\; \frac{\partial\mathsf{z}}{\partial\bar{\mathsf{r}}^{\mathsf{T}}} \;=\; \frac{\partial\mathsf{z}}{\partial\mathsf{r}^{\mathsf{T}}}\frac{\partial\mathsf{r}}{\partial\bar{\mathsf{r}}^{\mathsf{T}}} \;=\; \frac{\partial\mathsf{z}}{\partial\mathsf{r}^{\mathsf{T}}}\frac{\partial(\mathsf{P}\bar{\mathsf{r}})}{\partial\bar{\mathsf{r}}^{\mathsf{T}}} \;=\; \frac{\partial\mathsf{z}}{\partial\mathsf{r}^{\mathsf{T}}}\mathsf{P}, \tag{A.39}$$

which is the desired formula.

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