Dynamics

(Fall Semester 2021)

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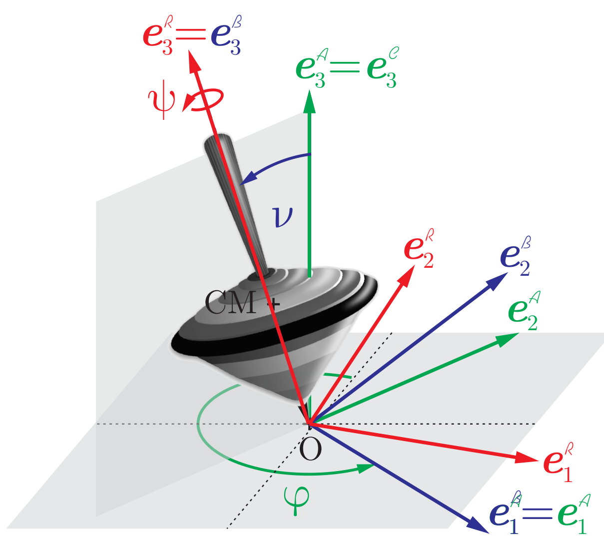


These lecture notes cover the concepts and most examples discussed during lectures. They are not a complete textbook, but they do provide a thorough introduction to all course topics as well as some extra background reading, extended explanations, and various examples beyond what can be discussed in class.

Especially in times of covid-19, you are strongly recommended to visit the (virtual or in-person) lectures and to take your own notes, while studying these lecture notes alongside for support and further reading. The weekly exercise sessions will help deepen the acquired knowledge and practice problem solving techniques.

The lecture topics covered during each week are announced at the beginning of the semester in the course syllabus, so you are welcome read up on the topics ahead of time.

Contents

0 Preface

These course notes summarize the contents of the course called DYNAMICS, which constitutes the third part of the series of fundamental mechanics courses taught within ETH Zürich's Mechanical and Civil Engineering programs. Following MECHANICS 1 (which introduced kinematics and statics) and MECHANICS 2 (which focused on concepts of stresses and strains and the static deformation of linear elastic bodies), this course studies the *time-dependent behavior* of mechanical systems – from individual particles to systems of particles to rigid and, ultimately, deformable bodies. We will discuss how to describe and understand the time-dependent motion of systems (generally referred to as the **kinematics**), followed by the relation between a system's motion and those externally applied forces and torques that cause the motion (generally referred to as the **kinetics**). Besides presenting the underlying theory, we will study numerous examples to highlight the usefulness of the derived mathematical relations and assess their practical relevance. It is assumed that participants have a proper understanding of the contents of MECHANICS 1 and 2 (so we will keep repetitions to a minimum) as well as of ANALYSIS 1 and 2 (so we can exploit those mathematical concepts in our derivations). Where necessary, we will introduce new mathematical and physical principles along the way to the extent necessary.

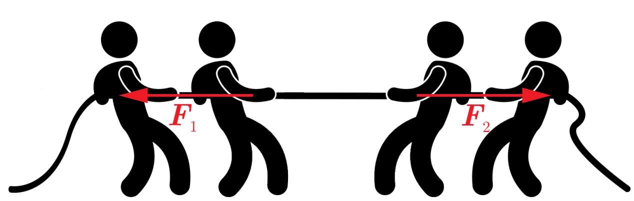
At the end of each section, you will find a *Summary of Key Relations*: a concise table with the most important equations derived and required to solve the exercise, homework and exam problems. The sum of all those tables will also serve as the *formula collection* you will be allowed to consult during the final exam of this course (as the only available supporting material). It is therefore advised to familiarize oneself with those summary tables during the course (e.g., by using those to solve the exercise problems), so that they form a familiar reference by the time of the exam. In case this is your first encounter of mechanics in *English*, you may find the Glossary (starting on page 235) at the end of these course notes helpful, which include German translations of the most important terminology as well as a brief description of those terms. Most terms highlighted in blue throughout the notes can be found in the glossary (just click on those in the PDF version).

Before we dive into dynamics, let us recap on just about one page and in a cartoon fashion the key concepts of mechanics to be expanded here:

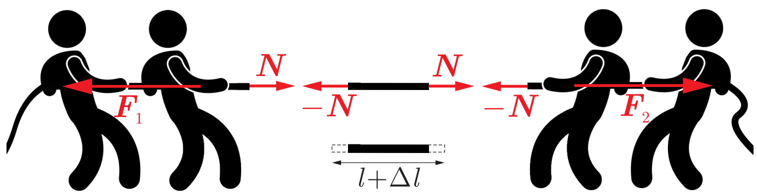
Mechanics 1 concentrated on static (i.e., non-moving) systems and established that the resultant force $\boldsymbol{R}$ and the resultant torque $\boldsymbol{M}$ acting on a body (or a system of bodies) vanish in **equilibrium**:

$$\boldsymbol{R} = \sum_{i=1}^{n} \boldsymbol{F}_i = \boldsymbol{0}, \qquad \boldsymbol{M} = \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_i = \boldsymbol{0}.$$

For the below example of "*Tauziehen*", we must have $\boldsymbol{F}_1 = -\boldsymbol{F}_2$ for the teams to be in equilibrium.



Mechanics 2 went a step further and showed that inner forces are responsible for deformation, and constitutive relations were discussed which link the deformation of a body (described by strains) to the causes of deformation (described by stresses). Throughout, inner and external forces were assumed to be in static equilibrium, so that the above relations of mechanical equilibrium (vanishing resultant forces and torques) still applied.



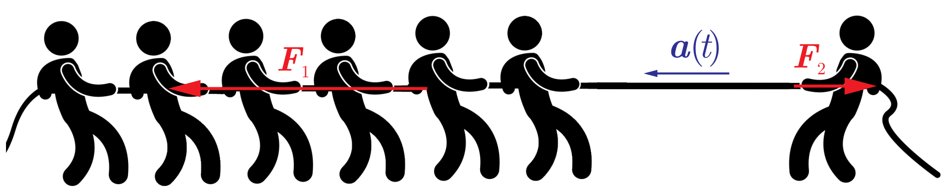
Finally, this course in DYNAMICS (which is essentially *Mechanics 3*) will address scenarios in which the resultant forces and torques are no longer zero:

$$\boldsymbol{R} = \sum_{i=1}^{n} \boldsymbol{F}_i \neq \boldsymbol{0}, \qquad \boldsymbol{M} = \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_i \neq \boldsymbol{0}.$$

In this case, the system is *not* in static equilibrium anymore but instead will respond to the applied forces and torques with motion, governed by – among others – Newton's famous second law,

$$\boldsymbol{F} = m\boldsymbol{a}.$$

Our goal will be to extend the concepts from MECHANICS 1 and 2 to bodies and systems in motion. To start simple, we will first discuss the dynamics of *particles* (i.e., bodies of negligible small size), which we gradually extend to systems of particles, which then naturally leads to rigid and finally deformable bodies.



I apologize in advance for any *typos* that may have found their way into these lecture notes. This is a truly evolving set of notes that was initiated in the fall semester of 2018 and has been extended and improved ever since. Though I made a great effort to ensure these notes are free of essential typos, I cannot rule out that some have remained. If you spot any mistakes, feel free to send me a highlighted PDF at the end of the semester, so I can make sure those typos are corrected for future years. I would like to thank Prof. George Haller and his team for their course slides, some of which served as the basis for these lecture notes. I am also grateful to Dr. Paolo Tiso for many helpful discussions and to all those students who have pointed out typos.

I hope you will find the course interesting and these notes supportive while studying DYNAMICS.

Dennis M. Kochmann
Zürich, November 2021

1 Single-Particle Dynamics

We begin simple – by discussing the mechanics of a single **particle**, which you may be familiar with from physics courses. We will review and explain the key concepts for our needs, as this will serve as the basis for our later discussion of the mechanics of a **rigid** or **deformable body**. We also use this introductory chapter to lay out our notation and terminology.

A **particle** is an idealized view of a (small) object whose size and shape have a negligibly influence on its motion. Of course, no object is negligibly small in reality. However, when the object's shape and size are such that they do not significantly affect the object's motion, then we may safely assume that the total mass m of the object is concentrated in a single point (without a particular shape or size), and that its motion can be described as a translation through space without considering rotations of the object about any of its axes.

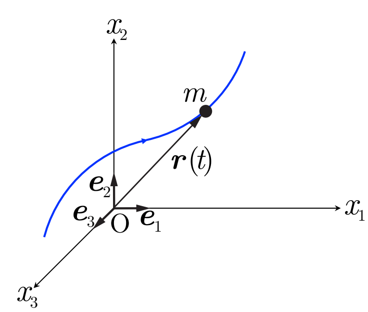
As an example, consider the flight of a golf ball over a long distance. To a good approximation, the golf ball is a point moving through 3D space and – unless one wants to study the intricate fluid mechanics around the ball's surface – the exact shape and size of the ball is of little relevance. Its flight can be described by its time-dependent position only. By contrast, consider throwing a book up in the air. Depending on whether or not you give the book an initial spin (and about which axis you spin it), the book's tumbling motion can change dramatically. This is a scenario in which the book cannot be viewed as a particle of irrelevant size and shape, but its rotation can significantly affect its motion, and its size and shape influence how it tumbles.

A key consequence of the above assumptions is that *the motion of a particle can be described uniquely by translational degrees of freedom only*, and we can safely neglect any rotational degrees of freedom. In the following, we will focus on a single particle and study its motion through space (which we call its **kinematics**), and we will discuss how its motion depends on the forces acting on the particle (which we call the **kinetics**).

1.1 Kinematics

1.1.1 Kinematics in a Cartesian reference frame

We describe the positions of particles in space within a **fixed Cartesian reference frame** $\mathcal{C}$ in 3D, defined by an **orthonormal basis** of vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and a fixed origin O. Consequently, any position in space can uniquely be described by the coordinates $\{x_1, x_2, x_3\}$. We denote the **position** of a particle by $\boldsymbol{r} \in \mathbb{R}^3$, writing



$$\boldsymbol{r} = \sum_{i=1}^{3} x_i \boldsymbol{e}_i = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 + x_3 \boldsymbol{e}_3. \tag{1.1}$$

Here and in the following, we use **bold** symbols to denote vector (and later matrix and tensor) quantities, while scalar variables are never bold. In the above 3D case, x_1, x_2, and x_3 are the three independent **degrees of freedom** (**DOF**s) that describe the particle's position. In general:

$$\textbf{A particle in } d \textbf{ dimensions has } d \textbf{ degrees of freedom.} \tag{1.2}$$

As the position of the particle changes with **time** t, we are interested in the trajectory $\boldsymbol{r}(t)$, which assigns to each time t a unique position $\boldsymbol{r} \in \mathbb{R}^d$. The motion of a single particle is hence described by its **position** $\boldsymbol{r}(t)$, from which we derive its **velocity** and **acceleration** as, respectively,[1]

$$\boxed{\boldsymbol{v}(t) = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}(t)} = \dot{\boldsymbol{r}}(t) \qquad \Rightarrow \qquad \boxed{\boldsymbol{a}(t) = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}(t) = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}(t)} = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{r}}(t) \tag{1.3}$$

Dots always denote derivatives with respect to time. The **speed** of particle is $v(t) = |\boldsymbol{v}(t)|$, and likewise the scalar acceleration is $a(t) = |\boldsymbol{a}(t)|$.

In most of our discussion throughout this course, we use a fixed Cartesian reference frame. We may hence exploit the fact that the basis $\mathcal{C} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is **time-invariant** (i.e., $\dot{\boldsymbol{e}}_i = \boldsymbol{0}$). Such a *fixed* reference frame is called an **inertial frame**, and it allows us to write

$$\boldsymbol{r}(t) = \sum_{i=1}^{d} x_i(t)\boldsymbol{e}_i \qquad \Rightarrow \qquad \boldsymbol{v}(t) = \sum_{i=1}^{d} \dot{x}_i(t)\boldsymbol{e}_i \qquad \Rightarrow \qquad \boldsymbol{a}(t) = \sum_{i=1}^{d} \ddot{x}_i(t)\boldsymbol{e}_i \tag{1.4}$$

It is important to keep in mind that this only holds as long as $\boldsymbol{e}_i(t) = \boldsymbol{e}_i = \text{const.}$, since otherwise we would require additional terms in the above time derivatives. Since the basis vectors form an **orthonormal basis**, the ith component of vector $\boldsymbol{r}$ is

$$x_i = \boldsymbol{r} \cdot \boldsymbol{e}_i \qquad \text{for } i = 1, 2, 3. \tag{1.5}$$

For a particular basis $\mathcal{C}$, the above vectors can also be expressed in their **component form** as

$$[\boldsymbol{r}(t)]_{\mathcal{C}} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \quad \Rightarrow \quad [\boldsymbol{v}(t)]_{\mathcal{C}} = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} \quad \Rightarrow \quad [\boldsymbol{a}(t)]_{\mathcal{C}} = \begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{pmatrix}. \tag{1.6}$$

As a notation convention, we write

$$[\boldsymbol{r}]_{\mathcal{C}} \tag{1.7}$$

to denote the components of a vector $\boldsymbol{r}$ in the Cartesian frame $\mathcal{C}$. Since for the most part, we only work with a single – fixed Cartesian – frame of reference, we may safely drop the subscript $_{\mathcal{C}}$ for convenience and simply write $[\boldsymbol{r}]$, $[\boldsymbol{v}]$, etc., whenever only a single, unique inertial coordinate system is being used. This differentiation between frames will become important later in the course when discussing moving reference frames.

Having established the above relations between position, velocity and acceleration, we are in place to describe the **motion of a particle** through space. In most problems only partial information is available (e.g., the position $\boldsymbol{r}(t)$ is known but the velocity and acceleration are not, or only the acceleration is known as a function of velocity, i.e., $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{v})$ or as a function of position, i.e., $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x})$). In such cases, the mathematical relations derived in Examples 1.1 and 1.2 below are helpful in finding the yet unknown kinematic quantities.

[1] All boxed equations in these lecture notes become part or the formula collection (see, e.g., Section 1.3).

Example 1.1. Finding a particle trajectory from a time-dependent acceleration

If the acceleration $\boldsymbol{a}(t)$ of a particle is a known function of time (starting from an initial time t_0), then integration with respect to time gives

$$\boldsymbol{a}(t) = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \quad \Leftrightarrow \quad \boldsymbol{a}(t)\,\mathrm{d}t = \mathrm{d}\boldsymbol{v} \quad \Rightarrow \quad \int_{t_0}^{t} \boldsymbol{a}(t)\,\mathrm{d}t = \int_{\boldsymbol{v}(t_0)}^{\boldsymbol{v}(t)} \mathrm{d}\boldsymbol{v}, \tag{1.8}$$

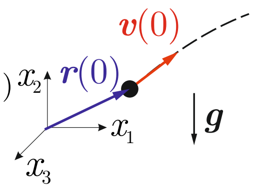
so that, with $\boldsymbol{v}(t_0)$ being the initial velocity at time t_0,

$$\boldsymbol{v}(t) = \int_{t_0}^{t} \boldsymbol{a}(t)\,\mathrm{d}t + \boldsymbol{v}(t_0). \tag{1.9}$$

Integrating once again with respect to time (and starting from the initial position $\boldsymbol{r}(t_0)$), we arrive at

$$\boldsymbol{v}(t) = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \quad \Leftrightarrow \quad \boldsymbol{v}(t)\,\mathrm{d}t = \mathrm{d}\boldsymbol{r} \quad \Rightarrow \quad \boldsymbol{r}(t) = \int_{t_0}^{t} \boldsymbol{v}(t)\,\mathrm{d}t + \boldsymbol{r}(t_0). \tag{1.10}$$

As an example, consider **gravity** g accelerating a particle in the negative x_2-direction. Releasing a particle at $t = 0$ from the initial position $\boldsymbol{r}(0)$ and with initial velocity $\boldsymbol{v}(0)$ gives



$$\boldsymbol{a}(t) = -g\boldsymbol{e}_2 = \text{const.} \quad \Rightarrow \quad \boldsymbol{v}(t) = -gt\boldsymbol{e}_2 + \boldsymbol{v}(0) \tag{1.11}$$

and

$$\boldsymbol{r}(t) = -\frac{gt^2}{2}\boldsymbol{e}_2 + \boldsymbol{v}(0)t + \boldsymbol{r}(0). \tag{1.12}$$

Notice that, because the acceleration acts only in the x_2-direction, the velocity in the perpendicular x_1- and x_3-directions remains constant throughout the flight (and hence remains the same as in the initial instance).

Example 1.2. Finding a particle trajectory from a velocity- or position-dependent acceleration (or from a position-dependent velocity)

In case of **drag** acting on a particle, the particle experiences a negative acceleration. For example, viscous drag due to a surrounding medium at low speeds results in a negative acceleration linearly proportional[2] to the particle velocity, i.e., $\boldsymbol{a} = -k\boldsymbol{v}$ (drag counteracts the particle motion, and the resistance grows linearly with the particle speed through a viscosity constant $k > 0$). In such a case, we know $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{v})$, i.e., the acceleration as a function of velocity.

[2] At low speeds (to be correct, at low Reynolds numbers) the viscous drag on a particle due to a surrounding medium is typically linearly proportional to the particle speed. This is so-called **Stokes' drag**, in 1D $a = -kv$. At higher Reynolds number, **Rayleigh drag** results in a deceleration that is quadratically proportional to the particle speed, in 1D $a = -kv^2$. We here limit ourselves to the former case of linear drag, though the latter results in a similar kinematic problem that can be solved in an analogous fashion.

For simplicity, consider the case of 1D motion, i.e., $\boldsymbol{r} = x\boldsymbol{e}$ and $\boldsymbol{v} = v\boldsymbol{e}$, $\boldsymbol{a} = a\boldsymbol{e}$ in a known direction $\boldsymbol{e}$. Here, we know $a = a(v) = -kv$. Separation of variables in this case lets us find the velocity via

$$a(v) = \frac{\mathrm{d}v}{\mathrm{d}t} \qquad \Leftrightarrow \qquad \frac{\mathrm{d}v}{a(v)} = \mathrm{d}t \qquad \Rightarrow \qquad \int_{v(t_0)}^{v(t)} \frac{\mathrm{d}v}{a(v)} = \int_{t_0}^{t} \mathrm{d}t = t - t_0. \tag{1.13}$$

For the specific case of linear viscous drag, i.e., $a(v) = -kv$, we thus obtain

$$-\frac{1}{k}\int_{v(t_0)}^{v(t)} \frac{\mathrm{d}v}{v} = t - t_0 \qquad \Rightarrow \qquad v(t) = v(t_0)\exp\left[-k(t-t_0)\right]. \tag{1.14}$$

The above can be generalized for 3D motion. Writing $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{v}) = -k\boldsymbol{v}$ for each Cartesian component ($i = 1, 2, 3$) becomes $a_i = -kv_i$. Separation of variables in this case lets us find each velocity component via

$$a_i = \frac{\mathrm{d}v_i}{\mathrm{d}t} \qquad \Rightarrow \qquad \int_{v_i(t_0)}^{v_i(t)} \frac{\mathrm{d}v_i}{-kv_i} = t - t_0 \qquad \Rightarrow \qquad v_i(t) = v_i(t_0)\exp\left[-k(t-t_0)\right]. \tag{1.15}$$

A similar strategy can be applied if the velocity is known in the form $v = v(x)$. Again exploiting the definition of v and using separation of variables, we find (in 1D) that

$$v = v(x) = \frac{\mathrm{d}x}{\mathrm{d}t} \qquad \Rightarrow \qquad \int_{x(t_0)}^{x(t)} \frac{\mathrm{d}x}{v(x)} = \int_{t_0}^{t} \mathrm{d}t = t - t_0. \tag{1.16}$$

Alternatively, if the acceleration is known as a function of position, i.e., $a = a(x)$ (e.g., when a rocket launch from earth is considered, where the gravitational acceleration is a function of the rocket's height above ground), then

$$a = a(x) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}v \qquad \Rightarrow \qquad \int_{v(x_0)}^{v(x)} v\,\mathrm{d}v = \int_{x_0}^{x} a(x)\,\mathrm{d}x. \tag{1.17}$$

Integration yields $v = v(x)$, so that (1.16) can again be applied to find $x = x(t)$.

Example 1.3. Particle on a circular trajectory

As a simple closing example, consider a particle attached to a string of fixed length $r > 0$ and rotating in the x_1-x_2-plane around the coordinate origin. The particle position is given by

$$\boldsymbol{r}(t) = r\cos\varphi(t)\boldsymbol{e}_1 + r\sin\varphi(t)\boldsymbol{e}_2 \tag{1.18}$$

with a known time-dependent angle $\varphi(t)$. The particle velocity and speed follow as, respectively,

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = r\left[-\sin\varphi(t)\boldsymbol{e}_1 + \cos\varphi(t)\boldsymbol{e}_2\right]\dot{\varphi}(t) \qquad \Rightarrow \qquad v = |\boldsymbol{v}| = r\dot{\varphi}(t), \tag{1.19}$$

and its acceleration is

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = -r\left[\cos\varphi(t)\boldsymbol{e}_1 + \sin\varphi(t)\boldsymbol{e}_2\right]\dot{\varphi}^2(t) + r\left[-\sin\varphi(t)\boldsymbol{e}_1 + \cos\varphi(t)\boldsymbol{e}_2\right]\ddot{\varphi}(t). \tag{1.20}$$

1.1.2 Constrained motion

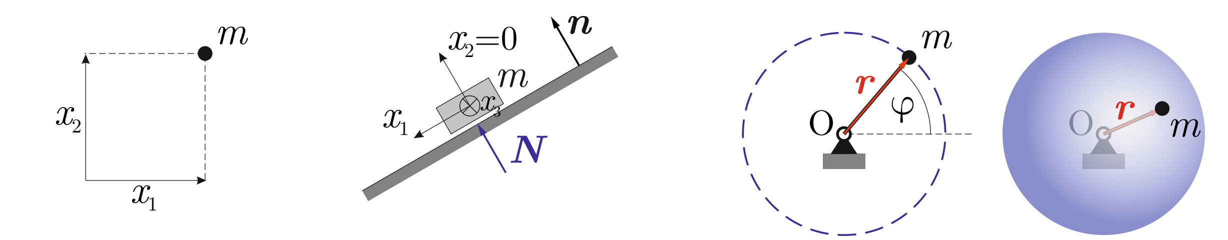
An unconstrained particle moving freely in d dimensions has a total of d degrees of freedom (DOFs), as it can translate in each of the d directions (while rotational DOFs are neglected). In many scenarios, the motion of a particle is kinematically **constrained** (e.g., if the particle is moving on a **rigid** ground, or if it is attached to a rigid rod or rope – in such cases, the particle can move but its trajectory is restricted). Generally speaking:

$$\textbf{In case of } k \textbf{ kinematic constraints, a particle has } (d-k) \textbf{ degrees of freedom.} \tag{1.21}$$

When dealing with constraints, it is beneficial to choose the frame of reference and the associated coordinates wisely to minimize complications and efforts (see Example 1.4 below).

Although we will discuss this point in more detail later, we point out that any kinematic constraint requires a **constraint force** (or **reaction force**) to enforce that the particle stays on the restricted trajectory. For examples, see Example 1.4(c) and (d).

Example 1.4. Unconstrained and constrained motion



(a,b) free motion in 2D (c) particle sliding on the ground (d) particle rotating around point O in 2D/3D

(a) A **particle moving freely** through space is unconstrained and therefore has d translational DOFs in d dimensions: $\boldsymbol{r} = x_1\boldsymbol{e}_1 + \ldots + x_d\boldsymbol{e}_d$ (as pointed out before, particles have a negligible size, so that no rotational DOFs are considered).

(b) A **particle moving in 2D** (which is a common assumption in many of the following examples) can be uniquely described by two translational DOFs only, e.g. $\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$.

(c) A **particle sliding on a flat ground** can only move within the 2D plane of the ground (which is possibly inclined). A single constraint ($k = 1$) in 3D spaces leads to the particle having $(3-1) = 2$ DOFs. If the ground has a unit normal vector $\boldsymbol{n}$ (and the coordinate origin O lies in the ground plane), then we must have $\boldsymbol{r}(t) \cdot \boldsymbol{n} = 0$. (This scalar equation implies that the particle indeed has only two independent DOFs.) In such cases, it is wise to choose a coordinate system that is aligned with the ground, e.g., as shown above with $\boldsymbol{e}_2 || \boldsymbol{n}$ such that $x_2 = 0$ and $\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_3\boldsymbol{e}_3$. As an important consequence (to be discussed later), a **constraint** or **reaction force** $\boldsymbol{N}$ from the ground onto the particle is needed to keep the particle on the ground. Its magnitude is yet unknown, but we know that such a force must exist to prevent the particle from falling into the ground. This will become important when dealing with forces on particles, and we should keep in mind that *any constraint always implies the existence of (possibly hidden) reaction forces.*

(d) A **particle moving around a fixed point O on a rigid link** is constrained in its motion, since the distance to point O, $|\boldsymbol{r}(t) - \boldsymbol{r}_\mathrm{O}|$, must remain constant at all times. In practice, this is realized by a rigid bar or rope which imposes the kinematic constraint ($k = 1$). To keep the particle on its trajectory, a **constraint** or **reaction force** is needed, which is the force in the rigid link.

In 3D, the particle has $(3 - 1) = 2$ DOFs, and it is moving on a sphere of constant radius $|\boldsymbol{r}(t) - \boldsymbol{r}_\mathrm{O}|$. The particle's position can be described, e.g., by two spherical angles (φ, θ).

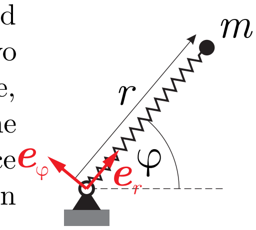
In 2D, the particle has $(2 - 1) = 1$ DOF, so the particle's motion is described by only a single independent DOF. It is moving on a circle of constant radius $|\boldsymbol{r}(t) - \boldsymbol{r}_\mathrm{O}|$, described, e.g., by the angle φ with the x_1-axis (as in Example 1.2).

If the rigid link was replaced by an **elastic** spring, the particle could move in all three directions. The spring will generate a force, if it is stretched or compressed, but it does not restrict the motion of the particle kinematically. The particular arrangement may still make it convenient to use polar coordinates (r, φ) in 2D, or spherical coordinates (r, φ, θ) in 3D, to describe the particle's position in space; yet, this is a choice of convenience and does not imply a kinematic constraint.

1.1.3 Kinematics in polar coordinates

In many problems, it will be convenient to not use Cartesian coordinates to describe the motion of a particle. For example, for a particle rotating around a fixed pole, it may be more convenient to use polar coordinates in 2D or spherical coordinates in 3D.

As an example, consider the rotation of a particle on a spring around a fixed hinge in 2D, as shown on the right. Rather than using $\{x_1(t), x_2(t)\}$ as the two independent DOFs to describe the particle position, we can alternatively use, e.g., **polar coordinates** $\{r(t), \varphi(t)\}$, where r denotes the distance from the hinge and φ the angle with respect to the a fixed axis. After all, it is our choice to introduce d reasonable, independent DOFs to describe the particle motion in d dimensions.



The polar description in terms of $\{r(t), \varphi(t)\}$ comes with an orthonormal basis $\{\boldsymbol{e}_r(t), \boldsymbol{e}_\varphi(t)\}$, which – unlike in a Cartesian frame of reference – is *time-dependent*. Such a moving frame is called a **non-inertial frame**. We can still use the kinematic relations (1.3) between position, velocity and acceleration vectors, but we must be careful when formulating those quantities in component form, since we no longer have $\dot{\boldsymbol{e}}_i = \boldsymbol{0}$. Let us discuss the example of kinematics in polar coordinates in detail in the following.

We describe the position of a particle in 2D by polar coordinates $r(t)$ (the distance from the origin) and $\varphi(t)$ (the angle measured counter-clockwise from the x_1-axis), such that

$$\boldsymbol{r}(t) = r(t)\cos\varphi(t)\boldsymbol{e}_1 + r(t)\sin\varphi(t)\boldsymbol{e}_2. \tag{1.22}$$

Such a description is convenient, e.g., when describing the motion of a particle attached a (massless) stretchable spring whose length is $r(t)$ and whose orientation is $\varphi(t)$ (as shown on the right).

Noting that both r and φ depend on time, the velocity of the particle is

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = (\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi)\boldsymbol{e}_1 + (\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi)\boldsymbol{e}_2, \tag{1.23}$$

where we dropped the explicit dependence of r and φ on t for conciseness.

Similarly, the acceleration follows as

$$\begin{aligned}\boldsymbol{a} = &\left(\ddot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi - r\ddot{\varphi}\sin\varphi\right)\boldsymbol{e}_1 \\ &+ \left(\ddot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos\varphi - r\dot{\varphi}^2\sin\varphi + r\ddot{\varphi}\cos\varphi\right)\boldsymbol{e}_2.\end{aligned} \tag{1.24}$$

Recall that all those vectors were expressed based on a Cartesian reference frame $\mathcal{C}$, so that in that frame we also write the vector components as

$$[\boldsymbol{r}]_{\mathcal{C}} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}, \qquad [\boldsymbol{v}]_{\mathcal{C}} = \begin{pmatrix} \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \end{pmatrix}, \tag{1.25}$$

and

$$[\boldsymbol{a}]_{\mathcal{C}} = \begin{pmatrix} \ddot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi - r\ddot{\varphi}\sin\varphi \\ \ddot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos\varphi - r\dot{\varphi}^2\sin\varphi + r\ddot{\varphi}\cos\varphi \end{pmatrix}. \tag{1.26}$$

Alternatively, we may define a basis that rotates with the particle. To this end we introduce a co-rotating frame $\mathcal{R}$, whose basis vectors point in the radial (r) and perpendicular circular (φ) directions at any time t. Specifically we define

$$\boldsymbol{e}_r(t) = \cos\varphi(t)\boldsymbol{e}_1 + \sin\varphi(t)\boldsymbol{e}_2, \qquad \boldsymbol{e}_\varphi(t) = -\sin\varphi(t)\boldsymbol{e}_1 + \cos\varphi(t)\boldsymbol{e}_2 \tag{1.27}$$

or, in component form with respect to the Cartesian frame $\mathcal{C}$,

$$[\boldsymbol{e}_r]_{\mathcal{C}} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \quad [\boldsymbol{e}_\varphi]_{\mathcal{C}} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}. \tag{1.28}$$

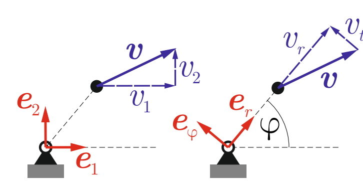
It is important to note that the new basis $\{\boldsymbol{e}_r, \boldsymbol{e}_\varphi\}$ is *time-dependent*. Further, notice that the basis is *orthonormal* since $\boldsymbol{e}_r \cdot \boldsymbol{e}_\varphi = 0$ and $|\boldsymbol{e}_r| = |\boldsymbol{e}_\varphi| = 1$.

We may now express any vector $\boldsymbol{x}$ with respect to this co-rotating basis $\mathcal{R}$:

$$\boldsymbol{x} = x_r\boldsymbol{e}_r + x_\varphi\boldsymbol{e}_\varphi \qquad \text{with} \qquad x_r = \boldsymbol{x}\cdot\boldsymbol{e}_r, \quad \text{and} \quad x_\varphi = \boldsymbol{x}\cdot\boldsymbol{e}_\varphi. \tag{1.29}$$

For example, the velocity vector $\boldsymbol{v}$ can be written as $\boldsymbol{v} = v_r\boldsymbol{e}_r + v_\varphi\boldsymbol{e}_\varphi$, whose components are obtained by a projection of (1.23) onto the basis vectors (1.27):

$$\boldsymbol{e}_r \cdot \boldsymbol{v} = v_r = \dot{r} \qquad \text{and} \qquad \boldsymbol{e}_\varphi \cdot \boldsymbol{v} = v_\varphi = r\dot{\varphi}. \tag{1.30}$$



We here exploit that the co-rotating basis is orthonormal, i.e., $\boldsymbol{e}_r \cdot \boldsymbol{e}_\varphi = 0$ and $\boldsymbol{e}_r \cdot \boldsymbol{e}_r = \boldsymbol{e}_\varphi \cdot \boldsymbol{e}_\varphi = 1$, so that $\boldsymbol{v} \cdot \boldsymbol{e}_r = (v_r \boldsymbol{e}_r + v_\varphi \boldsymbol{e}_\varphi) \cdot \boldsymbol{e}_r = v_r$ and $\boldsymbol{v} \cdot \boldsymbol{e}_\varphi = (v_r \boldsymbol{e}_r + v_\varphi \boldsymbol{e}_\varphi) \cdot \boldsymbol{e}_\varphi = v_\varphi$.

The velocity vector in the rotating polar frame $\mathcal{R}$ hence becomes

$$\boxed{\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\varphi}\boldsymbol{e}_\varphi} \tag{1.31}$$

In words, the velocity vector at each instance of time has a component $\dot{r}$ into the radial outward $\boldsymbol{e}_r$-direction and a component $r\dot{\varphi}$ in the tangential, circular $\boldsymbol{e}_\varphi$-direction.

Analogously, the components of the acceleration vector $\boldsymbol{a}$ in the co-rotating $\mathcal{R}$-frame are

$$\boldsymbol{e}_r \cdot \boldsymbol{a} = a_r = \ddot{r} - r\dot{\varphi}^2 \qquad \text{and} \qquad \boldsymbol{e}_\varphi \cdot \boldsymbol{a} = a_\varphi = 2\dot{r}\dot{\varphi} + r\ddot{\varphi}, \tag{1.32}$$

so that

$$\boxed{\boldsymbol{a} = \left(\ddot{r} - r\dot{\varphi}^2\right)\boldsymbol{e}_r + \left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right)\boldsymbol{e}_\varphi} \tag{1.33}$$

We have thus identified the radial and tangential components of the velocity and acceleration vectors. When formulating all vectors in the rotating, non-inertial frame $\mathcal{R}$, the components of the velocity and acceleration vectors are

$$[\boldsymbol{v}]_\mathcal{R} = \begin{pmatrix} \dot{r} \\ r\dot{\varphi} \end{pmatrix}, \qquad [\boldsymbol{a}]_\mathcal{R} = \begin{pmatrix} \ddot{r} - r\dot{\varphi}^2 \\ 2\dot{r}\dot{\varphi} + r\ddot{\varphi} \end{pmatrix}. \tag{1.34}$$

It is important to realize that these components apply in a *rotating, non-inertial* frame of reference. As a consequence, we generally have

$$a_r \neq \dot{v}_r \qquad \text{and} \qquad a_\varphi \neq \dot{v}_\varphi, \tag{1.35}$$

because these lack the additional terms that stem from differentiating the basis vectors with respect to time. Therefore, keep in mind that *the kinematic relations* (1.3) *apply to the position, velocity and acceleration* ***vectors*** *– and* ***not their components****.*

To introduce some further definitions, $\omega = \dot{\varphi}$ is usually referred to as the **angular velocity** [rad/s], while $\dot{\omega} = \ddot{\varphi}$ is known as the **angular acceleration** [rad/s^2] of the particle.

We close by noting that the same theoretical treatment can be applied to *cylindrical coordinates* and *spherical coordinates* in 3D, the details of which are skipped here (but can be found in dynamics textbooks; see, e.g., this website for spherical coordinates).

Example 1.5. Particle rotating on a taut, inextensible string

Consider a particle that is attached to a taut, inextensible[3] string of constant length l, which is rotating about a fixed point O in 2D. The kinematics of the particle is best described using polar coordinates. In this case we have $r(t) = l =$ const. so that $\dot{r} = 0$ and $\ddot{r} = 0$. Consequently,

$$\boldsymbol{v}(t) = l\dot{\varphi}(t)\boldsymbol{e}_\varphi(t), \qquad \boldsymbol{a}(t) = -l\dot{\varphi}^2(t)\boldsymbol{e}_r(t) + l\ddot{\varphi}(t)\boldsymbol{e}_\varphi(t), \tag{1.36}$$

[3] *Taut* means that the string always remains under tension (is not slack). For an inextensible string, i.e., one which cannot stretch, this means that the string maintains a constant length.

i.e., at any instance of time the particle velocity is tangential to its circular motion, while the acceleration contains both radial and tangential components. If the rotation is of constant angular velocity, i.e., $\omega = \dot{\varphi} = \text{const.}$, then $v = l|\omega|$ and

$$\boldsymbol{v}(t) = l\omega \boldsymbol{e}_\varphi(t), \qquad \boldsymbol{a}(t) = -l\omega^2 \boldsymbol{e}_r(t) = -\frac{v^2}{l}\boldsymbol{e}_r(t), \tag{1.37}$$

i.e., the velocity is of constant magnitude and only changes its direction as the particle moves on the circular trajectory (always being perpendicular to the string). The angular acceleration also has a constant magnitude and points radially inward (accelerating the particle in a **centripetal** fashion towards the center of rotation). It grows quadratically with angular velocity.

1.1.4 Kinematics of space curves

Having discussed Cartesian and polar coordinates, we here introduce a general description of a 3D particle trajectory, which is sometimes more convenient and known as the description of a **space curve**. As in the polar scenario discussed above, we do *not* use Cartesian coordinates to describe the particle trajectory but we introduce a convenient **non-inertial frame** of reference.

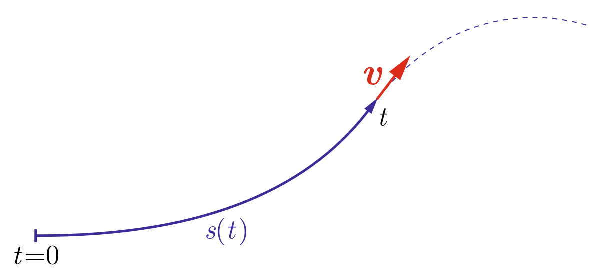
To this end, we introduce the unique coordinate $s(t)$ which measures the **path length** traveled by the particle since time $t = 0$. For example, if a particle is travelling on a straight line in the x_1-direction (starting at $x_1(0) = 0$), then $s(t) = x_1(t)$. If the particle is traveling on a circular trajectory of constant radius R, then $s(t) = R\varphi(t)$ with $\varphi(t)$ being the angle traveled since time $t = 0$. For general trajectories, the description naturally becomes more complex.

The position of a particle is now described as

$$\boldsymbol{r} = \boldsymbol{r}(s) \qquad \text{with} \qquad s = s(t). \tag{1.38}$$

For convenience, let us introduce the unit tangent vector along the particle trajectory as

$$\boldsymbol{e}_t = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}, \tag{1.39}$$



which may be re-interpreted by introducing the speed

$$v = |\boldsymbol{v}| = \dot{s} \tag{1.40}$$

and noticing that

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-1}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\boldsymbol{v}}{v} \qquad \Rightarrow \qquad \boxed{\boldsymbol{e}_t = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\boldsymbol{v}}{v}} \tag{1.41}$$

In other words, $\boldsymbol{e}_t(t)$ is always tangential to the trajectory of the particle and it points into the direction of the instantaneous velocity $\boldsymbol{v}$ at any point on the trajectory (and it has unit length).

This allows us to write the velocity vector in general as

$$\boxed{\boldsymbol{v} = \dot{s}\boldsymbol{e}_t} \tag{1.42}$$

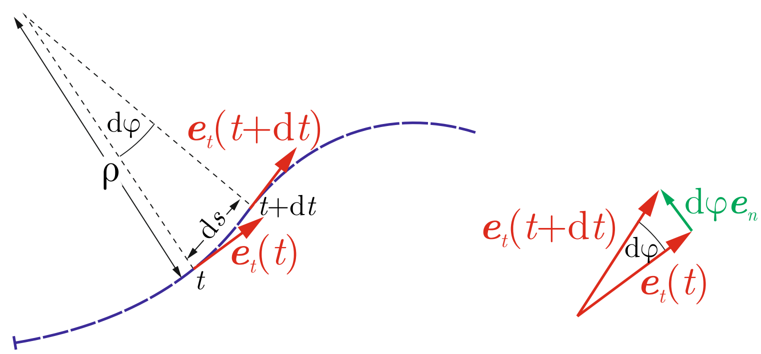
In order to arrive at the acceleration vector, we differentiate with respect to time, arriving at

$$\boldsymbol{a} = \ddot{s}\boldsymbol{e}_t + \dot{s}\dot{\boldsymbol{e}}_t. \tag{1.43}$$

To find $\dot{\boldsymbol{e}}_t$, let us first realize that, since $|\boldsymbol{e}_t| = 1$,

$$\boldsymbol{e}_t \cdot \boldsymbol{e}_t = 1 \qquad \Rightarrow \qquad 2\boldsymbol{e}_t \cdot \dot{\boldsymbol{e}}_t = 0 \qquad \Rightarrow \qquad \dot{\boldsymbol{e}}_t \perp \boldsymbol{e}_t, \tag{1.44}$$

i.e., $\dot{\boldsymbol{e}}_t$ must always be oriented perpendicular to $\boldsymbol{e}_t$ and hence to the particle trajectory. This motivates the introduction of the **principal normal vector** $\boldsymbol{e}_n$ as the unit vector pointing into the direction of $\dot{\boldsymbol{e}}_t$. In order to determine the magnitude of $\dot{\boldsymbol{e}}_t$, we inspect two times t and $t + \mathrm{d}t$, separate by an infinitesimal time increment $\mathrm{d}t$, and compare their $\boldsymbol{e}_t$-vectors, as shown schematically below.



From geometry, we conclude that (approximately)

$$\dot{\boldsymbol{e}}_t = \frac{\boldsymbol{e}_t(t + \mathrm{d}t) - \boldsymbol{e}_t(t)}{\mathrm{d}t} = \frac{\mathrm{d}\varphi\, \boldsymbol{e}_n}{\mathrm{d}t}, \tag{1.45}$$

where we exploited that both $\boldsymbol{e}_t(t)$ and $\boldsymbol{e}_t(t + \mathrm{d}t)$ are unit vectors of length 1, and the angle $\mathrm{d}\varphi$ is so small that $\boldsymbol{e}_t(t + \mathrm{d}t) - \boldsymbol{e}_t(t)$ is approximately the length of the arc of a circle of radius 1 and angle $\mathrm{d}\varphi$. We then replace $\mathrm{d}\varphi$ via the relation

$$\mathrm{d}s = \rho\, \mathrm{d}\varphi, \tag{1.46}$$

where $\mathrm{d}s$ is an infinitesimally small segment of the trajectory (approximated as a circular arc), and ρ denotes the **radius of curvature** of the so-called **osculating circle**[4]. It is related to the local **curvature** κ via

$$\kappa = 1/\rho. \tag{1.47}$$

[4]The *osculating circle* of any point on a trajectory is the circle going through that point and having the same curvature as the trajectory at that point. For a more intuitive interpretation, imagine you are driving a vehicle on a curved path when suddenly the wheels lock in their current orientation. Unless that orientation is straight, the car will leave the path and continue in a circle. That circle is the osculating circle and its radius the radius of curvature.

For example, for a **trajectory in 2D** described by $x_2 = f(x_1)$, the radius of curvature is obtained from

$$\frac{1}{\rho} = \frac{f''}{\sqrt{1+(f')^2}^3} \qquad \text{with} \qquad f' = \frac{\mathrm{d}f}{\mathrm{d}x_1}, \quad f' = \frac{\mathrm{d}^2 f}{\mathrm{d}x_1^2}. \tag{1.48}$$

For a circular motion with constant radius R, we obviously have $\rho = R$. For rectilinear motion on a straight line $\rho \to \infty$. If the particle travels on a quadratic parabola, then the curvature depends on position following (1.48), e.g.,

$$x_2 = f(x_1) = c\,x_1^2 \qquad \Rightarrow \qquad \rho = \frac{\sqrt{1+(2cx_1)^2}^3}{2c}. \tag{1.49}$$

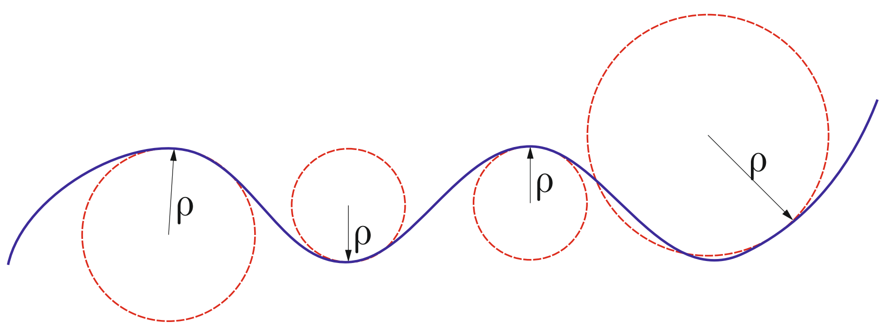
Insertion of (1.46) into (1.45) results in

$$\dot{\boldsymbol{e}}_t = \frac{\boldsymbol{e}_t(t+\mathrm{d}t) - \boldsymbol{e}_t(t)}{\mathrm{d}t} = \frac{\mathrm{d}\varphi\,\boldsymbol{e}_n}{\mathrm{d}t} = \frac{1}{\rho}\frac{\mathrm{d}s\,\boldsymbol{e}_n}{\mathrm{d}t} = \frac{\dot{s}}{\rho}\boldsymbol{e}_n = \frac{v}{\rho}\boldsymbol{e}_n, \tag{1.50}$$

which finally yields the acceleration vector in the space curve frame as

$$\boxed{\boldsymbol{a} = \ddot{s}\boldsymbol{e}_t + \frac{v^2}{\rho}\boldsymbol{e}_n} \tag{1.51}$$

As already seen for circular motion (cf. (1.37)), the second term represents the **centripetal acceleration** which accelerates any particle with non-zero speed and finite curvature radius towards the center of the (instantaneous) **osculating circle**. Note that (1.51) lets us re-interpret the curvature $\kappa = 1/\rho$ as the magnitude of acceleration experienced by a particle traveling with unit speed $v = \dot{s} = 1 = \text{const.}$ along the space curve. Shown below are examples of osculating circles at various points on a trajectory in 2D.



To arrive at a 3D formulation, one may define the (unit) **binormal vector**

$$\boldsymbol{e}_b = \boldsymbol{e}_t \times \boldsymbol{e}_n, \tag{1.52}$$

so that $\{\boldsymbol{e}_t, \boldsymbol{e}_n, \boldsymbol{e}_b\}$ forms an orthonormal triad for each point on the space curve. However, the acceleration $\boldsymbol{a}$ has a vanishing component in this third direction, so (1.51) remains valid.

Example 1.6. Circular motion

For a particle moving on a circular path with constant radius R, the length of the path traveled up to time $t \geq 0$ (assuming that $\varphi = 0$ at time $t = 0$) is $s(t) = R\varphi(t)$. Applying (1.42) leads to the velocity

$$\boldsymbol{v} = \dot{s}\boldsymbol{e}_t = R\dot{\varphi}\boldsymbol{e}_t, \tag{1.53}$$

and comparison with (1.31) shows that $\boldsymbol{e}_t = \boldsymbol{e}_\varphi$ (which makes sense: the $\boldsymbol{e}_\varphi$-vector defined in (1.28) always points into the tangential direction of the circular motion). Also, $v = \dot{s} = R|\dot{\varphi}| = R|\omega|$.

Next, applying (1.51) with $\rho = R = \text{const.}$ yields

$$\boldsymbol{a} = R\ddot{\varphi}\,\boldsymbol{e}_t + \frac{(R\dot{\varphi})^2}{R}\boldsymbol{e}_n = R\ddot{\varphi}\,\boldsymbol{e}_t + R\dot{\varphi}^2\boldsymbol{e}_n. \tag{1.54}$$

Comparison with (1.33) shows that $\boldsymbol{e}_n = -\boldsymbol{e}_r$, which could also have been expected ($\boldsymbol{e}_r$ was defined radially outwards, while $\boldsymbol{e}_n$ is defined such that it points towards the center of curvature).

We note that for a general 2D motion expressed in polar coordinates (r, φ) with varying radius $r(t)$ and angle $\varphi(t)$, we have $\mathrm{d}s = \sqrt{\mathrm{d}r^2 + r\,\mathrm{d}\varphi^2}$, so the above derivation becomes significantly more complex (e.g., $\boldsymbol{e}_t$ and $\boldsymbol{e}_n$ will no longer align with $\boldsymbol{e}_\varphi$ and $-\boldsymbol{e}_r$, respectively).

1.1.5 A brief summary of particle kinematics

The kinematics of a particle in d dimensions is described by d independent DOFs, the exact choice of which can be made by convenience. In the presence of k kinematic constraints, that number reduces to $(d - k)$ DOFs. Irrespective of the choice of description, we always have the vector relations

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}, \qquad \boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}, \tag{1.55}$$

whereas their components depend on the chosen reference frame. For example, in

- Cartesian coordinates in d dimensions:

$$\boldsymbol{v} = \sum_{i=1}^{d} \dot{x}_i\boldsymbol{e}_i, \qquad \boldsymbol{a} = \sum_{i=1}^{d} \ddot{x}_i\boldsymbol{e}_i \tag{1.56}$$

- polar coordinates in 2D:

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\varphi}\boldsymbol{e}_\varphi, \qquad \boldsymbol{a} = \left(\ddot{r} - r\dot{\varphi}^2\right)\boldsymbol{e}_r + \left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right)\boldsymbol{e}_\varphi \tag{1.57}$$

- space curve description:

$$\boldsymbol{v} = \dot{s}\boldsymbol{e}_t, \qquad \boldsymbol{a} = \ddot{s}\boldsymbol{e}_t + \frac{v^2}{\rho}\boldsymbol{e}_n \tag{1.58}$$

Importantly, from the above three only the Cartesian description with a fixed origin is an *inertial* frame, while the polar and space curve descriptions involve *non-inertial* reference frames.

1.2 Kinetics

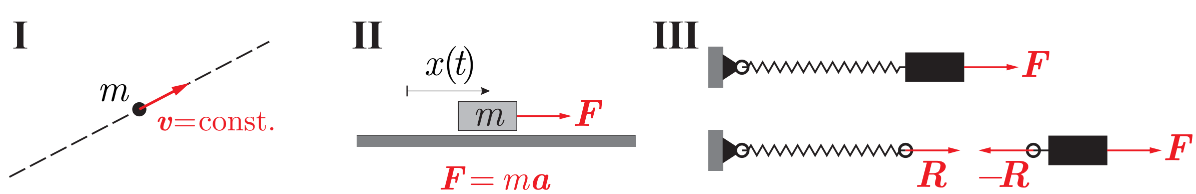
So far, we have described the motion of a particle in terms of its position, velocity, and acceleration (and we have shown how those are linked to each other). All those considerations were linked purely to the **geometry** of the system and of the particle's motion, without caring about what caused the motion. All of that is known as the **kinematics** of a system (and it is quite analogous to the kinematic description of deformation in Mechanics 2, where, e.g., the displacements and strains were linked through kinematic relations to describe the deformation of bodies). In order to link a particle's motion to the causes of its motion – viz., any applied forces – we need to go a step further and study the **kinetics** of a particle. The latter is summarized by the following three famous axioms, which were first published by Sir Issac Newton in 1687.

1.2.1 Newton's axioms and balance of linear momentum

Before presenting Newton's axioms, let us introduce the **linear momentum** vector $\boldsymbol{P}$ of a particle as the product of its velocity vector $\boldsymbol{v}$ and the particle mass m, i.e.,

$$\boxed{\boldsymbol{P} = m\boldsymbol{v}} \tag{1.59}$$

A particle's linear momentum always points in the direction of its velocity. With this definition, we are in place to discuss Newton's three axioms, which are schematically shown below. We point out that all three axioms apply only in an **inertial frame** (i.e., in a non-moving frame – we will get back to this point later in Section 3.4).



I. **Newton's first axiom**:

In the absence of any force acting on a particle (so the **resultant force** $\boldsymbol{F} = \sum_i \boldsymbol{F}_i$ vanishes), its **linear momentum** is conserved (i.e., remains constant) over time:

$$\boldsymbol{F} = \sum_i \boldsymbol{F}_i = \boldsymbol{0} \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}) = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{P} = m\boldsymbol{v} = \text{const.} \tag{1.60}$$

Note that, here and in the following, we often simply write $\boldsymbol{F}$ for the applied (resultant/net) force[5] acting on a particle (i.e., this is the sum of all individual forces being applied, and we do not explicitly write out the sum over all applied forces $\boldsymbol{F}_i$).

[5] In Mechanics 1, the symbol $\boldsymbol{R}$ was introduced for the *resultant* force acting on a body (i.e., the sum of all applied forces). Here, we prefer to use $\boldsymbol{F}$, since we will reserve $\boldsymbol{R}$ for frictional forces that appear in many examples (also, the majority of textbooks uses $\boldsymbol{F}$).

II. **Newton's second axiom**:

The rate of change of linear momentum equals the applied (net) force:

$$\boxed{\sum_i \boldsymbol{F}_i = \dot{\boldsymbol{P}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v})} \tag{1.61}$$

We will refer to this relation as **linear momentum balance** (abbreviated as **LMB**). It is one of the key equations we will use frequently throughout the course. Simply put, *to change the linear momentum of a particle, it takes a force.*

Let us consider three special cases. First, in case of a static problem (no particle motion, so $\boldsymbol{v} = \boldsymbol{0}$), Newton's second axiom reduces to static equilibrium:

$$\text{special case } \textbf{statics}: \quad \boldsymbol{v} = \boldsymbol{0} \quad \Rightarrow \quad \sum_i \boldsymbol{F}_i = \boldsymbol{0} \tag{1.62}$$

The case of equilibrium in statics is hence recovered as a special case. Second, consider a particle of constant mass (i.e., $\dot{m} = 0$), in which case Newton's second axiom becomes

$$\text{special case } \textbf{constant mass}: \quad \dot{m} = 0 \quad \Rightarrow \quad \boxed{\sum_i \boldsymbol{F}_i = m\boldsymbol{a}} \tag{1.63}$$

To be correct, Newton – who did not consider particles with changing mass – formulated his second axiom as $\boldsymbol{F} = m\boldsymbol{a}$, which is equivalent to the above if $m = \text{const}$. Third, in case of no applied forces, we automatically recover the first axiom since

$$\text{special case } \textbf{zero forces}: \quad \sum_i \boldsymbol{F}_i = \boldsymbol{0} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}) = 0 \quad \Rightarrow \quad \boldsymbol{P} = m\boldsymbol{v} = \text{const.} \tag{1.64}$$

III. **Newton's third axiom**:

When a particle exerts a force on a second particle, the second particle simultaneously exerts a force equal in magnitude and opposite in direction onto the first one:

$$\boxed{\text{Action} = \text{Reaction}} \tag{1.65}$$

This axiom was already introduced in Mechanics 1 and is simply restated here. There is no change needed in the context of dynamics.

It is important to note that, in the third example (III) in the above figure we have $\boldsymbol{F} \neq \boldsymbol{R}$ in general (unlike in statics) because, by Newton's second axiom,

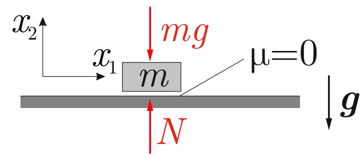
$$m\boldsymbol{a} = \boldsymbol{F} - \boldsymbol{R} \qquad \Rightarrow \qquad \boldsymbol{F} - \boldsymbol{R} = \boldsymbol{0} \quad \text{only if} \quad \begin{cases} \boldsymbol{a} = \boldsymbol{0} & \text{(in statics)}, \\ m = 0 & \text{(for massless bodies)}. \end{cases} \tag{1.66}$$

The latter case will become important in some examples later: whenever the *mass of a body is negligibly small* (e.g., the mass of a string compared to the mass of a heavy particle attached to the string), then the net force on the string must vanish – as in statics.

Example 1.7. Sliding block on a frictionless surface

Consider a particle of mass m sliding on a frictionless horizontal ground in the x_1-direction, such that the particle *kinematics* are simply described by

$$\boldsymbol{r}(t) = x_1(t)\boldsymbol{e}_1, \quad \boldsymbol{v}(t) = \dot{x}_1(t)\boldsymbol{e}_1, \quad \boldsymbol{a}(t) = \ddot{x}_1(t)\boldsymbol{e}_1. \tag{1.67}$$



The constraint $x_2(t) = 0$ entails a reaction force $\boldsymbol{N} = N\boldsymbol{e}_2$ to keep the particle on the constrained path, i.e., on the ground (without such a force, gravity would accelerate the particle downwards so that $x_2(t) \neq 0$). Since there is no friction, no horizontal force acts.

When it comes to the *kinetics* of the sliding particle, linear momentum balance hence reads

$$m\boldsymbol{a} = m\ddot{x}_1\boldsymbol{e}_1 = (N - mg)\boldsymbol{e}_2. \tag{1.68}$$

Note that linear momentum balance can also be interpreted component-wise:

$$\begin{matrix} \sum_i F_{i,1} = m\ddot{x}_1 \\ \sum_i F_{i,2} = m\ddot{x}_2 \end{matrix} \quad \Rightarrow \quad \begin{matrix} 0 = m\ddot{x}_1 \\ N - mg = 0 \end{matrix} \tag{1.69}$$

so that

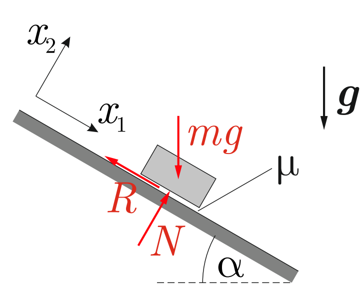
$$\ddot{x}_1 = 0 \quad \Rightarrow \quad v_1 = \dot{x}_1 = \text{const.} \qquad \text{and} \qquad N = mg. \tag{1.70}$$

In other words, since no forces act in the x_1-direction, linear momentum in the x_1-direction is conserved and hence the velocity in the x_1-direction remains constant at all times:

$$m\dot{x}_1 = \text{const.} \qquad \Rightarrow \qquad v_1 = \text{const.} \tag{1.71}$$

Example 1.8. Sliding block on an inclined surface

A block of mass m is sliding down an inclined surface (under an angle α). The block is subjected to gravity acting vertically down. The sliding friction is characterized by a **kinetic friction coefficient** μ, so that during sliding $R = \mu N$ (where R and N denote the magnitudes of the friction and normal forces, respectively) with $\boldsymbol{R}$ oriented opposite to the velocity of the sliding particle (as long as $|\boldsymbol{v}| \neq 0$).



Here, it is wise to align the coordinate system with the inclined surface (as shown). In this inclined frame of reference, the particle's *kinematics* is simply described by its position $\boldsymbol{r}(t) = x_1(t)\boldsymbol{e}_1$, so that its velocity is $\boldsymbol{v}(t) = \dot{x}_1(t)\boldsymbol{e}_1$ and acceleration $\boldsymbol{a}(t) = \ddot{x}_1(t)\boldsymbol{e}_1$.

When it comes to the *kinetics*, linear momentum balance reads

$$\begin{aligned} \textstyle\sum_i F_{i,1} &= -R + mg\sin\alpha = m\ddot{x}_1, \\ \textstyle\sum_i F_{i,2} &= N - mg\cos\alpha = 0, \end{aligned} \tag{1.72}$$

so that

$$R = \mu N = \mu m g \cos\alpha \quad \Rightarrow \quad \ddot{x}_1 = g(\sin\alpha - \mu\cos\alpha). \tag{1.73}$$

Notice that only in the special case $\sin\alpha - \mu\cos\alpha = 0$ – which is equivalent to $\mu = \tan\alpha$ – the particle is sliding at a constant velocity $\dot{x}_1 = \text{const.}$ since $\ddot{x}_1 = 0$. If $\mu > \tan\alpha$, then $\ddot{x}_1 < 0$ so that the particle will decelerate and eventually come to rest. If $\mu < \tan\alpha$, then $\ddot{x}_1 > 0$ so the particle will accelerate downward and never come to rest.

Recall for comparison that within **statics** we have $|\boldsymbol{R}| < \mu_0|\boldsymbol{N}|$ with μ_0 the coefficient of static friction. In dynamics, **sliding friction** is characterized by the **kinetic friction coefficient** μ and the relations

$$\boxed{\boldsymbol{R} = -\mu|\boldsymbol{N}|\frac{\boldsymbol{v}}{|\boldsymbol{v}|}} \qquad \text{such that} \qquad |\boldsymbol{R}| = \mu|\boldsymbol{N}|, \tag{1.74}$$

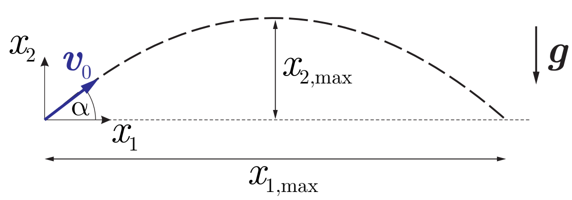
and $\boldsymbol{R}$ always points into the direction opposite to $\boldsymbol{v}$. Note that μ is not a property of a material, but it depends on the *combination of the two materials in contact* and we usually have $\mu \le \mu_0$. Therefore, the force needed to overcome static friction ($|R| = \mu_0|N|$) is typically larger than the force required to maintain sliding friction ($|R| = \mu|N|$).

Example 1.9. Projectile motion (throwing a particle)

Throwing a particle of mass m from the ground ($\boldsymbol{x} = \boldsymbol{0}$ at time $t = 0$) with an initial velocity

$$\boldsymbol{v}_0 = v_0\cos\alpha\,\boldsymbol{e}_1 + v_0\sin\alpha\,\boldsymbol{e}_2 \qquad \text{at time } t = 0 \tag{1.75}$$

results in a parabola of flight. Let us find the motion of the flying ball in 2D.



For a free flight in 2D, the *kinematics* are given by

$$\boldsymbol{r}(t) = x_1(t)\boldsymbol{e}_1 + x_2(t)\boldsymbol{e}_2, \tag{1.76}$$

and velocity and acceleration follow by differentiation with respect to time.

Kinetics is governed by the balance of linear momentum in the two coordinate directions, which yields the accelerations whose integration with respect to time provides the velocity and position:

$$\begin{matrix} \sum_i F_{i,1} = 0 = m\ddot{x}_1 \\ \sum_i F_{i,2} = -mg = m\ddot{x}_2 \end{matrix} \quad \Rightarrow \quad \begin{matrix} \dot{x}_1 = c_1 \\ \dot{x}_2 = -gt + d_1 \end{matrix} \quad \Rightarrow \quad \begin{matrix} x_1 = c_1 t + c_0 \\ x_2 = -\frac{g}{2}t^2 + d_1 t + d_0 \end{matrix} \tag{1.77}$$

Using the initial conditions to find the constants of integration yields the balls' motion as

$$x_1(t) = v_0 t \cos\alpha, \qquad x_2(t) = -\frac{g}{2}t^2 + v_0 t \sin\alpha. \tag{1.78}$$

The particle hits the ground again at point $\boldsymbol{x}_{1,\max} = x_{1,\max}\boldsymbol{e}_1$ when

$$x_2(t_{1,\max}) = -\frac{g}{2}t_{1,\max}^2 + v_0 \sin\alpha\, t_{1,\max} = 0 \qquad \Leftrightarrow \qquad t_{1,\max} = \frac{2v_0\sin\alpha}{g}, \tag{1.79}$$

so that

$$x_{1,\max} = x_1(t_{1,\max}) = \frac{2v_0^2 \sin\alpha\, \cos\alpha}{g}. \tag{1.80}$$

Note that the maximum distance is achieved when the ball is thrown under an angle of $\alpha = \pi/4$ ($45°$) when $x_{1,\max} = v_0^2/g$.

The maximum height during flight is reached when

$$\dot{x}_2(t_{2,\max}) = -g t_{2,\max} + v_0 \sin\alpha = 0 \qquad \Leftrightarrow \qquad t_{2,\max} = \frac{v_0 \sin\alpha}{g}, \tag{1.81}$$

so that

$$x_{2,\max} = x_2(t_{2,\max}) = \frac{v_0^2 \sin^2\alpha}{2g}. \tag{1.82}$$

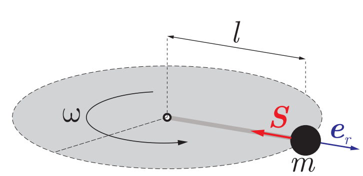
Example 1.10. Particle on a string

In Example 1.5 we showed that the acceleration of a particle following a circular trajectory of radius l at constant angular velocity ω is given by

$$\boldsymbol{a} = -l\omega^2 \boldsymbol{e}_r, \tag{1.83}$$

where $\boldsymbol{e}_r$ is the unit vector pointing radially outward.

As a simple example, consider a particle of mass m attached to a string of fixed length l, which remains taut at all times and rotates at a constant angular velocity ω. In this scenario, if we neglect gravity, the only force acting on the particle is the force $\boldsymbol{S}$ in the string. Linear momentum balance in this case becomes



$$\boldsymbol{S} = m\boldsymbol{a} = -ml\omega^2 \boldsymbol{e}_r. \tag{1.84}$$

Consequently, the force $\boldsymbol{S}$ in the string must always point radially inward (along the rope), and it has a constant magnitude of $|\boldsymbol{S}| = ml\omega^2$.

1.2.2 Work–energy balance

For many systems it is convenient to invoke so-called *energy principles* rather than working with the momentum balance laws. The main idea is to compare two states of the particle rather than considering the entire trajectory of the particle in between those two states. Starting from the balance of linear momentum, we integrate along a particle's trajectory $\boldsymbol{r}(t)$ over time to arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}) = \sum_i \boldsymbol{F}_i \qquad \Rightarrow \qquad \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{r} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \sum_i \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}. \tag{1.85}$$

Note that with $\mathrm{d}\boldsymbol{r} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\,\mathrm{d}t = \boldsymbol{v}\,\mathrm{d}t$ and assuming $m = \text{const.}$, this leads to

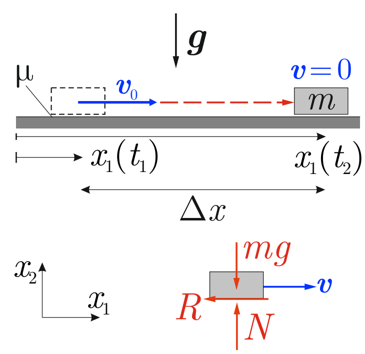
$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{r} = \int_{t_1}^{t_2} m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \boldsymbol{v}\,\mathrm{d}t = m\int_{\boldsymbol{v}(t_1)}^{\boldsymbol{v}(t_2)} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{v} = m\frac{|\boldsymbol{v}(t_2)|^2}{2} - m\frac{|\boldsymbol{v}(t_1)|^2}{2}, \tag{1.86}$$

where we identify $m|\boldsymbol{v}|^2/2$ as the **kinetic energy** T of the particle. Therefore, the change in kinetic energy when going from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ equals the **work** W_{12} performed on the particle by external forces $\boldsymbol{F}_i$, which is the **work–energy balance** for a *particle of constant mass*:

$$\boxed{T(t_2) - T(t_1) = W_{12}, \qquad T(t) = \frac{m}{2}|\boldsymbol{v}(t)|^2, \qquad W_{12} = \int_{t_1}^{t_2} \sum_i \boldsymbol{F}_i \cdot \boldsymbol{v}\,\mathrm{d}t = \sum_i \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}} \tag{1.87}$$

Example 1.11. Frictional sliding

Consider a particle of mass m with initial velocity v_0 slipping in the x_1-direction over a flat surface with kinetic friction coefficient μ. Besides gravity and normal forces acting on the particle in the x_2-direction (like in the frictionless Example 1.7), the particle here experiences additionally a frictional force $\boldsymbol{R} = -\mu m g \boldsymbol{e}_1$ (as long as $\boldsymbol{v} \neq \boldsymbol{0}$), which opposes the 1D particle motion. When does the particle come to a stop?



This is an ideal example for using the work–energy balance, since we are only interested in the initial state and final state of the particle.

The kinetic energy at the start (initial velocity $\dot{x}_1 = v_0$) and end (final velocity $\dot{x}_1 = 0$) is, respectively,

$$T(t_1) = \frac{1}{2}mv_0^2, \qquad T(t_2) = 0. \tag{1.88}$$

Linear momentum balance in the x_2-direction gives $N = mg$ so that $\boldsymbol{R} = -R\boldsymbol{e}_1$ with $R = \mu N = \mu m g$. Therefore, the work done on the particle between times t_1 and t_2 is

$$W_{12} = \int_{t_1}^{t_2} [(N - mg)\boldsymbol{e}_2 - R\boldsymbol{e}_1)] \cdot \dot{x}_1\boldsymbol{e}_1\,\mathrm{d}t = -R\int_{t_1}^{t_2} \dot{x}_1\,\mathrm{d}t = -\mu m g \int_{x_1(t_1)}^{x_1(t_2)} \mathrm{d}x_1. \tag{1.89}$$

Note that $\boldsymbol{N}$ and $m\boldsymbol{g}$ do not perform any work on the particle, as those forces are acting perpendicular to the motion of the particle.

Defining the travel distance until the particle comes to a halt as

$$\Delta x = x_1(t_2) - x_1(t_1), \tag{1.90}$$

we finally obtain from the *work–energy balance*

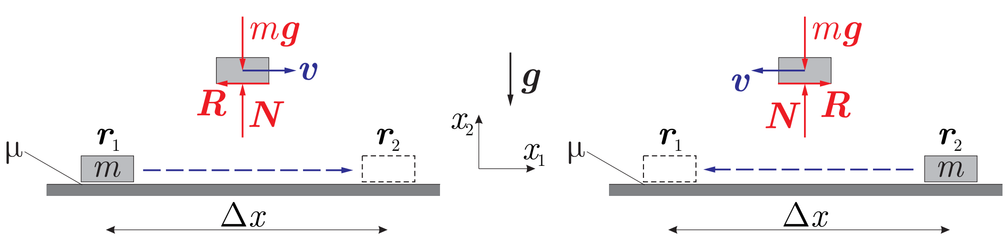
$$W_{12} = -\mu mg\,\Delta x = T(t_2) - T(t_1) = 0 - \frac{1}{2}mv_0^2 \qquad \Leftrightarrow \qquad \Delta x = \frac{v_0^2}{2\mu g}. \tag{1.91}$$

The complete initial kinetic energy of the particle is hence consumed by the frictional force over the distance Δx. (As a sanity check, in case of negligible friction, $\mu \to 0$, the travel distance correctly approaches $\Delta x \to \infty$, as linear momentum is conserved in the absence of a friction force, so the particle maintains a constant velocity and never comes to rest).

The above sliding friction example is a **non-conservative system**. To understand this, let us repeat the above example by first sliding on a frictional surface from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$, and then pushing the particle back with the same initial velocity so it moves back from $\boldsymbol{r}_2$ to $\boldsymbol{r}_1$ (over a distance $\Delta x = |\boldsymbol{r}_2 - \boldsymbol{r}_1|$). The work done on the entire path (noting that $\boldsymbol{R} = -\mu mg\boldsymbol{e}_1$ when going from 1 to 2, and $\boldsymbol{R} = \mu mg\boldsymbol{e}_1$ on the return path, as shown in the schematic below) is

$$\begin{aligned} W_{12} + W_{21} &= \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{R}\cdot \mathrm{d}\boldsymbol{r} + \int_{\boldsymbol{r}_2}^{\boldsymbol{r}_1} \boldsymbol{R}\cdot \mathrm{d}\boldsymbol{r} = \int_{x_1(t_1)}^{x_1(t_2)} -\mu mg\boldsymbol{e}_1 \cdot \mathrm{d}x_1\boldsymbol{e}_1 + \int_{x_1(t_2)}^{x_1(t_1)} \mu mg\boldsymbol{e}_1 \cdot \mathrm{d}x_1\boldsymbol{e}_1 \\ &= -\mu mg \int_{x_1(t_1)}^{x_1(t_2)} \mathrm{d}x_1 + \mu mg \int_{x_1(t_2)}^{x_1(t_1)} \mathrm{d}x_1 = -\mu mg\,\Delta x + \mu mg(-\Delta x) = -2\mu mg\,\Delta x < 0, \end{aligned}$$

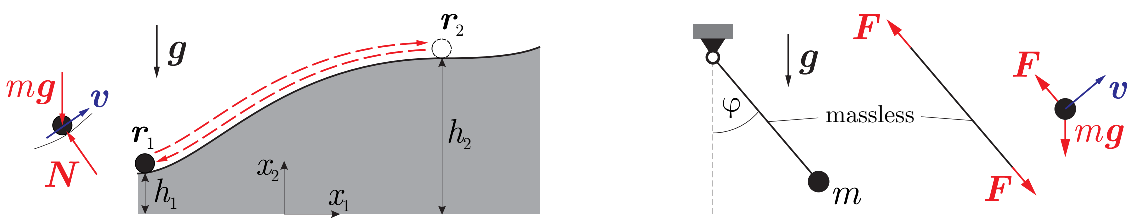
i.e., the particle is losing kinetic energy due to friction (like in Example 1.11). Recall from the previous example that normal and gravitational forces do not perform work in this scenario, so they are omitted here in the work calculation.



By contrast, consider now a particle subjected to gravity, being moved from $\boldsymbol{r}_1$ up to $\boldsymbol{r}_2$ and back down to $\boldsymbol{r}_1$ (*without frictional losses*). For example, we kick the particle up a hill with some initial velocity and, once it has reached its maximum height, we let it slide down the hill to the point from where we started – all without friction. It is important to notice that the normal force $\boldsymbol{N}$ from the ground does not perform any work on the particle since $\boldsymbol{N} \perp \mathrm{d}\boldsymbol{r}$ (the same applies, e.g., if the particle is attached to a massless string, where the force in the string is always perpendicular to the particle motion; see the schematic below). In the absence of friction, the only force performing work on the particle is hence that due to gravity. The total work performed on the particle hence evaluates to

$$\begin{aligned} W_{12} + W_{21} &= \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} + \int_{\boldsymbol{r}_2}^{\boldsymbol{r}_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} -mg\boldsymbol{e}_2 \cdot \mathrm{d}\boldsymbol{r} + \int_{\boldsymbol{r}_2}^{\boldsymbol{r}_1} -mg\boldsymbol{e}_2 \cdot \mathrm{d}\boldsymbol{r} \\ &= -mg \int_{h_1}^{h_2} \mathrm{d}x_2 - mg \int_{h_2}^{h_1} \mathrm{d}x_2 = -mg \int_{h_1}^{h_2} \mathrm{d}x_2 + mg \int_{h_1}^{h_2} \mathrm{d}x_2 = 0, \end{aligned} \tag{1.92}$$

where we defined $h_1 = \boldsymbol{r}_1 \cdot \boldsymbol{e}_2$ and $h_2 = \boldsymbol{r}_2 \cdot \boldsymbol{e}_2$ as the altitudes (x_2-coordinates) of points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively. Overall, the particle apparently conserves its kinetic energy during the cycle; we conclude that gravity is a *conservative force*.



The latter example defines a **conservative system**: the energy of a particle is constant when returning to its initial point along an arbitrary closed path. In mathematical terms, the work done by a conservative force $\boldsymbol{F}$ along any closed path Γ (starting at some point $\boldsymbol{r}_1$ and ending at the same point) vanishes, i.e.,

$$W_\Gamma = \oint_\Gamma \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0. \tag{1.93}$$

This implies that the path itself is irrelevant to the work done by a conservative force $\boldsymbol{F}$, only the initial and end points matter. This in turn makes us conclude that we should not even need to carry out the integration along a (non-closed) path from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ in order to calculate the work done by force $\boldsymbol{F}$. Instead, there must be a way to evaluate the work by only considering the two end points (since the path is irrelevant anyways). This is indeed the case for conservative forces, since they can be derived from a **potential energy** $V(\boldsymbol{r})$ such that

$$\boxed{\boldsymbol{F} = -\frac{\mathrm{d}V}{\mathrm{d}\boldsymbol{r}}} \tag{1.94}$$

We note that vector derivatives are carried out component-wise, i.e.,

$$V = V(\boldsymbol{r}) \qquad \Rightarrow \qquad \boldsymbol{F} = -\frac{\mathrm{d}V}{\mathrm{d}\boldsymbol{r}} = -\sum_{i=1}^{d} \frac{\mathrm{d}V}{\mathrm{d}x_i}\boldsymbol{e}_i \qquad \Leftrightarrow \qquad [\boldsymbol{F}] = \begin{pmatrix} -\mathrm{d}V/\mathrm{d}x_1 \\ -\mathrm{d}V/\mathrm{d}x_2 \\ -\mathrm{d}V/\mathrm{d}x_3 \end{pmatrix}. \tag{1.95}$$

Examples of a **conservative force** are

- **Gravity**: for a gravitational acceleration $\boldsymbol{g}$, the potential energy is

$$V(\boldsymbol{r}) = -m\boldsymbol{r}\cdot\boldsymbol{g} \qquad \text{and} \qquad \boldsymbol{F} = -\,\mathrm{d}V/\,\mathrm{d}\boldsymbol{r} = m\boldsymbol{g}. \tag{1.96}$$

  Specifically, for $\boldsymbol{g} = -g\boldsymbol{e}_3$ this gives $V(\boldsymbol{r}) = mgx_3$ and $\boldsymbol{F} = -mg\boldsymbol{e}_3$.

- **Elastic spring**: for an elastic spring stretched in the x_i-direction with unstretched spring position x_0 and stiffness k, the potential energy reads

$$V(\boldsymbol{r}) = \frac{1}{2}k(x_i - x_0)^2 \qquad \text{and} \qquad \boldsymbol{F} = -\,\mathrm{d}V/\,\mathrm{d}\boldsymbol{r} = -k(x_i - x_0)\boldsymbol{e}_i \tag{1.97}$$

Since we know that for any conservative force $\boldsymbol{F}$ there exists a potential energy $V(\boldsymbol{r})$ which is unique for each point $\boldsymbol{r}$, the work W_{12} performed by the conservative force between points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ now becomes

$$W_{12} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = -\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \frac{\mathrm{d}V}{\mathrm{d}\boldsymbol{r}}\cdot\mathrm{d}\boldsymbol{r} = -\int_{V(\boldsymbol{r}_1)}^{V(\boldsymbol{r}_2)} \mathrm{d}V = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2). \tag{1.98}$$

In other words, the work along any path between two points is given by the (negative) potential energy difference:

$$\boxed{W_{12} = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2)} \tag{1.99}$$

This is a convenient shortcut to computing the work W_{12} for any conservative forces. If *mixed forces act*, some of which being conservative and some non-conservative, then we can compute the work done by the conservative forces in the above fashion, whereas the work done by any non-conservative force is generally computed by integration (recall that the work W_{12} in (1.87) was additive in case of several forces $\boldsymbol{F}_i$).

In the special case of a **conservative system**, i.e., one where *only conservative forces* act, we know that the total work is given by

$$W_{12} = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2), \tag{1.100}$$

with V being the total potential energy. Substituting the work–energy balance (1.87) for W_{12} immediately leads to

$$T(t_2) - T(t_1) = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2). \tag{1.101}$$

This indicates that *conservative systems* obey the **conservation of energy** (i.e., the total energy including kinetic and potential contributions is conserved along any continuous path):

$$T(t_2) + V(\boldsymbol{r}_2) = T(t_1) + V(\boldsymbol{r}_1) \qquad \Leftrightarrow \qquad \boxed{T + V = \text{const.}} \tag{1.102}$$

Summary:

Let us quickly recap. The work–energy balance (1.87) holds for both conservative and non-conservative systems (the work done by external forces must always balance the change in kinetic energy of a particle). For the special case of *conservative forces*, the work W_{12} can conveniently be calculated as a potential difference that is path-independent, cf. (1.99). Finally, if all externally applied forces are conservative, then we are dealing with a *conservative system* and we can take a shortcut by exploiting the *conservation of energy*, cf. (1.102). For a *non-conservative force*, the work W_{12} is obtained by integrating along the path, cf. (1.87).

Examples of a **non-conservative force** are

- **Friction**: in case of sliding friction with a kinetic friction coefficient $\mu > 0$, the frictional force acting on a particle is

$$\boldsymbol{R} = -\mu|\boldsymbol{N}|\frac{\boldsymbol{v}}{|\boldsymbol{v}|} \qquad \text{for} \quad |\boldsymbol{v}| \neq 0. \tag{1.103}$$

  As discussed above, the direction of the friction force $\boldsymbol{R}$ depends on the direction of the particle velocity, which is why it cannot derive from a potential $V(\boldsymbol{r})$ that depends only on position (the path matters as well) – hence, the friction is not conservative.

- **Viscous drag**: viscous drag results, e.g., from a particle moving through a fluid. At low speeds (low Reyonlds number), the resulting drag force on the particle is linearly proportional to the particle velocity:

$$\boldsymbol{F}_\text{d} = -k\boldsymbol{v} \tag{1.104}$$

  with a viscosity $k > 0$. Obviously, this force depends on $\boldsymbol{v}$ and hence on the particle's trajectory, which is why the drag force cannot derive from a potential $V(\boldsymbol{r})$ and is therefore non-conservative.

Example 1.12. Maximum height during flight parabola (revisited)

Let us reconsider a particle of mass m, thrown with an initial velocity v_0 under an angle α and subjected to gravity. In Example 1.9 we solved for the trajectory of the particle by applying linear momentum balance and integrating the kinematic equations. Here, we revisit the problem, using the work–energy balance.

The starting point is characterized by kinetic energy $T_1 = T(t_1)$ and zero potential energy[6], whereas the point of maximum altitude $x_{2,\max}$ is the point where the vertical velocity component of the particle is zero (at this point the particle is no longer moving upward and will in the next instance start to move downward, so the instantaneous vertical velocity is zero). Since no forces act in the horizontal direction, the horizontal velocity component is constant during flight and hence agrees with the initial horizontal velocity $v = \dot{x}_1 = v_0 \cos\alpha$. Therefore, the kinetic and potential energies of initiation (state 1) and at the maximum height (state 2) are, respectively,

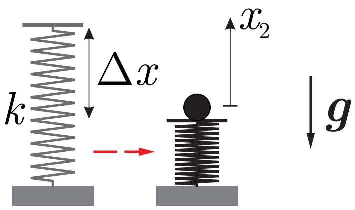
$$T_1 = \frac{1}{2}mv_0^2, \quad V_1 = 0, \qquad T_2 = \frac{1}{2}m(v_0\cos\alpha)^2, \quad V_2 = mgx_{2,\max}. \tag{1.105}$$

[6] The work–energy balance considers only differences between states, which is why the absolute altitude of the particle is irrelevant and only the distance in altitude matters. For that reason, we can choose the zero altitude level (the coordinate origin) arbitrarily and here pick the initial point.

Conservation of energy for this conservative systems yields directly

$$\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv_0^2\cos^2\alpha + mgx_{2,\max} \qquad \Leftrightarrow \qquad x_{2,\max} = \frac{v_0^2}{2g}(1-\cos^2\alpha) = \frac{v_0^2\sin^2\alpha}{2g}. \quad (1.106)$$

As a variation, assume that the particle is accelerated vertically upwards from rest by being **released from a compressed spring** (stiffness k). With the spring energy $V(\Delta x) = \frac{k}{2}(\Delta x)^2$ for a compression by Δx, the maximum height during flight becomes



$$T_1 = 0,\ V_1 = \frac{k}{2}(\Delta x)^2, \quad T_2 = 0,\ V_2 = mgx_{2,\max} \quad (1.107)$$

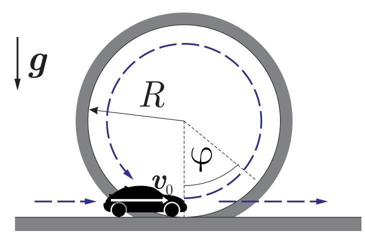
so that the conservation of energy yields the maximum height as

$$\frac{k}{2}(\Delta x)^2 = mgx_{2,\max} \qquad \Leftrightarrow \qquad x_{2,\max} = \frac{k(\Delta x)^2}{2gm}. \quad (1.108)$$

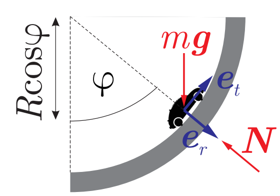
We thus obtain the maximum height quickly and without having to integrate the kinematic relations as in Example 1.9. Note that the same solution applies to frictionless sliding up on a foundation of arbitrary inclination.

Example 1.13. Car in a looping

We consider a car of mass m which enters a circular looping of radius R with an initial horizontal velocity v_0. We assume that the car engine is off (i.e., the foot is off the gas pedal) and friction is negligible. Further, we assume the car is small compared to R, so it may be approximated as a particle.



What is the minimal entry velocity v_0 required for the car to not fall off? What is the maximal force acting on the driver?



Let us first use the work–energy balance, which here in the absence of friction reduces to the *conservation of energy*, in order to find the minimum velocity v_0 such that the car reaches the top of the looping.

We know the speed v_0 at entry (at $\varphi = 0$, which we define as the altitude 0 so that the potential energy vanishes). Since the only other force that does work on the car is gravity, we can use the conservation of energy to determine the car's speed $v(\varphi)$ under any other angle $\varphi \in [0, 2\pi)$ within the looping by equating the total energy:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2(\varphi) + mgR(1-\cos\varphi), \quad (1.109)$$

which we solve for the speed

$$v(\varphi) = \sqrt{v_0^2 - 2gR(1-\cos\varphi)}. \quad (1.110)$$

As can be expected, the velocity is minimal when $\cos\varphi = -1$, i.e., at $\varphi = \pi$, where

$$v_{\min} = v(\pi) = \sqrt{v_0^2 - 4gR}. \tag{1.111}$$

Apparently, a *necessary condition* for the car to achieve the looping is hence

$$v_0^2 - 4gR > 0 \qquad \text{or} \qquad v_0 > 2\sqrt{gR}. \tag{1.112}$$

If this condition is not satisfied, the car will never reach the top and simply slide down backwards after having reached its maximum altitude.

In order to determine under what conditions the car does not fall off, we need to go one step further. During the looping, while there is no friction force, there is a normal force N (see the schematic above) which keeps the car on the ground (this is the reaction force due to the constrained motion on the circular looping). At any location in the looping, we can calculate N from the system kinetics. *Falling off* implies that the reaction force N becomes negative – a solution with $N < 0$ would imply that the reaction force no longer pushes the car off the ground, but the reaction force N would be required to pull the car towards the ground (to prevent it from falling off). Since the car is freely sliding on the ground, there is no such attractive force and the car would indeed fall off. Therefore, the *sufficient condition* for the car to not fall off is

$$N(\varphi) \geq 0 \qquad \forall\ \varphi \in [0, 2\pi). \tag{1.113}$$

Reaction force N is most easily obtained from the balance of linear momentum in polar coordinates; the radius $r = R = \text{const.}$, only φ is a free coordinate. As derived earlier for a circular path (see Section 1.1.3), we know that in polar coordinates

$$\boldsymbol{v} = v_t \boldsymbol{e}_t = R\dot{\varphi}\boldsymbol{e}_t \qquad \Rightarrow \qquad v(\varphi) = R\dot{\varphi}(\varphi). \tag{1.114}$$

Moreover, from a free-body diagram we obtain the balance of linear momentum in the radial direction with $a_r = -R\dot{\varphi}^2 = -(R\dot{\varphi})^2/R = -v^2/R$ as

$$ma_r = -N(\varphi) + mg\cos\varphi \qquad \Rightarrow \qquad -m\frac{v^2(\varphi)}{R} = -N(\varphi) + mg\cos\varphi \tag{1.115}$$

so that, inserting (1.110) for $v(\varphi)$,

$$N(\varphi) = \frac{mv^2(\varphi)}{R} + mg\cos\varphi = \frac{mv_0^2}{R} - 2mg(1-\cos\varphi) + mg\cos\varphi = \frac{mv_0^2}{R} + mg(3\cos\varphi - 2). \tag{1.116}$$

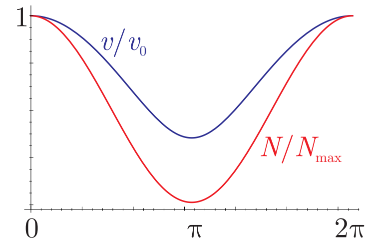
Preventing lift-off means

$$N(\varphi) \geq 0 \quad \forall \quad \varphi \in (0, 2\pi] \qquad \Leftrightarrow \qquad \frac{mv_0^2}{R} + mg(3\cos\varphi - 2) \geq 0 \quad \forall \quad \varphi \in (0, 2\pi]. \tag{1.117}$$

The worst-case scenario is reached when $\cos\varphi = -1$ or $\varphi = \pi$ (at the maximum height, as could have been expected), from which follows the *sufficient condition to prevent lift-off*:

$$\frac{mv_0^2}{R} + mg(3\cos\pi - 2) \geq 0 \qquad \Leftrightarrow \qquad v_0 \geq \sqrt{5gR}. \tag{1.118}$$

Note that this condition is stricter than the necessary condition (1.112), hence $v_0 \geq \sqrt{5gR}$ ensures both that the car reaches the top and does not fall off.

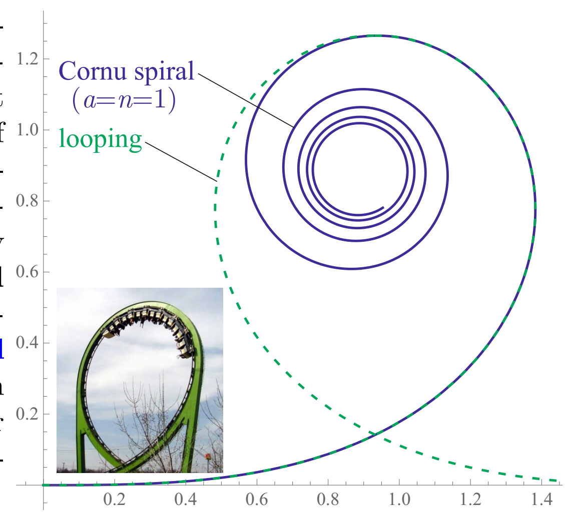


The *maximum N-force* on the car and hence on the driver is attained when $\cos\varphi = 1$ or $\varphi = 2\pi$, i.e., when the car has completed the looping. In that case, using as an example the minimum required entry velocity of $v_0 = \sqrt{5gR}$, we obtain from (1.116)

$$N(2\pi) = \frac{m5gR}{R} + mg(3\cos(2\pi) - 2) = 6mg. \tag{1.119}$$

That is, the driver experiences an acceleration of $6g$ upon completion of the looping. As a rule of thumb, an average person loses consciousness at about $5g$. To avoid such high accelerations, typical roller coasters never have circular loopings. Recall that we showed (see Section 1.1.4) the relation for the particle acceleration

$$\boldsymbol{a} = \dot{s}\boldsymbol{e}_t + \frac{v^2}{\rho}\boldsymbol{e}_n \tag{1.120}$$



for a general trajectory through space (not necessarily following a circular motion). If one wants the centripetal acceleration, the second term in (1.120), to not grow significantly, one must choose a varying radius of curvature ρ. For example, noting that v^2 decreases linearly with altitude across the looping (by the conservation of energy), we may choose ρ to decrease linearly from the entry point of the looping until its top (and then ascend in a symmetric fashion). The result is a so-called **Euler spiral** (also known as the **Cornu spiral** or **clothoid**) whose curvature increases linearly with path length. This design is typically used for roller coasters in amusement parks as well as for the transition from straight to curved segments on highways.

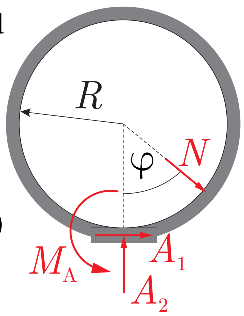
A possible parametrization (only mentioned without proof here) is

$$\boldsymbol{r}(t) = a\int_0^t \sin\frac{\tau^{n+1}}{n+1}\,\mathrm{d}\tau\boldsymbol{e}_1 + a\int_0^t \cos\frac{\tau^{n+1}}{n+1}\,\mathrm{d}\tau\boldsymbol{e}_1, \tag{1.121}$$

resulting in a trajectory of curvature $\kappa = t^n/a$; so the choice $n = 1$ results in a linearly increasing curvature.

Finally, note that we can also evaluate the support reactions at the base of a looping, assuming that it is clamped to its foundation. The looping itself is not moving at all (only the car in contact with the looping is). Therefore, we have a *static* system where the classical equilibrium relations apply with a moving normal force $\boldsymbol{N}(\varphi)$.

Drawing a free-body diagram of the (circular) looping without the car, which is assumed under an angle φ, and applying the *static* balance of forces and torques yields:



$$\begin{aligned} A_1 &= -N\sin\varphi = -\left[\frac{mv_0^2}{R} + mg(3\cos\varphi - 2)\right]\sin\varphi, \\ A_2 &= N\cos\varphi = \left[\frac{mv_0^2}{R} + mg(3\cos\varphi - 2)\right]\cos\varphi, \\ M_\mathrm{A} &= NR\sin\varphi = \left[\frac{mv_0^2}{R} + mg(3\cos\varphi - 2)\right]R\sin\varphi. \end{aligned} \tag{1.122}$$

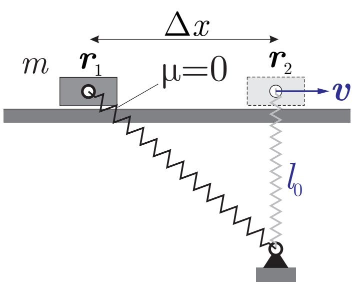
By comparison, if we had only placed the particle inside the looping and fixed it at an angle φ under static conditions (so only $-mg\boldsymbol{e}_3$ is acting on the foundation), the support reactions would have been

$$A_1 = 0, \qquad A_2 = mg, \qquad M_\mathrm{A} = mgR\sin\varphi. \tag{1.123}$$

The moving car results is significantly altered support reactions, which must be considered when desining the support of the looping.

Example 1.14. Elastic slingshot

Consider a slingshot composed of a linear spring (stiffness k, undeformed length l_0) which is stretched by moving a particle of mass m by Δx to the left on a frictionless ground, as shown on the right. If the particle is released in that position from rest, what is its speed v when it reaches the unstretched spring length at position $\boldsymbol{r}_2$?



The problem can best be solved by exploiting the *conservation of energy* in this conservative system (no friction forces, gravity is perpendicular to the motion of the particle, so only the spring force must be considered, which is conservative).

Comparing the two states' kinetic and potential energies,

$$T(t_1) = 0, \quad V(\boldsymbol{r}_1) = \frac{k}{2}\left(\sqrt{(\Delta x)^2 + l_0^2} - l_0\right)^2, \qquad T(t_2) = \frac{m}{2}v^2, \quad V(\boldsymbol{r}_2) = 0, \tag{1.124}$$

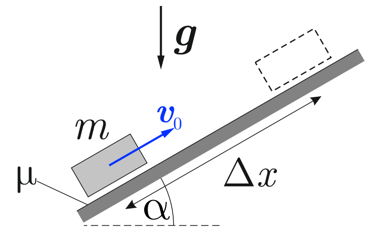
leads to

$$T(t_1) + V(\boldsymbol{r}_1) = T(t_2) + V(\boldsymbol{r}_2) \qquad \Rightarrow \qquad v = \sqrt{\frac{k}{m}}\left(\sqrt{(\Delta x)^2 + l_0^2} - l_0\right). \tag{1.125}$$

We note that the same result could have been obtained from application of the balance of linear momentum. However, that would have been significantly more complex since the spring force varies nonlinearly with position (and so does the acceleration on the mass to be integrated for the velocity). The work–energy principle is therefore a convenient shortcut.

Example 1.15. Frictional sliding up a slope

Consider a particle of mass m, which is sliding up an inclined slope (angle φ), starting with an initial speed v_0 and experiencing friction with the ground. What distance Δx does the particle travel up the slope?



Since we again know the initial state (speed v_0, potential energy chosen to vanish) and only care about the final state (speed 0, altitude $h = \Delta x \sin\alpha$), this example can beneficially be solved by the work–energy balance. However, we encounter both *conservative forces* (gravity) and *non-conservative* forces (friction).

In this case the work–energy balance states that:

$$0 - \frac{1}{2} m v_0^2 = W_{12}^{\text{friction}} + W_{12}^{\text{gravity}} \tag{1.126}$$

with

$$W_{12}^{\text{friction}} = \int_0^{\Delta x} -R \,\mathrm{d}x = -\mu m g \cos\alpha\,\Delta x, \qquad W_{12}^{\text{gravity}} = V_1 - V_2 = 0 - mgh. \tag{1.127}$$

Altogether, with $h = \Delta x \sin\alpha$ we arrive at

$$\frac{1}{2} m v_0^2 = mg\Delta x \sin\alpha + \mu m g \cos\alpha\,\Delta x \qquad \Rightarrow \qquad \Delta x = \frac{v_0^2}{2g(\sin\alpha + \mu\cos\alpha)}. \tag{1.128}$$

Note that for the special case $\alpha = 0$ we recover the solution $\Delta x = v_0^2/2\mu g$ for horizontal sliding from Example 1.11, whereas for $\alpha = 90°$ we recover the solution of the maximum height of a free vertical flight (without normal force and hence without friction) previously derived in Example 1.9 as $\Delta x = v_0^2/2g$.

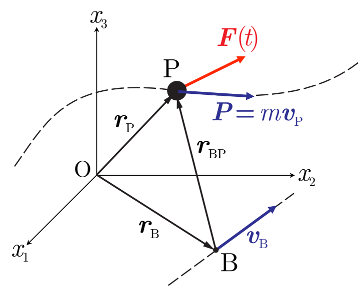
———

1.2.3 Balance of angular momentum

With the balance of linear momentum, we derived in Section 1.2.1 the dynamic version of force equilibrium in statics. We saw that, under dynamic conditions, the sum of all forces is no longer necessarily zero, and that a non-zero resultant force leads to an acceleration of the particle. We further saw that the proportionality between applied force and particle acceleration is the particle's mass. Following Newton's axioms, in order to change the motion of a particle, we must apply a force due to the particle's inertia which aims to maintain a constant velocity. The heavier the particle, the larger the required force to impose a certain acceleration of the particle.

We next seek a principle that replaces the equilibrium of all *torques*[7] in statics, which will lead us to the balance of angular momentum.

[7] Here and in the following, we use the term "*torque*" rather than "*moment*" (in German both correspond to "*Moment*") to avoid confusion with "*momentum*" ("*Impuls*" in German).

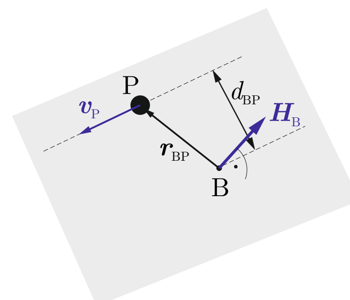


Let us first introduce the **angular momentum** $\boldsymbol{H}_\mathrm{B}$ *of a particle* P with respect to an arbitrary point B as the vector

$$\boxed{\boldsymbol{H}_\mathrm{B} = \boldsymbol{r}_\mathrm{BP} \times \boldsymbol{P}} \qquad \text{with linear momentum} \quad \boldsymbol{P} = m\boldsymbol{v}_\mathrm{P}, \tag{1.129}$$

where we denote the particle velocity by $\boldsymbol{v}_\mathrm{P}$ for clarity.

To interpret the angular momentum vector introduced above, we recognize that, by definition, $\boldsymbol{H}_\mathrm{B}$ is perpendicular to the plane spanned by $\boldsymbol{r}_\mathrm{BP}$ and the particle velocity $\boldsymbol{v}_\mathrm{P}$.



Further, taking the norm of the above definition shows that $|\boldsymbol{H}_\mathrm{B}| = m|\boldsymbol{r}_\mathrm{BP} \times \boldsymbol{v}_\mathrm{P}| = mv_\mathrm{P}d_\mathrm{BP}$, where d_BP is the distance between the linear momentum vector (and hence $\boldsymbol{v}_\mathrm{P}$) through particle P and point B (as sketched on the right).

Inserting $\boldsymbol{r}_\mathrm{BP} = \boldsymbol{r}_\mathrm{P} - \boldsymbol{r}_\mathrm{B}$ and therefore $\dot{\boldsymbol{r}}_\mathrm{BP} = \boldsymbol{v}_\mathrm{P} - \boldsymbol{v}_\mathrm{B}$ and using linear momentum balance (i.e., $\dot{\boldsymbol{P}} = \sum_i \boldsymbol{F}_i$), we obtain

$$\dot{\boldsymbol{H}}_\mathrm{B} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_\mathrm{BP} \times \boldsymbol{P}) = \dot{\boldsymbol{r}}_\mathrm{BP} \times \boldsymbol{P} + \boldsymbol{r}_\mathrm{BP} \times \dot{\boldsymbol{P}} = (\boldsymbol{v}_\mathrm{P} - \boldsymbol{v}_\mathrm{B}) \times \boldsymbol{P} + \boldsymbol{r}_\mathrm{BP} \times \sum_i \boldsymbol{F}_i. \tag{1.130}$$

We next introduce the **resultant torque** (i.e., the sum of all **torques**) with respect to B as

$$\boldsymbol{M}_\mathrm{B} = \boldsymbol{r}_\mathrm{BP} \times \sum_i \boldsymbol{F}_i. \tag{1.131}$$

We insert (1.131) into (1.130) and exploit that $\boldsymbol{v}_\mathrm{P} \times \boldsymbol{P} = \boldsymbol{v}_\mathrm{P} \times m\boldsymbol{v}_\mathrm{P} = \boldsymbol{0}$, so that we finally arrive at the **angular momentum balance** (which we abbreviate as **AMB**):

$$\boxed{\boldsymbol{M}_\mathrm{B} = \dot{\boldsymbol{H}}_\mathrm{B} + \boldsymbol{v}_\mathrm{B} \times \boldsymbol{P}} \tag{1.132}$$

This equation is the analog to the vanishing sum of all torques in the static equilibrium case.

Let us consider four special cases:

- First, we automatically recover the **static** case: without motion, we have $\dot{\boldsymbol{H}}_\mathrm{B} = \boldsymbol{0}$ and $\boldsymbol{P} = \boldsymbol{0}$, so that (1.132) reduces to

  $$\boldsymbol{M}_\mathrm{B} = \boldsymbol{0}, \tag{1.133}$$

  so the *sum of all torques must vanish for static equilibrium* (as we know from Mechanics 1).

- An often used special case occurs when **point B is fixed** or if B and P **move in parallel**:

  $$\text{if } \boldsymbol{v}_\mathrm{B} = \boldsymbol{0} \;\wedge\; \boldsymbol{v}_\mathrm{B} || \boldsymbol{v}_\mathrm{P} \qquad \Rightarrow \qquad \boldsymbol{v}_\mathrm{B} \times \boldsymbol{P} = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{M}_\mathrm{B} = \dot{\boldsymbol{H}}_\mathrm{B}. \tag{1.134}$$

  This is analogous to linear momentum balance: an applied external torque results in a change of the particle's angular momentum (like an applied force results in a change of its linear momentum). As a general guideline, we should *aim for this special case whenever possible by choosing* B *wisely* (viz. either as a fixed point or one that is moving parallel to the particle) to simplify the relations.

- If **no external torque** acts on the system ($\boldsymbol{M}_\mathrm{B} = \boldsymbol{0}$) with a **fixed point** B (or parallel motion of P and B), then we arrive at

$$\dot{\boldsymbol{H}}_\mathrm{B} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{H}_\mathrm{B} = \boldsymbol{r}_\mathrm{BP} \times m\boldsymbol{v}_\mathrm{P} = \text{const.} \quad \Rightarrow \quad |\boldsymbol{r}_\mathrm{BP}|\,|\boldsymbol{v}| = r_\mathrm{BP} v_\mathrm{P} = \text{const.} \tag{1.135}$$

  Therefore, the product of distance r_BP and particle speed remains constant over time.

  This is known as the **conservation of angular momentum** (analogous to the conservation of linear momentum by Newton's first axiom).

- Finally, for a **massless body**, $m = 0$ leads to $\boldsymbol{P} = \boldsymbol{0}$ so that $\boldsymbol{H}_\mathrm{B} = \boldsymbol{0} = \text{const.}$ and $\dot{\boldsymbol{H}}_\mathrm{B} = \boldsymbol{0}$, so

$$\text{if } m = 0 \quad \Rightarrow \quad \sum_i \boldsymbol{F}_i = \boldsymbol{0}, \qquad \boldsymbol{M}_\mathrm{B} = \boldsymbol{0}, \tag{1.136}$$

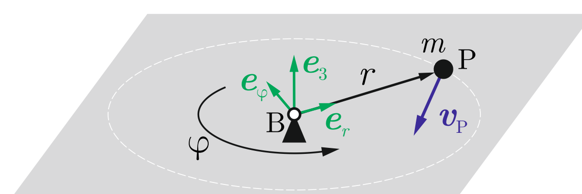
  where we included both LMB and AMB for a comparison. That is, *for massless bodies we recover the static equilibrium relations of a vanishing net force and torque.*

Special case 2D:

Consider a particle of mass m rotating around a fixed point B in a 2D plane with instantaneous angle $\varphi(t)$ and distance $r(t)$.

Using co-rotating polar coordinates, the above 3D formulation here reduces to

$$\begin{aligned}\boldsymbol{H}_\mathrm{B} = \boldsymbol{r}_\mathrm{BP} \times \boldsymbol{P} &= r\boldsymbol{e}_r \times (m\boldsymbol{v}_\mathrm{P}) \\ &= r\boldsymbol{e}_r \times m(\dot{r}\boldsymbol{e}_r + r\dot{\varphi}\boldsymbol{e}_\varphi) \\ &= mr^2\dot{\varphi}(\boldsymbol{e}_r \times \boldsymbol{e}_\varphi) = mr^2\dot{\varphi}\boldsymbol{e}_3,\end{aligned}$$



where $\boldsymbol{e}_3$ denotes the out-of-plane unit vector.

Applying the balance of angular momentum (1.132) with an applied in-plane torque $\boldsymbol{M}_\mathrm{B} = M_\mathrm{B}\boldsymbol{e}_3$ leads to

$$\boldsymbol{M}_\mathrm{B} = M_\mathrm{B}\boldsymbol{e}_3 = \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\varphi}\boldsymbol{e}_3\right) = m(r^2\ddot{\varphi} + 2r\dot{r}\dot{\varphi})\boldsymbol{e}_3 = mr(r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\boldsymbol{e}_3. \tag{1.137}$$

Therefore, we arrive at

$$\boldsymbol{M}_\mathrm{B} = M_\mathrm{B}\boldsymbol{e}_3 \qquad \Rightarrow \qquad M_\mathrm{B} = mr(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) = ma_\varphi r. \tag{1.138}$$

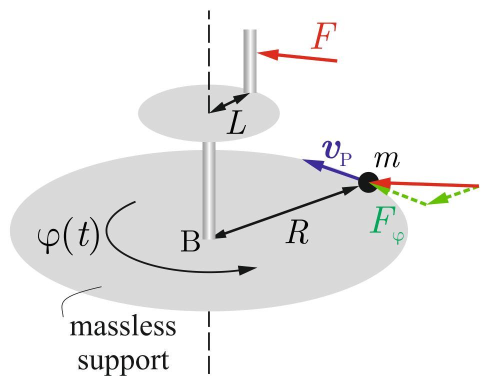
In many subsequent examples, we consider a rotation at a *fixed distance* $r(t) = R = \text{const.}$, e.g., when the particle is attached to a rotating disk or to a rigid string or stick. In such cases the above relations simplify since $\dot{r} = 0$, so that

$$\text{if } r(t) = R = \text{const.} \quad \Rightarrow \quad \boxed{M_\mathrm{B} = I_\mathrm{B}\ddot{\varphi}} \quad \text{with} \quad \boxed{I_\mathrm{B} = mR^2} \tag{1.139}$$

We call the constant I_B the **moment of inertia** (of a particle of mass m with respect to a point B in 2D, rotating at a distance R).

Angular vs. linear momentum balance:

The above 2D formulation lets us reinterpret the balance of angular momentum (AMB) in comparison to the balance of linear momentum (LMB).



Recall that LMB set into relation the force applied to a particle and the resulting acceleration of the particle via $\boldsymbol{F} = m\boldsymbol{a}$, with the mass m acting as the proportionality constant (affecting the *inertia* of the particle as a resistance against changes of its velocity).
Let us rewrite (1.139) as

$$M_B = I_B\ddot{\varphi} = mR^2\ddot{\varphi}. \tag{1.140}$$

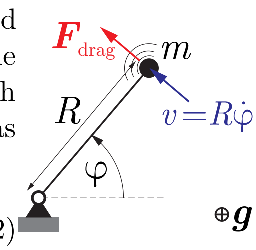
Let us further assume that a force F is applied at a distance L perpendicular to the *massless* support frame to introduce the external torque M_B, so $M_B = FL$.

The tangential force component F_φ accelerating (or decelerating) the particle's circular motion can be obtained from AMB on the massless support frame, for which the equilibrium of torques yields $FL = F_\varphi R$. Overall, we obtain

$$FL = F_\varphi R = mR^2\ddot{\varphi} \qquad \Leftrightarrow \qquad F_\varphi = mR\ddot{\varphi} = m\frac{\mathrm{d}}{\mathrm{d}t}(R\dot{\varphi}) = m\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} = ma_\varphi, \tag{1.141}$$

where v_φ and a_φ denote the tangential velocity and acceleration of the particle, respectively (and $\mathrm{d}v_\varphi/\mathrm{d}t = a_\varphi$ since $r = \text{const.}$). We see that AMB is indeed nothing else but an alternative statement of LMB (in fact, we did derive AMB from LMB). The force on the particle is related to its acceleration via its mass. Here, we convert the force into a torque and the constant of proportionality between torque and acceleration must then involve the lever arm. Finally, converting acceleration into angular acceleration – the moment of inertia I_B is nothing else but the resultant constant of proportionality between the torque and the *angular* acceleration (when rotating about a fixed point).

Example 1.16. Rotating particle in air



Consider a particle of mass m that is rotating at constant distance R around a fixed point. The surrounding air results in a viscous drag force opposing the particle's motion (assumed linearly proportional to the particle's velocity with viscosity $k > 0$). If gravity is acting perpendicular to the plane of rotation (as shown), then the only force acting on the particle is the drag force

$$\boldsymbol{F}_d = -k\boldsymbol{v} \qquad \Rightarrow \qquad \boldsymbol{M}_B = \boldsymbol{r}_{BP} \times \boldsymbol{F}_d = -Rkv\boldsymbol{e}_3. \tag{1.142}$$

For a particle rotating at constant radius R, we know that $v = v_t = R\dot{\varphi}$ (see Section 1.1.3), so that

$$\boldsymbol{M}_B = -kR^2\dot{\varphi}\boldsymbol{e}_3. \tag{1.143}$$

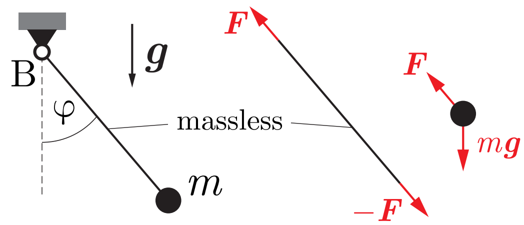
Finally, applying the balance of angular momentum for a 2D rotation at constant radius R yields

$$M_{\mathrm{B}} = mR^2\ddot{\varphi} \qquad \Rightarrow \qquad \ddot{\varphi} + \frac{k}{m}\dot{\varphi} = 0. \tag{1.144}$$

This last form is the so-called **equation of motion** of the particle, a differential equation to be solved for its time-dependent degree of freedom $\varphi(t)$, given appropriate initial conditions.

Example 1.17. Massless pendulum with an attached mass

We consider a particle of mass m attached to a rigid massless string of length R subject to gravity. Here, angular momentum balance is preferable since it avoids having to introduce and calculate the force $\boldsymbol{F}$ in the string. Using angular momentum balance around B avoids the force in the string and yields



$$mR^2\ddot{\varphi} = -mgR\sin\varphi. \qquad \Leftrightarrow \qquad \ddot{\varphi} + \frac{g}{R}\sin\varphi = 0. \tag{1.145}$$

For small amplitudes ($|\varphi| \ll 1$), we may use a Taylor expansion about $\varphi = 0$, viz.

$$\sin(0+\varphi) = \sin 0 + \varphi\cos 0 - \frac{\varphi^2}{2}\sin 0 + O(|\varphi|^3) = \varphi + O(|\varphi|^3) \tag{1.146}$$

to simplify the equation above of motion into a linear ordinary differential equation (ODE):

$$\ddot{\varphi} + \frac{g}{R}\varphi \approx 0. \tag{1.147}$$

It is important to keep in mind that this last step is valid only for small angles $|\varphi| \ll 1$.

Note that the same equation of motion, of course, could also have been obtained from linear momentum balance in the tangential direction:

$$ma_\varphi = -mg\sin\varphi \qquad \text{and} \qquad a_\varphi = R\ddot{\varphi} \qquad \Rightarrow \qquad mR\ddot{\varphi} = -mg\sin\varphi, \tag{1.148}$$

which again leads to (1.145).

Finally, if the pendulum experiences drag due to the surrounding medium, then we may introduce the same drag force as in Example 1.16, leading to the equation of motion of a particle pendulum subjected to gravity and air drag:

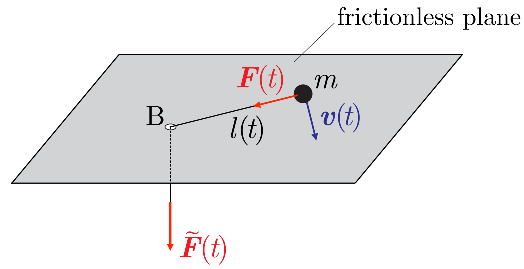
$$\ddot{\varphi} + \frac{k}{m}\dot{\varphi} + \frac{g}{R}\sin\varphi = 0 \qquad \Rightarrow \qquad \ddot{\varphi} + \frac{k}{m}\dot{\varphi} + \frac{g}{R}\varphi \approx 0. \tag{1.149}$$

This is a good example of the assumption of *massless bodies* (such as the string here). As discussed before, for a massless body all inertial terms are neglected, so that one recovers the static equilibrium equations

$$\text{if a particle or body is massless} \qquad \Rightarrow \qquad \sum_i \boldsymbol{F}_i = \boldsymbol{0}, \qquad \boldsymbol{M}_{\mathrm{B}} = \boldsymbol{0}. \tag{1.150}$$

Example 1.18. Circling mass on a string pulled inward in 2D

A particle of mass m is circling around a hole, attached to a taught string of changing but known length $l(t)$. The string is held steady at a length $l(0)$ until time $t = 0$. It is then pulled inward. The force needed to pull the particle inward is unknown (it involves, among others, frictional forces in the hole). Known are the initial length $l(0)$, the initial particle speed $v(0)$, and the changing length $l(t)$. Assuming that the particle is sliding on a frictionless ground, what is the particle speed $v(t)$?



Using the balance of angular momentum around the hole (point B) is ideal here, since there are no applied torques to consider in this case (neglecting frictional forces), i.e., $\boldsymbol{M}_{\mathrm{B}} = \boldsymbol{0}$. Since also $\boldsymbol{v}_{\mathrm{B}} = \boldsymbol{0}$, we have

$$\dot{\boldsymbol{H}}_{\mathrm{B}} + \boldsymbol{v}_{\mathrm{B}} \times \boldsymbol{P} = \boldsymbol{M}_{\mathrm{B}} \qquad \Rightarrow \qquad \dot{\boldsymbol{H}}_{\mathrm{B}} = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{H}_{\mathrm{B}} = \text{const.} \tag{1.151}$$

i.e., *angular momentum is conserved*. Inserting the definition of the angular momentum vector leads to

$$\boldsymbol{H}_{\mathrm{B}} = \boldsymbol{r}_{\mathrm{BP}} \times \boldsymbol{P} = \boldsymbol{r}_{\mathrm{BP}} \times m\boldsymbol{v}_{\mathrm{P}} \qquad \Rightarrow \qquad \boldsymbol{r}_{\mathrm{BP}}(t) \times m\boldsymbol{v}_{\mathrm{P}}(t) = \text{const.} \tag{1.152}$$

Using polar coordinates and $\boldsymbol{r}_{\mathrm{BP}}(t) = l(t)\boldsymbol{e}_r(t)$, we realize that

$$\boldsymbol{r}_{\mathrm{BP}} \times \boldsymbol{v}_{\mathrm{P}} = l(t)\boldsymbol{e}_r(t) \times [v_r(t)\boldsymbol{e}_r(t) + v_\varphi(t)\boldsymbol{e}_\varphi(t)] = l(t)v_\varphi(t)\boldsymbol{e}_r(t) \times \boldsymbol{e}_\varphi(t) = l(t)v_\varphi(t)\boldsymbol{e}_3 \tag{1.153}$$

at all times t (with $\boldsymbol{e}_3$ being the out-of-plane unit vector). Therefore, the conservation of angular momentum, assuming a constant mass m, yields

$$\boldsymbol{r}_{\mathrm{BP}}(0) \times \boldsymbol{v}(0) = \boldsymbol{r}_{\mathrm{BP}}(t) \times \boldsymbol{v}(t) \quad \Rightarrow \quad l(0)\,v_t(0) = l(t)\,v_t(t) \quad \Leftrightarrow \quad v_t(t) = \frac{l(0)}{l(t)}v_t(0). \tag{1.154}$$

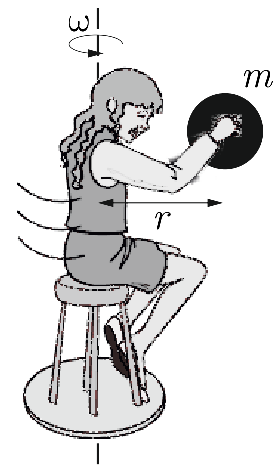
Finally, exploiting that $v_t(t) = l(t)\omega(t)$, the angular velocity $\omega(t) = \dot{\varphi}(t)$ scales as

$$\omega(t) = \left(\frac{l(0)}{l(t)}\right)^2 \omega(0) \tag{1.155}$$

Note that in this example energy is not necessarily conserved and the work done by the pulling force is unknown, so the work–energy balance does not help in this scenario.

Example 1.19. Heavy mass held on a rotating chair

The conservation of angular momentum is most easily visualized on a spinning chair. (Try this out at home, if you have a rotating desk chair!) Let us consider an approximately massless person sitting on a rotating chair that is spinning at a constant angular velocity ω_0 (without any frictional losses or air drag, we may assume that no torques act on the person, so they would spin at ω_0 forever by angular momentum conservation – analogous to a particle forever continuing at constant velocity if no forces act by linear momentum conservation). Let the person hold in their hands a heavy object, approximated as a particle of mass m, at a distance r_0 from the axis of rotation. How does the angular velocity change, if the distance is changed from r_0 to r?



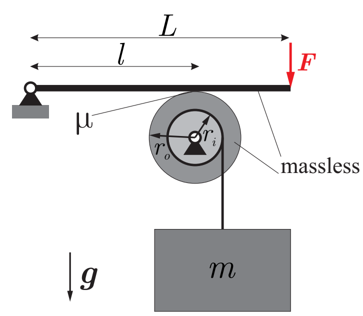
Exploiting the conservation of angular momentum (in the absence of any torques that could accelerate or decelerate the rotation), we know from Example 1.18 that

$$r_0^2\omega_0 = r^2\omega \qquad \Rightarrow \qquad \omega = \left(\frac{r_0}{r}\right)^2 \omega_0. \tag{1.156}$$

Therefore, bringing m closer to the body (i.e., closer to the axis of rotation) will increase the angular velocity, while moving it further away will decrease the angular velocity. Note that since a force is needed to move the mass m (and the force does work on the mass), *energy is not conserved.*

Example 1.20. Handle brake

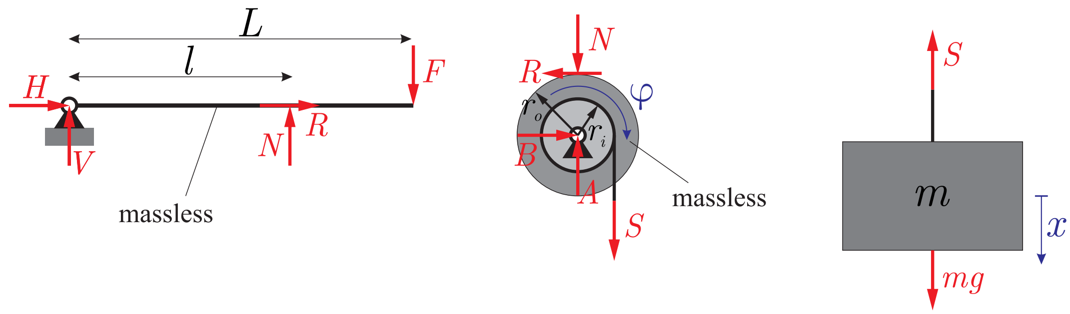
The shown construction is used to control the vertical motion of a heavy particle of mass m. The particle is attached to a rigid, massless rope coiled around a massless disk (inner radius r_i, outer radius r_o). The downward motion of the heavy particle can be controlled by applying a force F to the shown massless handle brake, resulting in kinetic friction (coefficient μ) between the brake lever and disk.



What force F must be applied to let the particle move downwards at a constant speed v_c?

This is a classical problem which in statics was solved by drawing free-body diagrams and applying the equilibrium relations of vanishing net forces and torques on each subsystem. Here, we proceed in exactly the same manner – but we replace the static equilibrium equations by the balance equations of linear and angular momentum.

We start with free-body diagrams of all three system components, the heavy particle, the disk, and the brake, as shown below. For each system we can formulate a balance law of convenience: LMB for the particle m in the shown x-direction, AMB for the disk to avoid the support reactions A and B at the center, and again AMB for the brake lever for the same reason (avoiding the support reactions H and V). Note that for each *massless* component we do not need to account for inertial effects.



For the *massless* lever arm, static equilibrium of torques gives

$$0 = Nl - FL \qquad \Rightarrow \qquad N = F\frac{L}{l}. \tag{1.157}$$

Note that friction always counteracts the motion, so that – knowing the mass will move downwards and hence the disk will rotate clockwise, we drew the friction force R in the above free-body diagram in the correct direction to counteract the motion of the rotating disk. Similarly, we know that N is positive in the shown direction, since a compressive force pressed the lever against the disk. Therefore, we may write, assuming kinetic friction between the lever and the disk,

$$R = \mu N = \mu F\frac{L}{l}, \tag{1.158}$$

without further worrying about the directions of the forces.

For the *massless* disk we obtain from static equilibrium of torques

$$0 = Sr_i - Rr_o, \qquad \Rightarrow \qquad S = R\frac{r_o}{r_i} = \mu F\frac{L}{l}\frac{r_o}{r_i}. \tag{1.159}$$

Finally, for the heavy particle, linear momentum balance reveals

$$m\ddot{x} = mg - S = mg - \mu F\frac{L}{l}\frac{r_o}{r_i} \qquad \Leftrightarrow \qquad \ddot{x} = g - \mu\frac{F}{m}\frac{L}{l}\frac{r_o}{r_i}. \tag{1.160}$$

Note that, for convenience, we here define the x-coordinate to point downwards (so that, when formulating $m\ddot{x} = \sum_i F_i$ all forces pointing into the x-direction downward are positive, while upward forces are appear as negative). This choice is arbitrary and we could as well have written $m\ddot{x} = -mg + S$ with x defined as positive upwards.

For the mass to move downwards at a *constant velocity* v_c, we need its acceleration to vanish, i.e.,

$$\ddot{x} = 0 \qquad \Leftrightarrow \qquad g - \mu\frac{F}{m}\frac{L}{l}\frac{r_o}{r_i} = 0 \qquad \Rightarrow \qquad F = \frac{mg}{\mu}\frac{l}{L}\frac{r_i}{r_o} = F_{\text{crit}}. \tag{1.161}$$

However, this force cannot be applied right away. Otherwise, the mass would never start to move. If the system is first considered without any applied force ($F = 0$), then the particle is falling freely with $\ddot{x} = g$. Then, the motion is described by

$$\ddot{x}(t) = g \quad \Rightarrow \quad \dot{x}(t) = gt \quad \Rightarrow \quad x(t) = \frac{gt^2}{2}. \tag{1.162}$$

The velocity v_c is reached after a time period t_c:

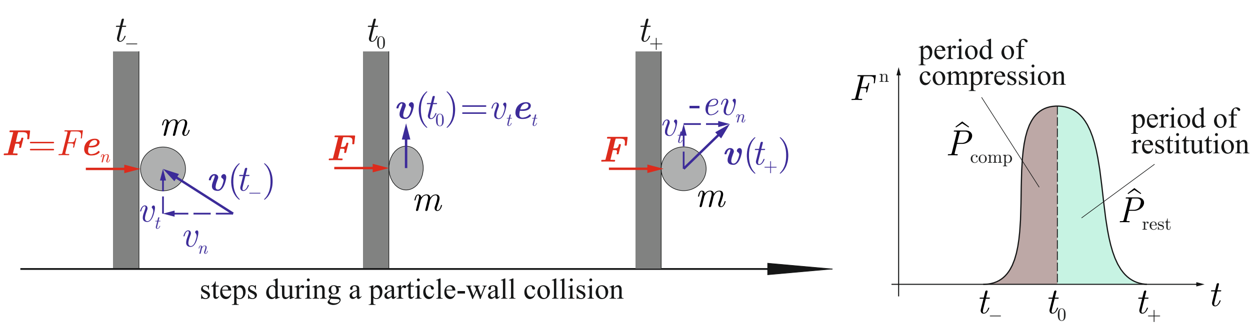
$$\dot{x}(t_c) = v_c \qquad \Leftrightarrow \qquad t_c = \frac{v_c}{g}. \tag{1.163}$$

Therefore, a constant downward speed of v_c is achieved by applying the force F_{crit} from time t_c on (neglecting any transient effects).

1.2.4 Particle impact

Particle **impact** is an interesting example of applying the momentum balance laws derived so far. It does not require much new physics, but it is worth discussing this example in detail, as it will lead us to simple relations about the particle speeds before and after impact.

Let us consider a particle of mass m, which is impacting a rigid wall with an initial velocity $\boldsymbol{v}_-$. The wall is significantly heavier and larger than the particle, so we may treat it as a rigid object with constant zero velocity. The **collision** event between the particle and the wall hence results in a change in the velocity of the particle, which we would like to analyze. During the collision, the wall exerts a time-dependent force $\boldsymbol{F}(t)$ onto the particle (initially being zero until the particle first touches the wall at time t_-, then increasing to a maximum, and finally decaying to zero again as the particle separates from the wall after impact at time $t_+ > t_-$, as shown below).



Since the balance equation of linear momentum must hold at every instance of time during the collision, we may integrate it over the time period of the collision, denoted by $[t_-, t_+]$, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}) = \boldsymbol{F}(t) \quad \Leftrightarrow \quad \mathrm{d}(m\boldsymbol{v}) = \boldsymbol{F}(t)\,\mathrm{d}t \qquad \Rightarrow \qquad \int_{m\boldsymbol{v}(t_-)}^{m\boldsymbol{v}(t_+)} \mathrm{d}(m\boldsymbol{v}) = \int_{t_-}^{t_+} \boldsymbol{F}(t)\,\mathrm{d}t. \tag{1.164}$$

If we denote the integrated force received by the particle by the **impulsive force $\hat{\boldsymbol{P}}$**, then

$$m\boldsymbol{v}(t_+) - m\boldsymbol{v}(t_-) = \hat{\boldsymbol{P}} = \int_{t_+}^{t_-} \boldsymbol{F}(t)\,\mathrm{d}t. \tag{1.165}$$

Note that $\hat{\boldsymbol{P}}$ $[Ns]$ does not have the units of a force, but it is a force integrated over time (so, roughly speaking, the 'accumulated force' applied to the particle during a specific period).

In order to better understand the nature of this impulsive force, we decompose the particle collision into a **period of compression** $[t_-, t_0)$ and a **period of restitution** $[t_0, t_+]$, as schematically shown below, such that

$$\hat{\boldsymbol{P}} = \hat{\boldsymbol{P}}_{\text{comp}} + \hat{\boldsymbol{P}}_{\text{res}}, \qquad \hat{\boldsymbol{P}}_{\text{comp}} = \int_{t_-}^{t_0} \boldsymbol{F}(t)\,\mathrm{d}t, \quad \hat{\boldsymbol{P}}_{\text{rest}} = \int_{t_0}^{t_+} \boldsymbol{F}(t)\,\mathrm{d}t, \tag{1.166}$$

where time t_0 is taken as the time at which the particle velocity normal to the wall vanishes before changing its sign (the normal velocity changes direction, so there must exist such time t_0 at which the normal velocity component is zero).

Let us assume that the wall is **frictionless**, so we know that there is no frictional force and, consequently, the force on the particle must be acting *perpendicular to the wall* at all times. For convenience, we introduce a local coordinate system such that in 2D the basis vectors $\{\boldsymbol{e}_\text{n}, \boldsymbol{e}_\text{t}\}$ point in the direction normal and tangential to the wall, respectively, and $\boldsymbol{e}_n \perp \boldsymbol{e}_t$. This allows us to write $\boldsymbol{F}(t) = F(t)\boldsymbol{e}_n$ and consequently

$$\hat{\boldsymbol{P}} = \hat{P}\boldsymbol{e}_\text{n} \qquad \text{and} \qquad \hat{\boldsymbol{P}}_{\text{comp}} = \hat{P}_{\text{comp}}\boldsymbol{e}_n, \qquad \hat{\boldsymbol{P}}_{\text{rest}} = \hat{P}_{\text{rest}}\boldsymbol{e}_n. \tag{1.167}$$

Now formulating the momentum balance (1.165) for each of the two phases leads to

$$m\boldsymbol{v}(t_0) - m\boldsymbol{v}(t_-) = \hat{P}_{\text{comp}}\boldsymbol{e}_n, \qquad m\boldsymbol{v}(t_+) - m\boldsymbol{v}(t_0) = \hat{P}_{\text{rest}}\boldsymbol{e}_n. \tag{1.168}$$

Inserting $\boldsymbol{v} = v_n\boldsymbol{e}_n + v_t\boldsymbol{e}_t$ and exploiting that $\boldsymbol{e}_n \cdot \boldsymbol{e}_t = 0$ lets us conclude that the tangential velocity component must stay constant during the entire collision:

$$\boxed{v_t(t_-) = v_t(t_0) = v_t(t_+)} \tag{1.169}$$

Furthermore, the normal components in (1.168) must satisfy

$$mv_n(t_0) - mv_n(t_-) = \hat{P}_{\text{comp}}, \qquad mv_n(t_+) - mv_n(t_0) = \hat{P}_{\text{rest}}. \tag{1.170}$$

Recall that we had defined t_0 such that $v_n(t_0) = 0$ (since t_0 marks the turning point for the particle velocity during the collision). and was the definition for time t_0. (1.170) thus reduces to

$$-mv_n(t_-) = \hat{P}_{\text{comp}}, \qquad mv_n(t_+) = \hat{P}_{\text{rest}}. \tag{1.171}$$

These are two equations with three unknowns, $\hat{P}_{\text{comp}}$, $\hat{P}_{\text{rest}}$, and $v_n(t_+)$ (while we assumed that the initial particle velocity $\boldsymbol{v}(t_-) = \boldsymbol{v}_-$ is known, so that $v_n(t_-) = \boldsymbol{v}(t_-) \cdot \boldsymbol{e}_n$ is known as well).

Consequently, one additional equation is required to solve the above system of equations. To this end, we introduce the **coefficient of restitution** e as

$$e = \frac{\hat{P}_{\text{rest}}}{\hat{P}_{\text{comp}}} \qquad \Rightarrow \qquad \hat{P}_{\text{rest}} = e\,\hat{P}_{\text{comp}}, \qquad e \in [0,1]. \tag{1.172}$$

e characterizes how much of the impulsive force "injected into" the particle during the initial compression phase of the collision is available for recovery of the particle and separation of particle and wall during the subsequent restitution phase. If $e = 1$, the impact is **purely elastic** and the particle is fully restored during the restitution phase and separates from the wall. For the **perfectly plastic** case $e = 0$, no restitution occurs and the particle remains at the wall (there is no impulsive force driving the particle to separate from the wall). For intermediate cases $0 < e < 1$, often referred to as **partially elastic**, the particle is only partially restored.

Inserting (1.172) into (1.170) finally yields

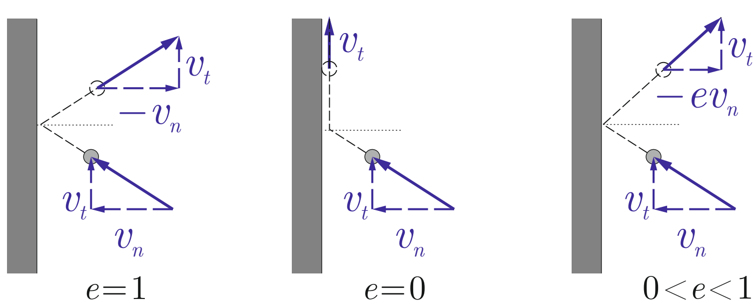
$$-mv_n(t_-) = \hat{P}_{\text{comp}}, \quad mv_n(t_+) = \hat{P}_{\text{rest}} = e\,\hat{P}_{\text{comp}} \qquad \Rightarrow \qquad \boxed{v_n(t_+) = -e\,v_n(t_-)} \tag{1.173}$$

As a consequence, the particle's velocity changes sign (as could have been expected) and it also changes its magnitude unless $e = 1$. Overall, the particle velocity during a *frictionless impact* thus changes according to

$$\boldsymbol{v}(t_-) = v_n\boldsymbol{e}_n + v_t\boldsymbol{e}_t \qquad \Rightarrow \qquad \boldsymbol{v}(t_+) = -ev_n\boldsymbol{e}_n + v_t\boldsymbol{e}_t. \tag{1.174}$$

The coefficient of restitution e generally depends on the particle's material properties. Linear elastic materials have $e = 1$ (e.g., a rubbery bouncy ball), whereas $0 \le e < 1$ implies that some of the kinetic energy of the particle before impact is used to permanently deform or change the particle, thus dissipating energy (e.g., a hacky sack is an example for $e \approx 0$, changing its shape and showing little to no rebound).

Let us finally investigate the change in angle of the particle velocity during impact. While the tangential component of the velocity vector remains the same for a frictionless impact, the normal component changes depending on the value of e. As shown schematically below, this affects the direction of the particle's flight in the following manner. For a purely elastic impact, the angle of the incoming particle (measured against the normal) is also the angle of the rebounding particle (with the normal) since $|v_t(t_-)/v_n(t_-)| = |v_t(t_+)/v_n(t_+)|$. In case of a perfectly plastic collision, the particle retains no normal component after impact ($v_n(t_+) = 0$) and hence slides down the wall with $\boldsymbol{v}(t_+) = v_t\boldsymbol{e}_t$. For any case $0 < e < 1$, the particle rebounds with an angle between the incoming angle and the vertical wall since $|v_t(t_+)/v_n(t_+)| = |v_t(t_-)/ev_n(t_-)| \ge |v_t(t_-)/v_n(t_-)|$.

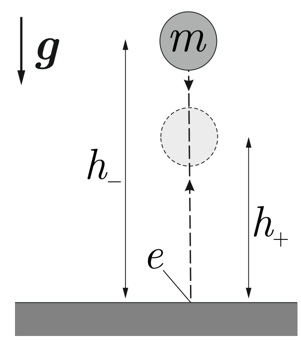


As a concluding remark, notice that the *kinetic energy of the particle* always changes during impact, unless the collision is purely elastic, since (with $0 \leq e \leq 1$)

$$T(t_-) = \frac{m}{2}\left[v_n^2(t_-) + v_t^2(t_-)\right] \qquad \text{and} \qquad T(t_+) = \frac{m}{2}\left[\big(ev_n(t_-)\big)^2 + v_t^2(t_+)\right] \leq T(t_-). \quad (1.175)$$

Example 1.21. Particle dropped vertically onto a rigid ground

Consider a particle of mass m dropped vertically onto a rigid ground, released from rest at an initial height h_- and rebounding to a final height h_+. Let us find out how we can use the difference in height (and later the rebound period) to determine the coefficient of restitution e.



Since the particle is dropped in the normal direction onto the ground, only normal velocity components are to be considered and $\boldsymbol{v} \cdot \boldsymbol{e}_t = v_t = 0$ at all times t. Except for the purely elastic case (for which $e = 1$ and the particle bounces back to the initial height), we cannot use the conservation of energy for the entire experiment. However, we can use it for each flight of the particle (from release to impact, and from rebound to the final height).

This gives for the flight from release at h_- to the ground with impact velocity $v_- = v(t_-)$

$$mgh_- = \frac{m}{2}v_-^2 \qquad \Rightarrow \qquad |v_-| = \sqrt{2gh_-} \quad (1.176)$$

and for the flight from rebound with velocity $v_+ = v(t_+)$ to the maximum rebound height h_+

$$\frac{m}{2}v_+^2 = mgh_+ \qquad \Rightarrow \qquad |v_+| = \sqrt{2gh_+}. \quad (1.177)$$

Since v_- and v_+ are the normal velocities immediately before and after the impact, they are related by the coefficient of restitution:

$$|v_+| = e|v_-| \quad \Rightarrow \quad \sqrt{2gh_+} = e\sqrt{2gh_-} \qquad \Leftrightarrow \qquad e = \sqrt{\frac{h_+}{h_-}}. \quad (1.178)$$

Hence, the coefficient of restitution can be obtained from the heights before and after impact.

As a variation, let us find out how the coefficient of restitution can alternatively be determined from the *flight times* of the particle in the air between two ground contacts. For a particle of mass m, flying upwards with some initial velocity $\boldsymbol{v} = v_0\boldsymbol{e}_2$ (and only gravity is acting on the particle, so $\boldsymbol{F} = -mg\boldsymbol{e}_2$), the flight starting from $x_2 = 0$ is described by

$$m\ddot{x}_2(t) = -mg \qquad \Rightarrow \qquad \dot{x}_2(t) = -gt + v_0 \qquad \Rightarrow \qquad x_2(t) = -\frac{gt^2}{2} + v_0 t. \quad (1.179)$$

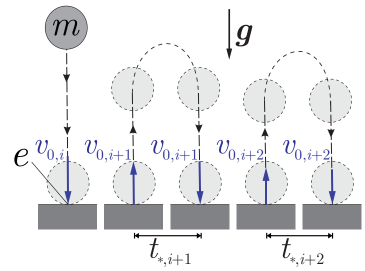
The time t_* from launching the particle with v_0 to reaching the ground again is therefore given by

$$x_2(t_*) = -\frac{gt_*^2}{2} + v_0 t_* = t_*\left(-\frac{gt_*}{2} + v_0\right) \stackrel{!}{=} 0 \qquad \Rightarrow \qquad t_* = \frac{2v_0}{g} \quad (\text{or } t_* = 0). \quad (1.180)$$

Furthermore, note that – by the conservation of energy during the particle flight – the particle speed at launch is the same as the particle speed when it reaches the ground again.

Now consider the particle of mass m dropped onto the rigid ground. As shown above, the particle reaches the ground with $|v_-|$ and leaves the ground again with $|v_+| = e|v_-|$. By the above energy conservation argument, the particle again has velocity $e|v_-|$ when it reaches the ground the next time. Hence, the upward particle velocity after rebound varies from one rebound to the next according to

$$v_{0,i+1} = ev_{0,i}, \tag{1.181}$$



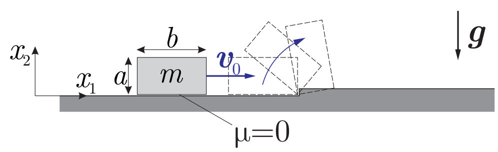
By inserting the latter into (1.180), we obtain the following relation for the duration of the particle flight between two subsequent rebounds:

$$\frac{t_{*,i+1}}{t_{*,i}} = \frac{v_{0,i+1}}{v_{0,i}} = e. \tag{1.182}$$

In conclusion, we have found two ways to *measure the coefficient of restitution* for a particle–wall impact. We can either measure the change in height or the change in flight times between rebounds to find the coefficient of restitution e. Note that the above is independent of the particle mass m. Finally, recall that e is a material property that defines (losely speaking) how much of the kinetic energy is consumed by a permanent change of the particle by, e.g., plasticity or damage.

Example 1.22. Car roll-over

A car of mass m and cross-section $a \times b$ slides across a frictionless ground horizontally with an initial velocity v_0 against a shallow curb, resulting in an impulsive collision with the curb. At what velocity v_0 does the car start to roll over?



We need to make a few approximations and assumptions to solve this problem with those methods available to us. To this end, let us approximate the car as a particle of mass m, located at the center of mass of the car. We assume that during the collision an impulsive force $\hat{\boldsymbol{F}}$ acts onto the car at the bottom of the curb, directed towards the center of mass (i.e., acting onto the particle). Since the curb is shallow, its height is negligible and we may assume that force $\boldsymbol{F}$ acts (and the care trips over) a fixed point B, as shown below.

We consider the balance of angular momentum around the curb at the fixed point B (so $\boldsymbol{v}_\mathrm{B} = \boldsymbol{0}$):

$$\boldsymbol{M}_\mathrm{B} = \dot{\boldsymbol{H}}_\mathrm{B} + \boldsymbol{v}_\mathrm{B} \times \boldsymbol{P} \qquad \Rightarrow \qquad \boldsymbol{M}_\mathrm{B} = \dot{\boldsymbol{H}}_\mathrm{B}. \tag{1.183}$$

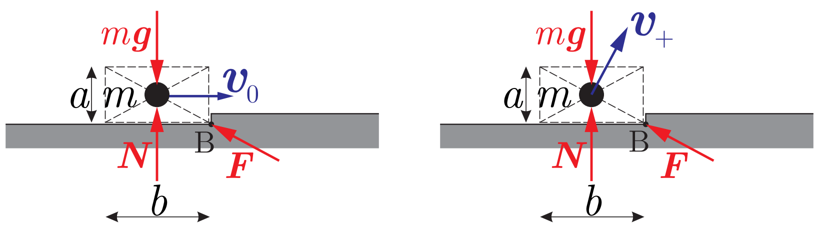
For a short collision time period $[t_-, t_+]$ the above can be integrated as

$$\boldsymbol{H}_\mathrm{B}(t_+) - \boldsymbol{H}_\mathrm{B}(t_-) = \int_{t_-}^{t_+} \boldsymbol{M}_\mathrm{B}\,\mathrm{d}t \qquad \text{and} \qquad \boldsymbol{M}_\mathrm{B} = \boldsymbol{r}_\mathrm{BP} \times (\boldsymbol{N} - m\boldsymbol{g}). \tag{1.184}$$

Note that this relation is analogous to (1.165), which was derived based on linear momentum balance during a collision. The impulsive force $\hat{\boldsymbol{F}}$ plays no role here since it goes through point B and hence produces no torque with respect to B.

We further assume that the time period of the collision is so short (and the impulsive force so dominant), that the normal and gravity forces remain approximately constant during $[t_-, t_+]$. This in turn leads to the conclusion that the angular momentum *remains constant* during the collision in this problem:

$$\boldsymbol{H}_\mathrm{B}(t_+) - \boldsymbol{H}_\mathrm{B}(t_-) = \int_{t_-}^{t_+} \boldsymbol{M}_\mathrm{B}\,\mathrm{d}t = \int_{t_-}^{t_+} \boldsymbol{r}_\mathrm{BP} \times (\boldsymbol{N} - m\boldsymbol{g})\,\mathrm{d}t \approx \boldsymbol{0}. \tag{1.185}$$



before the collision

immediately after the collision

The angular momentum *before the collision* is

$$\boldsymbol{H}_\mathrm{B}(t_-) = \boldsymbol{r}_\mathrm{BP} \times m\boldsymbol{v}_0 = \left(-\frac{b}{2}\boldsymbol{e}_1 + \frac{a}{2}\boldsymbol{e}_2\right) \times mv_0\boldsymbol{e}_1 = -mv_0\frac{a}{2}\boldsymbol{e}_3, \tag{1.186}$$

and the angular momentum immediately *after the collision* (when the car is assumed to have a speed v_+ rotating about point B with radius $\sqrt{a^2+b^2}/2$, as shown above) is

$$\boldsymbol{H}_\mathrm{B}(t_+) = \boldsymbol{r}_\mathrm{BP} \times m\boldsymbol{v}_+ = -\frac{\sqrt{a^2+b^2}}{2}mv_+\boldsymbol{e}_3. \tag{1.187}$$

Conservation of angular momentum thus yields the velocity v_+ immediately after the impact as

$$-mv_0\frac{a}{2}\boldsymbol{e}_3 \stackrel{!}{=} -\frac{\sqrt{a^2+b^2}}{2}mv_+\boldsymbol{e}_3 \qquad \Rightarrow \qquad v_+ = v_0\frac{a}{\sqrt{a^2+b^2}}. \tag{1.188}$$

To *determine whether or not the car rolls over*, we need to consider the car's motion after it hits the curb. We here assume for simplicity that, immediately following the collision, the car will perform a rotation around point B and the car remains attached to point B during the rotation (as shown below). The question is now, given an initial velocity V_+, how far will the car rotate? Under the action of gravity (and by the conservation of energy in this conservative system), the velocity will decrease as the car rolls over and the center of mass gains altitude. We thus determine if the center of mass has a non-zero speed when it has rotated right above the curb vertical above point B – if the velocity is positive, the car will continue to rotate onto the curb; if the velocity has gone to zero before that point, then the car will rotate back onto the street.

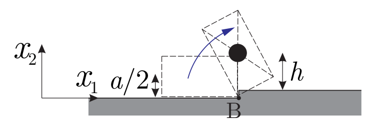
After the collision, the car's rotation is a conservative rigid-body motion, so that we may use the conservation of energy. Right after the collision we have

$$E(t_1) = V(t_1) + T(t_1) \qquad \text{with} \qquad V(t_1) = mg\frac{a}{2}, \quad T(t_1) = \frac{m}{2}v_+^2, \tag{1.189}$$

whereas the instance when the center of mass is right above point B (at an altitude of $h = \sqrt{a^2+b^2}/2$ above B) is characterized by

$$E(t_2) = V(t_2) + T(t_2) \qquad \text{with} \qquad V(t_2) = mgh = mg\frac{\sqrt{a^2+b^2}}{2}, \quad T(t_2) = \frac{m}{2}v_{\text{crit}}^2. \tag{1.190}$$

As discussed, a roll-over occurs if $v_{\text{crit}} \geq 0$, so we insert the limiting case $v_{\text{crit}} = 0$. Conservation of energy along with v_+ from (1.188) thus gives



critical state for roll-over

$$mg\frac{a}{2} + \frac{m}{2}v_+^2 = mg\frac{\sqrt{a^2+b^2}}{2} \qquad \text{with} \qquad v_+^2 = v_0^2\frac{a^2}{a^2+b^2}, \tag{1.191}$$

and solving for the critical initial velocity leads to

$$v_0 = \sqrt{g\left(\sqrt{a^2+b^2} - a\right)\left(1 + \frac{b^2}{a^2}\right)}. \tag{1.192}$$

We close by verifying two special cases: if $b \gg a$ (a very long and flat car), then $v_0^{\text{crit}} \to \infty$ so an extremely large initial speed v_0 is required for a roll-over. By contrast, for $a \gg b$ (a very high and thin car) $v_0^{\text{crit}} \to 0$, so a very tall and narrow car will roll over already at very low initial speeds.

1.3 Summary of Key Relations

At the end of our discussion of the dynamics of a single particle, we summarize all key relations – for a single particle of constant mass m and position $\boldsymbol{r}(t)$ – in the following box. This summary box (together with all other such summary boxes that we will add at the end of each topic that we discuss in the following) will grow into the *formula collection* to be used during exams.

kinematics of a single particle:

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}, \qquad \boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$$

velocity and acceleration components in **polar coordinates** (r, φ):

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\varphi}\boldsymbol{e}_\varphi, \qquad \boldsymbol{a} = (\ddot{r} - r\dot{\varphi}^2)\boldsymbol{e}_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\boldsymbol{e}_\varphi$$

velocity and acceleration components using the **space curve** description:

$$\boldsymbol{v} = \dot{s}\boldsymbol{e}_t, \qquad \boldsymbol{a} = \ddot{s}\boldsymbol{e}_t + \frac{v^2}{\rho}\boldsymbol{e}_n \qquad \text{with} \qquad \boldsymbol{e}_t = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\boldsymbol{v}}{v}, \quad \boldsymbol{e}_n = \frac{\dot{\boldsymbol{e}}_t}{|\dot{\boldsymbol{e}}_t|} = \frac{\rho}{v}\dot{\boldsymbol{e}}_t$$

balance of linear momentum for a particle of mass m:

$$\sum_i \boldsymbol{F}_i = \dot{\boldsymbol{P}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v})$$

kinetic friction:

$$\boldsymbol{R} = -\mu|\boldsymbol{N}|\frac{\boldsymbol{v}}{|\boldsymbol{v}|}, \qquad |\boldsymbol{R}| = \mu|\boldsymbol{N}|$$

work–energy balance for a particle of constant mass m:

$$T(t_2) - T(t_1) = W_{12}, \qquad T(t) = \frac{1}{2}m|\boldsymbol{v}(t)|^2, \qquad W_{12} = \sum_i \int_{t_1}^{t_2} \boldsymbol{F}_i \cdot \boldsymbol{v}\,\mathrm{d}t = \sum_i \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}$$

for a **conservative force**:

$$\boldsymbol{F} = -\frac{\mathrm{d}V}{\mathrm{d}\boldsymbol{r}} \qquad \Rightarrow \qquad W_{12} = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2)$$

conservation of energy for a conservative system:

$$T + V = \text{const.}$$

balance of angular momentum of a particle with respect to point B:

$$\boldsymbol{M}_\mathrm{B} = \dot{\boldsymbol{H}}_\mathrm{B} + \boldsymbol{v}_\mathrm{B} \times \boldsymbol{P} \qquad \text{with} \qquad \boldsymbol{H}_\mathrm{B} = \boldsymbol{r}_\mathrm{BP} \times \boldsymbol{P}, \quad \boldsymbol{P} = m\boldsymbol{v}_\mathrm{P}$$

special case of a **rotation in 2D** around a fixed point B at a distance R:

$$M_\mathrm{B} = I_\mathrm{B}\ddot{\varphi} \qquad \text{with} \qquad I_\mathrm{B} = mR^2$$

particle **collision** with a **frictionless rigid wall**:

$$v_n(t_+) = -e\,v_n(t_-), \qquad v_t(t_+) = v_t(t_-)$$

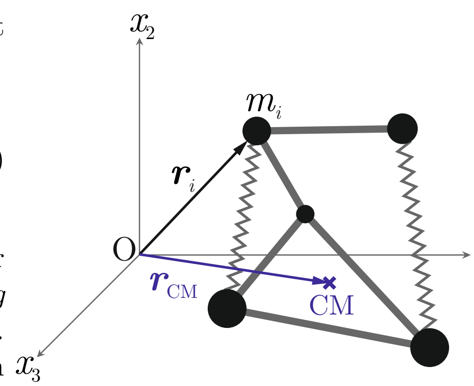
2 Dynamics of Systems of Particles

Having discussed the dynamics of a single particle, we proceed to consider a system of n particles at positions $\boldsymbol{r}_i(t)$, (where $i = 1, \ldots, n$), described in a Cartesian, inertial reference frame. Systems of multiple particles are not only of importance in many practical situations, but the relations derived here will also form the basis for our subsequent discussion of the dynamics of bodies of finite size. As for a single particle, we first discuss the kinematics and then the kinetics of systems of particles.

2.1 Kinematics

We begin by defining the **total mass** M and the position of the **center of mass** $\boldsymbol{r}_{\text{CM}}(t)$ of a system of n particles at positions $\boldsymbol{r}_i(t)$ and with masses m_i (for $i = 1, \ldots, n$) as

$$M = \sum_{i=1}^{n} m_i, \qquad \boldsymbol{r}_{\text{CM}}(t) = \frac{1}{M}\sum_{i=1}^{n} m_i \boldsymbol{r}_i(t) \tag{2.1}$$



Note that, while the total mass M remains constant over time, the center of mass $\boldsymbol{r}_{\text{CM}}(t)$ is *instantaneously changing* based on the changing positions of the individual particles. Also, the center of mass generally does not coincide with the location of any particle.

In an unconstrained system, all particles can move freely, so the system has a total of $d \times n$ DOFs in d dimensions[8]. In most mechanical systems, particles are connected by means of, e.g., rigid rod or stretchable springs. Like for a single particle, we can impose constraints on the particle motion and, in particular, we can constrain the motion of particles relative to each other by introducing **kinematic constraint**s (as already discussed for a single particle in Section 1.1.2).

An important kinematic constraint is a **rigid** link between particles i and j, which enforces that $|\boldsymbol{r}_i(t) - \boldsymbol{r}_j(t)| = \text{const.}$ for all times t. If all particles are rigidly connected, i.e., if $|\boldsymbol{r}_i(t) - \boldsymbol{r}_j(t)| = \text{const.}$ for all $i, j = 1, \ldots, n$ at all times t, then we speak of a **rigid system** of particles. In simply terms, in a rigid system the distances between all particles remain constant.

A rigid link $|\boldsymbol{r}_i - \boldsymbol{r}_j| = \text{const.}$ is an example of a so-called a *holonomic constraint*, which can most generally be expressed as

$$f(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n, t) = 0. \tag{2.2}$$

That is, a kinematic constraint which depends only on the particle positions and time is called a **holonomic constraint**. We further say two constraints are *independent*, if the two relations (2.2) associated with the two constraints are linearly independent.

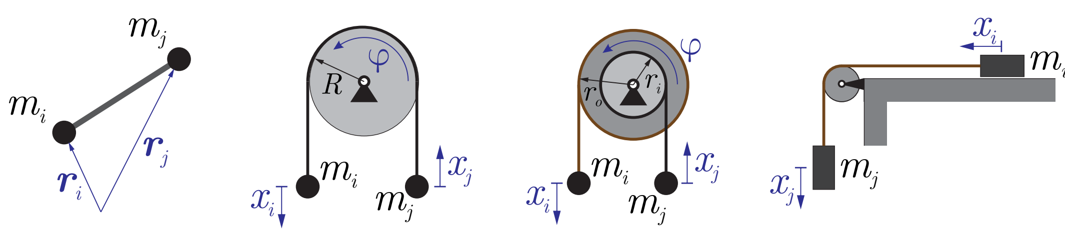
[8]For us, all particles in a particle system are *distinguishable* (unlike, e.g., in atomic particle ensembles). Therefore, the total number of DOFs is $d \times n$.

In an extension of our observation for a single particle, we state:

$$\textbf{If a system of } n \textbf{ particles has } r \textbf{ independent holonomic constraints, then the system in } d \textbf{ dimensions has } (n \times d - r) \textbf{ degrees of freedom.} \tag{2.3}$$

As already discussed for a single particle, each constraint produces a **constraint force** (or constraint forces): in order to maintain the kinematic constraint, forces are required (and shall not be forgotten when drawing free-body diagrams). Summarized below are four examples of holonomic kinematic constraints.

Example 2.1. Kinematic constraints



(a) **(b)** **(c)** **(d)**

(a) A **rigid link** (e.g., a rigid bar) is a kinematic constraint defined by $|\boldsymbol{r}_i(t) - \boldsymbol{r}_j(t)| = \text{const}$. The constraint results in a force in the bar and hence in constraint forces acting onto the two particles parallel to the bar, which prevent the particles from increasing or decreasing their distance. In 3D this system has $2 \times 3 - 1 = 5$ DOFs.

(b) Two particles connected by an **inextensible rope** (i.e., a rope of constant length) also adhere to a kinematic constraint. If we assume that the particles move only vertically, then it is convenient to introduce only vertical DOFs, such as x_i and x_j shown in the graphic. In principle, we can introduce those arbitrarily up or down, and the rotation angle of the pulley clockwise or counterclockwise. It is oftentimes convenient to introduce them in a fashion that is 'kinematically reasonable', as done here (if the left particle moves down, the pulley rotates counter-clockwise, and the right particle moves up). Yet, this is not required.

In the shown example of two particles connected by a rope over a pulley of radius R, the two vertical positions of particles i and j and the rotation angle of the pulley are connected via

$$x_i(t) = x_j(t) = R\,\varphi(t). \tag{2.4}$$

The constraint results in a force in the rope, leading to forces acting onto the two particles (and the pulley). It is important to *define the constraints consistently with the DOFs*. In example **(b)** shown above, x_i points downwards, while x_j points upwards; therefore $x_i = x_j$. If we instead chose both x_i and x_j to point downwards, then would we instead have $x_i = -x_j$.

When we consider that both particles can only move vertically, then each particle has 1 DOF, so that with 1 constraint the system has only a single DOF ($2 \times 1 - 1 = 1$).

(c) Two particles connected to two **inextensible ropes** wrapped around a rigid cylinder with inner and outer radii r_i and r_o, respectively, satisfy the kinematic constraints (again, using the specific DOF definitions of the above schematic)

$$x_i(t) = r_o\varphi(t), \qquad x_j(t) = r_i\varphi(t) \qquad \Rightarrow \qquad x_i(t) = x_j(t)\frac{r_o}{r_i}. \tag{2.5}$$

Again, when we consider that both particles can only move vertically, then the system has only a single DOF.

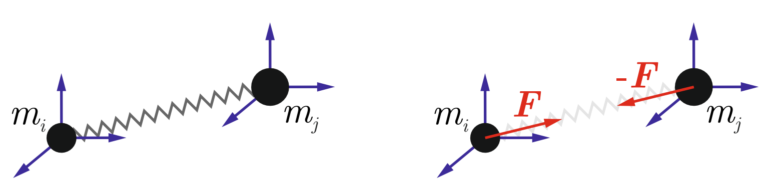
(d) The last schematic shows two particles that are connected by an inextensible rope, as in case **(b)**. Therefore, they must satisfy the kinematic constraint

$$x_i(t) = x_j(t), \tag{2.6}$$

when we introduce the DOFs conveniently, as shown above. That is, rather than introducing horizontal and vertical components for each particle position, we exploit that we know their direction motion, so that x_i and x_j can conveniently be introduced as shown to admit the simple constraint $x_i = x_j$. Of course, one could also introduce alternative DOFs and change the constraint accordingly. As before, this system has only a single DOF.

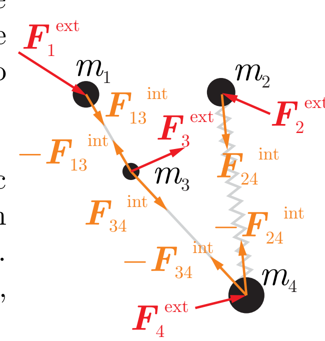
We re-iterate that the choice of the definition of the DOFs in the above examples is arbitrary. As already mentioned, we could as well have defined the translation DOFs x_i and x_j in opposite directions, the same applies to the rotational DOFs – as long as we formulate the constraints appropriately. E.g., if in case (b) both x_i and x_j pointed upwards, then we had $x_i = -x_j = -R\varphi$. Or if in case (b) φ was defined clockwise (and x_i, x_j as shown), then we had $x_i = x_j = -R\varphi$. We may hence choose the DOFs arbitrarily, but it makes sense to *define DOFs in agreement with known or anticipated motion*. For example, in cases (b) and (c) we drew all DOFs based on the assumption that the pulleys were to rotate counterclockwise. Again, this is arbitrary (and the solution would be exactly the same if defined otherwise), as long as we define all constraints consistently (and the same applies to the system kinetics, discussed next in Section 2.2).

We point out that *not every connection between particles is necessarily a kinematic constraint*. For example, when two particles are connected by an elastic spring, then that spring does indeed induce a force onto the particles, as soon as they change their relative distance (and hence stretch or compress the spring). However, the particles are still free to move in a kinematically unconstrained fashion. So, two particles in 3D still retain their $2\times 3 = 6$ DOF, even though connected by a spring.



2.2 Kinetics

For a system of particles $i = 1, \dots, n$ that are connected (e.g., through rigid links, deformable springs, ropes, etc.), we decompose all forces in the system into **internal forces** (i.e., forces stemming from particle-particle interactions) and **external forces** (i.e., forces applied externally to individual particles in the system).



As we generally assume that particle links (e.g., rigid bars or elastic springs) are *massless*, internal forces always act as force pairs between particles (same magnitude, opposite direction), as shown on the right. When denoting the force acting on particle i due to particle j by $\boldsymbol{F}_{ij}$, Newton's third law (action = reaction) for a massless link requires

$$\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji} \qquad \text{for all} \qquad i \neq j. \tag{2.7}$$

Therefore, the net force acting on particle i is

$$\boldsymbol{F}_i = \boldsymbol{F}_i^{\text{ext}} + \sum_{j \neq i} \boldsymbol{F}_{ij} = \boldsymbol{F}_i^{\text{ext}} + \boldsymbol{F}_i^{\text{int}} \qquad \text{where} \qquad \boldsymbol{F}_i^{\text{int}} = \sum_{j \neq i} \boldsymbol{F}_{ij}^{\text{int}} \tag{2.8}$$

is the total internal force on particle i, and $\boldsymbol{F}_i^{\text{ext}}$ denotes the external force applied to particle i.

When summing over all internal forces acting on the system of particles, we observe that the internal forces cancel pairwise, since

$$\begin{aligned} \sum_{i=1}^n \boldsymbol{F}_i^{\text{int}} &= \sum_{i=1}^n \sum_{j \neq i} \boldsymbol{F}_{ij}^{\text{int}} = \sum_{i=1}^n \left[\sum_{j>i} \boldsymbol{F}_{ij}^{\text{int}} + \sum_{i>j} \boldsymbol{F}_{ij}^{\text{int}} \right] \\ &= \sum_{i=1}^n \left[\sum_{j>i} \boldsymbol{F}_{ij}^{\text{int}} + \sum_{j>i} \boldsymbol{F}_{ji}^{\text{int}} \right] = \sum_{i=1}^n \left[\sum_{j>i} \boldsymbol{F}_{ij}^{\text{int}} + \sum_{j>i} (-\boldsymbol{F}_{ij}^{\text{int}}) \right] = \boldsymbol{0}. \end{aligned} \tag{2.9}$$

Consequently, the sum over all forces on all particles is the same as the sum over all external forces only (which will be beneficial in the formulation of the balance laws):

$$\boxed{\sum_{i=1}^n \boldsymbol{F}_i^{\text{int}} = \boldsymbol{0}} \qquad \Rightarrow \qquad \boxed{\sum_i \boldsymbol{F}_i = \sum_{i=1}^n \boldsymbol{F}_i^{\text{ext}}} \tag{2.10}$$

In the following, we will extend the balance laws previously derived for a single particle to a system of particles.

2.2.1 Balance of linear momentum

Assuming that the number of particles remains constant as does the total mass (i.e., $M = \text{const.}$), we start with the balance of linear momentum for each particle in an inertial frame, i.e.,

$$\boldsymbol{F}_i = \frac{\mathrm{d}}{\mathrm{d}t}(m_i \dot{\boldsymbol{r}}_i) \qquad \text{for } i = 1, \dots, n. \tag{2.11}$$

Summing the above linear momentum balance over all particles, using (2.10), gives

$$\sum_{i=1}^{n} \boldsymbol{F}_i = \sum_{i=1}^{n} \boldsymbol{F}_i^{\text{ext}} = \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}(m_i \dot{\boldsymbol{r}}_i) = M \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{M} \sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_i = M \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{v}_{\text{CM}}, \tag{2.12}$$

where

$$\boldsymbol{v}_{\text{CM}} = \frac{1}{M} \sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_i = \frac{1}{M} \sum_{i=1}^{n} m_i \boldsymbol{v}_i \qquad \text{and} \qquad \boldsymbol{a}_{\text{CM}} = \frac{1}{M} \sum_{i=1}^{n} m_i \ddot{\boldsymbol{r}}_i = \frac{1}{M} \sum_{i=1}^{n} m_i \boldsymbol{a}_i \tag{2.13}$$

denote, respectively, the velocity and acceleration of the center of mass of the system of particles.

Realizing that the total linear momentum of a system of particles is given by

$$\boxed{\boldsymbol{P} = M \boldsymbol{v}_{\text{CM}}} \tag{2.14}$$

(2.12) gives us the **linear momentum balance** *for a system of n particles* as

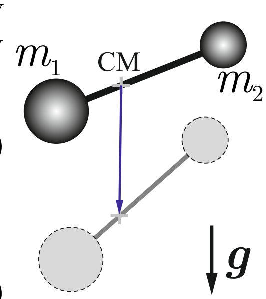
$$\boxed{\sum_{i=1}^{n} \boldsymbol{F}_i^{\text{ext}} = \dot{\boldsymbol{P}} = \frac{\mathrm{d}}{\mathrm{d}t}(M \boldsymbol{v}_{\text{CM}})} \qquad \Rightarrow \qquad \boxed{\sum_{i=1}^{n} \boldsymbol{F}_i^{\text{ext}} = M \boldsymbol{a}_{\text{CM}}} \tag{2.15}$$

This allows us to interpret the system of particles, loosely speaking, as a "*super-particle*" having mass M and whose motion is described by the position $\boldsymbol{r}_{\text{CM}}(t)$ of its center of mass. Note that these relations apply *not only for rigid* collections of particles but for general systems of n particles, irrespective of the (symmetric) particle interactions, since the inner forces do not enter the equation.

The above relations of linear momentum balance are analogous to Eqs. (1.61) and (1.63), respectively, for a single particle – here replacing the particle mass m by the total mass M of the system of particles, and replacing the particle velocity by the velocity of the center of mass of the system of particles.

Of course, we may still also – additionally – apply the balance of linear momentum equations (1.61) and (1.63) to each particle individually, if needed. However, when formulating balance laws for an individual particle, we may *not* drop the internal forces (since those only cancel in the above relations for the full system).

Example 2.2. Falling dumbbell



Consider a dumbbell, i.e., two particles of masses m_1 and m_2 connected by a rigid, massless bar. When the system is subjected to gravity as the only external forces, linear momentum balance on the whole system reads

$$(m_1 + m_2)\boldsymbol{a}_{\text{CM}} = m_1 \boldsymbol{g} + m_2 \boldsymbol{g} \qquad \Rightarrow \qquad \boldsymbol{a}_{\text{CM}} = \boldsymbol{g}, \tag{2.16}$$

so that integration with suitable initial conditions leads to

$$\boldsymbol{r}_{\text{CM}}(t) = -\frac{\boldsymbol{g}t^2}{2} + \boldsymbol{v}_{\text{CM}}(0)t + \boldsymbol{r}_{\text{CM}}(0). \tag{2.17}$$

Therefore the dumbbell will accelerate downwards under the action of gravity and thereby, as expected, behave like a "*super-particle*" of mass $m_1 + m_2$ (cf. our derivation in Example1.1 for a single particle). However, when we consider linear momentum balance for each particle individually, we obtain for particle 1 and 2, respectively,

$$m_1 \boldsymbol{a}_1 = m_1 \boldsymbol{g} + \boldsymbol{F}_s, \qquad m_2 \boldsymbol{a}_2 = m_2 \boldsymbol{g} - \boldsymbol{F}_s, \tag{2.18}$$

where $\boldsymbol{F}_s$ is the constraint force (acting onto each particle parallel to the rigid link). Therefore, while the system as a whole is undergoing a free fall, accelerated by $\boldsymbol{g}$, the two particles generally experience different accelerations ($\boldsymbol{a}_1, \boldsymbol{a}_2 \neq \boldsymbol{g}$ in general), which depend on the current orientation of the dumbbell and are more complex to calculate. As expected, adding up the two individual LMB equations in (2.18) yields the system LMB in (2.16).

2.2.2 Work–energy balance

Starting with the work–energy balance for each individual particle, which we derived in Eq. (1.87), we now sum the kinetic energies of all n particles as well as the work done on all n particles. This leads to the **work–energy balance** *for a system of particles* as

$$T(t) = \frac{1}{2} \sum_{i=1}^{n} m_i |\boldsymbol{v}_i(t)|^2, \quad W_{12} = \sum_{i=1}^{n} \int_{\boldsymbol{r}_i(t_1)}^{\boldsymbol{r}_i(t_2)} \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_i \qquad \Rightarrow \qquad T(t_2) - T(t_1) = W_{12} \tag{2.19}$$

Hence, the work–energy balance for a system of particles is completely analogous to a single particle – the kinetic energy and the work are replaced by those of the system.

We consider the following special cases:

- As before, in case of a **conservative force**, $W_{12} = V_1 - V_2$ can be derived from a potential V for each force. If all forces are conservative, then the **conservative system** is governed by the conservation of the total energy, i.e.,

  $$T + V = \text{const.} \tag{2.20}$$

  for the whole system of n particles.

- If the link between two particles is **rigid**, one can show that the *internal forces perform no work*. To see this, notice that a rigid link between particles i and j implies

  $$|\boldsymbol{r}_i(t) - \boldsymbol{r}_j(t)| = \text{const.} \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t} |\boldsymbol{r}_i - \boldsymbol{r}_j|^2 = 0 \qquad \Rightarrow \qquad (\boldsymbol{r}_i - \boldsymbol{r}_j) \cdot (\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{r}}_j) = 0. \tag{2.21}$$

  As discussed before, the interaction force between particles i and j is always parallel to the distance vector: $\boldsymbol{F}_{ij} || \boldsymbol{r}_{ij} = \boldsymbol{r}_j - \boldsymbol{r}_i$. Therefore, from (2.21) it follows that

  $$\boldsymbol{r}_{ij} \perp (\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{r}}_j) \qquad \Rightarrow \qquad \boldsymbol{F}_{ij} \perp (\boldsymbol{v}_i - \boldsymbol{v}_j). \tag{2.22}$$

Looking at the work done by the interaction forces $\boldsymbol{F}_{ij}$ and $\boldsymbol{F}_{ji} = \boldsymbol{F}_{ij}$ onto the two particles i and j, we realize that

$$\int_{\boldsymbol{r}_i(t_1)}^{\boldsymbol{r}_i(t_2)} \boldsymbol{F}_{ij} \cdot \mathrm{d}\boldsymbol{r}_i + \int_{\boldsymbol{r}_j(t_1)}^{\boldsymbol{r}_j(t_2)} \boldsymbol{F}_{ji} \cdot \mathrm{d}\boldsymbol{r}_j = \int_{t_1}^{t_2} \boldsymbol{F}_{ij} \cdot \boldsymbol{v}_i \, \mathrm{d}t + \int_{t_1}^{t_2} -\boldsymbol{F}_{ij} \cdot \boldsymbol{v}_j \, \mathrm{d}t = \int_{t_1}^{t_2} \boldsymbol{F}_{ij} \cdot (\boldsymbol{v}_i - \boldsymbol{v}_j) \, \mathrm{d}t = 0. \tag{2.23}$$

Hence, the forces arising from a rigid link in sum do not perform any work.

- We can extend the previous point to a **rigid system**: *if a system of particles is rigid, all internal forces perform no work.* This can easily be established by summing over all particles:

$$\sum_{i=1}^{n} \int_{\boldsymbol{r}_i(t_1)}^{\boldsymbol{r}_i(t_2)} \boldsymbol{F}_i^{\mathrm{int}} \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i=1}^{n} \sum_{j \neq i} \int_{t_1}^{t_2} \boldsymbol{F}_{ij} \cdot \boldsymbol{v}_i \, \mathrm{d}t = \sum_{i=1}^{n} \sum_{j > i} \int_{t_1}^{t_2} \boldsymbol{F}_{ij} \cdot (\boldsymbol{v}_i - \boldsymbol{v}_j) \, \mathrm{d}t = 0. \tag{2.24}$$

  Therefore, all interaction forces together perform no work on the rigid system of particles. Note that this allows us to replace the total particle force $\boldsymbol{F}_i$ in (2.19) by only the external force $\boldsymbol{F}_i^{\mathrm{ext}}$. Keep in mind that this is true only for *rigid* systems. If, e.g., two particles are connected by an elastic spring, the spring may perform (conservative) work on the particles, as evidenced by the change in potential energy when the distance between the particles changes.

We highlight the important fact that, even though the linear momentum of a system of particles was simply $\boldsymbol{P} = M\boldsymbol{v}_{\mathrm{CM}}$, the analogous does *not* hold for its kinetic energy, i.e.,

$$T = \sum_{i=1}^{n} \frac{m_i}{2} |\boldsymbol{v}_i|^2 \;\neq\; \frac{M}{2} |\boldsymbol{v}_{\mathrm{CM}}|^2. \tag{2.25}$$

Physically speaking, this is because the center of mass' translation alone does not capture all kinetic energy in the system (an additional contribution to the kinetic energy comes from the rotation of the system about the center of mass, which is not captured by $\boldsymbol{v}_{\mathrm{CM}}$).

2.2.3 Balance of angular momentum

Like for linear momentum balance, we start with the balance law of angular momentum for each individual particle (see Eq. (1.132)). The total angular momentum is then given by the sum over all particles, i.e.,

$$\boldsymbol{H}_{\mathrm{B}} = \sum_{i=1}^{n} \boldsymbol{r}_{\mathrm{BP}_i} \times \boldsymbol{P}_i \qquad \text{with} \qquad \boldsymbol{P}_i = m_i \boldsymbol{v}_{\mathrm{P}_i}. \tag{2.26}$$

Following the analogous derivation as for a single particle (cf. Eq. (1.130)), we now obtain

$$\begin{aligned} \dot{\boldsymbol{H}}_{\mathrm{B}} &= \sum_{i=1}^{n} \dot{\boldsymbol{r}}_{\mathrm{BP}_i} \times \boldsymbol{P}_i + \sum_{i=1}^{n} \boldsymbol{r}_{\mathrm{BP}_i} \times \dot{\boldsymbol{P}}_i = \sum_{i=1}^{n} (\boldsymbol{v}_{\mathrm{P}_i} - \boldsymbol{v}_{\mathrm{B}}) \times \boldsymbol{P}_i + \sum_{i=1}^{n} \boldsymbol{r}_{\mathrm{BP}_i} \times \boldsymbol{F}_i \\ &= \sum_{i=1}^{n} \boldsymbol{v}_{\mathrm{P}_i} \times \boldsymbol{P}_i - \boldsymbol{v}_{\mathrm{B}} \times \sum_{i=1}^{n} \boldsymbol{P}_i + \sum_{i=1}^{n} \boldsymbol{r}_{\mathrm{BP}_i} \times \boldsymbol{F}_i \qquad \text{and} \qquad \sum_{i=1}^{n} \boldsymbol{P}_i = \boldsymbol{P} = M \boldsymbol{v}_{\mathrm{CM}}. \end{aligned} \tag{2.27}$$

Note that only the external forces survive summation in the last term (all internal forces cancel since they are pairs of equal magnitude and opposite direction and same lever am with respect to point B). Further, we may use that $\boldsymbol{v}_{\mathrm{P}_i} || \boldsymbol{P}_i$ since $\boldsymbol{P}_i = m_i \boldsymbol{v}_{\mathrm{P}_i}$.

This leads to the **angular momentum balance** *for a system of n particles*:

$$\boxed{\boldsymbol{M}_{\mathrm{B}}^{\mathrm{ext}} = \dot{\boldsymbol{H}}_{\mathrm{B}} + \boldsymbol{v}_{\mathrm{B}} \times \boldsymbol{P}} \qquad \text{with} \qquad \boldsymbol{P} = M \boldsymbol{v}_{\mathrm{CM}} \tag{2.28}$$

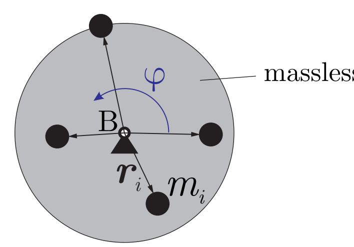
with the **resultant torque** of all *external* forces,

$$\boldsymbol{M}_{\mathrm{B}}^{\mathrm{ext}} = \sum_{i=1}^{n} \boldsymbol{r}_{\mathrm{BP}_i} \times \boldsymbol{F}_i^{\mathrm{ext}}. \tag{2.29}$$

As already discussed for a single particle, the term $\boldsymbol{v}_{\mathrm{B}} \times \boldsymbol{P}$ vanishes, if (i) point B is fixed ($\boldsymbol{v}_{\mathrm{B}} = \boldsymbol{0}$) or (ii) if B moves parallel to the center of mass ($\boldsymbol{v}_{\mathrm{B}} || \boldsymbol{v}_{\mathrm{CM}}$). It is therefore convenient to pick one of those two options when choosing point B.

Example 2.3. Particles on a rotating disk

We consider a massless disk that is rotating with angular velocity $\omega(t) = \dot{\varphi}(t)$ around its center B. n particles of masses $m_1, \ldots, m_n$ are rigidly attached to the disk at fixed radii $r_1, \ldots, r_n$. In this case the above formulation reduces to (using co-rotating polar coordinates, as in the case for a single particle; see Section 1.2.3):



$$\begin{aligned} \boldsymbol{H}_{\mathrm{B}} &= \sum_{i=1}^{n} \boldsymbol{r}_{\mathrm{BP}_i} \times \boldsymbol{P}_i = \sum_{i=1}^{n} r_i \boldsymbol{e}_{r_i} \times (m_i \boldsymbol{v}_{\mathrm{P}_i}) \\ &= \sum_{i=1}^{n} m_i r_i^2 \omega \boldsymbol{e}_3 = \sum_{i=1}^{n} m_i r_i^2 \dot{\varphi} \boldsymbol{e}_3, \end{aligned} \tag{2.30}$$

where we used that $\boldsymbol{v}_{\mathrm{P}i} = r_i \omega \boldsymbol{e}_{\varphi_i}$ and $\boldsymbol{e}_{r_i} \times \boldsymbol{e}_{\varphi_i} = \boldsymbol{e}_3$. Hence, for this special case the balance of angular momentum and the moment of inertia I_{B} become, respectively,

$$M_{\mathrm{B}}^{\mathrm{ext}} = I_{\mathrm{B}} \ddot{\varphi}, \qquad \text{with} \qquad I_{\mathrm{B}} = \sum_{i=1}^{n} m_i r_i^2. \tag{2.31}$$

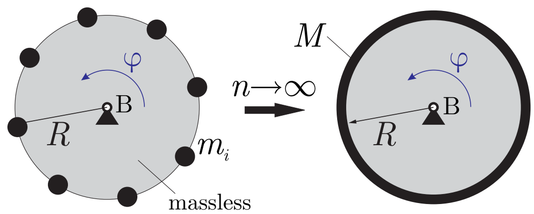
If the only force acting on the system is gravity (besides the support reactions at the center), then the total external moment is (assuming the coordinate origin at point B)

$$\boldsymbol{M}_{\mathrm{B}}^{\mathrm{ext}} = \sum_{i=1}^{n} \boldsymbol{r}_{\mathrm{BP}_i} \times m_i \boldsymbol{g} = \left(\sum_{i=1}^{n} m_i \boldsymbol{r}_{\mathrm{BP}_i} \right) \times \boldsymbol{g} = M \boldsymbol{r}_{\mathrm{CM}} \times \boldsymbol{g}. \tag{2.32}$$

This shows that, if the particles are arranged such that their center of mass is point B, then gravity does not produce an external moment on the system of particles and hence does not lead to a rotation of the disk.

Example 2.4. Moment of inertia of a thin wheel and of a slender rod

Let us consider the special case of n particles distributed uniformly along the circumference of an otherwise massless disk of radius R. We keep the total mass M constant while increasing the number of particles, so that the mass of each particle becomes $m_i = M/n$. This leads to

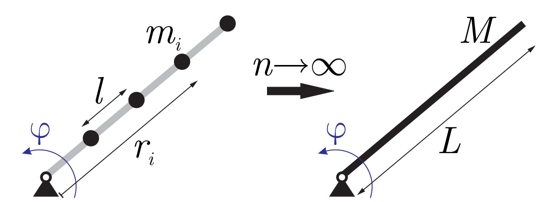


$$I_{\mathrm{B}} = \sum_{i=1}^{n} m_i r_i^2 = \sum_{i=1}^{n} \frac{M}{n} R^2 = MR^2 \sum_{i=1}^{n} \frac{1}{n} = MR^2. \tag{2.33}$$

This shows that the result is independent of the number of particles and, in particular, also holds for the case $n \to \infty$, which corresponds to a wheel whose mass is distributed evenly around the perimeter (e.g., a bike's tire rim). We have thus derived the *2D moment of inertia of a thin wheel of radius R and mass M, rotating about its center*, as $I_{\mathrm{B}} = MR^2$.

Analogously, we consider a slender rod of total mass M and length L, which is rotating about one ofits end points. Following the above example, we interpret the rod as a distribution of particles, each of mass $m_i = M/n$, which are equally spaced at distances $l = L/n$, so that the ith particle is located a distance $r_i = il = iL/n$ from the center of rotation. This leads to

$$I_{\mathrm{B}} = \sum_{i=1}^{n} m_i r_i^2 = \sum_{i=1}^{n} \frac{M}{n} \left(i\frac{L}{n}\right)^2 = \frac{ML^2}{n^3} \sum_{i=1}^{n} i^2. \tag{2.34}$$



Inserting the square sum formula

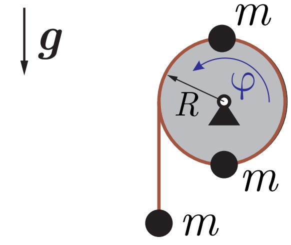
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{2.35}$$

finally gives the *2D moment of inertia of a slender rod of mass M and length L, rotating about one of its end points*:

$$I_{\mathrm{B}} = ML^2 \frac{n(n+1)(2n+1)}{6n^3} = ML^2 \frac{(n+1)(n+2)}{6n^2} \qquad \Rightarrow \qquad \lim_{n\to\infty} I_{\mathrm{B}} = \frac{ML^2}{3}. \tag{2.36}$$

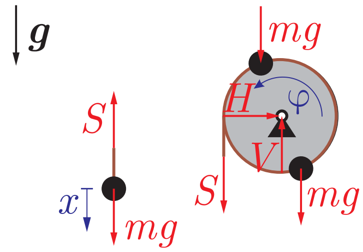
Example 2.5. Particle on a rope attached to a rotating disk

The construction shown on the right consists of three particles of equal masses $m_i = m$, two of which are rigidly attached to a rotating massless disk of radius R, the third one is connected to a rigid rope wrapped around the disk. The system is released from rest under the influence of gravity.



What is the downward acceleration of the particle on the rope?

We describe the kinematics of the system by the two DOFs x and φ, shown below. To find the acceleration of the particle on the rope, we need to use linear momentum balance on that particle only. This end, we introduce free-body diagrams for both components of the system (see Example 1.20). Linear momentum balance on the particle then reads



$$m\ddot{x} = mg - S. \tag{2.37}$$

For the rotating disk, angular momentum balance is advantageous to avoid the support reactions. With moment of inertia $I_B = 2mR^2$, this gives

$$2mR^2\ddot{\varphi} = SR + mgR\sin\varphi - mgR\sin\varphi = SR, \tag{2.38}$$

where we observe that the external moment resulting from gravity acting on the two masses on the disk cancels by symmetry for any rotation angle: $\boldsymbol{r}_1 \times (-mg)\boldsymbol{e}_2 + \boldsymbol{r}_2 \times (-mg)\boldsymbol{e}_2 = \boldsymbol{0}$ since $\boldsymbol{r}_1 = -\boldsymbol{r}_2$ (where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ denote the vectors from the center of rotation to the two particles). This confirms our derivation in (2.32) (the center of mass of the two particles here rests at the center of the disk).

Finally, we need a kinematic constraint to solve the system of (so far) two equations with three unknowns ($\ddot{x}$, $\ddot{\varphi}$ and S). The rigid rope implies that

$$x(t) = R\varphi(t) \qquad \Rightarrow \qquad \ddot{x}(t) = R\ddot{\varphi}(t). \tag{2.39}$$

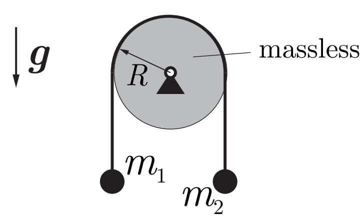
The three equations (2.37), (2.38), and (2.39) can be solved for

$$m\ddot{x} = mg - S = mg - 2mR\ddot{\varphi} = mg - 2m\ddot{x} \qquad \Rightarrow \qquad \ddot{x} = \frac{g}{3}. \tag{2.40}$$

Hence, the particle accelerates downwards with only a third of the gravitational acceleration (because it additionally needs to overcome the inertia of the rotating particles on the disk).

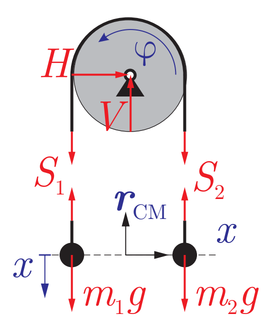
Example 2.6. Two particles connected by a rigid rope

Two particles of masses m_1 and m_2 are connected by a rigid rope that revolves around a massless disk of radius R. Considering gravity, what are the accelerations of the two particles and what is the reaction force on the disk's hinge?



Using the kinematic constraint $x_1 = x_2 = x$ (as drawn) we can find the solution from linear momentum balance on the two particles and angular momentum balance on the disk (which is massless):

$$m_1\ddot{x} = m_1 g - S_1, \qquad m_2\ddot{x} = S_2 - m_2 g, \qquad (S_1 - S_2)R = 0. \tag{2.41}$$



Solving the three equations leads to the particle acceleration

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g. \tag{2.42}$$

As a quick sanity check, we verify that for equal masses $m_1 = m_2$, the system is in static equilibrium and $\ddot{x} = 0$. Further, if $m_1 \gg m_2$ then $\ddot{x} \approx g$, while for $m_1 \ll m_2$ we have $\ddot{x} \approx -g$.

The reaction forces on the hinge are obtained from linear momentum balance as $H = 0$ and

$$V = S_1 + S_2 = m_1(g - \ddot{x}) + m_2(g + \ddot{x}) = (m_1 + m_2)g + (m_2 - m_1)\ddot{x} = 4g\frac{m_1 m_2}{m_1 + m_2}. \tag{2.43}$$

Notice the difference to the static case where $V = (m_1 + m_2)g$, which is recovered if $m_1 = m_2$.

Let us also verify the *balance of linear momentum on the global (complete) system.* We know that

$$(m_1 + m_2)\boldsymbol{a}_{\text{CM}} = \sum_i \boldsymbol{F}_i^{\text{ext}} = -(m_1 + m_2)g\boldsymbol{e}_2 + V\boldsymbol{e}_2 + H\boldsymbol{e}_1. \tag{2.44}$$

The location and acceleration of the center of mass, respectively, can be written as

$$\boldsymbol{r}_{\text{CM}} = \frac{m_1(-x\boldsymbol{e}_2) + m_2(x\boldsymbol{e}_2)}{m_1 + m_2} = -\frac{m_1 - m_2}{m_1 + m_2}x\boldsymbol{e}_2 \qquad \Rightarrow \qquad \boldsymbol{a}_{\text{CM}} = -\frac{m_1 - m_2}{m_1 + m_2}\ddot{x}\boldsymbol{e}_2. \tag{2.45}$$

Note that there is, in general, also an $\boldsymbol{e}_1$-component of the location of the center of mass (which depends on the two masses m_1 and m_2 as well as on the radius of the disk). However, both particles move only vertically, so the $\boldsymbol{e}_1$-component of the location of the center of mass does not change over time, and we chose to place our coordinate system such that the $\boldsymbol{e}_1$-component is zero. In addition, the second derivative of the constant $\boldsymbol{e}_1$-component with respect to time vanishes, so the acceleration $\boldsymbol{a}_{\text{CM}}$ remains unaffected irrespective of the coordinate origin.

Insertion of the center of mass' acceleration $\boldsymbol{a}_{\text{CM}}$ into (2.44) gives

$$-(m_1 + m_2)\frac{m_1 - m_2}{m_1 + m_2}\ddot{x}\boldsymbol{e}_2 = -(m_1 + m_2)g\boldsymbol{e}_2 + V\boldsymbol{e}_2 + H\boldsymbol{e}_1 \qquad \Rightarrow \qquad H = 0. \tag{2.46}$$

We can easily verify that $\ddot{x}$ from (2.42) and V from (2.43) satisfy this relation. Alternatively, if only one of those two unknowns had been determined, we could have used (2.46) to find the missing one.

We can also exploit the *conservation of energy of the global system* since this is a conservative system (the support forces in the hinge do no work). Choosing the potential energy with respect to the initial configuration (so the initial kinetic and potential energy is zero) gives

$$-m_1 g x + m_2 g x + \frac{m_1}{2}\dot{x}^2 + \frac{m_2}{2}\dot{x}^2 = 0 \qquad \Leftrightarrow \qquad \frac{m_1 - m_2}{m_1 + m_2}2gx = \dot{x}^2. \tag{2.47}$$

Differentiating both sides of this equation with respect to time (and assuming that $\dot{x} \neq 0$ during motion) yields directly

$$\frac{m_1 - m_2}{m_1 + m_2}2g\dot{x} = 2\dot{x}\ddot{x} \qquad \Rightarrow \qquad \ddot{x} = \frac{m_1 - m_2}{m_1 + m_2}g. \tag{2.48}$$

Finally, note that each particle does *not* conserve its energy, since the force in the rope performs work on each particle. That is, even though the complete system conserves its energy, each particle individually does not in general.

2.3 Particle Collisions

2.3.1 Collision of two particles

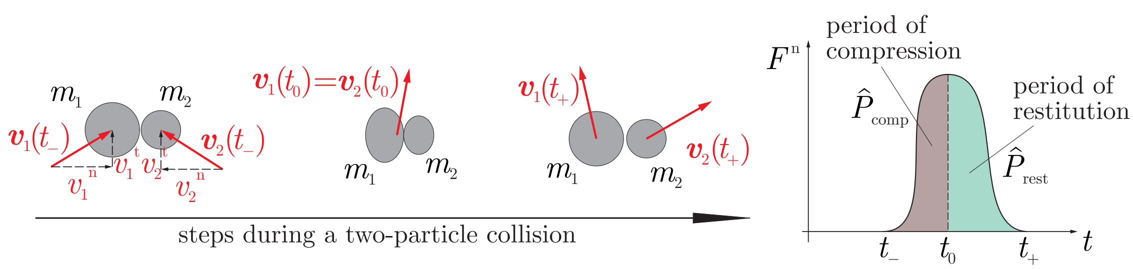
In an extension of the particle-wall impact of Section (1.2.4), let us consider two particles of masses m_1 and m_2, colliding at initial velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively. We seek the velocities of the two particles after the **collision**.

We follow an analogous procedure here as for the particle-wall impact in Section (1.2.4). There we showed that linear momentum balance for a particle of mass m_i and velocity $\boldsymbol{v}_i$, subjected to an impulsive force over a time interval $[t_-, t_+]$, leads to

$$m_i \boldsymbol{v}_i(t_+) - m_i \boldsymbol{v}_i(t_-) = \hat{\boldsymbol{P}}_i \qquad \text{with} \qquad \hat{\boldsymbol{P}}_i = \int_{t_-}^{t_+} \boldsymbol{F}_i(t)\,\mathrm{d}t, \tag{2.49}$$

which we may apply to each particle now ($i = 1, 2$). Furthermore, as for the single particle we decompose the collision into a **period of compression** $[t_-, t_0)$ and a **period of restitution** $[t_0, t_+]$ such that any impulsive force $\hat{\boldsymbol{P}}$ is decomposed into

$$\hat{\boldsymbol{P}}_i = \hat{\boldsymbol{P}}_{i,\text{com}} + \hat{\boldsymbol{P}}_{i,\text{res}}, \quad \hat{\boldsymbol{P}}_{i,\text{comp}} = \int_{t_-}^{t_0} \boldsymbol{F}_i(t)\,\mathrm{d}t, \quad \hat{\boldsymbol{P}}_{i,\text{rest}} = \int_{t_0}^{t_+} \boldsymbol{F}_i(t)\,\mathrm{d}t. \tag{2.50}$$



While the collision process (schematically shown above) is analogous to that of a particle impacting a wall, we emphasize a few key differences. First, we may no longer assume that the velocity of any particle at time t_0 vanishes. Instead, we assume that $\boldsymbol{v}_1(t_0) = \boldsymbol{v}_2(t_0) = \boldsymbol{v}(t_0)$; i.e., the two particles are first coming closer, than attain a common velocity $\boldsymbol{v}$ at time t_0, and then – depending on the coefficient of restitution – may separate again. Second, there is now an impulsive force acting onto each of the two particles and, by Newton's third axiom, they are of the same magnitude but opposite sign: $\boldsymbol{F}_1 = -\boldsymbol{F}_2$ and therefore $\hat{\boldsymbol{P}}_1 = -\hat{\boldsymbol{P}}_2$ (and the same holds true for the compression and restitution phases).

Applying linear momentum balance to each particle for the compression and restitution phases separately (and writing $\hat{\boldsymbol{P}}_1 = \hat{\boldsymbol{P}}$ and $\hat{\boldsymbol{P}}_2 = -\hat{\boldsymbol{P}}$ for convenience) leads to

$$\begin{aligned} m_1\boldsymbol{v}(t_0) - m_1\boldsymbol{v}_1(t_-) &= \hat{\boldsymbol{P}}_{\text{comp}}, \qquad & m_2\boldsymbol{v}(t_0) - m_2\boldsymbol{v}_2(t_-) &= -\hat{\boldsymbol{P}}_{\text{comp}}, \\ m_1\boldsymbol{v}_1(t_+) - m_1\boldsymbol{v}(t_0) &= \hat{\boldsymbol{P}}_{\text{rest}} & m_2\boldsymbol{v}_2(t_+) - m_2\boldsymbol{v}(t_0) &= -\hat{\boldsymbol{P}}_{\text{rest}}. \end{aligned} \tag{2.51}$$

For a **frictionless** collision, momentum is again exchanged only in the *normal direction*[9] ($^{\mathrm{n}}$), while no interaction forces are generated in the *tangential direction* ($^{\mathrm{t}}$), i.e., $\hat{\boldsymbol{P}} = \hat{P}\boldsymbol{e}_{\mathrm{n}}$ and hence

$$\boxed{v_1^{\mathrm{t}}(t_+) = v_1^{\mathrm{t}}(t_-), \qquad v_2^{\mathrm{t}}(t_+) = v_2^{\mathrm{t}}(t_-)}. \tag{2.52}$$

Having found the tangential velocity components, we proceed to evaluate (2.51) in the normal direction to find

$$\begin{aligned} m_1 v^{\mathrm{n}}(t_0) - m_1 v_1^{\mathrm{n}}(t_-) &= \hat{P}_{\mathrm{comp}}, \qquad & m_2 v^{\mathrm{n}}(t_0) - m_2 v_2^{\mathrm{n}}(t_-) &= -\hat{P}_{\mathrm{comp}}, \\ m_1 v_1^{\mathrm{n}}(t_+) - m_1 v^{\mathrm{n}}(t_0) &= \hat{P}_{\mathrm{rest}} & m_2 v_2^{\mathrm{n}}(t_+) - m_2 v^{\mathrm{n}}(t_0) &= -\hat{P}_{\mathrm{rest}}. \end{aligned} \tag{2.53}$$

These are four equations for five unknowns (the two final velocities, the velocity at t_0, and the impulsive forces during the compression and restitution periods). One additional equation is needed.

Like for the particle–wall impact, we exploit the definition of the **coefficient of restitution**,

$$e = \frac{\hat{P}_{\mathrm{rest}}}{\hat{P}_{\mathrm{comp}}}, \qquad e \in [0,1] \tag{2.54}$$

to solve the above system of equations, which yields

$$\boxed{v_1^{\mathrm{n}}(t_+) = \frac{m_1 v_1^{\mathrm{n}}(t_-) + m_2 v_2^{\mathrm{n}}(t_-) + e\, m_2 \left[v_2^{\mathrm{n}}(t_-) - v_1^{\mathrm{n}}(t_-)\right]}{m_1 + m_2}} \tag{2.55}$$

and

$$\boxed{v_2^{\mathrm{n}}(t_+) = \frac{m_1 v_1^{\mathrm{n}}(t_-) + m_2 v_2^{\mathrm{n}}(t_-) + e\, m_1 \left[v_1^{\mathrm{n}}(t_-) - v_2^{\mathrm{n}}(t_-)\right]}{m_1 + m_2}} \tag{2.56}$$

As before, we distinguish **three cases**:

- **elastic collision**: the impulsive force is fully recovered. In this case $\hat{P}_{\mathrm{rest}} = \hat{P}_{\mathrm{comp}}$ and the solution of the complete system of equations yields

  $$v_1^{\mathrm{n}}(t_+) = \frac{m_1 v_1^{\mathrm{n}}(t_-) + m_2 v_2^{\mathrm{n}}(t_-) + m_2 \left[v_2^{\mathrm{n}}(t_-) - v_1^{\mathrm{n}}(t_-)\right]}{m_1 + m_2} = \frac{m_1 - m_2}{m_1 + m_2} v_1^{\mathrm{n}}(t_-) + \frac{2m_2}{m_1 + m_2} v_2^{\mathrm{n}}(t_-).$$

  and

  $$v_2^{\mathrm{n}}(t_+) = \frac{m_1 v_1^{\mathrm{n}}(t_-) + m_2 v_2^{\mathrm{n}}(t_-) + m_1 \left[v_1^{\mathrm{n}}(t_-) - v_2^{\mathrm{n}}(t_-)\right]}{m_1 + m_2} = \frac{2m_1}{m_1 + m_2} v_1^{\mathrm{n}}(t_-) + \frac{m_2 - m_1}{m_1 + m_2} v_2^{\mathrm{n}}(t_-).$$

  For example in the special case of *equal masses* $m_1 = m_2$, we conclude $v_1^{\mathrm{n}}(t_+) = v_2^{\mathrm{n}}(t_-)$ and $v_2^{\mathrm{n}}(t_+) = v_1^{\mathrm{n}}(t_-)$.

[9] Note that here and in the following we use *superscripts* $^{\mathrm{n}}$ and $^{\mathrm{t}}$ to avoid confusion with the subscripts 1 and 2.

- **plastic collision**: the impulsive force during the restitution phase vanishes. In this case $\hat{P}_{\text{rest}} = 0$ and
$$v_1^{\text{n}}(t_+) = v_2^{\text{n}}(t_+) = \frac{m_1 v_1^{\text{n}}(t_-) + m_2 v_2^{\text{n}}(t_-)}{m_1 + m_2}. \tag{2.57}$$
In this case the two masses continue with the same *normal* velocity (whereas their tangential velocities may still be different). Also, in case of *equal masses*, the resulting normal velocity is simply the average of the two initial velocities.

- **partially elastic collision**: collisions with $0 \leq e \leq 1$ represent a partially elastic collision (only some of the compressive impulse force is recovered during the restitution phase) and relations (2.55) and (2.56) apply.

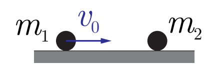
Note that we can rearrange the relations (2.55) and (2.56) for the general case $0 \leq e \leq 1$ and use conservation of momentum on the global system for the total time period of collision (i.e., adding the compression and restitution phases in (2.53)):
$$\boxed{m_1 v_1^{\text{n}}(t_+) + m_2 v_2^{\text{n}}(t_+) = m_1 v_1^{\text{n}}(t_-) + m_2 v_2^{\text{n}}(t_-)} \tag{2.58}$$
This is legitimate since no external forces act on the system of particles (only inner interaction forces are present during the collision). This altogether leads to a re-interpretation of the coefficient of restitution,
$$\boxed{e = -\frac{v_2^n(t_+) - v_1^n(t_+)}{v_2^n(t_-) - v_1^n(t_-)}} \tag{2.59}$$
as the ratio of the relative normal particle velocities after and before the collision. Solving (2.58) and (2.59) for the normal velocities $v_1^{\text{n}}(t_+)$ and $v_2^{\text{n}}(t_+)$ after the collision again leads to (2.55) and (2.56).

Finally, note that particle collisions are *not conservative in general.* Energy conservation only applies for elastic collisions ($e = 1$). Otherwise, energy is dissipated (i.e., kinetic energy is consumed by the permanent deformation of the particles).

Example 2.7. Particle collision with a rigid wall (revisited)

The collision of a single particle of mass m with a rigid wall (discussed in Section 1.2.4) can be interpreted as a special case of a two-body collision, taking $m_1 = m$ and $m_2 \to \infty$ as well as $\boldsymbol{v}_2(t) = \boldsymbol{0}$ for all times t. Equation (2.55) here specializes to
$$v_1^{\text{n}}(t_+) = \lim_{m_2 \to \infty} \left. \frac{m_1 v_1^{\text{n}}(t_-) + m_2 v_2^{\text{n}}(t_-) + e\, m_2 \left[v_2^{\text{n}}(t_-) - v_1^{\text{n}}(t_-)\right]}{m_1 + m_2} \right|_{v_2(t_-)=0} = -e\, v_1(t_-), \tag{2.60}$$
while we have for a frictionless wall
$$v_1^{\text{t}}(t_+) = v_1^{\text{t}}(t_-). \tag{2.61}$$
These are the same relations already derived in Section 1.2.4.

Example 2.8. Collision with a particle at rest



Consider two particles of masses m_1 and m_2 which are colliding *elastically*. m_2 is initially at rest, while m_1 hits with a normal velocity of v_0. What are the velocities after the collision?

Using $v_1^{\rm n}(t_-) = v_0$ and and $v_2^{\rm n}(t_-) = 0$ as well as $e = 1$, relations (2.55) and (2.56) reduce to

$$v_1^{\rm n}(t_+) = \frac{m_1 - m_2}{m_1 + m_2} v_0, \qquad v_2^{\rm n}(t_+) = \frac{2m_1}{m_1 + m_2} v_0. \tag{2.62}$$

Therefore, particle 2 after the collision will always have a velocity in the same direction as the initial velocity of particle 1 (thus moving away from the collision). The orientation of the velocity of particle 1 after the collision depends on the two masses. If $m_1 > m_2$, then both masses will continue in the same direction. If $m_1 < m_2$, then m_2 will bounce back in the opposite direction. In case of equal masses, we obtain

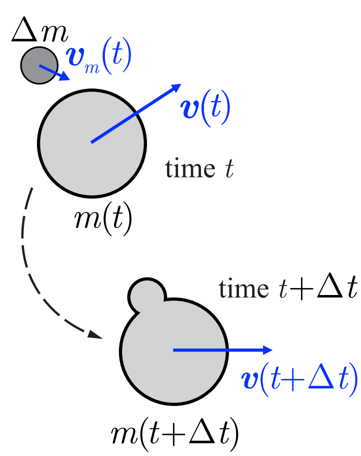
$$v_1^{\rm n}(t_+) = 0 \qquad v_2^{\rm n}(t_+) = v_0, \tag{2.63}$$

i.e., the complete linear momentum is transferred from particle 1 to particle 2. Notice that, independent of the two masses, the kinetic energy is conserved since $\frac{m_1}{2}[v_1^{\rm n}(t_+)]^2 + \frac{m_2}{2}[v_2^{\rm n}(t_+)]^2 = \frac{m_1}{2}v_0^2$.

2.3.2 Particles of variables mass: mass accretion and mass loss

Having discussed systems of particles, we are also in a position to study the kinetics of a particle with a changing mass $m(t)$. (Recall that so far we have always assumed the mass of each particle remains constant over time.) This scenario is in fact a special case of the above two-particle collision. If a particle is continuously accumulating mass by picking up infinitesimally small particles from the surroundings, we speak of *mass accretion*. The opposite case is *mass loss* where a particle is continuously losing mass by ejecting infinitesimally small particles to its environment.

The process of **mass accretion** is important when particles gain or lose mass during their motion, e.g., when a rocket is propelled by exhausting gas, or when droplets grow while moving through a saturated medium. Consider a particle of mass $m(t)$ and velocity $\boldsymbol{v}(t)$, which continually acquires (i.e., *accretes*) mass.



Assume that, at time t, over a small time increment Δt an additional mass Δm is gained by a particle which has an initial velocity $\boldsymbol{v}(t)$ and mass $m(t)$, while the small additional mass Δm has the initial velocity $\boldsymbol{v}_m(t)$. Let us compare the linear momentum before and after mass accretion, $\boldsymbol{P}(t)$ and $\boldsymbol{P}(t+\Delta t)$, respectively. We start with the initial linear momentum at time t, which is the sum of the linear momentum of the particle and of the additional mass increment:

$$\boldsymbol{P}(t) = m(t)\boldsymbol{v}(t) + \Delta m\, \boldsymbol{v}_m(t). \tag{2.64}$$

After the small increment Δt, the particle has linear momentum $\boldsymbol{P}(t+\Delta t) = m(t+\Delta t)\boldsymbol{v}(t+\Delta t)$. Since the time increment Δt is small, we may apply a Taylor expansion (truncated to quadratic order), so that the linear momentum at time $t+\Delta t$ is approximated by

$$\begin{aligned}\boldsymbol{P}(t+\Delta t) = m(t+\Delta t)\boldsymbol{v}(t+\Delta t) &= \left[m(t)+\dot{m}(t)\Delta t + O(\Delta t^2)\right]\left[\boldsymbol{v}(t)+\dot{\boldsymbol{v}}(t)\Delta t + O(\Delta t^2)\right] \\ &= m(t)\boldsymbol{v}(t) + \Delta t\left[\dot{m}(t)\boldsymbol{v}(t)+m(t)\dot{\boldsymbol{v}}(t)\right] + O(\Delta t^2).\end{aligned} \tag{2.65}$$

The rate of change of linear momentum is hence (dropping the t-dependence for brevity)

$$\begin{aligned}\dot{\boldsymbol{P}} &= \lim_{\Delta t\to 0}\frac{\boldsymbol{P}(t+\Delta t)-\boldsymbol{P}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{m\boldsymbol{v}+\Delta t\,(\dot{m}\boldsymbol{v}+m\dot{\boldsymbol{v}})+O(\Delta t^2)-(m\boldsymbol{v}+\Delta m\,\boldsymbol{v}_m)}{\Delta t} \\ &= \dot{m}\boldsymbol{v}+m\dot{\boldsymbol{v}}-\dot{m}\boldsymbol{v}_m = \dot{m}(\boldsymbol{v}-\boldsymbol{v}_m)+m\dot{\boldsymbol{v}},\end{aligned} \tag{2.66}$$

where we used that $\lim_{\Delta t\to 0}\Delta m\,\boldsymbol{v}_m/\Delta t = \dot{m}\boldsymbol{v}_m$. Further, note that $\dot{\boldsymbol{v}} = \boldsymbol{a}$.

From Newton's second axiom we know that $\sum_i \boldsymbol{F}_i = \dot{\boldsymbol{P}}$. Therefore, equating the rate of linear momentum change to the applied forces yields

$$\boxed{\sum_i \boldsymbol{F}_i = m\boldsymbol{a}+\dot{m}(\boldsymbol{v}-\boldsymbol{v}_m)} \qquad \Leftrightarrow \qquad m\boldsymbol{a} = \sum_i \boldsymbol{F}_i + \dot{m}\boldsymbol{v}_{\text{rel}} \quad \text{with} \quad \boldsymbol{v}_{\text{rel}} = \boldsymbol{v}_m - \boldsymbol{v}, \tag{2.67}$$

which is also known as the **rocket equation** (the reason will become apparent in Example 2.9). Of course, in the case $\dot{m} = 0$ the rocket equation reduces to the classical form of Newton's second law.

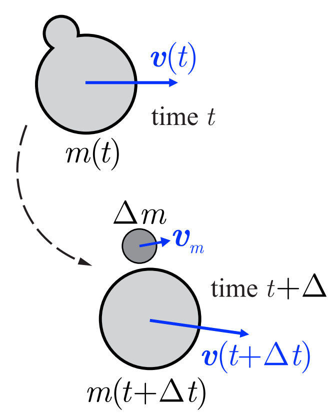
While the above derivation applies for mass accretion, we can extend it to the case of **mass loss**. We start with a linear momentum of

$$\boldsymbol{P}(t) = m(t)\boldsymbol{v}(t) \tag{2.68}$$

and, after an infinitesimal time increment Δt, end up with

$$\boldsymbol{P}(t+\Delta t) = m(t+\Delta t)\boldsymbol{v}(t+\Delta t) + \Delta m\,\boldsymbol{v}_m, \tag{2.69}$$

where we have ejected a small particle of mass Δm at velocity $\boldsymbol{v}_m$, so that $\dot{m} < 0$. Performing the analogous Taylor expansion as in (2.66) again leads to (2.67).



Therefore, *the rocket equation* (2.67) *applies equally to mass accretion* ($\dot{m} > 0$) *and mass loss* ($\dot{m} < 0$), and of course also for $\dot{m} = 0$.

Example 2.9. Rocket motion (mass loss)

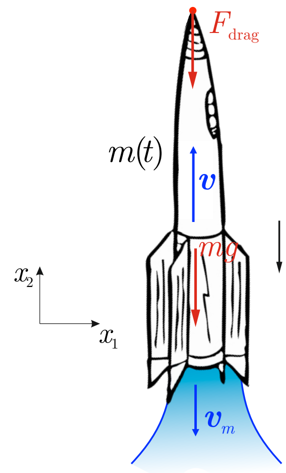
A rocket is moving into the x_2-direction and propelled by hot exhaust gas leaving the rocket downward at a constant velocity $v_{\text{rel}} = c$ (relative to the rocket), i.e.,

$$\boldsymbol{v}_m - \boldsymbol{v} = c\boldsymbol{e}_2 \qquad \text{with} \qquad c < 0. \tag{2.70}$$

Since the hot gas is leaving the rocket, the rocket is losing mass over time ($\dot{m} < 0$). Therefore, linear momentum balance in the x_2-direction is written as the rocket equation:

$$F_2 = m\ddot{x}_2 - \dot{m}c \qquad \Rightarrow \qquad m\ddot{x}_2 = F_2 + \dot{m}c, \tag{2.71}$$

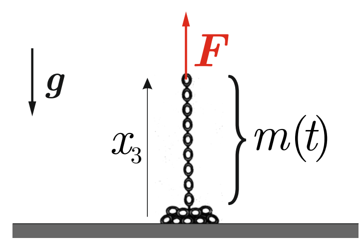
where F_2 is the externally applied force in the vertical direction (which usually contains the affects of gravity and drag, i.e., $F_2 = -mg - F_{\text{drag}}$).
Note that $\dot{m} < 0$ and $c < 0$, so that $\dot{m}c > 0$ is indeed accelerating the rocket upwards. The term $\dot{m}c > 0$ therefore represents the **thrust** (perceived as an additional upward force). The *lift-off condition* is $|\dot{m}c| > mg$, since the initial drag force vanishes at zero initial velocity.



Example 2.10. Lifting a heavy rope off the ground (mass accretion)

Consider a long and heavy rope (or chain) of total mass m_0 and total length l, which is initially lying on the ground. A force F is applied to lift the rope gradually off the ground, such that it is moving upwards (in the x_3-direction) with a constant acceleration a_0. What is the required magnitude of force F?

Let us denote by $x_3(t)$ the current length of the rope that has lifted off the ground at time t. We will use $x_3(t)$ to conveniently describe the motion of the rope.



The known constant acceleration can be integrated along with the initial conditions that the rope is lying statically on the ground initially, i.e., $x_3(0) = 0$ and $\dot{x}_3(0) = 0$. Hence, the 1D *kinematics* of the rope is given by

$$\ddot{x}_3(t) = a_0 = \text{const.} \qquad \Rightarrow \qquad \dot{x}_3(t) = a_0 t \qquad \Rightarrow \qquad x_3(t) = \frac{a_0}{2}t^2. \tag{2.72}$$

To calculate the force F, we invoke the balance of linear momentum, using its specific form of the rocket equation since the mass of the moving rope changes over time. Specifically, at a height of x_3 and time t, the mass $m(t)$ that has lifted off the ground at time t and is moving is given by

$$m(x_3) = \frac{m_0}{l}x_3 \qquad \Rightarrow \qquad m(t) = \frac{m_0 a_0}{2l}t^2 \tag{2.73}$$

Using linear momentum balance in this scenario implies the *kinetic* relation

$$m(t)\ddot{x}_3(t) = F(t) - m(t)g + \dot{m}(t)\left(v_m - v\right), \tag{2.74}$$

where the last term stems from the changing mass. Since the part of the chain on the ground remains at rest while the other part is moving at a constant velocity of $\dot{x}_3$, we have

$$v_m - v = 0 - \dot{x}_3 = -\dot{x}_3 \qquad \text{and also} \qquad \dot{m}(t) = \frac{m_0 a_0}{l}t. \tag{2.75}$$

Overall, this leads to

$$F(t) = m(t)\ddot{x}_3(t) + m(t)g + \dot{m}(t)\dot{x}_3(t). \tag{2.76}$$

Inserting the known velocity and acceleration from (2.72) into this equation yields

$$\begin{aligned} F(t) &= \frac{m_0 a_0^2}{2l}t^2 + \frac{m_0 a_0 g}{2l}t^2 + \frac{m_0 a_0^2}{l}t^2 \\ &= \frac{m_0 a_0}{2l}t^2(3a_0 + g) = m_0\frac{x_3}{l}(3a_0 + g). \end{aligned} \tag{2.77}$$

Therefore, the force F must grow linearly with chain length x_3. As a sanity check, note that under static conditions (no chain motion, so $a_0 = 0$), the force F must only balance gravity for a given chain length, which is recovered here as a special case ($F = mg = m_0 g x_3/l$).

2.4 Summary of Key Relations

The dynamics of systems of particles is an extension of single-particle dynamics. Importantly, internal forces do not enter LMB and AMB, which may be written in terms of the motion of the center of mass. The following box, which becomes part of our *formula collection*, summarizes all key relations.

center of mass and **total mass** of a system of n particles:

$$\boldsymbol{r}_{\mathrm{CM}}(t) = \frac{1}{M}\sum_{i=1}^{n} m_i \boldsymbol{r}_i(t), \qquad M = \sum_{i=1}^{n} m_i$$

balance of linear momentum for $M = \text{const.}$:

$$\sum_{i=1}^{n} \boldsymbol{F}_i^{\mathrm{int}} = \boldsymbol{0}, \qquad \sum_{i=1}^{n} \boldsymbol{F}_i^{\mathrm{ext}} = \dot{\boldsymbol{P}} = \frac{\mathrm{d}}{\mathrm{d}t}(M\boldsymbol{v}_{\mathrm{CM}}) \qquad \Rightarrow \qquad \sum_{i=1}^{n} \boldsymbol{F}_i^{\mathrm{ext}} = M\boldsymbol{a}_{\mathrm{CM}}$$

balance of linear momentum for a particle of varying mass $m(t)$:

$$\sum_i \boldsymbol{F}_i = m\boldsymbol{a} + \dot{m}(\boldsymbol{v} - \boldsymbol{v}_m)$$

conservation of linear momentum:

$$\text{if } \sum_{i=1}^{n} \boldsymbol{F}_i^{\mathrm{ext}} = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{P} = \sum_{i=1}^{n} m_i \boldsymbol{v}_i = \text{const.}$$

work–energy balance:

$$T(t_2) - T(t_1) = W_{12} \qquad \text{with} \qquad T(t) = \frac{1}{2}\sum_{i=1}^{n} m_i |\boldsymbol{v}_i(t)|^2, \quad W_{12} = \sum_{i=1}^{n} \int_{\boldsymbol{r}_i(t_1)}^{\boldsymbol{r}_i(t_2)} \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_i$$

in the special case of **rigid** connections:

$$W_{12} = \sum_{i=1}^{n} \int_{\boldsymbol{r}_i(t_1)}^{\boldsymbol{r}_i(t_2)} \boldsymbol{F}_i^{\mathrm{ext}} \cdot \mathrm{d}\boldsymbol{r}_i$$

balance of angular momentum with respect to point B:

$$\boldsymbol{M}_{\mathrm{B}}^{\mathrm{ext}} = \dot{\boldsymbol{H}}_{\mathrm{B}} + \boldsymbol{v}_{\mathrm{B}} \times \boldsymbol{P} \qquad \text{with} \qquad \boldsymbol{P} = M\boldsymbol{v}_{\mathrm{CM}}, \qquad \boldsymbol{H}_{\mathrm{B}} = \sum_{i=1}^{n} \boldsymbol{r}_{\mathrm{BP}_i} \times \boldsymbol{P}_i.$$

coefficient of restitution for a **two-particle collision**:

$$e = -\frac{v_2^n(t_+) - v_1^n(t_+)}{v_2^n(t_-) - v_1^n(t_-)} = \frac{\hat{P}_{\mathrm{rest}}}{\hat{P}_{\mathrm{comp}}}$$

particle velocities after a **two-particle collision**:

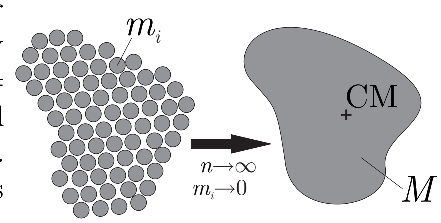
$$v_1^{\mathrm{n}}(t_+) = \frac{m_1 v_1^{\mathrm{n}}(t_-) + m_2 v_2^{\mathrm{n}}(t_-) + e\, m_2 \left[v_2^{\mathrm{n}}(t_-) - v_1^{\mathrm{n}}(t_-)\right]}{m_1 + m_2}, \qquad v_1^{\mathrm{t}}(t_+) = v_1^{\mathrm{t}}(t_-)$$

$$v_2^{\mathrm{n}}(t_+) = \frac{m_1 v_1^{\mathrm{n}}(t_-) + m_2 v_2^{\mathrm{n}}(t_-) + e\, m_1 \left[v_1^{\mathrm{n}}(t_-) - v_2^{\mathrm{n}}(t_-)\right]}{m_1 + m_2}, \qquad v_2^{\mathrm{t}}(t_+) = v_2^{\mathrm{t}}(t_-)$$

3 Dynamics of Rigid Bodies

All examples and principles so far have relied on the assumption that the bodies of interest were so small that they could be approximated as particles. This unfortunately does not apply to many or most engineering applications. Consider, e.g., a wheel rolling down a hill – a particle has no rotational DOFs, so rolling and sliding make no difference to a particle, yet there obviously is a difference in the dynamics of a sliding vs. rolling tyre. Similarly, throw a book up in the air and observe the influence of rotations about all three axes during the book's flight – this cannot be described by the flight of a particle. In such cases, we need to account for the *size* and *shape* of the object of interest; we hence need to transition from a particle to a **body**. To begin simple, in this section we only consider rigid bodies. A **rigid body** is *undeformable* (i.e., it can move freely but cannot change its shape or size).

With a bit of imagination, we picture a rigid body as a collection of infinitely many particles (or material points), packed together infinitely close and having infinitesimally small masses. We may hence consider a system of n particles with masses Δm_i ($i = 1, \dots, n$), and pass to the limit $n \to \infty$ and $\Delta m_i \to 0$ (at fixed relative particle distances) – the limit is a *continuous body* $\mathcal{B}$. If the distance between particles (or material points) remains constant (consider rigid links between all pairs of particles), then we have a **rigid body**.



In the following, we will revisit the dynamics of systems of particles for this special case of a continuous body. In this limit, all discrete sums over particles turn into integrals ($\sum_{i=1}^n \to \int_\mathcal{B}$), so we will adjust our kinematic and kinetic definitions and relations accordingly.

3.1 Kinematics

The **total mass** of a rigid body $\mathcal{B}$ is defined as the limit of summing over infinitely many particles of infinitesimal mass, or as the integral over the body:

$$M = \lim_{\substack{n\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^n \Delta m_i = \int_\mathcal{B} \mathrm{d}m = \int_\mathcal{B} \rho \, \mathrm{d}V, \tag{3.1}$$

where

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{\mathrm{d}m}{\mathrm{d}V} \tag{3.2}$$

denotes the **mass density** of body $\mathcal{B}$. In the following, we will usually assume a spatially constant mass density ($\rho(\boldsymbol{x}) = \rho = \text{const.}$), although we may also consider cases of an inhomogeneous mass distribution.

The **center of mass** follows analogously as

$$\boldsymbol{r}_\mathrm{CM}(t) = \lim_{\substack{n\to\infty \\ \Delta m_i \to 0}} \frac{1}{M} \sum_{i=1}^n \boldsymbol{r}_i(t)\, \Delta m_i \quad \Rightarrow \quad \boxed{\boldsymbol{r}_\mathrm{CM}(t) = \frac{1}{M} \int_\mathcal{B} \boldsymbol{r}(t)\, \mathrm{d}m = \frac{1}{M} \int_\mathcal{B} \boldsymbol{r}(t) \rho \, \mathrm{d}V} \tag{3.3}$$

The kinematic description of continuous bodies is more complex than that of particles. Instead of assigning the positions $\boldsymbol{r}_i$, velocities $\boldsymbol{v}_i$, and accelerations $\boldsymbol{a}_i$ of n independent particles, we now introduce **continuous fields** including, e.g, the position vector field $\boldsymbol{r}(\boldsymbol{x},t)$, the velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ and, analogously, the acceleration field $\boldsymbol{a}(\boldsymbol{x},t)$, which all depend on the position $\boldsymbol{x} \in \mathcal{B}$ and time t. In principle, each point of the body $\mathcal{B}$ could undergo a different motion through space and time – however, points in a *rigid* body are physically coupled by the kinematic constraints of rigidity of the body.

The motion of a rigid body as a whole – unlike that of particles – involves both translations and rotations. Therefore, the motion of a rigid body in

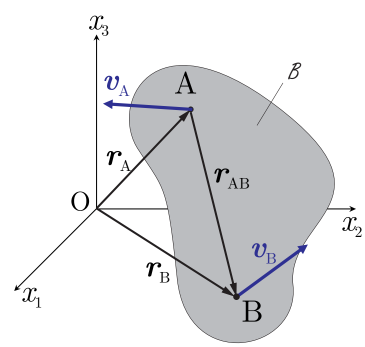
2D is described by **3 degrees of freedom**, including translations in the two in-plane directions, e.g., x_1 and x_2, and an in-plane rotation φ about the out-of-plane x_3-axis.

3D is described by **6 degrees of freedom**, including translations in the three x_i-directions as well as three rotations about the three x_i-axes.

It is important to note that every **point** P on the body is still described uniquely by its position vector $\boldsymbol{r}_\mathrm{P}(t)$. One of the key tasks of rigid-body kinematics is to answer how the trajectories of different points on a body are related to each other, as the body is moving through space and time. For example, we may seek relations between the kinematic variables at two distinct points on the body.

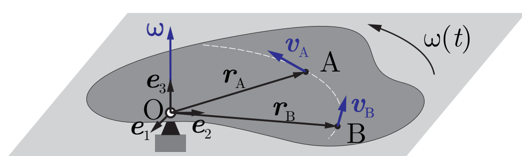
Consider a rigid body $\mathcal{B}$ that undergoes both translation and rotation, and let us go from a point $\mathrm{A} \in \mathcal{B}$ to a point $\mathrm{B} \in \mathcal{B}$, whose motion is linked by

$$\begin{aligned} \boldsymbol{r}_\mathrm{B} &= \boldsymbol{r}_\mathrm{A} + \boldsymbol{r}_\mathrm{AB} \\ \dot{\boldsymbol{r}}_\mathrm{B} &= \dot{\boldsymbol{r}}_\mathrm{A} + \dot{\boldsymbol{r}}_\mathrm{AB} \\ \ddot{\boldsymbol{r}}_\mathrm{B} &= \ddot{\boldsymbol{r}}_\mathrm{A} + \ddot{\boldsymbol{r}}_\mathrm{AB} \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \boldsymbol{r}_\mathrm{B} &= \boldsymbol{r}_\mathrm{A} + \boldsymbol{r}_\mathrm{AB}, \\ \boldsymbol{v}_\mathrm{B} &= \boldsymbol{v}_\mathrm{A} + \dot{\boldsymbol{r}}_\mathrm{AB}, \\ \boldsymbol{a}_\mathrm{B} &= \boldsymbol{a}_\mathrm{A} + \ddot{\boldsymbol{r}}_\mathrm{AB}. \end{aligned} \tag{3.4}$$



The term $\boldsymbol{r}_\mathrm{AB}$ and its time derivatives must be related to how the rigid body is moving through space. Note that if the *rigid* body $\mathcal{B}$ is purely translating, then all points are undergoing the exact same motion, so that $\boldsymbol{v}_\mathrm{A}(t) = \boldsymbol{v}_\mathrm{B}(t)$ and $\boldsymbol{a}_\mathrm{A}(t) = \boldsymbol{a}_\mathrm{B}(t)$ and the time derivatives of $\boldsymbol{r}_\mathrm{AB}$ vanish. Consequently, the terms $\dot{\boldsymbol{r}}_\mathrm{AB}$ and $\ddot{\boldsymbol{r}}_\mathrm{AB}$ must be associated with a *rotation* of body $\mathcal{B}$.

3.1.1 Rigid-body kinematics in 2D



Assume that a rigid body $\mathcal{B}$ is moving in 2D around a fixed point O ($\boldsymbol{v}_\mathrm{O} = \boldsymbol{0}$). The velocity vector of any point A on the body must lie in the 2D plane. In addition, since the body is rigid, we know from multiple-particle dynamics (see Section 2.2.2) that we must have

$$\boldsymbol{r}_\mathrm{OA} \cdot (\boldsymbol{v}_\mathrm{A} - \boldsymbol{v}_\mathrm{O}) = \boldsymbol{r}_\mathrm{OA} \cdot \boldsymbol{v}_\mathrm{A} = 0 \quad \Rightarrow \quad \boldsymbol{r}_\mathrm{OA} \perp \boldsymbol{v}_\mathrm{A}.$$

We may thus write the (yet unknown) velocity of a point A on body $\mathcal{B}$ as

$$\boldsymbol{v}_\mathrm{A} = \boldsymbol{\omega}_\mathrm{A} \times \boldsymbol{r}_\mathrm{OA} \qquad \text{with some angular velocity vector} \quad \boldsymbol{\omega}_\mathrm{A} = \omega_\mathrm{A}\boldsymbol{e}_3. \tag{3.5}$$

Since this holds for any arbitrary point A $\in \mathcal{B}$, we may pick another point B $\in \mathcal{B}$, whose velocity may be written as

$$\boldsymbol{v}_\mathrm{B} = \boldsymbol{\omega}_\mathrm{B} \times \boldsymbol{r}_\mathrm{OB} \qquad \text{with} \qquad \boldsymbol{\omega}_\mathrm{B} = \omega_\mathrm{B}\boldsymbol{e}_3. \tag{3.6}$$

Since points A and B belong to the same *rigid* body $\mathcal{B}$, their kinematics are coupled by

$$|\boldsymbol{r}_\mathrm{AB}| = \text{const.} \qquad \Rightarrow \qquad \boldsymbol{r}_\mathrm{AB} \cdot (\boldsymbol{v}_\mathrm{A} - \boldsymbol{v}_\mathrm{B}) = 0. \tag{3.7}$$

When inserting (3.5) and (3.6) into (3.7), we obtain

$$\begin{aligned} 0 &= \boldsymbol{r}_\mathrm{AB} \cdot (\boldsymbol{\omega}_\mathrm{A} \times \boldsymbol{r}_\mathrm{OA} - \boldsymbol{\omega}_\mathrm{B} \times \boldsymbol{r}_\mathrm{OB}) = (\boldsymbol{r}_\mathrm{OB} - \boldsymbol{r}_\mathrm{OA}) \cdot (\boldsymbol{\omega}_\mathrm{A} \times \boldsymbol{r}_\mathrm{OA} - \boldsymbol{\omega}_\mathrm{B} \times \boldsymbol{r}_\mathrm{OB}) \\ &= \boldsymbol{r}_\mathrm{OB} \cdot (\boldsymbol{\omega}_\mathrm{A} \times \boldsymbol{r}_\mathrm{OA}) + \boldsymbol{r}_\mathrm{OA} \cdot (\boldsymbol{\omega}_\mathrm{B} \times \boldsymbol{r}_\mathrm{OB}) = (\omega_\mathrm{A} - \omega_\mathrm{B})\, \boldsymbol{r}_\mathrm{OB} \cdot (\boldsymbol{e}_3 \times \boldsymbol{r}_\mathrm{OA})\,. \end{aligned} \tag{3.8}$$

Since the above needs to hold for *any* points A and B $\in \mathcal{B}$, we conclude that

$$\omega_\mathrm{A} = \omega_\mathrm{B} = \omega. \tag{3.9}$$

That is, at any instance of time t, there is a *unique* **angular velocity** $\boldsymbol{\omega}(t)$ for every point on the rigid body, and *every point on the body $\mathcal{B}$ rotates around* O *with the same angular velocity $\boldsymbol{\omega}(t)$*.

Now, consider two points A and B whose velocities are

$$\boldsymbol{v}_\mathrm{A} = \boldsymbol{\omega} \times \boldsymbol{r}_\mathrm{OA} \qquad \text{and} \qquad \boldsymbol{v}_\mathrm{B} = \boldsymbol{\omega} \times \boldsymbol{r}_\mathrm{OB},$$

so that

$$\boldsymbol{v}_\mathrm{AB} = \boldsymbol{v}_\mathrm{B} - \boldsymbol{v}_\mathrm{A} = \boldsymbol{\omega} \times (\boldsymbol{r}_\mathrm{OB} - \boldsymbol{r}_\mathrm{OA}) = \boldsymbol{\omega} \times \boldsymbol{r}_\mathrm{AB}.$$

Remarkably, this implies that the relative velocity between A and B is independent of O, as given by the so-called **velocity transfer formula**:

$$\boxed{\boldsymbol{v}_\mathrm{B} = \boldsymbol{v}_\mathrm{A} + \boldsymbol{\omega} \times \boldsymbol{r}_\mathrm{AB}} \tag{3.10}$$

In words, **every point in $\mathcal{B}$ is rotating around every other point in $\mathcal{B}$ with the same unique (instantaneous) angular velocity vector $\boldsymbol{\omega}(t)$**. Therefore, if we know ω and the velocity at any point within $\mathcal{B}$, then we can uniquely determine the velocity at any other point of the same body $\mathcal{B}$.

Moreover, notice that we do not even need to assume that body $\mathcal{B}$ is rotating around a *fixed* point O $\in \mathcal{B}$. The above derivation also holds if point O is moving, since changing (3.5) and (3.6) to

$$\boldsymbol{v}_\mathrm{A} = \boldsymbol{v}_\mathrm{O} + \boldsymbol{\omega} \times \boldsymbol{r}_\mathrm{OA}, \qquad \boldsymbol{v}_\mathrm{B} = \boldsymbol{v}_\mathrm{O} + \boldsymbol{\omega} \times \boldsymbol{r}_\mathrm{OB} \tag{3.11}$$

has no effect on (3.8) and (3.10) nor any of the conclusions drawn above, because $\boldsymbol{v}_\mathrm{A} - \boldsymbol{v}_\mathrm{B}$ remains unaffected and unchanged.

Now that point O does not need to be fixed, this brings up a question: *can the motion of a body $\mathcal{B}$* (that is rotating and translating through space) *always be interpreted as a pure rotation around a stationary point* Π *such that* $\boldsymbol{v}_\Pi = \boldsymbol{0}$. If so, it may be beneficial to locate point Π in order to more easily calculate the velocity (and later acceleration) of points on the body.

To this end, assume that we know the velocity of an arbitrary point $\mathrm{P} \in \mathcal{B}$ as $\boldsymbol{v}_\mathrm{P} = v_\mathrm{P}\boldsymbol{e}_\mathrm{P}$ along with the instantaneous angular velocity $\boldsymbol{\omega} = \omega\boldsymbol{e}_3$ of body $\mathcal{B}$. We may identify the location of our sought center of rotation Π as follows. If Π exists, we must be able to write

$$\boldsymbol{v}_\mathrm{P} = \boldsymbol{\omega} \times \boldsymbol{r}_{\Pi\mathrm{P}} \qquad \Rightarrow \qquad \boldsymbol{\omega} \times \boldsymbol{v}_\mathrm{P} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{\Pi\mathrm{P}})\,. \tag{3.12}$$

Exploiting that for a 2D rotation $\boldsymbol{r}_{\Pi\mathrm{P}} \cdot \boldsymbol{e}_3 = \boldsymbol{0}$, we have $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{\Pi\mathrm{P}}) = -\omega^2 \boldsymbol{r}_{\Pi\mathrm{P}}$, so that

$$\boldsymbol{\omega} \times \boldsymbol{v}_\mathrm{P} = \omega\boldsymbol{e}_3 \times \boldsymbol{v}_\mathrm{P} = -\omega^2\boldsymbol{r}_{\Pi\mathrm{P}} = \omega^2\boldsymbol{r}_{\mathrm{P}\Pi} \qquad \Rightarrow \qquad \boxed{\boldsymbol{r}_{\mathrm{P}\Pi} = \frac{1}{\omega}\boldsymbol{e}_3 \times \boldsymbol{v}_\mathrm{P}} \tag{3.13}$$

This tells us where point Π is located relative to point P. If we repeat the analogous construction starting from a second point $\mathrm{Q} \in \mathcal{B}$, then by analogy

$$\boldsymbol{r}_\Pi = \boldsymbol{r}_\mathrm{P} + \boldsymbol{r}_{\mathrm{P}\Pi} \stackrel{!}{=} \boldsymbol{r}_\mathrm{Q} + \boldsymbol{r}_{\mathrm{Q}\Pi} \quad \Rightarrow \quad \boldsymbol{r}_\mathrm{P} + \frac{1}{\omega}\boldsymbol{e}_3 \times \boldsymbol{v}_\mathrm{P} \stackrel{!}{=} \boldsymbol{r}_\mathrm{Q} + \frac{1}{\omega}\boldsymbol{e}_3 \times \boldsymbol{v}_\mathrm{Q}. \tag{3.14}$$

This relation, however, holds for any choice of $\mathrm{P}, \mathrm{Q} \in \mathcal{B}$. To see this, we rearrange the above into

$$\boldsymbol{r}_\mathrm{P} - \boldsymbol{r}_\mathrm{Q} = \frac{1}{\omega}\boldsymbol{e}_3 \times \boldsymbol{v}_\mathrm{Q} - \frac{1}{\omega}\boldsymbol{e}_3 \times \boldsymbol{v}_\mathrm{P} \qquad \Leftrightarrow \qquad \boldsymbol{r}_{\mathrm{QP}} = -\frac{1}{\omega}\boldsymbol{e}_3 \times (\boldsymbol{v}_\mathrm{P} - \boldsymbol{v}_\mathrm{Q}). \tag{3.15}$$

Next, using the velocity transfer formula $\boldsymbol{v}_\mathrm{P} = \boldsymbol{v}_\mathrm{Q} + \boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{QP}}$ gives

$$\boldsymbol{r}_{\mathrm{QP}} = -\frac{1}{\omega}\boldsymbol{e}_3 \times (\omega\boldsymbol{e}_3 \times \boldsymbol{r}_{\mathrm{QP}}) = -\boldsymbol{e}_3 \times (\boldsymbol{e}_3 \times \boldsymbol{r}_{\mathrm{QP}}) = -(-\boldsymbol{r}_{\mathrm{QP}}), \tag{3.16}$$

which is obviously satisfied for any points P and Q on body $\mathcal{B}$. Therefore, *for any body $\mathcal{B}$ in 2D there exists a unique point* Π *with respect to which each point on $\mathcal{B}$ undergoes a pure rotation.* Note that this point Π *does not need to lie within* $\mathcal{B}$ (no such assumption was made), and the location of this point may change with time (so it is only instantaneous).

Point Π is called the **instantaneous center of rotation** (**ICR**) which may *change with time* and which *may lie inside or outside body $\mathcal{B}$.* The ICR is further characterized by the fact that its velocity vanishes, i.e.,

$$\boxed{\boldsymbol{v}_\Pi = \boldsymbol{v}_\mathrm{ICR} = \boldsymbol{0}} \tag{3.17}$$

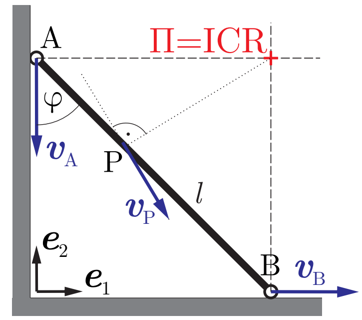
We remark that relation (3.13) also allows us to *identify the ICR graphically.* If the velocity vector $\boldsymbol{v}_\mathrm{P}$ at a point P is known, then we know from (3.13) that

$$\boldsymbol{r}_{\mathrm{P}\Pi} \perp \boldsymbol{v}_\mathrm{P}. \tag{3.18}$$

That is, Π must lie on a ray perpendicular to the velocity $\boldsymbol{v}_\mathrm{P}$. For example, if we only know the (non-parallel) velocity directions of two distinct points A and B, then the ICR can be found as the intersection of the two rays going through A and B and being perpendicular to the two respective velocity vectors (see Example 3.1 below).

Example 3.1. Ladder sliding down a wall

A ladder of length l is leaning against a rigid wall under an angle φ, as shown. Imagine that the ladder can slide frictionlessly, which results in the ladder sliding down under the action gravity. Since the two end points A and B can move only vertically and horizontally, respectively, the ICR of the ladder's motion can easily be found by a graphical construction: by drawing the two lines perpendicular to the velocities and finding their intersection, which is point Π, the ICR.



To be quantitative, we use the velocity transfer formula with $\boldsymbol{v}_\Pi = \boldsymbol{0}$, which gives the two velocities as

$$\boldsymbol{v}_\mathrm{A} = \boldsymbol{v}_\Pi + \boldsymbol{\omega} \times \boldsymbol{r}_{\Pi\mathrm{A}} = \dot\varphi \boldsymbol{e}_3 \times (-l\sin\varphi)\boldsymbol{e}_1 = -\dot\varphi\, l \sin\varphi\, \boldsymbol{e}_2,$$

and

$$\boldsymbol{v}_\mathrm{B} = \boldsymbol{v}_\Pi + \boldsymbol{\omega} \times \boldsymbol{r}_{\Pi\mathrm{B}} = \dot\varphi \boldsymbol{e}_3 \times (-l\cos\varphi)\boldsymbol{e}_2 = \dot\varphi\, l \cos\varphi\, \boldsymbol{e}_1. \tag{3.19}$$

We can also derive the relation between the two speeds for any angle φ (assuming $\dot\varphi \neq 0$) as

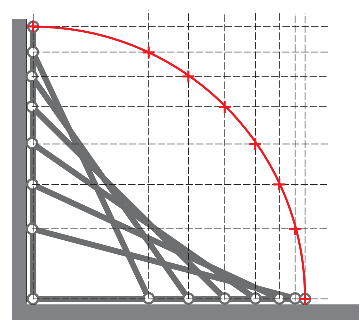
$$\frac{v_\mathrm{A}}{v_\mathrm{B}} = \frac{|\boldsymbol{v}_\mathrm{A}|}{|\boldsymbol{v}_\mathrm{B}|} = \tan\varphi. \tag{3.20}$$

In addition, we can find the velocity of any point P on the ladder as

$$v_\mathrm{P} = \dot\varphi |\boldsymbol{r}_\mathrm{P} - \boldsymbol{r}_\mathrm{ICR}|, \tag{3.21}$$

whose direction can be constructed graphically, knowing that $\boldsymbol{v}_\mathrm{P} \perp (\boldsymbol{r}_\mathrm{P} - \boldsymbol{r}_\mathrm{ICR})$, as shown above.

When graphically finding the ICR for each angle $\varphi \in [0, \pi/2]$, we obtain the so-called **centrode** as the line connecting all ICRs during the course of the ladder sliding from the vertical to the horizontal configuration. The centrode is shown on the right (red line).



The velocity transfer formula thus allows us to find the velocity of any point on the ladder if, e.g., the angular velocity $\dot\varphi = \omega$ is known. Consider, e.g., that points A and B carry each a particle of mass m and the ladder (which is assumed massless and rigid) is released from rest close to the vertical position. Then $\dot\varphi = \omega$ can be calculated by using the conservation of energy for frictionless sliding (we will come back to this problem when discussing energy principles for rigid bodies).

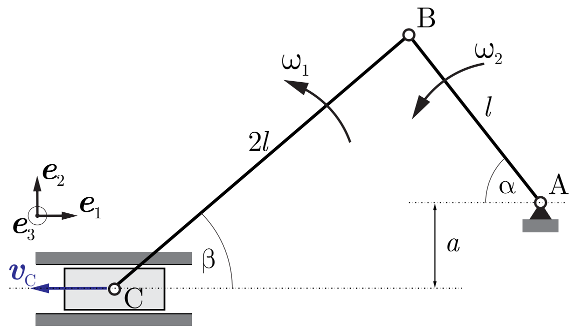
Example 3.2. Slider crank mechanism

A **slider crank** mechanism transforms linear motion (e.g., of a piston) into circular motion (e.g. of a wheel), or the other way around. It is frequently used, e.g., in engines. Shown below is an example, in which the linear motion of point C is causing a circular motion of point B.

Points A and B, and B and C are each connected by *rigid*, massless rods linked through hinges.

If the instantaneous velocity v_C of point C in the shown configuration (with known angles α, β) is known, what is the velocity of point B?

To solve this problem, we could in principle use the velocity transfer formula to determine $\boldsymbol{v}_B$, starting from the known velocity $\boldsymbol{v}_C$.



Unfortunately, the angular velocities of the two rods are unknown (and ω_1 those are needed for the velocity transfer formula). Furthermore, note that the two rods have, in general, two different angular velocities, since points A, B and C do *not* lie on the same rigid body. Here, we use the following strategy.

We may use the velocity transfer formula *twice* for point B, viz.

$$\boldsymbol{v}_B = \boldsymbol{v}_C + \boldsymbol{\omega}_1 \times \boldsymbol{r}_{CB} \qquad \text{and} \qquad \boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega}_2 \times \boldsymbol{r}_{AB}, \tag{3.22}$$

where ω_i denotes the unknown angular velocity of rod i. The known distance vectors are

$$\boldsymbol{r}_{CB} = 2l\cos\beta\,\boldsymbol{e}_1 + 2l\sin\beta\,\boldsymbol{e}_2, \qquad \boldsymbol{r}_{AB} = -l\cos\alpha\,\boldsymbol{e}_1 + l\sin\alpha\,\boldsymbol{e}_2. \tag{3.23}$$

The two equations in (3.22) provide two expressions for the sought velocity of point B, but either equation contains one unknown angular velocity – hence, we need to find those angular velocities.

From the problem definition, we see that

$$\boldsymbol{v}_C = -v_C\boldsymbol{e}_1, \qquad \boldsymbol{v}_A = \boldsymbol{0}, \qquad \boldsymbol{\omega}_1 = \dot{\beta}\boldsymbol{e}_3, \qquad \boldsymbol{\omega}_2 = -\dot{\alpha}\boldsymbol{e}_3. \tag{3.24}$$

Note the correct signs of the angular velocities due to the definition of α and β (e.g., $\omega_2 > 0$ implies that angle α becomes smaller, hence $\dot{\alpha} < 0$, whereas $\omega_1 > 0$ implies $\dot{\beta} > 0$). Inserting the distance vectors (3.23) and the velocity and angular velocity vectors (3.24) into (3.22) and equating the two expressions for $\boldsymbol{v}_B$, i.e., imposing $\boldsymbol{v}_C + \boldsymbol{\omega}_1 \times \boldsymbol{r}_{CB} = \boldsymbol{v}_A + \boldsymbol{\omega}_2 \times \boldsymbol{r}_{AB}$, gives

$$-v_C\boldsymbol{e}_1 + \dot{\beta}\boldsymbol{e}_3 \times (2l\cos\beta\,\boldsymbol{e}_1 + 2l\sin\beta\,\boldsymbol{e}_2) = -\dot{\alpha}\boldsymbol{e}_3 \times (-l\cos\alpha\,\boldsymbol{e}_1 + l\sin\alpha\,\boldsymbol{e}_2)\,, \tag{3.25}$$

which simplifies to

$$-v_C\boldsymbol{e}_1 + 2l\dot{\beta}\,(\cos\beta\,\boldsymbol{e}_2 - \sin\beta\,\boldsymbol{e}_1) = l\dot{\alpha}\,(\cos\alpha\,\boldsymbol{e}_2 + \sin\alpha\,\boldsymbol{e}_1)\,. \tag{3.26}$$

This in fact implies two scalar equations:

$$2l\dot{\beta}\cos\beta = l\dot{\alpha}\cos\alpha \qquad \text{and} \qquad -v_C - 2l\dot{\beta}\sin\beta = l\dot{\alpha}\sin\alpha. \tag{3.27}$$

Note that the first of these equations could also have been obtained from geometry directly since

$$2l\sin\beta = a + l\sin\alpha \qquad \Rightarrow \qquad 2l\dot{\beta}\cos\beta = l\dot{\alpha}\cos\alpha. \tag{3.28}$$

Equations (3.27) can be solved for

$$\dot{\alpha} = \frac{2\dot{\beta}\cos\beta}{\cos\alpha} \qquad \Rightarrow \qquad v_C = -\left(l\dot{\alpha}\sin\alpha - 2l\dot{\beta}\sin\beta\right) = -2l\dot{\beta}\left(\sin\beta + \cos\beta\tan\alpha\right), \tag{3.29}$$

so that

$$\dot{\beta} = -\frac{v_C}{2l\left(\sin\beta + \cos\beta\tan\alpha\right)} \qquad \Rightarrow \qquad \dot{\alpha} = \frac{2\cos\beta}{\cos\alpha}\frac{-v_C}{2l\left(\sin\beta + \cos\beta\tan\alpha\right)}. \tag{3.30}$$

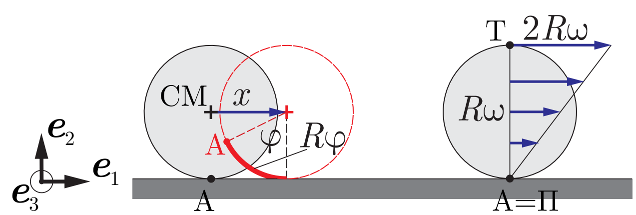
Indeed, as drawn we should obtain $\omega_1 = \dot{\beta} < 0$ and $\omega_2 = -\dot{\alpha} > 0$ for $v_C > 0$.

Finally, the sought velocity of the joint at B is obtained from either of the two expressions in (3.22), e.g.,

$$\boldsymbol{v}_B = \boldsymbol{\omega}_2 \times \boldsymbol{r}_{AB} = l\dot{\alpha}\left(\cos\alpha\,\boldsymbol{e}_2 + \sin\alpha\,\boldsymbol{e}_1\right) = -v_C\frac{\cos\beta}{\cos\alpha}\frac{\left(\cos\alpha\,\boldsymbol{e}_2 + \sin\alpha\,\boldsymbol{e}_1\right)}{\left(\sin\beta + \cos\beta\tan\alpha\right)}. \tag{3.31}$$

Example 3.3. Two-dimensional rolling disk

Consider a disk of radius R that is rolling without sliding. Due to the kinematic constraint of rolling, the rolled length $R\varphi$ must match the distance x traveled by the disk, hence $x = R\varphi$, where φ is measured clockwise in our example:



Since the disk's center of mass undergoes pure translation, we have $\boldsymbol{x}_{CM} = x\boldsymbol{e}_1 = R\varphi\boldsymbol{e}_1$ and consequently $\boldsymbol{v}_{CM} = R\omega\boldsymbol{e}_1$ where $\omega = \dot{\varphi}$. The top point of the disk is moving with

$$\boldsymbol{v}_T = \boldsymbol{v}_{CM} + (-\omega)\boldsymbol{e}_3 \times R\boldsymbol{e}_2 = R\omega\boldsymbol{e}_1 + (-R\omega)(-\boldsymbol{e}_1) = 2R\omega\boldsymbol{e}_1 \tag{3.32}$$

and the bottom point in contact with the ground is moving with

$$\boldsymbol{v}_A = \boldsymbol{v}_{CM} + (-\omega)\boldsymbol{e}_3 \times (-R)\boldsymbol{e}_2 = R\omega\boldsymbol{e}_1 - R\omega\boldsymbol{e}_1 = \boldsymbol{0}. \tag{3.33}$$

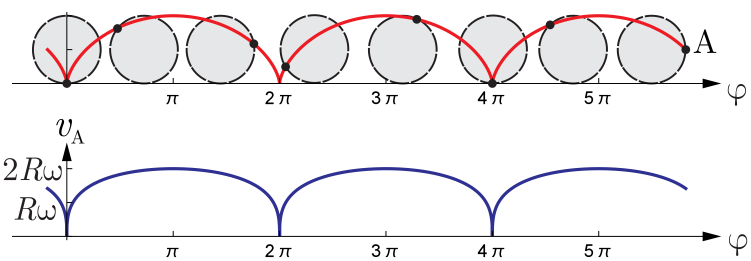
Hence the contact point A is also the *instantaneous center of rotation* (A = Π). The velocity of point A after a rotation by φ is given by

$$\begin{aligned}\boldsymbol{v}_A &= \boldsymbol{v}_{CM} + (-\omega)\boldsymbol{e}_3 \times \left[-R\sin\varphi\,\boldsymbol{e}_1 - R\cos\varphi\,\boldsymbol{e}_2\right] = R\omega\boldsymbol{e}_1 + R\omega\boldsymbol{e}_3 \times \left[\sin\varphi\,\boldsymbol{e}_1 + \cos\varphi\,\boldsymbol{e}_2\right] \\ &= R\omega\boldsymbol{e}_1 + R\omega\left[\sin\varphi\,\boldsymbol{e}_2 - \cos\varphi\,\boldsymbol{e}_1\right] = R\omega\left[(1-\cos\varphi)\boldsymbol{e}_1 + \sin\varphi\,\boldsymbol{e}_2\right]\end{aligned} \tag{3.34}$$

so that

$$v_A = |\boldsymbol{v}_A| = R\omega\sqrt{(1-\cos\varphi)^2 + \sin\varphi^2} = R\omega\sqrt{2(1-\cos\varphi)}. \tag{3.35}$$

As expected, the velocity v_A vanishes periodically whenever $\varphi = n \cdot 2\pi$ with $n \in \mathbb{Z}$ (i.e., whenever A is the bottom point and hence the instantaneous center of rotation in contact with the ground). Its motion and velocity are shown below:



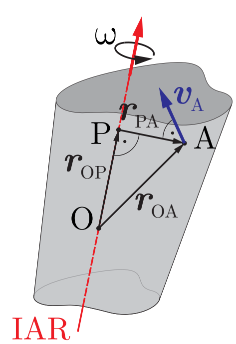
Note that the velocity of any other point P on the disk can easily be obtained both graphically and analytically as

$$\boldsymbol{v}_P = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{AP} = (-\omega)\boldsymbol{e}_3 \times \boldsymbol{r}_{AP}. \tag{3.36}$$

It is important to emphasize that the relation (3.36) *applies only to points* P *that are on the disk* (e.g., the transfer formula cannot be used to compute the velocity between a point on the disk and a point that is not on the disk but, e.g., on the ground).

3.1.2 Rigid-body kinematics in 3D

A rotation in 3D is uniquely defined by an axis of rotation (whose orientation we define by a unit vector $\boldsymbol{e}_\omega$ with $|\boldsymbol{e}_\omega| = 1$) and an angle of rotation φ around the axis of rotation in the sense defined by the orientation of $\boldsymbol{e}_\omega$ (using the *right-hand rule*). Consequently, a rotational motion of a body in 3D is defined by an angular velocity vector



$$\boldsymbol{\omega}(t) = \omega(t)\boldsymbol{e}_\omega(t) \qquad \text{with} \qquad \omega(t) \in \mathbb{R}, \quad |\boldsymbol{e}_\omega(t)| = 1. \tag{3.37}$$

We note that for general rigid-body motions in 3D $\boldsymbol{e}_\omega(t)$ changes with time, so we say that $\boldsymbol{e}_\omega$ defines the **instantaneous axis of rotation** (**IAR**).

The extension to 3D of the kinematic relations derived in Section 3.1.1 is now relatively straightforward. Instead of the instantaneous *center* of rotation (ICR), we now have an instantaneous *axis* of rotation (IAR), around which the body is instantaneously rotating with an angular velocity $\omega(t)$. Rotations in 2D are included as a special case for which $\boldsymbol{e}_\omega(t) = \boldsymbol{e}_3 = \text{const}$.

Pick any point O that lies on the IAR and consider a rotation with angular velocity $\boldsymbol{\omega} = \omega\boldsymbol{e}_\omega$. For any point $A \in \mathcal{B}$ we can identify an associated point $P \in \mathcal{B}$ which is the closest point on the IAR to A (see the schematic above). Then, we may interpret the 3D rotation as an instantaneous 2D rotation of point A around point P in the plane perpendicular to $\boldsymbol{e}_\omega$ and going through A and P. Hence, we may re-use the velocity transfer formula derived for 2D, and the velocity of point A is

$$\boldsymbol{v}_A = \omega\boldsymbol{e}_\omega \times \boldsymbol{r}_{PA}. \tag{3.38}$$

Note that $\boldsymbol{e}_\omega \times \boldsymbol{r}_{\mathrm{OP}} = |\boldsymbol{r}_{\mathrm{OP}}|(\boldsymbol{e}_\omega \times \boldsymbol{e}_\omega) = \mathbf{0}$, so that we may equivalently write

$$\boldsymbol{v}_{\mathrm{A}} = \omega \boldsymbol{e}_\omega \times (\boldsymbol{r}_{\mathrm{OP}} + \boldsymbol{r}_{\mathrm{PA}}) = \boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{OA}}. \tag{3.39}$$

This shows that, in case of a 3D rotation with angular velocity vector $\boldsymbol{\omega}$, the velocity of *any* point $\mathrm{A} \in \mathcal{B}$ can be expressed as $\boldsymbol{v}_{\mathrm{A}} = \boldsymbol{\omega} \times \boldsymbol{r}_{\Pi\mathrm{A}}$ (like in 2D) with Π being any point that lies on the instantaneous axis of rotation.

As a new complication in 3D, the IAR need not have vanishing velocity but it can translate with a velocity $\boldsymbol{v}_\Pi = v_\Pi \boldsymbol{e}_\omega$ (since in any plane perpendicular to $\boldsymbol{e}_\omega$ the body undergoes a pure rotation, the IAR cannot translate in the plane – but, of course, it is free to move parallel to $\boldsymbol{e}_\omega$, which does not affect the rotations). Because the body is rigid, every point on the IAR must move with the same velocity $\boldsymbol{v}_\Pi$. Hence, *points on the IAR may move only parallel to the IAR*. This, however, does not alter any of our prior derivations for the following reason. Take $\boldsymbol{v}_{\mathrm{A}} = \boldsymbol{v}_\Pi + \boldsymbol{\omega} \times \boldsymbol{r}_{\Pi\mathrm{A}}$ and $\boldsymbol{v}_{\mathrm{B}} = \boldsymbol{v}_\Pi + \boldsymbol{\omega} \times \boldsymbol{r}_{\Pi\mathrm{B}}$, with $\boldsymbol{v}_\Pi = v_\Pi \boldsymbol{e}_\omega$. Subtraction of the two equations eliminates $\boldsymbol{v}_\Pi$ – like $\boldsymbol{v}_{\mathrm{O}}$ vanished in the 2D derivation (cf. Eq. (3.11) and its discussion). As a consequence, all of the expressions derived above, especially (3.10), also hold in 3D, i.e., we may still apply the velocity transfer formula

$$\boldsymbol{v}_{\mathrm{B}} = \boldsymbol{v}_{\mathrm{A}} + \boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{AB}} \tag{3.40}$$

This shows that, in 3D as in 2D, *every point on $\mathcal{B}$ is rotating around every other point on $\mathcal{B}$ with $\boldsymbol{\omega}$*.

Also, if the velocity $\boldsymbol{v}_{\mathrm{A}}$ of a point $\mathrm{A} \in \mathcal{B}$ as well as ω is known, then the vector pointing from A to the axis of rotation (yielding the vector from A to P in the above schematic) is given by (3.13) (which is unaffected by any non-zero $\boldsymbol{v}_\Pi = v_\Pi \boldsymbol{e}_\omega$ since $\boldsymbol{e}_\omega \times \boldsymbol{v}_\Pi = \mathbf{0}$), so that

$$\boldsymbol{r}_{\mathrm{AP}} = \frac{1}{\omega} \boldsymbol{e}_\omega \times \boldsymbol{v}_{\mathrm{A}}, \tag{3.41}$$

which may again aid in the graphical construction of points on the IAR.

Having established the velocity transfer expression (which is identical in 2D and 3D), we can also apply the very same transfer relation to the accelerations of two points A and B on the same body $\mathcal{B}$ as follows. Starting with $\boldsymbol{v}_{\mathrm{B}} = \boldsymbol{v}_{\mathrm{A}} + \boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{AB}}$, differentiation with respect to time yields

$$\begin{aligned} \boldsymbol{a}_{\mathrm{B}} &= \dot{\boldsymbol{v}}_{\mathrm{A}} + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{AB}}) = \boldsymbol{a}_{\mathrm{A}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{\mathrm{AB}} + \boldsymbol{\omega} \times \dot{\boldsymbol{r}}_{\mathrm{AB}} \\ &= \boldsymbol{a}_{\mathrm{A}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{\mathrm{AB}} + \boldsymbol{\omega} \times (\boldsymbol{v}_{\mathrm{B}} - \boldsymbol{v}_{\mathrm{A}}) = \boldsymbol{a}_{\mathrm{A}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{\mathrm{AB}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{AB}}). \end{aligned} \tag{3.42}$$

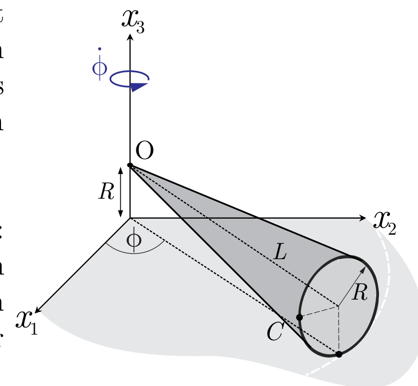
Altogether, the **velocity transfer formula** and **acceleration transfer formula** for two points A and B *on the same rigid body*, valid in 2D and in 3D, have been obtained as, respectively:

$$\boxed{\begin{aligned} \boldsymbol{v}_{\mathrm{B}} &= \boldsymbol{v}_{\mathrm{A}} + \boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{AB}} \\ \boldsymbol{a}_{\mathrm{B}} &= \boldsymbol{a}_{\mathrm{A}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{\mathrm{AB}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{AB}}) \end{aligned}} \tag{3.43}$$

Thus, *if we know the velocity (acceleration) at any point $A \in \mathcal{B}$ along with the angular velocity vector $\boldsymbol{\omega}$, we can uniquely calculate the velocity (acceleration) at any other point $B \in \mathcal{B}$.*

Example 3.4. Rolling cone

A rigid cone of length L and end radius R is anchored by a joint at height R and rolling on the ground without slipping with an angular velocity $\dot{\phi}$ around the x_3-axis (as shown). What is the (instantaneous) velocity of point C within the Cartesian reference frame, for a given orientation ϕ of the cone?



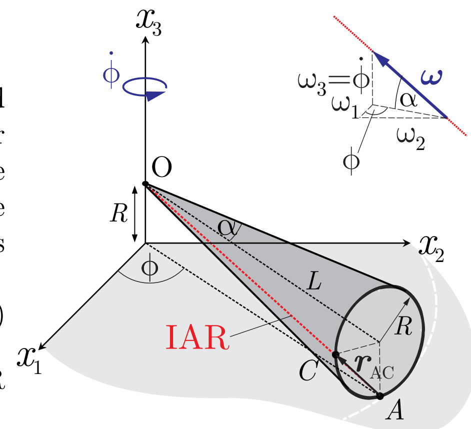
The motion of the cone is complex as it involves two rotations: on the one hand, it rotates around the x_3-axis with known angular velocity $\dot{\phi}$; simultaneously, it rotates about its own central axis as it rolls over the ground with unknown angular velocity.

We will solve this problem in two ways to illustrate the concepts at play. *First*, we will solve the problem in a general fashion, using the fixed Cartesian coordinate system shown above with a given angle ϕ within the x_1-x_2-plane (Approach I). *Second*, we will re-solve the problem using a more convenient coordinate system that rotates with the cone (Approach II).

Approach I:

We know that we can always describe the motion of a rigid body in 3D by a pure rotation about the IAR with an angular velocity vector $\boldsymbol{\omega}$, plus possibly a translation that can only be parallel to the IAR. Therefore, if we know any point Π on the IAR, we may write the velocity of any point P on the cone as

$$\boldsymbol{v}_\mathrm{P} = \boldsymbol{v}_\Pi + \boldsymbol{\omega} \times \boldsymbol{r}_{\Pi P}. \tag{3.44}$$



Unfortunately, we do not yet know the location of the IAR nor the angular velocity vector $\boldsymbol{\omega}$.

First, we know that point O is stationary: $\boldsymbol{v}_\mathrm{O} = \mathbf{0}$. Second, we have learned from the rotating disk (Example 3.3) that point A also has a vanishing velocity, $\boldsymbol{v}_\mathrm{A} = \mathbf{0}$, since it is the ICR of the rolling end of the cone. The velocity transfer formula between A and C tells us that

$$\boldsymbol{v}_\mathrm{A} = \boldsymbol{v}_\mathrm{O} + \boldsymbol{\omega} \times \boldsymbol{r}_\mathrm{OA} \qquad \Rightarrow \qquad \boldsymbol{\omega} \times \boldsymbol{r}_\mathrm{OA} = \mathbf{0} \qquad \Rightarrow \qquad \boldsymbol{\omega} \parallel \boldsymbol{r}_\mathrm{OA}. \tag{3.45}$$

Now we know at least the orientation of $\boldsymbol{\omega}$. Further, applying the velocity transfer formula again for any other point P on the line connecting points A and O yields

$$\boldsymbol{v}_\mathrm{P} = \boldsymbol{v}_\mathrm{O} + \boldsymbol{\omega} \times \boldsymbol{r}_\mathrm{OP} = \boldsymbol{\omega} \times \boldsymbol{r}_\mathrm{OP} = \mathbf{0} \qquad \text{because} \qquad \boldsymbol{r}_\mathrm{OP} \parallel \boldsymbol{r}_\mathrm{OA} \parallel \boldsymbol{\omega}. \tag{3.46}$$

From this it follows that all points on the line connecting points A and O have a zero velocity. Hence, *the line through* OA *is the IAR of the cone* (and, since $\boldsymbol{v}_\mathrm{O} = \mathbf{0}$, the IAR has no parallel velocity). We may draw the general conclusion:

If two distinct points on the same (rotating) body have zero velocity, then the body's IAR goes through those two points. (3.47)

Now that we know the orientation of the IAR, we may exploit that we know the angular velocity component around the x_3-axis, viz.

$$\boldsymbol{\omega} \cdot \boldsymbol{e}_3 = \dot{\phi}, \tag{3.48}$$

so that

$$\boldsymbol{\omega} = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \dot{\phi} \boldsymbol{e}_3 \qquad \text{or} \qquad [\boldsymbol{\omega}]_{\mathcal{C}} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \dot{\phi} \end{pmatrix} \tag{3.49}$$

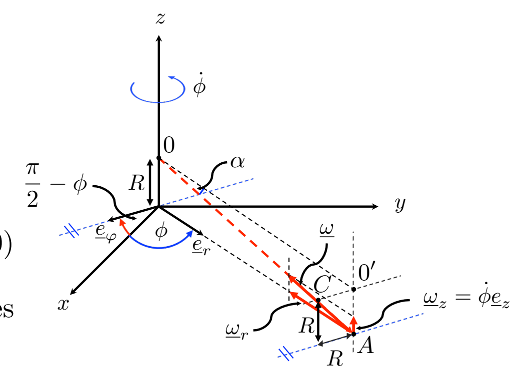
with yet unknown components $\omega_1, \omega_2 \in \mathbb{R}$.

Since we know the direction of $\boldsymbol{\omega}$, we may write

$$\boldsymbol{\omega} = \omega \boldsymbol{e}_\omega \qquad \text{with} \qquad \boldsymbol{e}_\omega = \frac{\boldsymbol{r}_{\text{OA}}}{|\boldsymbol{r}_{\text{OA}}|}$$

and

$$\boldsymbol{r}_{\text{OA}} = L \cos\phi\, \boldsymbol{e}_1 + L \sin\phi\, \boldsymbol{e}_2 - R \boldsymbol{e}_3. \tag{3.50}$$



Evaluating the components of vector $\boldsymbol{r}_{\text{OA}}/|\boldsymbol{r}_{\text{OA}}|$ for angles $0 \leq \phi, \alpha \leq \frac{\pi}{2}$ (as drawn) and defining $\tan\alpha = R/L$, gives

$$[\boldsymbol{\omega}]_{\mathcal{C}} = \omega \begin{pmatrix} \cos\phi\, \cos\alpha \\ \sin\phi\, \cos\alpha \\ -\sin\alpha \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \dot{\phi} \end{pmatrix}, \tag{3.51}$$

which we may solve for

$$\omega = -\frac{\dot{\phi}}{\sin\alpha} \qquad \text{and} \qquad [\boldsymbol{\omega}]_{\mathcal{C}} = \dot{\phi} \begin{pmatrix} -\cos\phi\, \cot\alpha \\ -\sin\phi\, \cot\alpha \\ 1 \end{pmatrix}. \tag{3.52}$$

Now that we know $\boldsymbol{\omega}$, the velocity of point C follows from the velocity transfer formula, e.g.

$$\boldsymbol{v}_{\text{C}} = \boldsymbol{v}_{\text{A}} + \boldsymbol{\omega} \times \boldsymbol{r}_{\text{AC}} = \boldsymbol{\omega} \times \boldsymbol{r}_{\text{AC}} \qquad \text{with} \qquad \boldsymbol{r}_{\text{AC}} = R\boldsymbol{e}_3 + R(\sin\phi\, \boldsymbol{e}_1 - \cos\phi\, \boldsymbol{e}_2) \tag{3.53}$$

so that

$$\begin{aligned} \boldsymbol{v}_{\text{C}} &= \dot{\phi}\,(-\cos\phi\, \cot\alpha\, \boldsymbol{e}_1 - \sin\phi\, \cot\alpha\, \boldsymbol{e}_2 + \boldsymbol{e}_3) \times [R\boldsymbol{e}_3 + R(\sin\phi\, \boldsymbol{e}_1 - \cos\phi\, \boldsymbol{e}_2)] \\ &= R\dot{\phi}\big[(\cos\phi - \sin\phi\, \cot\alpha)\boldsymbol{e}_1 + (\cos\phi\, \cot\alpha + \sin\phi)\boldsymbol{e}_2 + \cot\alpha\, \boldsymbol{e}_3\big]. \end{aligned} \tag{3.54}$$

This completes the problem. The velocity of any other point on the cone can be obtained analogously. Finally, keep in mind that these are all *instantaneous* velocities only valid in the current configuration of the cone. For example, as the cone rotates, point A will lift off the ground and assume a non-zero velocity while the new point in contact with the ground (A') will define the new IAR as the line connectin OA', etc. Point C's velocity will also change and, e.g., turn to zero once point C touches the ground.

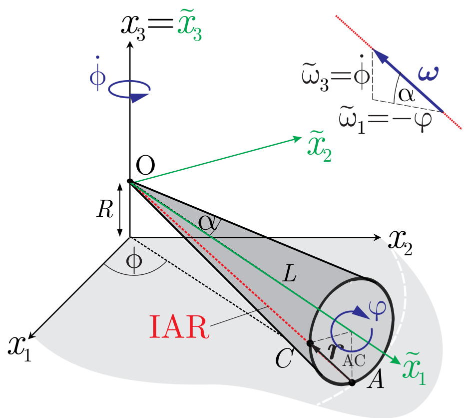
Approach II:

We can solve the above problem also in a more convenient fashion (the above is more general and therefore a helpful reference; the following uses a simpler approach).

We know that the cone rotates simultaneously around the $\boldsymbol{e}_3$-axis with angular velocity $\dot{\phi}$ and around its central axis with an orientation

$$\boldsymbol{e}_1^{\mathcal{M}} = \cos\phi\,\boldsymbol{e}_1 + \sin\phi\,\boldsymbol{e}_2 \tag{3.55}$$

and a yet unknown angular velocity $\dot{\varphi}$ (where φ denotes the angle of rotation of the cone about its central axis of symmetry in the shown orientation).



We may use kinematics to relate the angles of rotation. Since the cone is rolling in a frictionless manner, the cone must satisfy the kinematic constraint

$$\phi\,L = \varphi R \qquad \Rightarrow \qquad \dot{\phi}\,L = \dot{\varphi}R. \tag{3.56}$$

In other words, a rotation by ϕ implies that the cone's end cross-section has traveled a distance ϕL on the ground, and the motion of the center of mass of a disk was shown to move as φR (we chose the orientation of φ so it is positive). In our chosen coordinate system, the total angular velocity vector is thus

$$\boldsymbol{\omega} = \dot{\phi}\,\boldsymbol{e}_3 + (-\dot{\varphi})\boldsymbol{e}_1^{\mathcal{M}} = \dot{\phi}\,\boldsymbol{e}_3 - \dot{\phi}\frac{L}{R}\boldsymbol{e}_1^{\mathcal{M}} = \dot{\phi}\,\boldsymbol{e}_3 - \dot{\phi}\cot\alpha\,\boldsymbol{e}_1^{\mathcal{M}} \qquad \text{with} \qquad \tan\alpha = R/L. \tag{3.57}$$

Inserting (3.55) leads to

$$\boldsymbol{\omega} = \dot{\phi}\,\boldsymbol{e}_3 - \dot{\phi}\cot\alpha\,(\cos\phi\,\boldsymbol{e}_1 + \sin\phi\,\boldsymbol{e}_2) \qquad \Rightarrow \qquad [\boldsymbol{\omega}]_{\mathcal{C}} = \dot{\phi}\begin{pmatrix} -\cos\phi\,\cot\alpha \\ -\sin\phi\,\cot\alpha \\ 1 \end{pmatrix}, \tag{3.58}$$

which is the same angular velocity found in Approach I above, see Eq. (3.51). The velocity $\boldsymbol{v}_{\mathrm{C}}$ follows as in Approach I from applying the velocity transfer formula once again.

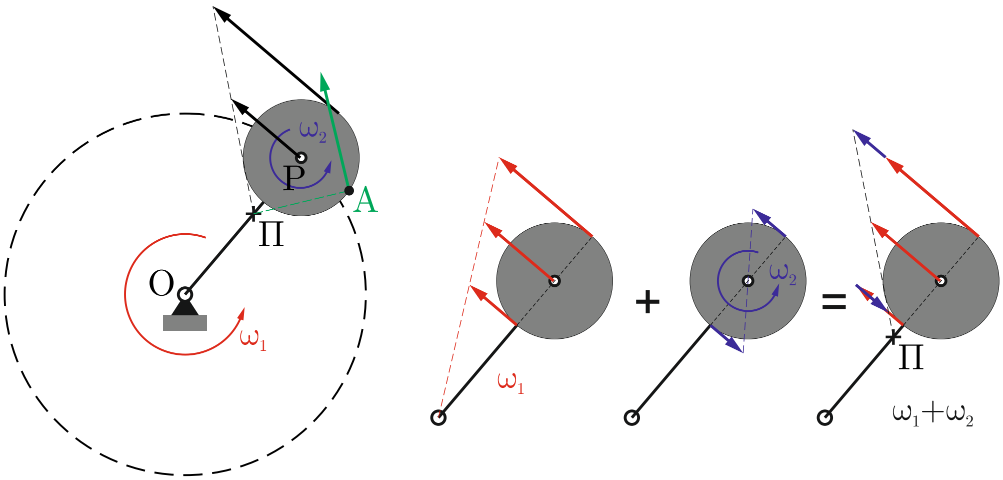
In Approach II of the above example we exploited one important observation, viz. that **rotation vectors are additive**. That is, if a body is *simultaneously rotating about multiple axes* with unit vectors $\boldsymbol{g}_i$ and angular velocities ω_i $(i = 1, \ldots, n)$, then the **total angular velocity vector** is

$$\boldsymbol{\omega} = \sum_{i=1}^{n} \omega_i \boldsymbol{g}_i. \tag{3.59}$$

So, we may add the individual angular velocity vectors to arrive at the total angular velocity vector (as done, e.g., in Approach II of Example 3.4).

Example 3.5. Double-rotating funfair ride

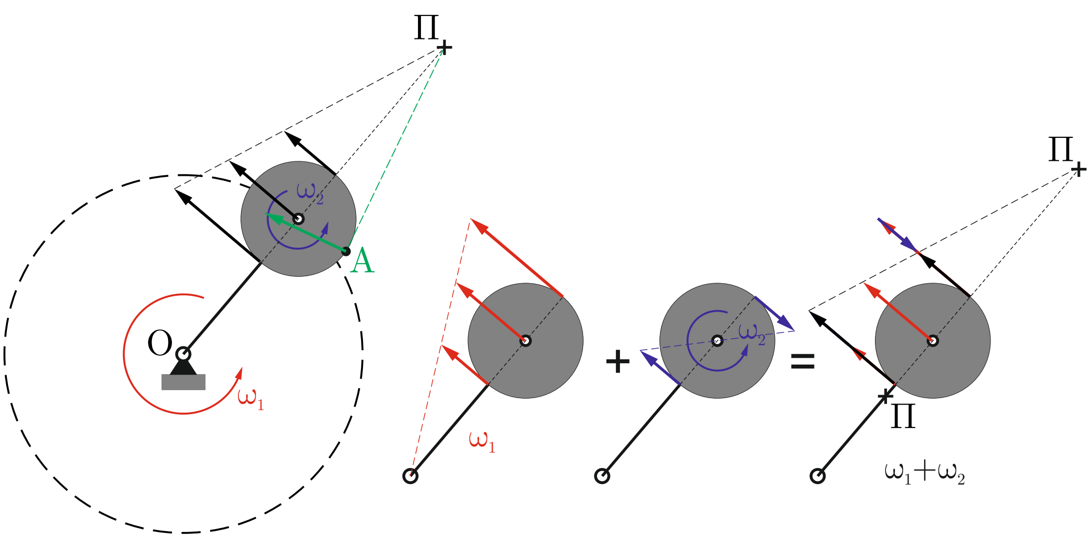
A classical example of two simultaneous rotations is a funfair ride, in which a small circular cart is rotating with a *relative* angular velocity ω_2 around a joint P which itself is rotating with angular velocity ω_1 around a fixed pole O, as shown below. This case is simple since both rotations are in the plane (hence the concept of an instantaneous center of rotation in 2D applies).



The combined rotation produces a velocity field on the cart that can indeed be described as a rotation with angular velocity $\omega_1 + \omega_2$ around the ICR Π (which is neither point O nor point P); instead it can be obtained graphically as shown above. The velocity of any other point A on the body then follows as a rotation around Π with $\omega_1 + \omega_2$. When using the velocity transfer formula, we obtain the velocity of A as

$$\boldsymbol{v}_{\mathrm{A}} = \boldsymbol{v}_{\mathrm{O}} + \boldsymbol{\omega}_1 \times \boldsymbol{r}_{\mathrm{OP}} + (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \boldsymbol{r}_{\mathrm{PA}}. \tag{3.60}$$

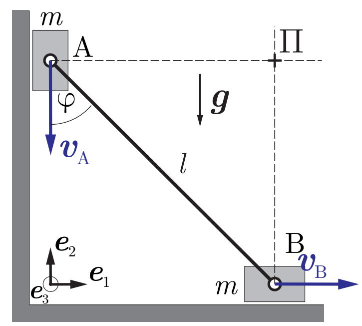
Shown below is an analogous scenario with counter-rotating angular velocities (assuming $\omega_1 > 0$ and $\omega_2 < 0$).



It is important to point out that $\boldsymbol{\omega}_2$ here is the *relative* angular velocity *around the rotating pole* P. That is, $\boldsymbol{\omega}_2$ is the angular velocity generated by a motor that is sitting on the arm OA and rotating with $\boldsymbol{\omega}_1$ around O. The *absolute* angular velocity of the cart is, as demonstrated above, $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. Hence, if the absolute angular velocity of the rotating cart (around its ICR) was given as, say, $\boldsymbol{\omega}^*$, then $(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)$ in (3.60) would have to be replaced by $\boldsymbol{\omega}^*$.

Example 3.6. Sliding ladder revisited

Let us revisit Example 3.1 of the ladder sliding down a wall with some modifications. Assume that points A and B carry particles, each of mass m, which slide without friction on the walls under the action of gravity. If the massless rod of length l is initially at rest close to the vertical position, what are the accelerations of points A and B, and what is the force in the rigid rod as a function of the angle φ?



In order to find the accelerations of the two particles (or the force in the rod, which is then obtained from linear momentum balance also requiring the force in the rod) we could use the acceleration transfer formula. Unfortunately, we however do not yet know the acceleration of any point in the system.

In Example 3.1 we already found the two velocities as

$$\boldsymbol{v}_{\mathrm{A}} = -\dot{\varphi}\, l \sin\varphi\, \boldsymbol{e}_2, \qquad \boldsymbol{v}_{\mathrm{B}} = \dot{\varphi}\, l \cos\varphi\, \boldsymbol{e}_1. \tag{3.61}$$

To find the yet unknown angular velocity $\omega = \dot{\varphi}$, we may use the conservation of energy, since the system is conservative. Comparing the energy at some angle φ with angular velocity $\dot{\varphi}$ with the initial state at rest at $\varphi \approx 0$ having only the potential energy of mass A at altitude l, we have

$$mgl = \frac{m}{2} v_{\mathrm{A}}^2 + \frac{m}{2} v_{\mathrm{B}}^2 + mgl\cos\varphi. \tag{3.62}$$

Inserting the two velocities from (3.61) leads to

$$mgl = \frac{m}{2}\dot{\varphi}^2 l^2 \sin^2\varphi + \frac{m}{2}\dot{\varphi}^2 l^2 \cos^2\varphi + mgl\cos\varphi, \tag{3.63}$$

so that the angular velocity of the system is

$$\omega^2 = \dot{\varphi}^2 = 2\frac{g}{l}(1 - \cos\varphi). \tag{3.64}$$

In order to calculate the accelerations and (ultimately) the inner force in the rod, we use the acceleration transfer formula:

$$\boldsymbol{a}_{\mathrm{B}} = \boldsymbol{a}_{\mathrm{A}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{\mathrm{AB}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{AB}}) \tag{3.65}$$

with

$$\boldsymbol{r}_{\mathrm{AB}} = l \sin\varphi\, \boldsymbol{e}_1 - l \cos\varphi\, \boldsymbol{e}_2, \qquad \text{and} \qquad \boldsymbol{\omega} = \omega\, \boldsymbol{e}_3, \quad \dot{\boldsymbol{\omega}} = \dot{\omega}\, \boldsymbol{e}_3 \tag{3.66}$$

so that

$$\boldsymbol{a}_{\mathrm{B}} = \boldsymbol{a}_{\mathrm{A}} + \dot{\omega} l \left(\sin\varphi\, \boldsymbol{e}_2 + \cos\varphi\, \boldsymbol{e}_1\right) - \omega^2 l \left(\sin\varphi\, \boldsymbol{e}_1 - \cos\varphi\, \boldsymbol{e}_2\right). \tag{3.67}$$

We now exploit that the directions of motion of the two masses are known, so we may define

$$\boldsymbol{a}_{\mathrm{A}} = -a_{\mathrm{A}} \boldsymbol{e}_2, \qquad \boldsymbol{a}_{\mathrm{B}} = a_{\mathrm{B}} \boldsymbol{e}_1. \tag{3.68}$$

Insertion into (3.67) and decomposing the vector equation into its two components in the $\boldsymbol{e}_1$- and $\boldsymbol{e}_2$-directions yields

$$a_{\mathrm{A}} = \dot{\omega} l \sin\varphi + \omega^2 l \cos\varphi, \qquad a_{\mathrm{B}} = \dot{\omega} l \cos\varphi - \omega^2 l \sin\varphi. \tag{3.69}$$

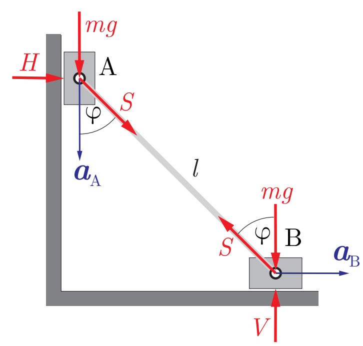
The yet unknown angular acceleration $\dot{\omega}$ can be obtained by time differentiation of (3.64), giving

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega^2 = \frac{\mathrm{d}}{\mathrm{d}t}\left[2\frac{g}{l}(1-\cos\varphi)\right] \qquad \Rightarrow \qquad 2\omega\dot{\omega} = 2\frac{g}{l}\sin\varphi\,\omega \qquad \Rightarrow \qquad \dot{\omega} = \frac{g}{l}\sin\varphi \tag{3.70}$$

(assuming that $\omega \neq 0$ during the sliding of the ladder). Finally, inserting $\dot{\omega}$ and ω^2 from (3.64) and (3.70), respectively, into (3.69) yields the sought accelerations as

$$a_{\mathrm{A}}(\varphi) = g\left(1 + 2\cos\varphi - 3\cos^2\varphi\right), \qquad a_{\mathrm{B}}(\varphi) = g\left(3\cos\varphi - 2\right)\sin\varphi. \tag{3.71}$$

From these two, the force in the rigid, massless rod can be calculated from the balance of linear momentum applied to either of the two masses. From free-body diagrams on each of the two masses we obtain the linear momentum balance in the respective direction of motion as



$$m a_{\mathrm{A}} = S\cos\varphi + mg \qquad \Rightarrow \qquad S = m\frac{a_{\mathrm{A}}(\varphi) - g}{\cos\varphi} \tag{3.72}$$

or

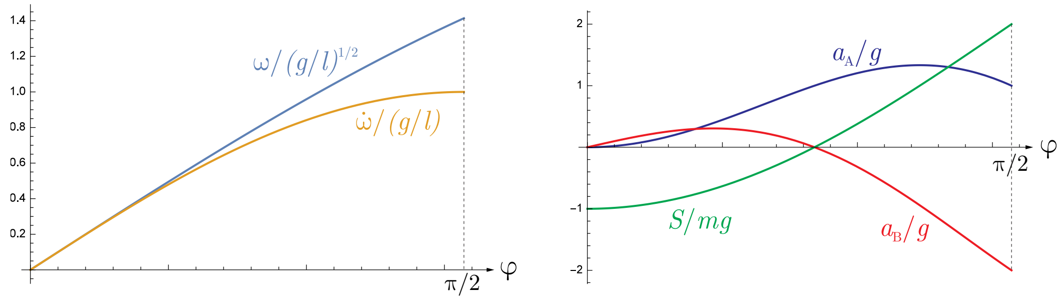
$$m a_{\mathrm{B}} = -S\sin\varphi \qquad \Rightarrow \qquad S = -m\frac{a_{\mathrm{B}}(\varphi)}{\sin\varphi}. \tag{3.73}$$

Inserting the above accelerations a_{A} and a_{B} from (3.71) shows that both equations indeed result in the same inner force in the rod, viz.

$$S(\varphi) = mg\left(2 - 3\cos\varphi\right). \tag{3.74}$$

As a quick sanity check, the force in the rod is $S = -mg$ when $\varphi = 0$, i.e., the rod is under a compressive load that equals the weight of mass A. Once the ladder slides, the force in the rod grows continuously and turns positive. When $\varphi = \pi/2$ (the ladder has fully come down), then $S = 2mg$, so the rod is under tension. The force vanishes in between when $\varphi = \arccos(2/3) = \varphi_0$ (in this configuration $a_{\mathrm{A}}(\varphi_0) = g$, $a_{\mathrm{B}}(\varphi_0) = 0$, and $S(\varphi_0) = 0$).

The two accelerations as well as the force in the rod, along with the angular velocity and acceleration of the system, are shown below for $\varphi \in (0, \pi/2)$ (all quantities are normalized as labeled).



Let us finally note that we can also formulate linear momentum balance *on the complete system*, which can be used to compute, e.g., the two reaction forces from the ground, since

$$M\boldsymbol{a}_{\mathrm{CM}} = \sum_i \boldsymbol{F}_i^{\mathrm{ext}} = -2mg\boldsymbol{e}_2 + V\boldsymbol{e}_2 + H\boldsymbol{e}_1.$$

with

$$M\boldsymbol{a}_{\mathrm{CM}} = 2m\frac{1}{2m}\left(m\boldsymbol{a}_{\mathrm{A}} + m\boldsymbol{a}_{\mathrm{B}}\right) = m\left(-a_{\mathrm{A}}\boldsymbol{e}_2 + a_{\mathrm{B}}\boldsymbol{e}_1\right), \tag{3.75}$$

which leads to

$$V = m(2g - a_{\mathrm{A}}), \qquad H = ma_{\mathrm{B}}. \tag{3.76}$$

The latter two could, of course, also have been obtained from linear momentum balance on the two individual masses (using the respective either component not used above to compute the force in the rod).

3.2 Kinetics

Having discussed the kinematics of rigid bodies in 2D and 3D, it is time to review the kinetic balance laws of rigid bodies, which follow quite analogously from those derived for systems of rigidly linked particles (see Section 2).

3.2.1 Balance of linear momentum

The total **linear momentum** vector of a body $\mathcal{B}$ is defined as

$$\boldsymbol{P} = \lim_{\substack{N\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{N} \boldsymbol{v}_i\,\Delta m_i = \int_{\mathcal{B}} \boldsymbol{v}\,\mathrm{d}m = \int_{\mathcal{B}} \rho\boldsymbol{v}\;\mathrm{d}V. \tag{3.77}$$

By invoking the definition of the center of mass of a rigid body, cf. (3.3), we also have

$$\boldsymbol{P} = M\left(\frac{1}{M}\int_{\mathcal{B}} \boldsymbol{v}\,\mathrm{d}m\right) = M\boldsymbol{v}_{\mathrm{CM}}. \tag{3.78}$$

Hence, as already discussed for systems of particles, we may interpret a rigid body as a "*super-particle*" with an effective mass M and an effective velocity $\boldsymbol{v}_{\mathrm{CM}}$.

As shown for systems of particles, the sum of all inner forces cancels in a rigid body (where all material points are linked by pairwise rigid constraints), so that the **linear momentum balance** for a rigid body reads

$$\sum_i \boldsymbol{F}_i^{\mathrm{ext}} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P}. \tag{3.79}$$

If the total mass M of $\mathcal{B}$ remains constant (which we usually assume), then the above becomes

$$\boxed{\sum_i \boldsymbol{F}_i^{\mathrm{ext}} = M\boldsymbol{a}_{\mathrm{CM}}} \qquad \text{if } M = \text{const.} \tag{3.80}$$

Note that in the following we usually consider only externally applied forces, so that we may omit the superscript "ext" for convenience (and for analogy with the static case).

If no external forces act on a rigid body, we observe the **conservation of linear momentum**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P} = \frac{\mathrm{d}}{\mathrm{d}t}(M\boldsymbol{v}_{\mathrm{CM}}) = \boldsymbol{0} \qquad \Leftrightarrow \qquad M\boldsymbol{v}_{\mathrm{CM}} = \text{const.} \tag{3.81}$$

3.2.2 Balance of angular momentum

The balance equation of angular momentum around a point B is, in principle, a straight-forward extension of the relations derived in Section 2 for systems of particles. However, the devil is in the detail, as we will see soon (and relations become more complex here).

The **angular momentum** vector of a rigid body with respect to a point B is defined as

$$\boldsymbol{H}_{\mathrm{B}} = \lim_{\substack{N\to\infty\\ \Delta m_i\to 0}} \sum_{i=1}^{n} \boldsymbol{r}_{\mathrm{BP}_i} \times (\Delta m_i \boldsymbol{v}_i) = \int_{\mathcal{B}} (\boldsymbol{r}-\boldsymbol{r}_{\mathrm{B}}) \times \boldsymbol{v}\;\mathrm{d}m = \int_{\mathcal{B}} (\boldsymbol{r}-\boldsymbol{r}_{\mathrm{B}}) \times \rho\boldsymbol{v}\;\mathrm{d}V. \tag{3.82}$$

Taking the analogous limit – from a system of rigidly connected particles to a rigid body – yields the balance of angular momentum for rigid bodies as

$$\boldsymbol{M}_{\mathrm{B}} = \dot{\boldsymbol{H}}_{\mathrm{B}} + \boldsymbol{v}_{\mathrm{B}} \times \boldsymbol{P}, \tag{3.83}$$

where we need to consider *only the resultant torque due to external forces acting on the rigid body* (for the same reasons discussed for systems of particles).

Unfortunately, calculating the integral definition of $\boldsymbol{H}_{\mathrm{B}}$ in (3.82) is in practice non-trivial in general and calls for some discussion. *Assuming that point B is part of body $\mathcal{B}$*, we can use the velocity transfer formula to expand the above integral into

$$\begin{aligned}\boldsymbol{H}_{\mathrm{B}} &= \int_{\mathcal{B}} (\boldsymbol{r}-\boldsymbol{r}_{\mathrm{B}}) \times \boldsymbol{v}\,\mathrm{d}m = \int_{\mathcal{B}} (\boldsymbol{r}-\boldsymbol{r}_{\mathrm{B}}) \times [\boldsymbol{v}_{\mathrm{B}} + \boldsymbol{\omega} \times (\boldsymbol{r}-\boldsymbol{r}_{\mathrm{B}})]\,\mathrm{d}m \\ &= \int_{\mathcal{B}} (\boldsymbol{r}-\boldsymbol{r}_{\mathrm{B}})\,\mathrm{d}m \times \boldsymbol{v}_{\mathrm{B}} + \int_{\mathcal{B}} (\boldsymbol{r}-\boldsymbol{r}_{\mathrm{B}}) \times [\boldsymbol{\omega} \times (\boldsymbol{r}-\boldsymbol{r}_{\mathrm{B}})]\,\mathrm{d}m.\end{aligned} \tag{3.84}$$

Note that the first integral can be re-written as

$$\int_{\mathcal{B}}(\boldsymbol{r}-\boldsymbol{r}_{\mathrm{B}})\,\mathrm{d}m\times\boldsymbol{v}_{\mathrm{B}}=\left[\int_{\mathcal{B}}\boldsymbol{r}\,\mathrm{d}m-\boldsymbol{r}_{\mathrm{B}}\int_{\mathcal{B}}\mathrm{d}m\right]\times\boldsymbol{v}_{\mathrm{B}}=M\,(\boldsymbol{r}_{\mathrm{CM}}-\boldsymbol{r}_{\mathrm{B}})\times\boldsymbol{v}_{\mathrm{B}}. \tag{3.85}$$

Therefore, if we choose point B to be either a fixed point (so that $\boldsymbol{v}_{\mathrm{B}}=\mathbf{0}$) or the center of mass of body $\mathcal{B}$ (i.e., B = CM), then (3.85) vanishes and

$$\boldsymbol{H}_{\mathrm{B}}=\int_{\mathcal{B}}\overline{\boldsymbol{r}}\times(\boldsymbol{\omega}\times\overline{\boldsymbol{r}})\,\mathrm{d}m\qquad\text{with}\qquad\overline{\boldsymbol{r}}=\boldsymbol{r}-\boldsymbol{r}_{\mathrm{B}}. \tag{3.86}$$

The above may be reinterpreted as an integral of $\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})$ over body $\mathcal{B}$, where $\boldsymbol{r}$ is now measured in a coordinate system whose origin has been moved to point B (by moving all Cartesian axes in parallel). By using the vector identity $\overline{\boldsymbol{r}}\times(\boldsymbol{\omega}\times\overline{\boldsymbol{r}})=(\overline{\boldsymbol{r}}\cdot\overline{\boldsymbol{r}})\boldsymbol{\omega}-(\overline{\boldsymbol{r}}\cdot\boldsymbol{\omega})\overline{\boldsymbol{r}}$, we obtain

$$\overline{\boldsymbol{r}}\times(\boldsymbol{\omega}\times\overline{\boldsymbol{r}})=(\overline{\boldsymbol{r}}\cdot\overline{\boldsymbol{r}})\boldsymbol{\omega}-(\overline{\boldsymbol{r}}\cdot\boldsymbol{\omega})\overline{\boldsymbol{r}}=\left[(\overline{\boldsymbol{r}}\cdot\overline{\boldsymbol{r}})\boldsymbol{I}-\overline{\boldsymbol{r}}\otimes\overline{\boldsymbol{r}}\right]\boldsymbol{\omega}, \tag{3.87}$$

where we used the tensor/dyadic product[10]. Note that $\boldsymbol{\omega}$ is constant for the entire body $\mathcal{B}$, so that – when inserting (3.87) into (3.86) – $\boldsymbol{\omega}$ can be taken outside the integral, so

$$\boldsymbol{H}_{\mathrm{B}}=\int_{\mathcal{B}}\left[(\overline{\boldsymbol{r}}\cdot\overline{\boldsymbol{r}})\boldsymbol{I}-\overline{\boldsymbol{r}}\otimes\overline{\boldsymbol{r}}\right]\mathrm{d}m\,\boldsymbol{\omega}=\boldsymbol{I}_{\mathrm{B}}\boldsymbol{\omega}\qquad\text{with}\qquad\boldsymbol{I}_{\mathrm{B}}=\int_{\mathcal{B}}\left[(\overline{\boldsymbol{r}}\cdot\overline{\boldsymbol{r}})\boldsymbol{I}-\overline{\boldsymbol{r}}\otimes\overline{\boldsymbol{r}}\right]\mathrm{d}m. \tag{3.88}$$

$\boldsymbol{I}_{\mathrm{B}}$ is the **moment of inertia tensor**. In component form, the above becomes

$$\begin{aligned}[\overline{\boldsymbol{r}}\times(\boldsymbol{\omega}\times\overline{\boldsymbol{r}})]_{\mathcal{C}}&=(\overline{x}_1^2+\overline{x}_2^2+\overline{x}_3^2)\begin{pmatrix}\omega_1\\\omega_2\\\omega_3\end{pmatrix}-(\overline{x}_1\omega_1+\overline{x}_2\omega_2+\overline{x}_3\omega_3)\begin{pmatrix}\overline{x}_1\\\overline{x}_2\\\overline{x}_3\end{pmatrix}\\&=\begin{pmatrix}\overline{x}_2^2+\overline{x}_3^2&-\overline{x}_1\overline{x}_2&-\overline{x}_1\overline{x}_3\\-\overline{x}_1\overline{x}_2&\overline{x}_1^2+\overline{x}_3^2&-\overline{x}_2\overline{x}_3\\-\overline{x}_1\overline{x}_3&-\overline{x}_2\overline{x}_3&\overline{x}_1^2+\overline{x}_2^2\end{pmatrix}\begin{pmatrix}\omega_1\\\omega_2\\\omega_3\end{pmatrix},\end{aligned} \tag{3.89}$$

where we denoted the components of the angular velocity and position vectors in Cartesian coordinates as, respectively,

$$[\boldsymbol{\omega}]_{\mathcal{C}}=(\omega_1,\omega_2,\omega_3)^{\mathrm{T}},\qquad[\overline{\boldsymbol{r}}]_{\mathcal{C}}=(\overline{x}_1,\overline{x}_2,\overline{x}_3)^{\mathrm{T}}. \tag{3.90}$$

Thus, **if B = CM or $\boldsymbol{v}_{\mathrm{B}}=\mathbf{0}$** and Cartesian coordinates $\overline{x}_i$ are measured with respect to point B, then the **angular momentum** vector in 3D can be expressed as

$$\boxed{\boldsymbol{H}_{\mathrm{B}}=\boldsymbol{I}_{\mathrm{B}}\boldsymbol{\omega}\qquad\text{with}\qquad[\boldsymbol{I}_{\mathrm{B}}]_{\mathcal{C}}=\int_{\mathcal{B}}\begin{pmatrix}\overline{x}_2^2+\overline{x}_3^2&-\overline{x}_1\overline{x}_2&-\overline{x}_1\overline{x}_3\\-\overline{x}_1\overline{x}_2&\overline{x}_1^2+\overline{x}_3^2&-\overline{x}_2\overline{x}_3\\-\overline{x}_1\overline{x}_3&-\overline{x}_2\overline{x}_3&\overline{x}_1^2+\overline{x}_2^2\end{pmatrix}\rho\,\mathrm{d}V} \tag{3.91}$$

It is important to Keep in mind that the integration in (3.91) must be carried out *with respect to point* B (following our definition of $\overline{\boldsymbol{r}}$), i.e., the coordinate origin must be B for integration.

[10]The *tensor product* (or *dyadic product*) is defined as $\boldsymbol{u}\otimes\boldsymbol{v}=\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$. It yields a second order tensor whose components $(\boldsymbol{u}\otimes\boldsymbol{v})_{ij}=u_iv_j$ form a square matrix.

When we insert (3.91) into (3.83) and recall (3.85), we arrive at the general form of *angular momentum balance for a rigid body* with respect to a *point B on the body*:

$$\boxed{\boldsymbol{M}_\mathrm{B} = \dot{\boldsymbol{H}}_\mathrm{B} + \boldsymbol{v}_\mathrm{B} \times \boldsymbol{P} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{I}_\mathrm{B}\boldsymbol{\omega} + M\left(\boldsymbol{r}_\mathrm{CM} - \boldsymbol{r}_\mathrm{B}\right) \times \boldsymbol{v}_\mathrm{B}\right] + \boldsymbol{v}_\mathrm{B} \times \boldsymbol{P}} \tag{3.92}$$

The above form of AMB simplifies significantly, if we choose point B wisely (and we are free to choose, as long as B is on the body). Note the following **special cases**:

- if B is the *center of mass*, then $\boldsymbol{r}_\mathrm{CM} - \boldsymbol{r}_\mathrm{B} = \boldsymbol{0}$ and $\boldsymbol{v}_\mathrm{B} || \boldsymbol{P}$ so that

$$\boxed{\boldsymbol{M}_\mathrm{B} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{I}_\mathrm{B}\boldsymbol{\omega}\right) \qquad \text{if B} = \text{CM}} \tag{3.93}$$

- if B is a *fixed point*, then $\boldsymbol{v}_\mathrm{B} = \boldsymbol{0}$ so that, again,

$$\boxed{\boldsymbol{M}_\mathrm{B} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{I}_\mathrm{B}\boldsymbol{\omega}\right) \qquad \text{if } \boldsymbol{v}_\mathrm{B} = \boldsymbol{0}} \tag{3.94}$$

- if the frame of reference is chosen such that $\dot{\boldsymbol{I}}_\mathrm{B} = \boldsymbol{0}$, then the above two cases reduce to

$$\boxed{\boldsymbol{M}_\mathrm{B} = \boldsymbol{I}_\mathrm{B}\dot{\boldsymbol{\omega}} \qquad \text{if B} = \text{CM or } \boldsymbol{v}_\mathrm{B} = \boldsymbol{0}, \text{ and if } \dot{\boldsymbol{I}}_\mathrm{B} = \boldsymbol{0}} \tag{3.95}$$

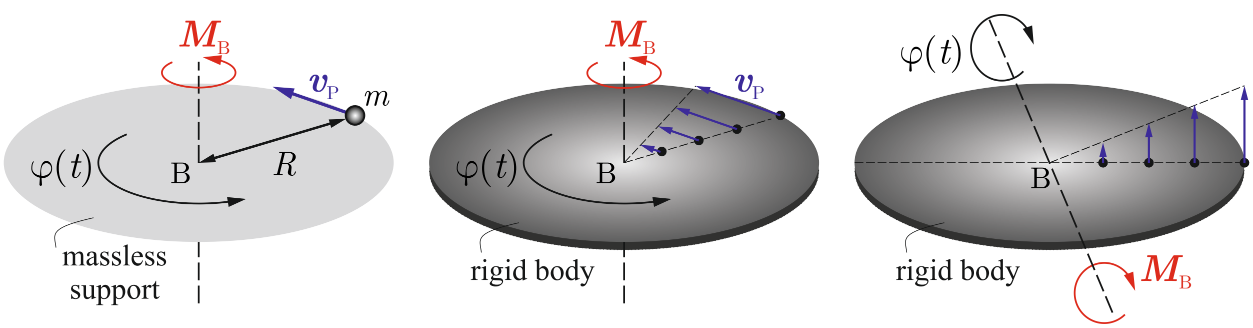
The integral definition of $\boldsymbol{I}_\mathrm{B}$ in (3.91) depends on the chosen coordinate system and is in general time-dependent. In the following, we will first consider only cases in which $\dot{\boldsymbol{I}}_\mathrm{B} = \boldsymbol{0}$ so that $\dot{\boldsymbol{H}}_\mathrm{B} = \boldsymbol{I}_\mathrm{B}\dot{\boldsymbol{\omega}}$. The case $\dot{\boldsymbol{I}}_\mathrm{B} \neq \boldsymbol{0}$ will be discussed in Section 3.4.

Recall that we had previously already encountered the 2D version of (3.95) for particles (see Eq. (1.139)), viz.

$$M_\mathrm{B} = I_\mathrm{B}\dot{\omega} = I_\mathrm{B}\ddot{\varphi}, \tag{3.96}$$

and for a particle of mass m rotating at a distance R around B we obtained $I_\mathrm{B} = mR^2$. The above presents the more general expression for rigid bodies in 3D.

To interpret relation (3.95), let us revisit our discussion for a single particle – shown below on the left is the scenario of a single particle on a massless support. As established in Section 1.2.3, AMB is indeed a reformulation of LMB, linking the torque to the angular acceleration of the particle (like LMB links forces to accelerations), and I_B was introduced back then as the constant of proportionality in between: $M_\mathrm{B} = I_\mathrm{B}\ddot{\varphi}$.



Here, we have arrive at the analogous version for rigid bodies in 3D, Eq. (3.95). We can picture a rigid body as a collection of infinitely many infinitesimally small material points that rotate about a given axis. Comparing a particle (shown above on the left) to a rigid body (shown in the middle), the principle is the same: for a rigid body, AMB tells us how an applied torque M_B is related to the collective angular acceleration $\ddot{\varphi}$ of all material points on the rigid body, whose shape and size define the constant I_B. As a novelty here, Eq. (3.95) is a vector/tensor equation, which comes from the nature of 3D rigid bodies: depending on the axis about which they rotate, the value of I_B is different. For example, the sketch on the right shows the same rigid disk, now rotating about a different axis. The mass inside the rigid body is distributed differently with respect to the two shown axes (and the velocity profiles across the body are also different), so I_B must be different when rotating about either of the two shown axis.

Consider the shown disk rotates about each of its three principal axes, and let us assume that there is a unique $I_{\mathrm{B},i}$ for each axis ($i = 1, 2, 3$). When we rotate with ω_i about each axis with a torque $M_{\mathrm{B},i}$, we arrive at

$$\begin{aligned} M_{\mathrm{B},1} &= I_{\mathrm{B},1}\dot{\omega}_1 \\ M_{\mathrm{B},2} &= I_{\mathrm{B},2}\dot{\omega}_2 \\ M_{\mathrm{B},3} &= I_{\mathrm{B},3}\dot{\omega}_3 \end{aligned} \qquad \Leftrightarrow \qquad \underbrace{\begin{pmatrix} M_{\mathrm{B},1} \\ M_{\mathrm{B},2} \\ M_{\mathrm{B},3} \end{pmatrix}}_{\boldsymbol{M}_\mathrm{B}} = \underbrace{\begin{pmatrix} I_{\mathrm{B},1} & & \\ & I_{\mathrm{B},2} & \\ & & I_{\mathrm{B},3} \end{pmatrix}}_{\boldsymbol{I}_\mathrm{B}} \underbrace{\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix}}_{\dot{\boldsymbol{\omega}}}, \tag{3.97}$$

so we arrive at $\boldsymbol{M}_\mathrm{B} = \boldsymbol{I}_\mathrm{B}\dot{\boldsymbol{\omega}}$, which is Eq. (3.95). Unfortunately, the full story is more complicated and I_B is not always diagonal (which we will discuss in Section 3.2.3). Yet, (3.97) gives a quick intuitive understanding for the reason of having a vector-valued relation in 3D.

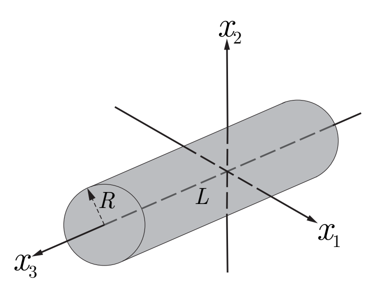
Example 3.7. Moment of inertia tensor of a cylinder

Consider a cylinder of length L and radius R with a constant mass density ρ. By symmetry, the center of mass is at the geometric center of the cylinder (i.e., the mid-point of the central axis of rotation). Let us calculate the moment of inertia tensor of the cylinder with respect to its center of mass, using a coordinate system with the x_3-axis aligned with its axis of rotational symmetry (as shown below) and writing $x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$ to ease integration over the circular cross-section:

$$[\boldsymbol{I}_\mathrm{CM}] = \rho \int_{r=0}^{R} \int_{\varphi=0}^{2\pi} \int_{x_3=-L/2}^{L/2} r^2 \begin{pmatrix} \sin^2\varphi + x_3^2/r^2 & -\cos\varphi\,\sin\varphi & -\sin\varphi\,x_3/r \\ -\sin\varphi\,\cos\varphi & \cos^2\varphi + x_3^2/r^2 & -\cos\varphi\,x_3/r \\ -\sin\varphi\,x_3/r & -\cos\varphi\,x_3/r & 1 \end{pmatrix} \mathrm{d}x_3\,r\,\mathrm{d}\varphi\,\mathrm{d}r. \tag{3.98}$$

Exploiting that $\sin\varphi$ and $\cos\varphi$ as well as $\sin\varphi\,\cos\varphi$ integrate to zero over $[0, 2\pi)$, we observe that all off-diagonal entries evaluate to zero. With the total mass $M = \rho\pi R^2 L$, the remaining components evaluate to

$$[\boldsymbol{I}_\mathrm{CM}] = \begin{pmatrix} \frac{M}{12}(L^2 + 3R^2) & 0 & 0 \\ 0 & \frac{M}{12}(L^2 + 3R^2) & 0 \\ 0 & 0 & \frac{MR^2}{2} \end{pmatrix}. \tag{3.99}$$



As can be expected, the moment of inertia I_{33} against rotation about the x_3-axis, i.e., about the cylinder's axis of rotational symmetry, is much smaller than when rotating the body about the length of the cylinder ($I_{33} \ll I_{11}, I_{22}$ when $R \ll L$).

Since the derived $[\boldsymbol{I}_{\text{CM}}]$ is diagonal, the three non-zero entries are also the principal values of tensor $\boldsymbol{I}_{\text{CM}}$, and our coordinate axes align with the principal directions of the cylinder. The above diagonal form of $\boldsymbol{I}_{\text{CM}}$ (which is a consequence of our choice of the coordinate system) is advantageous since it *decouples the equations of angular momentum balance*. This is a consequence of

$$\boldsymbol{H}_{\text{CM}} = \boldsymbol{I}_{\text{CM}}\boldsymbol{\omega} = I_{11}\omega_1\boldsymbol{e}_1 + I_{22}\omega_2\boldsymbol{e}_2 + I_{33}\omega_3\boldsymbol{e}_3 \qquad \text{and} \qquad \dot{\boldsymbol{I}}_{\text{CM}} = \boldsymbol{0}, \tag{3.100}$$

so that, with $\boldsymbol{M}_{\text{CM}} = \sum_{i=1}^{3} M_{\text{CM},i}\boldsymbol{e}_i$,

$$\boldsymbol{M}_{\text{CM}} = \dot{\boldsymbol{H}}_{\text{CM}} = \boldsymbol{I}_{\text{CM}}\dot{\boldsymbol{\omega}} \qquad \Rightarrow \qquad M_{\text{CM},i} = I_{(ii)}\dot{\omega}_i \quad \text{for} \quad i = 1,2,3. \tag{3.101}$$

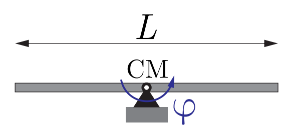
This confirms that the torque $[M_{\text{CM}}]_i$ needed to alter the angular velocity ω_i about the x_i-axis is directly proportional to I_{ii}.

In the special case of a **slender cylindrical rod** with $R \ll L$, the components of the moment of inertia tensor further simplify to

$$I_{11} = I_{22} = \frac{M}{12}(L^2 + 3R^2) \approx \frac{ML^2}{12}, \qquad \text{and} \qquad I_{33} = \frac{MR^2}{2}. \tag{3.102}$$

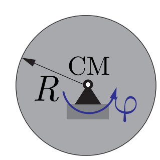
Notice that I_{33} is the **moment of inertia of a disk** of mass M and radius R in 2D rotating about its center of mass, while $I_{11} = I_{22}$ is the **moment of inertia of a slender rod** of mass M and length L in 2D rotating about its center of mass.

Thus, for a **slender rod** of length L and mass M, rotating about its center of mass in 2D, we have



$$M_{\text{CM}} = I_{\text{CM}}\ddot{\varphi} \qquad \text{with} \qquad \boxed{I_{\text{CM}} = \frac{ML^2}{12}} \tag{3.103}$$

while for a **disk or cylinder** of radius R and mass M, rotating about its center of mass in 2D, we have



$$M_{\text{CM}} = I_{\text{CM}}\ddot{\varphi} \qquad \text{with} \qquad \boxed{I_{\text{CM}} = \frac{MR^2}{2}} \tag{3.104}$$

I_{CM}, i.e., the moment of inertia around the center of mass in 2D is often referred to as the **centroidal moment of inertia**.

3.2.3 Moment of inertia tensor

As the **moment of inertia tensor** $\boldsymbol{I}_\mathrm{B}$ is crucial for the balance of angular momentum of rigid bodies, we will discuss some of its properties in more detail. Recall that a **second-order tensor** $\boldsymbol{T} = \sum_{i,j=1}^{3} T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ in 3D is generally characterized by its components T_{ij} (which form a 3×3 matrix) as well as the associated basis $B = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. Unlike a matrix (which is defined only by components and has no basis), the basis of a tensor is essential, since it allows us to apply certain mathematical concepts (e.g., we can compute principal axes and associated eigenvalues, or we can apply coordinate transforms to derive the components of the tensor in any other basis). All tensor relations derived for, e.g., the **stress**, **strain**, and **area moment of inertia** tensors in Mechanics 1 and 2 apply here.

As shown above, the components of the moment of inertia tensor in a Cartesian reference frame $\mathcal{C}$ (whose origin lies at the center of mass of body $\mathcal{B}$) with respect to a point B that is either the center of mass or not moving are given by

$$[\boldsymbol{I}_\mathrm{B}]_\mathcal{C} = \int_\mathcal{B} \begin{pmatrix} x_2^2 + x_3^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_1^2 + x_3^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_1^2 + x_2^2 \end{pmatrix} \rho \, \mathrm{d}V. \tag{3.105}$$

Irrespective of the shape of the rigid body under considerations, we can draw the following conclusions from the above form of the moment of inertia tensor:

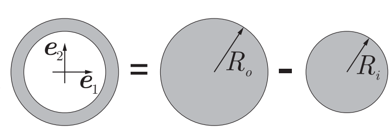
- **Symmetry**: by definition the moment of inertia tensor is *symmetric*: $\boldsymbol{I}_\mathrm{B} = \boldsymbol{I}_\mathrm{B}^\mathrm{T}$.
- **Additivity**: since the integral over body $\mathcal{B}$ can be split additively into separate integrals over various subdomains $\mathcal{B}^i \subset \mathcal{B}$, the moment of inertia tensor of a composite body is additive. That is, $\boldsymbol{I}_\mathrm{CM}$ of a composite body $\mathcal{B} = \mathcal{B}^1 \cup \mathcal{B}^2$ (and $\mathcal{B}^1 \cap \mathcal{B}^2 = \emptyset$) is the sum of the individual contributions, i.e.,

  $$\boldsymbol{I}_\mathrm{B} = \int_\mathcal{B} \overline{\boldsymbol{r}} \times (\boldsymbol{\omega} \times \overline{\boldsymbol{r}})\, \rho \mathrm{d}V = \int_{\mathcal{B}_1} \overline{\boldsymbol{r}} \times (\boldsymbol{\omega} \times \overline{\boldsymbol{r}})\, \rho \mathrm{d}V + \int_{\mathcal{B}_2} \overline{\boldsymbol{r}} \times (\boldsymbol{\omega} \times \overline{\boldsymbol{r}})\, \rho \mathrm{d}V = \boldsymbol{I}_\mathrm{B}^1 + \boldsymbol{I}_\mathrm{B}^2. \tag{3.106}$$

  This is equally true when considering hollow objects or bodies with inner holes, where $\boldsymbol{I}_\mathrm{B}$ equals the complete moment of inertia tensor (of the solid body) minus the moment of inertia tensor of the removed parts. It is important to note though that all added contributions $\boldsymbol{I}_\mathrm{B}^i$ must be *with respect to the same point* B.

  For example, for a **hollow cylinder** (inner radius R_i, outer radius R_o) the non-zero components of the moment of inertia tensor from Example 3.7 are

  $$I_{11} = I_{22} = \frac{M_o}{12}(L^2 + 3R_o^2) - \frac{M_i}{12}(L^2 + 3R_i^2), \qquad I_{33} = \frac{M_o R_o^2}{2} - \frac{M_i R_i^2}{2}, \qquad M_{o/i} = \rho\pi R_{o/i}^2 L. \tag{3.107}$$

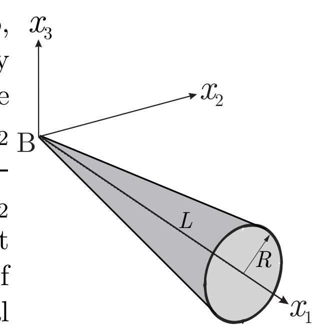
As a special case, let us take the limit of a thin-walled hollow cylinder of radius R, thickness $t \ll R$, and length L. In this case we may assume $R_i = R - \frac{t}{2}$ and $R_o = R + \frac{t}{2}$ as well as, for a homogeneous body of mass density ρ, $M_0 = \rho\pi R_o^2 L$ and $M_i = \rho\pi R_i^2 L$. In the limit of a thin cylinder ($t \ll R$), insertion of the radii and masses into (3.107) leads to

$$\begin{aligned} I_{11} = I_{22} &= \frac{\rho\pi RtL}{12}\left[2L^2 + 3(4R^2+t^2)\right] \xrightarrow{t\ll R} \frac{\rho\pi RtL}{12}(2L^2+12R^2) = \frac{M}{12}(L^2+6R^2), \\ I_{33} &= \frac{\rho\pi RtL}{2}(4R^2+t^2) \stackrel{t\ll R}{\to} \rho\, 2\pi RtL\, R^2 = MR^2 \end{aligned} \tag{3.108}$$

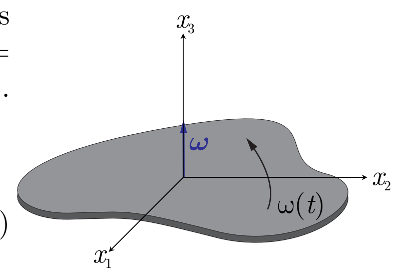
with the total (approximate) mass $M = 2\pi R\,t\,L$ of the thin cylinder. As a sanity check, notice that I_{33} agrees with the moment of inertia derived in Example 2.4 for a thin wheel in 2D, while $I_{11} = I_{22}$ correctly reduces to the moment of inertia of a slender bar in the limit of a long and slender cylinder ($R \ll L$), cf. (3.103).

- **Symmetry axes**: *if the x_i-axis is an axis of rotational symmetry*, then all off-diagonal terms of $\boldsymbol{I}$ with i-subscript vanish (see, e.g., the I_{13},I_{23}-components in Example 3.7). The same applies *if x_i stands normal on a symmetry plane (so the origin lies in the symmetry plane)* (see, e.g., the $I_{12} = I_{21}$-component in Example 3.7).

  For **example**, the cylindrical cone shown on the right has rotational symmetry about the x_1-axis, so $I_{12} = I_{13} = I_{21} = I_{31} = 0$. Also, the x_2-axis is perpendicular to the x_1-x_3-plane, which is a symmetry plane of the cone, so $I_{21} = I_{23} = I_{12} = I_{32} = 0$. Finally, the analogous applies to the x_3-axis (being perpendicular to the x_1-x_2 symmetry plane) so that $I_{31} = I_{32} = I_{13} = I_{23} = 0$. The only non-zero components in the shown coordinate frame are hence I_{11}, I_{22} and I_{33}. Note that the same also applies, e.g., if the origin lies at the center of mass of the cone (it generally holds as long as one of the Cartesian coordinate axes is aligned with an axis of rotational symmetry of the cone).



- **Reduction to 2D**: for 2D problems in the x_1-x_2-plane, the x_3-axis is naturally perpendicular to a plane of symmetry, so that $I_{13} = I_{23} = 0$. Further, the rotation is confined to the plane, so $\boldsymbol{\omega} = \omega \boldsymbol{e}_3$. Therefore, in this reference frame $\mathcal{C}$



$$[\boldsymbol{H}_{\text{CM}}]_{\mathcal{C}} = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I_{33}\omega \end{pmatrix}. \tag{3.109}$$

Hence, for 2D problems the angular momentum balance (if B = CM or $\boldsymbol{v}_\text{B} = \boldsymbol{0}$, and if $\dot{I}_{33} = 0$) becomes

$$M_\text{B} = I_{33}\dot{\omega}, \qquad \text{with} \qquad I_{33} = \int_{\mathcal{B}} (\overline{x}_1^2 + \overline{x}_2^2)\,\text{d}m = \int_{\mathcal{B}} \overline{r}^2 \rho\,\text{d}V. \tag{3.110}$$

I_{33} is often referred to as the **axial moment of inertia**.

As an **example**, let us revisit the moment of inertia of a slender bar in 2D (length L, density ρ, cross-sectional area A) rotating about its center of mass in the plane (with $\overline{x}_1$ pointing along the beam's length and $\overline{x}_3$ being the out-of-plane axis, the origin is at the center). Applying the definition of I_{33} from (3.110) again leads to the moment of inertia of a slender rod about its center of mass:

$$I_{33} = \int_{-L/2}^{L/2} \overline{x}_1^2 \rho\, A \,\mathrm{d}\overline{x}_1 = 2\frac{(L/2)^3}{3}\rho A = \frac{\rho A L^3}{12} = \frac{ML^2}{12} \qquad \text{with} \qquad M = \rho AL. \quad (3.111)$$

We note that for a 2D rotation about the center of mass, one may also define

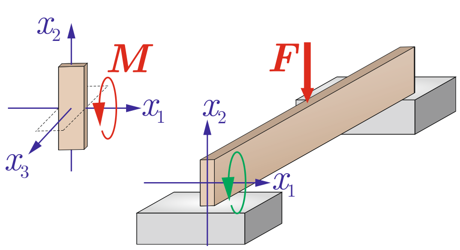
$$I_{33} = r_{\mathrm{g}}^2 M, \quad (3.112)$$

where M is the mass of the body and r_{g} is the so-called **radius of gyration**. By way of interpretation, the rigid body's rotational inertia is equivalent to that of a particle of mass M rotating at a radius r_{g}. For the above example of a slender bar rotating about its CM, we see that $r_{\mathrm{g}} = L/\sqrt{12}$.

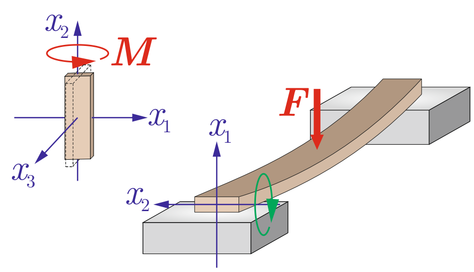
- **Relation to area moments**: The moment of inertia tensor bears great similarity to the **area moment of inertia** tensor introduced in Mechanics 2. For example, take the axial moment I_{33} in (3.110) and consider the case of a uniform mass density ρ and thickness t. In this case, the above reduces to

$$I_{33} = \rho \int_{\mathcal{B}} (\overline{x}_1^2 + \overline{x}_2^2) t \,\mathrm{d}A = \rho t \int_{\mathcal{B}} \overline{r}^2 \,\mathrm{d}A = \rho t I_P, \qquad \text{where} \qquad I_P = \int_{\mathcal{B}} \overline{r}^2 \,\mathrm{d}A \quad (3.113)$$

is the **polar moment of area** already introduced in Mechanics 2 (where and x-y-coordinate system was used, as shown for beam bending).



rotation vs. bending about the x_1-axis



rotation vs. bending about the x_2-axis

Similarly, if we consider a thin plate of uniform thickness t and density ρ that is rotating with $\boldsymbol{\omega}$ in 3D with the x_3-axis pointing out-of-plane (so $\overline{x}_3 \approx 0$) as shown in the two schematics above, then we observe that the non-zero components in the moment of inertia tensor in (3.109) become

$$\begin{aligned} I_{11} &= \int_{\mathcal{B}} (\overline{x}_2^2 + \overline{x}_3^2) \,\mathrm{d}m = \rho t \int_{\mathcal{B}} \overline{x}_2^2 \,\mathrm{d}A = \rho t\, I_x, \\ I_{22} &= \int_{\mathcal{B}} (\overline{x}_1^2 + \overline{x}_3^2) \,\mathrm{d}m = \rho t \int_{\mathcal{B}} \overline{x}_1^2 \,\mathrm{d}A = \rho t\, I_y, \\ I_{12} = I_{21} &= \int_{\mathcal{B}} -\overline{x}_1 \overline{x}_2 \,\mathrm{d}m = -\rho t \int_{\mathcal{B}} \overline{x}_1 \overline{x}_2 \,\mathrm{d}A = \rho t\, C_{xy}, \end{aligned} \quad (3.114)$$

where I_x and I_y are the area moments of inertia and C_{xy} is the **deviatoric moment** known from Mechanics 2. Therefore, the smaller (e.g.) the bending resistance about the x_1-axis (characterized by I_x), the smaller the rotational inertia about the x_1-axis (characterized by I_{11}). If $I_x < I_y$, then $I_{11} < I_{22}$. The above schematic illustrates the analogy between mass and area moments of inertia.

- **Principal axes**: working with the general form of $\boldsymbol{I}_\text{B}$ in (3.105) can be cumbersome, since it implies a coupling of the three equations of angular momentum balance in 3D in general (if all components of $[\boldsymbol{I}_\text{B}]_\mathcal{C}$ are non-zero, then $\boldsymbol{I}_\text{B}\boldsymbol{\omega} = (I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3)\boldsymbol{e}_1 + \ldots$ yields three equations in which ω_1, ω_2 and ω_3 are coupled). Example 3.7, by contrast, had a diagonal moment of inertia tensor, which decouples the equations of angular momentum balance (in this case $\boldsymbol{I}_\text{B}\boldsymbol{\omega} = I_{11}\omega_1\boldsymbol{e}_1 + I_{22}\omega_2\boldsymbol{e}_2 + I_{33}\omega_3\boldsymbol{e}_3$ decouples the angular velocities). We now understand that this was the case Example 3.7, because the x_3-axis was aligned with the axis of rotational symmetry (and because of the symmetries of the body).

  Analogous to the stress, strain, and area moment of inertia tensors introduced in Mechanics 2, we can always find a (principal) coordinate system $\hat{\mathcal{C}}$ in which the moment of inertia tensor is diagonal. This special system aligns with the **principal axes** of the body or, equivalently, with the **principal directions** $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ of the tensor. In the principal axes system with basis $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$, the moment of inertia tensor assumes the form

$$[\boldsymbol{I}_\text{B}]_{\hat{\mathcal{C}}} = \begin{pmatrix} \hat{I}_1 & & \\ & \hat{I}_2 & \\ & & \hat{I}_3 \end{pmatrix}, \tag{3.115}$$

  where the diagonal entries $\{\hat{I}_1, \hat{I}_2, \hat{I}_3\}$ are the tensor's **eigenvalues**. Rotation of the moment of inertia tensor can also be accomplished by the transformation rules (and, graphically, using Mohr's circle) as derived in Mechanics 2. As discussed above, *axes of rotational symmetry and axes perpendicular to planes of symmetry are always principal axes.*

  In practice, it is recommended to *always align the coordinate system with the principal axes or axes of symmetry*, so one can work with a diagonal moment of inertia tensor and decouple the angular moment components, whenever possible.

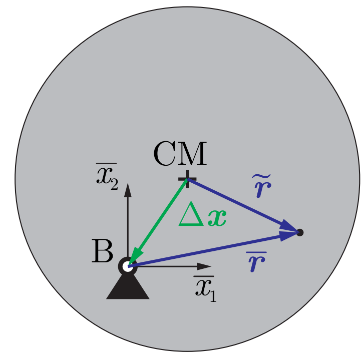
- **Parallel axes theorem** (**Steiner's theorem**): we have seen that the components of $\boldsymbol{I}_\text{B}$ within a given reference frame $\mathcal{C}$ generally depend both on the geometry of the body and on the reference point B. Since we cannot tabulate the components of $\boldsymbol{I}$ for a given geometry for all possible points B, one usually finds only the components of $\boldsymbol{I}_\text{CM}$ in textbooks. It will hence be convenient to have a simple formula that tells us the relation between $\boldsymbol{I}_\text{B}$ and $\boldsymbol{I}_\text{CM}$ for any arbitrary point $\text{B} \in \mathcal{B}$.

  Recall that $\boldsymbol{I}_\text{B}$ was generally defined by the integral in (3.105) with respect to a reference frame $\mathcal{C}$ having the coordinate origin at point B (assuming that $\boldsymbol{v}_\text{B} = \boldsymbol{0}$):

$$\boldsymbol{I}_\text{B} = \int_\mathcal{B} \left[(\overline{\boldsymbol{r}} \cdot \overline{\boldsymbol{r}})\boldsymbol{I} - \overline{\boldsymbol{r}} \otimes \overline{\boldsymbol{r}}\right] \rho \, \mathrm{d}V \qquad \text{with} \qquad \overline{\boldsymbol{r}} = \boldsymbol{r} - \boldsymbol{r}_\text{B}. \tag{3.116}$$

  If we shift the origin from $\text{B} \in \mathcal{B}$ to the CM of $\mathcal{B}$ by the distance vector

$$\Delta\boldsymbol{x} = \boldsymbol{r}_\text{B} - \boldsymbol{r}_\text{CM} \tag{3.117}$$



while *keeping all coordinate axes parallel* (i.e., we only move the origin by translation without changing the orientations of the coordinate axes, resulting in a new frame $\tilde{\mathcal{C}}$), then we may write for the new frame of reference

$$\overline{\boldsymbol{r}} = \boldsymbol{r} - \boldsymbol{r}_{\mathrm{B}} = \boldsymbol{r} - \boldsymbol{r}_{\mathrm{CM}} + \boldsymbol{r}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{B}} = \tilde{\boldsymbol{r}} - \Delta\boldsymbol{x} \tag{3.118}$$

with

$$\tilde{\boldsymbol{r}} = \boldsymbol{r} - \boldsymbol{r}_{\mathrm{CM}}. \tag{3.119}$$

Insertion into the integral yields

$$\begin{aligned} \boldsymbol{I}_{\mathrm{B}} &= \int_{\mathcal{B}} \left[(\tilde{\boldsymbol{r}} - \Delta\boldsymbol{x}) \cdot (\tilde{\boldsymbol{r}} - \Delta\boldsymbol{x})\, \boldsymbol{I} - (\tilde{\boldsymbol{r}} - \Delta\boldsymbol{x}) \otimes (\tilde{\boldsymbol{r}} - \Delta\boldsymbol{x})\right] \rho \,\mathrm{d}V \\ &= \int_{\mathcal{B}} \left[(\tilde{\boldsymbol{r}} \cdot \tilde{\boldsymbol{r}})\, \boldsymbol{I} - \tilde{\boldsymbol{r}} \otimes \tilde{\boldsymbol{r}}\right] \rho \,\mathrm{d}V + \int_{\mathcal{B}} \left[(\Delta\boldsymbol{x} \cdot \Delta\boldsymbol{x})\, \boldsymbol{I} - \Delta\boldsymbol{x} \otimes \Delta\boldsymbol{x}\right] \rho \,\mathrm{d}V \\ &\qquad - \int_{\mathcal{B}} \left[2(\tilde{\boldsymbol{r}} \cdot \Delta\boldsymbol{x})\, \boldsymbol{I} - (\tilde{\boldsymbol{r}} \otimes \Delta\boldsymbol{x} + \Delta\boldsymbol{x} \otimes \tilde{\boldsymbol{r}})\right] \rho \,\mathrm{d}V \\ &= \boldsymbol{I}_{\mathrm{CM}} + \left[(\Delta\boldsymbol{x} \cdot \Delta\boldsymbol{x})\, \boldsymbol{I} - \Delta\boldsymbol{x} \otimes \Delta\boldsymbol{x}\right] M, \end{aligned} \tag{3.120}$$

where the last integral vanished because $\Delta\boldsymbol{x} = \text{const.}$ and

$$\int_{\mathcal{B}} \tilde{\boldsymbol{r}} \rho \,\mathrm{d}V = \int_{\mathcal{B}} (\boldsymbol{r} - \boldsymbol{r}_{\mathrm{CM}}) \rho \,\mathrm{d}V = \int_{\mathcal{B}} \boldsymbol{r}\, \rho \,\mathrm{d}V - \boldsymbol{r}_{\mathrm{CM}} \int_{\mathcal{B}} \rho \,\mathrm{d}V = M \boldsymbol{r}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{CM}} M = \boldsymbol{0}, \tag{3.121}$$

so that

$$\begin{aligned} &\int_{\mathcal{B}} \left[2(\tilde{\boldsymbol{r}} \cdot \Delta\boldsymbol{x})\, \boldsymbol{I} - (\tilde{\boldsymbol{r}} \otimes \Delta\boldsymbol{x} + \Delta\boldsymbol{x} \otimes \tilde{\boldsymbol{r}})\right] \rho \,\mathrm{d}V \\ &\quad = 2 \left(\int_{\mathcal{B}} \tilde{\boldsymbol{r}} \rho \,\mathrm{d}V \cdot \Delta\boldsymbol{x} \right) \boldsymbol{I} - \int_{\mathcal{B}} \tilde{\boldsymbol{r}} \rho \,\mathrm{d}V \otimes \Delta\boldsymbol{x} - \Delta\boldsymbol{x} \otimes \int_{\mathcal{B}} \tilde{\boldsymbol{r}} \rho \,\mathrm{d}V = \boldsymbol{0}. \end{aligned} \tag{3.122}$$

Altogether, we have thus arrived at

$$\boldsymbol{I}_{\mathrm{B}} = \boldsymbol{I}_{\mathrm{CM}} + \left[(\Delta\boldsymbol{x} \cdot \Delta\boldsymbol{x})\, \boldsymbol{I} - \Delta\boldsymbol{x} \otimes \Delta\boldsymbol{x}\right] M, \tag{3.123}$$

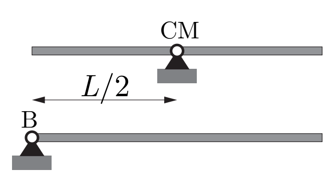
which in component from, with $[\Delta\boldsymbol{x}]_{\mathcal{C}} = (\Delta x_1, \Delta x_2, \Delta x_3)^{\mathrm{T}}$, evaluates to

$$\boxed{[\boldsymbol{I}_{\mathrm{B}}] = [\boldsymbol{I}_{\mathrm{CM}}] + M \begin{pmatrix} (\Delta x_2)^2 + (\Delta x_3)^2 & -\Delta x_1\, \Delta x_2 & -\Delta x_1\, \Delta x_3 \\ -\Delta x_1\, \Delta x_2 & (\Delta x_1)^2 + (\Delta x_3)^2 & -\Delta x_2\, \Delta x_3 \\ -\Delta x_1\, \Delta x_3 & -\Delta x_2\, \Delta x_3 & (\Delta x_1)^2 + (\Delta x_2)^2 \end{pmatrix}} \tag{3.124}$$

This relation is the so-called **parallel axis theorem**, also known as **Steiner's theorem** (or Hyugens' theorem). This is analogous to Steiner's theorem already introduced in Mechanics 2 for the area moments of inertia.

For **2D rotations** in the plane, Steiner's theorem simply reduces to

$$\boxed{I_{\mathrm{B}} = I_{\mathrm{CM}} + M(\Delta x)^2}$$



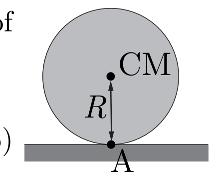
where $\Delta x = |\boldsymbol{r}_{\mathrm{B}} - \boldsymbol{r}_{\mathrm{CM}}|$ is the shifted distance in the plane.

For **example**, consider a rod ($R \ll L$) that is rotating about its end point B in 2D. Here,

$$I_{\mathrm{CM}} = \frac{ML^2}{12} \qquad \Rightarrow \qquad I_{\mathrm{B}} = I_{\mathrm{CM}} + M(\Delta x)^2 = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}. \tag{3.125}$$

As can be expected, the moment of inertia is larger when rotating about the rod's end point than when rotating about its center of mass. This indicates, by definition, that *the axial moment of inertia is always minimal with respect to the center of mass* ($I_{\mathrm{B}} \geq I_{\mathrm{CM}}$ for all $\mathrm{B} \in \mathcal{B}$).

As another **example**, a disk rotating about its instantaneous center of rotation A, while rolling on the ground, has the moment of inertia



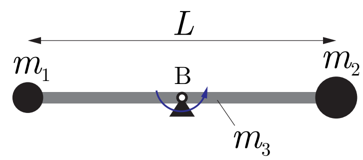
$$I_{\mathrm{A}} = I_{\mathrm{CM}} + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2. \tag{3.126}$$

If, by contrast, I_{A} is known, we can also use Steiner's theorem to find $I_{\mathrm{CM}} = I_{\mathrm{A}} - MR^2$.

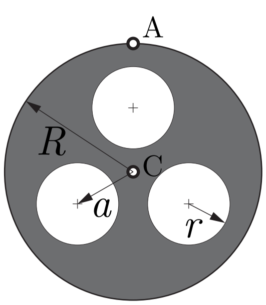
- **Examples**: Table 1 (on the next page) lists example moment of inertia tensors for a few frequently used geometric bodies with respect to the shown frames of reference.

Example 3.8. Moments of inertia in 2D

The rigid body shown on the right consists of a rod (length L, mass m_3) and two point masses (m_1 and m_2) attached to its end points. The combined moment of inertia around point B is easily obtained by exploiting the additivity of the moment of inertia:



$$I_{\mathrm{B}} = I_{\mathrm{B}}^{\mathrm{rod}} + I_{\mathrm{B}}^{1} + I_{\mathrm{B}}^{2} = \frac{m_3 L^2}{12} + (m_1 + m_2)\left(\frac{L}{2}\right)^2. \tag{3.127}$$

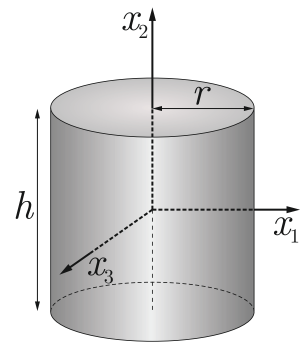


The second example is a rigid body in the shape of a circular disk (outer radius R) with three symmetrically arranged holes (each of radius r, distance from center to center a, as shown). The disk has a uniform mass density ρ and uniform thickness t.
The moment of inertia with respect to the center of mass (point C) is

$$\begin{aligned} I_{\mathrm{C}} &= I_{\mathrm{C}}^{\text{full disk}} - I_{\mathrm{C}}^{\text{holes}} = \frac{1}{2} m_{\text{full disk}} R^2 - 3\left(I_{\mathrm{CM}}^{\text{hole}} + m_{\text{hole}} a^2\right) \\ &= \frac{1}{2}(\pi R^2 t\rho)R^2 - 3\left[\frac{1}{2}(\pi r^2 t\rho)r^2 + (\pi r^2 t\rho)a^2\right] = \frac{\pi\rho t}{2}\left[R^4 - 3r^2(r^2 + 2a^2)\right]. \end{aligned}$$

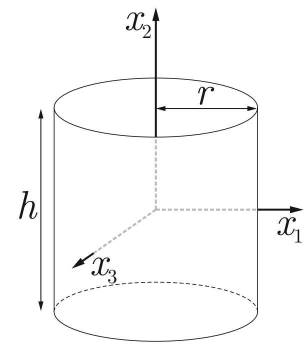
If the moment of inertia of the same disk with respect to point A is needed, the parallel axis theorem helps us shift the moment of inertia to

$$I_{\mathrm{A}} = I_{\mathrm{C}} + m_{\mathrm{disk}} R^2 = I_{\mathrm{C}} + (\pi R^2 t\rho - 3\pi r^2 \rho t)R^2. \tag{3.128}$$



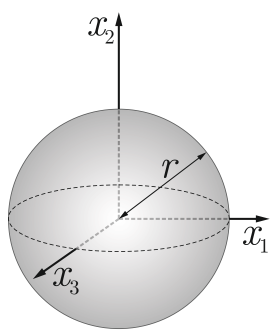
(a) solid cylinder:

$$[\boldsymbol{I}_{\mathrm{CM}}] = \frac{M}{12}\begin{pmatrix} 3r^3+h^2 & & \\ & 6r^2 & \\ & & 3r^3+h^2 \end{pmatrix}$$



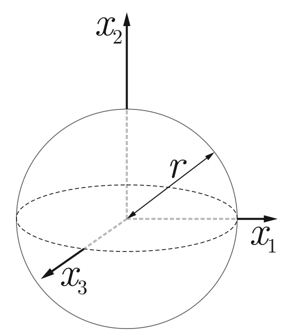
(b) hollow cylinder:

$$[\boldsymbol{I}_{\mathrm{CM}}] = \frac{M}{6}\begin{pmatrix} 6r^2+h^2 & & \\ & 6r^2 & \\ & & 6r^2+h^2 \end{pmatrix}$$



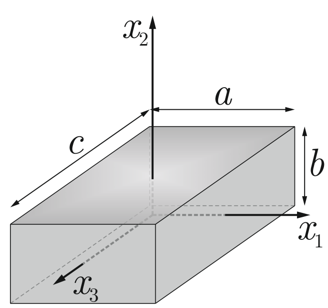
(c) solid sphere:

$$[\boldsymbol{I}_{\mathrm{CM}}] = \frac{2Mr^2}{5}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$



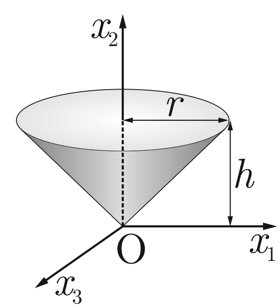
(d) hollow sphere:

$$[\boldsymbol{I}_{\mathrm{CM}}] = \frac{2Mr^2}{3}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$



(e) solid cuboid:

$$[\boldsymbol{I}_{\mathrm{CM}}] = \frac{M}{12}\begin{pmatrix} b^2+c^2 & & \\ & a^2+c^2 & \\ & & a^2+b^2 \end{pmatrix}$$



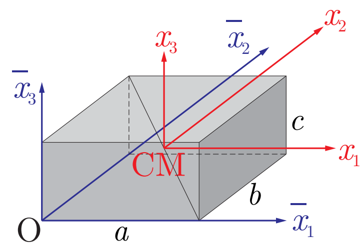
(f) solid cone

$$[\boldsymbol{I}_{\mathrm{O}}] = \frac{M}{20}\begin{pmatrix} 3r^2+12h^2 & & \\ & 6r^2 & \\ & & 3r^2+12h^2 \end{pmatrix}$$

Table 1: List of moment of inertia tensors of frequently used bodies of total mass M.

Example 3.9. Moment of inertia of a cuboidal block

Consider a block of extensions a, b and c with a uniform mass density ρ. If we compute the moment of inertia tensor with respect to point O shown on the right, then our coordinate axes are *not* aligned with the symmetry axes of the body (none of the three symmetry planes of the body falls onto $x_i = 0$ for any of the coordinate axes). Therefore, we may expect that $\boldsymbol{I}_\mathrm{B}$ is not diagonal.



In fact, we have, e.g.,

$$I_{\mathrm{O},12} = -\int_\mathcal{B} \overline{x}_1 \overline{x}_2 \rho \,\mathrm{d}V = -\rho \int_0^c \int_0^b \int_0^a \overline{x}_1 \overline{x}_2 \,\mathrm{d}\overline{x}_1 \,\mathrm{d}\overline{x}_2 \,\mathrm{d}\overline{x}_3 = -\rho \frac{a^2}{2}\frac{b^2}{2} c = -\frac{M}{4} ab \tag{3.129}$$

with the total mass $M = \rho abc$. If, by contrast, we move the coordinate origin to the center of mass (CM) while keeping all axes parallel (as shown), then we see

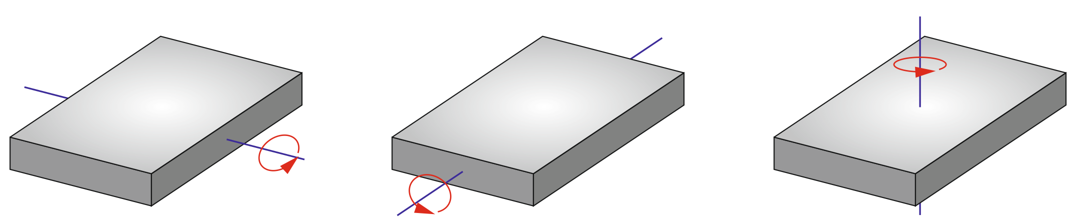
$$I_{\mathrm{CM},12} = -\int_\mathcal{B} \overline{x}_1 \overline{x}_2 \,\mathrm{d}m = -\rho \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \overline{x}_1 \overline{x}_2 \,\mathrm{d}\overline{x}_1 \,\mathrm{d}\overline{x}_2 \,\mathrm{d}\overline{x}_3 = 0. \tag{3.130}$$

The latter can also be verified by Steiner's theorem since (using $I_{\mathrm{CM},12} = 0$)

$$I_{\mathrm{O},12} = I_{\mathrm{CM},12} - M \Delta\overline{x}_1 \,\Delta\overline{x}_2 = -M\left(-\frac{a}{2}\right)\left(-\frac{b}{2}\right) = -\frac{M}{4} ab. \tag{3.131}$$

With respect to the center of mass, the axes are aligned with the symmetries of the cuboid, which is why $[\boldsymbol{I}_\mathrm{CM}]$ is diagonal and we have

$$I_{\mathrm{CM},11} = \frac{M}{12}(b^2 + c^2), \qquad I_{\mathrm{CM},22} = \frac{M}{12}(a^2 + c^2), \qquad I_{\mathrm{CM},33} = \frac{M}{12}(a^2 + b^2). \tag{3.132}$$



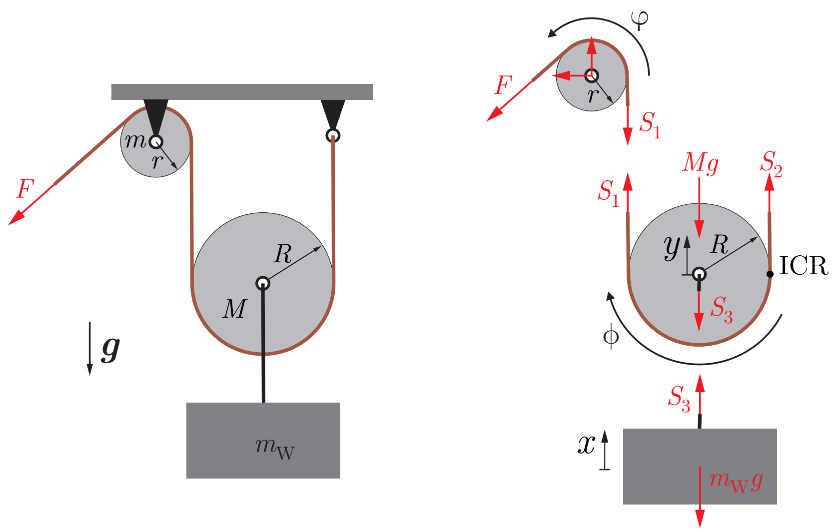
Note that, if the cuboid has two very short sides (e.g., $a, b \ll c$ and writing $c = L$), then we recover the moment of inertia tensor for a slender rod around the x_1- and x_2-axes, since in this case

$$I_{\mathrm{CM},11} \approx \frac{M}{12} L^2, \qquad I_{\mathrm{CM},22} \approx \frac{M}{12} L^2, \qquad I_{\mathrm{CM},33} = \frac{M}{12}(a^2 + b^2). \tag{3.133}$$

Here, $I_{\mathrm{CM},33}$ is the **moment of inertia of a rectangular body** in 2D rotating about its center of mass.

Example 3.10. Pulley system

The system of two pulleys shown below is used to lift a heavy mass m_w by applying a force F. The pulleys have masses M and m, and radii R and r, as indicated, and the inextensible ropes are approximately massless. What is the upward acceleration of m_w for a given force F?



We begin by drawing free-body diagrams (see above); noting that the ropes are assumed massless, the forces in the ropes (labeled S_i) are considered as shown.

We describe the *kinematics* of the system have the shown degrees of freedom $x(t)$, $y(t)$, $\phi(t)$ and $\varphi(t)$. We recognize that the ICR of the larger pulley is on the right, as shown. Further exploiting the inextensibility of the rope leads to the kinematic constraints

$$r\varphi = 2R\phi, \qquad y = R\phi, \qquad y = x, \tag{3.134}$$

and hence

$$\ddot{x} = R\ddot{\phi}, \qquad r\ddot{\varphi} = R\ddot{\phi} = 2\ddot{x}, \qquad \ddot{y} = \ddot{x}. \tag{3.135}$$

Thus, having four degrees of freedom and three independent kinematic constraints implies that we can reduce the kinematic description to a single degree of freedom only, e.g., $x(t)$.

Next investigating the *kinetics* of the system, linear momentum balance on the heavy weight leads to

$$m_\text{w}\ddot{x} = S_3 - m_\text{w}g. \tag{3.136}$$

To avoid the reaction forces on the small pulley as well as force $\boldsymbol{S}_2$, we use angular momentum balance for the two pulleys (around its center for the small pulley, and around the ICR for the large pulley):

$$I_\text{CM}\ddot{\varphi} = \frac{mr^2}{2}\ddot{\varphi} = (F - S_1)r, \qquad \text{and} \qquad I_\text{ICR}\ddot{\phi} = \frac{3MR^2}{2}\ddot{\phi} = S_1 2R - (S_3 + Mg)R. \tag{3.137}$$

Inserting the kinematic relations from (3.135) leads to

$$m\ddot{x} = F - S_1, \qquad \text{and} \qquad \frac{3M}{2}\ddot{x} = 2S_1 - (S_3 + Mg) \qquad \text{and} \qquad m_\mathrm{w}\ddot{x} = S_3 - m_\mathrm{w}g, \quad (3.138)$$

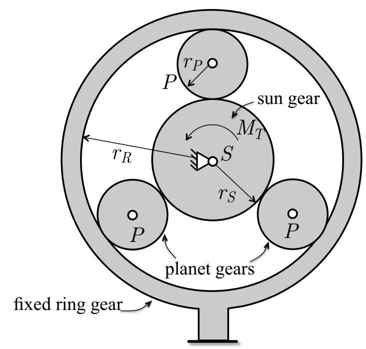
which can be solved for the sought acceleration

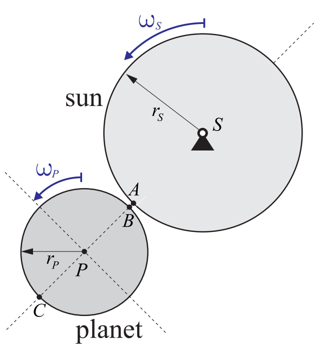
$$\ddot{x} = \frac{4F - 2(m_\mathrm{w} + M)g}{3M + 4m + 2m_\mathrm{w}}. \quad (3.139)$$

As a sanity check, equilibrium in the static case (assuming that all accelerations are zero) leads to $\ddot{x} = 0$ and hence $4F = 2(m_\mathrm{w} + M)g$ (i.e., in equilibrium the force F must only balance gravity acting on the heavy weight and the larger pulley, while the smaller pulley is fixed to the cealing and hence has no impact). Also, if the pulleys were massless ($m = M = 0$), then $\ddot{x} = -g + 2F/m_\mathrm{w}$.

Example 3.11. Planetary gear (epicyclic gear)

Shown on the right is a **planetary gear** system. The outer *ring* gear of radius r_R is fixed. The inner *sun* gear of radius r_S rotates about its center of mass S. Its mass and centroidal moment of inertia are m_S and I_S, respectively. The mass and centroidal moment of the *planet* gears (each of radius r_P) are m_P and I_P, respectively. All gears are rolling on each other without slipping. If a torque M_T is applied to the sun gear, what is its angular acceleration $\dot{\omega}_S$, if the planet gears are free to move without load? Gravity may be neglected.





Let us first describe the *kinematics* of the system. Shown on the left is the sun gear (fixed center of mass S) with only one of the planet gears (center of mass P). Since the center of mass of the sun gear is hinged and since the outer ring is fixed (cf. Example 3.3), we know that

$$\boldsymbol{v}_\mathrm{S} = \boldsymbol{0}, \qquad \boldsymbol{v}_\mathrm{C} = \boldsymbol{0}, \quad (3.140)$$

where the latter follows from rolling without slipping of the planet gear on the outer ring gear. Contact points A and B of the two gears must be moving with the same velocity for rolling without slipping, so

$$\boldsymbol{v}_\mathrm{A} = \boldsymbol{v}_\mathrm{B}. \quad (3.141)$$

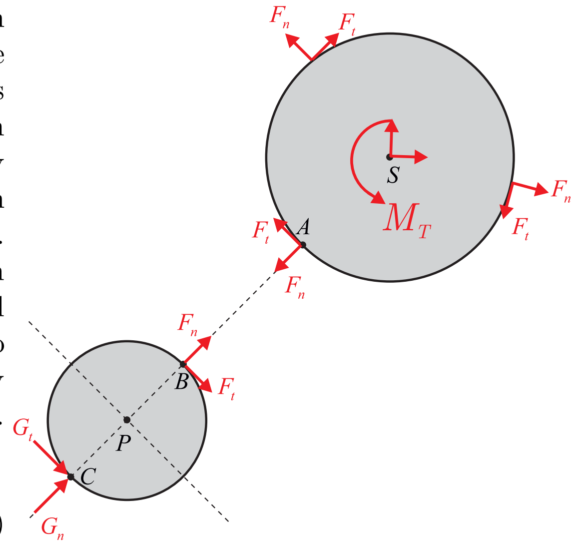
This allows us to use the velocity transfer formula, leading to

$$\boldsymbol{v}_\mathrm{B} = \boldsymbol{v}_\mathrm{C} + \boldsymbol{\omega}_P \times \boldsymbol{r}_\mathrm{CB} \stackrel{!}{=} \boldsymbol{v}_\mathrm{A} = \boldsymbol{v}_\mathrm{S} + \boldsymbol{\omega}_S \times \boldsymbol{r}_\mathrm{SA} \qquad \Rightarrow \qquad \boldsymbol{\omega}_P \times \boldsymbol{r}_\mathrm{CB} = \boldsymbol{\omega}_S \times \boldsymbol{r}_\mathrm{SA}. \quad (3.142)$$

Note that the exact location of the planet gear is irrevelant since points S-A-B-C always lie on a straight line (of some orientation $\boldsymbol{e}$). Therefore, we may write, e.g., $\boldsymbol{r}_\mathrm{CB} = 2r_P\boldsymbol{e}$ and $\boldsymbol{r}_\mathrm{SA} = -r_S\boldsymbol{e}$, which gives (with $\boldsymbol{e}_3$ denoting the out-of-plane basis vector)

$$\omega_P\boldsymbol{e}_3 \times 2r_P\boldsymbol{e} = \omega_S\boldsymbol{e}_3 \times (-r_S\boldsymbol{e}) \qquad \Rightarrow \qquad 2r_P\omega_P = -r_S\omega_S \quad \Leftrightarrow \quad \omega_P = -\frac{r_S}{2r_P}\omega_S. \quad (3.143)$$

Next, we inspect the *kinetics* of the system which is driven by the torque $\boldsymbol{M} = M_T \boldsymbol{e}_3$ applied to the sun gear. To this end, we draw free-body diagrams of the sun gear and one of the planet gears (shown on the right). For reasons of symmetry, we may assume that the contact forces F_i between the sun gear and each planet gear are the same (as drawn). Rolling without slipping means that the gears in contact may exchange both normal and tangential forces (for which we must have $|F_t| \leq \mu_0 |F_n|$ to avoid slipping for smooth surfaces; note that in reality the gears have teeth, so slipping is prevented anyways). Angular momentum balance on the sun wheel gives



$$M_T - 3F_t r_S = I_S \dot{\omega}_S. \tag{3.144}$$

Angular momentum balance on the planet wheel is best formulated around point C, so that the contact forces at point C vanish from the equation which becomes, using (3.143),

$$-F_t \cdot 2r_P = I_\text{C} \dot{\omega}_P \qquad \Rightarrow \qquad -F_t \cdot 2r_P = -I_\text{C} \frac{r_S}{2r_P} \dot{\omega}_S \quad \Leftrightarrow \quad F_t = I_\text{C} \frac{r_S}{4r_P^2} \dot{\omega}_S, \tag{3.145}$$

with $I_\text{C} = I_P + m_P r_P^2$ by Steiner's theorem. Insertion into (3.144) finally yields the sought angular acceleration:

$$M_T - 3I_\text{C} \frac{r_S^2}{4r_P^2} \dot{\omega}_S = I_S \dot{\omega}_S \qquad \Leftrightarrow \qquad \dot{\omega}_S = \frac{4r_P^2 M_T}{4r_P^2 I_S + 3(I_P + m_P r_P^2) r_S^2}. \tag{3.146}$$

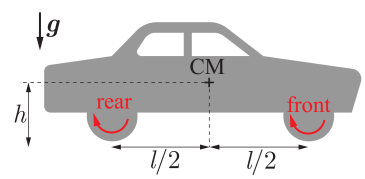
The angular acceleration of the planet gear is obtained as

$$\dot{\omega}_P = -\frac{r_S}{2r_P} \dot{\omega}_S = -\frac{2r_P r_S M_T}{4r_P^2 I_S + 3(I_P + m_P r_P^2) r_S^2}. \tag{3.147}$$

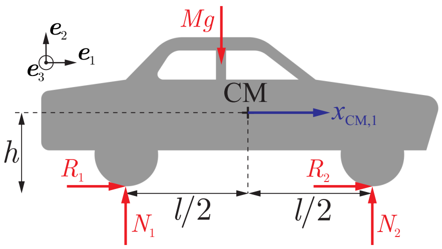
Based on the above results, we can also determine, e.g., the velocity of points P by the velocity transfer formula, viz. $\boldsymbol{v}_\text{P} = \boldsymbol{v}_\text{C} + \boldsymbol{\omega}_P \times \boldsymbol{r}_\text{CP} = \boldsymbol{\omega}_P \times \boldsymbol{r}_\text{CP}$.

Example 3.12. Maximum acceleration of a car

Consider a car with either front or rear drive (i.e., only one axis is being driven by the engine), whose simplified version is shown on the right. The car has a total mass M and the shown approximate extensions. For each of the two cases (front vs. rear drive), what is the maximum acceleration of the car so the wheels roll without slipping?



To identify the kinetics of the vehicle, we start with a free-body diagram (see below).



For the case of a rear drive, we have $R_2 \approx 0$ (no torque is driving the front wheel, and we may assume the moment of inertia of the wheels is small) and $R_1 \neq 0$ (the rear wheel is driven by a torque). Then, linear momentum balance for the car on the ground yields

$$M\ddot{x}_{\text{CM},1} = R_1, \qquad M\ddot{x}_{\text{CM},2} = 0 = N_1 + N_2 - Mg. \quad (3.148)$$

Balance of angular momentum around the center of mass of the car contributes the equation

$$I_{\text{CM}}\dot{\omega} = 0 = -N_1\frac{l}{2} + N_2\frac{l}{2} + hR_1 \quad \Leftrightarrow \quad N_1 - N_2 = R_1\frac{2h}{l}, \quad (3.149)$$

where we used that the car is not supposed to rotate about its CM, so $\dot{\omega} = 0$ (i.e., no wheel is allowed to get off the ground). For rolling without slipping we must have

$$|R_1| \leq \mu_0|N_1| \qquad \Rightarrow \qquad R_1 \leq \mu_0 N_1 \quad (3.150)$$

with the shown directions of the forces. From (3.148) and (3.149) we obtain

$$2N_1 = Mg + R_1\frac{2h}{l} \quad \Rightarrow \quad N_1 = \frac{Mg}{2} + R_1\frac{h}{l} \geq \frac{R_1}{\mu_0} \quad \Leftrightarrow \quad R_1 \leq \frac{\mu_0 Mg}{2(1-\mu_0 h/l)}, \quad (3.151)$$

where we assumed $1 - \mu_0 h/l > 0$. Insertion into linear momentum balance finally yields

$$\ddot{x}_{\text{CM},1} = \frac{R_1}{M} \leq \frac{\mu_0 g}{2(1-\mu_0 h/l)} = \ddot{x}_{\text{max,rear}}. \quad (3.152)$$

Note that we may also check that $N_2 \geq 0$ to avoid lift-off of the front wheel of the car, which is satisfied as long as (using the limiting case $R_1 = \mu_0 N_1$ as a worst-case scenario)

$$N_2 = Mg - N_1 \geq 0 \qquad \Rightarrow \qquad Mg - \frac{Mg}{2(1-\mu_0 h/l)} \geq 0 \qquad \Rightarrow \qquad l \geq 2\mu_0 h. \quad (3.153)$$

Of course, we can repeat the entire procedure for a front-driven car, in which case $R_1 \approx 0$ and $R_2 \neq 0$. Without showing the full derivation (which is analogous to the above), the maximum acceleration in this case is obtained as

$$\ddot{x}_{\text{CM},1} = \frac{R_2}{M} \leq \frac{\mu_0 g}{2(1+\mu_0 h/l)} = \ddot{x}_{\text{max,front}}. \quad (3.154)$$

A comparison shows that $\ddot{x}_{\text{max,front}} < \ddot{x}_{\text{max,rear}}$, so the front-driven car can tolerate less acceleration before slipping occurs (when driving forward).

3.2.4 Angular momentum transfer formula

In our previous definition of $\boldsymbol{I}_{\mathrm{B}}$ we always assumed that point B was on the body. Let us also consider the case that B *is an arbitrary point*. In this case we can exploit the center of mass relations (writing $\boldsymbol{r}_{\mathrm{CB}} = \boldsymbol{r}_{\mathrm{B}} - \boldsymbol{r}_{\mathrm{CM}}$ for conciseness) to find that

$$\begin{aligned}
\boldsymbol{H}_{\mathrm{B}} &= \int_{\mathcal{B}} (\boldsymbol{r} - \boldsymbol{r}_{\mathrm{B}}) \times \boldsymbol{v} \, \mathrm{d}m = \int_{\mathcal{B}} [\boldsymbol{r} - (\boldsymbol{r}_{\mathrm{CM}} + \boldsymbol{r}_{\mathrm{CB}})] \times \boldsymbol{v} \, \mathrm{d}m \\
&= \int_{\mathcal{B}} (\boldsymbol{r} - \boldsymbol{r}_{\mathrm{CM}}) \times \boldsymbol{v} \, \mathrm{d}m - \int_{\mathcal{B}} \boldsymbol{r}_{\mathrm{CB}} \times \boldsymbol{v} \, \mathrm{d}m = \boldsymbol{H}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{CB}} \times \int_{\mathcal{B}} [\boldsymbol{v}_{\mathrm{CM}} + \boldsymbol{\omega} \times (\boldsymbol{r} - \boldsymbol{r}_{\mathrm{CM}})] \, \mathrm{d}m \\
&= \boldsymbol{H}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{CB}} \times \boldsymbol{v}_{\mathrm{CM}} \int_{\mathcal{B}} \mathrm{d}m - \boldsymbol{r}_{\mathrm{CB}} \times \int_{\mathcal{B}} \boldsymbol{\omega} \times (\boldsymbol{r} - \boldsymbol{r}_{\mathrm{CM}}) \, \mathrm{d}m \\
&= \boldsymbol{H}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{CB}} \times M \boldsymbol{v}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{CB}} \times \boldsymbol{\omega} \left[\times \int_{\mathcal{B}} (\boldsymbol{r} - \boldsymbol{r}_{\mathrm{CM}}) \, \mathrm{d}m \right] = \boldsymbol{H}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{CB}} \times M \boldsymbol{v}_{\mathrm{CM}},
\end{aligned}$$

where we re-used relation (3.121). Next, inserting $\boldsymbol{P} = M\boldsymbol{v}_{\mathrm{CM}}$, we find for any *arbitrary point* B that

$$\boldsymbol{H}_{\mathrm{B}} = \boldsymbol{H}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{CB}} \times \boldsymbol{P} = \boldsymbol{H}_{\mathrm{CM}} + \boldsymbol{P} \times \boldsymbol{r}_{\mathrm{CB}}. \tag{3.155}$$

Likewise, for another point A we have

$$\boldsymbol{H}_{\mathrm{A}} = \boldsymbol{H}_{\mathrm{CM}} + \boldsymbol{P} \times \boldsymbol{r}_{\mathrm{CA}}. \tag{3.156}$$

Subtracting the two hence leads to

$$\boldsymbol{H}_{\mathrm{B}} - \boldsymbol{H}_{\mathrm{A}} = \boldsymbol{P} \times (\boldsymbol{r}_{\mathrm{CB}} - \boldsymbol{r}_{\mathrm{CA}}) = \boldsymbol{P} \times \boldsymbol{r}_{\mathrm{AB}} \qquad \Rightarrow \qquad \boxed{\boldsymbol{H}_{\mathrm{B}} = \boldsymbol{H}_{\mathrm{A}} + \boldsymbol{P} \times \boldsymbol{r}_{\mathrm{AB}}} \tag{3.157}$$

This is the **angular momentum transfer formula** for arbitrary points A and B, which do *not* need to lie on the same body $\mathcal{B}$. This has an important consequence: if we know $\boldsymbol{H}_{\mathrm{A}}$ for any point A on the body, we can use (3.157) to find $\boldsymbol{H}_{\mathrm{B}}$ for any arbitrary other point B that is not necessarily on the body. We highlight this principle in Example 3.13 below.

Example 3.13. Angular momentum transfer relations for a rotating disk

Consider a disk of radius R and mass M, which is rolling without slipping on the ground. We previously derived the angular momentum of the disk in this scenario with respect to the center of mass as (cf. (3.104))

$$H_{\mathrm{CM}} = I_{\mathrm{CM}} \omega = \frac{MR^2}{2} \omega. \tag{3.158}$$

We further used Steiner's theorem to show that angular momentum balance with respect to point A on the disk in contact with the ground becomes (cf. (3.126))

$$H_{\mathrm{A}} = I_{\mathrm{A}} \omega = \frac{3MR^2}{2} \omega. \tag{3.159}$$

The angular momentum transfer formula (3.157) confirms this since

$$\boldsymbol{H}_{\mathrm{A}} = \boldsymbol{H}_{\mathrm{CM}} + \boldsymbol{P} \times (\boldsymbol{r}_{\mathrm{A}} - \boldsymbol{r}_{\mathrm{CM}}) \tag{3.160}$$

so that, with $\boldsymbol{v}_{\mathrm{CM}} = \boldsymbol{v}_{\mathrm{A}} + \boldsymbol{\omega} \times (\boldsymbol{r}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{A}})$ and $\boldsymbol{v}_{\mathrm{A}} = \boldsymbol{0}$, we write

$$\begin{aligned} \boldsymbol{H}_{\mathrm{A}} &= I_{\mathrm{CM}}\omega \boldsymbol{e}_3 + M\boldsymbol{v}_{\mathrm{CM}} \times (\boldsymbol{r}_{\mathrm{A}} - \boldsymbol{r}_{\mathrm{CM}}) = I_{\mathrm{CM}}\omega \boldsymbol{e}_3 + M\left[\omega \boldsymbol{e}_3 \times (\boldsymbol{r}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{A}})\right] \times (\boldsymbol{r}_{\mathrm{A}} - \boldsymbol{r}_{\mathrm{CM}}) \\ &= I_{\mathrm{CM}}\omega \boldsymbol{e}_3 + M\omega \boldsymbol{e}_3 |\boldsymbol{r}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{A}}|^2 = \left(I_{\mathrm{CM}} + MR^2\right)\omega \boldsymbol{e}_3 = \frac{3MR^2}{2}\omega \boldsymbol{e}_3. \end{aligned} \tag{3.161}$$

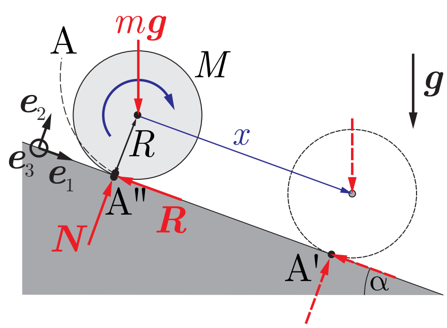
Writing $\boldsymbol{H}_{\mathrm{A}} = I_{\mathrm{A}}\omega \boldsymbol{e}_3$, this agrees with (3.160) as expected.

It is essential to recall that Steiner's theorem only applies to points *on the body* but not e.g. to points in contact with the body. The angular momentum transfer formula (3.157) gives us a convenient way to involve *points that are not on the body*. For example, let us pick point A′ which is the point on the ground instantaneously in contact with the disk (as shown in the above schematic). For this point we have

$$\boldsymbol{H}_{\mathrm{A}'} = \boldsymbol{H}_{\mathrm{CM}} + \boldsymbol{P} \times (\boldsymbol{r}_{\mathrm{A}'} - \boldsymbol{r}_{\mathrm{CM}}) = \boldsymbol{H}_{\mathrm{CM}} + \boldsymbol{P} \times (\boldsymbol{r}_{\mathrm{A}} - \boldsymbol{r}_{\mathrm{CM}}) = \boldsymbol{H}_{\mathrm{A}}. \tag{3.162}$$

Therefore, we can also calculate the angular momentum of the disk with respect to contact point A′ on the ground, and it agrees with the one with respect to contact point A on the disk.

Be careful when using *angular momentum balance with respect to these two points*! While the externally applied torque (due to gravity on an inclined slope) is constant for point A, one must carefully define which point A′ is chosen on the ground. Angular momentum balance generally reads



$$\boldsymbol{M}_{\mathrm{A}'} = \dot{\boldsymbol{H}}_{\mathrm{A}'} + \boldsymbol{v}_{\mathrm{A}'} \times \boldsymbol{P}, \tag{3.163}$$

where

$$\boldsymbol{M}_{\mathrm{A}'} = (\boldsymbol{r}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{A}'}) \times m\boldsymbol{g} \qquad \text{with} \qquad \boldsymbol{g} = g\,(\sin\alpha\,\boldsymbol{e}_1 - \cos\alpha\,\boldsymbol{e}_2) \tag{3.164}$$

in the shown reference frame, and

$$\boldsymbol{P} = M\boldsymbol{v}_{\mathrm{CM}} = M\dot{x}\boldsymbol{e}_1, \tag{3.165}$$

where x denotes the disk's travel distance down the slope, as shown.

We now need to make a choice. If we choose point A′ to be the *moving* point on the ground always in contact with the disk (i.e., at each instance of time pick A′ as the contact point instead of a fixed point in space), then

$$\boldsymbol{r}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{A}'} = R\boldsymbol{e}_2, \qquad \boldsymbol{v}_{\mathrm{A}'} = \dot{x}\boldsymbol{e}_1 \neq \boldsymbol{0}, \qquad \boldsymbol{M}_{\mathrm{A}'} = -MgR\sin\alpha\,\boldsymbol{e}_3 = \text{const.}, \tag{3.166}$$

so that angular momentum balance (3.163) with $\boldsymbol{H}_{\mathrm{A}'} = \boldsymbol{H}_{\mathrm{A}}$ from (3.162) and $\boldsymbol{P} = M\dot{x}\boldsymbol{e}_1$ gives

$$-MgR\sin\alpha\,\boldsymbol{e}_3 = \dot{\boldsymbol{H}}_{\mathrm{A}'} + \boldsymbol{v}_{\mathrm{A}'} \times \boldsymbol{P} = \dot{\boldsymbol{H}}_{\mathrm{A}} + \dot{x}\boldsymbol{e}_1 \times \boldsymbol{P} = \dot{\boldsymbol{H}}_{\mathrm{A}}. \tag{3.167}$$

Hence, the torque and the angular momentum with respect to A$'$ remain constant over time.

By contrast, if we choose a *fixed* point A$''$ on the ground, then the body moves away from A$''$ over time. In this case,

$$\boldsymbol{r}_{\text{CM}} - \boldsymbol{r}_{\text{A}''} = x\boldsymbol{e}_1 + R\boldsymbol{e}_2, \qquad \boldsymbol{v}_{\text{A}''} = \boldsymbol{0}, \tag{3.168}$$

and

$$\boldsymbol{M}_{\text{A}''} = (\boldsymbol{r}_{\text{CM}} - \boldsymbol{r}_{\text{A}'}) \times (m\boldsymbol{g} + \boldsymbol{N}) = -Mg(R\sin\alpha + x\cos\alpha)\boldsymbol{e}_3 + xN\boldsymbol{e}_3, \tag{3.169}$$

where $\boldsymbol{N} = N\boldsymbol{e}_2$ is the normal force from the ground onto the disk. The angular momentum transfer formula here yields

$$\boldsymbol{H}_{\text{A}''} = \boldsymbol{H}_{\text{CM}} + \boldsymbol{P} \times (\boldsymbol{r}_{\text{A}''} - \boldsymbol{r}_{\text{CM}}) = I_{\text{CM}}\omega\boldsymbol{e}_3 + \boldsymbol{P} \times (\boldsymbol{r}_{\text{A}'} - \boldsymbol{r}_{\text{CM}} - x\boldsymbol{e}_1) = \boldsymbol{H}_{\text{A}'} - \boldsymbol{P} \times x\boldsymbol{e}_1. \tag{3.170}$$

When exploiting $\boldsymbol{H}_{\text{A}} = \boldsymbol{H}_{\text{A}'}$ and $\boldsymbol{P} \parallel \boldsymbol{e}_1$, angular momentum balance (3.163) with respect to the fixed point A$''$ becomes

$$-Mg(R\sin\alpha + x\cos\alpha)\boldsymbol{e}_3 + xN\boldsymbol{e}_3 = \dot{\boldsymbol{H}}_{\text{A}}. \tag{3.171}$$

If we further exploit linear momentum balance (in the direction normal to the ground), we obtain

$$N = Mg\cos\alpha \qquad \Rightarrow \qquad -MgR\sin\alpha\boldsymbol{e}_3 = \dot{\boldsymbol{H}}_{\text{A}}, \tag{3.172}$$

so that (3.171) again reduces to (3.167). Hence, we arrive at the same angular momentum balance when using point A on the disk, point A' on the ground moving with the disk, and a fixed point A" on the ground.

Finally, note that we could alternatively formulate angular momentum balance with respect to the disk's center of mass, which would avoid the complications stemming form the angular momentum discussed above. Yet, this would necessarily involve the unknown tangential force $\boldsymbol{R}$ on the ground (responsible for the rolling motion), so one would need an additional equation (e.g., linear momentum balance.)

Example 3.14. Angular momentum balance with respect to the ICR

In many applications, like the rolling disk in Example 3.13, we may want to formulate angular momentum balance with respect to the instantaneous center of rotation (ICR) of a body in 2D. In this case we need to exercise caution since that point may not lie on the body (recall, e.g., the sliding ladder of Example 3.1).

Angular momentum balance with respect to any point, therefore also with respect to the ICR, reads

$$\boldsymbol{M}_{\text{ICR}} = \dot{\boldsymbol{H}}_{\text{ICR}} + \boldsymbol{v}_{\text{ICR}} \times \boldsymbol{P} = \dot{\boldsymbol{H}}_{\text{ICR}}, \tag{3.173}$$

since the instantaneous velocity of the ICR vanishes by definition ($\boldsymbol{v}_{\text{ICR}} = \boldsymbol{0}$). As the ICR does not necessarily lie on the moving body, we invoke the angular momentum transfer formula to write

$$\boldsymbol{H}_{\text{ICR}} = \boldsymbol{H}_{\text{CM}} + \boldsymbol{P} \times \boldsymbol{r}_{\text{CM}-\text{ICR}}, \tag{3.174}$$

where $\boldsymbol{r}_{\text{CM}-\text{ICR}} = \boldsymbol{r}_{\text{ICR}} - \boldsymbol{r}_{\text{CM}}$ is the vector from the bodies center of mass to the ICR. We further know that

$$\boldsymbol{H}_{\text{CM}} = \boldsymbol{I}_{\text{CM}}\boldsymbol{\omega} \qquad \text{and} \qquad \boldsymbol{P} = M\boldsymbol{v}_{\text{CM}} = M\,\boldsymbol{\omega} \times \boldsymbol{r}_{\text{ICR}-\text{CM}} = -M\,\boldsymbol{\omega} \times \boldsymbol{r}_{\text{CM}-\text{ICR}}, \tag{3.175}$$

where we exploited that the velocity of any point on the body in 2D can be formulated as a pure rotation about the center of mass. Insertion of (3.175) into (3.174) yields

$$\boldsymbol{H}_{\text{ICR}} = \boldsymbol{I}_{\text{CM}}\boldsymbol{\omega} - (M\,\boldsymbol{\omega} \times \boldsymbol{r}_{\text{CM}-\text{ICR}}) \times \boldsymbol{r}_{\text{CM}-\text{ICR}}. \tag{3.176}$$

For a 2D rotation in the x_1-x_2-plane we know $\boldsymbol{\omega} = \omega \boldsymbol{e}_3$ and $\boldsymbol{\omega} \perp \boldsymbol{r}_{\text{CM}-\text{ICR}}$ and can further exploit (3.109) to arrive at

$$\boldsymbol{H}_{\text{ICR}} = \left(I_{\text{CM}} + M r^2_{\text{CM}-\text{ICR}}\right)\omega \boldsymbol{e}_3 \qquad \Rightarrow \qquad \boldsymbol{M}_{\text{ICR}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(I_{\text{CM}} + M r^2_{\text{CM}-\text{ICR}}\right)\omega \boldsymbol{e}_3 \tag{3.177}$$

with $r_{\text{CM}-\text{ICR}} = |\boldsymbol{r}_{\text{CM}-\text{ICR}}|$. For example, in the case of a rolling disk of radius R, we have $r_{\text{CM}-\text{ICR}} = R = \text{const.}$, so that angular momentum balance with respect to the ICR becomes

$$M_{\text{ICR}} = \left(I_{\text{CM}} + MR^2\right)\dot{\omega}, \tag{3.178}$$

which agrees with are findings in Example 3.13. Note that, even though (3.178) looks like a simple application of Steiner's theorem, it is not in general, since the ICR does not lie on the body, so *Steiner's theorem cannot be applied.* What we have shown here is that it can indeed be applied even to the ICR in 2D, as long as the distance $r_{\text{CM}-\text{ICR}}$ remains constant.

3.2.5 Work–energy balance

Like for particles, we can also formulate the work–energy balance law for rigid bodies. The total kinetic energy of a rigid body $\mathcal{B}$ is obtained by integration:

$$\begin{aligned} T &= \int_{\mathcal{B}} \frac{1}{2}|\boldsymbol{v}|^2\,\mathrm{d}m = \frac{1}{2}\int_{\mathcal{B}} |\boldsymbol{v}_{\text{C}} + \boldsymbol{\omega} \times (\boldsymbol{r} - \boldsymbol{r}_{\text{C}})|^2\,\mathrm{d}m \\ &= \frac{1}{2}\int_{\mathcal{B}} |\boldsymbol{v}_{\text{C}}|^2\,\mathrm{d}m + \boldsymbol{v}_{\text{C}} \cdot \left(\boldsymbol{\omega} \times \int_{\mathcal{B}} (\boldsymbol{r} - \boldsymbol{r}_{\text{C}})\,\mathrm{d}m\right) + \frac{1}{2}\int_{\mathcal{B}} |\boldsymbol{\omega} \times (\boldsymbol{r} - \boldsymbol{r}_{\text{C}})|^2\,\mathrm{d}m. \end{aligned} \tag{3.179}$$

If we assume that $\text{C} \in \mathcal{B}$ and that point C is either fixed ($\boldsymbol{v}_{\text{C}} = \boldsymbol{0}$) or the center of mass ($\text{C} = \text{CM}$), then the second term above vanishes so that

$$T = \frac{1}{2}\int_{\mathcal{B}} |\boldsymbol{v}_{\text{C}}|^2\,\mathrm{d}m + \frac{1}{2}\int_{\mathcal{B}} |\boldsymbol{\omega} \times (\boldsymbol{r} - \boldsymbol{r}_{\text{C}})|^2\,\mathrm{d}m = \frac{1}{2}M|\boldsymbol{v}_{\text{C}}|^2 + \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{I}_{\text{C}}\,\boldsymbol{\omega}, \tag{3.180}$$

where the latter expression follows from $(\boldsymbol{\omega}\times\boldsymbol{r})\cdot(\boldsymbol{\omega}\times\boldsymbol{r}) = [\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})]\cdot\boldsymbol{\omega}$ and using the definition of $\boldsymbol{H}_{\mathrm{C}}$, cf. Eq. (3.86). Hence, the **kinetic energy of a rigid body** $\mathcal{B}$ can be expressed solely through the kinematics of a point $\mathrm{C}\in\mathcal{B}$, viz.

$$\boxed{T = \frac{1}{2}M|\boldsymbol{v}_{\mathrm{C}}|^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{I}_{\mathrm{C}}\,\boldsymbol{\omega} \qquad \text{if} \qquad \boldsymbol{v}_{\mathrm{C}} = \boldsymbol{0} \quad \text{or} \quad \mathrm{C} = \mathrm{CM}} \tag{3.181}$$

We see that T has a **translational kinetic energy** contribution $T_{\mathrm{trans}} = \frac{1}{2}M|\boldsymbol{v}_{\mathrm{C}}|^2$ as well as a **rotational kinetic energy** contribution $T_{\mathrm{rot}} = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{I}_{\mathrm{C}}\,\boldsymbol{\omega}$.

Again taking the limit of infinitely many material points (or particles) stacked into body $\mathcal{B}$, the **work–energy balance** (2.19) for a rigid body becomes

$$\boxed{T(t_2) - T(t_1) = W_{12}, \qquad W_{12} = \sum_i \int_{\boldsymbol{r}_i(t_1)}^{\boldsymbol{r}_i(t_2)} \boldsymbol{F}_i^{\mathrm{ext}}\cdot\mathrm{d}\boldsymbol{r}_i = \sum_i \int_{t_1}^{t_2} \boldsymbol{F}_i^{\mathrm{ext}}\cdot\boldsymbol{v}_i\,\mathrm{d}t,} \tag{3.182}$$

where $\boldsymbol{r}_i$ and $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i$ denote, respectively, the position and velocity of the point of attack of the external forces $\boldsymbol{F}_i^{\mathrm{ext}}$. Further, we exploited that in a *rigid* body the internal forces perform no work (as discussed for rigid links within systems of particles).

As before, if an external force is **conservative**, then we can exploit that the force derives from a potential V and the work done by the force can be expressed as a potential difference (see Section 2.2.2 for a detailed discussion). If both *conservative and non-conservative forces* act simultaneously on the body, then we may again exploit the additive structure of W_{12}. Specifically, we may still use the difference in potential energy to compute $W_{12} = V_1 - V_2$ for each conservative force in the system, while we only need to integrate the work done by those forces that are not conservative, so $T(t_2) + V(t_2) - [T(t_1) + V(t_1)] = W_{12}^{\text{non-cons.}}$.

If *all external forces are conservative*, then the system is conservative and the overall (kinetic plus potential) energy is conserved. The **conservation of energy** of a rigid body in the special case of *only conservative forces* implies

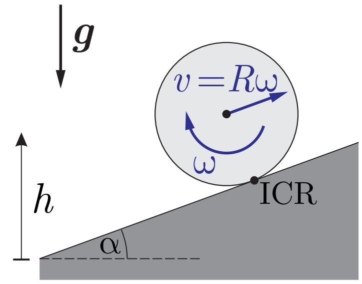
$$\boxed{T + V = \text{const.}} \tag{3.183}$$

Note that if gravity acts on a homogeneous rigid body $\mathcal{B}$ with center of mass CM, then the potential energy due to gravity can conveniently be expressed as $V = Mgh_{\mathrm{CM}}$,

$$V = \int_{\mathcal{B}} \boldsymbol{g}\cdot\boldsymbol{r}\rho\,\mathrm{d}V = \boldsymbol{g}\cdot\int_{\mathcal{B}} \boldsymbol{r}\rho\,\mathrm{d}V = M\boldsymbol{g}\cdot\boldsymbol{r}_{\mathrm{CM}}, \tag{3.184}$$

where M is the total mass and $\boldsymbol{r}_{\mathrm{CM}}$ denotes the location of CM (the shape and current orientation of the body are irrelevant here). For a 2D problem, the above simply becomes $V = Mgh_{\mathrm{CM}}$ with h_{CM} being the altitude of CM above the chosen zero level.

Example 3.15. Rolling disk on an inclined slope



We consider a circular disk (or cylinder) of mass M and radius R, rolling without slipping on an inclined slope of angle α. Given an initial rolling velocity v_0, how high does the body roll, and how does the answer vary if the body is sliding frictionlessly instead of rolling?

This problem is ideal to solve with the work–energy balance, which allows us to compare the initial and final stages. The system is conservative, so we may apply the *conservation of energy.*

The kinetic energy of the disk is most conveniently obtained either with respect to the center of mass (CM) or with respect to the instantaneous center of rotation (ICR), both yielding the same kinetic energy. Specifically, with $|\boldsymbol{v}_{\text{CM}}| = |\omega|R$ we obtain with respect to CM or ICR, respectively,

$$T = \frac{1}{2}M(\omega R)^2 + \frac{1}{2}\frac{MR^2}{2}\omega^2 \qquad \text{or} \qquad T = \frac{1}{2}\frac{3MR^2}{2}\omega^2. \tag{3.185}$$

These two are obviously the same, as expected. Since we consider rolling without slipping, the system is conservative. To find the maximum height to which the body will roll if it starts to roll up the slope with an initial velocity $|\boldsymbol{v}_{\text{CM}}| = v_0 = R\omega_0$, we consider the potential energy due to from gravity only, so that

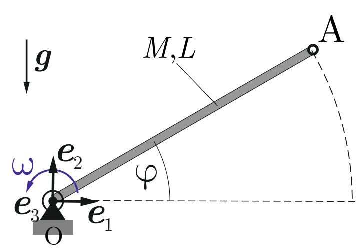
$$V = Mgh \qquad \Rightarrow \qquad \frac{3MR^2}{4}\omega_0^2 = \frac{3M}{4}v_0^2 = Mgh_{\max} \qquad \Leftrightarrow \qquad h_{\max} = \frac{3v_0^2}{4g}. \tag{3.186}$$

For comparison, we consider the case of a particle of mass m sliding up an inclined slope without friction and without rolling. We saw in Example 1.15 that in this case

$$\frac{m}{2}v_0^2 = mgh_{\max} \qquad \Leftrightarrow \qquad h_{\max} = \frac{v_0^2}{2g}. \tag{3.187}$$

Hence, the travel distance $s_{\max} = h_{\max}\cot\alpha$ is larger in case of a rolling cylinder as compared to a frictionlessly sliding one because of the access kinetic energy that the rolling cylinder initially has.

Example 3.16. Falling hinged stick



Consider a rod of length L and mass M, which is hinged to rotate freely (without friction) at one end (O) while the other end (A) is free. Starting from the initial position $\varphi = \varphi_0$, the rod falls downward under the action of gravity. Assuming that point O remains hinged, what is the acceleration of point A during the fall? How does it compare to a freely falling (not hinged) rod of mass M?

The acceleration of point A can be obtained from the acceleration transfer formula, viz.

$$\boldsymbol{a}_{\text{A}} = \boldsymbol{a}_{\text{O}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{\text{OA}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{\text{OA}}) \tag{3.188}$$

where $\boldsymbol{a}_\mathrm{O} = \boldsymbol{0}$ and $\boldsymbol{r}_\mathrm{OA} = L\cos\varphi\,\boldsymbol{e}_1 + L\sin\varphi\,\boldsymbol{e}_2$ (where we defined φ to be positive counter-clockwise and with respect to the horizontal $\boldsymbol{e}_1$-axis). When inserting $\boldsymbol{\omega} = \dot\varphi\boldsymbol{e}_3$, this gives

$$\begin{aligned}\boldsymbol{a}_\mathrm{A} &= \ddot\varphi\boldsymbol{e}_3 \times \boldsymbol{r}_\mathrm{OA} + \dot\varphi\boldsymbol{e}_3 \times (\dot\varphi\boldsymbol{e}_3 \times \boldsymbol{r}_\mathrm{OA})\\ &= \ddot\varphi\boldsymbol{e}_3 \times (L\cos\varphi\boldsymbol{e}_1 + L\sin\varphi\boldsymbol{e}_2) + \dot\varphi\boldsymbol{e}_3 \times (\dot\varphi\boldsymbol{e}_3 \times (L\cos\varphi\boldsymbol{e}_1 + L\sin\varphi\boldsymbol{e}_2))\\ &= L\ddot\varphi(\cos\varphi\boldsymbol{e}_2 - \sin\varphi\boldsymbol{e}_1) - L\dot\varphi^2(\cos\varphi\boldsymbol{e}_1 + \sin\varphi\boldsymbol{e}_2)\\ &= -L(\dot\varphi^2\cos\varphi + \ddot\varphi\sin\varphi)\boldsymbol{e}_1 + L(-\dot\varphi^2\sin\varphi + \ddot\varphi\cos\varphi)\boldsymbol{e}_2.\end{aligned} \tag{3.189}$$

So far we have only used kinematics, but we still need to find $\dot\varphi$ and $\ddot\varphi$ as functions of φ. The former can be found by using the conservation of energy (no external forces do any work on the system; gravity is conservative). Comparing the initial state with $\dot\varphi_0$ at φ_0 gives:

$$\frac{1}{2}I_\mathrm{O}\dot\varphi_0^2 + Mg\frac{L}{2}\sin\varphi_0 = \frac{1}{2}I_\mathrm{O}\dot\varphi^2 + Mg\frac{L}{2}\sin\varphi \qquad \text{with} \qquad I_\mathrm{O} = \frac{1}{3}ML^2. \tag{3.190}$$

If the pendulum is released from rest ($\dot\varphi_0 = 0$), then

$$\frac{1}{2}\frac{1}{3}ML^2\dot\varphi^2 = Mg\frac{L}{2}(\sin\varphi_0 - \sin\varphi) \qquad \Rightarrow \qquad \dot\varphi^2 = \frac{3g}{L}(\sin\varphi_0 - \sin\varphi). \tag{3.191}$$

Next, we need to find the angular acceleration $\ddot\varphi$. For example, we may use angular momentum balance (which is free of the reaction forces when taken around O):

$$\frac{1}{3}ML^2\ddot\varphi = -Mg\frac{L}{2}\cos\varphi \qquad \Rightarrow \qquad \ddot\varphi = -\frac{3g}{2L}\cos\varphi. \tag{3.192}$$

Note that the same could also have been obtained from differentiating $\dot\varphi^2$ in (3.191) with respect to time. Inserting the angular velocity and acceleration into $\boldsymbol{a}_\mathrm{A}$ finally gives

$$\boldsymbol{a}_\mathrm{A} = -\frac{3g}{2}\cos\varphi\,[2(\sin\varphi_0 - \sin\varphi) - \sin\varphi]\,\boldsymbol{e}_1 - \frac{3g}{2}\left[2(\sin\varphi_0\sin\varphi - \sin^2\varphi) + \cos\varphi^2\right]\boldsymbol{e}_2. \tag{3.193}$$

For example, when starting from the horizontal position ($\varphi_0 = 0$), this reduces to

$$\boldsymbol{a}_\mathrm{A} = \frac{9g}{2}\cos\varphi\sin\varphi\,\boldsymbol{e}_1 - \frac{3g}{2}\left(1 - 3\sin^2\varphi\right)\boldsymbol{e}_2. \tag{3.194}$$

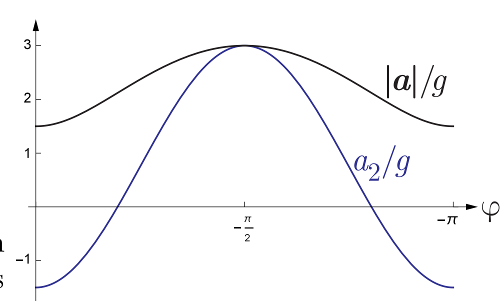
The magnitude of the acceleration can be rearranged to

$$a_\mathrm{A} = |\boldsymbol{a}_\mathrm{A}| = \frac{3g}{2}\sqrt{\frac{1}{2}(5 - 3\cos 2\varphi)},$$

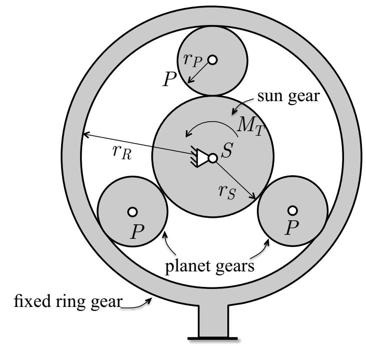
while the vertical component $a_{\mathrm{A},2} = \boldsymbol{a}_\mathrm{A}\cdot\boldsymbol{e}_2$ is

$$a_{\mathrm{A},2} = -\frac{3g}{2}\left(1 - 3\sin^2\varphi\right).$$

Therefore, the acceleration of point A is $|-3g/2| > g$ upon release (at $\varphi - 0$), after which it decreases until it inverts its sign before reaching the bottom at $\varphi = -\frac{\pi}{2}$ (see the plot on the right). Note that there are indeed points that accelerate faster than with gravity.



Example 3.17. Planetary gear revisited



As a final example of the work–energy balance of rigid bodies, let us revisit Example 3.11 and solve it by using the work–energy balance. The total kinetic energy of the gear at any time t is

$$T(t) = \frac{1}{2} I_S \omega_S^2(t) + \frac{3}{2} I_\mathrm{C} \omega_P^2(t), \tag{3.195}$$

where we picked the *fixed* points S (on the sun wheel) and C (on the planet) wheel, so that only rotational energy is to be considered.

Assuming that the gear is accelerated from rest (so $\omega_S = \omega_P = 0$ at $t = 0$) and equating the energy difference to the work done by the constant external torque M_T, work–energy balance becomes

$$T(t) - T(0) = \frac{1}{2} I_S \omega_S^2(t) + \frac{3}{2} I_\mathrm{C} \omega_P^2(t) = M_T \int_0^t \mathrm{d}\varphi_S \quad \Rightarrow \quad \frac{1}{2} I_S \omega_S^2(t) + \frac{3}{2} I_\mathrm{C} \omega_P^2(t) = M_T \varphi_S \tag{3.196}$$

Here, we needed the work done by an external torque. The latter can be interpreted as the work done by a tangentially applied force $\boldsymbol{F}$ such that $\boldsymbol{M}_T = \boldsymbol{F} \times \boldsymbol{r}_\mathrm{CT}$ and $\boldsymbol{F} \perp \boldsymbol{r}_\mathrm{CT}$ so $M_T = F r_\mathrm{CT}$ with $r_\mathrm{CT} = |\boldsymbol{r}_\mathrm{CT}|$. Exploiting the circular movement with $\mathrm{d}r = r_\mathrm{CT}\,\mathrm{d}\varphi$ and $\boldsymbol{F} \parallel \mathrm{d}\boldsymbol{r}$:

$$W_{12} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = F \int_0^{\varphi_S} r_\mathrm{CT}\,\mathrm{d}\varphi = F r_\mathrm{CT} \int_0^{\varphi_S} \mathrm{d}\varphi = M_T \int_0^{\varphi_S} \mathrm{d}\varphi = M_T \varphi_S. \tag{3.197}$$

In general, we note that the **work done by a torque** $\boldsymbol{M}_T(t)$ on a body rotating with $\boldsymbol{\varphi}(t)$ is

$$\boxed{W_{12} = \int_{\boldsymbol{\varphi}(t_1)}^{\boldsymbol{\varphi}(t_2)} \boldsymbol{M}_T \cdot \mathrm{d}\boldsymbol{\varphi} = \int_{t_1}^{t_2} \boldsymbol{M}_T \cdot \boldsymbol{\omega}\,\mathrm{d}t} \tag{3.198}$$

This relation is analogous to the work derived for forces.

Differentiating (3.196) with respect to time and using $\omega_S = \dot{\varphi}_S$ leads to (for $\omega_S \neq 0$)

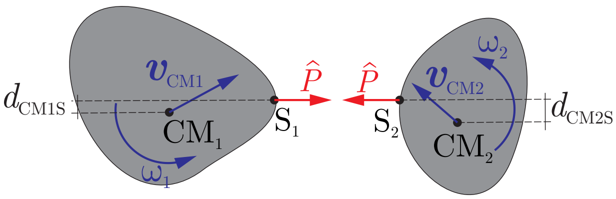
$$I_S \omega_S \dot{\omega}_S + 3 I_\mathrm{C} \omega_P \dot{\omega}_P = M_T \omega_S \quad \Leftrightarrow \quad \left(I_S + 3 I_\mathrm{C} \frac{\omega_P}{\omega_S} \frac{\dot{\omega}_P}{\dot{\omega}_S} \right) \dot{\omega}_S = M_T, \tag{3.199}$$

which, upon insertion of

$$\omega_P = -\omega_S \frac{r_S}{2 r_P} \quad \text{and} \quad \dot{\omega}_P = -\dot{\omega}_S \frac{r_S}{2 r_P} \quad \Rightarrow \quad \frac{\omega_P}{\omega_S} = \frac{\dot{\omega}_P}{\dot{\omega}_S} = -\frac{r_S}{2 r_P} \tag{3.200}$$

from the kinematics of Example 3.11, yields the same solution for the sun gear's angular acceleration as derived there, cf. Eq. (3.147).

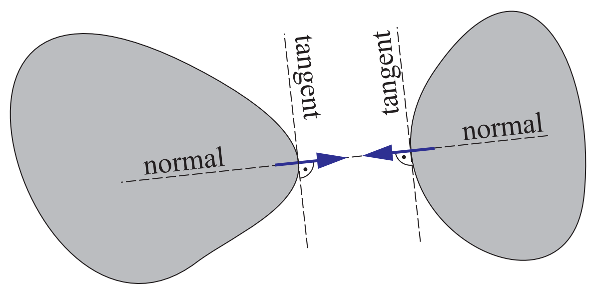
3.3 Collision of Rigid Bodies



In Section 2.3 we discussed collisions of particles. Here, we extend the relations found there to rigid bodies. Specifically, we here treat the general case of an *eccentric, oblique collision.* A collision is called an **eccentric collision** if the impulsive forces exchanged between the bodies are not on the line connecting the centers of mass of the participating bodies.

By definition, *particle* collisions were always *centric*, as particles have negligible dimensions, so any force must be directed towards the center of mass of the particles. For rigid bodies, that constraint is no longer in place. A collision is further called an **oblique collision** if the impulsive force exchange is at an angle to the line of impact (e.g., shown above are two particles colliding with initial velocities that are not aligned with the impulsive forces).

To keep things simple, we here and in the following assume that the colliding bodies are *convex*. In this case, every collision – irrespective of the initial velocities of the bodies – has unique normal and tangent directions in 2D (and a unique normal direction and tangent plane in 3D). The normal is given by the local (common) normal vector perpendicular to each body's surface, while the tangent direction (or plane) is defined by being perpendicular to that normal.



Recall that a **collision** was defined as a contact over a short time window $[t_-, t_+]$, decomposed into compression and restitution phases. For particles, we used the conservation of linear momentum during the collision along with the coefficient of restitution to find the particle velocities after the collision (see Section 2.3).

Here, that concept is extended to rigid bodies, which requires accounting for rotations. For simplicity, let us consider the 2D scenario schematically shown above, where the collision between two bodies 1 and 2 happens at a point S in the plane, so that there is a pair of forces $\boldsymbol{P}(t)$ and $-\boldsymbol{P}(t)$ (having the same magnitudes but opposite directions by Newton's third axiom) which act onto the two bodies, respectively, at points S_1 and S_2, as shown. For an eccentric collision, S_1, S_2 and the centers of mass CM_1 and CM_2 do not fall onto a straight line (as shown above; the collision is eccentric only if $d_{CM_1S} = d_{CM_2S} = 0$).

Like for particles, forces $\pm\boldsymbol{P}(t)$ are time-dependent, and we can eliminate the time-dependence in the equations by considering the complete collision period $[t_-, t_+]$. To this end, we integrate the equations of linear momentum balance and (new here) angular momentum balance in time analogously to Section 2.3 over the complete collision period, again defining an impulsive force as

$$\hat{\boldsymbol{P}} = \int_{t_-}^{t_+} \boldsymbol{P}(t)\,\mathrm{d}t. \tag{3.201}$$

Integration of linear and angular momentum balance for body 1 then yields

$$\boxed{\hat{\boldsymbol{P}} = m_1\left[\boldsymbol{v}_{\mathrm{CM}_1}(t_+) - \boldsymbol{v}_{\mathrm{CM}_1}(t_-)\right], \qquad \boldsymbol{r}_{\mathrm{CM}_1\mathrm{S}} \times \hat{\boldsymbol{P}} = \boldsymbol{I}_{\mathrm{CM}_1}\left[\boldsymbol{\omega}_1(t_+) - \boldsymbol{\omega}_1(t_-)\right]} \tag{3.202}$$

where $\boldsymbol{r}_{\mathrm{CM}_1\mathrm{S}}$ denotes the vector pointing from body 1's center of mass to the collision point S_1. Note that we assume the collision is sufficiently short, so we may assume that $\dot{\boldsymbol{I}}_{\mathrm{CM}} = \boldsymbol{0}$ for both bodies.

For body 2, the integrated balance laws of linear and angular momentum require

$$-\hat{\boldsymbol{P}} = m_2\left[\boldsymbol{v}_{\mathrm{CM}_2}(t_+) - \boldsymbol{v}_{\mathrm{CM}_2}(t_-)\right], \qquad \boldsymbol{r}_{\mathrm{CM}_2\mathrm{S}} \times (-\hat{\boldsymbol{P}}) = \boldsymbol{I}_{\mathrm{CM}_2}\left[\boldsymbol{\omega}_2(t_+) - \boldsymbol{\omega}_2(t_-)\right]. \tag{3.203}$$

For the 2D scenario shown above, the conservation of angular momentum for both bodies further reduces to

$$-d_{\mathrm{CM}_1\mathrm{S}}\hat{P} = I_{\mathrm{CM}_1}\left[\omega_1(t_+) - \omega_1(t_-)\right], \qquad d_{\mathrm{CM}_2\mathrm{S}}\hat{P} = I_{\mathrm{CM}_2}\left[\omega_2(t_+) - \omega_2(t_-)\right]. \tag{3.204}$$

In case of **smooth**, **frictionless** surfaces, the same assumption applies as for particles, viz. that forces are exchanged and hence momentum is transferred between the two bodies only in the direction normal to the plane of impact ($^{\mathrm{n}}$), so that $\hat{\boldsymbol{P}} = \hat{P}\boldsymbol{e}_n$ and the tangential components of $\boldsymbol{v}_{\mathrm{CM}_1}$ and $\boldsymbol{v}_{\mathrm{CM}_2}$ remain unchanged during the collision[11]:

$$\boxed{v^t_{\mathrm{CM}_1}(t_+) = v^t_{\mathrm{CM}_1}(t_-), \qquad v^t_{\mathrm{CM}_2}(t_+) = v^t_{\mathrm{CM}_2}(t_-)} \tag{3.205}$$

By contrast, in the normal direction we have

$$\hat{P} = m_1\left[v^n_{\mathrm{CM}_1}(t_+) - v^n_{\mathrm{CM}_1}(t_-)\right], \qquad -\hat{P} = m_2\left[v^n_{\mathrm{CM}_2}(t_+) - v^n_{\mathrm{CM}_2}(t_-)\right]. \tag{3.206}$$

Equations (3.204) and (3.206) provide four equations for five unknowns: $\omega_1(t_+)$, $\omega_2(t_+)$, $v^n_{\mathrm{CM}_1}(t_+)$, $v^n_{\mathrm{CM}_2}(t_+)$, and $\hat{P}$. The missing equation is given by the definition of the **coefficient of restitution**, which is expressed – analogously to the case of particles – in terms of the relative normal velocities of the contact point S (seen as S_1 and S_2 from body 1 and 2, respectively):

$$\boxed{e = -\frac{v^n_{\mathrm{S}_2}(t_+) - v^n_{\mathrm{S}_1}(t_+)}{v^n_{\mathrm{S}_2}(t_-) - v^n_{\mathrm{S}_1}(t_-)}} \tag{3.207}$$

This is the (negative) ratio of the *relative separation velocities* of the contact points immediatley after and before the collision (cf. (2.59) for particles). The involved velocities of the two contact points are related to the motion of the two bodies via the velocity transfer formula:

$$\boldsymbol{v}_{\mathrm{S}_1} = \boldsymbol{v}_{\mathrm{CM}_1} + \boldsymbol{\omega}_1 \times \boldsymbol{r}_{\mathrm{CM}_1\mathrm{S}}, \qquad \boldsymbol{v}_{\mathrm{S}_2} = \boldsymbol{v}_{\mathrm{CM}_2} + \boldsymbol{\omega}_2 \times \boldsymbol{r}_{\mathrm{CM}_2\mathrm{S}}, \tag{3.208}$$

which in our 2D scenario becomes (as drawn)

$$v^n_{\mathrm{S}_1}(t) = v^n_{\mathrm{CM}_1}(t) - d_{\mathrm{CM}_1\mathrm{S}}\omega_1(t), \qquad v^n_{\mathrm{S}_2}(t) = v^n_{\mathrm{CM}_2}(t) + d_{\mathrm{CM}_2\mathrm{S}}\omega_2(t). \tag{3.209}$$

[11] To avoid confusion with the other indices used here, we use superscripts $^{\mathrm{n}}$ and $^{\mathrm{t}}$ instead of subscripts for the normal and tangential directions, respectively.

Evaluating the above velocities at times t_- and t_+ altogether yields a system of equations that can be solved for the velocities and angular velocities of the two bodies after the collision.

If, by contrast, the surfaces are **rough** and hence **not frictionless** so no slipping occurs, then the two bodies stick together at point S during contact. As a consequence, there is a non-zero tangential component of $\hat{\boldsymbol{P}}$ and we can no longer assume that impulsive forces act only in the normal direction. In this case, we have the equations analogous to (3.206) in both the normal and tangential directions, viz.,

$$\begin{aligned} \hat{P}^n &= m_1 \left[v^n_{\text{CM}_1}(t_+) - v^n_{\text{CM}_1}(t_-) \right], \qquad -\hat{P}^n = m_2 \left[v^n_{\text{CM}_2}(t_+) - v^n_{\text{CM}_2}(t_-) \right], \\ \hat{P}^t &= m_1 \left[v^t_{\text{CM}_1}(t_+) - v^t_{\text{CM}_1}(t_-) \right], \qquad -\hat{P}^t = m_2 \left[v^t_{\text{CM}_2}(t_+) - v^t_{\text{CM}_2}(t_-) \right]. \end{aligned} \tag{3.210}$$

We need one further equation (since we also have the unknown $\hat{P}^t$-component), which is given by the **no-slip condition**

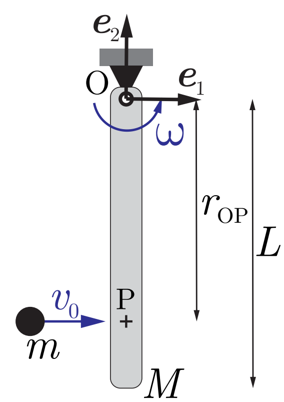
$$v^t_{\text{S}_1}(t_+) = v^t_{\text{S}_2}(t_+). \tag{3.211}$$

This again closes the system of equations.

We close by noting that the above scenario applies to general collisions between two rigid bodies, and the integrated balance equations of linear and angular momentum may vary based on the particular scenario of investigation. We here outlined the general approach to arrive at the required governing equations to solve a collision problem, while we do not present general solutions (as done for particles in Section 1.2.4), since those are specific to a given problem; see, e.g., Examples 3.18 and 3.19 below.

Example 3.18. Hitting a hanging pendulum with a ball

We consider a hanging pendulum of mass M and length L, which has the shape of a long, slender rod and is initially at rest. The pendulum is hit by a ball (a particle of mass m) with a horizontal velocity v_0 immediately before the collision. As sketched on the right, the ball hits the pendulum at point P at a distance r_{OP} from the hinge O. Let us assume a *frictionless* collision with a known coefficient of restitution e and neglect gravity. What are the velocity v_+ of the ball and the angular velocity ω_+ of the pendulum immediately after the collision?



Since there is no external torque acting on the system with respect to O, angular momentum with respect to O is conserved during the collision. Equating the angular momentum before and after the collision with respect to O gives

$$\boldsymbol{r}_{\text{m}}(t_-) \times \boldsymbol{P}_{\text{m}}(t_-) + \boldsymbol{I}_{\text{O}}\boldsymbol{\omega}(t_-) = \boldsymbol{r}_{\text{m}}(t_+) \times \boldsymbol{P}_{\text{m}}(t_+) + \boldsymbol{I}_{\text{O}}\boldsymbol{\omega}(t_+) \tag{3.212}$$

where $\boldsymbol{r}_{\text{m}}$ and $\boldsymbol{P}_{\text{m}}$ denote, respectively, the position and (linear) momentum of the ball, $\boldsymbol{\omega}$ is the angular velocity of the pendulum, whose moment of inertia tensor with respect to point O is denoted by $\boldsymbol{I}_{\text{O}}$. We furthermore know that

$$\boldsymbol{r}_{\text{m}}(t_-) = \boldsymbol{r}_{\text{m}}(t_+) = -r_{\text{OP}}\boldsymbol{e}_2, \qquad \boldsymbol{P}_{\text{m}}(t_-) = mv_0\boldsymbol{e}_1, \qquad \boldsymbol{P}_{\text{m}}(t_+) = mv_+\boldsymbol{e}_1 \tag{3.213}$$

and

$$\boldsymbol{\omega}(t_-) = \mathbf{0}, \qquad \boldsymbol{\omega}(t_+) = \omega_+ \boldsymbol{e}_3. \tag{3.214}$$

Inserting these relations as well as the moment of inertia tensor for a slender rod (e.g. from Example 3.7 with $R \ll L$) transforms (3.212) into

$$-r_{\mathrm{OP}} \boldsymbol{e}_2 \times m v_0 \boldsymbol{e}_1 = -r_{\mathrm{OP}} \boldsymbol{e}_2 \times m v_+ \boldsymbol{e}_1 + \boldsymbol{I}_{\mathrm{O}} \omega_+ \boldsymbol{e}_3 \qquad \Rightarrow \qquad m r_{\mathrm{OP}} v_0 = m r_{\mathrm{OP}} v_+ + \frac{ML^2}{3} \omega_+ \tag{3.215}$$

As we need a second equation to determine v_+ and ω_+, we introduce the definition of the coefficient of restitution for a frictionless collision from (3.207). Here, we take contact point S_2 as point P on the pendulum, while S_1 is the opposite contact point on the ball. This yields

$$e = -\frac{r_{\mathrm{OP}} \omega_+ - v_+}{0 - v_0} \qquad \Rightarrow \qquad v_+ = r_{\mathrm{OP}} \omega_+ - e v_0. \tag{3.216}$$

Insertion into (3.215) gives

$$m r_{\mathrm{OP}} v_0 = m r_{\mathrm{OP}} \left(r_{\mathrm{OP}} \omega_+ - e v_0 \right) + \frac{ML^2}{3} \omega_+ \qquad \Leftrightarrow \qquad r_{\mathrm{OP}} (1+e) v_0 = \left(r_{\mathrm{OP}}^2 + \frac{ML^2}{3m} \right) \omega_+ \tag{3.217}$$

and after some rearrangement

$$\omega_+ = \frac{1+e}{1 + \frac{M}{3m} \left(\frac{L}{r_{\mathrm{OP}}} \right)^2} \frac{v_0}{r_{\mathrm{OP}}} \qquad \Rightarrow \qquad v_+ = r_{\mathrm{OP}} \omega_+ - e v_0 = \frac{1 - e \frac{M}{3m} \left(\frac{L}{r_{\mathrm{OP}}} \right)^2}{1 + \frac{M}{3m} \left(\frac{L}{r_{\mathrm{OP}}} \right)^2} v_0. \tag{3.218}$$

Note that, if $e = 0$ (fully plastic collision, cf. Section 1.2.4), then $v_+ = r_{\mathrm{OP}} \omega_+$, so the ball continues after the collision at the same speed as the contact point P on the pendulum. As a further sanity check, we observe that ω_+ is always positive for $v_0 > 0$ (the pendulum rotates to the right after being hit), whereas the sign of v_+ depends on the ratios M/m and L/r_{OP} as well as on e.

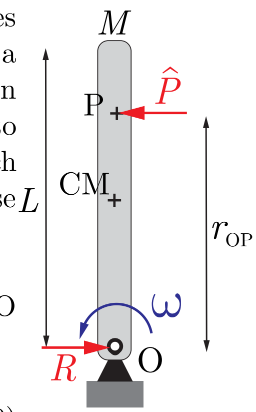
For example, for an elastic collision ($e = 1$) and a hit at $r_{\mathrm{OP}} = 2L/3$, we obtain

$$\omega_+ = \frac{3}{1 + \frac{3M}{4m}} \frac{v_0}{L} \qquad \text{and} \qquad v_+ = \frac{1 - \frac{3M}{4m}}{1 + \frac{3M}{4m}} v_0. \tag{3.219}$$

If the pendulum is heavy ($M > \frac{4}{3}m$), then the ball bounces back ($v_+ < 0$). If the pendulum is light ($M < \frac{4}{3}m$), the ball continues in the same direction at slower speed ($0 < v_+ < v_0$). In the special case $M = \frac{4}{3}m$ the ball transfers its complete kinetic energy onto the pendulum and remains at rest after the collision. Finally, in case of a very heavy pendulum ($M \to \infty$), the pendulum remains at rest ($\omega_+ \to 0$), while the ball bounces back with $v_+ = -v_0$, as expected from an elastic collision with a rigid wall.

We close by noting that the analogous solution can be found when the ball is initially at rest and hit by a pendulum of an initial angular velocity.

Example 3.19. Sweet spot of a baseball bat (or door stopper)



The above example of hitting a hanging pendulum is found in practice, e.g., when a baseball bat is hit by a baseball at point P. A similar problem arises when a swinging door hits a door stopper at point P. Let us reconsider a rod of mass M and length L, which is hinged at its end point O, as shown on the right, and hit at a point P by an impulsive force $\hat{P}$ perpendicular to the axis of the rod. As it turns out, there exists a "*sweep spot*" P for which point O experiences no reaction force (of interest for baseball players and those installing door stoppers alike). Let us find this sweet spot.

We begin by formulating the angular momentum balance in 2D about point O *integrated over the time of the collision*. This leads to

$$I_{\mathrm{O}}\left[\omega(t_+)-\omega(t_-)\right]=r_{\mathrm{OP}}\hat{P} \qquad \text{with} \qquad I_{\mathrm{O}}=\frac{1}{3}ML^2, \tag{3.220}$$

where $\omega(t_+)$ and $\omega(t_-)$ are the rod's angular velocities immediately before and after the impact, respectively. Next, we could also integrate linear momentum balance, however this would involve the reaction force in the hinge. Instead, we integrate the balance of angular momentum over the collision time again, now with respect to the center of mass of the rod. This yields the second equation

$$I_{\mathrm{CM}}\left[\omega(t_+)-\omega(t_-)\right]=r_{\mathrm{CM\,P}}\hat{P}+r_{\mathrm{CM\,O}}\hat{R} \qquad \text{with} \qquad I_{\mathrm{CM}}=\frac{1}{12}ML^2, \tag{3.221}$$

where R denotes the unknown horizontal reaction force at the hinge. Note that no vertical reaction force arises since the body changes the velocity of its center of mass only in the horizontal direction during the collision and no other vertical forces act.

By writing $r_{\mathrm{CM\,P}}=r_{\mathrm{OP}}-L/2$ and $r_{\mathrm{CM\,O}}=L/2$ by geometry, we arrive at the two equations

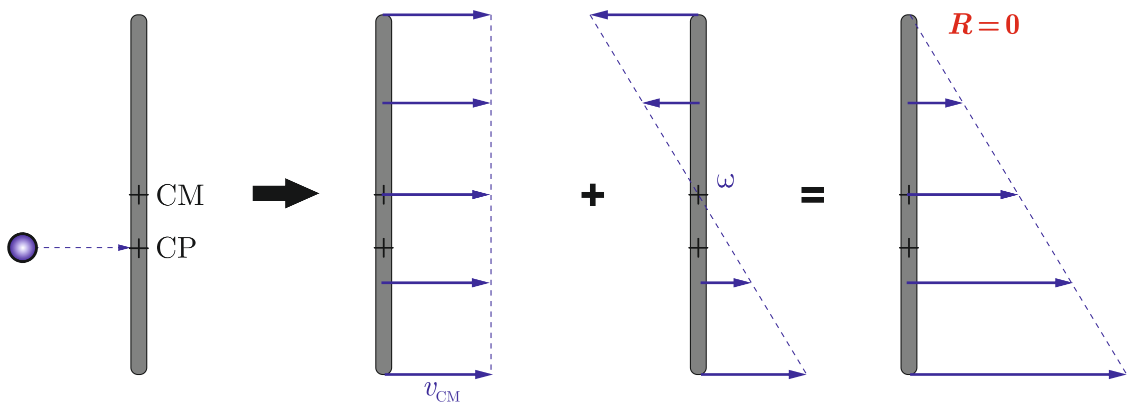
$$\frac{ML^2}{3}\left[\omega(t_+)-\omega(t_-)\right]=r_{\mathrm{OP}}\hat{P}, \qquad \frac{ML^2}{12}\left[\omega(t_+)-\omega(t_-)\right]=(r_{\mathrm{OP}}-L/2)\hat{P}+L\hat{R}/2. \tag{3.222}$$

Now, let us consider the special case in which the reaction force is assumed to vanish (i.e., point P is the sweet spot such that $\hat{R}=0$). Setting $\hat{R}=0$ and dividing the two equations in (3.222) by each other (the first by the second) gives

$$4=\frac{r_{\mathrm{OP}}}{r_{\mathrm{OP}}-L/2} \qquad \Leftrightarrow \qquad r_{\mathrm{OP}}=\frac{2}{3}L, \tag{3.223}$$

i.e., the sweep spot is at exactly 2/3 of the length of the rod. This is the so-called center of percussion. The **center of percussion** of a rigid body attached to a pivot/hinge is that point for which a perpendicular impact will produce *no reactive forces at the pivot/hinge* (here at point O). Translational and rotational motions cancel at the hinge when an impulsive force is applied at the center of percussion. This sweet spot is of importance in practice as it minimizes, e.g., the force acting on the wrist of a baseball player or the force in the hinge of a door.

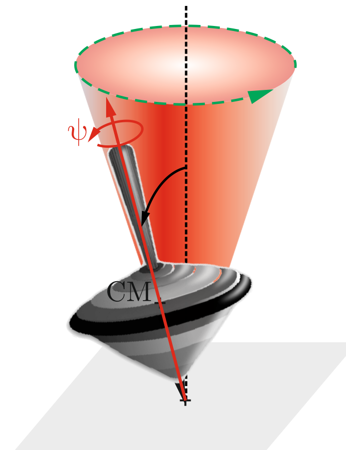
To understand the concept of the center of percussion, consider a long rod hit by an impulsive force (e.g., due to a collision with an approach ball), as shown below. The collision changes both the linear and the angular momentum of the rod: because of the impulsive force acting on the rod in the horizontal direction, the center of mass (CM) of the rod will start moving to the right in a purely horizontal fashion. If the impulse occurs exactly at CM, then the rod will move without rotation. By contrast, if the hit occurs above or below CM, the balance of angular momentum (or conservation of angular momentum for the whole system) will lead to a rotation of the rod. If the hit happens above the CM, then the rod rotates clockwise; if the rod is hit below its CM, then it rotates counter-clockwise. The superposition of translation plus rotation defines the overall motion of the rod.



Now, consider the top point of the rod to be hinged. If this top point due to the collision is supposed to move to the right or to the left, then there must be a force in the hinge to suppress that motion – hence, there is a reaction in the hinge. Only if the velocity of the top point is zero throughout the collision (as shown in the special case in the above schematic), then there is no reaction force ($\boldsymbol{R} = \mathbf{0}$). This requires the hit to occur at the special point which leads to rotations and translations to cancel in the top point of the rod; and this defines the **center of percussion**.

3.4 Non-Inertial Frames

All of the kinetic balance laws derived so far (which trace back to Newton's famous three axioms) apply only in an **inertial reference frame**. As soon as we switch our description to a moving, **non-inertial frame**, those kinetic balance laws no longer apply for an observer in the moving frame. Therefore, we proceed to establish general relations between the kinematics and kinetics of inertial and non-inertial frames for rigid bodies in 3D – and we will argue that those equally apply to (systems of) particles. Finally, we also need to take into account that many of our assumptions made so far cease to hold in non-inertial frames. For example, as soon as the axis of rotation of a body is no longer fixed (such as in a moving frame attached to a rotating **spinning top**, shown on the right), we cannot assume $\dot{\boldsymbol{I}} = \boldsymbol{0}$, so we need to revisit our relations derived, e.g., for angular momentum balance.



We generally need to exercise caution when formulating component equations (rather than symbolic vector equations), because $\dot{\boldsymbol{e}}_i \neq \boldsymbol{0}$ in a moving frame in general. As an added complication, we sometimes have to deal with several different reference frames in the following, so we here introduce a framework that is sufficiently general to work with arbitrarily many (inertial or non-inertial) reference frames.

3.4.1 Active and passive rotations

We begin by considering two reference frames (and extend the concept to more frames as needed):

- an *inertial frame* $\mathcal{C}$ with a fixed Cartesian basis $\{\boldsymbol{e}_1^{\mathcal{C}}, \boldsymbol{e}_2^{\mathcal{C}}, \boldsymbol{e}_3^{\mathcal{C}}\}$ and origin O,
- a *non-inertial frame* $\mathcal{M}$ with a rotating basis $\{\boldsymbol{e}_1^{\mathcal{M}}(t), \boldsymbol{e}_2^{\mathcal{M}}(t), \boldsymbol{e}_3^{\mathcal{M}}(t)\}$ and with the same fixed origin O (we will later discuss moving origins as well).

We assume that both bases are *orthonormal*, which implies that $\{\boldsymbol{e}_1^{\mathcal{M}}(t), \boldsymbol{e}_2^{\mathcal{M}}(t), \boldsymbol{e}_3^{\mathcal{M}}(t)\}$ must be related to $\{\boldsymbol{e}_1^{\mathcal{C}}, \boldsymbol{e}_2^{\mathcal{C}}, \boldsymbol{e}_3^{\mathcal{C}}\}$ by a rotation (generally in 3D).

Here and in the following, we use superscripts $^{\mathcal{C}}$ and $^{\mathcal{M}}$ to denote vectors associate with frames $\mathcal{C}$ and $\mathcal{M}$, respectively (and we can easily define more frames in this fashion). Since in most cases we use only a single inertial frame, we may also *drop the superscripts* $\mathcal{C}$ when referring to a unique inertial frame and simply write $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ for the basis of the fixed Cartesian frame. As before, we refer to the *components of a vector* $\boldsymbol{v}$ in the $\mathcal{M}$-frame as $[\boldsymbol{v}]_{\mathcal{M}}$, and the ith component as $[v_i]_{\mathcal{M}}$.

By projecting the basis vectors of the $\mathcal{M}$-frame onto the Cartesian basis vectors of the $\mathcal{C}$-frame, we establish a relation between the components of a vector $\boldsymbol{v}$ in the two frames (see the schematic above). Specifically, we start with a basis vector $\boldsymbol{e}_j^{\mathcal{M}}$ and its representation

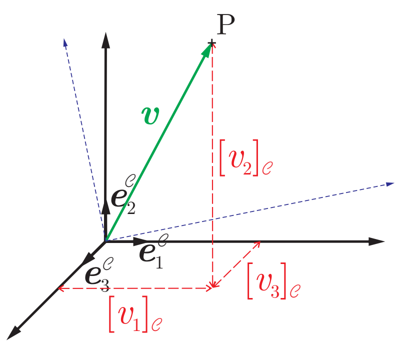
$$\boldsymbol{e}_j^{\mathcal{M}} = \sum_{i=1}^{3} \left(\boldsymbol{e}_j^{\mathcal{M}} \cdot \boldsymbol{e}_i^{\mathcal{C}}\right) \boldsymbol{e}_i^{\mathcal{C}}, \tag{3.224}$$

which allows us to express a vector $\boldsymbol{v}$ in both bases as

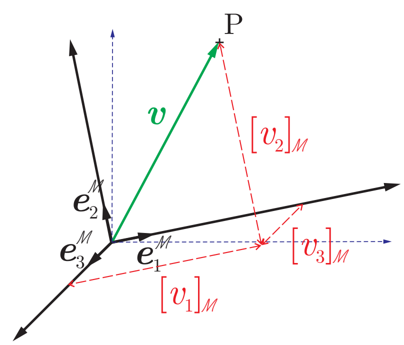
$$\boldsymbol{v} = \sum_{i=1}^{3} [v_i]_{\mathcal{C}} \boldsymbol{e}_i^{\mathcal{C}} \tag{3.225}$$

and, equivalently,

$$\boldsymbol{v} = \sum_{j=1}^{3} [v_j]_{\mathcal{M}} \boldsymbol{e}_j^{\mathcal{M}} = \sum_{j=1}^{3} [v_j]_{\mathcal{M}} \sum_{i=1}^{3} \left(\boldsymbol{e}_j^{\mathcal{M}} \cdot \boldsymbol{e}_i^{\mathcal{C}} \right) \boldsymbol{e}_i^{\mathcal{C}} = \sum_{i=1}^{3} \underbrace{\left[\sum_{j=1}^{3} \boldsymbol{e}_j^{\mathcal{M}} \cdot \boldsymbol{e}_i^{\mathcal{C}} \, [v_j]_{\mathcal{M}} \right]}_{\overset{!}{=}[v_i]_{\mathcal{C}}} \boldsymbol{e}_i^{\mathcal{C}}. \tag{3.226}$$



components of $\boldsymbol{v}$ in the $\mathscr{C}$-frame



components of $\boldsymbol{v}$ in the $\mathscr{M}$-frame

By comparison, the components of vector $\boldsymbol{v}$ in the two frames $\mathcal{C}$ and $\mathcal{M}$ are related via

$$[v_i]_{\mathcal{C}} = \sum_{j=1}^{3} [T_{ij}^{\mathcal{CM}}][v_j]_{\mathcal{M}} \qquad \text{with} \qquad [T_{ij}^{\mathcal{CM}}] = \boldsymbol{e}_i^{\mathcal{C}} \cdot \boldsymbol{e}_j^{\mathcal{M}}. \tag{3.227}$$

$[T_{ij}^{\mathcal{CM}}]$ are the components of a **rotation matrix** that converts vector components in the $\mathcal{M}$-basis into their corresponding components in the $\mathcal{C}$-basis; therefore, the superscripts are to be read as $\boldsymbol{T}^{\mathcal{CM}} = \boldsymbol{T}^{\mathcal{C}\leftarrow\mathcal{M}}$, i.e., from $\mathcal{M}$ to $\mathcal{C}$ (but we usually omit the arrow for better visibility and conciseness).

Note that, if $\mathcal{C}$ is the Cartesian basis, then the ith column of the matrix $\boldsymbol{T}^{\mathcal{CM}}$ contains simply the three vector components of $\boldsymbol{e}_j^{\mathcal{M}}$ $(i = 1, 2, 3)$ in that Cartesian basis.

The inverse relation can be found analogously, starting with

$$\boldsymbol{e}_j^{\mathcal{C}} = \sum_{i=1}^{3} \left(\boldsymbol{e}_j^{\mathcal{C}} \cdot \boldsymbol{e}_i^{\mathcal{M}} \right) \boldsymbol{e}_i^{\mathcal{M}} \quad \Rightarrow \quad \boldsymbol{v} = \sum_{j=1}^{3} [v_j]_{\mathcal{C}} \boldsymbol{e}_j^{\mathcal{C}} = \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\boldsymbol{e}_j^{\mathcal{C}} \cdot \boldsymbol{e}_i^{\mathcal{M}} \right) [v_j]_{\mathcal{C}} \boldsymbol{e}_i^{\mathcal{M}} \overset{!}{=} \sum_{i=1}^{3} [v_i]_{\mathcal{M}} \boldsymbol{e}_i^{\mathcal{M}}, \tag{3.228}$$

so that

$$[v_i]_{\mathcal{M}} = \sum_{j=1}^{3} [T_{ij}^{\mathcal{MC}}][v_j]_{\mathcal{C}} \qquad \text{with} \qquad [T_{ij}^{\mathcal{MC}}] = \boldsymbol{e}_i^{\mathcal{M}} \cdot \boldsymbol{e}_j^{\mathcal{C}}. \tag{3.229}$$

Analogous to the prior observation, we conclude that, if $\mathcal{C}$ is the Cartesian basis, then the ith row of the rotation matrix $\boldsymbol{T}^{\mathcal{MC}}$ contains the three vector components of $\boldsymbol{e}_j^{\mathcal{M}}$ ($i = 1, 2, 3$) in that Cartesian basis.

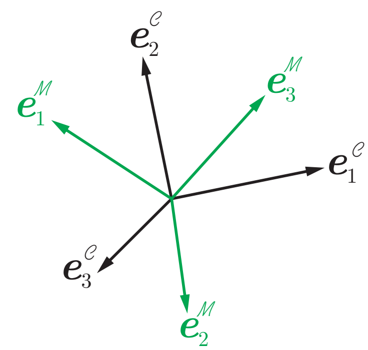
Comparing their definitions shows that the two rotation matrices are each other's transpose:

$$[T_{ij}^{\mathcal{MC}}] = \boldsymbol{e}_i^{\mathcal{M}} \cdot \boldsymbol{e}_j^{\mathcal{C}} = \left(\boldsymbol{e}_j^{\mathcal{M}} \cdot \boldsymbol{e}_i^{\mathcal{C}}\right)^{\mathrm{T}} = [T_{ij}^{\mathcal{CM}}]^{\mathrm{T}} \qquad \Rightarrow \qquad \boxed{\boldsymbol{T}^{\mathcal{MC}} = \left(\boldsymbol{T}^{\mathcal{CM}}\right)^{\mathrm{T}}} \tag{3.230}$$

The thus defined rotations are often referred to as **passive rotations**, as they do not actively move anything but are merely instantaneous relations between vector components in two different frames of reference (vector $\boldsymbol{v}$ does not move, we only express its components in different bases).

As a next step, let us introduce **active rotations** that take a vector $\boldsymbol{v}$ and rotate it onto another vector $\boldsymbol{u} = \boldsymbol{R}\boldsymbol{v}$. Note that this operation is independent of a basis or frame of reference, so $\boldsymbol{R}$ is a *second-order tensor*[12] defined by its components R_{ij} along with a particular basis.

For example, we may look for the rotation $\boldsymbol{R}^{\mathcal{MC}} = \boldsymbol{R}^{\mathcal{M}\leftarrow\mathcal{C}}$ that takes the basis vectors $\{\boldsymbol{e}_1^{\mathcal{C}}, \boldsymbol{e}_3^{\mathcal{C}}, \boldsymbol{e}_3^{\mathcal{C}}\}$ and rotates them onto the basis vectors $\{\boldsymbol{e}_1^{\mathcal{M}}, \boldsymbol{e}_3^{\mathcal{M}}, \boldsymbol{e}_3^{\mathcal{M}}\}$. This implies that



$$\boldsymbol{e}_1^{\mathcal{M}} = \boldsymbol{R}^{\mathcal{MC}}\boldsymbol{e}_1^{\mathcal{C}}, \qquad \boldsymbol{e}_2^{\mathcal{M}} = \boldsymbol{R}^{\mathcal{MC}}\boldsymbol{e}_2^{\mathcal{C}}, \qquad \boldsymbol{e}_3^{\mathcal{M}} = \boldsymbol{R}^{\mathcal{MC}}\boldsymbol{e}_3^{\mathcal{C}}, \tag{3.231}$$

which are nine equations for the nine unknown components of $\boldsymbol{R}^{\mathcal{MC}}$ (still assuming that both bases are orthonormal, so a unique solution exists).

To compute the components of $\boldsymbol{R}^{\mathcal{MC}}$ within the basis $\mathcal{C}$, we may exploit the relation

$$[R_{ij}^{\mathcal{MC}}]_{\mathcal{C}} = \boldsymbol{e}_i^{\mathcal{C}} \cdot \boldsymbol{R}^{\mathcal{MC}}\boldsymbol{e}_j^{\mathcal{C}} = \boldsymbol{e}_i^{\mathcal{C}} \cdot \boldsymbol{e}_j^{\mathcal{M}} \qquad \Rightarrow \qquad [R_{ij}^{\mathcal{MC}}]_{\mathcal{C}} = \boldsymbol{e}_i^{\mathcal{C}} \cdot \boldsymbol{e}_j^{\mathcal{M}}. \tag{3.232}$$

Hence, we observe that the components of this tensor $\boldsymbol{R}^{\mathcal{MC}}$ are identical to those of the transformation matrix $\boldsymbol{T}^{\mathcal{CM}}$ introduced above (which makes sense: we either rotate the vector within the a given basis, or we look at the same vector within two bases – these two rotations are opposite).

Analogously, we can also find the components of $\boldsymbol{R}^{\mathcal{CM}}$ in the $\mathcal{C}$-frame: the relations

$$\boldsymbol{e}_1^{\mathcal{C}} = \boldsymbol{R}^{\mathcal{CM}}\boldsymbol{e}_1^{\mathcal{M}}, \qquad \boldsymbol{e}_2^{\mathcal{C}} = \boldsymbol{R}^{\mathcal{CM}}\boldsymbol{e}_2^{\mathcal{M}}, \qquad \boldsymbol{e}_3^{\mathcal{C}} = \boldsymbol{R}^{\mathcal{CM}}\boldsymbol{e}_3^{\mathcal{M}} \tag{3.233}$$

lead to

$$[R_{ij}^{\mathcal{CM}}]_{\mathcal{C}} = \boldsymbol{e}_i^{\mathcal{C}} \cdot \boldsymbol{R}^{\mathcal{CM}}\boldsymbol{e}_j^{\mathcal{C}} = \boldsymbol{e}_j^{\mathcal{C}} \cdot \left(\boldsymbol{R}^{\mathcal{CM}}\right)^{\mathrm{T}}\boldsymbol{e}_i^{\mathcal{C}}. \tag{3.234}$$

From the discussion of the rotation matrices above, we may guess that the components $R_{ij}^{\mathcal{CM}}$ should be the transpose of $R_{ij}^{\mathcal{MC}}$. Inserting this relation here results in

$$[R_{ij}^{\mathcal{CM}}]_{\mathcal{C}} = \boldsymbol{e}_j^{\mathcal{C}} \cdot \left(\boldsymbol{R}^{\mathcal{CM}}\right)^{\mathrm{T}}\boldsymbol{e}_i^{\mathcal{C}} = \boldsymbol{e}_j^{\mathcal{C}} \cdot \boldsymbol{R}^{\mathcal{MC}}\boldsymbol{e}_i^{\mathcal{C}} = \boldsymbol{e}_j^{\mathcal{C}} \cdot \boldsymbol{e}_i^{\mathcal{M}} \qquad \Rightarrow \qquad [R_{ij}^{\mathcal{CM}}]_{\mathcal{C}} = [R_{ji}^{\mathcal{MC}}]_{\mathcal{C}}, \tag{3.235}$$

[12] Second-order tensors, like vectors, are defined by components and a basis. While vectors (which in fact are first-order tensors) are written as $\boldsymbol{v} = \sum_i v_i \boldsymbol{e}_i$, second-order tensors are defined as $\boldsymbol{R} = \sum_{i,j} R_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ with the dyadic/tensor product $\boldsymbol{a}\otimes\boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^{\mathrm{T}} = \sum_{i,j} a_i b_j \boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Tensors act on vectors as a linear mapping, meaning that $\boldsymbol{R}\boldsymbol{v}$ is a vector $\boldsymbol{u}$ and their components are related by $u_i = \sum_j R_{ij} v_j$ within a given basis; i.e., $\boldsymbol{u} = \sum_i u_i \boldsymbol{e}_i = \sum_{i,j} R_{ij} v_j \boldsymbol{e}_i$. The components of a tensor $\boldsymbol{T}$ within a basis $\mathcal{C}$ are identified as $T_{ij} = \boldsymbol{e}_i^{\mathcal{C}} \cdot \boldsymbol{T}\boldsymbol{e}_j^{\mathcal{C}}$.

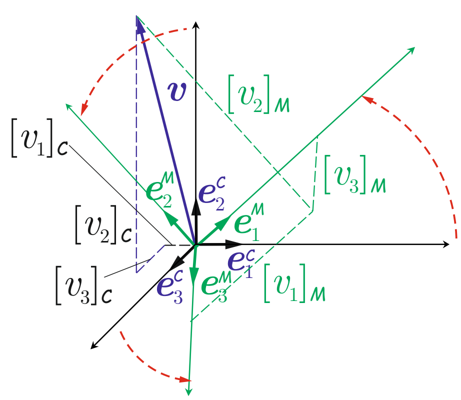
which indeed confirms that $\boldsymbol{R}^{\mathcal{CM}} = (\boldsymbol{R}^{\mathcal{MC}})^{\mathrm{T}}$. In summary, we have the following coordinate transformation rules for **passive rotations** (replacing $\boldsymbol{T}$ by the respective $\boldsymbol{R}^{\mathrm{T}}$ components):

$$\boxed{[v_i]_{\mathcal{C}} = \sum_{j=1}^{3}[R_{ij}^{\mathcal{CM}}]^{\mathrm{T}}[v_j]_{\mathcal{M}}, \quad [v_i]_{\mathcal{M}} = \sum_{j=1}^{3}[R_{ij}^{\mathcal{MC}}]^{\mathrm{T}}[v_j]_{\mathcal{C}} \qquad \text{with} \qquad [R_{ij}^{\mathcal{MC}}] = [R_{ji}^{\mathcal{CM}}] = \boldsymbol{e}_i^{\mathcal{C}} \cdot \boldsymbol{e}_j^{\mathcal{M}}} \tag{3.236}$$

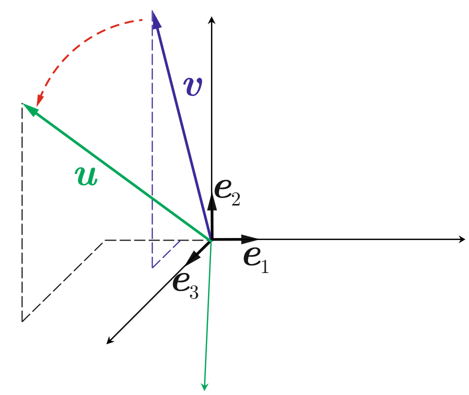
while for **active rotations** we have

$$\boxed{\boldsymbol{e}_i^{\mathcal{M}} = \boldsymbol{R}^{\mathcal{MC}}\boldsymbol{e}_i^{\mathcal{C}}, \qquad \boldsymbol{e}_i^{\mathcal{C}} = \boldsymbol{R}^{\mathcal{CM}}\boldsymbol{e}_i^{\mathcal{M}} \qquad \text{with} \qquad \boldsymbol{R}^{\mathcal{MC}} = \left(\boldsymbol{R}^{\mathcal{CM}}\right)^{\mathrm{T}}} \tag{3.237}$$

where the same tensor components $[R_{ij}^{\mathcal{MC}}]_{\mathcal{C}} = [R_{ji}^{\mathcal{CM}}]_{\mathcal{C}}$ are the same as defined in (3.236), i.e., $[R_{ij}^{\mathcal{MC}}]_{\mathcal{C}} = [R_{ji}^{\mathcal{CM}}]_{\mathcal{C}} = \boldsymbol{e}_i^{\mathcal{C}} \cdot \boldsymbol{e}_j^{\mathcal{M}}$.



passive rotation: same vector $\boldsymbol{v}$ measured in two frames



active rotation: vector $\boldsymbol{v}$ is rotated onto vector $\boldsymbol{u}$

Note that two subsequent opposite (active) rotations applied to a vector $\boldsymbol{v}$ must result in the identity, i.e.,

$$\boldsymbol{v} \stackrel{!}{=} \boldsymbol{R}^{\mathcal{CM}}(\boldsymbol{R}^{\mathcal{MC}}\boldsymbol{v}) = (\boldsymbol{R}^{\mathcal{CM}}\boldsymbol{R}^{\mathcal{MC}})\boldsymbol{v} \qquad \Rightarrow \qquad \boldsymbol{R}^{\mathcal{CM}}\boldsymbol{R}^{\mathcal{MC}} = \boldsymbol{R}^{\mathcal{CM}}\left(\boldsymbol{R}^{\mathcal{CM}}\right)^{\mathrm{T}} = \boldsymbol{I}. \tag{3.238}$$

This shows that, in fact, any second-order tensor $\boldsymbol{R}$ that describes a rotation must satisfy

$$\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{I} \quad \Rightarrow \quad \boldsymbol{R}^{-1} = \boldsymbol{R}^{\mathrm{T}} \qquad \text{and} \qquad \det \boldsymbol{R} = 1. \tag{3.239}$$

All such rotation tensors form a group, the so-called **special orthogonal group** SO(d) in d dimensions. (Without proof, we mention that the condition $\det \boldsymbol{R} = 1$ ensures that $\boldsymbol{R}$ is a *rotation* as opposed to a *reflection* which is characterized by $\det \boldsymbol{R} = -1$.).

As an **example**, let us identify the three **elementary rotations** about each coordinate axis $\boldsymbol{e}_i^{\mathcal{C}}$. For example, a rotation about the $\boldsymbol{e}_3^{\mathcal{C}}$-axis by φ_3 defines the frame $\mathcal{M}_3$ as

$$\boldsymbol{e}_1^{\mathcal{M}_3} = \cos\varphi_3\,\boldsymbol{e}_1^{\mathcal{C}} + \sin\varphi_3\,\boldsymbol{e}_2^{\mathcal{C}}, \qquad \boldsymbol{e}_2^{\mathcal{M}_3} = -\sin\varphi_3\,\boldsymbol{e}_1^{\mathcal{C}} + \cos\varphi_3\,\boldsymbol{e}_2^{\mathcal{C}}, \qquad \boldsymbol{e}_3^{\mathcal{M}_3} = \boldsymbol{e}_3^{\mathcal{C}}, \tag{3.240}$$

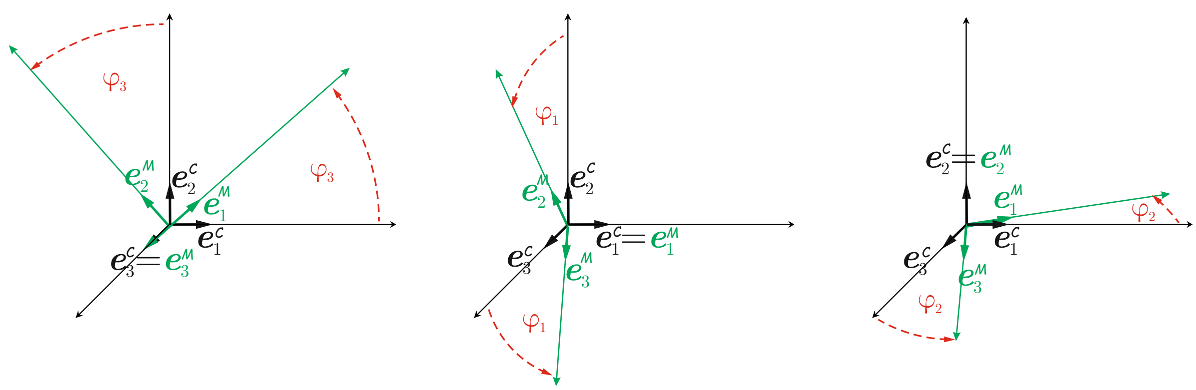
which leads to the components of the rotation tensor as

$$[\boldsymbol{R}^{\mathcal{M}_3\mathcal{C}}]_\mathcal{C} = \begin{pmatrix} \cos\varphi_3 & -\sin\varphi_3 & 0 \\ \sin\varphi_3 & \cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.241}$$

so we see that the ith column of $\boldsymbol{R}^{\mathcal{M}_3\mathcal{C}}$ contains the components of vector $\boldsymbol{e}_i^{\mathcal{M}_3}$. Analogously, the rotations about the $\boldsymbol{e}_1^\mathcal{C}$- and $\boldsymbol{e}_2^\mathcal{C}$-axes follow as

$$[\boldsymbol{R}^{\mathcal{M}_1\mathcal{C}}]_\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_1 & -\sin\varphi_1 \\ 0 & \sin\varphi_1 & \cos\varphi_1 \end{pmatrix}, \qquad [\boldsymbol{R}^{\mathcal{M}_2\mathcal{C}}]_\mathcal{C} = \begin{pmatrix} \cos\varphi_2 & 0 & \sin\varphi_2 \\ 0 & 1 & 0 \\ -\sin\varphi_2 & 0 & \cos\varphi_2 \end{pmatrix}. \tag{3.242}$$

Note the flipped sign in $\boldsymbol{R}^{\mathcal{M}_2\mathcal{C}}$, which is due to the definition of a positive rotation angle when rotating about the $\boldsymbol{e}_3^\mathcal{C}$-axis. The three rotations are shown below.



We can easily verify that inverting any of the above rotations (i.e., replacing φ_i by $-\varphi_i$) immediately confirms that $\boldsymbol{R}^{-1} = \boldsymbol{R}^\mathrm{T}$ for each of the above tensors, since $\cos(-\varphi) = \cos\varphi$ while $\sin(-\varphi) = -\sin\varphi$.

Any general rotation in 3D can be defined as a *subsequent composition* of the above three elementary rotations, e.g., if a body is first rotated by φ_1 about the $\boldsymbol{e}_1^\mathcal{C}$-axis (into the $\mathcal{M}_1$-frame), then by φ_2 about the $\boldsymbol{e}_2^\mathcal{C}$-axis (into the $\mathcal{M}_2$-frame), and finally by φ_3 about the $\boldsymbol{e}_3^\mathcal{C}$-axis (into the $\mathcal{M}_3$-frame), then the complete rotation is characterized by

$$\boldsymbol{R}^{\mathcal{M}_3\mathcal{C}} = \boldsymbol{R}^{\mathcal{M}_3\mathcal{M}_2}\boldsymbol{R}^{\mathcal{M}_2\mathcal{M}_1}\boldsymbol{R}^{\mathcal{M}_1\mathcal{C}} \tag{3.243}$$

with components

$$[\boldsymbol{R}^{\mathcal{M}_3\mathcal{C}}]_\mathcal{C} = \begin{pmatrix} \cos\varphi_3 & -\sin\varphi_3 & 0 \\ \sin\varphi_3 & \cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi_2 & 0 & \sin\varphi_2 \\ 0 & 1 & 0 \\ -\sin\varphi_2 & 0 & \cos\varphi_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_1 & -\sin\varphi_1 \\ 0 & \sin\varphi_1 & \cos\varphi_1 \end{pmatrix}. \tag{3.244}$$

Note that the order of the rotation tensors is reversed when compared to the order of applying the rotations, because the action of a tensor $\boldsymbol{R}_1$ on a vector $\boldsymbol{v}$ defined as $\boldsymbol{R}_1\boldsymbol{v}$, so a subsequent application of $\boldsymbol{R}_2$ yields $\boldsymbol{R}_2(\boldsymbol{R}_1\boldsymbol{v}) = (\boldsymbol{R}_2\boldsymbol{R}_1)\boldsymbol{v}$.

We note without proof here that a rotation about an axis with unit vector $\boldsymbol{e}$ by an angle φ is generally described by a rotation tensor $\boldsymbol{R}$ whose Cartesian components are given by

$$[R_{ij}]_{\mathcal{C}} = e_i e_j + (\delta_{ij} - e_i e_j)\cos\varphi - \epsilon_{ijk} e_k \sin\varphi \qquad \text{for} \qquad i,j = 1,\dots,3, \tag{3.245}$$

where ϵ_{ijk} denotes the permutation symbol, and δ_{ik} is Kronecker's delta. This reduces to the above elementary rotations when choosing $\boldsymbol{e} = \boldsymbol{e}_i^{\mathcal{C}}$ and $\varphi = \varphi_i$ for each of the cases $i = 1,2,3$.

Example 3.20. Passive and active rotations

As an example of passive and active rotations let us consider two frames of reference defined as follows:

- **frame** $\mathcal{C}$ has a fixed, Cartesian basis $\{\boldsymbol{e}_1^{\mathcal{C}}, \boldsymbol{e}_2^{\mathcal{C}}, \boldsymbol{e}_3^{\mathcal{C}}\} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and origin O,
- **frame** $\mathcal{M}$ is obtained from $\mathcal{C}$ by first rotating the basis vectors by $\pi/4$ about the $\boldsymbol{e}_3^{\mathcal{C}}$-axis, followed by a rotation by $-\pi/2$ about the $\boldsymbol{e}_2^{\mathcal{C}}$-axis, with the same origin O.

The rotation from $\mathcal{C}$ to $\mathcal{M}$ is hence described by

$$[\boldsymbol{R}^{\mathcal{MC}}]_{\mathcal{C}} = \begin{pmatrix} \cos\frac{-\pi}{2} & 0 & \sin\frac{-\pi}{2} \\ 0 & 1 & 0 \\ -\sin\frac{-\pi}{2} & 0 & \cos\frac{-\pi}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}, \tag{3.246}$$

for which it is a simple exercise to verify that $(\boldsymbol{R}^{\mathcal{MC}})^{\mathrm{T}} = (\boldsymbol{R}^{\mathcal{MC}})^{-1} = \boldsymbol{R}^{\mathcal{CM}}$: the inverse rotation is found by first rotating by $+\pi/2$ about the $\boldsymbol{e}_2^{\mathcal{C}}$-axis, then by $-\pi/4$ about the $\boldsymbol{e}_3^{\mathcal{C}}$-axis:

$$[\boldsymbol{R}^{\mathcal{CM}}]_{\mathcal{C}} = \begin{pmatrix} \cos\frac{-\pi}{4} & -\sin\frac{-\pi}{4} & 0 \\ \sin\frac{-\pi}{4} & \cos\frac{-\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{2} & 0 & -\sin\frac{\pi}{2} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{2} & 0 & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1 & 0 & 0 \end{pmatrix}. \tag{3.247}$$

The basis vectors of the $\mathcal{M}$-frame are $\boldsymbol{e}_i^{\mathcal{M}} = \boldsymbol{R}^{\mathcal{MC}} \boldsymbol{e}_i^{\mathcal{C}}$ whose components are the columns of $\boldsymbol{R}^{\mathcal{MC}}$:

$$[\boldsymbol{e}_1^{\mathcal{M}}]_{\mathcal{C}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad [\boldsymbol{e}_2^{\mathcal{M}}]_{\mathcal{C}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad [\boldsymbol{e}_3^{\mathcal{M}}]_{\mathcal{C}} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.248}$$

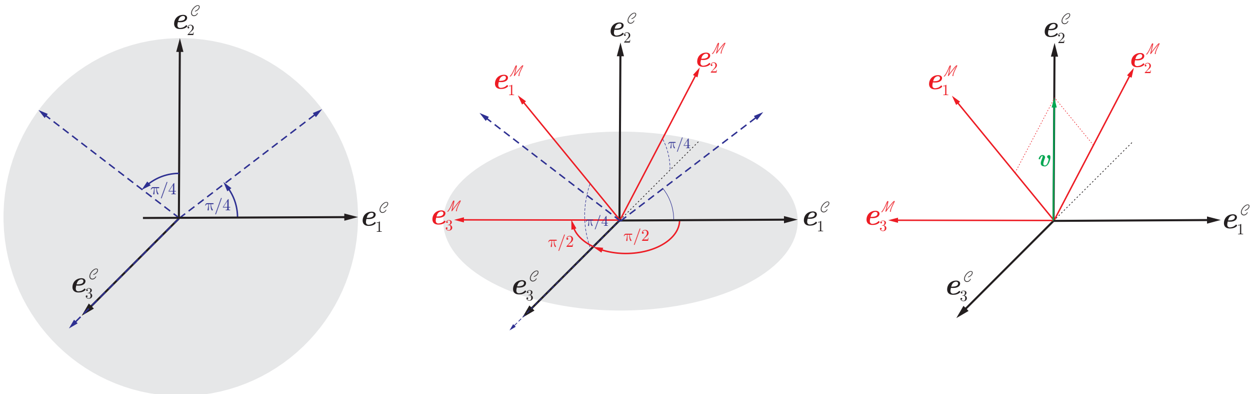
As an example vector, $\boldsymbol{v} = \boldsymbol{e}_2$ has components $[v_1]_{\mathcal{C}} = [v_3]_{\mathcal{C}} = 0$ and $[v_2]_{\mathcal{C}} = 1$, whereas in the $\mathcal{M}$-frame we obtain

$$[v_i]_{\mathcal{M}} = \sum_{j=1}^{3} [R_{ij}^{\mathcal{CM}}][v_j]_{\mathcal{C}} \qquad \Rightarrow \qquad [\boldsymbol{v}]_{\mathcal{M}} = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \tag{3.249}$$

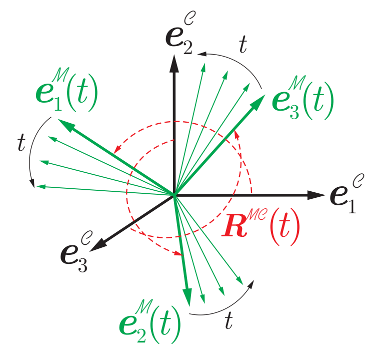
As a final check, we can also revert the coordinate transform according to

$$[v_i]_{\mathcal{C}} = \sum_{j=1}^{3} [R_{ij}^{\mathcal{MC}}][v_j]_{\mathcal{M}} \qquad \Rightarrow \qquad [\boldsymbol{v}]_{\mathcal{C}} = \begin{pmatrix} 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{3.250}$$

The two bases along with vector $\boldsymbol{v}$ and its $\mathcal{M}$-components are illustrated below.



3.4.2 Rotating frames of reference



Active rotations act on vectors and perform a rotation (e.g., from the $\mathcal{C}$-basis to the $\mathcal{M}$-basis). This rotation can be time-dependent, e.g., if the $\mathcal{C}$-basis is time-invariant (*inertial frame*) while the $\mathcal{M}$-basis is rotating as a function of time (*non-inertial frame*). When expressing a vector in the $\mathcal{M}$-frame as $\boldsymbol{r}(t) = \sum_{i=1}^{d} r_i(t)\boldsymbol{e}_i^{\mathcal{M}}(t)$, it is important to compute the time derivatives of the basis vectors $\boldsymbol{e}_i^{\mathcal{M}}$ in order to compute $\dot{\boldsymbol{r}}(t)$. Differentiating the $\mathcal{M}$-basis with respect to time (and exploiting that the Cartesian frame is time-invariant) leads to

$$\boldsymbol{e}_i^{\mathcal{M}}(t) = \boldsymbol{R}^{\mathcal{MC}}(t)\,\boldsymbol{e}_i^{\mathcal{C}} \qquad \Rightarrow \qquad \dot{\boldsymbol{e}}_i^{\mathcal{M}}(t) = \dot{\boldsymbol{R}}^{\mathcal{MC}}(t)\,\boldsymbol{e}_i^{\mathcal{C}}. \tag{3.251}$$

From (3.239) it follows that (dropping the t-dependence for brevity)

$$\boldsymbol{R}^{\mathcal{MC}}\left(\boldsymbol{R}^{\mathcal{MC}}\right)^{\mathrm{T}} = \boldsymbol{I} \quad \Rightarrow \quad \dot{\boldsymbol{R}}^{\mathcal{MC}}\left(\boldsymbol{R}^{\mathcal{MC}}\right)^{\mathrm{T}} = -\boldsymbol{R}^{\mathcal{MC}}\left(\dot{\boldsymbol{R}}^{\mathcal{MC}}\right)^{\mathrm{T}} = -\left[\dot{\boldsymbol{R}}^{\mathcal{MC}}\left(\boldsymbol{R}^{\mathcal{MC}}\right)^{\mathrm{T}}\right]^{\mathrm{T}}. \tag{3.252}$$

This motivates the definition of the **spin tensor**

$$\boldsymbol{W}^{\mathcal{MC}} = \dot{\boldsymbol{R}}^{\mathcal{MC}}\left(\boldsymbol{R}^{\mathcal{MC}}\right)^{\mathrm{T}} \qquad \text{with} \qquad [\boldsymbol{W}^{\mathcal{MC}}]_{\mathcal{C}} = \begin{pmatrix} 0 & -\Omega_3^{\mathcal{M}} & \Omega_2^{\mathcal{M}} \\ \Omega_3^{\mathcal{M}} & 0 & -\Omega_1^{\mathcal{M}} \\ -\Omega_2^{\mathcal{M}} & \Omega_1^{\mathcal{M}} & 0 \end{pmatrix}. \tag{3.253}$$

The latter component form follows directly from the relation (3.252), which shows that the spin tensor $\boldsymbol{W}^{\mathcal{MC}} = -\left(\boldsymbol{W}^{\mathcal{MC}}\right)^{\mathrm{T}}$ is skew-symmetric (having zero diagonal components and anti-symmetric off-diagonal components) with only three independent components.

Notice that for any vector $\boldsymbol{v}$

$$\boldsymbol{W}^{\mathcal{MC}}\boldsymbol{v} = \boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{v} \qquad \text{with} \qquad [\boldsymbol{\Omega}^{\mathcal{M}}]_{\mathcal{C}} = \begin{pmatrix} \Omega_1^{\mathcal{M}} \\ \Omega_2^{\mathcal{M}} \\ \Omega_3^{\mathcal{M}} \end{pmatrix}, \tag{3.254}$$

where $\boldsymbol{\Omega}^{\mathcal{M}}$ is the **angular velocity** vector of frame $\mathcal{M}$. Here and in the following, we use the superscript $^{\mathcal{M}}$ to indicate that the rotation is that of the $\mathcal{M}$-frame.

From this derivation, it follows – starting with (3.251) – that

$$\dot{\boldsymbol{e}}_i^{\mathcal{M}} = \dot{\boldsymbol{R}}^{\mathcal{MC}}\boldsymbol{e}_i^{\mathcal{C}} = \dot{\boldsymbol{R}}^{\mathcal{MC}}(\boldsymbol{R}^{\mathcal{MC}})^{-1}\boldsymbol{e}_i^{\mathcal{M}} = \left(\dot{\boldsymbol{R}}^{\mathcal{MC}}(\boldsymbol{R}^{\mathcal{MC}})^{\mathrm{T}}\right)\boldsymbol{e}_i^{\mathcal{M}} = \boldsymbol{W}^{\mathcal{MC}}\boldsymbol{e}_i^{\mathcal{M}} = \boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{e}_i^{\mathcal{M}}. \tag{3.255}$$

The same relation can be re-interpreted as follows. We have seen that, for a rigid body moving in 3D with an instantaneous angular velocity vector $\boldsymbol{\Omega}(t)$, the velocity of any point on the body can be calculated by the velocity transfer formula (3.10). Let us treat the reference frame $\mathcal{M}$ like a moving body with basis vectors $\{\boldsymbol{e}_1^{\mathcal{M}}, \boldsymbol{e}_2^{\mathcal{M}}, \boldsymbol{e}_3^{\mathcal{M}}\}$ which rotate with an instantaneous angular velocity vector $\boldsymbol{\Omega}^{\mathcal{M}}(t)$. In this case, using a fixed origin, the velocity transfer formula gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_i^{\mathcal{M}}(t) = \dot{\boldsymbol{e}}_i^{\mathcal{M}}(t) = \boldsymbol{\Omega}^{\mathcal{M}}(t) \times \boldsymbol{e}_i^{\mathcal{M}}(t), \tag{3.256}$$

which is equivalent to (3.255). This simplifies our derivations significantly. The absolute, inertial time derivative of any vector $\boldsymbol{y}$ in the $\mathcal{M}$-frame, when measured by an *inertial observer* in the $\mathcal{C}$-frame, is

$$\dot{\boldsymbol{y}}^{\mathcal{C}} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{3}[y_i]_{\mathcal{M}}\boldsymbol{e}_i^{\mathcal{M}} = \sum_{i=1}^{3}[\dot{y}_i]_{\mathcal{M}}\boldsymbol{e}_i^{\mathcal{M}} + \sum_{i=1}^{3}[y_i]_{\mathcal{M}}\dot{\boldsymbol{e}}_i^{\mathcal{M}} = \sum_{i=1}^{3}[\dot{y}_i]_{\mathcal{M}}\boldsymbol{e}_i^{\mathcal{M}} + \boldsymbol{\Omega}^{\mathcal{M}} \times \sum_{i=1}^{3}[y_i]_{\mathcal{M}}\boldsymbol{e}_i^{\mathcal{M}}. \tag{3.257}$$

The above obviously deviates from the time derivative measured by an *observer moving with frame* $\mathcal{M}$, which is

$$\mathring{\boldsymbol{y}}^{\mathcal{M}} = \sum_{i=1}^{3}[\dot{y}_i]_{\mathcal{M}}\boldsymbol{e}_i^{\mathcal{M}}, \tag{3.258}$$

since for the moving observer the basis vectors $\boldsymbol{e}_i^{\mathcal{M}}$ appear to be time-invariant. Note the difference in notation: a *superscript* $^{\mathcal{M}}$ refers to a quantity associated with the $\mathcal{M}$-frame (such as the angular velocity $\boldsymbol{\Omega}^{\mathcal{M}}$, or the basis vectors $\boldsymbol{e}_i^{\mathcal{M}}$, or – here – the time derivative with respect to frame $\mathcal{M}$), while $[\cdot]_{\mathcal{M}}$ refers to the components of a vector measured in the $\mathcal{M}$-frame.

The relation between the two time derivatives follows from the above two expressions, viz.

$$\underbrace{\dot{\boldsymbol{y}}^{\mathcal{C}}}_{\substack{\text{(absolute) time}\\ \text{derivative in } \mathcal{C}}} = \underbrace{\mathring{\boldsymbol{y}}^{\mathcal{M}}}_{\substack{\text{(relative) time}\\ \text{derivative with}\\ \text{respect to } \mathcal{M}}} + \boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{y} \qquad \Rightarrow \qquad \boxed{\dot{\boldsymbol{y}}^{\mathcal{C}} = \mathring{\boldsymbol{y}}^{\mathcal{M}} + \boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{y}} \tag{3.259}$$

This relation is quite important. It means that, if we are in a moving frame of reference $\mathcal{M}$, we do not need to measure our velocity with respect to some inertial basis $\mathcal{C}$. All we need to know is our velocity $\mathring{\boldsymbol{y}}^{\mathcal{M}}$ with respect to the moving frame, the angular velocity $\boldsymbol{\Omega}^{\mathcal{M}}$ with which our frame is rotating, and our current position $\boldsymbol{y}$ to uniquely calculate our velocity $\dot{\boldsymbol{y}}^{\mathcal{C}}$ in the inertial frame. (Note that $\boldsymbol{y}$ is a vector that can be expressed in either frame, $\mathcal{C}$ or $\mathcal{M}$, which is why it has no superscript in the above relation.)

In practice, Eq. (3.259) can only be evaluated if *all vectors are expressed in the same frame*. Therefore, it makes sense to *express the right-hand side in* (3.259) *exclusively in the* $\mathcal{M}$*-frame* (including the vectors $\boldsymbol{y}$ and also $\boldsymbol{\Omega}^{\mathcal{M}}$). To this end, we point out that any vector, such as an angular velocity $\boldsymbol{\Omega}$, can be expressed with respect to both bases and the relations (3.236) and (3.237) apply as well, e.g.,

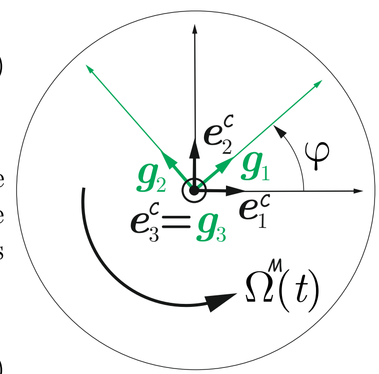
$$\boldsymbol{\Omega} = \sum_{i=1}^{3} [\Omega_i]_{\mathcal{C}} \boldsymbol{e}_i^{\mathcal{C}} = \sum_{i=1}^{3} [\Omega_i]_{\mathcal{M}} \boldsymbol{e}_i^{\mathcal{M}} \qquad \text{and} \qquad [\Omega_i]_{\mathcal{C}} = [R_{ij}^{\mathcal{CM}}]^{\mathrm{T}} [\Omega_j]_{\mathcal{M}} \tag{3.260}$$

with the rotation defined by (3.236) and (3.237).

Example 3.21. Rotations in 2D

As a special case, let us consider two reference frames related by a **rotation in 2D**, so we may be interested in a rotating basis (see the schematic below)

$$\begin{aligned} \boldsymbol{g}_1(t) &= \cos\varphi(t)\, \boldsymbol{e}_1^{\mathcal{C}} + \sin\varphi(t)\, \boldsymbol{e}_2^{\mathcal{C}}, \\ \boldsymbol{g}_2(t) &= -\sin\varphi(t)\, \boldsymbol{e}_1^{\mathcal{C}} + \cos\varphi(t)\, \boldsymbol{e}_2^{\mathcal{C}}, \\ \boldsymbol{g}_3(t) &= \boldsymbol{e}_3^{\mathcal{C}}, \end{aligned} \tag{3.261}$$



where the time-varying angle $\varphi(t)$ describes the rotation from the Cartesian to the co-rotating basis vectors in the plane (see the schematic on the right). Notice that the (absolute) time derivatives measured in the $\mathcal{C}$-frame are

$$\begin{aligned} \dot{\boldsymbol{g}}_1^{\mathcal{C}} &= -\dot\varphi \sin\varphi\, \boldsymbol{e}_1^{\mathcal{C}} + \dot\varphi \cos\varphi\, \boldsymbol{e}_2^{\mathcal{C}} = \dot\varphi\, \boldsymbol{g}_2, \\ \dot{\boldsymbol{g}}_2^{\mathcal{C}} &= -\dot\varphi \cos\varphi\, \boldsymbol{e}_1^{\mathcal{C}} - \dot\varphi \sin\varphi\, \boldsymbol{e}_2^{\mathcal{C}} = -\dot\varphi\, \boldsymbol{g}_1, \end{aligned} \tag{3.262}$$

so that, writing $\Omega^{\mathcal{M}}(t) = \dot\varphi(t)$, we conclude

$$\dot{\boldsymbol{g}}_1^{\mathcal{C}} = \Omega^{\mathcal{M}} \boldsymbol{g}_2, \qquad \dot{\boldsymbol{g}}_2^{\mathcal{C}} = -\Omega^{\mathcal{M}} \boldsymbol{g}_1, \qquad \dot{\boldsymbol{g}}_3^{\mathcal{C}} = \boldsymbol{0}. \tag{3.263}$$

The above relations may conveniently be abbreviated by introducing the angular velocity vector

$$\boldsymbol{\Omega}^{\mathcal{M}}(t) = \Omega^{\mathcal{M}}(t) \boldsymbol{e}_3^{\mathcal{C}} = \Omega^{\mathcal{M}}(t) \boldsymbol{g}_3 \tag{3.264}$$

such that indeed

$$\dot{\boldsymbol{g}}_i^{\mathcal{C}} = \boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{g}_i \qquad \text{for } i = 1, 2, 3. \tag{3.265}$$

This confirms relation (3.259), since the time derivative of a vector (expressed in the co-rotating basis vectors)

$$\boldsymbol{v} = \sum_{i=1}^{3} [v_i]_{\mathcal{M}} \boldsymbol{g}_i \tag{3.266}$$

becomes

$$\begin{aligned} \dot{\boldsymbol{v}}^{\mathcal{C}} &= \sum_{i=1}^{3} \left([\dot v_i]_{\mathcal{M}}\, \boldsymbol{g}_i + [v_i]_{\mathcal{M}} \dot{\boldsymbol{g}}_i \right) = \mathring{\boldsymbol{v}} + \sum_{i=1}^{3} [v_i]_{\mathcal{M}}\, \boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{g}_i = \mathring{\boldsymbol{v}} + \boldsymbol{\Omega}^{\mathcal{M}} \times \sum_{i=1}^{3} [v_i]_{\mathcal{M}} \boldsymbol{g}_i \\ &= \mathring{\boldsymbol{v}} + \boldsymbol{\Omega} \times \boldsymbol{v}. \end{aligned} \tag{3.267}$$

Example 3.22. Polar coordinates revisited

Recall that we had derived in Section 1.1.3 the velocity and acceleration components of a rotating particle in polar coordinates (measured with respect to the rotating $\mathcal{M}$-frame) as

$$[\boldsymbol{v}]_{\mathcal{M}} = \begin{pmatrix} \dot{r} \\ r\dot{\varphi} \\ 0 \end{pmatrix}, \qquad [\boldsymbol{a}]_{\mathcal{M}} = \begin{pmatrix} \ddot{r} - r\dot{\varphi}^2 \\ 2\dot{r}\dot{\varphi} + r\ddot{\varphi} \\ 0 \end{pmatrix}, \tag{3.268}$$

and we made the important conclusion (as evident from the above vector components) that

$$[\boldsymbol{a}]_{\mathcal{M}} \neq \frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{v}]_{\mathcal{M}}. \tag{3.269}$$

In other words, we cannot simply differentiate the vector components to obtain the acceleration components:

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{r} \neq \ddot{r} - r\dot{\varphi}^2 \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(r\dot{\varphi}\right) \neq 2\dot{r}\dot{\varphi} + r\ddot{\varphi}. \tag{3.270}$$

Equation (3.259) immediately explains why:

$$\dot{\boldsymbol{v}} = \mathring{\boldsymbol{v}}^{\mathcal{M}} + \boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{v} \qquad \Rightarrow \qquad [\dot{\boldsymbol{v}}]_{\mathcal{M}} = \begin{pmatrix} \ddot{r} \\ \dot{r}\dot{\varphi} + r\ddot{\varphi} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} \times \begin{pmatrix} \dot{r} \\ r\dot{\varphi} \\ 0 \end{pmatrix} = \begin{pmatrix} \ddot{r} - r\dot{\varphi}^2 \\ 2\dot{r}\dot{\varphi} + r\ddot{\varphi} \\ 0 \end{pmatrix}. \tag{3.271}$$

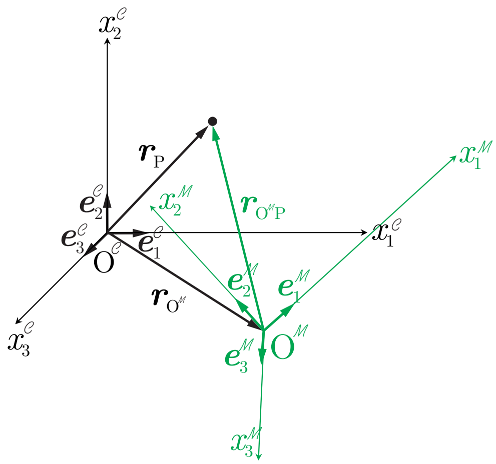
In other words, we may not only differentiate the components of $[\boldsymbol{v}]_{\mathcal{M}}$ in the rotating frame, but we must also account for the fact that the basis vectors in polar coordinates rotate and hence have a non-zero time derivative (as also discussed in Example 3.21 above). Using the correct relation (3.259) hence yields the correct time derivative (here expressed in components $[\boldsymbol{a}]_{\mathcal{M}}$ measured with respect to the rotating frame).

3.4.3 Balance of linear momentum

Now that we understand the kinematic relations between inertial and non-inertial frames, let us revisit the kinetic balance laws and begin with linear momentum balance. For generality, we consider the following two frames:

- **frame** $\mathcal{C}$ has a fixed, Cartesian basis $\{\boldsymbol{e}_1^{\mathcal{C}}, \boldsymbol{e}_2^{\mathcal{C}}, \boldsymbol{e}_3^{\mathcal{C}}\} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and origin $\mathrm{O}^{\mathcal{C}}$,
- **frame** $\mathcal{M}$ has a rotating basis $\{\boldsymbol{e}_1^{\mathcal{M}}, \boldsymbol{e}_2^{\mathcal{M}}, \boldsymbol{e}_3^{\mathcal{M}}\}$ and a moving origin $\mathrm{O}^{\mathcal{M}}$.

Note that, unlike in our previous discussions, we here also consider distinct origins of the two reference frames. This does not affect the relations between the two basis vectors (which are still governed by the rotation relations derived above).



The position of any point P in space can be expressed with respect to both origins, the fixed Cartesian one and the moving origin $\mathrm{O}^{\mathcal{M}}$, viz.

$$\boldsymbol{r}_{\mathrm{P}}(t) = \boldsymbol{r}_{\mathrm{O}^{\mathcal{M}}}(t) + \boldsymbol{r}_{\mathrm{O}^{\mathcal{M}}\mathrm{P}}(t), \tag{3.272}$$

where $\boldsymbol{r}_{O^{\mathcal{M}}P}$ is the position of point P relative to the moving frame. Taking a time derivative on both sides of the equation with (3.259) gives

$$\dot{\boldsymbol{r}}_{\mathrm{P}}^{\mathcal{C}}(t) = \dot{\boldsymbol{r}}_{\mathrm{O}^{\mathcal{M}}}(t) + \overset{\circ}{\boldsymbol{r}}{}_{\mathrm{O}^{\mathcal{M}}\mathrm{P}}^{\mathcal{M}}(t) + \boldsymbol{\Omega}^{\mathcal{M}}(t) \times \boldsymbol{r}_{\mathrm{O}^{\mathcal{M}}\mathrm{P}}(t), \tag{3.273}$$

while the second time derivative becomes (dropping the t-dependence for simplicity)

$$\ddot{\boldsymbol{r}}_{\mathrm{P}}^{\mathcal{C}} = \ddot{\boldsymbol{r}}_{\mathrm{O}^{\mathcal{M}}} + \overset{\circ\circ}{\boldsymbol{r}}{}_{\mathrm{O}^{\mathcal{M}}\mathrm{P}}^{\mathcal{M}} + \underbrace{\frac{\mathrm{d}\boldsymbol{\Omega}^{\mathcal{M}}}{\mathrm{d}t} \times \boldsymbol{r}_{\mathrm{O}^{\mathcal{M}}\mathrm{P}}}_{\text{Euler acceleration}} + \underbrace{2\boldsymbol{\Omega}^{\mathcal{M}} \times \overset{\circ}{\boldsymbol{r}}{}_{\mathrm{O}^{\mathcal{M}}\mathrm{P}}^{\mathcal{M}}}_{\text{Coriolis acceleration}} + \underbrace{\boldsymbol{\Omega}^{\mathcal{M}} \times (\boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{r}_{\mathrm{O}^{\mathcal{M}}\mathrm{P}})}_{\text{centripetal acceleration}}. \tag{3.274}$$

We notice that – beyond the first two terms – three additional terms have appeared because of the non-inertial basis. These three new acceleration terms are referred to as the **Coriolis acceleration** $\boldsymbol{a}_{\text{Coriolis}}$, **centripetal acceleration** $\boldsymbol{a}_{\text{centripetal}}$ and **Euler acceleration** $\boldsymbol{a}_{\text{Euler}}$, as shown above.

Having established a relation between the accelerations measured in both systems, let us return to Newton's axioms which, as we discussed, apply only in inertial reference frames. For a rigid body, we may make the particular choice $\boldsymbol{r}_{\mathrm{P}} = \boldsymbol{r}_{\mathrm{CM}}$, so that linear momentum balance implies

$$M\ddot{\boldsymbol{r}}_{\mathrm{CM}}^{\mathcal{C}} = \sum_i \boldsymbol{F}_i^{\text{ext}}. \tag{3.275}$$

If we denote the position, velocity and acceleration of the center of mass *measured with respect to the moving $\mathcal{M}$-frame* by, respectively,

$$\boldsymbol{r}_{\mathrm{CM}}^{\mathcal{M}} = \boldsymbol{r}_{\mathrm{O}^{\mathcal{M}}\mathrm{CM}}, \qquad \boldsymbol{v}_{\mathrm{CM}}^{\mathcal{M}} = \overset{\circ}{\boldsymbol{r}}{}_{\mathrm{O}^{\mathcal{M}}\mathrm{CM}}^{\mathcal{M}}, \qquad \boldsymbol{a}_{\mathrm{CM}}^{\mathcal{M}} = \overset{\circ\circ}{\boldsymbol{r}}{}_{\mathrm{O}^{\mathcal{M}}\mathrm{CM}}^{\mathcal{M}}, \tag{3.276}$$

then **linear momentum balance** *for a rigid body in a moving frame* becomes

$$\boxed{M\boldsymbol{a}_{\mathrm{CM}}^{\mathcal{M}} = \sum_i \boldsymbol{F}_i^{\text{ext}} + \boldsymbol{F}_{\text{Coriolis}} + \boldsymbol{F}_{\text{Euler}} + \boldsymbol{F}_{\text{centrifugal}} - M\boldsymbol{a}_{\mathrm{O}^{\mathcal{M}}}} \tag{3.277}$$

Besides the externally applied forces, the right-hand side includes new additional forces, viz.

- the **Coriolis force**: $\boxed{\boldsymbol{F}_{\text{Coriolis}} = -2M\boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{v}_{\mathrm{CM}}^{\mathcal{M}}}$
- the **centrifugal force**: $\boxed{\boldsymbol{F}_{\text{centrifugal}} = -M\boldsymbol{\Omega}^{\mathcal{M}} \times (\boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{r}_{\mathrm{CM}}^{\mathcal{M}})}$
- the **Euler force**: $\boxed{\boldsymbol{F}_{\text{Euler}} = -M\frac{\mathrm{d}\boldsymbol{\Omega}^{\mathcal{M}}}{\mathrm{d}t} \times \boldsymbol{r}_{\mathrm{CM}}^{\mathcal{M}}}$
- the (fictitious) force $-M\boldsymbol{a}_{\mathrm{O}^{\mathcal{M}}}$ due to the **moving origin**.

Note the changes in sign resulting in the centri*petal* acceleration becoming centri*fugal* force. The three above forces are **apparent forces** (or **fictitious forces**) and not external, physical forces; i.e., to an observer in the moving frame objects look like they are moving due to these forces (because their frame of reference is moving), but those forces are not observed in an inertial frame.

It is helpful to keep in mind that the

- *Coriolis force* is present only if the body is *moving with respect to the rotating frame*, i.e., if $\boldsymbol{v}^{\mathcal{M}} = \mathring{\boldsymbol{r}}^{\mathcal{M}} \neq \boldsymbol{0}$. If a body is fixed onto a rotating frame, a co-rotating observer notices no Coriolis force.
- *Euler force* appears only if the *angular velocity of the reference frame is changing with time*, i.e., $\dot{\Omega}^{\mathcal{M}} \neq 0$. If a rotating system rotates with constant angular velocity, no Euler force is experienced.
- *centrifugal force* always acts in a rotating system. Note that

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = (\boldsymbol{\Omega} \cdot \boldsymbol{r})\boldsymbol{\Omega} - \Omega^2 \boldsymbol{r} \qquad \text{with} \qquad \Omega = |\boldsymbol{\Omega}|, \tag{3.278}$$

which is helpful in particular in 2D rotations where $\boldsymbol{\Omega} \cdot \boldsymbol{r} = \boldsymbol{0}$, so the centrifugal force in 2D simplifies to

$$\boldsymbol{F}_{\text{centrifugal,2D}} = M(\Omega^{\mathcal{M}})^2 \boldsymbol{r}^{\mathcal{M}}_{\text{CM}}. \tag{3.279}$$

In most practical cases we will be able to use the same fixed origin, so $\mathrm{O}^{\mathcal{C}} = \mathrm{O}^{\mathcal{M}}$ and the final term in (3.277) vanishes.

In summary, when formulating the balance of linear momentum in a moving frame $\mathcal{M}$, then we must not forget – in addition to the physical forces $\boldsymbol{F}^{\text{ext}}$ – the three apparent/fictitious/inertial forces given above.

Example 3.23. Rotation in 2D revisited

We consider a 2D rotation about the $\boldsymbol{e}_3$-axis, so the motion is constrained to the x_1-x_2-plane and the reference frame $\mathcal{M}$ is rotating with angular velocity

$$\boldsymbol{\Omega}^{\mathcal{M}} = \Omega \boldsymbol{e}_3, \tag{3.280}$$

while the origin is fixed. Assume that we are an observer within the rotating system, so we measure the 2D position and velocity of any particle or body in the rotating system as, respectively

$$\boldsymbol{r}^{\mathcal{M}}(t) = \sum_{i=1}^{2} y_i(t)\boldsymbol{e}_i^{\mathcal{M}}(t) \qquad \text{and} \qquad \boldsymbol{v}^{\mathcal{M}}(t) = \mathring{\boldsymbol{r}}^{\mathcal{M}}(t) = \sum_{i=1}^{2} \dot{y}_i(t)\boldsymbol{e}_i^{\mathcal{M}}(t), \tag{3.281}$$

and the acceleration measured in the $\mathcal{M}$-frame follows analogously as

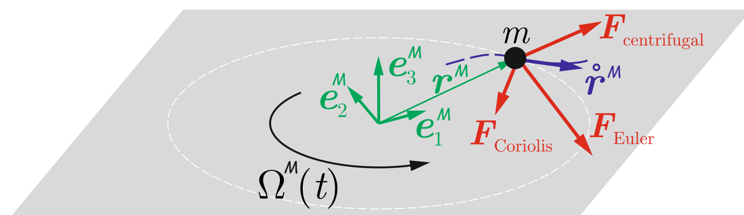
$$\boldsymbol{a}^{\mathcal{M}}(t) = \mathring{\mathring{\boldsymbol{r}}}^{\mathcal{M}}(t) = \sum_{i=1}^{2} \ddot{y}_i(t)\boldsymbol{e}_i^{\mathcal{M}}(t) \tag{3.282}$$

Note that we here labeled the components of the position as $y_i(t)$ instead of $[r_i]_{\mathcal{M}}(t)$ for simplicity and brevity.

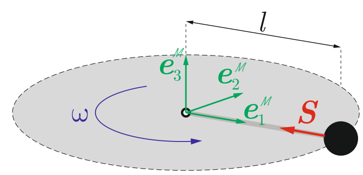
Writing out the components of the individual terms in (3.277) in the moving frame for a particle (or body) of mass m gives

$$m\begin{pmatrix}\ddot{y}_1\\ \ddot{y}_2\\ 0\end{pmatrix} = [\boldsymbol{F}^{\text{ext}}]_{\mathcal{M}} \underbrace{-2m\Omega^{\mathcal{M}}\begin{pmatrix}-\dot{y}_2\\ \dot{y}_1\\ 0\end{pmatrix}}_{\text{Coriolis force}} + \underbrace{m\Omega^2\begin{pmatrix}y_1\\ y_2\\ 0\end{pmatrix}}_{\text{centrifugal force}} \underbrace{-m\dot{\Omega}\begin{pmatrix}-y_2\\ y_1\\ 0\end{pmatrix}}_{\text{Euler force}}, \tag{3.283}$$

where the components of $\boldsymbol{F}^{\text{ext}}$ must be those *with respect to the rotating basis* $\mathcal{M}$, as indicated. Here, it is apparent that the centri*fugal* force is acting radially outward. All fictitious forces are schematically shown below for a particle moving along the dashed trajectory in the rotating frame (all forces are in the plane).



Example 3.24. Particle on a taut string revisited



In Example 1.10 we derived that the force in a taut string (of length l) due to a particle of mass m attached to its end and rotating at a constant angular velocity ω is given by

$$\boldsymbol{S} = m\boldsymbol{a} = -ml\omega^2\boldsymbol{e}_r. \tag{3.284}$$

What we derived back then can now easily be retrieved by our formulation of a rotating frame of reference, in which linear momentum balance takes the form (3.277). We pick a moving $\mathcal{M}$-frame which rotates with the particle and string and whose origin is fixed at the center of rotation, as shown. In this frame, the position of the particle is

$$\boldsymbol{r}_{\text{CM}}^{\mathcal{M}} = l\boldsymbol{e}_1^{\mathcal{M}} = \text{const.} \qquad \Rightarrow \qquad \boldsymbol{a}_{\text{CM}}^{\mathcal{M}} = \boldsymbol{0}, \qquad \boldsymbol{a}_{\text{CM}}^{\mathcal{M}} = \boldsymbol{0}. \tag{3.285}$$

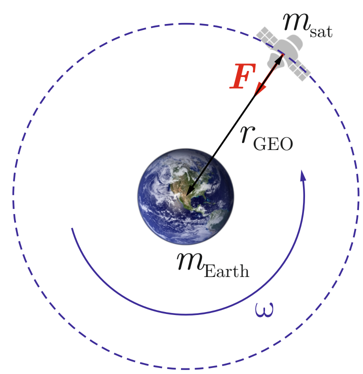
Consequently, $\boldsymbol{F}_{\text{Coriolis}} = \boldsymbol{0}$. Further, we have $\boldsymbol{\Omega}^{\mathcal{M}} = \omega\boldsymbol{e}_3^{\mathcal{M}} = \text{const.}$ so $\boldsymbol{F}_{\text{Euler}} = \boldsymbol{0}$. Altogether, linear momentum in the moving frame (3.277) simplifies to

$$\boldsymbol{0} = -\boldsymbol{S} - m\boldsymbol{\Omega}^{\mathcal{M}} \times (\boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{r}_{\text{CM}}^{\mathcal{M}}) \qquad \Leftrightarrow \qquad \boldsymbol{S} = -m\boldsymbol{\Omega}^{\mathcal{M}} \times (\boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{r}_{\text{CM}}^{\mathcal{M}}) = -m\omega^2 l\boldsymbol{e}_1^{\mathcal{M}}. \tag{3.286}$$

We hence recover the solution (3.284)

Example 3.25. Satellite in geostationary orbit

A satellite in so-called geostationary or geosynchronous equatorial orbit (GEO) is surrounding the Earth with a radius of rotation r_{GEO} such that it remains *geostationary* (i.e., above exactly the same location on the Earth's ground) if no external forces act on the satellite except for gravity. What is r_{GEO}?



This is a particular case of the scenario of Example 3.24. Instead of a taut string, gravity is attracting the satellite of mass m_{sat} towards the Earth's center, the magnitude of the gravitational force being

$$F = G\frac{m_{\mathrm{sat}}m_{\mathrm{Earth}}}{r^2}, \tag{3.287}$$

where $G \approx 6.674 \cdot 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ is the gravitational constant, $m_{\mathrm{Earth}} \approx 5.974 \cdot 10^{24}\mathrm{kg}$ the Earth's mass, and r the radius of rotation of the satellite.

In geostationary orbit, the above gravitational force balances the centrifugal force in the absence of any other forces acting on the satellite. Therefore, linear momentum balance (3.286) with $\boldsymbol{S} = -F\boldsymbol{e}_1^{\mathcal{M}}$ becomes

$$G\frac{m_{\mathrm{sat}}m_{\mathrm{Earth}}}{r_{\mathrm{GEO}}^2} = m_{\mathrm{sat}}\omega^2 r_{\mathrm{GEO}} \qquad \Leftrightarrow \qquad r_{\mathrm{GEO}} = \left(\frac{Gm_{\mathrm{Earth}}}{\omega^2}\right)^{1/3}. \tag{3.288}$$

For geostationarity, the satellite must rotate (above the equator) with the Earth's angular velocity, which is

$$\omega = \frac{2\pi}{1\mathrm{d}} = \frac{2\pi}{24 \cdot 60 \cdot 60\mathrm{s}} \approx 7.27 \cdot 10^{-5}\mathrm{s}^{-1}. \tag{3.289}$$

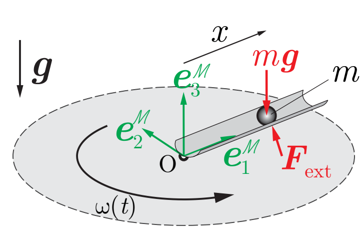
This leads to

$$r_{\mathrm{GEO}} = \left(\frac{G\, m_{\mathrm{Earth}}}{\omega^2}\right)^{1/3} \approx 42.3 \cdot 10^3 \text{ km}, \tag{3.290}$$

which is approximately $35.8 \cdot 10^3$ km above the Earth's surface. The satellite moves at a constant speed of $v_{\mathrm{sat}} = \omega r_{\mathrm{GEO}} \approx 3.07$ km/s.

Example 3.26. Particle in a rotating barrel

Consider a cylindrical barrel that is rotating with angular velocity $\omega(t)$ in the x_1-x_2-plane, i.e., $\boldsymbol{\omega}(t) = \omega(t)\boldsymbol{e}_3$. Inside the barrel slides a particle of mass m (assume that the barrel constrains the particle motion along the barrel's axis, i.e., the particle fits right into the barrel so it can only move along the barrel). If released from rest at an initial distance x_0 from the center, what is the motion of the particle and what forces are acting on the particle?



Here, it is natural to use a moving frame with basis $\{\boldsymbol{e}_1^{\mathcal{M}}, \boldsymbol{e}_2^{\mathcal{M}}, \boldsymbol{e}_3^{\mathcal{M}}\}$ that is rotating with the barrel, so that $\boldsymbol{\Omega}^{\mathcal{M}} = \boldsymbol{\omega}$ (and we take $\mathrm{O} = \mathrm{O}^{\mathcal{M}}$, so the origin is fixed and identical in both frames and $\boldsymbol{a}_{\mathrm{O}^{\mathcal{M}}} = \boldsymbol{0}$).

In the rotating frame, only a single degree of freedom x along the axis of the barrel is needed to describe the position of the particle relative to the moving frame as[13] $\boldsymbol{r}_{\mathrm{CM}}^{\mathcal{M}} = x\boldsymbol{e}_1^{\mathcal{M}}$. The kinematics of the particle motion *in the rotating $\mathcal{M}$-frame* then follows as

$$\boldsymbol{r}_{\mathrm{CM}}^{\mathcal{M}} = x\boldsymbol{e}_1^{\mathcal{M}} \qquad \Rightarrow \qquad \boldsymbol{v}_{\mathrm{CM}}^{\mathcal{M}} = \dot{x}\boldsymbol{e}_1^{\mathcal{M}} \qquad \Rightarrow \qquad \boldsymbol{a}_{\mathrm{CM}}^{\mathcal{M}} = \ddot{x}\boldsymbol{e}_1^{\mathcal{M}}. \tag{3.291}$$

Linear momentum balance *in the non-inertial frame $\mathcal{M}$* is described by Eq. (3.277), viz.

$$\begin{aligned} m\boldsymbol{a}_{\mathrm{CM}}^{\mathcal{M}} &= \boldsymbol{F}_{\mathrm{ext}} + \boldsymbol{F}_{\mathrm{Coriolis}} + \boldsymbol{F}_{\mathrm{Euler}} + \boldsymbol{F}_{\mathrm{centrifugal}} \\ &= \boldsymbol{F}_{\mathrm{ext}} - 2m\boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{v}_{\mathrm{CM}}^{\mathcal{M}} - m\dot{\boldsymbol{\Omega}}^{\mathcal{M}} \times \boldsymbol{r}_{\mathrm{CM}}^{\mathcal{M}} - m\boldsymbol{\Omega}^{\mathcal{M}} \times (\boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{r}_{\mathrm{CM}}^{\mathcal{M}}). \end{aligned} \tag{3.292}$$

Insertion of (3.291) along with $\boldsymbol{\Omega}^{\mathcal{M}}(t) = \boldsymbol{\omega}(t) = \omega(t)\boldsymbol{e}_3^{\mathcal{M}}$ yields

$$\begin{aligned} m\ddot{x}\boldsymbol{e}_1^{\mathcal{M}} &= \boldsymbol{F}_{\mathrm{ext}} - 2m\omega\boldsymbol{e}_3^{\mathcal{M}} \times \dot{x}\boldsymbol{e}_1^{\mathcal{M}} - m\dot{\omega}\boldsymbol{e}_3^{\mathcal{M}} \times x\boldsymbol{e}_1^{\mathcal{M}} + m\omega^2 x\boldsymbol{e}_1^{\mathcal{M}} \\ &= \boldsymbol{F}_{\mathrm{ext}} - 2m\omega\dot{x}\boldsymbol{e}_2^{\mathcal{M}} - m\dot{\omega}x\boldsymbol{e}_2^{\mathcal{M}} + m\omega^2 x\boldsymbol{e}_1^{\mathcal{M}} \end{aligned} \tag{3.293}$$

Here, the resultant external force is the sum of gravity and the force exerted by the barrel onto the particle (which can only act perpendicular to the direction of motion for frictionless sliding of the particle), therefore we may write $\boldsymbol{F}_{\mathrm{ext}} = F_2\boldsymbol{e}_2^{\mathcal{M}} + (F_3 - mg)\boldsymbol{e}_3^{\mathcal{M}}$ with yet unknown F_2 and F_3 components. This allows us to decouple the above linear momentum balance into its components:

$$\begin{aligned} m\ddot{x} &= m\omega^2 x & & & \ddot{x} - \omega^2 x &= 0, \\ 0 &= F_2 - 2m\omega\dot{x} - m\dot{\omega}x & &\Rightarrow & m(2\omega\dot{x} + \dot{\omega}x) &= F_2, \\ 0 &= F_3 - mg & & & mg &= F_3. \end{aligned} \tag{3.294}$$

The first of these equations is the equation of motion to be solved for $x(t)$ with suitable initial conditions. For example, for a particle initially at rest at $x(0) = x_0$ at time $t = 0$ and if $\omega(t) = \omega = \text{const.}$, then the particle is accelerating outward exponentially according to

$$x(t) = \frac{x_0}{2}\left[\exp(\omega t) + \exp(-\omega t)\right]. \tag{3.295}$$

The second and third equations in (3.294) provide the components of the reaction force from the barrel onto the particle. For the example solution (3.295) with $\omega(t) = \omega = \text{const.}$ those become

$$F_2(t) = 2m\omega\dot{x}(t) = m\omega^2 x_0\left[\exp(\omega t) - \exp(-\omega t)\right] \qquad \text{and} \qquad F_3 = mg. \tag{3.296}$$

Note that we could also have obtained the reaction force F_2 from the balance of angular momentum around the origin O in an *inertial* system. We have

$$M_{\mathrm{O},3} = \dot{H}_{\mathrm{O}} \qquad \text{with} \qquad H_{\mathrm{O}} = I_{\mathrm{O}}\omega = mx^2\omega \quad \text{and} \quad M_{\mathrm{O},3} = F_2 x, \tag{3.297}$$

[13] Even though for a particle we do not need to work with the center of mass (CM), we use this notation $\boldsymbol{r}_{\mathrm{CM}}$ here for consistency with the above derivation.

so that we again obtain

$$F_2 x = \frac{\mathrm{d}}{\mathrm{d}t}\left(mx^2\omega\right) = 2mx\dot{x}\omega + mx^2\dot{\omega} \qquad \Rightarrow \qquad F_2 = 2m\dot{x}\omega + mx\dot{\omega}. \tag{3.298}$$

As a **variation**, consider the barrel being *inclined* by an angle α against the horizontal plane, so the particles moves upwards under an angle α in the barrel. In this case, we may want to introduce a new frame $\mathcal{R}$, which is obtained by rotating the $\mathcal{M}$-frame with $\boldsymbol{R}^{\mathcal{R}\mathcal{M}}$ about the $\boldsymbol{e}_2^{\mathcal{M}}$-axis by α, such that the $\boldsymbol{e}_1^{\mathcal{R}}$-axis aligns with the barrel. Here, the same derivations as above applies with

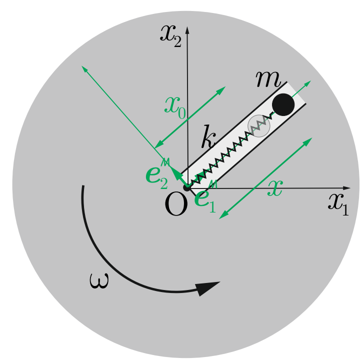
$$\boldsymbol{\Omega}^{\mathcal{R}} = \boldsymbol{\omega} \qquad \text{where, expressed in the } \mathcal{R}\text{-frame,} \quad \boldsymbol{\Omega}^{\mathcal{R}} = \omega \sin\alpha\, \boldsymbol{e}_1^{\mathcal{R}} + \omega\cos\alpha\, \boldsymbol{e}_2^{\mathcal{R}}. \tag{3.299}$$

With the external force $\boldsymbol{F}_{\text{ext}} = -mg(\sin\alpha \boldsymbol{e}_1^{\mathcal{R}} + \cos\alpha \boldsymbol{e}_2^{\mathcal{R}}) + F_2\boldsymbol{e}_2^{\mathcal{R}} + F_3\boldsymbol{e}_3^{\mathcal{R}}$, we now arrive at

$$m\ddot{x}\boldsymbol{e}_1^{\mathcal{R}} = \boldsymbol{F}_{\text{ext}} - 2m\omega\dot{x}\cos\alpha\boldsymbol{e}_2^{\mathcal{R}} - m\dot{\omega}x\cos\alpha\boldsymbol{e}_2^{\mathcal{R}} + m\omega^2\cos^2\alpha x\boldsymbol{e}_1^{\mathcal{R}}, \tag{3.300}$$

which admits analogous conclusions as the derivation above.

As a further **variation**, assume that the particle is attached to a spring of stiffness k, which is attached to the origin and unstretched in initial position x_0. Let us find the steady-state location of the particle under a constant angular velocity ω (where the apparent forces balance the spring force).



We align the spring with the rotating x_1-axis as before, so that the radial motion of the particle is again described *in the moving frame* by

$$\boldsymbol{r}_{\text{CM}}^{\mathcal{M}}(t) = x(t)\boldsymbol{e}_1^{\mathcal{M}}. \tag{3.301}$$

When assuming a steady-state solution with respect to the moving frame (i.e., $\mathring{\boldsymbol{r}}_{\text{CM}}^{\mathcal{M}} = \boldsymbol{0}$ and $\overset{\circ\circ}{\boldsymbol{r}}{}_{\text{CM}}^{\mathcal{M}} = \boldsymbol{0}$) and a constant angular velocity ($\dot{\omega} = 0$), the equations of linear momentum balance in the rotating frame simplify significantly. Linear momentum balance in the x_1-direction in the moving frame under these conditions gives

$$m\ddot{x}_1 = F_1 + m\omega^2 x = 0. \tag{3.302}$$

Since the force component F_1 is the resultant of the stretched spring, we have

$$F_1 = -k(x - x_0) \qquad \Rightarrow \qquad k(x - x_0) = m\omega^2 x \qquad \Leftrightarrow \qquad x = \frac{kx_0}{k - m\omega^2}. \tag{3.303}$$

Note that this solution only applies if $\omega \neq \sqrt{k/m}$. Under static conditions ($\omega = 0$), we recover the equilibrium solution $x = x_0$. When ω is increased from zero, the denominator decreases, so that the spring is increasingly stretched due to the centrifugal force.

Example 3.27. Throwing a particle on a merry-go-round

In Example 1.9 we discussed how to compute the flight trajectory of a particle that is only subjected to gravity. For example, if we throw a particle from the initial position $\boldsymbol{r}(0) = h_0\boldsymbol{e}_3$ with an initial velocity $\boldsymbol{v}(0) = v_0\boldsymbol{e}_1$ in the presence of a constant gravitational acceleration $\boldsymbol{g} = -g\boldsymbol{e}_3$, then we know the particle follows a parabola described by

$$\boldsymbol{r}(t) = \boldsymbol{r}(0) + \boldsymbol{v}(0)t + \frac{\boldsymbol{g}}{2}t^2 = \left(h_0 - \frac{gt^2}{2}\right)\boldsymbol{e}_3 + v_0 t\boldsymbol{e}_1. \tag{3.304}$$

Now, assume that the same particle is thrown with exactly the same conditions, but we are sitting on a rotating system that is rotating at a constant angular velocity Ω (such as on a merry-go-round), and observe the flight of the particle as a *rotating observer*. As shown above, linear momentum balance for the particle of mass m in the frame rotating with $\boldsymbol{\Omega}^{\mathcal{M}} = \Omega\boldsymbol{e}_3 =$ const. around a fixed origin O *in the moving frame* reads

$$m\begin{pmatrix}\ddot{y}_1\\ \ddot{y}_2\\ \ddot{y}_3\end{pmatrix} = \begin{pmatrix}F_1^{\text{ext}}\\ F_2^{\text{ext}}\\ F_3^{\text{ext}}\end{pmatrix} - 2m\Omega\begin{pmatrix}-\dot{y}_2\\ \dot{y}_1\\ 0\end{pmatrix} + m\Omega^2\begin{pmatrix}y_1\\ y_2\\ 0\end{pmatrix}, \tag{3.305}$$

where $F_i^{\text{ext}} = [F_i^{\text{ext}}]_{\mathcal{M}}$ are the external force components with respect to the rotating basis, and we again denote by y_i the ith component of the particle's position as observed by the rotating observer in the rotating $\mathcal{M}$-frame with basis $\left\{\boldsymbol{e}_1^{\mathcal{M}}(t), \boldsymbol{e}_2^{\mathcal{M}}(t), \boldsymbol{e}_3\right\}$.

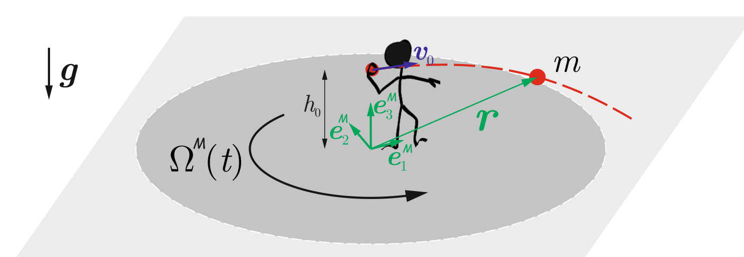
Note that the y_3-component in (3.305) is decoupled from the other two components (if the external forces do not depend on position, which we assume here). If the particle is thrown, as above, under the action of gravity as the only external force, then $[F_1^{\text{ext}}]_{\mathcal{M}} = [F_2^{\text{ext}}]_{\mathcal{M}} = 0$ and $[F_3]_{\mathcal{M}} = -mg$, so that the third component of (3.305) admits integration as before:

$$\ddot{y}_3(t) = -g \quad\Rightarrow\quad y_3(t) = -\frac{gt^2}{2} + \dot{y}_3(0)\,t + y_3(0) = h_0 - \frac{gt^2}{2}. \tag{3.306}$$

The y_1- and y_2-components must be obtained by solving a system of ordinary differential equations (ODEs), viz. the first two components of (3.305):

$$\ddot{y}_1 = 2\Omega\dot{y}_2 + \Omega^2 y_1, \qquad \ddot{y}_2 = -2\Omega\dot{y}_1 + \Omega^2 y_2. \tag{3.307}$$

The solution of the system of ODEs is nontrivial in general. Therefore, let us again consider the special case of a particle thrown with initial position $\boldsymbol{r}(0) = \boldsymbol{y}(0) = h_0\boldsymbol{e}_3$ with an initial velocity $\boldsymbol{v}(0) = \mathring{\boldsymbol{y}}(0) = v_0\boldsymbol{e}_1$.



The initial conditions for the motion of the particle in the plane are expressed in the rotating frame as

$$[\boldsymbol{y}(0)]_{\mathcal{M}} = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad [\mathring{\boldsymbol{y}}(0)]_{\mathcal{M}} = \begin{pmatrix} \dot{y}_1(0) \\ \dot{y}_2(0)) \end{pmatrix} = \begin{pmatrix} v_0 \\ 0 \end{pmatrix}. \tag{3.308}$$

As we can easily verify by insertion, (3.307) admits the following solution:

$$y_1(t) = v_0 t \cos(\Omega t), \qquad y_2(t) = -v_0 t \sin(\Omega t). \tag{3.309}$$

That is, the rotating observer does not see the particle flying along a straight line in the x_1-x_2-plane and only losing altitude, as was the case for an outside, inertial observer (cf. (3.304)). Instead the rotating observer watches the particle on a curved trajectory described by (3.309). Note that inserting the initial conditions into (3.306) further yields $y_3(t) = h_0 - \frac{1}{2}gt^2$.

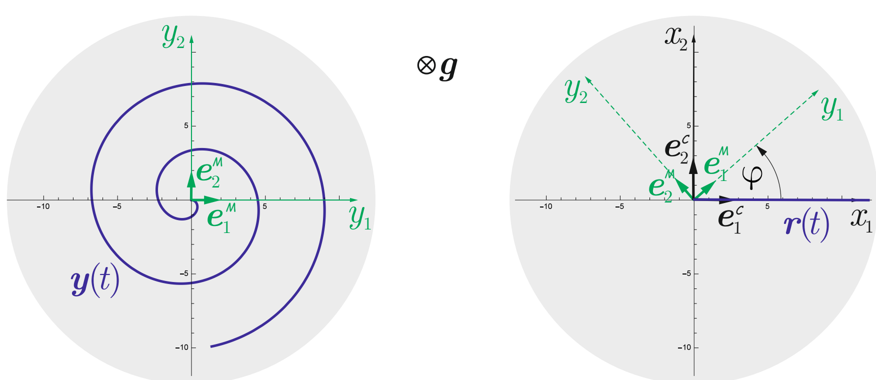
We can verify that those two descriptions agree by converting the *inertial* observer's trajectory (3.304) into the *rotating* observer's basis. To this end, we exploit that

$$\boldsymbol{e}_1^{\mathcal{C}} = \cos\varphi(t)\, \boldsymbol{e}_1^{\mathcal{M}}(t) - \sin\varphi(t)\, \boldsymbol{e}_2^{\mathcal{M}}(t), \qquad \boldsymbol{e}_3^{\mathcal{C}} = \boldsymbol{e}_3^{\mathcal{M}} \qquad \text{with} \quad \varphi(t) = \Omega t, \tag{3.310}$$

so that

$$\begin{aligned} \boldsymbol{r} &= \left(h_0 - \frac{gt^2}{2}\right) \boldsymbol{e}_3^{\mathcal{C}} + v_0 t \boldsymbol{e}_1^{\mathcal{C}} \\ &= \left(h_0 - \frac{gt^2}{2}\right) \boldsymbol{e}_3^{\mathcal{M}} + v_0 t \left[\cos(\Omega t)\boldsymbol{e}_1^{\mathcal{M}}(t) - \sin(\Omega t)\boldsymbol{e}_2^{\mathcal{M}}(t)\right] \\ &= \underbrace{\left(h_0 - \frac{gt^2}{2}\right)}_{y_3(t)} \boldsymbol{e}_3^{\mathcal{M}} + \underbrace{v_0 t \cos(\Omega t)}_{y_1(t)}\, \boldsymbol{e}_1^{\mathcal{M}}(t) \underbrace{-v_0 t \sin(\Omega t)}_{y_2(t)}\, \boldsymbol{e}_2^{\mathcal{M}}(t). \end{aligned} \tag{3.311}$$

The latter agrees with (3.309) and (3.306), so the solutions $\boldsymbol{y}(t)$ and $\boldsymbol{r}(t)$ are indeed identical – only described in two different reference frames. Shown below are the two solutions as seen in the rotating and inertial frames (plotted is the projection into the x_1-x_2- or y_1-y_2-plane as seen from above).



trajectory observed by a rotating observer   trajectory observed by an inertial observer

Let us also verify the apparent forces. Obviously, the *inertial* observer sees the flight trajectory (3.304) with

$$\boldsymbol{a}(t) = -\boldsymbol{g}, \tag{3.312}$$

so the observer concludes that

$$\boldsymbol{F} = m\boldsymbol{a} \qquad \Rightarrow \qquad \boldsymbol{F} = -m\boldsymbol{g}. \tag{3.313}$$

In other words, to the inertial observer the particle flies as if (correctly) only the gravitational force is acting on the particle.

By contrast, the *rotating* observer witnesses the curved trajectory (3.311) and measures the (relative) particle velocity

$$\begin{aligned}\mathring{\boldsymbol{y}}(t) &= \dot{y}_1(t)\boldsymbol{e}_1^{\mathcal{M}} + \dot{y}_2(t)\boldsymbol{e}_2^{\mathcal{M}} + \dot{y}_3(t)\boldsymbol{e}_3^{\mathcal{M}} \\ &= v_0\left[\cos\varphi(t)\boldsymbol{e}_1^{\mathcal{M}} - \sin\varphi(t)\boldsymbol{e}_2^{\mathcal{M}}\right] + v_0 t\left[-\dot{\varphi}\sin\varphi(t)\boldsymbol{e}_1^{\mathcal{M}} - \dot{\varphi}\cos\varphi(t)\boldsymbol{e}_2^{\mathcal{M}}\right] - gt\boldsymbol{e}_3^{\mathcal{M}} \\ &= v_0\left[\cos\varphi(t)\boldsymbol{e}_1^{\mathcal{M}} - \sin\varphi(t)\boldsymbol{e}_2^{\mathcal{M}}\right] - \Omega\, v_0 t\left[\sin\varphi(t)\boldsymbol{e}_1^{\mathcal{M}} + \cos\varphi(t)\boldsymbol{e}_2^{\mathcal{M}}\right] - gt\boldsymbol{e}_3^{\mathcal{M}},\end{aligned} \tag{3.314}$$

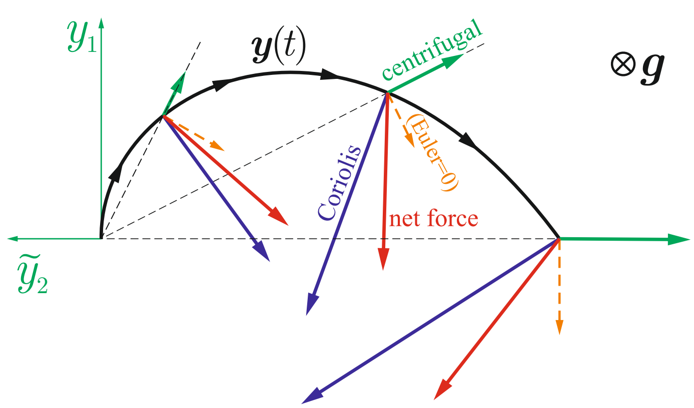
where we used $\varphi(t) = \Omega t$ and $\Omega = \dot{\varphi}$. Taking one more time derivative in the rotating frame and exploiting that $\dot{\Omega} = \ddot{\varphi} = 0$ yields

$$\begin{aligned}\mathring{\mathring{\boldsymbol{y}}}(t) &= v_0\Omega\left[-\sin\varphi\boldsymbol{e}_1^{\mathcal{M}} - \cos\varphi\boldsymbol{e}_2^{\mathcal{M}}\right] + v_0\Omega\left[-\sin\varphi\boldsymbol{e}_1^{\mathcal{M}} - \cos\varphi\boldsymbol{e}_2^{\mathcal{M}}\right] - \Omega^2 v_0 t\left[\cos\varphi\boldsymbol{e}_1^{\mathcal{M}} - \sin\varphi\boldsymbol{e}_2^{\mathcal{M}}\right] - g\boldsymbol{e}_3^{\mathcal{M}} \\ &= -2\Omega v_0\left[\sin\varphi\boldsymbol{e}_1^{\mathcal{M}} + \cos\varphi\boldsymbol{e}_2^{\mathcal{M}}\right] + \Omega^2 v_0 t\left[\cos\varphi\boldsymbol{e}_1^{\mathcal{M}} - \sin\varphi\boldsymbol{e}_2^{\mathcal{M}}\right] - 2\Omega^2 v_0 t\left[\cos\varphi\boldsymbol{e}_1^{\mathcal{M}} - \sin\varphi\boldsymbol{e}_2^{\mathcal{M}}\right] - g\boldsymbol{e}_3^{\mathcal{M}} \\ &= -2\Omega\big(v_0\left[\sin\varphi\boldsymbol{e}_1^{\mathcal{M}} + \cos\varphi\boldsymbol{e}_2^{\mathcal{M}}\right] + \Omega v_0 t\left[\cos\varphi\boldsymbol{e}_1^{\mathcal{M}} - \sin\varphi\boldsymbol{e}_2^{\mathcal{M}}\right]\big) + \Omega^2 v_0 t\left[\cos\varphi\boldsymbol{e}_1^{\mathcal{M}} - \sin\varphi\boldsymbol{e}_2^{\mathcal{M}}\right] - g\boldsymbol{e}_3^{\mathcal{M}} \\ &= -2\boldsymbol{\Omega}\times\mathring{\boldsymbol{y}} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{y}) + \boldsymbol{g}.\end{aligned}$$

Therefore, the co-rotating observer "sees" that the particle is accelerated as if it obeyed the linear momentum balance with fictitious forces included:

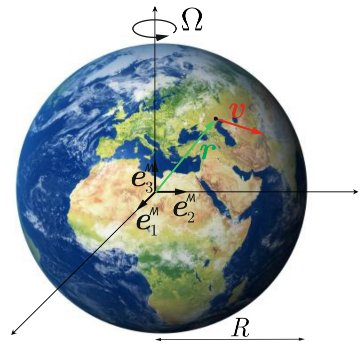
$$m\mathring{\mathring{\boldsymbol{y}}} = m\boldsymbol{g} - 2m\boldsymbol{\Omega}\times\mathring{\boldsymbol{y}} - m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{y}) = \boldsymbol{F}^{\text{ext}} + \boldsymbol{F}_{\text{Coriolis}} + \boldsymbol{F}_{\text{centrifugal}}. \tag{3.315}$$

The following schematic shows the apparent forces during the particle's flight in the plane, i.e., omitting the y_3-component (note that the Euler force vanishes since $\dot{\Omega} = 0$, but its hypothetical direction is included only for reference).



Example 3.28. Is the Earth an inertial frame?

Considering the Earth's rotation about its own axis, one may question if the Earth can be treated as an inertial frame of reference. Let us verify the influence of the apparent forces on particles moving on the Earth's surface. The Earth's radius and angular velocity are, respectively,



$$R \approx 6.4 \cdot 10^6\mathrm{m}, \qquad \Omega \approx \frac{2\pi}{24h} = \frac{2\pi}{24 \cdot 3600s} = 7.27 \cdot 10^{-5}\mathrm{s}^{-1}.$$

Considering the individual apparent forces, we conclude that:

- **Euler force**: since $\Omega = \text{const.}$, there is no Euler force: $\dot{\boldsymbol{\Omega}}^{\mathcal{M}} = 0$ so that $\boldsymbol{F}_{\text{Euler}} = \mathbf{0}$.
- **Coriolis force**: for a particle of mass m moving with velocity $\boldsymbol{v}$ on the Earth's surface:

$$|\boldsymbol{F}_{\text{Coriolis}}| = 2m|\boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{v}^{\mathcal{M}}| \leq 2m|\boldsymbol{\Omega}^{\mathcal{M}}|\,|\boldsymbol{v}^{\mathcal{M}}| \approx 14.58 \cdot 10^{-5}s^{-1} \cdot mv. \tag{3.316}$$

  Therefore, in most applications the Coriolis force only matters when m is large (e.g., ocean water) or if $|\boldsymbol{v}|$ is large (e.g., bullets, missiles). (It is for this reason that toilets flushing clockwise vs. counter-clockwise on the northern vs. southern hemisphere is only a myth.)

- **centrifugal force**: for a particle of mass m on the Earth's surface, we have

$$|\boldsymbol{F}_{\text{centrifugal}}| = m|\boldsymbol{\Omega}^{\mathcal{M}} \times (\boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{r}^{\mathcal{M}})| \leq m\Omega^2 R \approx 3.4 \cdot 10^{-2}\mathrm{m/s}^2 \cdot m. \tag{3.317}$$

  This force/acceleration, which is acting in radial direction (i.e., perpendicular to the Earth's axis of rotation into space) is essentially a correction to the gravitational acceleration. However, it is orders of magnitude smaller than gravity ($9.81\mathrm{m/s}^2$) and therefore usually negligible.

In conclusion, the Earth is close to an inertial frame for ordinary masses and velocities. However, inertial forces may become important in case of fast traveling objects (see Example 3.29) or of large masses moving on the Earth (see Example 3.30).

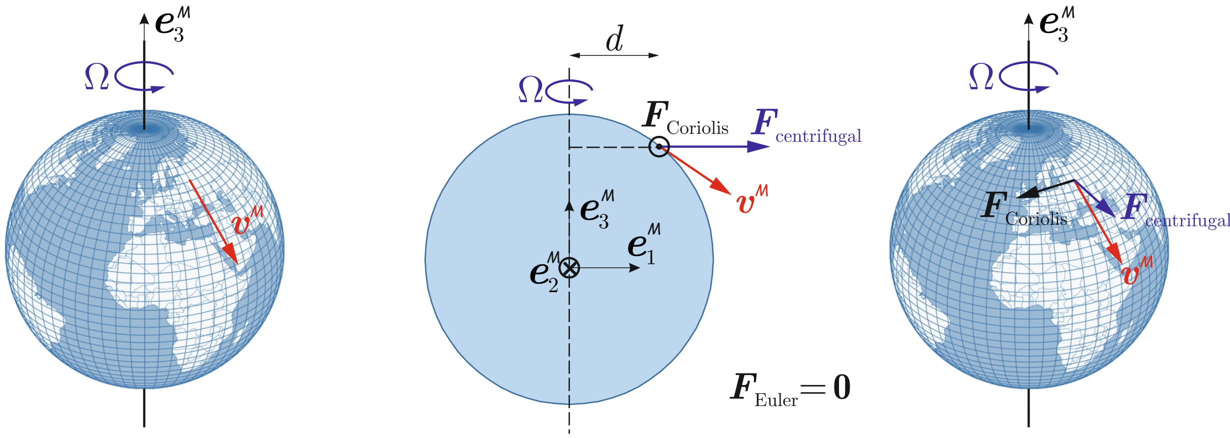
Example 3.29. Inertial forces on a fast traveling vehicle

Consider a vehicle (e.g., an airplane or a missile) traveling fast with velocity $\boldsymbol{v}^{\mathcal{M}}$ (measured relative to the Earth's surface) in straight Southern direction on the Northern hemisphere of the Earth, as shown below schematically. What are the inertial forces acting on the vehicle?

The Earth is rotating with $\boldsymbol{\Omega}^{\mathcal{M}} = \Omega \boldsymbol{e}_3$, so that the motion of the vehicle (on the rotating Earth in the co-rotating coordinate system shown below) results in inertial forces. The centrifugal force is

$$\boldsymbol{F}_{\text{centrifugal}} = -m\boldsymbol{\Omega}^{\mathcal{M}} \times (\boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{r}^{\mathcal{M}}) = -m\Omega\boldsymbol{e}_3^{\mathcal{M}} \times (\Omega\boldsymbol{e}_3^{\mathcal{M}} \times d\boldsymbol{e}_1^{\mathcal{M}}) = m\Omega^2 d\boldsymbol{e}_1^{\mathcal{M}}, \tag{3.318}$$

pointing radially away from the axis of rotation, as shown below.

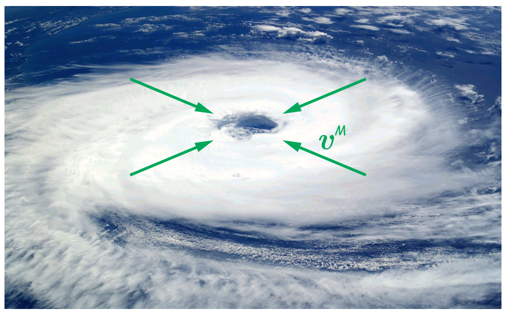


Finally, the Coriolis force evaluates to

$$\boldsymbol{F}_{\text{Coriolis}} = -2m\boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{v}^{\mathcal{M}} = -2m\Omega\boldsymbol{e}_3^{\mathcal{M}} \times \boldsymbol{v}^{\mathcal{M}} = -2m\Omega|\boldsymbol{e}_3^{\mathcal{M}} \times \boldsymbol{v}^{\mathcal{M}}|\boldsymbol{e}_2^{\mathcal{M}}, \tag{3.319}$$

perpendicular to both the axis of rotation and the relative velocity vector, as shown below. Finally, since $\boldsymbol{\Omega} = \text{const.}$, we know that $\boldsymbol{F}_{\text{Euler}} = \boldsymbol{0}$.

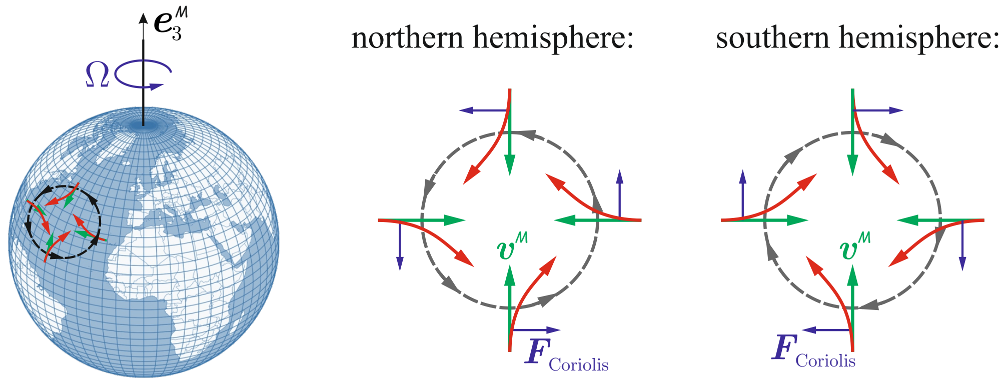
Example 3.30. The spinning of cyclones



A cyclone is a storm system whose spiral arrangement is rapidly rotating and produces thunderstorms with heavy rain and strong winds. (Depending on its location and strength, a (tropical) cyclone is also known as hurricane, typhoon, tropical or cyclonic storm.) The term "*hurricane*" usually refers to a tropical cyclone in the Atlantic or northeastern Pacific Ocean. Any cyclone is characterized by a low-pressure center (the "*eye*"), which leads to strong humid winds blowing towards the center.
This, in turn, leads to the migration of large masses of humid air towards the center of the cyclone in a radial fashion, as demonstrated above and below (see the shown velocity $\boldsymbol{v}^{\mathcal{M}}$).

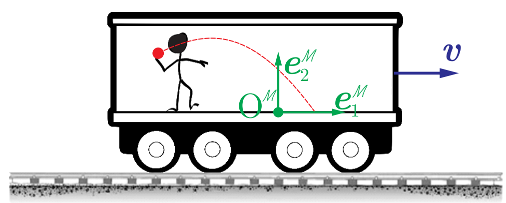
Let us follow the humid air masses dragged towards the center of the cyclone with some speed $\boldsymbol{v}^{\mathcal{M}}$ (measured relative to the Earth's surface which is rotating with the Earth at a constant angular velocity $\boldsymbol{\Omega}^{\mathcal{M}} = \Omega\boldsymbol{e}_3^{\mathcal{M}}$). On the northern hemisphere, the relative velocity $\boldsymbol{v}^{\mathcal{M}}$ along with the Earth's rotation $\boldsymbol{\Omega}^{\mathcal{M}}$ results in a Coriolis force that deflects the air masses in a counter-clockwise fashion (cf. the red arrows in the schematic below). This results in the shown counter-clockwise rotation of hurricanes on the northern hemisphere (such as those forming over the Atlantic and frequently moving towards the U.S. East Coast). By contrast, on the southern hemisphere (not illustrated here) the Coriolis force results in the opposite (clockwise) rotation of cyclones (such as those forming over the Pacific and threatening East Asia)[14].

[14] As pointed out before, toilets flushing clockwise vs. counter-clockwise on the northern vs. southern hemisphere is only a myth and not comparable, as the involved linear momentum mv is negligible.



Example 3.31. Aboard a moving train

Consider a passenger on a high-sped train (moving with a velocity $\boldsymbol{v}(t)$), and assume the passenger is observing a mechanical system; e.g., he is throwing a ball or watching a mechanical pendulum on the moving train. In this case, a convenient reference frame $\mathcal{M}$ is the one attached to the train. $\mathcal{M}$ is characterized by a moving coordinate origin $\mathrm{O}^{\mathcal{M}}$ (which moves at the same speed $\boldsymbol{v}(t)$ as the train) but its Cartesian basis does *not* rotate ($\boldsymbol{\Omega}^{\mathcal{M}} = \boldsymbol{0}$).



As a result, the balance of linear momentum (3.277) for the moving frame $\mathcal{M}$ reduces to

$$M\boldsymbol{a}_{\mathrm{CM}}^{\mathcal{M}} = \sum_i \boldsymbol{F}_i^{\mathrm{ext}} - M\boldsymbol{a}_{\mathrm{O}^{\mathcal{M}}}. \tag{3.320}$$

We realize that this equation is identical to that of an inertial, non-moving frame except for the last term involving the acceleration of the moving coordinate origin. Further, notice that (3.320) does not depend on the velocity $\boldsymbol{v}_{\mathrm{O}^{\mathcal{M}}} = \boldsymbol{v}(t)$ of the origin, but it only depends on the acceleration $\boldsymbol{a}_{\mathrm{O}^{\mathcal{M}}} = \dot{\boldsymbol{v}}(t)$ of the origin. Therefore, a train moving at *constant* speed $\boldsymbol{v}(t) = \mathrm{const.}$ (so $\boldsymbol{a}_{\mathrm{O}^{\mathcal{M}}} = \boldsymbol{0}$) is in fact an inertial frame: the same laws of mechanics apply as in a non-moving frame. By contrast, if the train's acceleration is non-zero $\boldsymbol{a}_{\mathrm{O}^{\mathcal{M}}} \neq \boldsymbol{0}$), it is a non-inertial frame and the last term in (3.320) may become significant.

For example, throwing a particle of mass m on the moving train is governed by linear momentum balance $m\boldsymbol{a}_{\mathrm{CM}}^{\mathcal{M}} = m(\boldsymbol{g} - \boldsymbol{a}_{\mathrm{O}^{\mathcal{M}}})$, so the particle's flight trajectory – as viewed from within the train – is affected by the train's acceleration. Specifically, $\boldsymbol{a}_{\mathrm{CM}}^{\mathcal{M}} = \boldsymbol{g} - \boldsymbol{a}_{\mathrm{O}^{\mathcal{M}}}$ implies that the particle's apparent acceleration changes in the direction of the train's movement. Recall that, of course, the particle obeys the same physical laws on the train as on an inertial ground. The flight parabola of the particle is exactly the same; it is just that the moving observer on the train *sees* a different flight path when the train is accelerating or decelerating.

3.4.4 Balance of angular momentum, Euler equations

Having discussed linear momentum balance, let us proceed to derive the analogous relation for the balance of angular momentum for a rigid body in a moving frame $\mathcal{M}$ in 3D. As discussed previously, relation (3.259) for time derivatives in a moving frame applies to all vectors, so it specifically also applies to the angular momentum vector:

$$\dot{\boldsymbol{H}}_\mathrm{B}^\mathcal{C} = \mathring{\boldsymbol{H}}_\mathrm{B}^\mathcal{M} + \boldsymbol{\Omega}^\mathcal{M} \times \boldsymbol{H}_\mathrm{B} \qquad \text{with} \qquad \boldsymbol{H}_\mathrm{B} = \boldsymbol{I}_\mathrm{B}\boldsymbol{\omega} + M(\boldsymbol{r}_\mathrm{CM} - \boldsymbol{r}_\mathrm{B}) \times \boldsymbol{v}_\mathrm{B} \tag{3.321}$$

If we choose point B such that B = CM or $\boldsymbol{v}_\mathrm{B} = \boldsymbol{0}$, then

$$\boldsymbol{H}_\mathrm{B} = \boldsymbol{I}_\mathrm{B}\boldsymbol{\omega} \qquad \Rightarrow \qquad \dot{\boldsymbol{H}}_\mathrm{B}^\mathcal{C} = (\boldsymbol{I}_\mathrm{B}\boldsymbol{\omega})^{\circ\mathcal{M}} + \boldsymbol{\Omega}^\mathcal{M} \times \boldsymbol{I}_\mathrm{B}\boldsymbol{\omega} \tag{3.322}$$

Note that $\boldsymbol{\omega}$ (the angular velocity vector of the rigid body) and $\boldsymbol{\Omega}^\mathcal{M}$ (the angular velocity vector of the chosen rotating coordinate system $\mathcal{M}$) do not have to be identical. We will see in the following that it is convenient to choose $\boldsymbol{\Omega}^\mathcal{M} = \boldsymbol{\omega}$ but this is *not* a requirement for those relations derived here to hold (and sometimes we may indeed want to choose $\boldsymbol{\Omega}^\mathcal{M} \neq \boldsymbol{\omega}$, e.g., if multiple rotations are involved).

The balance of angular momentum for a rigid body, formulated *in the moving frame* $\mathcal{M}$, thus follows as

$$\boxed{\boldsymbol{M}_\mathrm{B} = (\boldsymbol{I}_\mathrm{B}\boldsymbol{\omega})^{\circ\mathcal{M}} + \boldsymbol{\Omega}^\mathcal{M} \times \boldsymbol{I}_\mathrm{B}\boldsymbol{\omega} \qquad \text{if} \qquad \mathrm{B} = \mathrm{CM} \quad \text{or} \quad \boldsymbol{v}_\mathrm{B} = \boldsymbol{0}} \tag{3.323}$$

If the *$\mathcal{M}$-frame rotates with the rigid body*, i.e., if $\boldsymbol{\Omega}^\mathcal{M} = \boldsymbol{\omega}$, then

$$\boldsymbol{M}_\mathrm{B} = (\boldsymbol{I}_\mathrm{B}\boldsymbol{\omega})^{\circ\mathcal{M}} + \boldsymbol{\omega} \times \boldsymbol{I}_\mathrm{B}\boldsymbol{\omega} = \boldsymbol{I}_\mathrm{B}\mathring{\boldsymbol{\omega}}^\mathcal{M} + \boldsymbol{\omega} \times \boldsymbol{I}_\mathrm{B}\boldsymbol{\omega}, \tag{3.324}$$

where we exploited that $\mathring{\boldsymbol{I}}_\mathrm{B}^\mathcal{M} = \boldsymbol{0}$, since we compute the time derivative of the rigid body's moment of inertia tensor in its co-rotating reference frame (i.e., the coordinate system is rotating with the body, so the moment of inertia tensor with respect to that reference frame is time-invariant).

In order to further simplify the last term in (3.324), it is wise to *align the basis of* $\mathcal{M}$ *with the principal directions of the rotating body* $\mathcal{B}$ (which we denote by the principal frame $\hat{\mathcal{M}}$), so that the moment of inertia tensor in this frame is diagonal:

$$[\boldsymbol{I}_\mathrm{B}]_{\hat{\mathcal{M}}} = \begin{pmatrix} \hat{I}_1 & 0 & 0 \\ 0 & \hat{I}_2 & 0 \\ 0 & 0 & \hat{I}_3 \end{pmatrix} \quad \Rightarrow \quad [\boldsymbol{\omega} \times \boldsymbol{I}_\mathrm{B}\boldsymbol{\omega}]_{\hat{\mathcal{M}}} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} \hat{I}_1\omega_1 \\ \hat{I}_2\omega_2 \\ \hat{I}_3\omega_3 \end{pmatrix} = \begin{pmatrix} (\hat{I}_3 - \hat{I}_2)\omega_3\omega_2 \\ (\hat{I}_1 - \hat{I}_3)\omega_1\omega_3 \\ (\hat{I}_2 - \hat{I}_1)\omega_2\omega_1 \end{pmatrix}. \tag{3.325}$$

Therefore, if $\hat{\mathcal{M}}$ aligns with the principal axes of $\mathcal{B}$, then *in the $\hat{\mathcal{M}}$-frame*

$$\boxed{\left.\begin{aligned} \hat{I}_1\dot{\omega}_1 + (\hat{I}_3 - \hat{I}_2)\omega_3\omega_2 &= [M_{\mathrm{B},1}]_{\hat{\mathcal{M}}} \\ \hat{I}_2\dot{\omega}_2 + (\hat{I}_1 - \hat{I}_3)\omega_1\omega_3 &= [M_{\mathrm{B},2}]_{\hat{\mathcal{M}}} \\ \hat{I}_3\dot{\omega}_3 + (\hat{I}_2 - \hat{I}_1)\omega_2\omega_1 &= [M_{\mathrm{B},3}]_{\hat{\mathcal{M}}} \end{aligned}\right\} \quad \text{if} \quad \boldsymbol{\Omega}^\mathcal{M} = \boldsymbol{\omega}, \quad \text{and} \quad \begin{aligned} \omega_i &= [\omega_i]_{\hat{\mathcal{M}}} \\ \dot{\omega}_i &= [\mathring{\omega}_i]_{\hat{\mathcal{M}}} \end{aligned}} \tag{3.326}$$

Equations (3.326) are known as **Euler's equations** of motion, valid in the so-called **body frame** and for an arbitrarily-shaped body.

It is important to recall that in (3.326) **all quantities are measured with respect to the moving frame**: $[M_{\mathrm{B},i}]_{\hat{\mathcal{M}}}$ is the resultant torques around $\boldsymbol{e}_i^{\hat{\mathcal{M}}}$ with respect to point B measured in the $\hat{\mathcal{M}}$-frame, $\hat{I}_i$ is the ith eigenvalue of $\boldsymbol{I}_\mathrm{B}$ in the $\hat{\mathcal{M}}$-frame, and ω_i are the components $[\boldsymbol{\omega}]_{\hat{\mathcal{M}}}$ of the angular velocity vector measured in the moving frame $\hat{\mathcal{M}}$. This particular frame $\hat{\mathcal{M}}$, which aligns with the principal axes of body $\mathcal{B}$ and rotates with the same angular velocity ($\boldsymbol{\Omega}^{\mathcal{M}} = \boldsymbol{\omega}$) is also referred to as the **body frame**.

Let us point out a *common misconception*. We introduced the following notation to compute time derivatives with respect to moving and non-moving frames, which also applies to $\boldsymbol{\omega}$:

$$\boldsymbol{\omega}(t) = \sum_{i=1}^{3} \omega_i(t) \boldsymbol{e}_i^{\mathcal{M}}(t) \qquad \Rightarrow \qquad \dot{\boldsymbol{\omega}} = \sum_{i=1}^{3} \left(\dot{\omega}_i \boldsymbol{e}_i^{\mathcal{M}} + \omega_i \dot{\boldsymbol{e}}_i^{\mathcal{M}} \right), \qquad \mathring{\boldsymbol{\omega}}^{\mathcal{M}} = \sum_{i=1}^{3} \dot{\omega}_i \boldsymbol{e}_i^{\mathcal{M}}. \tag{3.327}$$

While $\dot{\boldsymbol{\omega}}$ represents the absolute angular velocity (with respect to an inertial frame), $\mathring{\boldsymbol{\omega}}^{\mathcal{M}}$ is the angular velocity measured with respect to the moving frame. The latter is the one which enters the Euler equations (3.326), and its components are $[\mathring{\omega}_i]_{\hat{\mathcal{M}}} = \dot{\omega}_i$. It is important to realize that, even though $\boldsymbol{\Omega}^{\mathcal{M}} = \boldsymbol{\omega}$ (so the $\mathcal{M}$-frame is rotating with the body), the time derivative $\mathring{\boldsymbol{\omega}}$ is not necessarily zero. When we say "*measured in the moving frame*" or "*measured with respect to the moving frame*", we mean a quantity measured with respect to the moving coordinate basis. For example, an object *fixed* on a rotating disk has a non-zero velocity $\boldsymbol{v} = \dot{\boldsymbol{x}}$ with respect to an inertial frame, but it has a vanishing velocity $\mathring{\boldsymbol{v}}$ with respect to the corotating coordinate system $\mathcal{M}$ (an observer sitting on the disk will not see any (relative) motion of the other points on the disk). However, even though the observer is rotating with $\mathcal{M}$, they will still feel a change in the angular velocity. Therefore, even with respect to the corotating observer, the angular velocity of the disk is changing and $\mathring{\boldsymbol{\omega}} \neq \boldsymbol{0}$ unless $\boldsymbol{\omega} = \text{const}$.

Special case: axisymmetric bodies

In various problems, we will be dealing with rotating *axisymmetric* bodies. These present a special case because the moment of inertia does not change, if the body is spinning about its axis of rotational symmetry. Therefore, one may introduce a moving frame $\mathcal{B}$ which aligns with the principal axes of the body but which does not spin with the body (so $\boldsymbol{\Omega}^{\mathcal{B}} \neq \boldsymbol{\omega}$). In this scenario, we cannot use Euler's equations directly but instead use (3.323), which here becomes

$$\boldsymbol{M}_\mathrm{O} = (\boldsymbol{I}_\mathrm{O}\boldsymbol{\omega})^{\circ\mathcal{B}} + \boldsymbol{\Omega}^{\mathcal{B}} \times \boldsymbol{I}_\mathrm{O}\boldsymbol{\omega} = \boldsymbol{I}_\mathrm{O}\mathring{\boldsymbol{\omega}} + \boldsymbol{\Omega}^{\mathcal{B}} \times \boldsymbol{I}_\mathrm{O}\boldsymbol{\omega}, \tag{3.328}$$

where we exploited that both $\boldsymbol{I}_\mathrm{O}$ is time-invariant in the $\mathcal{B}$-frame by axisymmetry (so $\mathring{\boldsymbol{I}}_\mathrm{O}^{\mathcal{B}} = \boldsymbol{0}$). In components we have

$$[\boldsymbol{\Omega}^{\mathcal{B}} \times \boldsymbol{I}_\mathrm{O}\boldsymbol{\omega}]_{\mathcal{B}} = \begin{pmatrix} \Omega_1^{\mathcal{B}} \\ \Omega_2^{\mathcal{B}} \\ \Omega_3^{\mathcal{B}} \end{pmatrix} \times \begin{pmatrix} \hat{I}_1 & & \\ & \hat{I}_2 & \\ & & \hat{I}_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \hat{I}_3 \Omega_2^{\mathcal{B}} \omega_3 - \hat{I}_2 \Omega_3^{\mathcal{B}} \omega_2 \\ \hat{I}_1 \Omega_3^{\mathcal{B}} \omega_1 - \hat{I}_3 \Omega_1^{\mathcal{B}} \omega_3 \\ \hat{I}_2 \Omega_1^{\mathcal{B}} \omega_2 - \hat{I}_1 \Omega_2^{\mathcal{B}} \omega_1 \end{pmatrix}, \tag{3.329}$$

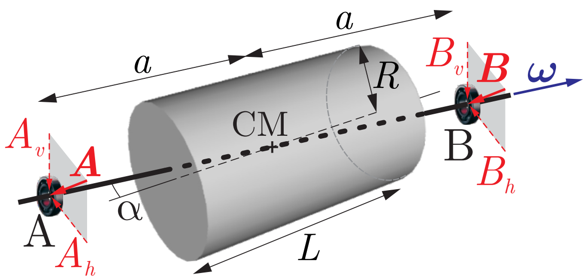
so that the balance of angular momentum in the $\mathcal{B}$-frame of the *axisymmetric body* becomes

$$\boxed{\left.\begin{aligned} \hat{I}_1\dot{\omega}_1 + \hat{I}_3\Omega_2^{\mathcal{B}}\omega_3 - \hat{I}_2\Omega_3^{\mathcal{B}}\omega_2 &= [M_{\mathrm{O},1}]_{\mathcal{B}} \\ \hat{I}_2\dot{\omega}_2 + \hat{I}_1\Omega_3^{\mathcal{B}}\omega_1 - \hat{I}_3\Omega_1^{\mathcal{B}}\omega_3 &= [M_{\mathrm{O},2}]_{\mathcal{B}} \\ \hat{I}_3\dot{\omega}_3 + \hat{I}_2\Omega_1^{\mathcal{B}}\omega_2 - \hat{I}_1\Omega_2^{\mathcal{B}}\omega_1 &= [M_{\mathrm{O},3}]_{\mathcal{B}} \end{aligned}\right\} \quad \text{for} \quad \begin{aligned} \omega_i &= [\omega_i]_{\mathcal{B}} \\ \dot{\omega}_i &= [\mathring{\omega}_i]_{\mathcal{B}} \end{aligned}} \tag{3.330}$$

These equations apply if the $\mathcal{B}$-frame aligns with the principal axes of an *axisymmetric* body but does not necessarily spin with the body about its axis of axisymmetry. If $\boldsymbol{\Omega}^{\mathcal{B}} = \boldsymbol{\omega}$, then $\mathcal{B} = \hat{\mathcal{M}}$ and (3.330) reduces to Euler's equations (3.326). This special case will become important when discussing the dynamics of a **spinning top** in Section 3.5.

Example 3.32. Poorly aligned rotating shaft

A cylindrical mechanical shaft of length L and radius R is supposed to rotate with a constant angular velocity ω about its longitudinal axis, supported by two floating bearings. Unfortunately, the shaft and the bearings were poorly aligned, so that the shaft's symmetry axis is offset by an angle α from the axis of rotation, as shown on the right. What are the resultant forces acting on the two bearings at locations A and B?



Since the floating bearings allow for axial motion, they do not support any axial forces (along the direction of the axis of rotation). Therefore, the forces $\boldsymbol{A}$ and $\boldsymbol{B}$ onto the bearings can only lie in the plane perpendicular to the axis of rotation (as indicated by the shown horizontal and vertical components). Since the center of mass of the cylindrical shaft can move only along the axis of rotation, its acceleration is of the form $\boldsymbol{a} = \boldsymbol{a}_{\text{CM},||}$ with $\boldsymbol{a}_{\text{CM},||} || \boldsymbol{\omega}$. Because there are no other external forces, linear momentum balance tells that

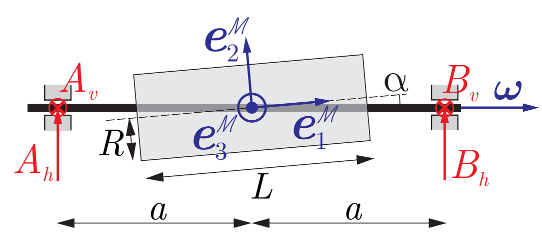
$$\boldsymbol{A} + \boldsymbol{B} = M\boldsymbol{a}_{\text{CM},||} \quad \text{but} \quad \boldsymbol{A}, \boldsymbol{B} \perp \boldsymbol{a}_{\text{CM},||} \quad \Rightarrow \quad \boldsymbol{A} = -\boldsymbol{B} \quad \text{and} \quad \boldsymbol{a} = \boldsymbol{a}_{\text{CM},||} = \boldsymbol{0}. \tag{3.331}$$

That is, forces $\boldsymbol{A}$ and $\boldsymbol{B}$ are of the same magnitude and opposite orientation, so

$$A_h = -B_h, \qquad A_v = -B_v, \tag{3.332}$$

and the (linear) acceleration of the cylinder is zero. With this we have exhausted linear momentum balance and turn to angular momentum balance to find the force magnitude(s).

Let us choose a rotating frame $\mathcal{M}$ that is aligned with the principal directions of the cylindrical shaft, specifically with the $\boldsymbol{e}_1^{\mathcal{M}}$-axis aligned with the longitudinal axis of the cylinder, as sketched. We here show a projection of the cylinder in the plane of angle α (i.e., a top view of the above 3D drawing).



In this frame, the forces in the bearings are assumed to point in the shown directions; i.e., they rotate with the body and the frame (so they always point in the shown direction relative to the tilt axis). The moment of inertia tensor is diagonal and the components are those from Example 3.7, viz.

$$\hat{I}_1 = \frac{MR^2}{2}, \qquad \hat{I}_2 = \hat{I}_3 = \frac{M}{12}\left(L^2 + 3R^2\right). \tag{3.333}$$

Also expressed in the moving $\mathcal{M}$-frame, the angular velocity vector $\boldsymbol{\omega}$ is expressed as

$$\boldsymbol{\omega} = \omega \cos\alpha \, \boldsymbol{e}_1^{\mathcal{M}} - \omega \sin\alpha \, \boldsymbol{e}_2^{\mathcal{M}} \qquad \Rightarrow \qquad \omega_1 = \omega\cos\alpha, \quad \omega_2 = -\omega\sin\alpha, \quad \omega_3 = 0. \tag{3.334}$$

To compute the reaction forces in the bearings, we turn to Euler's equations *in the $\mathcal{M}$-frame* (which is a *body frame* since it aligns with the principal axes and rotates with the body):

$$\begin{aligned}\hat{I}_1\dot{\omega}_1 + (\hat{I}_3 - \hat{I}_2)\omega_3\omega_2 &= M_{\mathrm{B},1}\\ \hat{I}_2\dot{\omega}_2 + (\hat{I}_1 - \hat{I}_3)\omega_1\omega_3 &= M_{\mathrm{B},2}\\ \hat{I}_3\dot{\omega}_3 + (\hat{I}_2 - \hat{I}_1)\omega_2\omega_1 &= M_{\mathrm{B},3},\end{aligned} \tag{3.335}$$

In the corotating frame $\omega_1, \omega_2, \omega_3$ from (3.334) are constant, so $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$. Further using $\hat{I}_2 = \hat{I}_3$ from (3.333), Euler's equations reduce to

$$M_{\mathrm{B},1} = 0, \qquad M_{\mathrm{B},2} = 0, \qquad M_{\mathrm{B},3} = (\hat{I}_2 - \hat{I}_1)\omega_2\omega_1. \tag{3.336}$$

The external torque components with respect to the center of mass follow from the forces onto the bearings, shown above:

$$M_{\mathrm{B},1} = -A_v a\sin\alpha + B_v a\sin\alpha, \qquad M_{\mathrm{B},2} = -A_v a\cos\alpha + B_v a\cos\alpha, \qquad M_{\mathrm{B},3} = -A_h a + B_h a. \tag{3.337}$$

Combining (3.337) with (3.332) immediately shows that

$$A_v = B_v = 0, \qquad M_{\mathrm{B},3} = 2B_h a, \tag{3.338}$$

i.e., the forces onto the bearings act only in the plane of misalignment angle α.

Finally, inserting the external moment $M_{\mathrm{B},3}$ from (3.338) into the third of Euler's equations yields

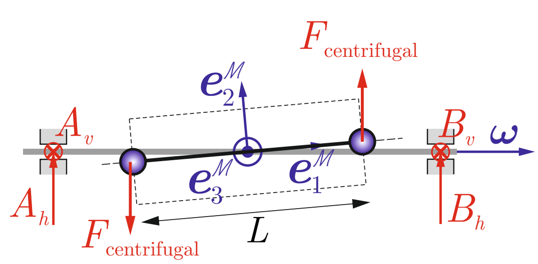
$$M_{\mathrm{B},3} = 2B_h a = (\hat{I}_2 - \hat{I}_1)\omega_2\omega_1 = -\frac{M}{12}\left(L^2 - 3R^2\right)\omega^2\sin\alpha\cos\alpha. \tag{3.339}$$

and therefore (with $2\sin\alpha\,\cos\alpha = \sin 2\alpha$)

$$B_h = -A_h = -\frac{M}{24a}\left(L^2 - 3R^2\right)\omega^2\sin\alpha\cos\alpha \quad\Rightarrow\quad |\boldsymbol{A}| = |\boldsymbol{B}| = \frac{M}{48a}\left(L^2 - 3R^2\right)\omega^2\sin 2\alpha. \tag{3.340}$$

This shows that the misalignment of the shaft results in reaction forces in the bearings that grow with misalignment angle α and *quadratically* with the angular velocity ω. Since significant forces can arise in this fashion, proper alignment of the shaft is essential. Note that these reaction forces $\boldsymbol{A}$ and $\boldsymbol{B}$ *rotate with the shaft* at angular velocity ω, as they were defined in the rotating $\mathcal{M}$-frame.

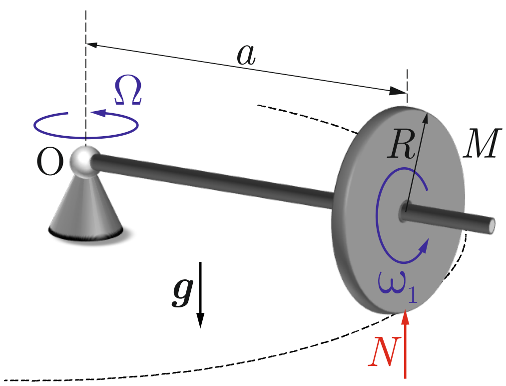
But **where do those reaction forces come from**? Simply put, the misalignment of the shaft leads to pairs of centrifugal forces acting in opposite directions. As an intuitive example, imagine the misaligned shaft as two particles, as shown on the right. The centrifugal forces onto the two particles are equal, so no net force arises.

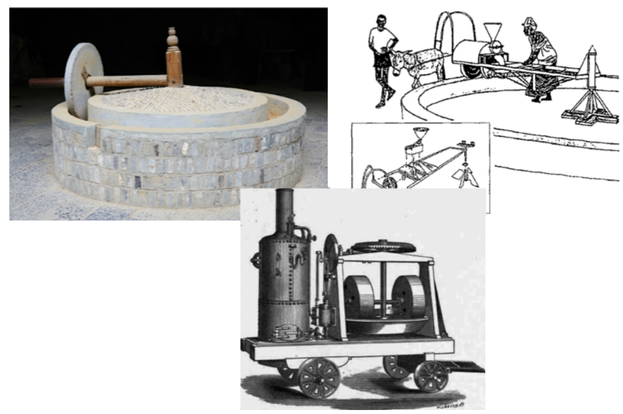


However, the misalignment leads to a torque with respect to the center of mass, and that torque must be balanced by the reaction forces. (Notice that no such torque would exist, if the shaft was perfectly lined.) As discussed before, the extra terms in the Euler equations stem from inertial forces. In fact, here the centrifugal forces are responsible.

Example 3.33. Pan mill ("Kollergang")

A pan mill (*Kollergang* in German) is a circular wheel (radius R, mass M) whose axis (length a, assumed massless) makes it rotate around a fixed pole, as shown below. For simplicity, let us assume that the wheel rolls without slipping and that it rotates with a constant angular velocity Ω around the fixed hinge or pole. What is the reaction force N from the ground onto the wheel?





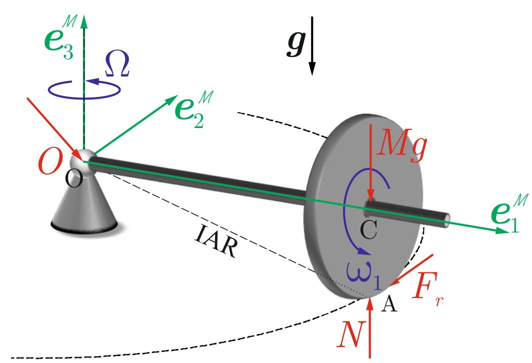
In a static problem, the answer would simply be $N = Mg$. Here, however, dynamic effects contribute. For convenience, let us choose a co-rotating basis $\{\boldsymbol{e}_1^{\mathcal{M}}, \boldsymbol{e}_2^{\mathcal{M}}, \boldsymbol{e}_3^{\mathcal{M}}\}$ whose origin $\mathrm{O}^{\mathcal{M}}$ is fixed and coincides with the inertial frame's origin O and whose coordinate axes align with the symmetry axes, as shown below (with $\boldsymbol{e}_3^{\mathcal{M}} = \boldsymbol{e}_3^{\mathcal{C}}$ normal to the plane of rotation). We formulate all governing equations in the moving frame $\mathcal{M}$ which rotates with the body (with Ω) around the fixed pole.

From Example 3.3 we know the instantaneous axis of rotation is OA, so that we must have $\boldsymbol{\omega} || \boldsymbol{r}_{\mathrm{OA}}$, and we know its component $[\omega_3]_{\mathcal{M}} = [\omega_3]_{\mathcal{C}} = \Omega$ (since $\boldsymbol{e}_3^{\mathcal{M}} = \boldsymbol{e}_3^{\mathcal{C}}$). In the co-rotating frame, we thus have

$$\boldsymbol{\omega} = \Omega \boldsymbol{e}_3^{\mathcal{M}} + \omega_1 \boldsymbol{e}_1^{\mathcal{M}}. \tag{3.341}$$

Angular velocity component ω_1 is obtained from the kinematic constraint between the two rotations (accounting for the proper signs by inspecting the rotation directions):

$$\frac{\Omega}{\omega_1} = -\frac{R}{a} \qquad \Rightarrow \qquad \boldsymbol{\omega} = \Omega \boldsymbol{e}_3^{\mathcal{M}} - \frac{\Omega a}{R} \boldsymbol{e}_1^{\mathcal{M}}. \tag{3.342}$$



The kinematics of the problem is now sufficiently described by the known constant angular velocity vector $\boldsymbol{\omega}$. To compute the normal force N, we need a kinetic balance equation. Linear momentum balance introduces unnecessary reaction forces (e.g., the unknown reaction force $\boldsymbol{O}$ in the hinge in the above sketch as well as the friction force F_r between the wheel and the ground), while angular momentum balance around O can avoid the forces in the hinge. O is also fixed ($\boldsymbol{v}_O = \boldsymbol{0}$), which avoids complicating terms in the angular momentum balance.

In the $\mathcal{M}$-frame, the coordinate axes align with the symmetry axes of the cylinder, so we know, using Steiner's theorem, that the moment of inertia tensor of the wheel has components

$$[\boldsymbol{I}_\text{O}]_\mathcal{M} = [\boldsymbol{I}_\text{C}]_\mathcal{M} + M \begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix} = \begin{pmatrix} \frac{MR^2}{2} & 0 & 0 \\ 0 & \frac{M}{12}(L^2+3R^2)+Ma^2 & 0 \\ 0 & 0 & \frac{M}{12}(L^2+3R^2)+Ma^2 \end{pmatrix}, \tag{3.343}$$

where we reused the components of $\boldsymbol{I}_\text{C}$ computed in Example 3.7. Since $\boldsymbol{I}_\text{O}$ is diagonal in the chosen frame, we conclude that $\mathcal{M}$ is a principal frame of the rotating disk (i.e., the coordinate axis coincide with the principal axes of the disk).

However, it is important to realize that the chosen $\mathcal{M}$-frame (which rotates about the $\boldsymbol{e}_3^\mathcal{C}$-axis with a fixed $\boldsymbol{e}_3^\mathcal{M} = \boldsymbol{e}_3^\mathcal{C}$) does *not* rotate with the disk; specifically, we have

$$\boldsymbol{\Omega}^\mathcal{M} = \Omega \boldsymbol{e}_3^\mathcal{M} \qquad \text{whereas} \qquad \boldsymbol{\omega} = -\frac{\Omega a}{R}\boldsymbol{e}_1^\mathcal{M} + \Omega \boldsymbol{e}_3^\mathcal{M}. \tag{3.344}$$

Therefore, we unfortunately cannot use Euler's equations but need to return to their original form (3.323), which here becomes

$$\boldsymbol{M}_\text{O} = (\boldsymbol{I}_\text{O}\boldsymbol{\omega})^{\circ\mathcal{M}} + \boldsymbol{\Omega}^\mathcal{M} \times \boldsymbol{I}_\text{O}\boldsymbol{\omega} = \boldsymbol{\Omega}^\mathcal{M} \times \boldsymbol{I}_\text{O}\boldsymbol{\omega}, \tag{3.345}$$

where we used that $\boldsymbol{\omega} = \text{const.}$ and $\boldsymbol{I}_\text{O} = \text{const.}$ in the chosen frame. Due to the diagonal form of $\boldsymbol{I}_\text{O}$ in the chosen $\mathcal{M}$-frame and with $[\omega_2]_\mathcal{M} = 0$, this balance law of angular momentum reduces to, completely expressed in the $\mathcal{M}$-frame,

$$\begin{pmatrix} M_{\text{O},1} \\ M_{\text{O},2} \\ M_{\text{O},3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix} \times \begin{pmatrix} \hat{I}_{\text{O},11} & 0 & 0 \\ 0 & \hat{I}_{\text{O},22} & 0 \\ 0 & 0 & \hat{I}_{\text{O},33} \end{pmatrix} \begin{pmatrix} -\Omega\frac{a}{R} \\ 0 \\ \Omega \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{I}_{\text{O},11}\Omega^2\frac{a}{R} \\ 0 \end{pmatrix} \tag{3.346}$$

or

$$\boldsymbol{M}_\text{O} = \Omega \boldsymbol{e}_3 \times \left[I_{\text{O},11}(-\Omega\tfrac{a}{R})\boldsymbol{e}_1^\mathcal{M} + I_{\text{O},33}\Omega \boldsymbol{e}_3^\mathcal{M} \right] = -I_{\text{O},11}\Omega^2 \frac{a}{R}\boldsymbol{e}_2^\mathcal{M}. \tag{3.347}$$

For the torques with respect to point O, we may neglect the reaction force $\boldsymbol{O}$ at the hinge, but we must account for the torques due to the force N from the ground, gravity acting on the wheel, and due to friction force F_r acting from the ground onto the wheel, which may be required to ensure the rolling without slipping condition. From the above sketch we obtain

$$\boldsymbol{M}_\text{O} = -F_r R \boldsymbol{e}_1^\mathcal{M} + (Mg - N)a\, \boldsymbol{e}_2^\mathcal{M} - F_r a \boldsymbol{e}_3^\mathcal{M}, \tag{3.348}$$

Insertion into (3.347) lets us conclude that $F_r = 0$. Note that we assume $\Omega = \text{const.}$, which is why no applied external torque is needed to keep the (conservative) system going; to conserve energy, the wheel will keep rolling at constant angular velocity.

We hence obtain from (3.347) that

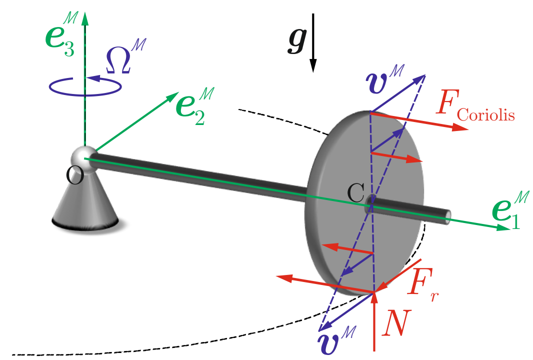
$$-I_{O,11}\Omega^2\frac{a}{R} = (Mg - N)a \qquad \Leftrightarrow \qquad -\frac{MR}{2}\Omega^2 = Mg - N. \tag{3.349}$$

The reaction force from the ground follows as

$$N = Mg + \frac{MR\Omega^2}{2}. \tag{3.350}$$

Therefore, we conclude that the reaction force N is larger than that in the static case ($N > Mg$). In fact, the extra force given by the second term (which grows quadratically with the angular velocity Ω) helps the milling process and is the rationale behind the pan mill design.

Finally, let us ask: **where does the normal force come from**? As discussed before, it is inertial forces that cause the extra terms in the Euler equations, which here are responsible for the force N from the ground. As the sketch illustrates, Coriolis forces must exist, because the wheel is rotating about its own axis, so that points on the wheel have a relative velocity $\boldsymbol{v}^{\mathcal{M}}$ with respect to the $\mathcal{M}$-frame's rotation about the $\boldsymbol{e}_3^{\mathcal{M}}$-axis. We know the Coriolis force scales as $\boldsymbol{F}_{\text{Coriolis}} \sim -\boldsymbol{\Omega}^{\mathcal{M}} \times \boldsymbol{v}^{\mathcal{M}}$, which gives the force directions shown in the sketch.

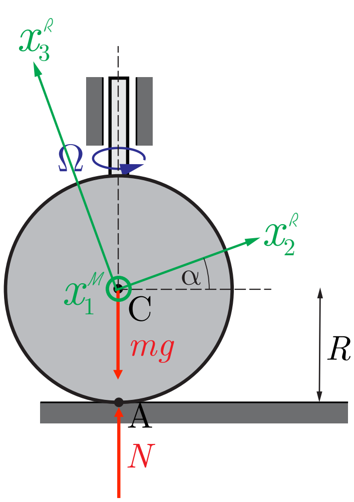


Altogether, these Coriolis forces produce a positive net torque with respect to point O about the $\boldsymbol{e}_2^{\mathcal{M}}$-axis, which must be balanced by force N from the ground.

Example 3.34. Pan mill revisited

As an alternative approach to the previous problem of the pan mill, we may indeed use Euler's equations but we need to choose a moving frame different from $\mathcal{M}$ defined above. Recall that the Euler equations can only be used if $\boldsymbol{\Omega} = \boldsymbol{\omega}$, so let us make this choice and denote the new moving frame by $\mathcal{R}$ with

$$\boldsymbol{\Omega}^{\mathcal{R}} = \boldsymbol{\omega}. \tag{3.351}$$



As a consequence, the *coordinate system now rotates with the milling wheel* (i.e., $\boldsymbol{e}_1^{\mathcal{R}} = \boldsymbol{e}_1^{\mathcal{M}}$ still points along the axis OA, but $\boldsymbol{e}_2^{\mathcal{R}}$ and $\boldsymbol{e}_3^{\mathcal{R}}$ must rotate with the wheel, as shown on the right). In this case we can use Euler's equations, but now we need to transform the angular velocity vector $\boldsymbol{\omega}$ (determined above in the $\mathcal{M}$-frame) into the $\mathcal{R}$-frame. This is accomplished by the coordinate transform

$$[\boldsymbol{\omega}]_{\mathcal{R}} = [\boldsymbol{R}_{\mathcal{R}\mathcal{M}}]^{\text{T}}[\boldsymbol{\omega}]_{\mathcal{M}}, \tag{3.352}$$

so that in the $\mathcal{R}$-frame we have (with α measured positive counter-clockwise about $\boldsymbol{e}_1^{\mathcal{R}}$, as shown)

$$[\boldsymbol{\omega}]_{\mathcal{R}} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}_{\mathcal{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} -\frac{a}{R}\Omega \\ 0 \\ \Omega \end{pmatrix} = \begin{pmatrix} -\frac{a}{R}\Omega \\ \Omega\sin\alpha \\ \Omega\cos\alpha \end{pmatrix}. \tag{3.353}$$

Time differentiation in the rotating $\mathcal{R}$-frame yields the angular acceleration vector (with $\dot{\alpha} = -\frac{a}{R}\Omega$) as

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix}_{\mathcal{R}} = \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}_{\mathcal{R}} = \begin{pmatrix} 0 \\ \Omega\dot{\alpha}\cos\alpha \\ -\Omega\dot{\alpha}\sin\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ -\Omega^2\frac{a}{R}\cos\alpha \\ \Omega^2\frac{a}{R}\sin\alpha \end{pmatrix}. \tag{3.354}$$

Now, we may use Euler's equations, when expressing all ingredients in the $\mathcal{R}$-frame. $[\boldsymbol{\omega}]_{\mathcal{R}}$ and $[\dot{\boldsymbol{\omega}}]_{\mathcal{R}}$ are given above. The moment of inertia tensor is identical to (3.343) due to rotational symmetry, i.e., $[\boldsymbol{I}_{\mathrm{O}}]_{\mathcal{M}} = [\boldsymbol{I}_{\mathrm{O}}]_{\mathcal{R}}$. We only need to compute the components of the resultant torque $\boldsymbol{M}_{\mathrm{O}}$ in the $\mathcal{R}$-frame, which is accomplished by a coordinate rotation, analogous to relation (3.352), giving

$$[\boldsymbol{M}_{\mathrm{O}}]_{\mathcal{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 0 \\ (mg-N)a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ (mg-N)a\cos\alpha \\ -(mg-N)a\sin\alpha \end{pmatrix}. \tag{3.355}$$

Overall, Euler's equations (3.326) now take the form

$$\begin{aligned} (\hat{I}_{\mathrm{O},33} - \hat{I}_{\mathrm{O},22})\Omega^2\cos\alpha\sin\alpha &= M_{\mathrm{O},1} = 0, \\ -\hat{I}_{\mathrm{O},22}\Omega^2\frac{a}{R}\cos\alpha - (\hat{I}_{\mathrm{O},11} - \hat{I}_{\mathrm{O},33})\frac{a}{R}\Omega^2\cos\alpha &= M_{\mathrm{O},2} = (mg-N)a\cos\alpha, \\ \hat{I}_{\mathrm{O},33}\Omega^2\frac{a}{R}\sin\alpha - (\hat{I}_{\mathrm{O},22} - \hat{I}_{\mathrm{O},11})\frac{a}{R}\Omega^2\sin\alpha &= M_{\mathrm{O},3} = -(mg-N)a\sin\alpha. \end{aligned}$$

The first equation is satisfied trivially since $\hat{I}_{\mathrm{O},33} = \hat{I}_{\mathrm{O},22}$. Inserting $\hat{I}_{\mathrm{O},33} = \hat{I}_{\mathrm{O},22}$ shows that the second and third equations reduce to a single (duplicated) equation:

$$\hat{I}_{\mathrm{O},33}\Omega^2\frac{a}{R} - (\hat{I}_{\mathrm{O},33} - \hat{I}_{\mathrm{O},11})\frac{a}{R}\Omega^2 = -(mg-N)a \qquad \Leftrightarrow \qquad \hat{I}_{\mathrm{O},11}\frac{1}{R}\Omega^2 = -(mg-N), \tag{3.356}$$

which is identical to (3.349). With $\hat{I}_{\mathrm{O},11} = \frac{1}{2}MR^2$ we again arrive at

$$\frac{MR^2}{2}\frac{\Omega^2}{R} = -(mg-N) \qquad \Leftrightarrow \qquad N = mg + \frac{1}{2}MR\Omega^2. \tag{3.357}$$

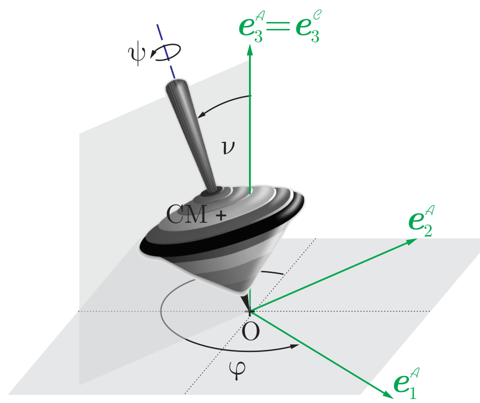
Hence, we obtain the same solution as in Example 3.33. As a summary, recall that we may choose any (co-rotating) reference frame convenient to us. In Example 3.33 we chose the $\mathcal{M}$-frame, in which the resultant torque was simple to compute, but we could not use Euler's equations. Here by contrast, the frame was oriented conveniently so we could use Euler's equations – at the cost of having to calculate the torques less conveniently. If correctly done, both approaches lead to the same solution, of course.

3.5 Application: Spinning Tops and Gyroscopes

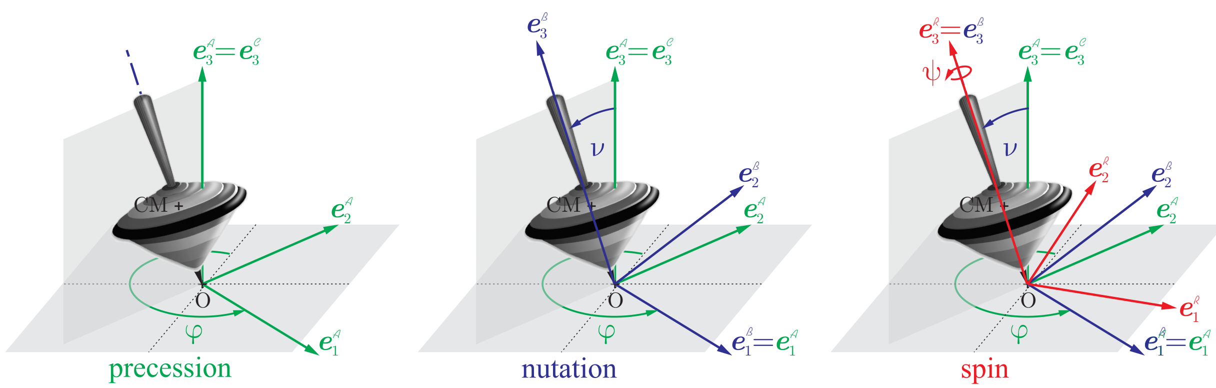
3.5.1 General description

Spinning tops are interesting applications of the above kinetic relations derived for co-rotating reference frames. A **spinning top** (*Kreisel* in German) is characterized by a free rotation in 3D about a stationary point O (i.e., although the body spins around multiple axis, point O is a stationary, fixed point for all times). Let us consider a spinning top of mass M with center of mass CM, whose tip is fixed at a point O (as shown below). The axis OC (through O and CM) is a principal axis but *not* necessarily an axis of rotational symmetry.

Since the body is fixed at point O and therefore cannot translate, the body's motion can be described by a total of $6-3=3$ independent DOFs. For example, we may describe its motion by three angles. Here we choose to use the so-called **Euler angles**, which follow the so-called **"3-1-3" construction** and characterize the motion of the body in 3D. For a spinning top on a flat ground (with normal $\boldsymbol{e}_3^{\mathcal{C}}$) we use the so-called **precession** angle φ (describing the rotation around the $\boldsymbol{e}_3^{\mathcal{C}}$-axis), the **nutation** angle ν (i.e., the tilt from the $\boldsymbol{e}_3^{\mathcal{C}}$-axis), and the **spin** angle ψ (rotation about its own axis). All three angles are illustrated in the schematic on the right.



In order to provide proper rotating frames to be used in the Euler equations later, let us define the above three angles rigorously. We start with a Cartesian basis $\{\boldsymbol{e}_1^{\mathcal{C}}, \boldsymbol{e}_2^{\mathcal{C}}, \boldsymbol{e}_3^{\mathcal{C}}\}$ (with $\boldsymbol{e}_3^{\mathcal{C}}$ normal to the ground) and describe each of the above angles by a new basis (shown in the figures below). Since point O is stationary, all reference frames use the same origin O.



1. First, we describe the **precession** of the body as a rotation from $\mathcal{C}$ to a new frame $\mathcal{A}$ with basis $\{\boldsymbol{e}_1^{\mathcal{A}}, \boldsymbol{e}_2^{\mathcal{A}}, \boldsymbol{e}_3^{\mathcal{A}} = \boldsymbol{e}_3^{\mathcal{C}}\}$, obtained by rotating the axes about $\boldsymbol{e}_3^{\mathcal{C}}$ to bring $\boldsymbol{e}_1^{\mathcal{A}}$ orthogonal to the plane of $\boldsymbol{r}_{\mathrm{OC}}$ and $\boldsymbol{e}_3^{\mathcal{C}}$. The angle $\varphi(t)$ of this rotation is the precession angle and results in the body rotating with an angular velocity
$$\boldsymbol{\omega}_\varphi = \dot{\varphi}\boldsymbol{e}_3^{\mathcal{C}} = \dot{\varphi}\boldsymbol{e}_3^{\mathcal{A}}. \tag{3.358}$$

2. Second, we describe the **nutation** of the body as a rotation from $\mathcal{A}$ to a new frame $\mathcal{B}$ with basis $\{\boldsymbol{e}_1^{\mathcal{B}} = \boldsymbol{e}_1^{\mathcal{A}}, \boldsymbol{e}_2^{\mathcal{B}}, \boldsymbol{e}_3^{\mathcal{B}}\}$, obtained by rotating the axes about $\boldsymbol{e}_1^{\mathcal{A}}$ to align $\boldsymbol{e}_3^{\mathcal{B}}$ with OC. The angle $\nu(t)$ of this rotation is the nutation angle, which contributes the angular velocity

$$\boldsymbol{\omega}_\nu = \dot{\nu}\boldsymbol{e}_1^{\mathcal{A}} = \dot{\nu}\boldsymbol{e}_1^{\mathcal{B}}. \tag{3.359}$$

3. Third, we describe the **spin** of the body about its own axis as a rotation from $\mathcal{B}$ to a new frame $\mathcal{R}$ with basis $\{\boldsymbol{e}_1^{\mathcal{R}}, \boldsymbol{e}_2^{\mathcal{R}}, \boldsymbol{e}_3^{\mathcal{R}} = \boldsymbol{e}_3^{\mathcal{B}}\}$, obtained by rotating the axes about $\boldsymbol{e}_3^{\mathcal{B}}$ to align $\boldsymbol{e}_1^{\mathcal{R}}$ and $\boldsymbol{e}_2^{\mathcal{R}}$ with the principal axes. The angle $\psi(t)$ of this rotation is the spin angle, which describes a rotation with angular velocity

$$\boldsymbol{\omega}_\psi = \dot{\psi}\boldsymbol{e}_3^{\mathcal{R}} = \dot{\psi}\boldsymbol{e}_3^{\mathcal{B}}. \tag{3.360}$$

We have thus created three rotating reference frames $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$, which we may conveniently use in the following. The total angular velocity of the rigid body is given by the superposition of all of the above three rotations, viz.

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\varphi + \boldsymbol{\omega}_\nu + \boldsymbol{\omega}_\psi = \dot{\varphi}\boldsymbol{e}_3^{\mathcal{A}} + \dot{\nu}\boldsymbol{e}_1^{\mathcal{B}} + \dot{\psi}\boldsymbol{e}_3^{\mathcal{R}}. \tag{3.361}$$

To understand the *kinetics*, we could formulate linear momentum balance in any of the rotating frames, e.g., in the $\mathcal{B}$-frame (which is rotating but not spinning with the body). This leads to

$$M\boldsymbol{a}_{\mathrm{CM}}^{\mathcal{B}} = \boldsymbol{F}_{\mathrm{ext}} - 2M\boldsymbol{\Omega}^{\mathcal{B}} \times \boldsymbol{v}_{\mathrm{CM}}^{\mathcal{B}} - M\frac{\mathrm{d}\boldsymbol{\Omega}^{\mathcal{B}}}{\mathrm{d}t} \times \boldsymbol{r}_{\mathrm{OC}}^{\mathcal{B}} - M\boldsymbol{\Omega}^{\mathcal{B}} \times (\boldsymbol{\Omega}^{\mathcal{B}} \times \boldsymbol{r}_{\mathrm{OC}}^{\mathcal{B}}). \tag{3.362}$$

For a freely spinning top, the only external forces (aside from gravity) can come through the fixed origin O. Therefore, it will be more convenient to exploit angular momentum balance with respect to O and formulate the associated Euler equations in one of the moving frames defined above, e.g., in $\mathcal{R}$ with basis $\{\boldsymbol{e}_1^{\mathcal{R}}, \boldsymbol{e}_2^{\mathcal{R}}, \boldsymbol{e}_3^{\mathcal{R}}\}$ and origin O. Since the $\mathcal{R}$-axes are aligned with the principal directions of the rotating body, its moment of inertia tensor $[\boldsymbol{I}_{\mathrm{O}}]_{\mathcal{R}}$ is diagonal in this frame. Also $\boldsymbol{\Omega}^{\mathcal{R}} = \boldsymbol{\omega}$. Therefore, the balance of angular momentum in the $\mathcal{R}$-frame reduces to Euler's equations – with all vector and tensor components measured with respect to the $\mathcal{R}$-frame:

$$\begin{aligned}
\hat{I}_1\dot{\omega}_1 + (\hat{I}_3 - \hat{I}_2)\omega_3\omega_2 &= M_{\mathrm{O},1}, \\
\hat{I}_2\dot{\omega}_2 + (\hat{I}_1 - \hat{I}_3)\omega_1\omega_3 &= M_{\mathrm{O},2}, \\
\hat{I}_3\dot{\omega}_3 + (\hat{I}_2 - \hat{I}_1)\omega_2\omega_1 &= M_{\mathrm{O},3}.
\end{aligned} \tag{3.363}$$

Note that the eigenvalues of the moment of inertia tensor are given by Steiner's theorem as

$$\hat{I}_1 = [I_{\mathrm{CM},11}]_{\mathcal{R}} + M\left|\boldsymbol{r}_{\mathrm{O-CM}}\right|^2, \qquad \hat{I}_2 = [I_{\mathrm{CM},22}]_{\mathcal{R}} + M\left|\boldsymbol{r}_{\mathrm{O-CM}}\right|^2, \qquad \hat{I}_3 = [I_{\mathrm{CM},33}]_{\mathcal{R}}, \tag{3.364}$$

while the resultant torque with respect to O is

$$\boldsymbol{M}_{\mathrm{O}} = \boldsymbol{M}^{\mathrm{ext}} + \boldsymbol{r}_{\mathrm{OC}} \times m\boldsymbol{g} \qquad \Rightarrow \qquad \begin{pmatrix} M_{\mathrm{O},1} \\ M_{\mathrm{O},2} \\ M_{\mathrm{O},3} \end{pmatrix} = [\boldsymbol{M}_{\mathrm{O}}]_{\mathcal{R}}. \tag{3.365}$$

Also measured in the $\mathcal{R}$-frame, the angular velocity vector is obtained via coordinate transformations as

$$\begin{aligned}[\boldsymbol{\omega}]_{\mathcal{R}} &= \dot{\varphi}[\boldsymbol{e}_3^{\mathcal{A}}]_{\mathcal{R}} + \dot{\nu}[\boldsymbol{e}_1^{\mathcal{B}}]_{\mathcal{R}} + \dot{\psi}[\boldsymbol{e}_3^{\mathcal{R}}]_{\mathcal{R}} \\ &= \dot{\varphi}\begin{pmatrix}\sin\nu\,\sin\psi\\ \sin\nu\,\cos\psi\\ \cos\nu\end{pmatrix} + \dot{\nu}\begin{pmatrix}\cos\psi\\ \sin\psi\\ 0\end{pmatrix} + \dot{\psi}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\dot{\varphi}\sin\nu\,\sin\psi + \dot{\nu}\cos\psi\\ \dot{\varphi}\sin\nu\,\cos\psi - \dot{\nu}\sin\psi\\ \dot{\varphi}\cos\nu + \dot{\psi}\end{pmatrix}.\end{aligned} \tag{3.366}$$

Insertion of (3.364), (3.365) and (3.366) into Euler's equations (3.363) yields a system of PDEs to be solved for $\varphi(t)$, $\nu(t)$ and $\psi(t)$ with suitable initial conditions. Unfortunately, a general solution for this problem is not feasible in closed form, which is why we proceed to study the most common special cases that do admit an analytical treatment.

3.5.2 Axisymmetric bodies

One special case of interest in many applications is that of an *axisymmetric body* spinning about its axis of rotational symmetry. In this case we can describe the rotation of the body conveniently in the $\mathcal{B}$-frame, since it is a principal frame of $\boldsymbol{I}_{\mathrm{O}}$ so $[\boldsymbol{I}_{\mathrm{O}}]_{\mathcal{B}}$ is diagonal and also $\mathring{\boldsymbol{I}}_{\mathrm{O}} = \boldsymbol{0}$ in the $\mathcal{B}$-frame (a spinning rotation of an axisymmetric body does not alter $[\boldsymbol{I}_{\mathrm{O}}]_{\mathcal{B}}$). Furthermore, in the $\mathcal{B}$-frame we have

$$\begin{aligned}[\boldsymbol{\omega}]_{\mathcal{B}} &= \dot{\varphi}[\boldsymbol{e}_3^{\mathcal{A}}]_{\mathcal{B}} + \dot{\nu}[\boldsymbol{e}_1^{\mathcal{B}}]_{\mathcal{B}} + \dot{\psi}[\boldsymbol{e}_3^{\mathcal{B}}]_{\mathcal{B}} \\ &= \dot{\varphi}\begin{pmatrix}0\\ \sin\nu\\ \cos\nu\end{pmatrix} + \dot{\nu}\begin{pmatrix}1\\0\\0\end{pmatrix} + \dot{\psi}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\dot{\nu}\\ \dot{\varphi}\sin\nu\\ \dot{\varphi}\cos\nu + \dot{\psi}\end{pmatrix} = \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}\end{aligned} \tag{3.367}$$

and

$$[\mathring{\boldsymbol{\omega}}]_{\mathcal{B}} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\dot{\nu}\\ \dot{\varphi}\sin\nu\\ \dot{\varphi}\cos\nu + \dot{\psi}\end{pmatrix} = \begin{pmatrix}\ddot{\nu}\\ \ddot{\varphi}\sin\nu + \dot{\varphi}\dot{\nu}\cos\nu\\ \ddot{\varphi}\cos\nu - \dot{\varphi}\dot{\nu}\sin\nu + \ddot{\psi}\end{pmatrix} = \begin{pmatrix}\dot{\omega}_1\\ \dot{\omega}_2\\ \dot{\omega}_3\end{pmatrix}. \tag{3.368}$$

Note that we now have $\boldsymbol{\Omega}^{\mathcal{B}} \neq \boldsymbol{\omega}$ since

$$[\boldsymbol{\Omega}^{\mathcal{B}}]_{\mathcal{B}} = \dot{\varphi}[\boldsymbol{e}_3^{\mathcal{A}}]_{\mathcal{B}} + \dot{\nu}[\boldsymbol{e}_1^{\mathcal{B}}]_{\mathcal{B}} = \begin{pmatrix}\dot{\nu}\\ \dot{\varphi}\sin\nu\\ \dot{\varphi}\cos\nu\end{pmatrix} = \begin{pmatrix}\Omega_1^{\mathcal{B}}\\ \Omega_2^{\mathcal{B}}\\ \Omega_3^{\mathcal{B}}\end{pmatrix}. \tag{3.369}$$

Therefore, we cannot use Euler's equations directly but instead use (3.323), which here becomes

$$\boldsymbol{M}_{\mathrm{O}} = (\boldsymbol{I}_{\mathrm{O}}\boldsymbol{\omega})^{\circ\mathcal{B}} + \boldsymbol{\Omega}^{\mathcal{B}} \times \boldsymbol{I}_{\mathrm{O}}\boldsymbol{\omega} = \boldsymbol{I}_{\mathrm{O}}\mathring{\boldsymbol{\omega}} + \boldsymbol{\Omega}^{\mathcal{B}} \times \boldsymbol{I}_{\mathrm{O}}\boldsymbol{\omega}, \tag{3.370}$$

as discussed in Section 3.4.4. Writing out angular momentum balance in the $\mathcal{B}$-frame, cf. (3.330) for an axisymmetric body with $\omega_i = [\omega_i]_{\mathcal{B}}$ and $\dot{\omega}_i = [\mathring{\omega}_i]_{\mathcal{B}}$ becomes

$$\begin{aligned}\hat{I}_1\dot{\omega}_1 + \hat{I}_3\Omega_2^{\mathcal{B}}\omega_3 - \hat{I}_2\Omega_3^{\mathcal{B}}\omega_2 &= [M_{\mathrm{O},1}]_{\mathcal{B}},\\ \hat{I}_2\dot{\omega}_2 + \hat{I}_1\Omega_3^{\mathcal{B}}\omega_1 - \hat{I}_3\Omega_1^{\mathcal{B}}\omega_3 &= [M_{\mathrm{O},2}]_{\mathcal{B}},\\ \hat{I}_3\dot{\omega}_3 + \hat{I}_2\Omega_1^{\mathcal{B}}\omega_2 - \hat{I}_1\Omega_2^{\mathcal{B}}\omega_1 &= [M_{\mathrm{O},3}]_{\mathcal{B}}.\end{aligned} \tag{3.371}$$

Notice from (3.369) and (3.367) that $\omega_1 = \Omega_1^{\mathcal{B}}$ and $\omega_2 = \Omega_2^{\mathcal{B}}$ whereas $\omega_3 = \Omega_3^{\mathcal{B}} + \dot{\psi}$. Axisymmetry further implies that $\hat{I}_1 = \hat{I}_2$. Therefore, the above angular momentum balance relations become

$$\begin{aligned}\hat{I}_1(\dot{\omega}_1 - \Omega_3^{\mathcal{B}}\Omega_2^{\mathcal{B}}) + \hat{I}_3\Omega_2^{\mathcal{B}}(\Omega_3^{\mathcal{B}} + \dot{\psi}) &= M_{\mathrm{O},1},\\ \hat{I}_1(\dot{\omega}_2 + \Omega_3^{\mathcal{B}}\Omega_1^{\mathcal{B}}) - \hat{I}_3\Omega_1^{\mathcal{B}}(\Omega_3^{\mathcal{B}} + \dot{\psi}) &= M_{\mathrm{O},2},\\ \hat{I}_3\dot{\omega}_3 &= M_{\mathrm{O},3}.\end{aligned}$$

Inserting the components $\Omega_i^{\mathcal{B}}$ from (3.369) and $\dot{\omega}_i$ from (3.368) finally yields

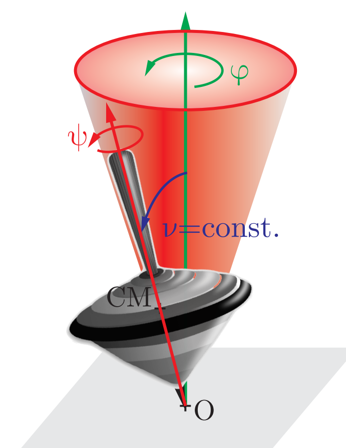
$$\begin{aligned}\hat{I}_1(\ddot{\nu} - \dot{\varphi}^2 \sin\nu\cos\nu) + \hat{I}_3\dot{\varphi}\sin\nu(\dot{\varphi}\cos\nu + \dot{\psi}) &= M_{\mathrm{O},1},\\ \hat{I}_1\left(\ddot{\varphi}\sin\nu + 2\dot{\varphi}\dot{\nu}\cos\nu\right) - \hat{I}_3\dot{\nu}(\dot{\varphi}\cos\nu + \dot{\psi}) &= M_{\mathrm{O},2},\\ \hat{I}_3\left(\ddot{\varphi}\cos\nu - \dot{\varphi}\dot{\nu}\sin\nu + \ddot{\psi}\right) &= M_{\mathrm{O},3}.\end{aligned} \tag{3.372}$$

In order to solve the system of ODEs in (3.372) for $\varphi(t)$, $\nu(t)$ and $\psi(t)$, we still need to apply further assumptions, as the general case does not admit closed-form solutions. One such example is the scenario of a steady precession, which occurs frequently (at least approximately) in practical applications and which will be discussed next.

Example 3.35. Steady precession of an axisymmetric spinning top

Let us assume the special case of a **steady precession**, which implies the spinning top's motion has a solution of the form

$$\nu = \text{const.}, \qquad \dot{\varphi} = \text{const.}, \qquad \dot{\psi} = \text{const.} \tag{3.373}$$



Such a solution is typically found for axisymmetric spinning tops moving without any externally applied torques other than gravity. In this case, applying (3.373) to (3.372) simplifies the equations dramatically since $\dot{\nu} = 0$, $\ddot{\varphi} = \ddot{\psi} = 0$. Consequently, (3.372) reduces to

$$\begin{aligned}-\hat{I}_1\dot{\varphi}^2\cos\nu\,\sin\nu + \hat{I}_3\dot{\varphi}\sin\nu\left(\dot{\varphi}\cos\nu + \dot{\psi}\right) &= M_{\mathrm{O},1},\\ 0 &= M_{\mathrm{O},2},\\ 0 &= M_{\mathrm{O},3}.\end{aligned} \tag{3.374}$$

This shows that steady precession is observed only if the external torques are parallel to the axis of nutation, since the above implies that $M_{\mathrm{O},2} = M_{\mathrm{O},3} = 0$, so we must have

$$\boldsymbol{M}_{\mathrm{O}} = M_{\mathrm{O},1}\boldsymbol{e}_1^{\mathcal{B}} \tag{3.375}$$

for a steady precession. The steady precession may thus be understood as a steady-state solution, in which the external torque due to gravity balances the gyroscopic angular acceleration about the $\boldsymbol{e}_1^{\mathcal{B}}$-axis computed above, viz.

$$-\hat{I}_1\dot{\varphi}^2\cos\nu\,\sin\nu + \hat{I}_3\dot{\varphi}\sin\nu\left(\dot{\varphi}\cos\nu + \dot{\psi}\right) = M_{\mathrm{O},1} \tag{3.376}$$

As an example, consider gravity as the only force acting on the spinning top's center of mass. This results in

$$\boldsymbol{M}_{\mathrm{O}} = \boldsymbol{r}_{\mathrm{OC}} \times m\boldsymbol{g} = mgr_{\mathrm{OC}} \sin\nu \, \boldsymbol{e}_1^{\mathcal{B}} \qquad \Rightarrow \qquad \begin{pmatrix} M_{\mathrm{O},1} \\ M_{\mathrm{O},2} \\ M_{\mathrm{O},3} \end{pmatrix} = \begin{pmatrix} mgr_{\mathrm{OC}} \sin\nu \\ 0 \\ 0 \end{pmatrix} \tag{3.377}$$

with $r_{\mathrm{OC}} = |\boldsymbol{r}_{\mathrm{OC}}|$. Hence, (3.376) presents a relation between the three constant kinematic variables $\dot{\varphi}$, $\dot{\psi}$ and ν:

$$\left(\hat{I}_3 - \hat{I}_1\right) \dot{\varphi}^2 \cos\nu + \hat{I}_3 \dot{\varphi}\dot{\psi} = mgr_{\mathrm{OC}}. \tag{3.378}$$

A steady-state solution of the above form is most likely for a *fast-spinning* top (implying $|\dot{\psi}| \gg |\dot{\varphi}|$). In this case, (3.378) provides the precession rate for a given spin:

$$|\dot{\psi}| \gg |\dot{\varphi}| \quad \Rightarrow \quad \dot{\varphi}^2 \ll |\dot{\varphi}\dot{\psi}| \qquad \Rightarrow \qquad \dot{\varphi} \approx \frac{mgr_{\mathrm{OC}}}{\hat{I}_3 \dot{\psi}} \tag{3.379}$$

The limit $|\dot{\psi}| \gg |\dot{\varphi}|$ is known as the **gyroscopic limit**, since gyroscopes are characterized by having one angular velocity component ($\dot{\psi}$) significantly larger than all other components. In this limit, one usually assumes that $\dot{\psi} = \text{const.}$, so that – given a spin rate $\dot{\psi}$ – the precession rate $\dot{\varphi}$ follows from (3.379).

We may find a similar relation more generally by considering the limiting case

$$|\hat{I}_3 \omega_3| \gg |\hat{I}_2 \omega_2| \qquad \Rightarrow \qquad \boldsymbol{\Omega}^{\mathcal{B}} \times \boldsymbol{I}_{\mathrm{O}} \boldsymbol{\omega} \approx \dot{\varphi} \boldsymbol{e}_3^{\mathcal{A}} \times \hat{I}_3 \dot{\psi} \boldsymbol{e}_3^{\mathcal{R}}. \tag{3.380}$$

For the *steady precession of an axisymmetric body* $(\boldsymbol{I}_{\mathrm{O}}\omega)^{\circ\mathcal{B}} = \boldsymbol{0}$, angular momentum balance becomes

$$\boldsymbol{M}_{\mathrm{O}} = (\boldsymbol{I}_{\mathrm{O}} \boldsymbol{\omega})^{\circ\mathcal{B}} + \boldsymbol{\Omega}^{\mathcal{B}} \times \boldsymbol{I}_{\mathrm{O}} \boldsymbol{\omega} = \boldsymbol{\Omega}^{\mathcal{B}} \times \boldsymbol{I}_{\mathrm{O}} \boldsymbol{\omega} \approx \dot{\varphi} \boldsymbol{e}_3^{\mathcal{A}} \times \hat{I}_3 \dot{\psi} \boldsymbol{e}_3^{\mathcal{R}}, \tag{3.381}$$

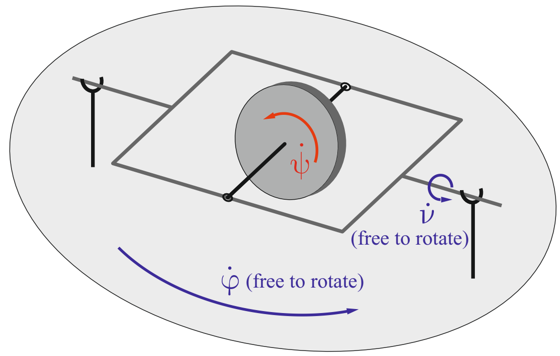
the orientations of the spin, precession, and torque must be related via

$$\dot{\varphi} \boldsymbol{e}_3^{\mathcal{A}} \times \dot{\psi} \boldsymbol{e}_3^{\mathcal{R}} \approx \frac{1}{\hat{I}_3} \boldsymbol{M}_{\mathrm{O}} \qquad \text{or, simply put,} \qquad \boxed{\dot{\boldsymbol{\varphi}} \times \dot{\boldsymbol{\psi}} \approx \frac{\boldsymbol{M}_{\mathrm{O}}}{\hat{I}_3}} \tag{3.382}$$

where we abbreviated $\dot{\boldsymbol{\varphi}} = \dot{\varphi} \boldsymbol{e}_3^{\mathcal{A}}$ and $\dot{\boldsymbol{\psi}} = \dot{\psi} \boldsymbol{e}_3^{\mathcal{R}}$. Relation (3.382) is sometimes referred to as the **TSP-rule**, linking the Torque axis, the Spin axis and the Precession axis of a fast-spinning axisymmetric rigid body.

Example 3.36. Gyroscopes

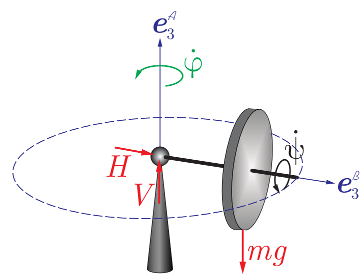
A **gyroscope** is a spinning wheel (or disc) mounted on a suspension such that its *spin axis can freely assume any orientation*. For it to reorient freely, the mount must be designed such that it cannot produce any torques onto the spinning body (an example is shown on the right). As a consequence, by angular momentum conservation the spin axis must be unaffected by tilting or rotating the mount; i.e., the mount can be re-oriented in an arbitrary fashion – the spinning body will main a constant spin axis. It is for this reason that gyroscopes are exploited, among others, in *navigation instruments* since they can indicate a unique, absolute direction independent of the orientation of the vehicle onto which they are mounted.



Gyroscopes usually operate at the gyroscopic limit, i.e., they spin at very high speed $\dot\psi$ compared to any precession or nutation speed (and the spin is approximately constant).

For example, consider the flywheel shown on the right, which spins with $\dot\psi$ about its axis of rotational symmetry lying in the horizontal plane ($\nu = \pi/2$). If the spinning wheel is initially held steady with $\dot\psi \neq 0$ and then released, we know from (3.376) and the above derivation that

$$\dot\psi \boldsymbol{e}_3^{\mathcal{R}} \perp \dot\varphi \boldsymbol{e}_3^{\mathcal{A}} \qquad \text{and} \qquad \dot\varphi = \frac{mgr_{\mathrm{OC}}}{I_{\mathrm{O},33}\dot\psi}. \tag{3.383}$$



Hence, if spun counterclockwise ($\dot\psi > 0$ as shown), then the body will start to precede with $\dot\varphi > 0$ (as shown) in a counter-clockwise fashion. If spun clockwise ($\dot\psi < 0$), then the flywheel will precede with $\dot\varphi < 0$ (as shown) in a clockwise fashion.

Interestingly, the flywheel is stable although the pin support at the center can only transmit (vertical and in-plane) forces. These forces of magnitudes H and V, respectively, can be obtained from the balance of linear momentum in the $\mathcal{B}$-frame (which rotates with the body, so the motion of the center of mass relative to the $\mathcal{B}$-frame vanishes). For the case of a steady precession, linear momentum balance gives

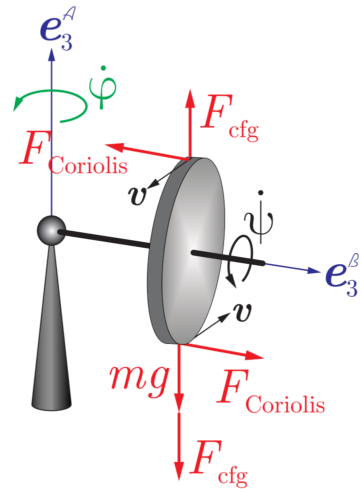
$$\boldsymbol{0} = \boldsymbol{F}_{\text{ext}} - M\boldsymbol{\Omega}^{\mathcal{B}} \times (\boldsymbol{\Omega}^{\mathcal{B}} \times \boldsymbol{r}_{\mathrm{OC}}^{\mathcal{B}}) \qquad \text{with} \qquad \boldsymbol{\Omega}^{\mathcal{B}} = \dot\varphi \boldsymbol{e}_e^{\mathcal{A}}. \tag{3.384}$$

For the particular case of a flywheel spinning around a horizontal axis of length r_{OC}, the only external forces are gravity and the support forces at O, so $\boldsymbol{F}_{\text{ext}} = -mg\boldsymbol{e}_3^{\mathcal{A}} + H\boldsymbol{e}_3^{\mathcal{B}} + V\boldsymbol{e}_3^{\mathcal{A}}$ (since no other force components act in the $\boldsymbol{e}_2$-direction, there is no support reaction in this direction either). This leads to

$$-M(\dot\varphi)^2 r_{\mathrm{OC}}\boldsymbol{e}_3^{\mathcal{B}} + mg\boldsymbol{e}_3^{\mathcal{A}} = H\boldsymbol{e}_3^{\mathcal{B}} + V\boldsymbol{e}_3^{\mathcal{A}} \quad \Rightarrow \quad H = -M(\dot\varphi)^2 r_{\mathrm{OC}}, \quad V = mg. \tag{3.385}$$

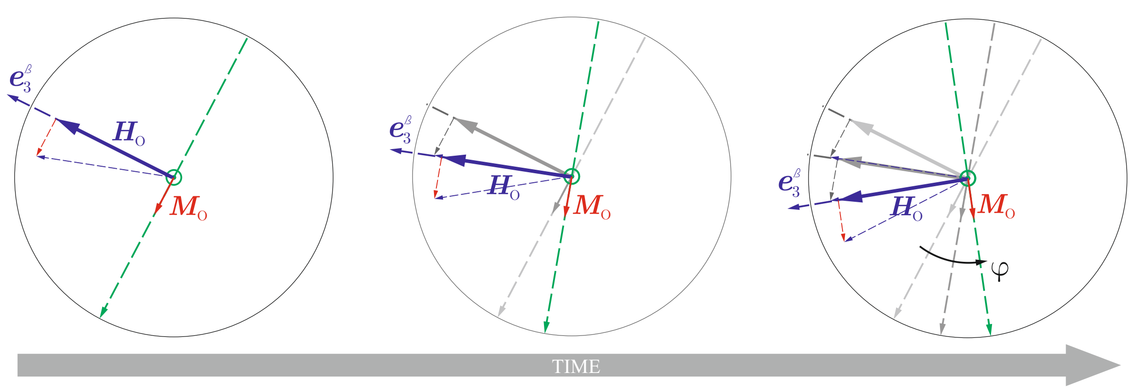
Therefore, V is a non-zero vertical force balancing the force due to gravity acting on the wheel. H is a non-zero horizontal force balancing the centrifugal force on the wheel. (Note that it is because of this latter horizontal force component that, when the wheel is supported by a string instead of a pivot, the string cannot be perfectly vertical.)

We still want to answer the question: **why does the flywheel not fall down**? Although forces are balanced, as discussed above, there is no apparent torque balancing the action of gravity on the wheel, yet it does not fall. Euler's equations give the answer by the additional $\boldsymbol{\Omega} \times \boldsymbol{I}\boldsymbol{\omega}$-term which balances the torque due to gravity – but what does it imply physically?



Recall that linear momentum balance (3.277) in the moving frame has additional components because of the frame's rotation, the inertial/fictitious forces. We obtained angular momentum balance originally from linear momentum balance and could, in principle, do the same here. As a result, the balance equations of angular momentum would have additional terms due to the torques produced by the inertial forces, specifically the Coriolis and centrifugal forces (the Euler force here is zero due to the constant angular velocities).

It is exactly these extra terms that make up those additional terms in the Euler equations. In this particular example, we assumed $|\dot{\psi}| \gg |\dot{\varphi}|$, so the dominant inertial forces are the Coriolis force due to the spinning wheel in the preceding frame ($\boldsymbol{F}_{\text{Coriolis}} = -2\boldsymbol{\Omega}^{\mathcal{B}} \times \boldsymbol{v}$ with $v = \dot{\psi}r$ and $\Omega^{\mathcal{B}} = \dot{\varphi}$) and the centrifugal force due to the fast spinning ($\boldsymbol{F}_{\text{cfg}} = -\dot{\psi}^2\boldsymbol{r}$), for any point at radius r on the wheel. Note that each point on the flywheel experiences a different force (and integration over the wheel is required to obtain the resultant forces). For a qualitative argument, we plotted above those two force components for the top and bottom points on the wheel. While the centrifugal forces cancel, the Coriolis force pair at the top and bottom of the wheel produces a torque which counteracts gravity. Therefore, the torque due to the Coriolis forces acting onto the fast spun wheel balance the torque due to gravity – and hence the wheel does not fall down.



We close by illustrating the reason for the precession of the flywheel. Recall that $\dot{\boldsymbol{H}}_O = \boldsymbol{M}_O$, so a change in angular momentum is related to a torque according to $\mathrm{d}\boldsymbol{H}_O = \boldsymbol{M}_O\,\mathrm{d}t$. Assume that the flywheel is initially spinning about the $\boldsymbol{e}_3^{\mathcal{B}}$-axis as shown above on the left (as a top view from above). The torque $\boldsymbol{M}_O$ changes $\boldsymbol{H}_O$ as shown. This in turn changes the orientation of $\boldsymbol{M}_O$, so the torque keeps chasing the angular momentum vector. Consequently, the flywheel precedes.

Example 3.37. Intermediate axis theorem (tennis racket theorem)

As a final example, let us discuss the **intermediate axis theorem** (also known as the **tennis racket theorem**). First seen in space by Russian cosmonaut Vladimir Dzhanibekov but theoretically known for more than a century, this theorem describes the intriguing motion of a 3D rigid body with *three distinct principal moments of inertia*. Instead of providing a rigorous derivation of the underlying stability conditions here, we provide a qualitative explanation and leave the detailed description to more advanced dynamics courses.

Consider a rigid-body in 3D without the action of any applied torques, so that Euler's equations become

$$\hat{I}_1\dot{\omega}_1 + (\hat{I}_3 - \hat{I}_2)\omega_3\omega_2 = 0, \tag{3.386a}$$
$$\hat{I}_2\dot{\omega}_2 + (\hat{I}_1 - \hat{I}_3)\omega_1\omega_3 = 0, \tag{3.386b}$$
$$\hat{I}_3\dot{\omega}_3 + (\hat{I}_2 - \hat{I}_1)\omega_2\omega_1 = 0. \tag{3.386c}$$

Let us assume that $\hat{I}_1 > \hat{I}_2 > \hat{I}_3$ and consider a rotation about ω_1 and small initial angular velocities $|\omega_2| \ll |\omega_1|$, $|\omega_3| \ll |\omega_1|$. In this case (3.386a) implies that $\dot{\omega}_1 \propto \omega_3\omega_2 \approx 0$. Differentiating (3.386b) with respect to time and inserting $\dot{\omega}_3$ from (3.386c) yields

$$\ddot{\omega}_2 = \frac{(\hat{I}_1 - \hat{I}_3)(\hat{I}_2 - \hat{I}_1)}{\hat{I}_2\hat{I}_3}\omega_1^2\omega_2 \tag{3.387}$$

which we rewrite as

$$\ddot{\omega}_2 = C\omega_2 \qquad \text{with} \qquad C = \frac{(\hat{I}_1 - \hat{I}_3)(\hat{I}_2 - \hat{I}_1)}{\hat{I}_2\hat{I}_3} < 0. \tag{3.388}$$

From analysis we know that this ODE has a solution $\omega_2(t)$ which is of harmonic (sine/cosine) type. Also, if the initial angular velocity ω_2 is small, this oscillation is expected to be small.

The same argument holds true when considering a rotation with ω_3 (we skip the analogous derivation).

By contrast, now consider a rotation about ω_2 with initially $|\omega_1| \ll |\omega_2|$, $|\omega_3| \ll |\omega_2|$. In this case (3.386b) implies that $\dot{\omega}_2 \approx 0$. Differentiating (3.386a) and inserting $\dot{\omega}_3$ from (3.386c) yields

$$\ddot{\omega}_1 = \frac{(\hat{I}_2 - \hat{I}_3)(\hat{I}_1 - \hat{I}_2)}{\hat{I}_1\hat{I}_3}\omega_2^2\omega_1 \tag{3.389}$$

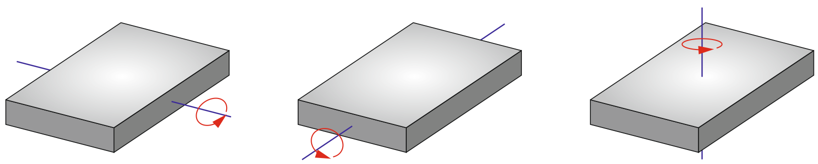
which we rewrite as

$$\ddot{\omega}_2 = D\omega_2 \qquad \text{with} \qquad D = \frac{(\hat{I}_2 - \hat{I}_3)(\hat{I}_1 - \hat{I}_2)}{\hat{I}_1\hat{I}_3}\omega_2^2 > 0. \tag{3.390}$$

From analysis we know that this ODE has a solution $\omega_1(t)$ of exponential type. Therefore, initially small values of ω_1 can quickly grow exponentially, implying an *instability*.

In summary, this shows that, for a rigid body with three distinct principal moments of inertia, an initial angular velocity around the principal axes corresponding to the largest or smallest moments of inertia leads to a stable rotating motion (possibly superimposed by stable oscillations of bounded amplitude). When rotating about the principal axis corresponding to the intermediate moment of inertia, however, any small initial perturbation results in an unstable *tumbling* motion.

Typical examples of bodies with distinct principal moments of inertia are books but also tennis rackets (therefore the eponymous theorem) and the famous T-handle in space.



A question for you: which of the three configurations is prone to show tumbling?

3.6 Summary of Key Relations

The following boxes summarize the key relations derived for rigid bodies in 3D (and special cases in 2D). We here extend our concepts from particles to continuous bodies and also include the case of non-inertial reference frames.

velocity and acceleration transfer formulae:

$$\boldsymbol{v}_{\mathrm{B}} = \boldsymbol{v}_{\mathrm{A}} + \boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{AB}}$$
$$\boldsymbol{a}_{\mathrm{B}} = \boldsymbol{a}_{\mathrm{A}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{\mathrm{AB}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{AB}})$$

center of mass and **instantaneous center/axis of rotation** (from a point P):

$$\boldsymbol{r}_{\mathrm{CM}} = \frac{1}{M} \int_{\mathcal{B}} \boldsymbol{r} \rho \, \mathrm{d}V, \qquad \boldsymbol{r}_{\mathrm{P\Pi}} = \frac{1}{\omega} \boldsymbol{e}_{\omega} \times \boldsymbol{v}_{\mathrm{P}} = \frac{\boldsymbol{\omega} \times \boldsymbol{v}_{\mathrm{P}}}{\omega^2}$$

balance of linear momentum if $M = \text{const.}$:

$$\sum_i \boldsymbol{F}_i^{\mathrm{ext}} = \dot{\boldsymbol{P}} = \frac{\mathrm{d}}{\mathrm{d}t}(M \boldsymbol{v}_{\mathrm{CM}}) \qquad \Rightarrow \qquad \sum_i \boldsymbol{F}_i^{\mathrm{ext}} = M \boldsymbol{a}_{\mathrm{CM}}$$

balance of angular momentum with respect to an arbitrary point B:

$$\boldsymbol{M}_{\mathrm{B}} = \dot{\boldsymbol{H}}_{\mathrm{B}} + \boldsymbol{v}_{\mathrm{B}} \times \boldsymbol{P}$$

angular momentum with respect to a point B on body $\mathcal{B}$:

$$\boldsymbol{H}_{\mathrm{B}} = \boldsymbol{I}_{\mathrm{B}} \boldsymbol{\omega} + M(\boldsymbol{r}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{B}}) \times \boldsymbol{v}_{\mathrm{B}}$$

moment of inertia tensor ($\mathrm{B} \in \mathcal{B}$ serves as coordinate origin):

$$[\boldsymbol{I}_{\mathrm{B}}] = \int_{\mathcal{B}} \begin{pmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ -x_1 x_2 & x_1^2 + x_3^2 & -x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & x_1^2 + x_2^2 \end{pmatrix} \rho \, \mathrm{d}V$$

parallel axes theorem (**Steiner's theorem**) with $\Delta \boldsymbol{x} = \boldsymbol{r}_{\mathrm{B}} - \boldsymbol{r}_{\mathrm{CM}}$:

$$[\boldsymbol{I}_{\mathrm{B}}] = [\boldsymbol{I}_{\mathrm{CM}}] + M \begin{pmatrix} (\Delta x_2)^2 + (\Delta x_3)^2 & -\Delta x_1 \, \Delta x_2 & -\Delta x_1 \, \Delta x_3 \\ -\Delta x_1 \, \Delta x_2 & (\Delta x_1)^2 + (\Delta x_3)^2 & -\Delta x_2 \, \Delta x_3 \\ -\Delta x_1 \, \Delta x_3 & -\Delta x_2 \, \Delta x_3 & (\Delta x_1)^2 + (\Delta x_2)^2 \end{pmatrix}$$

balance of angular momentum for $\mathrm{B} \in \mathcal{B}$ if $\mathrm{B} = \mathrm{CM}$ or $\boldsymbol{v}_{\mathrm{B}} = \boldsymbol{0}$ and if $\dot{\boldsymbol{I}}_{\mathrm{B}} = \boldsymbol{0}$:

$$\boldsymbol{M}_{\mathrm{B}} = \boldsymbol{I}_{\mathrm{B}} \dot{\boldsymbol{\omega}} \qquad \overset{\text{in 2D}}{\Rightarrow} \qquad M_{\mathrm{B}} = I_{\mathrm{B}} \dot{\omega} \quad \text{with} \quad I_{\mathrm{B}} = I_{\mathrm{CM}} + M(\Delta x)^2$$

angular momentum transfer formula for arbitrary points A and B:

$$\boldsymbol{H}_{\mathrm{B}} = \boldsymbol{H}_{\mathrm{A}} + \boldsymbol{P} \times \boldsymbol{r}_{\mathrm{AB}}$$

centroidal moments of inertia in 2D:

$$\text{slender rod}: \quad I_{\mathrm{CM}} = \frac{ML^2}{12}, \qquad \text{disk/cylinder}: \quad I_{\mathrm{CM}} = \frac{MR^2}{2}$$

kinetic energy of a rigid body for a point $\mathrm{C} \in \mathcal{B}$ with $\mathrm{C} = \mathrm{CM}$ or $\boldsymbol{v}_\mathrm{C} = \boldsymbol{0}$:

$$T = \frac{1}{2} M |\boldsymbol{v}_\mathrm{C}|^2 + \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{I}_\mathrm{C}\, \boldsymbol{\omega}$$

work–energy balance:

$$T(t_2) - T(t_1) = W_{12} \qquad W_{12} = \sum_i \int_{\boldsymbol{r}_i(t_1)}^{\boldsymbol{r}_i(t_2)} \boldsymbol{F}_i^\mathrm{ext} \cdot \mathrm{d}\boldsymbol{r}_i + \int_{\boldsymbol{\varphi}(t_1)}^{\boldsymbol{\varphi}(t_2)} \boldsymbol{M}^\mathrm{ext} \cdot \mathrm{d}\boldsymbol{\varphi}$$

time derivative relation between an **inertial frame** $\mathcal{C}$ and a **non-inertial frame** $\mathcal{M}$:

$$\dot{\boldsymbol{y}}^\mathcal{C} = \mathring{\boldsymbol{y}}^\mathcal{M} + \boldsymbol{\Omega}^\mathcal{M} \times \boldsymbol{y}$$

balance of linear momentum in a moving frame $\mathcal{M}$:

$$M \boldsymbol{a}_\mathrm{CM}^\mathcal{M} = \boldsymbol{F}_\mathrm{ext} + \boldsymbol{F}_\mathrm{Coriolis} + \boldsymbol{F}_\mathrm{Euler} + \boldsymbol{F}_\mathrm{centrifugal} - M \boldsymbol{a}_{\mathrm{O}^\mathcal{M}}$$

Coriolis, **Euler** and **centrifugal forces**:

$$\begin{aligned} \boldsymbol{F}_\mathrm{Coriolis} &= -2M \boldsymbol{\Omega}^\mathcal{M} \times \boldsymbol{v}_\mathrm{CM}^\mathcal{M}, \\ \boldsymbol{F}_\mathrm{Euler} &= -M \frac{\mathrm{d}\boldsymbol{\Omega}^\mathcal{M}}{\mathrm{d}t} \times \boldsymbol{r}_\mathrm{CM}^\mathcal{M}, \\ \boldsymbol{F}_\mathrm{centrifugal} &= -M \boldsymbol{\Omega}^\mathcal{M} \times (\boldsymbol{\Omega}^\mathcal{M} \times \boldsymbol{r}_\mathrm{CM}^\mathcal{M}) \end{aligned}$$

balance of angular momentum in a moving frame $\mathcal{M}$:

$$\boldsymbol{M}_\mathrm{B} = (\boldsymbol{I}_\mathrm{B} \boldsymbol{\omega})^{\circ\mathcal{M}} + \boldsymbol{\Omega}^\mathcal{M} \times \boldsymbol{I}_\mathrm{B} \boldsymbol{\omega} \qquad \text{if} \qquad \mathrm{B} = \mathrm{CM} \quad \text{or} \quad \boldsymbol{v}_\mathrm{B} = \boldsymbol{0}$$

Euler equations in a rotating (principal) **body frame** $\hat{\mathcal{M}}$ ($\boldsymbol{\Omega}^{\hat{\mathcal{M}}} = \boldsymbol{\omega}$):

$$\left.\begin{aligned} \hat{I}_1 \dot{\omega}_1 + (\hat{I}_3 - \hat{I}_2) \omega_3 \omega_2 &= [M_{\mathrm{B},1}]_{\hat{\mathcal{M}}} \\ \hat{I}_2 \dot{\omega}_2 + (\hat{I}_1 - \hat{I}_3) \omega_1 \omega_3 &= [M_{\mathrm{B},2}]_{\hat{\mathcal{M}}} \\ \hat{I}_3 \dot{\omega}_3 + (\hat{I}_2 - \hat{I}_1) \omega_2 \omega_1 &= [M_{\mathrm{B},3}]_{\hat{\mathcal{M}}} \end{aligned}\right\} \qquad \text{if} \quad \boldsymbol{\Omega}^\mathcal{M} = \boldsymbol{\omega} \quad \text{and} \quad \begin{aligned} \omega_i &= [\omega_i]_{\hat{\mathcal{M}}} \\ \dot{\omega}_i &= [\mathring{\omega}_i]_{\hat{\mathcal{M}}} \end{aligned}$$

angular momentum balance in a rotating (principal) frame $\mathcal{B}$ if $\mathring{\boldsymbol{I}} = \boldsymbol{0}$ ($\boldsymbol{\Omega}^{\hat{\mathcal{B}}} \neq \boldsymbol{\omega}$):

$$\left.\begin{aligned} \hat{I}_1 \dot{\omega}_1 + \hat{I}_3 \Omega_2^\mathcal{B} \omega_3 - \hat{I}_2 \Omega_3^\mathcal{B} \omega_2 &= [M_{\mathrm{B},1}]_{\hat{\mathcal{B}}} \\ \hat{I}_2 \dot{\omega}_2 + \hat{I}_1 \Omega_3^\mathcal{B} \omega_1 - \hat{I}_3 \Omega_1^\mathcal{B} \omega_3 &= [M_{\mathrm{B},2}]_{\hat{\mathcal{B}}} \\ \hat{I}_3 \dot{\omega}_3 + \hat{I}_2 \Omega_1^\mathcal{B} \omega_2 - \hat{I}_1 \Omega_2^\mathcal{B} \omega_1 &= [M_{\mathrm{B},3}]_{\hat{\mathcal{B}}} \end{aligned}\right\} \qquad \text{where} \qquad \begin{aligned} \omega_i &= [\omega_i]_{\hat{\mathcal{B}}} \\ \dot{\omega}_i &= [\mathring{\omega}_i]_{\hat{\mathcal{B}}} \end{aligned}$$

TSP-rule for a fast-spinning top:

$$\dot{\boldsymbol{\varphi}} \times \dot{\boldsymbol{\psi}} \approx \frac{\boldsymbol{M}_\mathrm{O}}{\hat{I}_3}$$

4 Vibrations

In several of our early examples we arrived at an equation of motion, viz. a partial differential equation which we said had to be solved with appropriate initial conditions – without actually solving those (see, e.g., Examples 1.16, 1.17 or 3.26). While the solution of such equations of motion may be quite complicated in general, we can indeed derive analytical solutions for some important classes of problems of engineering interest; one such example are mechanical vibrations. Before discussing vibrations, however, let us first introduce the helpful concept known as *Lagrange equations*.

4.1 Lagrange Equations

Let us consider a system of N particles whose positions are denoted by $\{\boldsymbol{r}_1, \dots, \boldsymbol{r}_N\}$ (we can later generalize our derivations to rigid bodies, which may be interpreted as collections of infinitely many rigidly connected particles, as discussed before). As mentioned for systems of N particles, we may introduce any **holonomic constraint** as

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, \dots, q_n, t) \qquad \text{for } i = 1, \dots, N, \tag{4.1}$$

where $(q_1, \dots, q_n)$ are so-called **generalized degrees of freedom** that are *independent* (and $n \leq d \times N$ in d dimensions). For a general system with k independent constraints and $d \times N$ degrees of freedom, we thus identify $n = d \times N - k$ generalized degrees of freedom. We can think of these generalized degrees of freedom as any set of (e.g., translational and/or rotational) independent kinematic variables that uniquely define the state of a system. Their choice may not necessarily be unique (and we should, as always, aim for a convenient choice), but the number of generalized degrees of freedom is fixed as $n = d \times N - k$ for a system of particles.

Lagrange's equations, which we derive in the following, are an oftentimes convenient shortcut to deriving the equations of motion for those generalized DOFs. We note that there are multiple ways to arrive at Lagrange's equations (and various approaches are found in textbooks). Here, we use a simple (but easy to follow) derivation as follows.

For a system of N particles, Newton's second law may be formulated as

$$m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i^{\text{cons.}} + \boldsymbol{F}_i^{\text{non-cons.}} = -\frac{\partial V}{\partial \boldsymbol{r}_i} + \boldsymbol{F}_i^{\text{non-cons.}} \qquad \text{for } i = 1, \dots, N, \tag{4.2}$$

where we accounted for both conservative forces $\boldsymbol{F}_i^{\text{cons.}}$, deriving from a potential $V = V(\boldsymbol{r}_1, \dots, \boldsymbol{r}_N)$, and non-conservative forces $\boldsymbol{F}_i^{\text{non-cons.}}$. Multiplying (4.2) by the respective particle velocity $\dot{\boldsymbol{r}}_i$ and summing over all particles leads to

$$0 = \sum_{i=1}^{N} \dot{\boldsymbol{r}}_i \cdot \left(m_i \ddot{\boldsymbol{r}}_i + \frac{\partial V}{\partial \boldsymbol{r}_i} - \boldsymbol{F}_i^{\text{non-cons.}} \right), \tag{4.3}$$

which must obviously hold for a system of particles satisfying the equations of motion (4.2). Note that we may reinterpret this latter equation by considering, for simplicity, a conservative system whose total energy must remain constant over time, so

$$T+V=\text{const.}\qquad\Rightarrow\qquad\frac{\mathrm{d}}{\mathrm{d}t}(T+V)=0\qquad\text{with}\qquad T=\sum_i\frac{m_i}{2}|\dot{\boldsymbol{r}}_i|^2.\tag{4.4}$$

Inserting the definition of the kinetic energy and applying the time derivative yields

$$0=\frac{\mathrm{d}}{\mathrm{d}t}(T+V)=\sum_{i=1}^N m_i\dot{\boldsymbol{r}}_i\cdot\ddot{\boldsymbol{r}}_i+\sum_{i=1}^N\frac{\partial V}{\partial\boldsymbol{r}_i}\cdot\dot{\boldsymbol{r}}_i=\sum_{i=1}^N\dot{\boldsymbol{r}}_i\cdot\left(m_i\ddot{\boldsymbol{r}}_i+\frac{\partial V}{\partial\boldsymbol{r}_i}\right),\tag{4.5}$$

so we recover (the conservative part of) (4.3) (which can be extended to non-conservative forces as shown above). One could also multiply (4.2) by a *virtual displacement* $\delta\boldsymbol{r}_i$ instead of $\dot{\boldsymbol{r}}_i$, which arrives at an equivalent form known as the principle of virtual work. Here, we simply accept that (4.3) holds.

Let us apply a few algebraic manipulations. First, we use the constraints (4.1) to write

$$\sum_{i=1}^N\frac{\partial V}{\partial\boldsymbol{r}_i}\cdot\dot{\boldsymbol{r}}_i=\sum_{i=1}^N\frac{\partial V}{\partial\boldsymbol{r}_i}\cdot\sum_{j=1}^n\frac{\mathrm{d}\dot{\boldsymbol{r}}_i}{\mathrm{d}\dot{q}_j}\dot{q}_j=\sum_{j=1}^n\sum_{i=1}^N\frac{\partial V}{\partial\boldsymbol{r}_i}\cdot\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}q_j}\dot{q}_j=\sum_{j=1}^n\frac{\partial V}{\partial q_j}\dot{q}_j\tag{4.6}$$

with the **generalized velocity** $\dot{q}_j$. Further, we use that

$$\begin{aligned}m_i\ddot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i&=m_i\frac{\mathrm{d}^2\boldsymbol{r}_i}{\mathrm{d}t^2}\cdot\sum_{j=1}^n\frac{\partial\boldsymbol{r}_i}{\partial q_j}\dot{q}_j=m_i\sum_{j=1}^n\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\boldsymbol{r}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_j}\right)-\dot{\boldsymbol{r}}_i\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\boldsymbol{r}_i}{\partial q_j}\right)\right]\dot{q}_j\\&=m_i\sum_{j=1}^n\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\boldsymbol{r}}_i\cdot\frac{\partial\dot{\boldsymbol{r}}_i}{\partial\dot{q}_j}\right)-\dot{\boldsymbol{r}}_i\cdot\frac{\partial\dot{\boldsymbol{r}}_i}{\partial q_j}\right]\dot{q}_j=m_i\sum_{j=1}^n\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_j}\frac{|\dot{\boldsymbol{r}}_i|^2}{2}-\frac{\partial}{\partial q_j}\frac{|\dot{\boldsymbol{r}}_i|^2}{2}\right]\dot{q}_j\\&=\sum_{j=1}^n\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_i}{\partial\dot{q}_j}-\frac{\partial T_i}{\partial q_j}\right)\dot{q}_j\qquad\text{with}\qquad T_i=m_i\frac{|\dot{\boldsymbol{r}}_i|^2}{2}\end{aligned}\tag{4.7}$$

being the kinetic energy of particle i. Next, we define the non-conservative **generalized force**

$$Q_j^{\mathrm{nc}}=\sum_{i=1}^N\boldsymbol{F}_i^{\text{non-cons.}}\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_j}\tag{4.8}$$

conjugate to the generalized coordinate q_j, such that

$$\sum_{i=1}^N\boldsymbol{F}_i^{\text{non-cons.}}\cdot\dot{\boldsymbol{r}}_i=\sum_{j=1}^n\sum_{i=1}^N\boldsymbol{F}_i^{\text{non-cons.}}\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_j}\dot{q}_j=\sum_{j=1}^n Q_j^{\mathrm{nc}}\dot{q}_j.\tag{4.9}$$

Altogether, we thus arrive at

$$0=\sum_{i=1}^N\dot{\boldsymbol{r}}_i\cdot\left(m_i\ddot{\boldsymbol{r}}_i+\frac{\partial V}{\partial\boldsymbol{r}_i}-\boldsymbol{F}_i^{\text{non-cons.}}\right)=\sum_{j=1}^n\left[\sum_{i=1}^N\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_i}{\partial\dot{q}_j}-\frac{\partial T_i}{\partial q_j}\right)+\frac{\partial V}{\partial q_j}-Q_j^{\mathrm{nc}}\right]\dot{q}_j.\tag{4.10}$$

Since (4.10) must be satisfied for any generalized velocities and since the generalized coordinates q_i (and hence velocities $\dot{q}_i$) are independent, we conclude that we must have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N}\frac{\partial T_i}{\partial \dot{q}_j}+\frac{\partial}{\partial q_j}\left(V-\sum_{i=1}^{N}T_i\right)-Q_j^{\mathrm{nc}}=0 \qquad \text{for } j=1,\dots,n. \tag{4.11}$$

If we exploit that $V=V(q_1,\dots,q_n)$ is independent of the generalized velocities $\dot{q}_i$ so $\partial V/\partial \dot{q}_j=0$ for all $j=1,\dots,n$, we may define the so-called **Lagrangian**

$$\mathcal{L}=T-V \qquad \text{with} \qquad T=\sum_{i=1}^{N}m_i\frac{|\dot{\boldsymbol{r}}_i|^2}{2} \tag{4.12}$$

such that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)-\frac{\partial \mathcal{L}}{\partial q_i}=Q_i^{\mathrm{nc}} \quad \text{for } i=1,\dots,n \qquad \text{with} \qquad \mathcal{L}=T-V} \tag{4.13}$$

These are the so-called **Lagrange equations**, which are equivalent to the equations of motion for the generalized degrees of freedom. If the system is *conservative*, then $Q_i^{\mathrm{nc}}=\mathbf{0}$ for each $i=1,\dots,n$ and the Lagrangian $\mathcal{L}$ alone determines the equations of motion. Otherwise, we recall that

$$\boxed{Q_i^{\mathrm{nc}}=\sum_{j=1}^{N}\boldsymbol{F}_j^{\text{non-cons.}}\cdot\frac{\partial \boldsymbol{r}_j}{\partial q_i}} \tag{4.14}$$

The Lagrange equations provide a convenient alternative to LMB and AMB, since they allow us to obtain the equations of motion of a mechanical system without the need to formulate balance laws. All that is needed are an admissible choice of the generalized degrees of freedom as well as the total potential and kinetic energies. Then, (4.13) yields the equations of motion directly by differentiation. Examples will follow after the discussion of a special case – mechanical equilibrium.

4.2 Mechanical Equilibrium

In the special case of **static** problems, the Lagrange equations with $T=0$ reduce to

$$\frac{\partial V}{\partial q_i}=Q_i^{\mathrm{nc}} \qquad \text{for } i=1,\dots,n, \tag{4.15}$$

which – when written out for a specific system – is equivalent to the vanishing sums of all forces and torques, hence describing mechanical **equilibrium**.

Furthermore, in the special case of a **static** and **conservative** system ($Q_i^{\mathrm{nc}}=0$) we have

$$\frac{\partial V}{\partial q_i}=0 \qquad \text{for } i=1,\dots,n. \tag{4.16}$$

This implies that an **equilibrium** *of a conservative system is a stationary point of its potential energy*. The simplest example to picture this relation is a particle of mass m exposed to gravity in a mountain landscape of varying altitude $h(\boldsymbol{x})$, so $V(\boldsymbol{x}) = mgh(\boldsymbol{x})$. The only places where the particle can come to rest (i.e., is in equilibrium) are mountain valleys and mountain tops, where the slope vanishes, so $\nabla V = mg\nabla h = \mathbf{0}$. Of course, we understand that mountain valleys are **stable** equilibria (after perturbation, the particle remains in the valley), whereas mountain tops are **unstable** equilibria (once perturbed, the particle will roll into the next valley and not return).

Whether or not an equilibrium is stable can generally be verified by inserting a found solution $\{\overline{q}_1, \dots, \overline{q}_n\}$ into the **Hessian** matrix with components

$$H_{ij} = \frac{\partial^2 V}{\partial q_i \, \partial q_j}, \tag{4.17}$$

to determine whether the found optimum is an energy minimum or maximum. A minimum implies a **stable** equilibrium and corresponds to a positive-definite Hessian. By contrast, a maximum is an **unstable** equilibrium and corresponds to a negative-definite Hessian. Saddle points are **metastable** equilibria.

For the special case of a system with a *single degree of freedom* q and total potential energy $V(q)$, we thus conclude:

$$\boxed{\begin{array}{ccccc} \text{stable equilibrium} & \Leftrightarrow & \text{energy minimum} & \Leftrightarrow & \dfrac{\partial^2 V}{\partial q^2} > 0 \\ \text{unstable equilibrium} & \Leftrightarrow & \text{energy maximum} & \Leftrightarrow & \dfrac{\partial^2 V}{\partial q^2} < 0 \end{array}} \tag{4.18}$$

Example 4.1. Vibrating mass–spring system

Consider a particle of mass m attached to a linear spring of stiffness k in 1D and a dashpot with velocity-proportional viscosity d, so there is only one DOF $q = x$. The dashpot produces a drag force $Q^{\text{nc}} = -d\dot{x} = -d\dot{q}$, while the kinetic and potential energy for the system shown below read

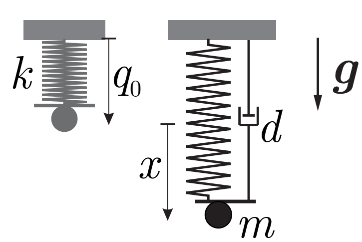
$$T = \frac{1}{2}m\dot{q}^2, \quad V = -mgq + \frac{1}{2}kq^2 \qquad \Rightarrow \qquad \mathcal{L} = T - V = \frac{1}{2}m\dot{q}^2 + mgq - \frac{1}{2}kq^2, \tag{4.19}$$

where we define q such that the spring is unstretched when $q = 0$.

The static equilibrium position q_0 is obtained as

$$0 = \left.\frac{\partial V}{\partial q}\right|_{q=q_0} = -mg + kq_0 \qquad \Rightarrow \qquad q_0 = \frac{mg}{k},$$

and we can easily verify that this is a stable equilibrium since $\partial^2 V/\partial q^2 = k > 0$.

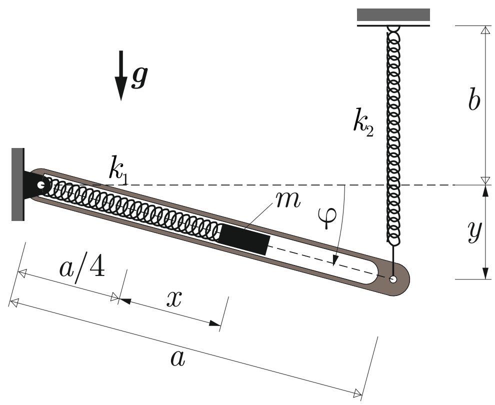


The equation of motion (e.o.m.) is obtained as Lagrange's equation, which here reduces to

$$-d\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = m\ddot{q} - mg + kq \qquad \Leftrightarrow \qquad m\ddot{q} + d\dot{q} + kq = mg. \tag{4.20}$$

This is a second-order, linear, inhomogeneous ordinary differential equation (ODE) to be solved for $q(t)$ with appropriate initial conditions. Note that if we introduce the degree of freedom as $x(t) = q(t) - q_0$ (so $x = 0$ is the equilibrium position), then the EoM becomes $m\ddot{x} + d\dot{x} + kx = 0$.

Example 4.2. Equilibrium of a particle in a rotating arm



The system shown on the right consists of a particle of mass m frictionlessly sliding in a massless arm of length a with a spring of stiffness k_1 and unstretched length $l_{0,1} = a/4$. The tip of the arm is connected to a vertical spring of stiffness k_2 and undeformed length $l_{0,2} = b$. Considering gravity acting downwards as shown, what are the equilibrium configurations of the system?

We start by formulating the total potential energy as a function of the angle φ and the elongation x of the spring within the arm (these are our chosen two DOFs that uniquely describe the system):

$$V = -mg\,(l_{0,1} + x)\sin(\varphi) + \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2\,(a\sin\varphi)^2 \tag{4.21}$$

In order to find the equilibria of this static, conservative system, we compute

$$\frac{\partial V}{\partial x} = -mg\sin\varphi + k_1 x = 0 \qquad \Rightarrow \qquad x = \frac{mg}{k_1}\sin\varphi$$
$$\frac{\partial V}{\partial \varphi} = -mg\,(l_{0,1} + x)\cos\varphi + k_2 a^2 \sin\varphi\cos\varphi = \left[-mg\left(l_{0,1} + \frac{mg}{k_1}\sin\varphi\right) + k_2 a^2 \sin\varphi\right]\cos\varphi = 0,$$

whose solution is

$$\varphi = \pm\frac{\pi}{2} \qquad \vee \qquad \sin\varphi = \frac{mgak_1}{4\,(k_1 k_2 a^2 - m^2 g^2)} \tag{4.22}$$

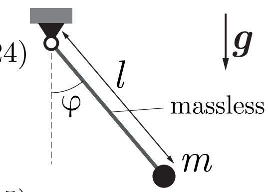
along with

$$x_0 = \pm\frac{mg}{c_1} \qquad \vee \qquad x_0 = \frac{m^2 g^2 a}{4\,(k_1 k_2 a^2 - m^2 g^2)}. \tag{4.23}$$

It is intuitive (and can be verified by second derivatives) that the $\varphi = \pm\frac{\pi}{2}$ solutions are unstable equilibria, while the second solution in (4.22) presents a stable equilibrium angle.

Example 4.3. Swinging pendulum

Consider pendulum consisting of a particle of mass m attached to a hinged massless rod of length l. What are the equilibria, and what is the equation of motion of the system?



Here, the kinetic and potential energy with the single DOF $q = \varphi$ are

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{q})^2, \qquad V = -mgl\cos q. \tag{4.24}$$

This leads to the Lagrangian

$$\mathcal{L} = T - V = \frac{1}{2}m(l\dot{q})^2 + mgl\cos q. \tag{4.25}$$

The equilibrium positions follow from

$$0 = \left.\frac{\partial V}{\partial q}\right|_{q=q_0} = mgl\sin q_0 \qquad \Rightarrow \qquad q_0 = 0 \quad \vee \quad q_0 = \pi. \tag{4.26}$$

Using second derivatives reveals whether or not these equilibria are stable:

$$\begin{aligned} \text{stable} \quad &\Leftrightarrow \quad \text{energy minimum} \quad \Leftrightarrow \quad \frac{\partial^2 V}{\partial q^2} > 0 \\ \text{unstable} \quad &\Leftrightarrow \quad \text{energy maximum} \quad \Leftrightarrow \quad \frac{\partial^2 V}{\partial q^2} < 0 \end{aligned} \tag{4.27}$$

Here, we have

$$\frac{\partial^2 V}{\partial q^2} = mgl\cos q \qquad \Rightarrow \qquad \left.\frac{\partial^2 V}{\partial q^2}\right|_{q=0} = mgl > 0, \qquad \left.\frac{\partial^2 V}{\partial q^2}\right|_{q=\pi} = -mgl < 0 \tag{4.28}$$

As expected, $q_0 = 0$ is a stable equilibrium, while $q_0 = \pi$ is an unstable equilibrium.

The equation of motion is derived as Lagrange's equation for the single DOF q, viz.

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = ml^2\ddot{q} + mgl\sin q \qquad \Rightarrow \qquad l\ddot{q} + g\sin q = 0. \tag{4.29}$$

This is a nonlinear ODE to be solved for the unknown function $q(t)$. Since this is difficult in general, one often turns to the special case of assuming small oscillations around a stable equilibrium (e.g., imagine the pendulum is initially at rest and then perturbed by a small deflection to cause oscillatory motion around the equilibrium). In such cases, we may assume $q(t) = q_0 + x(t)$ with a stable equilibrium q_0 and perturbation a $|x(t)| \ll 1$. Using a Taylor series to approximate the trigonometric term around $q_0 = 0$ yields

$$\sin(q_0 + x) = \sin q_0 + x\cos q_0 - \frac{x^2}{2}\sin q_0 + O(x^3) \qquad \overset{q_0=0}{\Rightarrow} \qquad \sin(q) \approx x + O(x^3). \tag{4.30}$$

Further using $\ddot{q}(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}[q_0 + x(t)] = \ddot{x}(t)$, the equation of motion reduces to a *linear*, homogeneous, second-order ODE (independent of the mass m):

$$\ddot{x} + \frac{g}{l}x \approx 0, \tag{4.31}$$

which is to be solved for $x(t)$ with appropriate initial conditions. We will discuss solutions to equations of this type in the next section.

4.3 Single-Degree-of-Freedom Vibrations

4.3.1 Definitions and equation of motion

Generally speaking, **vibrations** for our purposes are *small oscillations around a stable equilibrium.* We have seen that the equations of motion that govern such oscillations are usually differential equations. We hence need to solve those differential equations with suitable initial conditions in order to find the vibrational motion of a particle or body.

The **prototypical equation of motion** for a vibration with a single degree of freedom (see, e.g., Examples 4.1 and 4.3) has the form[15]

$$\boxed{m\ddot{x} + c\dot{x} + kx = F} \tag{4.32}$$

This is a second-order, linear ODE in time, which is a **homogeneous ODE** if $F = 0$ (this case is termed **free vibration**) or otherwise **inhomogeneous ODE** if $F \neq 0$ (this case is referred to as **forced vibration**). Solving the ODE for $x(t)$ results in the solution in its so-called **state space representation** (which refers to finding x as a function of time t).

For convenience, let us introduce the quantities

$$\delta = \frac{c}{2m} \geq 0, \qquad \omega_0 = \sqrt{\frac{k}{m}} \geq 0, \qquad f(t) = \frac{F(t)}{m} \tag{4.33}$$

to rewrite the second-order ODE (4.32) in its normalized form:

$$\boxed{\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = f} \tag{4.34}$$

Note that we may reduce any linear system of second-order to a first-order system as follows. We introduce the new degree-of-freedom vector $\boldsymbol{y}$ (so-called **phase space representation**) such that

$$\begin{array}{l} y_1 = x \\ y_2 = \dot{x} \end{array} \quad \Rightarrow \quad [\boldsymbol{y}] = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{4.35}$$

Consequently, the second-order ODE (4.34) becomes a first-order system of ODEs, viz.

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{F} \quad \text{with} \quad [\boldsymbol{A}] = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\delta \end{pmatrix}, \tag{4.36}$$

which is to be solved for $\boldsymbol{y}(t)$ and usually visualized in the (y_1, y_2)-phase space.

The *general solution* of the equation of motion (4.34) is obtained by starting with an exponential ansatz with unknowns $A, \lambda \in \mathbb{C}$, whose insertion into the equation of motion (4.34) yields

$$x(t) = Ae^{\lambda t} \qquad \Rightarrow \qquad \left(\lambda^2 + 2\delta\lambda + \omega_0^2\right) x(t) = f(t). \tag{4.37}$$

Depending on the choice of δ and f, we obtain different types of solutions. Several cases will be discussed in the following, including free vibrations ($f = 0$) and forced vibrations ($f \neq 0$).

[15] Even though the equation of motion may be much more complex in general (writing, e.g., $g(\ddot{x}, \dot{x}, x) = F(t)$ with some nonlinear function g), it can always be linearized to the form (4.32) through Taylor expansions of g. Therefore, (4.32) presents the general *linearized* equation of motion for small-amplitude oscillations.

4.3.2 Free vibrations

If $F(t) = 0$ and hence $f(t) = 0$, then we are concerned with a **free vibration**; i.e., the system vibration is caused only by the initial conditions, there is no applied time-dependent forcing that promotes vibrations. In this case, Eq. (4.37) simplifies to the so-called **characteristic equation**

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0 \qquad \Rightarrow \qquad \lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}. \tag{4.38}$$

Depending on the values of δ and ω_0, this admits four types of solutions, which will be discussed in the following in detail. It is convenient to introduce **Lehr's damping ratio**

$$D = \frac{\delta}{\omega_0} = \frac{c}{2m\omega_0}, \tag{4.39}$$

which defines the four cases discussed below.

- **Case 1**: **undamped vibration** ($D = 0$)

  If there is no damping, then $\delta = 0$ and the above solutions turn into a pair of purely imaginary values for λ, viz.

  $$\delta = 0 \qquad \Rightarrow \qquad \lambda_{1,2} = \pm\sqrt{-\omega_0^2} = \pm i\omega_0 \qquad \text{where} \qquad i = \sqrt{-1},\ \omega_0 \in \mathbb{R}. \tag{4.40}$$

  The solution is hence of the form (with complex coefficients $C_1, C_2 \in \mathbb{C}$)

  $$x(t) = C_1 \exp(i\omega_0 t) + C_2 \exp(-i\omega_0 t). \tag{4.41}$$

  Using Euler's identity, we transform the above into the alternative form

  $$\begin{aligned} x(t) &= C_1 \left[\cos(\omega_0 t) + i\sin(\omega_0 t)\right] + C_2 \left[\cos(\omega_0 t) - i\sin(\omega_0 t)\right] \\ &= (C_1 + C_2)\cos(\omega_0 t) + i(C_1 - C_2)\sin(\omega_0 t). \end{aligned} \tag{4.42}$$

  Since the solution must be real-valued (i.e., we seek $x(t) \in \mathbb{R}$), we must have

  $$\mathrm{Im}(C_1 + C_2) = 0, \quad \mathrm{Re}(C_1 - C_2) = 0 \qquad \text{s.t.} \qquad C_1 + C_2, i(C_1 - C_2) \in \mathbb{R}. \tag{4.43}$$

  This allows us to introduce two alternative (real-valued) coefficients

  $$A_1 = C_1 + C_2 \in \mathbb{R} \qquad \text{and} \qquad A_2 = i(C_1 - C_2) \in \mathbb{R}, \tag{4.44}$$

  so that

  $$x(t) = A_1 \cos(\omega_0 t) + A_2 \sin(\omega_0 t) \tag{4.45}$$

  with constants $A_1, A_2 \in \mathbb{R}$. Note that we may use the trigonometric identity $A\sin(\omega_0 t + \varphi_0) = A\cos(\varphi_0)\sin(\omega_0 t) + A\sin(\varphi_0)\cos(\omega_0 t)$ to introduce two alternative unknowns $A, \varphi_0 \in \mathbb{R}$ and define

  $$A_1 = A\sin(\varphi_0), \quad A_2 = A\cos(\varphi_0) \qquad \Rightarrow \qquad A = \sqrt{A_1^2 + A_2^2}, \quad \tan\varphi_0 = \frac{A_1}{A_2}. \tag{4.46}$$

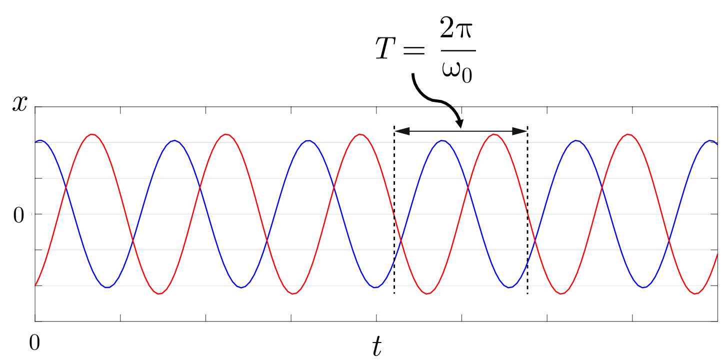
such that we now have the two alternative forms of the solution

$$\boxed{x(t) = A_1 \cos(\omega_0 t) + A_2 \sin(\omega_0 t) = A \sin(\omega_0 t + \varphi_0)} \qquad \text{with} \quad A_i, A, \varphi_0 \in \mathbb{R}. \tag{4.47}$$

ω_0 is the **eigenfrequency** (also known as **fundamental frequency** or **natural frequency**) of the system's free vibration. A is the **amplitude** of the vibration. The time **period** of a complete vibration cycle is

$$T = 2\pi/\omega_0. \tag{4.48}$$

The coefficients in the solution (4.47) are to be found from appropriate initial conditions; for example, if $x(0) = x_0$ and $\dot{x}(0) = v_0$, then $A_1 = x_0$ and $A_2 = v_0/\omega_0$. Note that, while A_1 and A_2 (or A and φ) depend on the initial conditions, the eigenfrequency ω_0 only depends on the system characteristics and not on the initial conditions. Shown below is an example of a free, undamped vibration.



Case 1: undamped vibration

- **Case 2**: **overdamped vibration** $(D > 1)$

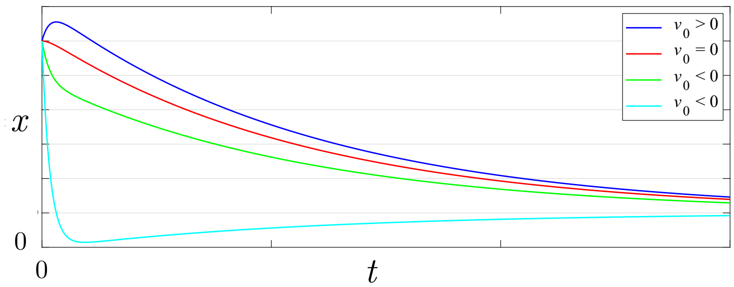
  In case of significant damping with $\delta > \omega_0$, two distinct, real eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ exist:

$$\delta > \omega_0 \qquad \Rightarrow \qquad \lambda_1 = -\delta - \sqrt{\delta^2 - \omega_0^2} < \lambda_2 = -\delta + \sqrt{\delta^2 - \omega_0^2} < 0. \tag{4.49}$$

  Consequently, the solution with constants $A_1, A_2 \in \mathbb{R}$ (again to be found from initial conditions) becomes

$$\boxed{x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}} \qquad \text{with} \qquad \boxed{\lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}} \quad \text{and} \quad A_i \in \mathbb{R}. \tag{4.50}$$

  This solution decays exponentially with time (since $\lambda_1, \lambda_2 < 0$) without displaying harmonic vibrations. For example, if $x(0) = x_0$ and $\dot{x}(0) = v_0$, then A_1, A_2 are determined from $A_1 + A_2 = x_0$ and $A_1\lambda_1 + A_2\lambda_2 = v_0$. Shown below is an example of a free, overdamped vibration for different cases of v_0.



Case 2: overdamped vibration

- **Case 3**: **critically damped vibration** ($D = 1$)

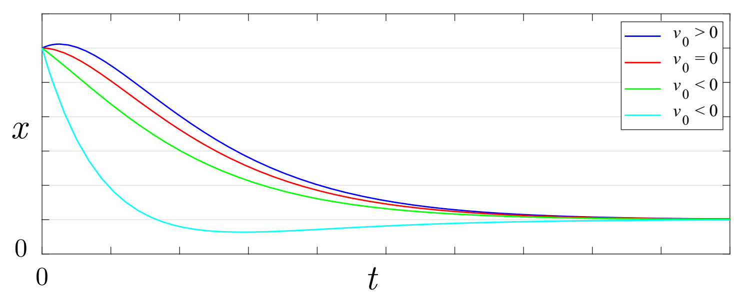
  In the special case of a critically damped system we have

$$\delta = \omega_0 \qquad \Rightarrow \qquad \lambda_1 = \lambda_2 = -\delta = -\omega_0, \tag{4.51}$$

  so only one unique, real eigenvalue exists. The solution is therefore of the form

$$\boxed{x(t) = A_1 e^{-\delta t} + A_2 t e^{-\delta t}} \qquad \text{with} \qquad A_1, A_2 \in \mathbb{R}. \tag{4.52}$$

  The constants are again to be found from initial conditions. Since the second law of thermodynamics requires $\delta > 0$ in general, this solution decays exponentially with time without displaying harmonic vibrations. Shown below is an example of a free, critically damped vibration for different cases of the initial velocity v_0.



Case 3: critically damped vibration

- **Case 4**: **underdamped vibration** ($0 < D < 1$)

  In this final case of low damping, we obtain two complex eigenvalues:

$$0 < \delta < \omega_0 \qquad \Rightarrow \qquad \lambda_{1,2} = -\delta \pm \sqrt{-(\omega_0^2 - \delta^2)} = -\delta \pm i\sqrt{\omega_0^2 - \delta^2} = -\delta \pm i\omega_d \tag{4.53}$$

  with $\omega_d = \sqrt{\omega_0^2 - \delta^2} \in \mathbb{R}^{>0}$. Expanding the exponentials in the solution and rewriting terms as done for undamped vibrations (see Case 1 above) leads to

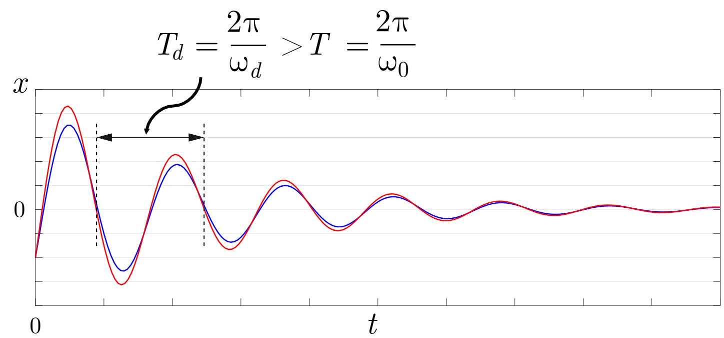
$$\boxed{x(t) = e^{-\delta t}\left[A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)\right]} \qquad \text{with} \qquad \boxed{\omega_d = \sqrt{\omega_0^2 - \delta^2}} \quad A_1, A_2 \in \mathbb{R}. \tag{4.54}$$

Coefficients A_1 and A_2 are again to be found from initial conditions.

In this case, we observe vibrations with the **damped natural frequency** $\omega_d < \omega_0$ so that the period extends to

$$T_d = 2\pi/\omega_d > T. \tag{4.55}$$

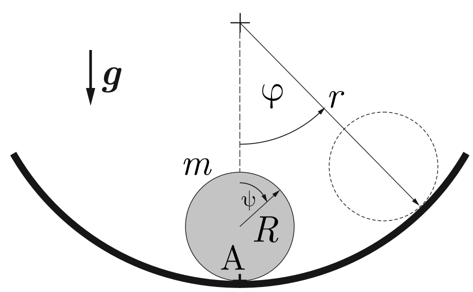
The amplitude of underdamped vibrations decays exponentially, and that decay grows with δ. Shown below is an example of a free, underdamped vibration along with the definition of T_d.



Case 4: underdamped vibration

Example 4.4. Vibration of a rolling cylinder

Consider a cylindrical body of mass m and radius R, which is rolling without slipping on a curved ground having the shape of a circular arch with radius r, as shown on the right. What is the eigenfrequency of the cylinder when oscillating about its equilibrium position?



Let us use Lagrange's equation to derive the equation of motion. The kinetic and potential energy and the resultant Lagrangian are (ψ being the rolling angle of the cylinder)

$$T = \frac{1}{2} I_\mathrm{A} \dot{\psi}^2, \qquad V = mg(r-R)(1-\cos\varphi) \quad \Rightarrow \quad \mathcal{L} = T - V = \frac{1}{2} I_\mathrm{A}\dot{\psi}^2 - mg(r-R)(1-\cos\varphi) \tag{4.56}$$

with $I_\mathrm{A} = \frac{3}{2}mR^2$ (where we exploit that contact point A is the ICR) and the kinematic constraint $R\psi = (r-R)\varphi$ due to rolling without slipping. Consequently, the equation of motion for the single DOF φ follows as

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \frac{\partial \mathcal{L}}{\partial \varphi} = I_\mathrm{A}\frac{(r-R)^2}{R^2}\ddot{\varphi} + mg(r-R)\sin\varphi = \frac{3}{2}m(r-R)^2\ddot{\varphi} + mg(r-R)\sin\varphi, \tag{4.57}$$

which we re-arrange into

$$\ddot{\varphi} + \frac{2g}{3(r-R)}\sin\varphi = 0. \tag{4.58}$$

Obviously, $\varphi = 0$ is a stable equilibrium in this problem (and a minimizer of the potential energy). Hence, we linearize the above equation of motion with $\varphi = 0 + x$ about $\varphi = 0$ for $|x| \ll 1$, which leads to

$$\ddot{x} + \frac{2g}{3(r-R)} x \approx 0. \tag{4.59}$$

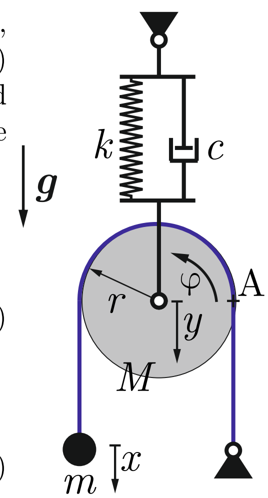
From comparing this equation of motion to the general form (4.34), we identify the eigenfrequency of this system as

$$\omega_0 = \sqrt{\frac{2g}{3(r-R)}}. \tag{4.60}$$

We further note that the system is undamped and without applied forcing, so we except an *undamped free vibration* with the above eigenfrequency.

Example 4.5. Vibration of a spring-mass-damper system

We consider the system shown on the right, consisting of a cylinder (mass M, radius r) connected to a linear spring (stiffness k) and dashpot (viscosity c) in series and, via a massless inextensible rope, to a particle of mass m and the ground. If the particle is given an initial downward speed v_0 (and if the viscosity c is very low), what is the resulting motion of the system?



To find the equation of motion, we define the kinetic and potential energy as

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_{\mathrm{A}} \dot{\varphi}^2, \qquad V = \frac{k}{2} y^2, \tag{4.61}$$

so the Lagrangian follows as

$$\mathcal{L} = T - V = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_{\mathrm{A}} \dot{\varphi}^2 - \frac{k}{2} y^2, \tag{4.62}$$

where we exploit that point A is the ICR, and $I_{\mathrm{A}} = \frac{3}{2} M r^2$. The kinematic constraints of the system reveal that

$$x = 2y, \quad x = 2r\varphi \qquad \Rightarrow \qquad y = \frac{x}{2}, \quad \dot{y} = \frac{\dot{x}}{2}, \quad r\dot{\varphi} = \frac{\dot{x}}{2}. \tag{4.63}$$

Insertion of those relations into the Lagrangian reduces the latter to depend only on a single DOF, here we choose x:

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2}\frac{3}{2} M r^2 \frac{\dot{x}^2}{4r^2} - \frac{k}{2}\frac{x^2}{4} = \frac{1}{2}\left(m + \frac{3}{8} M\right)\dot{x}^2 - \frac{k}{8} x^2. \tag{4.64}$$

The non-conservative force in the dashpot is

$$Q_x^{\mathrm{nc}} = -c\dot{y}\frac{\partial y}{\partial x}, \tag{4.65}$$

where the sign stems from the force acting against coordinate y, and the partial derivative from the definition (4.14).

Lagrange's equation yields

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = Q_x^{\mathrm{nc}} \qquad \Rightarrow \qquad \left(m + \frac{3}{8}M\right)\ddot{x} + \frac{k}{4}x = -c\dot{y}\frac{\partial y}{\partial x} = -\frac{c}{4}\dot{x} \tag{4.66}$$

or, after some re-arrangement,

$$\ddot{x} + \underbrace{\frac{c}{4m + \frac{3}{2}M}}_{=2\delta}\dot{x} + \underbrace{\frac{k}{4m + \frac{3}{2}M}}_{=\omega^2}x = 0. \tag{4.67}$$

Comparing this equation of motion with the general form (4.34), we find

$$\omega_0 = \sqrt{\frac{k}{4m + \frac{3}{2}M}}, \qquad \delta = \frac{c}{8m + 3M} \tag{4.68}$$

and further

$$D = \frac{\delta}{\omega_0} = c\sqrt{\frac{8m + 3M}{k}}. \tag{4.69}$$

As discussed above, depending on the value of D, the vibration can be underdamped ($D < 1$), overdamped ($D > 1$), or critically damped ($D = 1$).

For example, in case of low damping, the underdamped vibration manifests in a decaying oscillation with frequency $\omega_d = \sqrt{\omega_0^2 - \delta^2}$ and the general solution (4.54):

$$x(t) = e^{-\delta t}\left[A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)\right]. \tag{4.70}$$

Applying the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$ leads to

$$\begin{aligned} &x(0) = A_1 = 0, \\ &\dot{x}(0) = -\delta A_1 + \omega_d A_2 = v_0 \qquad \Rightarrow \qquad A_2 = v_0/\omega_d, \end{aligned} \tag{4.71}$$

so that the particle moves according to

$$x(t) = \frac{v_0}{\omega_d}e^{-\delta t}\sin(\omega_d t) \qquad \text{with} \qquad \omega_d = \sqrt{\omega_0^2 - \delta^2} \le \omega_0. \tag{4.72}$$

Notice that the vibration frequency in the damped case is lower than that of the undamped case. As a result, the period of vibration $T = 1/\omega$ is longer in the damped case than in the undamped case.

4.3.3 Forced vibrations

If the forcing $f(t)$ is non-zero, then we deal with a so-called **forced vibration**. In this case, the system is not free to vibrate but literally forced to vibrate through an applied force or torque that varies periodically in time (consider, e.g., an engine that imparts a cyclic load onto its surroundings). Here, the equation of motion is the inhomogeneous ODE

$$\ddot{x}(t) + 2\delta\dot{x}(t) + \omega_0^2 x(t) = f(t), \tag{4.73}$$

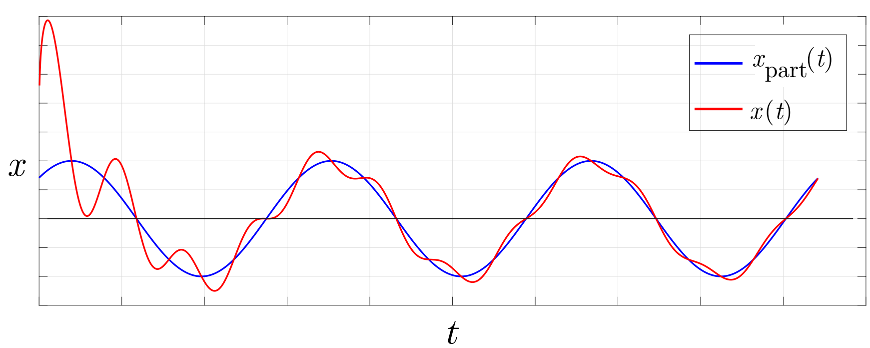
whose solution is to be found as

$$x(t) = x_{\text{hom}}(t) + x_{\text{part}}(t). \tag{4.74}$$

The **homogeneous solution** $x_{\text{hom}}(t)$ is the same as the solution derived above for a free vibration. Note that, except for the undamped case, $x_{\text{hom}}(t)$ always decays with time. The **particular solution** $x_{\text{part}}(t)$ depends on $f(t)$ and survives in the presence of damping for all times. Therefore, we may expect that for *vibrations with* $\delta > 0$ the solution after long times is

$$x(t) \approx x_{\text{part}}(t) \qquad \text{as} \qquad t \to \infty. \tag{4.75}$$

Shown below is an example of a solution $x(\mathrm{t}) = x_{\text{hom}}(t) + x_{\text{part}}(t)$, indicated in red, consisting of the homogeneous solution $x_{\text{hom}}(t)$ and the particular solution $x_{\text{part}}(t)$ in response to a sinusoidal loading, with $x_{\text{part}}(t)$ indicated in blue. For nonzero damping, the homogeneous solution vanishes exponentially over time, so that the general solution $x(t)$ approaches the particular solution $x_{\text{part}}(t)$.



In many real-world applications, the external forcing $f(t)$ is time-harmonic with a known excitation frequency. For example, consider vibrations caused to rotating machine or vehicle parts. Therefore (and because it allows for analytical solutions), we will consider an applied time-dependent force[16] $f(t)$ with an **excitation frequency** $\Omega \in \mathbb{R}$, the most general form of which is

$$f(t) = \hat{f}\cos(\Omega t) + \hat{g}\sin(\Omega t) \qquad \text{where} \qquad \hat{f}, \hat{g} \in \mathbb{R} \tag{4.76}$$

are constant coefficients that define the amplitude of the excitation force. Note that, here and in the following, we usually refer to $f(t)$ as the excitation *force*, even though the concepts apply equally to problems in which x is a rotational DOF, so that $f(t)$ is in fact an applied excitation *torque*.

[16]Here and in the following, we simply speak of the excitation *force* $f(t)$, even though it is, strictly speaking, force per mass as $f = F/m$.

Brief mathematical interlude:

Before we present the solution to the above problem of a forced vibration, let us quickly introduce (or review) a mathematical trick that will prove helpful here. For convenience (to avoid the use of trigonometric relations), we exploit Euler's identity

$$\exp(i\Omega t) = \cos(\Omega t) + i\sin(\Omega t), \tag{4.77}$$

so that we may write

$$f(t) = \hat{f}\cos(\Omega t) + \hat{g}\sin(\Omega t) = \text{Re}\left[(\hat{f} - i\hat{g})\exp(i\Omega t)\right] \tag{4.78}$$

since, by (4.77),

$$(\hat{f} - i\hat{g})\exp(i\Omega t) = \hat{f}\cos(\Omega t) + \hat{g}\sin(\Omega t) + i\left[-\hat{g}\cos(\Omega t) + \hat{f}\sin(\Omega t)\right]. \tag{4.79}$$

That is, we may rewrite (4.76) as

$$f(t) = \text{Re}\left[\hat{f}^*\exp(i\Omega t)\right] \qquad \text{with} \quad \hat{f}^* \in \mathbb{C}, \tag{4.80}$$

whereby we replace the coefficients $\hat{f}, \hat{g} \in \mathbb{R}$ by a complex-valued unknown $\hat{f}^* \in \mathbb{C}$, whose real and imaginary components define the amplitudes $\hat{f}$ and $\hat{g}$. To differentiate the two, here and in the following we use an asterisk (*) to denote all quantities in the complex domain.

An important aspect to realize is that we are solving a *linear* ODE:

$$\ddot{x}(t) + 2\delta\dot{x}(t) + \omega_0^2 x(t) = f(t) \quad \Leftrightarrow \quad \mathcal{L}[x(t)] = f(t) \quad \text{with} \quad \mathcal{L} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\delta\frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2 \tag{4.81}$$

being a linear differential operator. As a consequence, we may replace both $F(t)$ and $x(t)$ by complex-valued functions $F^*(t)$ and $x^*(t)$, so that, if $x^*(t)$ is a solution to

$$\mathcal{L}[x^*(t)] = f^*(t), \tag{4.82}$$

then we also know (from taking real and imaginary parts of this latter equation) that

$$\text{Re}\big[\mathcal{L}[x^*(t)]\big] = \mathcal{L}\big[\text{Re}[x^*(t)]\big] = \text{Re}[f^*(t)]. \tag{4.83}$$

In other words, when inserting $f^*(t)$ and solving for $x^*(t)$, then we know that $\text{Re}[x^*(t)]$ is a solution to the problem with excitation $\text{Re}[f^*(t)]$, whereas – analogously – $\text{Im}[x^*(t)]$ is a solution to the problem with excitation $\text{Im}[f^*(t)]$. Therefore, we will use the complex excitation

$$f^*(t) = \hat{f}^*\exp(i\Omega t) \tag{4.84}$$

and solve for the complex solution $x^*(t)$, from which we conclude that the sought solution is

$$x(t) = \text{Re}[x^*(t)]. \tag{4.85}$$

In the following, we will follow this strategy to avoid the use of trigonometric functions.

Let us insert the general *complex* function $f(t) = \hat{f}^* \exp(i\Omega t)$ into the governing ODE (4.73) and solve for the resulting *complex* solution $x^*(t)$, from which we may conclude that the sought particular solution for the forced-vibration problem is

$$x_{\text{part}}(t) = \text{Re}[x^*_{\text{part}}(t)]. \tag{4.86}$$

Insertion of $F^*(t) = \hat{F}^* \exp(i\Omega t)$ into the equation of motion (assuming the homogeneous solution decays) yields

$$\ddot{x}^*_{\text{part}}(t) + 2\delta\dot{x}^*_{\text{part}}(t) + \omega_0^2 x^*_{\text{part}}(t) = \hat{f}^* \exp(i\Omega t)t. \tag{4.87}$$

Since this equation must hold for all times t, the solution must be of the form

$$x^*_{\text{part}}(t) = \hat{x}^* \exp(i\Omega t) \qquad \text{with} \qquad \hat{x}^* \in \mathbb{C}. \tag{4.88}$$

Insertion into (4.87) leads to

$$\left(-\Omega^2 + 2i\Omega\delta + \omega_0^2\right) \hat{x}^* \exp(i\Omega t) = \hat{f}^* \exp(i\Omega t) \quad \forall\ t, \tag{4.89}$$

so that

$$\hat{x}^* = \frac{\hat{f}^*}{\left(-\Omega^2 + 2i\Omega\delta + \omega_0^2\right)} = \frac{\hat{f}^*\left(\omega_0^2 - \Omega^2\right)}{\left[(\omega_0^2 - \Omega^2)^2 + 4\Omega^2\delta^2\right]} + i\frac{\hat{f}^*\left(-2\Omega\delta\right)}{\left[(\omega_0^2 - \Omega^2)^2 + 4\Omega^2\delta^2\right]} \equiv \hat{x}_r + i\hat{x}_i.$$

Recall that, by Euler's identity, $(a + ib) = |a + ib| \exp[i \arctan(b/a)]$. Therefore, we write

$$\begin{aligned} x^*_{\text{part}}(t) &= \hat{x}^* \exp(i\Omega t) = (\hat{x}_r + i\hat{x}_i) \exp(i\Omega t) = |\hat{x}^*| \exp(i\Omega t) \exp\left[i \arctan\left(\frac{\hat{x}_i}{\hat{x}_r}\right)\right] \\ &= |\hat{x}^*| \exp\left[i(\Omega t - \varphi)\right] \end{aligned} \tag{4.90}$$

with the **phase delay**

$$\varphi = -\arctan\left(\frac{\hat{x}_i}{\hat{x}_r}\right) = \arctan\left(\frac{2\Omega\delta}{\omega_0^2 - \Omega^2}\right) \tag{4.91}$$

and the **amplitude**

$$|\hat{x}^*| = \sqrt{\hat{x}_r^2 + \hat{x}_i^2} = \frac{|\hat{f}^*|}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\Omega^2\delta^2}}. \tag{4.92}$$

The phase delay indicates that the motion $x_{\text{part}}(t)$ lags behind the applied forcing $f(t)$. As can be expected, the phase delay vanishes in case of zero damping (i.e., when $\delta = 0$).

It is important to notice that both phase delay and amplitude depend on the excitation frequency Ω. The amplitude peaks at **resonance** when $\Omega = \omega_0$, i.e., when the system is driven at the **natural frequency**. At resonance (and also near resonance), the amplitude grows significantly but remains bounded as long as $\delta \neq 0$. If there is no damping ($\delta = 0$), then the amplitude tends to infinity at resonance. As Ω is increased from below through the resonance, the phase delay jumps from 0 to π at resonance.

Introducing the frequency ratio η and re-using Lehr's damping ratio D, defined as

$$\boxed{\eta = \frac{\Omega}{\omega_0}, \qquad D = \frac{\delta}{\omega_0}} \tag{4.93}$$

we may write the amplitude of (the particular solution of) a forced vibration as

$$\boxed{|\hat{x}^*| = \frac{|\hat{f}^*|}{\omega_0^2} V(D,\eta) \qquad \text{with} \qquad V(D,\eta) = \frac{\omega_0^2}{\sqrt{(\omega_0^2-\Omega^2)^2 + 4\Omega^2\delta^2}} = \frac{1}{\sqrt{(1-\eta^2)^2 + 4D^2\eta^2}}} \tag{4.94}$$

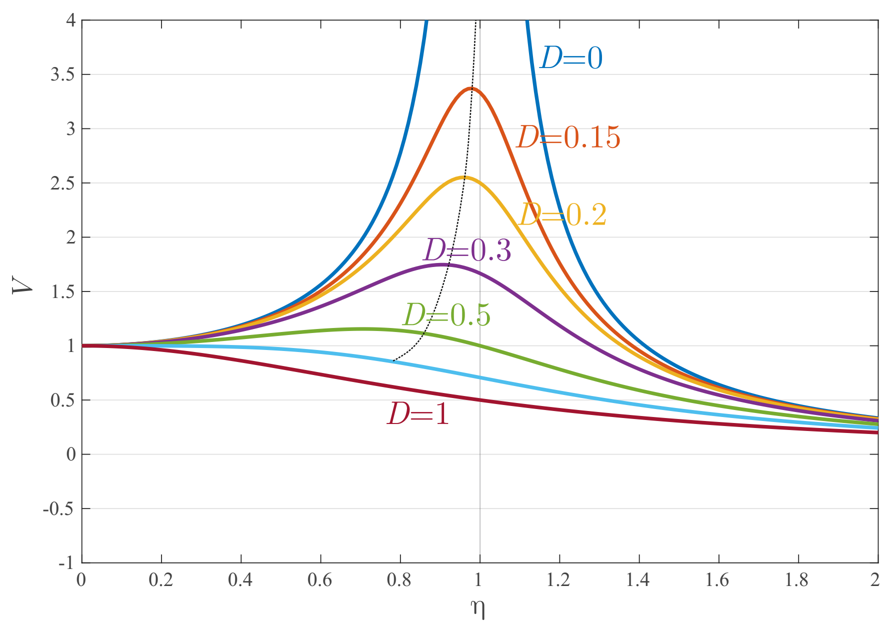
where V is the so-called **magnification factor** (if $V = 1$, then the amplitude of the static response with $\Omega = 0$ is recovered). Using the same dimensionless ratios, the phase delay becomes

$$\boxed{\varphi = \arctan\left(\frac{2\Omega\delta}{\omega_0^2 - \Omega^2}\right) = \arctan\left(\frac{2D\eta}{1-\eta^2}\right)} \tag{4.95}$$

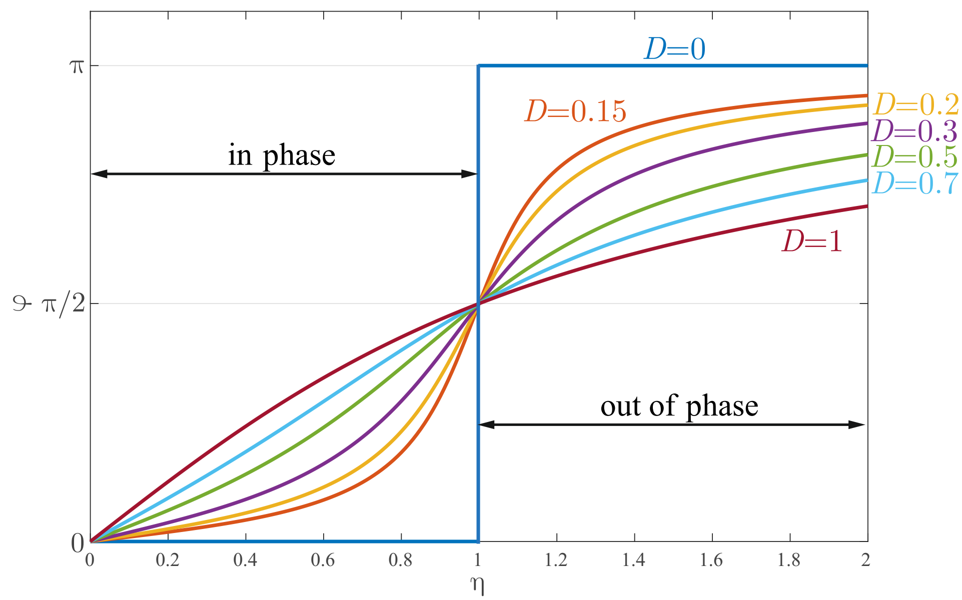
Shown below is the amplification factor $V(\eta; D)$, plotted as functions of the frequency ratio η for various dimensionless damping values D. The *maximum magnification* for a given D-value is achieved when

$$\eta = \sqrt{1-2D^2} \qquad \Leftrightarrow \qquad \Omega = \omega_0\sqrt{1-2D^2}. \tag{4.96}$$

The dashed line indicates the curve of the maximum amplification V for the shown values of D.



The next graphic below illustrates the phase shift $\varphi(\eta; D)$, plotted as functions of the frequency ratio η for various dimensionless damping values D. Note that for $D = 0$ the phase shift jumps at the natural frequency ω_0 (i.e., at $\eta = 1$) from 0 to π.



Finally, recall that the solution required the recovery of only the real part of $x^*_{\text{part}}(t)$. Therefore, e.g., if the applied excitation is of the form

$$\boxed{f(t) = \hat{f}\cos(\Omega t)} \qquad \Leftrightarrow \qquad \hat{f}_r = \hat{f}, \quad \hat{f}_i = 0, \tag{4.97}$$

then the particular solution is obtained as

$$x_{\text{part}}(t) = \text{Re}\left[x^*_{\text{part}}(t)\right] \qquad \Rightarrow \qquad \boxed{x_{\text{part}}(t) = \frac{\hat{f}}{\omega_0^2} V(D,\eta)\cos\left(\Omega t - \varphi\right)} \tag{4.98}$$

By analogy, we may conclude that

$$f(t) = \hat{f}\sin(\Omega t) \qquad \Rightarrow \qquad x_{\text{part}}(t) = \frac{\hat{f}}{\omega_0^2} V(D,\eta)\sin\left(\Omega t - \varphi\right). \tag{4.99}$$

Finally, recall that the complete solution of a forced vibration is

$$\boxed{x(t) = x_{\text{hom}}(t) + x_{\text{part}}(t)} \tag{4.100}$$

where, as discussed before, the homogeneous solution is that of a free vibration (which was derived in Section 4.3.2), which decays over time if $\delta > 0$.

We note that in the **special case of resonance** of an **undamped system**, i.e., if $\Omega = \omega_0$ for the case of $\delta = 0$, the above general solution does not apply. As long as $\delta \neq 0$, even if small, the above relations apply. Yet, if $\delta = 0$, we obtain the particular solution from solving the governing equation (written in its complex form)

$$\ddot{x}^*_{\text{part}}(t) + \omega_0^2 x^*_{\text{part}}(t) = f^*(t) = \hat{f}^* \exp(i\omega_0 t), \tag{4.101}$$

for which $x^*_{\text{part}}(t) = \hat{x}^* \exp(i\omega_0 t)$ is clearly *not* a solution (it renders the left-hand-side of (4.101) zero). In this special case, the solution has the form

$$x^*_{\text{part}}(t) = \hat{x}^* t \exp(i\omega_0 t). \tag{4.102}$$

To verify this, let us insert (4.102) into (4.101). Note that

$$\dot{x}^*_{\text{part}} = \hat{x}^* \left(1 + i\omega_0 t\right) \exp(i\omega_0 t) \qquad \text{and} \qquad \ddot{x}^*_{\text{part}} = \hat{x}^* \left(2i\omega_0 - \omega_0^2 t\right) \exp(i\omega_0 t), \tag{4.103}$$

which yields upon insertion into (4.101)

$$\hat{x}^* \left(2i\omega_0 - \omega_0^2 t + \omega_0^2 t\right) \exp(i\omega_0 t) = \hat{f}^* \exp(i\omega_0 t) \qquad \Rightarrow \qquad \hat{x}^* = \frac{\hat{f}^*}{2i\omega_0} = -i\frac{\hat{f}^*}{2\omega_0}. \tag{4.104}$$

Therefore, the special case of undamped resonance leads to, using $-i = \exp(-i\pi/2)$,

$$x^*_{\text{part}}(t) = -i\frac{\hat{f}^*}{2\omega_0} t \exp(i\omega_0 t) = \frac{\hat{f}^*}{2\omega_0} t \exp\left[i(\omega_0 t - \tfrac{\pi}{2})\right], \tag{4.105}$$

which is out-of-phase by $\varphi = \pi/2$ (as in the damped cases above at resonance), and the magnification is given by

$$V = \frac{|\hat{x}^*|\omega_0^2}{|\hat{f}^*|} = \frac{\omega_0}{2}. \tag{4.106}$$

As an example, if the system is excited by

$$f(t) = \hat{f} \cos(\omega_0 t), \tag{4.107}$$

then the undamped system response is

$$x(t) = x_{\text{hom}}(t) + \frac{\hat{f}}{2\omega_0} t \sin(\omega_0 t), \tag{4.108}$$

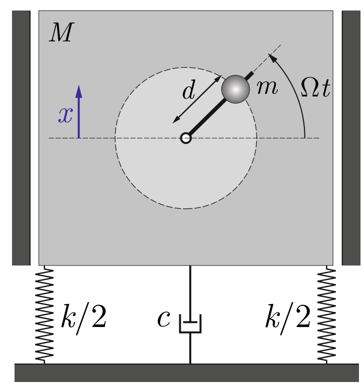
whose homogeneous contribution does *not* decay over time (in the absence of damping).

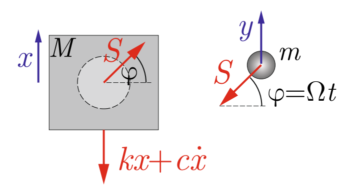
Importantly, the amplitude of the particular contribution in (4.108) *grows linearly with time* (of course, only within the assumptions made to arrive at the linearized equation of motion). Therefore, undamped forced vibrations quickly lead to excessive amplitudes at resonance (i.e., if the excitation frequency coincides with the resonant frequency).

Why does resonance lead to excessive vibrations? Consider the simple example of pushing a child on a swing-set. Without excitation it oscillates back and forth approximately at the natural frequency of a pendulum, $\omega_0 = \sqrt{g/l}$. In order to push the child higher, how would one optimally excite the system? Of course, it's intuitive to push the child at that exact same frequency – e.g., whenever it reaches the front- or back-most point (but one would not push at some random frequency while it is swinging in between). In order for the pushing to boost the child's amplitude, the forcing frequency should match the eigenfrequency. Admittedly, this is a simplistic analogy, but it illustrates the concept quite effectively.

Example 4.6. Unbalanced rotation

Let us consider a motor of mass M sitting on a machine bed (approximated as a linear spring of stiffness k and a linear dashpot of viscosity c). A particle of mass m is attached at a distance d eccentrically to the massless rotor arm of the motor, which is rotating with a constant angular velocity Ω, as schematically shown below. What is the system response?





In order to find the equation of motion of the system, we draw free-body diagrams, as shown below on the right. Linear momentum balance for the large motor with DOF x pointing upwards reads

$$M\ddot{x} = -kx - c\dot{x} + S\sin\varphi, \qquad \varphi = \Omega t, \tag{4.109}$$

where S denotes the magnitude of the force in the rotor arm, and $\varphi = \Omega t$ is the angle of the arm with the horizontal axis, as shown. To relate the motion of the two masses, let us formulate the kinematic relation

$$y = x + d\sin(\Omega t) \qquad \Rightarrow \qquad \ddot{y} = \ddot{x} - d\Omega^2\sin(\Omega t), \tag{4.110}$$

which indicates that the vertical motion $y(t)$ of the particle is the sum of the vertical motion $x(t)$ of the large mass and the relative motion due to the rotation of the particle. Linear momentum balance for the eccentric particle reveals

$$m\ddot{y} = -S\sin(\Omega t), \tag{4.111}$$

which may be solved for S, inserting (4.110), to yield

$$S\sin(\Omega t) = -m\ddot{x} + md\Omega^2\sin(\Omega t). \tag{4.112}$$

Finally, insertion into (4.109) leads to the equation of motion

$$(M+m)\ddot{x} + c\dot{x} + kx = md\Omega^2\sin(\Omega t) \tag{4.113}$$

or, equivalently,

$$\ddot{x} + \underbrace{\frac{c}{M+m}}_{=2\delta}\dot{x} + \underbrace{\frac{k}{M+m}}_{=\omega_0^2}x = \underbrace{\frac{m}{M+m}d\,\Omega^2\sin(\Omega t)}_{=f(t)=\hat{f}\sin(\Omega t)}. \tag{4.114}$$

Comparing this equation of motion with the general form (4.34) yields

$$\omega_0 = \sqrt{\frac{k}{M+m}}, \quad \delta = \frac{c}{2(M+m)} \qquad \Rightarrow \qquad D = \frac{\delta}{\omega_0} = \frac{c\sqrt{M+m}}{2\sqrt{k}}, \quad \eta = \frac{\Omega}{\omega_0}. \tag{4.115}$$

The general solution is

$$x(t) = x_{\text{hom}}(t) + x_{\text{part}}(t) \tag{4.116}$$

with the homogeneous solution being that of a free damped vibration (the exact nature of the solution depends on the value of D, as discussed in Section 4.3.2). The particular solution takes the form (4.99), here

$$x_{\text{part}}(t) = \frac{\hat{f}}{\omega_0^2} V(D,\eta) \sin(\Omega t - \varphi) = \frac{md\Omega^2}{k} V(D,\eta) \sin(\Omega t - \varphi). \tag{4.117}$$

The maximum vibration amplitude is reached when

$$\Omega = \omega_0 \sqrt{1 - 2D^2}. \tag{4.118}$$

4.4 Multi-DOF Vibrations

Systems of particles or bodies described by more than one generalized DOF may produce vibrations referred to as **multi-DOF vibrations**. Consider a system described by n generalized degrees of freedom. For convenience, we introduce vectors including all DOFs, including the vectors of

$$\begin{aligned} \text{generalized DOFs:} \quad & \boldsymbol{q}(t) \in \mathbb{R}^n, \\ \text{equilibrium positions:} \quad & \boldsymbol{q}_0 \in \mathbb{R}^n, \\ \text{deviations from equilibrium:} \quad & \boldsymbol{x}(t) = \boldsymbol{q}(t) - \boldsymbol{q}_0 \in \mathbb{R}^n. \end{aligned} \tag{4.119}$$

As for the single-DOF case, vibrations for our purposes are understood as small oscillations about a stable equilibrium $\boldsymbol{q}_0$. To this end, we may linearize the (generally nonlinear) equations of motion about a stable equilibrium and use the resulting linear equations of motion as the basis of our analysis.

4.4.1 Equations of motion for multi-DOF vibrations

To arrive at the linearized equations of motion, we generally have two alternatives:

- We derive the (nonlinear) equations of motion (e.g., via the Lagrange equations, or from linear/angular momentum balance) and then *linearize the nonlinear equations of motion* about the equilibrium.
- We expand the (nonlinear) kinetic and potential energies about the equilibrium to quadratic order and then *obtain the linearized equations of motion from the Lagrange equations.*

We will briefly outline both approaches here for completeness.

Following the **first approach**, let us assume the (generally nonlinear) equations of motion can be written as

$$\boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \boldsymbol{F}^{\text{ext}}(t), \tag{4.120}$$

for which a stable equilibrium is identified to satisfy $\boldsymbol{F}(\boldsymbol{q}_0,\boldsymbol{0},\boldsymbol{0})=\boldsymbol{0}$. $\boldsymbol{F}^{\text{ext}}(t)$ are small external forces applied after equilibrium to excite vibrations. A linearization of the equations of motion about the equilibrium via Taylor expansion leads to

$$\boldsymbol{F}(\boldsymbol{q}_0,\boldsymbol{0},\boldsymbol{0})+\frac{\partial\boldsymbol{F}}{\partial\boldsymbol{q}}(\boldsymbol{q}_0,\boldsymbol{0},\boldsymbol{0})\boldsymbol{x}+\frac{\partial\boldsymbol{F}}{\partial\dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0},\boldsymbol{0})\dot{\boldsymbol{x}}+\frac{\partial\boldsymbol{F}}{\partial\ddot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0},\boldsymbol{0})\ddot{\boldsymbol{x}}+\text{h.o.t.}=\boldsymbol{F}^{\text{ext}}(t) \tag{4.121}$$

or approximately (inserting $\boldsymbol{F}(\boldsymbol{q}_0,\boldsymbol{0},\boldsymbol{0})=\boldsymbol{0}$ by equilibrium and dropping all higher-order terms)

$$\boxed{\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{C}\dot{\boldsymbol{x}}+\boldsymbol{K}\boldsymbol{x}=\boldsymbol{F}^{\text{ext}}} \tag{4.122}$$

with the three *constant* matrices[17]

$$\boldsymbol{M}=\frac{\partial\boldsymbol{F}}{\partial\ddot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0},\boldsymbol{0}),\qquad \boldsymbol{C}=\frac{\partial\boldsymbol{F}}{\partial\dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0},\boldsymbol{0}),\qquad \boldsymbol{K}=\frac{\partial\boldsymbol{F}}{\partial\boldsymbol{q}}(\boldsymbol{q}_0,\boldsymbol{0},\boldsymbol{0}), \tag{4.123}$$

known as **mass matrix**, **damping matrix** and **stiffness matrix**, respectively. Note that these definitions, in general, do not reveal anything about the structure or properties of the three matrices. This will be different in the second approach demonstrated below, which accounts for the origin of the equations of motion.

The **second approach** is a bit more elaborate as we start at the energetic level. Consider a kinetic energy of the form $T=T(\boldsymbol{q},\dot{\boldsymbol{q}})$ as well as a potential energy $V=V(\boldsymbol{q})$. For small deviations $\boldsymbol{x}$ around a stable equilibrium $\boldsymbol{q}_0$ so that $\boldsymbol{q}(t)=\boldsymbol{q}_0+\boldsymbol{x}(t)$, we may Taylor expand the energies around an equilibrium $(\boldsymbol{q}_0,\dot{\boldsymbol{q}}_0)$ with $\dot{\boldsymbol{q}}_0=\boldsymbol{0}$. We expand up to *quadratic* order, so that differentiation in the Lagrange equations leads to linear equations of motion. For the kinetic energy this leads to

$$\begin{aligned}T(\boldsymbol{q},\dot{\boldsymbol{q}})=T(\boldsymbol{q}_0,\boldsymbol{0})&+\frac{\partial T}{\partial\boldsymbol{q}}(\boldsymbol{q}_0,\boldsymbol{0})\cdot\boldsymbol{x}+\frac{1}{2}\boldsymbol{x}\cdot\frac{\partial^2 T}{\partial\boldsymbol{q}\partial\boldsymbol{q}}(\boldsymbol{q}_0,\boldsymbol{0})\boldsymbol{x}+\frac{\partial T}{\partial\dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\cdot\dot{\boldsymbol{x}}\\&+\frac{1}{2}\dot{\boldsymbol{x}}\cdot\frac{\partial^2 T}{\partial\dot{\boldsymbol{q}}\partial\dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\dot{\boldsymbol{x}}+\frac{1}{2}\boldsymbol{x}\cdot\frac{\partial^2 T}{\partial\boldsymbol{q}\partial\dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\dot{\boldsymbol{x}}+\text{h.o.t.}\end{aligned} \tag{4.124}$$

Note that for physical reasons we may expect that $T(\boldsymbol{q},\boldsymbol{0})\equiv 0$ for vanishing velocities $\dot{\boldsymbol{q}}=\boldsymbol{0}$, i.e., if no motion occurs. Therefore, we may set

$$T(\boldsymbol{q}_0,\boldsymbol{0})=0,\qquad \frac{\partial T}{\partial\boldsymbol{q}}(\boldsymbol{q}_0,\boldsymbol{0})=\boldsymbol{0},\qquad \frac{\partial^2 T}{\partial\boldsymbol{q}\partial\boldsymbol{q}}(\boldsymbol{q}_0,\boldsymbol{0})=\boldsymbol{0}, \tag{4.125}$$

leading to

$$T(\boldsymbol{q},\dot{\boldsymbol{q}})\approx\frac{\partial T}{\partial\dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\cdot\dot{\boldsymbol{x}}+\frac{1}{2}\dot{\boldsymbol{x}}\cdot\frac{\partial^2 T}{\partial\dot{\boldsymbol{q}}\partial\dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\dot{\boldsymbol{x}}+\frac{1}{2}\boldsymbol{x}\cdot\frac{\partial^2 T}{\partial\boldsymbol{q}\partial\dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\dot{\boldsymbol{x}}. \tag{4.126}$$

Similarly, we consider conservative forces represented by a potential energy $V=V(\boldsymbol{q})$. Taylor expansion about equilibrium up to quadratic order gives

$$V\approx V(\boldsymbol{q}_0)+\frac{\partial V}{\partial\boldsymbol{q}}(\boldsymbol{q}_0)\cdot\boldsymbol{x}+\frac{1}{2}\boldsymbol{x}\cdot\frac{\partial^2 V}{\partial\boldsymbol{q}\partial\boldsymbol{q}}(\boldsymbol{q}_0)\boldsymbol{x}. \tag{4.127}$$

[17] The vector derivatives are to be evaluated component-wise; e.g., $\boldsymbol{M}=\partial\boldsymbol{F}/\ddot{\boldsymbol{q}}$ is a short, symbolic notation for the components $M_{ij}=\partial F_i/\partial\ddot{q}_j$. The other derivatives are defined analogously.

If non-conservative forces $\boldsymbol{Q}^{\mathrm{nc}} = \boldsymbol{Q}^{\mathrm{nc}}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{F}^{\mathrm{ext}}(t)$ act on the system, their approximation to leading order is obtained from a Taylor expansion up to linear order (since $\boldsymbol{Q}^{\mathrm{nc}}$ enters the Lagrange equations directly without differentiation, we truncate to linear order):

$$\boldsymbol{Q}^{\mathrm{nc}}(\boldsymbol{q},\dot{\boldsymbol{q}}) \approx \boldsymbol{Q}^{\mathrm{nc}}(\boldsymbol{q}_0,\boldsymbol{0}) + \frac{\partial \boldsymbol{Q}^{\mathrm{nc}}}{\partial \boldsymbol{q}}(\boldsymbol{q}_0,\boldsymbol{0})\boldsymbol{x} + \frac{\partial \boldsymbol{Q}^{\mathrm{nc}}}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\dot{\boldsymbol{x}} + \boldsymbol{F}^{\mathrm{ext}}(t). \tag{4.128}$$

Note that we include time-dependent external forces $\boldsymbol{F}^{\mathrm{ext}}(t)$ here, which excite the system and do not depend on $\boldsymbol{q}$ or $\dot{\boldsymbol{q}}$. Finally, we apply the Lagrange equations to the approximate forms (4.126), (4.127) and (4.128). The first term in the Lagrange equations evaluates to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\boldsymbol{x}}}\right) &\approx \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0}) + \frac{\partial^2 T}{\partial \dot{\boldsymbol{q}}\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\dot{\boldsymbol{x}} + \frac{1}{2}\frac{\partial^2 T}{\partial \boldsymbol{q}\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\boldsymbol{x}\right] \\ &= \frac{\partial^2 T}{\partial \dot{\boldsymbol{q}}\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\ddot{\boldsymbol{x}} + \frac{1}{2}\frac{\partial^2 T}{\partial \boldsymbol{q}\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\dot{\boldsymbol{x}},\end{aligned} \tag{4.129}$$

where we exploited that none of the derivatives depend on time, since they are evaluated at $(\boldsymbol{q}_0,\boldsymbol{0})$. Similarly, we obtain the second term in the Lagrange equations as

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \frac{\partial T}{\partial \boldsymbol{x}} - \frac{\partial V}{\partial \boldsymbol{x}} \approx \frac{1}{2}\frac{\partial^2 T}{\partial \boldsymbol{q}\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\dot{\boldsymbol{x}} - \frac{\partial V}{\partial \boldsymbol{q}}(\boldsymbol{q}_0) - \frac{\partial^2 V}{\partial \boldsymbol{q}\partial \boldsymbol{q}}(\boldsymbol{q}_0)\boldsymbol{x}.$$

Altogether, the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{Q}^{\mathrm{nc}} \tag{4.130}$$

now turn into

$$\begin{aligned}&\frac{\partial^2 T}{\partial \dot{\boldsymbol{q}}\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\ddot{\boldsymbol{x}} + \frac{\partial V}{\partial \boldsymbol{q}}(\boldsymbol{q}_0) + \frac{\partial^2 V}{\partial \boldsymbol{q}\partial \boldsymbol{q}}(\boldsymbol{q}_0)\boldsymbol{x} \\ &\qquad = \boldsymbol{Q}^{\mathrm{nc}}(\boldsymbol{q}_0,\boldsymbol{0}) + \frac{\partial \boldsymbol{Q}^{\mathrm{nc}}}{\partial \boldsymbol{q}}(\boldsymbol{q}_0,\boldsymbol{0})\boldsymbol{x} + \frac{\partial \boldsymbol{Q}^{\mathrm{nc}}}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\dot{\boldsymbol{x}} + \boldsymbol{F}^{\mathrm{ext}}(t).\end{aligned} \tag{4.131}$$

Since the Lagrange equations at equilibrium dictate that

$$\frac{\partial V}{\partial \boldsymbol{q}}(\boldsymbol{q}_0) = \boldsymbol{Q}^{\mathrm{nc}}(\boldsymbol{q}_0,\boldsymbol{0}), \tag{4.132}$$

the above further reduces to

$$\frac{\partial^2 T}{\partial \dot{\boldsymbol{q}}\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\ddot{\boldsymbol{x}} - \frac{\partial \boldsymbol{Q}^{\mathrm{nc}}}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0})\dot{\boldsymbol{x}} + \left[\frac{\partial^2 V}{\partial \boldsymbol{q}\partial \boldsymbol{q}}(\boldsymbol{q}_0) - \frac{\partial \boldsymbol{Q}^{\mathrm{nc}}}{\partial \boldsymbol{q}}(\boldsymbol{q}_0,\boldsymbol{0})\right]\boldsymbol{x} = \boldsymbol{F}^{\mathrm{ext}}(t). \tag{4.133}$$

Finally, we define the matrices

$$\boldsymbol{M} = \frac{\partial^2 T}{\partial \dot{\boldsymbol{q}}\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0}), \qquad \boldsymbol{C} = -\frac{\partial \boldsymbol{Q}^{\mathrm{nc}}}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0,\boldsymbol{0}), \qquad \boldsymbol{K} = \frac{\partial^2 V}{\partial \boldsymbol{q}\partial \boldsymbol{q}}(\boldsymbol{q}_0) - \frac{\partial \boldsymbol{Q}^{\mathrm{nc}}}{\partial \boldsymbol{q}}(\boldsymbol{q}_0,\boldsymbol{0}), \tag{4.134}$$

so that the linearized equations of motion become

$$\boxed{\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{F}^{\mathrm{ext}}} \tag{4.135}$$

which is identical to the equations of motion derived in (4.122) using the first approach.

No matter how we derive the linearized equations of motion, they involve the following quantities:

$$\begin{aligned}
\boldsymbol{M} \in \mathbb{R}^{n\times n}:&\quad \textbf{mass matrix} \text{ (pos.-definite, symmetric, constant)}\\
\boldsymbol{C} \in \mathbb{R}^{n\times n}:&\quad \textbf{damping matrix} \text{ (usually pos.-semi-definite, symmetric and constant)}\\
\boldsymbol{K} \in \mathbb{R}^{n\times n}:&\quad \textbf{stiffness matrix} \text{ (usually pos.-semi-definite, symmetric and constant)}\\
\boldsymbol{F}^{\text{ext}}(t) \in \mathbb{R}^{n}:&\quad \textbf{external force vector} \text{ (time-dependent).}
\end{aligned} \tag{4.136}$$

The matrix properties follow from their definitions (see, e.g., Eq. (4.134)). By *usually* in (4.136) we mean all cases considered here and in the following, i.e., we will assume that both $\boldsymbol{C}$ and $\boldsymbol{K}$ are symmetric, positive-semi-definite and constant. This is, e.g., the case if $\boldsymbol{Q}^{\text{nc}} = \boldsymbol{Q}^{\text{nc}}(\dot{\boldsymbol{q}}) = -\boldsymbol{C}\dot{\boldsymbol{q}}$ with a constant, symmetric, positive-semi-definite $\boldsymbol{C}$ as for linear viscous drag. However, this may not be the case in general (e.g., if complex non-conservative forces are involved, in case of contact problems or non-smooth dynamics, etc.). Yet, *we will not consider such cases in this course.*

Note that based on the above matrices, we can also express the *kinetic and potential energy of the linear(ized) system* with *constant* external forces $\boldsymbol{F}^{\text{ext}}$ as[18], respectively,

$$\boxed{T = \frac{1}{2}\dot{\boldsymbol{x}} \cdot \boldsymbol{M}\dot{\boldsymbol{x}}, \qquad V = \frac{1}{2}\boldsymbol{x} \cdot \boldsymbol{K}\boldsymbol{x} - \boldsymbol{F}^{\text{ext}} \cdot \boldsymbol{x}} \tag{4.137}$$

The structure of $\boldsymbol{C}$ depends on the nature of the non-conservative forces involved. As a simple example with damping, consider **velocity-proportional damping** as in the case of dashpots between particles or linear viscous drag in a fluid. In this case, we may introduce a dissipation potential, referred to as **Rayleigh's dissipation function** $\mathcal{D}$, such that

$$\boldsymbol{Q}_i^{\text{nc}} = -\frac{\partial \mathcal{D}}{\partial \dot{\boldsymbol{x}}_i} \qquad \text{with} \qquad \mathcal{D} = \frac{1}{2}\dot{\boldsymbol{x}} \cdot \boldsymbol{C}\dot{\boldsymbol{x}} \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}_i}\right) - \frac{\partial \mathcal{L}}{\partial \boldsymbol{x}_i} + \frac{\partial \mathcal{D}}{\partial \dot{\boldsymbol{x}}_i} = \boldsymbol{0}. \tag{4.138}$$

In many structural applications, structural damping is assumed as being velocity-proportional with the approximation, known as **Rayleigh damping**,

$$\boldsymbol{C} = \alpha\boldsymbol{M} + \beta\boldsymbol{K} \qquad \text{with} \qquad \alpha, \beta \geq 0. \tag{4.139}$$

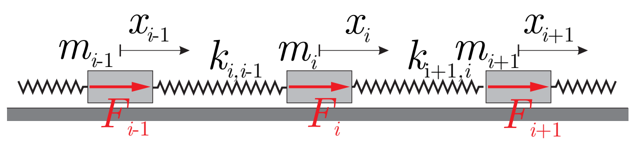
This is a first-order approximation; constants α and β can be tuned to damp low- and high-frequency vibrations, respectively. For example, if particles of equal mass m are submerged in a fluid to experience linear viscous drag, then we consider drag forces $\boldsymbol{Q}_i^{\text{nc}} = \boldsymbol{F}_{\text{drag},i} = -d\dot{\boldsymbol{x}}_i$ with some damping coefficient $d > 0$, so that $\boldsymbol{C} = d\boldsymbol{I} = \frac{d}{m}\boldsymbol{M} = \alpha\boldsymbol{M}$, since for identical particles $\boldsymbol{M} = m\boldsymbol{I}$. Alternatively, if elastically coupled particles are interconnected by dashpots (one dashpot in parallel to each spring connection, and the dashpot viscosity between particles i and j is denoted by $d_{ij} = \beta k_{ij}$), then we have $\boldsymbol{C} = \beta\boldsymbol{K}$. (4.139) presents a linera combination of these two cases.

[18] These are the exact kinetic and potential energy for linear systems, while they are a approximations about an equilibrium for nonlinear systems, following the Taylor expansions presented here.

Example 4.7. Elastic chain of particles

Consider a 1D chain of n particles of masses m_i $(i = 1, \dots, n)$ which are coupled through elastic springs of stiffnesses $k_{i,j}$ $(i, j = 1, \dots, n)$. In this case the equation of motion for each particle i (using the scalar coordinates x_i measured from equilibrium) can be obtained from a free-body diagram (valid for all particles except $i = 1$ and $i = n$):

$$m_i \ddot{x}_i = -k_{i,i-1}(x_i - x_{i-1}) + k_{i,i+1}(x_{i+1} - x_i) + F_i^{\text{ext}}(t). \tag{4.140}$$



For the first and last particle (with one missing spring each) we have the shortened equations

$$m_1 \ddot{x}_1 = k_{1,2}(x_2 - x_1) + F_1^{\text{ext}}(t), \qquad m_n \ddot{x}_n = -k_{n,n-1}(x_n - x_{n-1}) + F_n^{\text{ext}}(t). \tag{4.141}$$

Alternatively and more conveniently for more complex systems, the equations of motion can also be obtained from the Lagrange equations (4.13), using

$$T = \sum_{i=1}^{n} \frac{m_i}{2} \dot{x}_i^2, \quad V = \sum_{i=2}^{n} \frac{k_{i,i-1}}{2}(x_i - x_{i-1})^2 \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i} = F_i^{\text{ext}}, \tag{4.142}$$

which also yields the n equations given by (4.140) and (4.141).

Either way, the obtained equations of motion can be arranged into matrix form, viz.

$$\begin{pmatrix} m_1 & & & & & \\ & m_2 & & & & \\ & & \dots & & & \\ & & & m_i & & \\ & & & & \dots & \\ & & & & & m_n \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_i \\ \vdots \\ \ddot{x}_n \end{pmatrix} + \begin{pmatrix} k_{1,2} & -k_{1,2} & & & & \\ -k_{1,2} & k_{1,2} + k_{2,3} & -k_{2,3} & & & \\ & & \dots & & & \\ & & -k_{i-1,i} & k_{i-1,i} + k_{i,i+1} & -k_{i,i+1} & \\ & & & & \dots & \\ & & & & -k_{n,n-1} & k_{n,n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} F_1^{\text{ext}} \\ F_2^{\text{ext}} \\ \vdots \\ F_i^{\text{ext}} \\ \vdots \\ F_n^{\text{ext}} \end{pmatrix}.$$

This matrix equation is indeed of the form (4.135). Since the system is free of damping and non-conservative forces, we observe that $\boldsymbol{C} = \boldsymbol{0}$ in the above system of equations. The mass matrix and external force vector are, respectively,

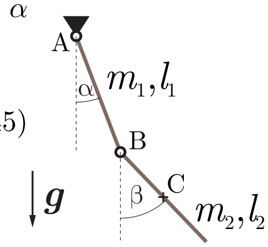
$$[\boldsymbol{M}] = \begin{pmatrix} m_1 & & & & & \\ & m_2 & & & & \\ & & \dots & & & \\ & & & m_i & & \\ & & & & \dots & \\ & & & & & m_n \end{pmatrix} \qquad \text{and} \qquad [\boldsymbol{F}^{\text{ext}}] = \begin{pmatrix} F_1^{\text{ext}} \\ F_2^{\text{ext}} \\ \vdots \\ F_i^{\text{ext}} \\ \vdots \\ F_n^{\text{ext}} \end{pmatrix}, \tag{4.143}$$

while the stiffness matrix reads

$$[\boldsymbol{K}] = \begin{pmatrix} k_{1,2} & -k_{1,2} & & & & \\ -k_{1,2} & k_{1,2}+k_{2,3} & -k_{2,3} & & & \\ & & \cdots & & & \\ & & -k_{i-1,i} & k_{i-1,i}+k_{i,i+1} & -k_{i,i+1} & \\ & & & & \cdots & \\ & & & & -k_{n,n-1} & k_{n,n-1} \end{pmatrix}. \tag{4.144}$$

Example 4.8. Motion of a double-pendulum

Consider a double-pendulum consisting of two rods of masses m_1, m_2 and lengths l_1, l_2, as shown. When describing the system by the two angles α and β as the generalized DOFs so $\boldsymbol{q} = \{\alpha, \beta\}$, the total kinetic energy is



$$T = \frac{1}{2} I_{\mathrm{A}}^1 \dot{\alpha}^2 + \frac{1}{2} I_{\mathrm{C}}^2 \dot{\beta}^2 + \frac{1}{2} m_2 v_{\mathrm{C}}^2 \tag{4.145}$$

with

$$I_{\mathrm{A}}^1 = \frac{m_1 l_1^2}{3}, \quad I_{\mathrm{C}}^2 = \frac{m_2 l_2^2}{12}. \tag{4.146}$$

Note that the last term in the kinetic energy is required since the center of mass of body 2 is moving. The required velocity of the CM of body 2 comes from the velocity transfer formula:

$$\begin{aligned} \boldsymbol{v}_{\mathrm{C}} &= \boldsymbol{v}_{\mathrm{B}} + \boldsymbol{\omega}_2 \times \boldsymbol{r}_{\mathrm{BC}} = \boldsymbol{\omega}_1 \times \boldsymbol{r}_{\mathrm{AB}} + \boldsymbol{\omega}_2 \times \boldsymbol{r}_{\mathrm{BC}} = (\dot{\alpha}\boldsymbol{e}_3) \times \boldsymbol{r}_{\mathrm{AB}} + (\dot{\beta}\boldsymbol{e}_3) \times \boldsymbol{r}_{\mathrm{BC}} \\ &= \left(\dot{\alpha} l_1 \cos\alpha + \dot{\beta}\frac{l_2}{2}\cos\beta\right)\boldsymbol{e}_1 + \left(\dot{\alpha} l_1 \sin\alpha + \dot{\beta}\frac{l_2}{2}\sin\beta\right)\boldsymbol{e}_2, \end{aligned} \tag{4.147}$$

so that altogether with $v_{\mathrm{C}} = |\boldsymbol{v}_{\mathrm{C}}|$

$$T = \frac{m_1 l_1^2}{6}\dot{\alpha}^2 + \frac{m_2 l_2^2{}^2}{24}\dot{\beta}^2 + \frac{1}{2} m_2 \left[\left(\dot{\alpha} l_1 \cos\alpha + \dot{\beta}\frac{l_2}{2}\cos\beta\right)^2 + \left(\dot{\alpha} l_1 \sin\alpha + \dot{\beta}\frac{l_2}{2}\sin\beta\right)^2\right]. \tag{4.148}$$

The potential energy (dependent on the positions of the two centers of mass) becomes

$$V = -m_1 g l_1 \frac{\cos\alpha}{2} - m_2 g\left(l_1\cos\alpha + l_2\frac{\cos\beta}{2}\right) = -g l_1 \cos\alpha\left(\frac{m_1}{2} + m_2\right) - g m_2 l_2 \frac{\cos\beta}{2}. \tag{4.149}$$

The Lagrange equations for this conservative system provide the (generally nonlinear) equations of motion with $\mathcal{L} = T - V$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial \alpha} = 0 \qquad \Leftrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) = \frac{\partial (T-V)}{\partial \alpha} \tag{4.150}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}}\right) - \frac{\partial \mathcal{L}}{\partial \beta} = 0 \qquad \Leftrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\beta}}\right) = \frac{\partial (T-V)}{\partial \beta}. \tag{4.151}$$

The exact equations are left out here for conciseness but can be obtained in closed form by differentatiating the above expressions of T and V. From the potential energy landscape one may easily find that there is one stable equilibrium at $\alpha = \beta = 0$ as well as unstable equilibria at $(\alpha, \beta) = (0, \pm\pi)$, at $(\alpha, \beta) = (\pm\pi, 0)$, and at $(\alpha, \beta) = (\pm\pi, \pm\pi)$.

We may also obtain the linearized system matrices from the above potentials, viz.

$$\boldsymbol{M} = \frac{\partial^2 T}{\partial \dot{\boldsymbol{q}} \partial \dot{\boldsymbol{q}}} \qquad \Rightarrow \qquad [\boldsymbol{M}] = \begin{pmatrix} \frac{\partial^2 T}{\partial \dot{\alpha}^2}, & \frac{\partial^2 T}{\partial \dot{\alpha} \partial \dot{\beta}} \\ \frac{\partial^2 T}{\partial \dot{\alpha} \partial \dot{\beta}}, & \frac{\partial^2 T}{\partial \dot{\beta}^2} \end{pmatrix} = \begin{pmatrix} \frac{(m_1 + 3m_2) l_1^2}{3} & \frac{l_1 l_2 m_2}{2} \cos(\alpha - \beta) \\ \frac{l_1 l_2 m_2}{2} \cos(\alpha - \beta) & \frac{l_2^2 m_2}{3} \end{pmatrix} \tag{4.152}$$

for the mass matrix, which for the stable equilibrium at $\alpha = \beta = 0$ simplifies to

$$[\boldsymbol{M}] = \frac{1}{6} \begin{pmatrix} 2(m_1 + 3m_2) l_1^2 & 3 l_1 l_2 m_2 \\ 3 l_1 l_2 m_2 & 2 l_2^2 m_2 \end{pmatrix}. \tag{4.153}$$

Similarly, we obtain the stiffness matrix as

$$\boldsymbol{K} = \frac{\partial^2 V}{\partial \boldsymbol{q} \partial \boldsymbol{q}} \qquad \Rightarrow \qquad [\boldsymbol{K}] = \begin{pmatrix} \frac{\partial^2 V}{\partial \alpha^2}, & \frac{\partial^2 V}{\partial \alpha \partial \beta} \\ \frac{\partial^2 V}{\partial \alpha \partial \beta}, & \frac{\partial^2 V}{\partial \beta^2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} l_1 (m_1 + 2m_2) g \cos\alpha & 0 \\ 0 & l_2 m_2 g \cos\beta \end{pmatrix} \tag{4.154}$$

and for the stable equilibrium at $\alpha = \beta = 0$

$$[\boldsymbol{K}] = \frac{1}{2} \begin{pmatrix} l_1 (m_1 + 2m_2) g & 0 \\ 0 & l_2 m_2 g \end{pmatrix}. \tag{4.155}$$

The same matrices could also have been obtained by first deriving the equations of motion via (4.150) and (4.151) and then linearizing about the equilibrium configuration. Either way we arrive at the above matrices and we see that $\boldsymbol{C} = \boldsymbol{0}$. Finally, note that both $\boldsymbol{M}$ and $\boldsymbol{K}$ are by definition symmetric in this case and (when evaluated at equilibrium) positive-definite.

4.4.2 Free, undamped vibrations

When investigating vibrations of multi-DOF systems, let us first consider the case of free vibrations (i.e., $\boldsymbol{F}^{\text{ext}}(t) = \boldsymbol{0}$) without damping (i.e., $\boldsymbol{C} = \boldsymbol{0}$). In this case the governing equations become

$$\boldsymbol{M} \ddot{\boldsymbol{x}} + \boldsymbol{K} \boldsymbol{x} = \boldsymbol{0}. \tag{4.156}$$

Analogous to the solution for single-DOF vibrations, we use the ansatz

$$\boldsymbol{x}(t) = \operatorname{Re}\left[\hat{\boldsymbol{x}}^* \exp(\lambda t)\right] \qquad \text{where} \qquad \hat{\boldsymbol{x}}^* \in \mathbb{C}^n, \quad \lambda \in \mathbb{C}. \tag{4.157}$$

Insertion of $\boldsymbol{x}^*(t) = \hat{\boldsymbol{x}}^* \exp(\lambda t)$ into (4.156) leads to

$$\lambda^2 \boldsymbol{M} \boldsymbol{x}^*(t) + \boldsymbol{K} \boldsymbol{x}^*(t) = \boldsymbol{0} \quad \forall\, t \qquad \Rightarrow \qquad (\boldsymbol{K} + \lambda^2 \boldsymbol{M}) \hat{\boldsymbol{x}}^* = \boldsymbol{0}. \tag{4.158}$$

This is an **eigenvalue problem**, which may alternatively be written $(\boldsymbol{M}^{-1} \boldsymbol{K} + \lambda^2 \boldsymbol{I}) \hat{\boldsymbol{x}}^* = \boldsymbol{0}$. Hence, for non-trivial solutions $(-\lambda^2)$ must be an eigenvalue of the matrix $\boldsymbol{M}^{-1} \boldsymbol{K}$. λ is obtained as a root of the **characteristic equation**

$$\det\left(\boldsymbol{K} + \lambda^2 \boldsymbol{M}\right) = 0 \tag{4.159}$$

Since $\boldsymbol{x}$ is n-dimensional, we have n (not necessarily distinct) roots λ_i^2.

Recall that, by construction, $\boldsymbol{M}$ is positive-definite (and therefore guaranteed to be invertible), while $\boldsymbol{K}$ is positive-semi-definite for those cases considered here. From linear algebra we know that this ensures *only real-valued eigenvalues* $(-\lambda_j^2) \in \mathbb{R}$ and, moreover, non-negative eigenvalues[19], i.e.,

$$-\lambda_j^2 \geq 0 \qquad \Rightarrow \qquad \lambda_j = \sqrt{-(-\lambda_j^2)} = i\omega_j \qquad \text{with} \qquad \omega_j = \pm\sqrt{-\lambda_j^2} \in \mathbb{R}. \tag{4.160}$$

Each of the $\omega_j \in \mathbb{R}$ $(j = 1, \dots, n)$ is an **eigenfrequency** of the system, whereas the corresponding eigenvector $\hat{\boldsymbol{x}}_j^*$ is known as its **mode shape** or **eigenmode** of the system. Note that, since $\boldsymbol{M}^{-1}\boldsymbol{K}$ is positive-semi-definite here, the eigenvalue problem also ensures that we have *real-valued eigenvectors* $\hat{\boldsymbol{x}}_j^* = \hat{\boldsymbol{x}}_j \in \mathbb{R}^n$ (so we drop the asterisk).

Further, if ω_j is a solution, then $-\omega_j$ must also be a solution, so we have pairs of eigenfrequencies $\pm\omega_j$ and consequently $2n$ solutions for ω_j. We now need to consider two cases for each pair of eigenfrequencies: either we have a pair of distinct non-zero eigenfrequencies $\pm\omega_j$, or the eigenfrequency vanishes, i.e., we have a duplicated eigenfrequency $\omega_j = 0$.

- **pair of non-zero eigenfrequencies**: in this case, the solution is analogous to the single-DOF free, undamped vibrations discussed in Section 4.3.2. It can be written for each eigenfrequency ω_j (defining ω_j as the positive root of the pair) as

$$\begin{aligned} \boldsymbol{x}_j(t) &= \mathrm{Re}\left[a_j \hat{\boldsymbol{x}}_j \exp(i\omega_j t) + b_j \hat{\boldsymbol{x}}_j \exp(-i\omega_j t)\right] \qquad \text{with} \qquad a_j, b_j \in \mathbb{C} \\ &= c_j \hat{\boldsymbol{x}}_j \cos(\omega_j t + \varphi_j) \qquad \text{with} \qquad c_j, \varphi_j \in \mathbb{R}. \end{aligned} \tag{4.161}$$

- **zero eigenfrequency**: if $\lambda_j = 0$, then the duplicated $\omega_j = 0$ yields the solution

$$\boldsymbol{x}_j(t) = a_j \hat{\boldsymbol{x}}_j \exp(0) + b_j \hat{\boldsymbol{x}}_j t \exp(0) = (a_j + b_j t)\hat{\boldsymbol{x}}_j \qquad \text{with} \qquad a_j, b_j \in \mathbb{R}. \tag{4.162}$$

  This solution implies not vibrational motion but **rigid-body motion**.

  To better understand this, let us insert $\lambda = 0$ into the equation of motion (4.158), leading to $\boldsymbol{K}\boldsymbol{x}_j = \boldsymbol{0}$ so that $\frac{1}{2}\boldsymbol{x}_j \cdot \boldsymbol{K}\boldsymbol{x}_j = V(\boldsymbol{x}_j) = 0$. This implies that displacing the particles according to $\boldsymbol{x}_j(t)$ produces zero potential energy. Therefore, $\boldsymbol{x}_j(t)$ represents rigid body motion, i.e., the particle system moves like a rigid body, leaving all particle distances unchanged.

We conclude that, since every rigid-body mode must result in a pair of zero eigenfrequency, **every system has as many pairs of zero eigenfrequencies as it has independent rigid-body modes**. For example, two elastically-coupled particles moving in 1D (as in Example 4.10 below) have one rigid-body mode (viz., translation in 1D). If the two particles are moving in 2D, then the system has three rigid-body modes (viz., translation in both directions and a rotation) and hence three zero eigenfrequencies, etc.

Finally, we identify the complete solution by summing over all eigenfrequencies (since each of them contributes a solution and the ODE is linear, the complete solution is a superposition of the

[19] Here and in the following we use index j to avoid confusion with the imaginary $i = \sqrt{-1}$.

eigenmodes). Assuming that there are $n-l \leq n$ rigid-body modes and that the eigenfrequencies are sorted so that the first l eigenfrequencies are non-zero (and all eigenfrequencies are non-negative), the general solution is

$$\boxed{\boldsymbol{x}(t) = \sum_{j=1}^{n} \boldsymbol{x}_j(t) = \sum_{j=1}^{l} c_j \cos(\omega_j t - \varphi_j)\hat{\boldsymbol{x}}_j + \sum_{j=l+1}^{n} (a_j + b_j t)\hat{\boldsymbol{x}}_j} \tag{4.163}$$

The $2n$ unknown coefficients $a_j, b_j, c_j, \varphi_j \in \mathbb{R}$ are to be determined from initial conditions, while the eigenfrequencies and eigenmodes follow from the eigenvalue problem

$$\boxed{\det\left(\boldsymbol{K} - \omega_j^2 \boldsymbol{M}\right) = 0} \qquad \text{and} \qquad \boxed{\left(\boldsymbol{K} - \omega_j^2 \boldsymbol{M}\right)\hat{\boldsymbol{x}}_j = \boldsymbol{0}} \tag{4.164}$$

We point out that the eigenmodes have a special orthogonal structure. Recall that the governing ODE is linear, so that each term j in the above solution must satisfy the equation

$$-\omega_j^2 \boldsymbol{M}\hat{\boldsymbol{x}}_j + \boldsymbol{K}\hat{\boldsymbol{x}}_j = \boldsymbol{0} \qquad \text{and hence also} \qquad -\omega_k^2 \boldsymbol{M}\hat{\boldsymbol{x}}_k + \boldsymbol{K}\hat{\boldsymbol{x}}_k = \boldsymbol{0} \tag{4.165}$$

for some $1 \leq j, k \leq n$ (for the rigid-body modes, the equations of motion are trivially satisfied, as discussed above[20]). Pre-multiplying the first equation by $\hat{\boldsymbol{x}}_k$ and subtracting the second pre-multiplied by $\hat{\boldsymbol{x}}_j$ yields (exploiting that $\boldsymbol{M}$ and $\boldsymbol{K}$ are symmetric here)

$$(\omega_k^2 - \omega_j^2)\hat{\boldsymbol{x}}_j \cdot \boldsymbol{M}\hat{\boldsymbol{x}}_k = 0 \qquad \Rightarrow \qquad \hat{\boldsymbol{x}}_j \cdot \boldsymbol{M}\hat{\boldsymbol{x}}_k = 0 \qquad \text{if} \qquad \omega_j \neq \omega_k. \tag{4.166}$$

Likewise, pre-multiplying the first equation by $\hat{\boldsymbol{x}}_k$ and adding the second pre-multiplied by $\hat{\boldsymbol{x}}_j$ (and inserting (4.166)) leads to

$$-(\omega_j^2 + \omega_k^2)\hat{\boldsymbol{x}}_j \cdot \boldsymbol{M}\hat{\boldsymbol{x}}_k + 2\hat{\boldsymbol{x}}_j \cdot \boldsymbol{K}\hat{\boldsymbol{x}}_k = 0 \qquad \Rightarrow \qquad \hat{\boldsymbol{x}}_j \cdot \boldsymbol{K}\hat{\boldsymbol{x}}_k = 0 \qquad \text{if} \qquad j \neq k. \tag{4.167}$$

Example 4.9. Eigenfrequencies of a double-pendulum

Having established the linearized equations of motion and the associated mass and stiffness matrices for a double-pendulum of two rods in Example 4.8, we may now compute the eigenfrequencies of that system. The system is undamped, so we expect two imaginary eigenvalues. Further, since no rigid body motion can occur (the upper beam is hinged, so the system cannot move through space without changing the potential energy), we expect distinct non-zero eigenvalues.

We use the mass and stiffness matrices derived in Example 4.8,

$$[\boldsymbol{M}] = \frac{1}{6}\begin{pmatrix} 2(m_1 + 3m_2)l_1^2 & 3l_1 l_2 m_2 \\ 3l_1 l_2 m_2 & 2l_2^2 m_2 \end{pmatrix}, \qquad [\boldsymbol{K}] = \frac{1}{2}\begin{pmatrix} l_1(m_1 + 2m_2)g & 0 \\ 0 & l_2 m_2 g \end{pmatrix}. \tag{4.168}$$

For the special case of equal rods, i.e., $m_1 = m_2 = m$ and $l_1 = l_2 = l$, the two matrices reduce to

$$[\boldsymbol{M}] = \frac{ml^2}{6}\begin{pmatrix} 8 & 3 \\ 3 & 2 \end{pmatrix}, \qquad [\boldsymbol{K}] = \frac{mgl}{2}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.169}$$

[20] The discussion here assumes distinct eigenvalues, which is not always the case. However, any set of eigenmodes can be turned into an orthonormal basis, so the general procedure discussed here can still be applied after possible orthogonalization.

Let us calculate the eigenvalues for this case, which leads to the characteristic equation

$$\begin{aligned}0 = \det\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right) &= \left(\frac{3mgl}{2} - \omega^2 \frac{8ml^2}{6}\right)\left(\frac{mgl}{2} - \omega^2 \frac{2ml^2}{6}\right) - \left(-\omega^2 \frac{3ml^2}{6}\right)^2 \\ &= \frac{m^2 l^4}{36}\left(7\omega^4 - 42\frac{g}{l}\omega^2 + 27\frac{g^2}{l^2}\right),\end{aligned} \tag{4.170}$$

whose roots are (noting that $6/\sqrt{7} < 3$)

$$\omega^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l} \qquad \Rightarrow \qquad \omega = \sqrt{3 \pm \frac{6}{\sqrt{7}}}\sqrt{\frac{g}{l}}. \tag{4.171}$$

We thus obtain two pairs of distinct eigenfrequencies:

$$\omega_1 = \pm\sqrt{3 - \frac{6}{\sqrt{7}}}\sqrt{\frac{g}{l}} \approx \pm 0.856\sqrt{g/l}, \qquad \omega_2 = \pm\sqrt{3 + \frac{6}{\sqrt{7}}}\sqrt{\frac{g}{l}} \approx \pm 2.295\sqrt{g/l}. \tag{4.172}$$

Dropping the negative eigenfrequencies for uniqueness, we conclude

$$\omega_1 \approx 0.856\sqrt{g/l}, \qquad \omega_2 \approx 2.295\sqrt{g/l}. \tag{4.173}$$

We can also compute the corresponding eigenmodes as the associated eigenvectors $\hat{\boldsymbol{x}}_j$ satisfying $(\boldsymbol{K} - \omega_j^2 \boldsymbol{M})\hat{\boldsymbol{x}}_j = \boldsymbol{0}$. Here, this leads to

$$[\hat{\boldsymbol{x}}_1] = \begin{pmatrix} \frac{1}{9}(1 + 2\sqrt{7}) \\ 1 \end{pmatrix}, \qquad [\hat{\boldsymbol{x}}_2] = \begin{pmatrix} \frac{1}{9}(1 - 2\sqrt{7}) \\ 1 \end{pmatrix}. \tag{4.174}$$

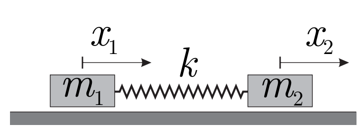
For comparison, the eigenfrequency of a simple (single-rod) pendulum is obtained from its linearized equations of motion as

$$\frac{ml^2}{3}\ddot{\varphi} + mg\frac{l}{2}\sin\varphi = 0 \quad \Rightarrow \quad \ddot{\varphi} + \frac{3g}{2l}\varphi \approx 0 \quad \Rightarrow \quad \omega_0 = \sqrt{\frac{3g}{2l}} \approx 1.225\sqrt{g/l}. \tag{4.175}$$

In both cases, the eigenfrequencies are independent of the masses and only depend on the pendulum length(s) and the gravitational acceleration.

Example 4.10. Elastic two-particle oscillator

Consider two particles of masses m_1 and m_2, which are connected by an elastic spring of stiffness k and slide frictionlessly on a planar ground in 1D. Let us find the eigenfrequencies and corresponding eigenmodes of the system.



We denote the 1D positions of the two masses by x_1 and x_2, and we write $\boldsymbol{x} = \{x_1, x_2\}$. The equations of motion in this case are a special form of those derived for the 1D elastic chain (see Example 4.7), viz.

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.176}$$

Insertion of $\boldsymbol{x}(t) = \hat{\boldsymbol{x}}\exp(i\omega t)$ yields

$$\left[-\omega^2\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} + \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\right]\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.177}$$

The characteristic equation $\det(\boldsymbol{K} - \omega^2\boldsymbol{M}) = 0$ hence becomes

$$\begin{aligned} 0 &= \det\left[\begin{pmatrix} k & -k \\ -k & k \end{pmatrix} - \omega^2\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\right] = \det\begin{bmatrix} -\omega^2 m_1 + k & -k \\ -k & -\omega^2 m_2 + k \end{bmatrix} \\ &= (-\omega^2 m_1 + k)(-\omega^2 m_2 + k) - k^2 = -\omega^2(m_1 k + m_2 k - \omega^2 m_1 m_2). \end{aligned} \tag{4.178}$$

The eigenfrequencies ω_j are obtained from the roots of this equation:

$$\omega^2 = 0 \quad \vee \quad \omega^2 = \frac{m_1 + m_2}{m_1 m_2}k \qquad \Rightarrow \qquad \omega_{1,2} = 0, \quad \omega_{3,4} = \pm\sqrt{\frac{m_1 + m_2}{m_1 m_2}k}. \tag{4.179}$$

As could have been expected, we have one rigid-body mode (identified by the repeated zero eigenfrequency $\omega_{1,2} = 0$). This becomes apparent when calculating the corresponding eigenmodes. To this end, we insert the solutions for λ into (4.177) to identify $\hat{\boldsymbol{x}}$. For $\omega = 0$ we thus obtain

$$\left[0\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} + \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\right]\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \hat{x}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{4.180}$$

which is defined up to an arbitrary coefficient $\hat{x} \in \mathbb{R}$. This is obviously rigid body motion since $(\hat{x}_1, \hat{x}_2)^{\mathrm{T}} = \hat{x}(1,1)^{\mathrm{T}}$ implies that both masses translate by the same amount $\hat{x}$. The rigid-body solution in this case of a repeated eigenvalue $\omega = 0$, with constants $a, b \in \mathbb{R}$, reads

$$[\boldsymbol{x}(t)] = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = [A\exp(0) + B\,t\exp(0)]\,\hat{x}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (a + bt)\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.181}$$

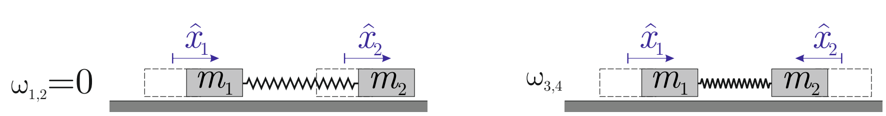
Analogously, we find the eigenmode associated with the non-zero eigenfrequency via

$$\left[-\frac{m_1 + m_2}{m_1 m_2}k\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} + \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\right]\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} -k\frac{m_1}{m_2} & -k \\ -k & -k\frac{m_2}{m_1} \end{pmatrix}\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which yields the eigenmode (again defined up to a constant $\hat{x} \in \mathbb{R}$)

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \hat{x}\begin{pmatrix} m_2 \\ -m_1 \end{pmatrix}. \tag{4.182}$$

In this case, the two particles move in opposite directions (which is clearly not rigid-body motion since this eigenmode changes the spring length and hence causes changes to the potential energy). The two eigenmodes are shown schematically below (recall that both mode shapes are only known up to a constant). The non-zero eigenfrequencies are associated with simultaneous counter-motion of the two particles.



The **complete solution** (4.163) thus becomes

$$[\boldsymbol{x}(t)] = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \hat{x} \begin{pmatrix} m_2 \\ -m_1 \end{pmatrix} \cos\left(\sqrt{\frac{m_1+m_2}{m_1 m_2}}k\,t + \varphi\right) + (a+bt)\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{4.183}$$

whose constants $\hat{x}, \varphi, a, b \in \mathbb{R}$ are to be obtained from initial conditions (e.g., the two known initial positions $x_i(0)$ and the two initial velocities $\dot{x}_i(t)$ provide four equations for the four unknown coefficients).

We point out that for arbitrary choices of $\hat{x}, \varphi, a, b$, we realize that the total linear momentum P of the two-particle system, evaluated as

$$\begin{aligned} P &= m_1\dot{x}_1 + m_2\dot{x}_2 = [m_1\hat{x}\, m_2 + m_2\hat{x}(-m_1)]\,(-\omega_3)\sin(\omega_3 t + \varphi) + (m_1+m_2)b \\ &= (m_1+m_2)b = \text{const.}, \end{aligned} \tag{4.184}$$

is conserved during vibrations. Specifically, the vibrational mode constitutes a *vibration about the center of mass* – the location of the center of mass of the two-particle system remains unchanged by the vibration and, of course, linear momentum is also conserved for the case of rigid-body translation with constant velocity. The observation of a constant linear momentum of the system makes indeed sense since, *for a free motion without external forcing nor damping, linear momentum is conserved* (see Section 2.2.1). This, of course, also includes vibrational motion.

4.4.3 Damped and forced vibrations

Systems of particles undergoing forced vibrations ($\boldsymbol{F}^{\text{ext}}(t) \neq \boldsymbol{0}$) in the presence of damping ($\boldsymbol{C} \neq \boldsymbol{0}$) are governed, in the linear(ized) case, by

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{F}^{\text{ext}}(t). \tag{4.185}$$

Like for single-DOF vibrations in Section 4.3.3, the solution is sought of the form

$$\boldsymbol{x}(t) = \boldsymbol{x}_{\text{hom}}(t) + \boldsymbol{x}_{\text{part}}(t), \tag{4.186}$$

whose homogeneous part $\boldsymbol{x}_{\text{hom}}(t)$ is the solution of a free vibration and, in the presence of any damping, decays with time.

The homogeneous solution of a **free vibration** is found by considering

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0}. \tag{4.187}$$

Analogous to the solution for undamped vibrations, we start with the ansatz

$$\boldsymbol{x}_{\text{hom}}(t) = \text{Re}\left[\hat{\boldsymbol{x}}^*_{\text{hom}}\exp(\lambda t)\right] \qquad \text{where} \qquad \hat{\boldsymbol{x}}^*_{\text{hom}} \in \mathbb{C}^n, \quad \lambda \in \mathbb{C}. \tag{4.188}$$

Insertion into (4.187) leads to

$$\lambda^2\boldsymbol{M}\boldsymbol{x}^*_{\text{hom}}(t) + \lambda\boldsymbol{C}\boldsymbol{x}^*_{\text{hom}}(t) + \boldsymbol{K}\boldsymbol{x}^*_{\text{hom}}(t) = \boldsymbol{0} \qquad \Rightarrow \qquad \left[\boldsymbol{K} + \lambda\boldsymbol{C} + \lambda^2\boldsymbol{M}\right]\hat{\boldsymbol{x}}^*_{\text{hom}} = \boldsymbol{0}. \tag{4.189}$$

The characteristic equation in this case reads

$$\det\left[\boldsymbol{K}+\lambda\boldsymbol{C}+\lambda^2\boldsymbol{M}\right]=0, \tag{4.190}$$

whose roots are the $2n$ (not necessarily distinct) eigenvalues λ_j. By construction, the case of non-zero damping leads to eigenvalues that are not necessarily non-negative, so that complex-valued eigenfrequencies $\omega_j\in\mathbb{C}$ are to be expected, leading to solutions that decay over time (analogous to the single-particle discussion).

For a **forced vibration**, the particular solution depends on the forcing. Consider, e.g.,

$$\boldsymbol{F}^{\text{ext}}(t)=\text{Re}\left[\hat{\boldsymbol{F}}^*\exp(i\Omega t)\right] \qquad \text{where} \qquad \hat{\boldsymbol{F}}^*\in\mathbb{C}^n, \tag{4.191}$$

with an excitation frequency $\Omega\in\mathbb{R}$. This implies that the solution must be of the form

$$\boldsymbol{x}_{\text{part}}(t)=\text{Re}\left[\boldsymbol{x}^*_{\text{part}}(t)\right]=\text{Re}\left[\hat{\boldsymbol{x}}^*_{\text{part}}\exp(i\Omega t)\right] \qquad \text{where} \qquad \hat{\boldsymbol{x}}^*_{\text{part}}\in\mathbb{C}^n. \tag{4.192}$$

Insertion of $\boldsymbol{x}^*_{\text{part}}(t)=\hat{\boldsymbol{x}}^*_{\text{part}}\exp(i\Omega t)$ into (4.185) gives the sought mode shape as

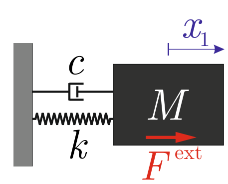
$$\left[-\Omega^2\boldsymbol{M}+i\Omega\boldsymbol{C}+\boldsymbol{K}\right]\hat{\boldsymbol{x}}^*_{\text{part}}=\hat{\boldsymbol{F}}^* \qquad \Rightarrow \qquad \hat{\boldsymbol{x}}^*_{\text{part}}=\left[-\Omega^2\boldsymbol{M}+i\Omega\boldsymbol{C}+\boldsymbol{K}\right]^{-1}\hat{\boldsymbol{F}}^*. \tag{4.193}$$

As before, the real part is to be taken as the final solution. Unfortunately, analytical solutions of both (4.189) and (4.193) are hard to find, which is why we introduce the concept of modal decomposition below.

Example 4.11. Tuned mass damping

A **tuned mass damper** is an engineering trick to avoid harmful vibrations at or near a resonant frequency, which is found especially in civil engineering applications – from bridges to skyscrapers. Rather than discussing a complicated engineering structure, let us study the simplest system possible, which illustrates the concept of a tuned mass damper.

Consider the shown system of a large mass M attached to a linear spring of stiffness k and a damper of viscosity c (this is our engineering structure). Limiting ourselves to 1D motion and neglecting gravity, this simple one-DOF system has an eigenfrequency of $\omega_0=\sqrt{k/M}$, and free vibrations in the underdamped case occur at the frequency



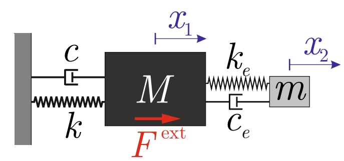
$$\omega_d=\sqrt{\omega_0^2-\delta^2}=\sqrt{\frac{k}{M}-\delta^2}\approx\sqrt{\frac{k}{M}} \quad \text{if } \delta=\frac{c}{2M}\ll 1 \tag{4.194}$$

for small damping. Now, assume the system is excited by an external force $F^{\text{ext}}(t)$, applied to the large mass M at the eigenfrequency:

$$F^{\text{ext}}(t)=\hat{F}\cos\left(\sqrt{\frac{k}{M}}t\right). \tag{4.195}$$

As discussed, this resonant excitation may lead to excessive vibration amplitudes that can harm the structure (e.g., leading to the failure of bridges or tall buildings, or at least to making people on/in those feel uncomfortable).

As a remedy, let us attach a small mass m through a second spring of stiffness k_e and a dashpot of viscosity c_e, as shown on the right – effectively turning the above single-DOF system into a two-DOF system. The vibrations of this two-DOF system are found from the equations of motion:



$$\underbrace{\begin{pmatrix} M & \\ & m \end{pmatrix}}_{=\boldsymbol{M}} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} c+c_e & -c_e \\ -c_e & c_e \end{pmatrix}}_{=\boldsymbol{C}} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} k+k_e & -k_e \\ -k_e & k_e \end{pmatrix}}_{=\boldsymbol{K}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} F^{\text{ext}}(t) \\ 0 \end{pmatrix}}_{=\boldsymbol{F}^{\text{ext}}(t)}. \tag{4.196}$$

Consider a harmonic excitation, so that

$$\boldsymbol{F}^{\text{ext}}(t) = \hat{\boldsymbol{F}}^* \exp(i\Omega t) = \begin{pmatrix} \hat{F} \\ 0 \end{pmatrix} \exp(i\Omega t) \qquad \Rightarrow \qquad \boldsymbol{x}_{\text{part}}(t) = \hat{\boldsymbol{x}}^* \exp(i\Omega t) = \begin{pmatrix} \hat{x}_1^* \\ \hat{x}_2^* \end{pmatrix} \exp(i\Omega t),$$

where for simplicity we work with complex-valued functions (tacitly assuming that taking real parts is required in the end). Insertion into the equations of motion yields

$$(-\Omega^2 \boldsymbol{M} + i\Omega \boldsymbol{C} + \boldsymbol{K})\hat{\boldsymbol{x}}^* = \hat{\boldsymbol{F}}^* \qquad \Rightarrow \qquad \hat{\boldsymbol{x}}^* = (-\Omega^2 \boldsymbol{M} + i\Omega \boldsymbol{C} + \boldsymbol{K})^{-1} \hat{\boldsymbol{F}}^*. \tag{4.197}$$

The (lengthy) expression for the amplitude vector $\hat{\boldsymbol{x}}^*$ can in fact be evaluated analytically. For example, the amplitude $\hat{x}_1^*$ of the big mass M for the special case of $\Omega = \sqrt{k/M}$ (the critical excitation frequency) evaluates to

$$\left| \hat{x}_1^* \Big|_{\Omega=\sqrt{\frac{k}{M}}} \right| = \frac{F}{k} \sqrt{\frac{\overline{c}^2\gamma^2 + (\overline{k} - \xi)^2}{\gamma^2 \left(\overline{c}^2\gamma^2 + \overline{k}^2 \right) + \xi^2 \left[(\overline{c}+1)^2\gamma^2 + \overline{k}^2 \right] - 2\gamma^2 \overline{k} \xi}} \tag{4.198}$$

with the abbreviations

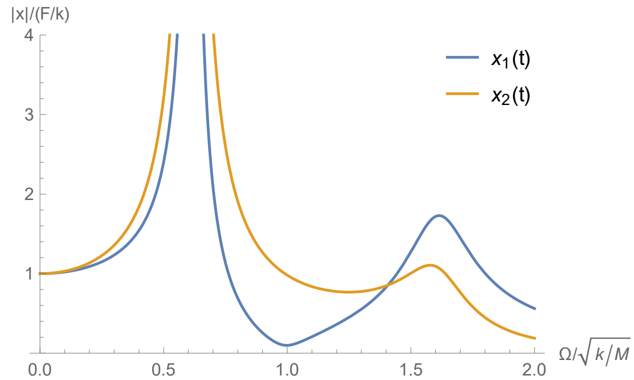
$$\xi = \frac{m}{M}, \qquad \overline{k} = \frac{k_e}{k}, \qquad \overline{c} = \frac{c_e}{c}, \qquad \gamma = \frac{c}{\sqrt{kM}}. \tag{4.199}$$

The idea is now to choose the added system parameters (m, k_e and c_e) such as to minize the above amplitude at the excitation frequency. Minimization of the amplitude with respect to, e.g., the attached mass m (which is usually simplest to adjust in practice) shows that

$$\left| \hat{x}_1^* \Big|_{\Omega=\sqrt{\frac{k}{M}}} \right| \to \min \qquad \text{if} \qquad m \to M \frac{k_e}{k} + \frac{c_e^2}{k_e}. \tag{4.200}$$

Hence, choosing an extra mass $m = M\frac{k_e}{k} + \frac{c_e^2}{k_e}$ effectively reduces the vibration amplitude of the large mass M to a minimum. Therefore, one speaks of the added (and carefully tuned) mass m as a *tuned mass damper*.

Of course, minimizing the amplitude at $\Omega = \sqrt{k/M}$ says little about what happens at other frequencies, yet it helps avoid an excessive vibrational amplitude at the target frequency of $\Omega = \sqrt{k/M}$. The plot below illustrates the amplitudes of the two masses M and m vs. the excitation frequency Ω (normalized by $\sqrt{k/M}$).



As the plot indicates, the amplitude of $x_1(t)$ becomes marginal at the target frequency of $\Omega = \sqrt{k/M}$, while the amplitude of $x_2(t)$ remains finite. Overall, we observe in the amplitude plot two resonances of this two-DOF system, yet none of them is close to the target frequency. This demonstrates the strategy of a *tuned mass damper*, i.e., an extra masses added to the system and tuned such that the system vibration amplitudes of interest are reduced at a given target frequency.

4.4.4 Modal Decomposition

Acknowledging that the above treatment of damped and forced vibrations does not generally lead to closed-form solutions (except for quite simple problems), we introduce the concept of **modal decomposition**, which will prove helpful in this context. The key idea is to exploit that the n eigenmodes $\hat{\boldsymbol{x}}_j$ can be turned into a basis of the n-dimensional space in which the sought solution lies. This allows us to decompose the solution of any vibration as follows:

$$\boxed{\boldsymbol{x}(t) = \sum_{j=1}^{n} \hat{\boldsymbol{x}}_j y_j(t) = \boldsymbol{X}\boldsymbol{y}(t)} \tag{4.201}$$

where we introduced for convenience the constant matrix $\boldsymbol{X}$ and vector $\boldsymbol{y}(t)$ as

$$\boldsymbol{X} = [\hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_n] \in \mathbb{R}^{n\times n}, \qquad \boldsymbol{y}(t) = [y_1(t), \ldots, y_n(t)]^{\mathrm{T}} \in \mathbb{R}^n. \tag{4.202}$$

Here, $\hat{\boldsymbol{x}}_j$ are the (possibly normalized) eigenmodes of the free vibration, and $y(t) \in \mathbb{R}$ are some unknown coefficients that are functions of time and to be determined. The strategy is thus to find $\boldsymbol{X}$ a-priori from the free-vibration eigenvalue problem, and then to identify the unknown functions $y_j(t)$.

Insertion of $\boldsymbol{x}(t) = \boldsymbol{X}\boldsymbol{y}(t)$ into the equations of motion (4.185) and pre-multiplication by $\boldsymbol{X}^{\mathrm{T}}$ yields

$$\boldsymbol{X}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{X}\ddot{\boldsymbol{y}} + \boldsymbol{X}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{X}\dot{\boldsymbol{y}} + \boldsymbol{X}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{X}\boldsymbol{y} = \boldsymbol{X}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{ext}}. \tag{4.203}$$

Recall that in (4.166) and (4.167) we showed

$$\hat{\boldsymbol{x}}_j \cdot \boldsymbol{M}\hat{\boldsymbol{x}}_k = 0 \qquad \text{and} \qquad \hat{\boldsymbol{x}}_j \cdot \boldsymbol{K}\hat{\boldsymbol{x}}_k = 0 \qquad \text{for} \qquad j \neq k. \tag{4.204}$$

We can further normalize the eigenvectors (knowing that $\hat{\boldsymbol{x}}_j \cdot \boldsymbol{M}\hat{\boldsymbol{x}}_j > 0$) according to

$$\hat{\boldsymbol{x}}_j \leftarrow \frac{\hat{\boldsymbol{x}}_j}{\sqrt{\hat{\boldsymbol{x}}_j \cdot \boldsymbol{M}\hat{\boldsymbol{x}}_j}} \qquad \Rightarrow \qquad \hat{\boldsymbol{x}}_j \cdot \boldsymbol{M}\hat{\boldsymbol{x}}_k = \delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{else.} \end{cases} \tag{4.205}$$

Using $-\omega_k^2 \boldsymbol{M}\hat{\boldsymbol{x}}_k + \boldsymbol{K}\hat{\boldsymbol{x}}_k = \boldsymbol{0}$, the above normalization also leads to

$$\hat{\boldsymbol{x}}_j \cdot \boldsymbol{K}\hat{\boldsymbol{x}}_k = \omega_k^2 \hat{\boldsymbol{x}}_j \cdot \boldsymbol{M}\hat{\boldsymbol{x}}_k \qquad \Rightarrow \qquad \hat{\boldsymbol{x}}_j \cdot \boldsymbol{K}\hat{\boldsymbol{x}}_k = \begin{cases} \omega_j^2 & \text{if } j = k, \\ 0 & \text{else.} \end{cases} \tag{4.206}$$

With this normalization, the components of the terms $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{X}$ and $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{X}$ simplify to

$$[\boldsymbol{X}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{X}]_{jk} = \hat{\boldsymbol{x}}_j \cdot \boldsymbol{M}\hat{\boldsymbol{x}}_k = \delta_{jk} \qquad \text{and} \qquad [\boldsymbol{X}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{X}]_{jk} = \hat{\boldsymbol{x}}_j \cdot \boldsymbol{K}\hat{\boldsymbol{x}}_k = \begin{cases} \omega_j^2 & \text{if } j = k, \\ 0 & \text{else.} \end{cases} \tag{4.207}$$

This shows that both matrices $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{X}$ and $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{X}$ are in fact diagonal:

$$[\boldsymbol{X}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{X}] = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \qquad \text{and} \qquad [\boldsymbol{X}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{X}] = \begin{pmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_n^2 \end{pmatrix}. \tag{4.208}$$

If we further deal with **structural damping**, i.e., $\boldsymbol{C} = \alpha\boldsymbol{M} + \beta\boldsymbol{K}$ with constants $\alpha, \beta \geq 0$, then (4.203) decouples into n equations

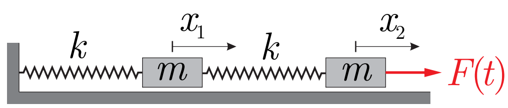
$$\boxed{\ddot{y}_j + (\alpha + \beta\omega_j^2)\dot{y}_j + \omega_j^2 y_j = \boldsymbol{F}^{\mathrm{ext}} \cdot \hat{\boldsymbol{x}}_j \qquad \text{for} \quad j = 1, \ldots, n} \tag{4.209}$$

which can be solved *independently* for each $y_j(t)$ ($j = 1, \ldots, n$) with suitable initial conditions. This is quite remarkable (and the key point of the method of modal decomposition): we started with a system of n coupled differential equations and decoupled it into n independent ODEs which can be solved one at a time. The only necessary step is the calculation of all eigenfrequencies and associated eigenmodes (and possibly their normalization).

It is essential to observe that (4.209) bears the same structure as the equation of motion discussed for single-DOF vibrations in Section 4.3. Therefore, all conclusions drawn there apply here as well for each mode j individually – including the decomposition into homogeneous and particular solutions for forced vibrations, the four cases of undamped, under-, over- and critically damped homogeneous vibrations, and the case of forced vibrations as well as resonance. For $\boldsymbol{F}^{\mathrm{ext}} = \hat{\boldsymbol{F}}\sin(\Omega t)$, each mode is excited at the frequency Ω.

Example 4.12. Forced vibration of a damped mass-spring system

Consider two particles of equal masses m connected by an elastic spring of stiffness k. The first particle is attached to the ground by another spring of stiffness k. The second particle is excited harmonically by an external force $F(t)$ of frequency Ω. The particles are subjected to structural damping. What is the vibrational response of the system?



The system matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ are obtained from the equations of motion, either via free-body diagrams and linear momentum balance or by using the Lagrange equations. Either way one arrives at (using structural damping for $\boldsymbol{C}$)

$$[\boldsymbol{M}] = \begin{pmatrix} m & \\ & m \end{pmatrix}, \qquad [\boldsymbol{K}] = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}, \qquad \boldsymbol{C} = \alpha \boldsymbol{M} + \beta \boldsymbol{K}. \tag{4.210}$$

Two distinct eigenfrequencies are obtained from the characteristic equation as

$$\omega_1 = \sqrt{\frac{1}{2}(3-\sqrt{5})}\,\omega_0, \qquad \omega_2 = \sqrt{\frac{1}{2}(3+\sqrt{5})}\,\omega_0 \qquad \text{with} \qquad \omega_0 = \sqrt{\frac{k}{m}}. \tag{4.211}$$

Note that no rigid-body mode exists since the first particle is elastically attached to the ground, so uniform translation of the two particles causes an increase in the potential energy and is therefore not rigid-body motion. The eigenmodes corresponding to the above two eigenfrequencies are

$$[\hat{\boldsymbol{x}}_1] = \frac{1}{\sqrt{\frac{m}{2}(5+\sqrt{5})}} \begin{pmatrix} -\frac{1}{2}(1+\sqrt{5}) \\ 1 \end{pmatrix}, \qquad [\hat{\boldsymbol{x}}_2] = \frac{1}{\sqrt{\frac{m}{2}(5-\sqrt{5})}} \begin{pmatrix} -\frac{1}{2}(1-\sqrt{5}) \\ 1 \end{pmatrix}. \tag{4.212}$$

Note that these have been normalized, as described above, such that $\hat{\boldsymbol{x}}_i \cdot \boldsymbol{M}\hat{\boldsymbol{x}}_j = \delta_{ij}$. Writing $\boldsymbol{X} = (\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2)$ yields

$$[\boldsymbol{X}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{X}] = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \qquad [\boldsymbol{X}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{X}] = \begin{pmatrix} \omega_1^2 & \\ & \omega_2^2 \end{pmatrix}, \tag{4.213}$$

while the structural damping assumption leads to

$$[\boldsymbol{X}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{X}] = \begin{pmatrix} \alpha + \beta\omega_1^2 & \\ & \alpha + \beta\omega_2^2 \end{pmatrix}. \tag{4.214}$$

Thus, the system of equations of motion decouples into two ODEs by applying the modal decomposition discussed above. Specifically, we write

$$\boldsymbol{x}(t) = y_1(t)\hat{\boldsymbol{x}}_1 + y_2(t)\hat{\boldsymbol{x}}_2, \tag{4.215}$$

which decouples the equations of motion into

$$\begin{aligned}\ddot{y}_1 + (\alpha + \beta\omega_1^2)\dot{y}_1 + \omega_1^2 y_1 &= \boldsymbol{F}\cdot\hat{\boldsymbol{x}}_1 = \frac{F(t)}{\sqrt{\frac{m}{2}(5+\sqrt{5})}},\\ \ddot{y}_2 + (\alpha + \beta\omega_2^2)\dot{y}_2 + \omega_2^2 y_2 &= \boldsymbol{F}\cdot\hat{\boldsymbol{x}}_2 = \frac{F(t)}{\sqrt{\frac{m}{2}(5-\sqrt{5})}},\end{aligned} \tag{4.216}$$

where we used that $\boldsymbol{F}(t) = [0, F(t)]^{\mathrm{T}}$. For the case $F(t) = \hat{F}^* \exp(i\Omega t)$, we make the ansatz

$$y_1(t) = \hat{y}_1^* \exp(i\Omega t), \qquad y_2(t) = \hat{y}_2^* \exp(i\Omega t) \qquad \text{with} \qquad \hat{y}_1^*, \hat{y}_2^* \in \mathbb{C}, \tag{4.217}$$

whose insertion into (4.216) gives

$$\begin{aligned}\hat{y}_1^* &= \frac{\hat{F}^*}{\sqrt{\frac{m}{2}(5+\sqrt{5})}\left(-\Omega^2 + (\alpha+\beta\omega_1^2)i\Omega + \omega_1^2\right)},\\ \hat{y}_2^* &= \frac{\hat{F}^*}{\sqrt{\frac{m}{2}(5-\sqrt{5})}\left(-\Omega^2 + (\alpha+\beta\omega_2^2)i\Omega + \omega_2^2\right)}.\end{aligned} \tag{4.218}$$

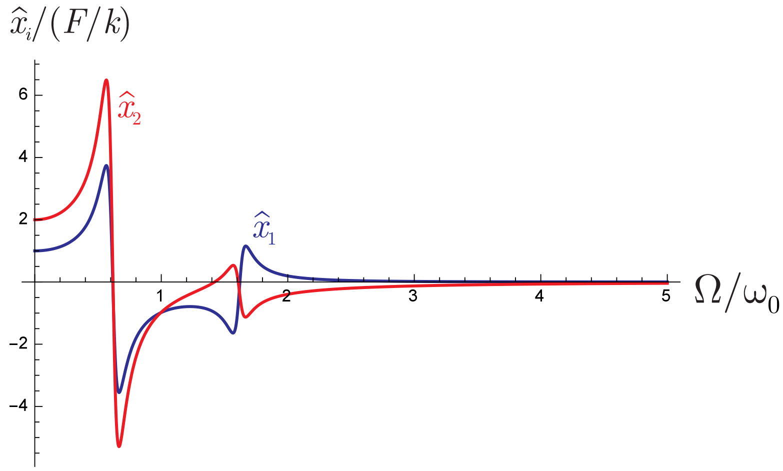
Therefore, we arrive at the complex-valued solution

$$\begin{aligned}[\boldsymbol{x}^*(t)] &= [\hat{\boldsymbol{x}}_1]y_1(t) + [\hat{\boldsymbol{x}}_2]y_2(t) = \frac{\hat{y}_1^*\exp(i\Omega t)}{\sqrt{\frac{m}{2}(5+\sqrt{5})}}\begin{pmatrix}-\frac{1}{2}(1+\sqrt{5})\\ 1\end{pmatrix} + \frac{\hat{y}_2^*\exp(i\Omega t)}{\sqrt{\frac{m}{2}(5-\sqrt{5})}}\begin{pmatrix}-\frac{1}{2}(1-\sqrt{5})\\ 1\end{pmatrix}\\ &= \left[\frac{2}{(5+\sqrt{5})\left(-\Omega^2 + (\alpha+\beta\omega_1^2)i\Omega + \omega_1^2\right)}\begin{pmatrix}-\frac{1}{2}(1+\sqrt{5})\\ 1\end{pmatrix}\right.\\ &\qquad \left. + \frac{2}{(5-\sqrt{5})\left(-\Omega^2 + (\alpha+\beta\omega_2^2)i\Omega + \omega_2^2\right)}\begin{pmatrix}-\frac{1}{2}(1-\sqrt{5})\\ 1\end{pmatrix}\right]\frac{\hat{F}^*\exp(i\Omega t)}{m}.\end{aligned} \tag{4.219}$$

If the forcing is $F(t) = \hat{F}\cos(\Omega t)$, then the forced vibration of the two particles is given by

$$\boldsymbol{x}(t) = \mathrm{Re}\left[\boldsymbol{x}^*(t)\right] \qquad \text{with} \qquad \hat{F}^* = \hat{F}, \tag{4.220}$$

whose tedious analytical evaluation is omitted here for brevity. Instead, we present in the figure below numerical results (for $\alpha = 0.1$ and $\beta = 0$). Specifically, we plot the *amplitudes* $\hat{x}_1$ and $\hat{x}_2$ as functions of the excitation frequency Ω. Clearly visible are the two **resonance** peaks near $\omega_1 \approx 0.618\omega_0$ and $\omega_2 \approx 1.618\omega_0$ (and also the associated anti-resonances), at which the amplitudes climax. Without damping, those peaks would tend to infinity; with damping, they remain finite. Also apparent is the switch in the motion of the particles from *in-phase* (amplitudes of same sign) to *out-of-phase* (amplitudes of opposite sign) below and above the first resonance, respectively. Here, and in general, each resonant frequency is noticeable in the motion of each DOF in a multi-DOF forced vibration.


$\hat{x}_i/(F/k)$
6
4
2
-2
-4
$\hat{x}_2$
$\hat{x}_1$
1
2
3
4
5
Ω/ω_0

4.5 Summary of Key Relations

Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i^{\mathrm{nc}} \qquad \text{with} \qquad \mathcal{L} = T - V, \qquad Q_i^{\mathrm{nc}} = \sum_{j=1}^{N} \boldsymbol{F}_j^{\text{non-cons.}} \cdot \frac{\partial \boldsymbol{r}_j}{\partial q_i}$$

equilibria of a **conservative, static system** with a single DOF q:

$$\text{stable equilibrium} \quad \Leftrightarrow \quad \text{energy minimum} \quad \Leftrightarrow \quad \frac{\partial^2 V}{\partial q^2} > 0$$

$$\text{unstable equilibrium} \quad \Leftrightarrow \quad \text{energy maximum} \quad \Leftrightarrow \quad \frac{\partial^2 V}{\partial q^2} < 0$$

general form of the **equation of motion** for a single DOF:

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = f(t) \qquad \text{with} \qquad \omega_0 = \sqrt{\frac{k}{m}}, \quad \delta = \frac{c}{2m}, \quad D = \frac{\delta}{\omega_0}, \quad T = \frac{2\pi}{\omega_0}$$

general solution for **undamped vibrations** ($D = 0$):

$$x(t) = A_1 \cos(\omega_0 t) + A_2 \sin(\omega_0 t) = A \sin(\omega_0 t + \varphi_0)$$

general solution for **overdamped vibrations** ($D > 1$):

$$x(t) = A_1 e^{-\left(\delta + \sqrt{\delta^2 - \omega_0^2}\right)t} + A_2 e^{-\left(\delta - \sqrt{\delta^2 - \omega_0^2}\right)t}$$

general solution for **critically damped vibrations** ($D = 1$):

$$x(t) = A_1 e^{-\delta t} + A_2 t e^{-\delta t}$$

general solution for **underdamped vibrations** ($0 < D < 1$):

$$x(t) = e^{-\delta t}\left[A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)\right], \qquad \omega_d = \sqrt{\omega_0^2 - \delta^2}$$

general solution for **forced vibrations**:

$$f(t) = \hat{f}\cos(\Omega t) \qquad \Rightarrow \qquad x(t) = x_{\mathrm{hom}}(t) + \frac{\hat{f}}{\omega_0^2} V \cos(\Omega t - \varphi)$$

with **magnification** and **phase**

$$V = \frac{1}{\sqrt{(1-\eta^2)^2 + 4D^2\eta^2}}, \qquad \varphi = \arctan\left(\frac{2D\eta}{1-\eta^2}\right) \qquad \text{where} \qquad \eta = \frac{\Omega}{\omega_0}$$

equations of motion for multiple-DOF vibrations:

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{F}(t)$$

kinetic and **potential energy** for multiple-DOF linear(ized) vibrations:

$$T = \frac{1}{2}\dot{\boldsymbol{x}} \cdot \boldsymbol{M}\dot{\boldsymbol{x}}, \qquad V = \frac{1}{2}\boldsymbol{x} \cdot \boldsymbol{K}\boldsymbol{x}$$

linearized system matrices around a stable equilibrium $\boldsymbol{q}_0$:

$$\boldsymbol{M} = \frac{\partial^2 T}{\partial \dot{\boldsymbol{q}} \partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0, \boldsymbol{0}), \qquad \boldsymbol{C} = -\frac{\partial \boldsymbol{Q}^{\text{nc}}}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0, \boldsymbol{0}), \qquad \boldsymbol{K} = \frac{\partial^2 V}{\partial \boldsymbol{q} \partial \boldsymbol{q}}(\boldsymbol{q}_0) - \frac{\partial \boldsymbol{Q}^{\text{nc}}}{\partial \boldsymbol{q}}(\boldsymbol{q}_0, \boldsymbol{0})$$

structural damping:

$$\boldsymbol{C} = \alpha \boldsymbol{M} + \beta \boldsymbol{K}, \qquad \alpha, \beta \geq 0$$

eigenfrequencies ω_j and **eigenmodes** $\hat{\boldsymbol{x}}_j$ are obtained from

$$\det\left(-\omega_j^2 \boldsymbol{M} + \boldsymbol{K}\right) = 0 \qquad \text{and} \qquad \left(-\omega_j^2 \boldsymbol{M} + \boldsymbol{K}\right)\hat{\boldsymbol{x}}_j = \boldsymbol{0}$$

general solution of **free vibrations** for multiple-DOF systems:

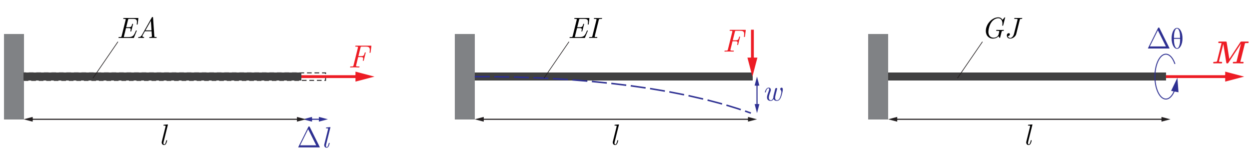
$$\boldsymbol{x}(t) = \sum_{j=1}^{n} \boldsymbol{x}_j(t) = \sum_{j=1}^{l} c_j \cos(\omega_j t + \varphi_j)\hat{\boldsymbol{x}}_j + \sum_{j=l+1}^{n} (a_j + b_j t)\hat{\boldsymbol{x}}_j$$

5 Dynamics of Deformable Bodies

So far, we have limited our consideration to particles and rigid bodies. In reality, however, systems are oftentimes not describable as systems of small particles nor as rigid bodies. In Mechanics 2 we saw that applied forces generally result in internal forces which, in turn, produce stresses that lead to deformation if the body is not rigid. Unfortunately, although in Mechanics 2 the foundation was laid to describe and predict the behavior of deformable bodies, that discussion was limited to (quasi)statics. Therefore, we will here reformulate what we learned in Mechanics 2 in the dynamic context, so that, e.g., accelerations (which we have seen to be interpretable as additional, inertial forces) may also lead to deformation. As a simple example, consider a quickly rotating turbine whose centrifugal forces may lead to internal tensile stresses and thus to an expansion of the turbine blades. Or consider a heavy mass attached at the tip of a slender beam, so that an excitation of the system may lead to vibrations that engage the beam in bending. Here, we will discuss the various ways in which dynamics affects the mechanics of **deformable** bodies, and we will restrict our study to **linear elastic** bodies (i.e., bodies whose stress–strain relation is linear).

5.1 Dynamics of Systems with Massless Deformable Bodies

Let us first consider the simple case of deformable bodies that are approximately *massless* (because their mass is significantly smaller than that of other system components). We have seen previously that *massless bodies still obey static equilibrium*. The deformation of such massless deformable bodies can therefore be described as in Mechanics 2 and, especially for the analysis of vibrations, can conveniently be reduced to simple models involving masses and springs. Let us review a few examples:



- **Bar**: As a first example, consider a slender bar of length l, Young's modulus E and cross-sectional area A, which is deforming under the action of an axial force, as shown above on the left. The relation between the change of length Δl of the 1D bar and the applied force F was found in Mechanics 2 as

$$\Delta l = \varepsilon l = \frac{\sigma}{E} l = \frac{Fl}{EA} \qquad \Leftrightarrow \qquad F = EA\frac{\Delta l}{l}. \tag{5.1}$$

  If we reinterpret the bar as an elastic spring, then its effective spring stiffness k_{eff} (resulting in the same relation between force and elongation) is

$$\boxed{k_{\text{eff}} = \frac{F}{\Delta l} = \frac{EA}{l}} \tag{5.2}$$

- **Beam**: Next, consider a slender cantilever Euler-Bernoulli beam of length l, Young's modulus E and area moment I, which is loaded by a transverse force F applied at its tip. The relation between the tip deflection w and the applied force F was found in Mechanics 2:

$$w = \frac{Fl^3}{3EI} \qquad \Rightarrow \qquad \boxed{k_{\text{eff}} = \frac{F}{w} = \frac{3EI}{l^3}} \tag{5.3}$$

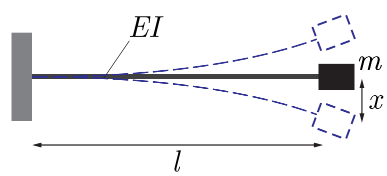
- **Torsional rod**: Finally, consider a cantilever rod of length l, shear modulus G and polar moment J, which is loaded by an applied torsional torque M. The relation between the relative twist angle $\Delta\theta$ between the two end points and the applied torque M was derived in Mechanics 2 as

$$\Delta\theta = \frac{Ml}{GJ} \qquad \Rightarrow \qquad \boxed{k_{\text{eff}} = \frac{M}{\Delta\theta} = \frac{GJ}{l}} \tag{5.4}$$

If the above deformable bodies are assumed *massless* (compared to those masses attached to them), then they can be replaced by elastic springs with the effective stiffness values derived above (since for massless bodies the static equilibrium relations still apply). The same general procedure can be applied to other deformable bodies and loading scenarios. As long as the relation between applied force or torque and resulting displacement or twist is known *and linear*, it can be replaced by an effective linear elastic spring.

Example 5.1. Flexural vibration of a particle on a beam

Consider an approximately massless beam (length l, Young's modulus E, area moment I) that is clamped at one end, while a heavy particle of mass m is attached to its free end, as shown.

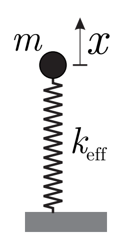


What is the **flexural** eigenfrequency of the system?

Using the effective stiffness (5.3) for beam bending, we may write the equation of motion of the particle as

$$m\ddot{x} = -F \qquad \text{with} \qquad F = k_{\text{eff}}x \qquad \text{and} \qquad k_{\text{eff}} = \frac{3EI}{l^3} \qquad \Rightarrow \qquad m\ddot{x} + k_{\text{eff}}x = 0, \tag{5.5}$$

where we replaced the beam deforming in bending by a linear spring of effective stiffness k_{eff}, as derived above. The free vibration of the system is thus governed by the 1D equation of motion, which yields the eigenfrequency as



$$\ddot{x} + \frac{k_{\text{eff}}}{m}x = 0 \qquad \Rightarrow \qquad \omega_0^2 = \frac{k_{\text{eff}}}{m} \qquad \Rightarrow \qquad \omega_0 = \sqrt{\frac{3EI}{ml^3}}. \tag{5.6}$$

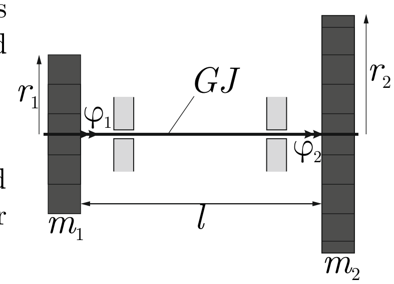
For example, for a circular cross-section of radius r, we have $I = \frac{\pi}{4}r^4$, so

$$\omega_0 = \sqrt{\frac{3EI}{ml^3}} = \sqrt{\frac{3E\pi r^4}{4ml^3}}. \tag{5.7}$$

Finally, recall that this solution based on the effective beam stiffness is only valid if the beam is approximately *massless*.

Example 5.2. Vibration of a shaft with two gears

Consider an approximately massless shaft of length l and torsional rigidity GJ (shear modulus G, polar area moment J), having gears attached to its free ends (approximated as a disk of radius r_1 and mass m_1 and a second disk of radius r_2 and mass m_2).



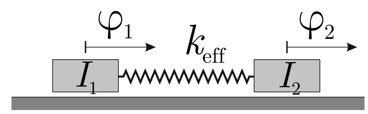
What are the eigenfrequencies of the system?

Let us first consider torsional deformation of the gear system and denote by φ_i the twist angle of disk i. We formulate the angular momentum balance equations for both disks as

$$I_1\ddot{\varphi}_1 = k_{\text{eff}}(\varphi_2 - \varphi_1), \qquad I_2\ddot{\varphi}_2 = -k_{\text{eff}}(\varphi_2 - \varphi_1), \tag{5.8}$$

with the two moments of inertia $I_1 = mr_1^2/2$ and $I_2 = mr_2^2/2$ with respect to the disks' centers, and the effective stiffness $k_{\text{eff}} = GJ/l$ from (5.4). The above equations can be written in matrix form as

$$\begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix}\begin{pmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{pmatrix} + k_{\text{eff}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.9}$$



We realize that this system of equations is equivalent to the equations of motion of the two-particle oscillator in Example 4.10 (as schematically shown above). As a consequence, we can re-use the eigenfrequencies derived there as Eq. (4.179), whose adaption here yields the **torsional** eigenfrequencies

$$\omega_1^{\text{tors.}} = 0, \qquad \omega_2^{\text{tors.}} = \sqrt{\frac{I_1 + I_2}{I_1 I_2} k_{\text{eff}}} = \sqrt{2\frac{m_1 r_1^2 + m_2 r_2^2}{m_1 r_1^2 m_2 r_2^2}\frac{GJ}{l}}. \tag{5.10}$$

The zero eigenfrequency corresponds to rigid-body motion. The rigid-body mode here is a pure rotation of the shaft without twisting, i.e., $\varphi_1 = \varphi_2$ (which is an eigenvector of the stiffness matrix).

Next, let us assume that the shaft can also undergo **axial deformation** (i.e., axial extension and compression). In this case, the gear system can be reduced to two masses connected through a linear spring with $k_{\text{eff}} = EA/l$, and the **longitudinal** eigenfrequencies are obtained analogously as

$$\omega_1^{\text{long.}} = 0, \qquad \omega_2^{\text{long.}} = \sqrt{\frac{m_1 + m_2}{m_1 m_2} k_{\text{eff}}} = \sqrt{\frac{m_1 + m_2}{m_1 m_2}\frac{EA}{l}}. \tag{5.11}$$

Here, the rigid-body mode is pure axial translation of the system.

Finally, without the two bearings, the shaft could also undergo **bending deformation**, so the *flexural eigenfrequencies* are

$$\omega_1^{\text{flex.}} = 0, \qquad \omega_2^{\text{flex.}} = \sqrt{\frac{m_1 + m_2}{m_1 m_2} k_{\text{eff}}} = \sqrt{\frac{m_1 + m_2}{m_1 m_2}\frac{3EI}{l^3}}, \tag{5.12}$$

where the rigid-body mode is translation perpendicular to the shaft's axis (which is impossible, however, in the presence of the two bearings).

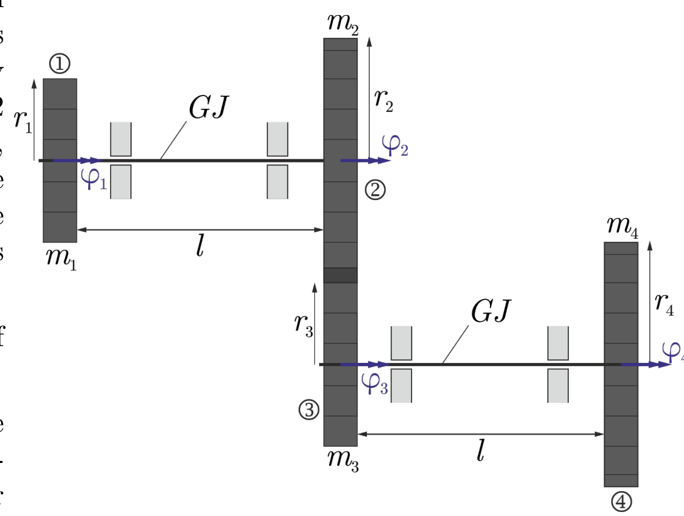
We close by considering the example of a circular cross-section of the shaft, so that $A = \pi r^2$, $I = \frac{\pi}{4} r^4$ and $J = 2I = \frac{\pi}{2} r^4$ (and $r \ll l, r_1, r_2$ for a slender shaft). Also, for an isotropic material $G = E/2(1+\nu)$ with Young's modulus E and Poisson's ratio ν. In this case

$$\begin{aligned}\omega_2^{\text{flex.}} &= \sqrt{\frac{m_1+m_2}{m_1 m_2}\frac{E\pi r^2}{l}\frac{3r^2}{4l^2}} \;<\; \omega_2^{\text{tors.}} = \sqrt{2\frac{m_1\frac{r_1^2}{r^2}+m_2}{m_1 m_2}\frac{E\pi r^2}{l}\frac{r^2}{4(1+\nu)r_1^2}} \\ &<\; \omega_2^{\text{long.}} = \sqrt{\frac{m_1+m_2}{m_1 m_2}\frac{E\pi r^2}{l}},\end{aligned} \tag{5.13}$$

so the lowest non-zero eigenfrequency (the **fundamental frequency**) is the flexural mode – which, however, can only engage without the two bearings. Otherwise, the torsional and longitudinal eigenfrequencies are of similar order with the torsional expected lower since $r < r_i$.

Example 5.3. Torsional vibration of a gear system

Let us consider a more complex system consisting of two approximately mass less shafts (each of length l and shear rigidity GJ), which are connected through gears 2 and 3 that rotate without slipping. Further, the two gears 1 and 4 are attached to the two free ends of the shafts, as shown on the right. All gears are approximated as disks of radii r_i and masses m_i as shown.



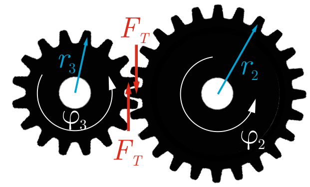
What are the torsional eigenfrequencies of the system for $I_1 = I_4$ and $r_2 = r_3$?

Since we are concerned with rotations of the shafts, let us formulate the equations of motion in this case as the equations of angular momentum balance.

We denote by φ_1, φ_2, φ_3 and φ_4 the (twist) rotation of each of the four gears (all with the same sense). Angular momentum balance for each gear with $I_i = \frac{1}{2} m_i r_i^2$ and $k_{\text{eff}} = GJ/l$ reads

$$\begin{aligned} I_1\ddot{\varphi}_1 &= k_{\text{eff}}(\varphi_2-\varphi_1), & I_3\ddot{\varphi}_3 &= k_{\text{eff}}(\varphi_4-\varphi_3) + F_T r_3, \\ I_2\ddot{\varphi}_2 &= -k_{\text{eff}}(\varphi_2-\varphi_1) + F_T r_2, & I_4\ddot{\varphi}_4 &= -k_{\text{eff}}(\varphi_4-\varphi_3), \end{aligned} \tag{5.14}$$

where F_T represents the unknown force transmitted between gears 2 and 3 (note that it appears positive in both AMB equations, since φ_2 and φ_3 are both defined with the same sense); see the schematic drawing of the two gears in question below.



From the gear kinematics we know that $r_2\varphi_2 = -r_3\varphi_3$ or $\varphi_3 = -\frac{r_2}{r_3}\varphi_2$, so that the equations for φ_2 and φ_3 can be combined. Let us introduce generalized coordinates $q_1 = \varphi_1$, $q_2 = \varphi_2$ and $q_3 = -\frac{r_3}{r_2}\varphi_4$ such that $q_1 = q_2 = q_3$ corresponds to rigid-body motion.

This turns the above four equations into

$$\begin{aligned} I_1\ddot{q}_1 &= k_{\text{eff}}(q_2 - q_1), \\ I_2\ddot{q}_2 &= -k_{\text{eff}}(q_2 - q_1) + F_T r_2, \\ -I_3\tfrac{r_2}{r_3}\ddot{q}_2 &= k_{\text{eff}}(-\tfrac{r_2}{r_3}q_3 + \tfrac{r_2}{r_3}q_2) + F_T r_3, \\ -I_4\tfrac{r_2}{r_3}\ddot{q}_3 &= -k_{\text{eff}}(-\tfrac{r_2}{r_3}q_3 + \tfrac{r_2}{r_3}q_2). \end{aligned} \tag{5.15}$$

Multiplying the third equation by r_2/r_3 and subtracting the third from the second equation eliminates the transmission force F_T and results in the reduced equations of motion:

$$\begin{aligned} I_1\ddot{q}_1 &= k_{\text{eff}}(q_2 - q_1), \\ \left[I_2 + I_3\left(\tfrac{r_2}{r_3}\right)^2\right]\ddot{q}_2 &= -k_{\text{eff}}\left[q_2 - q_1 - \left(\tfrac{r_2}{r_3}\right)^2 q_3 + \left(\tfrac{r_2}{r_3}\right)^2 q_2\right], \\ I_4\ddot{q}_3 &= -k_{\text{eff}}(q_3 - q_2). \end{aligned} \tag{5.16}$$

In matrix form, the equations of motion of a free vibration with $\boldsymbol{x} = (q_1, q_2, q_3)^{\mathrm{T}}$ become

$$\begin{pmatrix} I_1 & & \\ & I_2 + s^2 I_3 & \\ & & I_4 \end{pmatrix}\ddot{\boldsymbol{x}} + k_{\text{eff}}\begin{pmatrix} 1 & -1 & \\ -1 & 1+s^2 & -s^2 \\ & -1 & 1 \end{pmatrix}\boldsymbol{x} = \boldsymbol{0} \qquad \text{with} \qquad s = \frac{r_2}{r_3}. \tag{5.17}$$

We notice that in the special case $r_2 = r_3$ (i.e., $s = 1$) the above stiffness matrix is analogous to that of three masses coupled by elastic springs with the effective stiffness k_{eff} and moving in 1D (see Example 4.7).

The resulting free, undamped vibration defines three eigenfrequencies (and associated eigenmodes), which are obtained from solving $\det(\boldsymbol{K} + \lambda^2\boldsymbol{M}) = 0$ and using $\lambda = i\omega$. For example, for the *special case* $I_1 = I_4$ and $r_2 = r_3$, the three eigenfrequencies are obtained analytically as

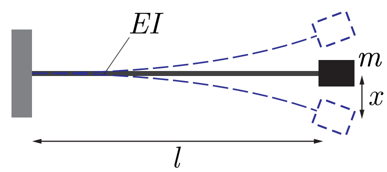
$$\omega_1 = 0, \qquad \omega_2 = \sqrt{\frac{k_{\text{eff}}}{I_1}}, \qquad \omega_3 = \sqrt{k_{\text{eff}}\left(\frac{1}{I_1} + \frac{2}{I_2 + I_3\left(\frac{r_2}{r_3}\right)^2}\right)} \qquad \text{with} \qquad k_{\text{eff}} = \frac{GJ}{l}. \tag{5.18}$$

As expected, there is one zero eigenfrequency due to the rigid-body mode $\hat{\boldsymbol{x}} = (1, 1, 1)^{\mathrm{T}}$.

5.2 Dynamics of Deformable Bodies with Non-Negligible Mass

If the mass of a body is no longer negligible, then all those relations derived under static conditions may no longer apply, including the balance equations as well as the force–displacement relations characterizing the deformation behavior. Here, the balance equations of linear and angular momentum must be re-formulated in the presence of inertial effects. Importantly, for deformable bodies, we may no longer assume that the body moves as a rigid body, easily described by the motion of its center of mass and the vectors of angular velocity and acceleration. Instead, each point on the body can undergo a very different trajectory.

As an example, consider the vibrating beam from Example 5.1: while its clamped end is not moving at all, its free end undergoes significant motion, and each point along the beam undergoes yet another motion. We need to account for this complication by considering spatially varying fields describing the motion of a deformable body.



As a consequence, formulating the kinetic relations can be done both at the *global, macroscopic level*, i.e., formulating the balance equations of linear and angular momentum for the entire body (as done in the previous sections), as well as at the *local, microscopic level*, i.e., at any point inside the body (as usually done in continuum mechanics and as was done in Mechanics 2 in the context of stresses in equilibrium).

The **global balance of linear momentum** of deformable bodies is identical to that of rigid bodies. Specifically, exploiting that internal forces in sum perform no work (as shown for systems of particles in Section 2), we still know that

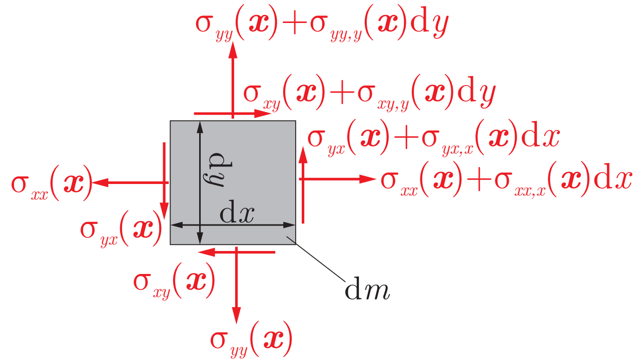
$$\boxed{\sum_i \boldsymbol{F}_i^{\text{ext}} = M\boldsymbol{a}_{\text{CM}}} \tag{5.19}$$

with the usual definitions of the total mass and the acceleration of the center of mass. To realize this, recall that we had interpreted rigid bodies as collections of infinitely many rigidly connected particles. If we now relax the latter constraint and picture a deformable body as a collection of infinitely many particles that are not rigidly connected (but, e.g., elastically coupled by linear spring connections between all pairs of particles), then we arrive at the description of a deformable body, and the same linear momentum balance (5.19) was shown to apply to both rigid and non-rigid collections of particles (in fact, we derived for the general case of systems of particles; see Section 2.2.1).

It is important to note that the balance equation (5.19) *also holds for subparts of the body* (e.g., when the body is cut into pieces to reveal internal forces), in which case the internal forces become external forces and must be considered in the balance equations (see Example 5.4 below). Angular momentum balance, of course, follows analogously and was derived in previous sections.

Deformable bodies move differently from rigid bodies because each material point of the body can now show an independent time-dependent motion. Unlike for rigid bodies where, e.g., the velocity and acceleration transfer formulae uniquely linked the velocity and acceleration, respectively, of each point on the body, here we may have both motion and deformation, and the latter leads to variations across the body.

Having established that (5.19) also applies to an arbitrary subset of a deformable (or rigid) body, we may apply it to an infinitesimal mass element $\mathrm{d}m$ cut from a body Ω at a position $\boldsymbol{x} \in \Omega$, as shown on the right in 2D (the 3D case follows analogously). On each of the cut surfaces, we introduce the normal stress and **shear stress** components acting on that respective surface, and we use a Taylor expansion to find the stresses on the right and top edges of the small element of size $\mathrm{d}x \times \mathrm{d}y$:



$$\boldsymbol{\sigma}(\boldsymbol{x} + \mathrm{d}\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x}) + \nabla\boldsymbol{\sigma}(\boldsymbol{x}) \cdot \mathrm{d}x + \text{h.o.t.} \tag{5.20}$$

For example, for the stress component σ_{xx} on the right surface this gives (writing $\boldsymbol{x} = (x, y)$)

$$\sigma_{xx}(x + \mathrm{d}x, y) \approx \sigma_{xx}(\boldsymbol{x}) + \sigma_{xx,x}(\boldsymbol{x})\,\mathrm{d}x. \tag{5.21}$$

Note that we here use coordinates (x, y, z) for consistency with Mechanics 1 and 2. Here and in the following we denote by $(\cdot)_{,x}$ a derivative of $(\cdot)$ with respect to x.

To apply (5.19), we multiply all stress components by respective length of the edge on which they are acting (thus turning stresses into forces, or to be specific, forces per thickness in 2D). Further considering a **body force** density $\boldsymbol{f}$ with components f_x,f_y in 2D (such as, e.g., gravity with $\boldsymbol{f} = \rho\boldsymbol{g}$), we obtain

$$\sum_i F_{i,x} = [\sigma_{xx}(\boldsymbol{x}) + \sigma_{xx,x}(\boldsymbol{x})\,\mathrm{d}x - \sigma_{xx}(\boldsymbol{x})]\;\mathrm{d}y + [\sigma_{xy}(\boldsymbol{x}) + \sigma_{xy,y}(\boldsymbol{x})\,\mathrm{d}y - \sigma_{xy}(\boldsymbol{x})]\;\mathrm{d}x + f_x\,\mathrm{d}x\,\mathrm{d}y = \mathrm{d}m\,a_x,$$
$$\sum_i F_{i,y} = [\sigma_{yy}(\boldsymbol{x}) + \sigma_{yy,y}(\boldsymbol{x})\,\mathrm{d}x - \sigma_{yy}(\boldsymbol{x})]\;\mathrm{d}x + [\sigma_{yx}(\boldsymbol{x}) + \sigma_{yx,x}(\boldsymbol{x})\,\mathrm{d}y - \sigma_{yx}(\boldsymbol{x})]\;\mathrm{d}x + f_y\,\mathrm{d}x\,\mathrm{d}y = \mathrm{d}m\,a_y,$$

which, using $\mathrm{d}m = \rho\,\mathrm{d}x\,\mathrm{d}y$ (per unit thickness in 2D), simplifies to

$$\begin{array}{l} \sigma_{xx,x}(\boldsymbol{x})\,\mathrm{d}x\,\mathrm{d}y + \sigma_{xy,y}(\boldsymbol{x})\,\mathrm{d}y\,\mathrm{d}x + f_x\,\mathrm{d}x\,\mathrm{d}y = \rho\,\mathrm{d}x\,\mathrm{d}y\,a_x \\ \sigma_{yy,y}(\boldsymbol{x})\,\mathrm{d}x\,\mathrm{d}x + \sigma_{yx,x}(\boldsymbol{x})\,\mathrm{d}y\,\mathrm{d}x + f_y\,\mathrm{d}x\,\mathrm{d}y = \rho\,\mathrm{d}x\,\mathrm{d}y\,a_y \end{array} \quad\Leftrightarrow\quad \begin{array}{l} \sigma_{xx,x} + \sigma_{xy,y} + f_x = \rho\,a_x \\ \sigma_{yx,x} + \sigma_{yy,y} + f_y = \rho\,a_y. \end{array} \tag{5.22}$$

Altogether, we have arrived at the **local balance of linear momentum** of deformable bodies as

$$\boxed{\sum_{j=1}^{3} \frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}x_j} + f_i = \rho a_i \qquad \text{for} \quad i = 1, 2, 3} \tag{5.23}$$

where we replaced (x, y, z) by (x_1, x_2, x_3) to allow for the summation. Recall that in Mechanics 2, equilibrium on an infinitesimal mass element was shown to obey the equation $\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}$. The above is the dynamic version thereof. Inserting $\boldsymbol{a} = \boldsymbol{0}$ automatically reduces (5.23) to the static version. (5.23) can be expressed even shorter in symbolic form as

$$\boxed{\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{x}, t) + \boldsymbol{f}(\boldsymbol{x}, t) = \rho\,\boldsymbol{a}(\boldsymbol{x}, t)} \tag{5.24}$$

Note that, as a novelty compared to Mechanics 2, all fields (including displacements, strains and stresses) are now dependent on both position and time.

We can alternatively derive the local balance law also from the following argument. We may write (5.19) for a body Ω with boundary $\partial\Omega$ as

$$\int_{\partial\Omega} \boldsymbol{\sigma}\boldsymbol{n}\,\mathrm{d}S + \int_{\Omega} \boldsymbol{f}\,\mathrm{d}V = \int_{\Omega} \rho\boldsymbol{a}\,\mathrm{d}V, \tag{5.25}$$

where the first term represents all external forces applied to the boundary $\partial\Omega$ of the body through tractions $\boldsymbol{\sigma}\boldsymbol{n}$, the second term denotes external forces in the form of distributed body forces, and the right-hand side stems from applying our definition of the center of mass and its acceleration, cf. (3.3). Applying the divergence theorem to the first term and rewriting the equation gives

$$\int_{\Omega} (\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f} - \rho\boldsymbol{a})\,\mathrm{d}V = \boldsymbol{0}. \tag{5.26}$$

We know that this relation must hold not only for the whole body Ω but also for any sub-body cut free from Ω. For the integral to vanish for any arbitrary subset of Ω, we must have

$$\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f} - \rho\boldsymbol{a} = \boldsymbol{0} \qquad \Leftrightarrow \qquad \operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f} = \rho\boldsymbol{a}, \tag{5.27}$$

which is exactly the local balance of linear momentum stated in (5.24).

In this section we went contrary our usual order of introducing the dynamic governing equations – we started by introducing the above kinetic relations linking accelerations and forces (or stresses). Let us point out that the *kinematics* of deformable bodies are the same as those discussed before for rigid bodies, enriched by those relations derived in Mechanics 2 for the deformation of bodies. Specifically, we use a **displacement** field $\boldsymbol{u}(\boldsymbol{x},t)$ to define the current position $\boldsymbol{y}$ of a material point originally at an undeformed position $\boldsymbol{x}$ at time t via

$$\boldsymbol{y}(\boldsymbol{x},t) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x},t). \tag{5.28}$$

For our purposes, we assume that $\boldsymbol{u}$ is continuously differentiable, so that we may work with the **strain** tensor

$$\boldsymbol{\varepsilon}(\boldsymbol{x},t) = \frac{1}{2}\left[\nabla\boldsymbol{u}(\boldsymbol{x},t) + \boldsymbol{u}(\boldsymbol{x},t)\nabla\right] \tag{5.29}$$

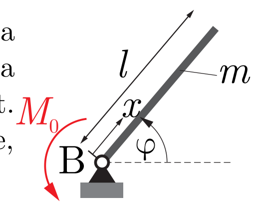
or, in component form,

$$\varepsilon_{ij}(\boldsymbol{x},t) = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}(\boldsymbol{x},t) + \frac{\partial u_j}{\partial x_i}(\boldsymbol{x},t)\right]. \tag{5.30}$$

The **constitutive law** governing the relation between stresses and strains (e.g., **Hooke's law** for **linear elasticity**) remains valid here – only that stresses and strains are now time-dependent.

Example 5.4. Rotating elastic beam (using global balance equations)

Consider an elastic beam (length l, mass m, cross-sectional area A), made of a linear elastic material with Young's modulus E. The beam is rotating about a hinge in 2D, driven by a constant external torque M_0 and starting from rest. Neglecting gravity, what are the inner reactions (axial force, transverse force, bending torque) due to the rotation of the system?



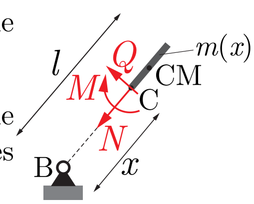
Let us first describe the motion of the system by relating the applied torque M_0 to the rotation. To this end (and to avoid the reaction forces at the hinge), we use the global balance of angular momentum about the hinge B, viz.

$$I_\mathrm{B}\ddot{\varphi} = M_0 \qquad \text{with} \quad I_\mathrm{B} = \frac{ml^2}{3} \qquad \Rightarrow \qquad \ddot{\varphi} = \frac{M_0}{I_\mathrm{B}}. \tag{5.31}$$

By inserting the initial conditions $\varphi(0) = 0$ and $\dot{\varphi}(0) = 0$, we find $\varphi(t)$ by integration:

$$\varphi(t) = \frac{M_0}{2I_\mathrm{B}}t^2 + at + b \qquad \text{and} \qquad a = b = 0 \qquad \Rightarrow \qquad \varphi(t) = \frac{M_0}{2I_\mathrm{B}}t^2. \tag{5.32}$$

To find the inner forces, we consider a free-body diagram of the free end of the rotating beam, cut at a distance $0 \le x \le l$ from the hinge, as shown.



We now apply the balance equations of linear and angular momentum to the shown subpart of the body, accounting for the fact that the drawn inner forces are now external forces to be considered in the balance equations.

For a cut at a distance x from the hinge, the cut end of the beam shown above has the mass

$$m(x) = m\frac{l-x}{l}. \tag{5.33}$$

Let us formulate linear momentum balance in polar coordinates, which in the radial direction applied to the cut end of the beam gives the axial force via

$$m(x)a_r(x) = -N(x) \quad \Rightarrow \quad N(x) = m\frac{l-x}{l}\frac{x+l}{2}\dot{\varphi}^2(t) = m\frac{l^2-x^2}{2l}\dot{\varphi}^2(t) = \frac{l^2-x^2}{l}\frac{M_0^2 m}{2I_\mathrm{B}^2}t^2, \tag{5.34}$$

where we exploited that the radial (centripetal) acceleration during the circular motion is

$$a_r = -r_\text{B-CM}\dot{\varphi}^2 = -\left(x + \frac{l-x}{2}\right)\dot{\varphi}^2 = -\frac{x+l}{2}\dot{\varphi}^2. \tag{5.35}$$

Similarly, linear momentum balance in the tangential direction with

$$a_\varphi = r_\text{B-CM}\ddot{\varphi} = \frac{x+l}{2}\ddot{\varphi} \tag{5.36}$$

gives

$$m(x)a_\varphi(x) = Q(x) \qquad \Rightarrow \qquad Q(x) = m\frac{l-x}{l}\frac{x+l}{2}\ddot{\varphi} = m\frac{l^2-x^2}{2l}\ddot{\varphi} = \frac{l^2-x^2}{l}\frac{M_0 m}{2I_\mathrm{B}}. \tag{5.37}$$

Finally, balance of angular momentum about the center of mass of the cut sub-body yields

$$I_{\mathrm{CM}}\ddot{\varphi} = -M(x) - Q(x)\frac{l-x}{2} \qquad \Rightarrow \qquad M(x) = -I_{\mathrm{CM}}(x)\frac{M_0}{I_{\mathrm{B}}} - Q(x)\frac{l-x}{2} \tag{5.38}$$

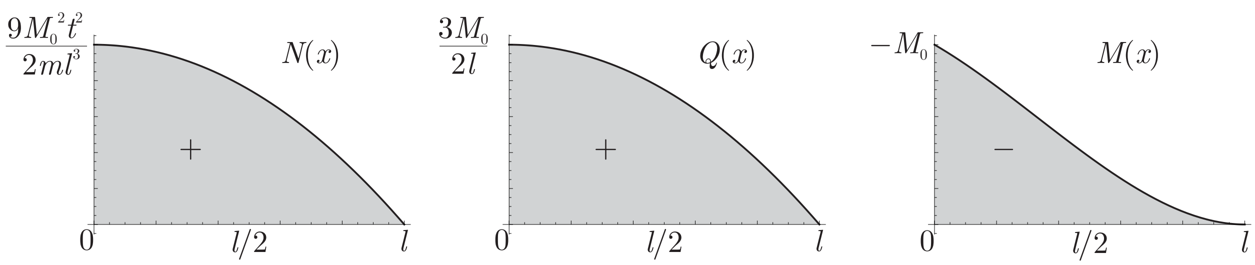
with the moment of inertia

$$I_{\mathrm{CM}}(x) = \frac{1}{12}m(x)\,(l-x)^2\,. \tag{5.39}$$

Altogether, this leads to

$$M(x) = -m\frac{(l-x)^3(4l+3x)}{12l^2}\ddot{\varphi}(t) = -\frac{(l-x)^3(4l+3x)}{12l^2}\frac{M_0 m}{I_{\mathrm{B}}}. \tag{5.40}$$

We can easily verify that all inner reactions, $N(x)$, $Q(x)$ and $M(x)$, vanish at the free end (at $x = l$). They are shown as functions of the radial coordinate x in the figures below (note that we inserted $I_{\mathrm{B}} = ml^2/3$). Under static conditions, all three reactions would vanish due to the absence of any external loads – *the dynamic effects are responsible for the inner forces and torque.*



From these inner forces and torque one can also determine the deformation of the rotating beam as a function of time by computing the axial elongation of the rod from $N(x)$, and the deflection due to bending from $Q(x)$ and/or $M(x)$. It is important to recall that we assume *small strains* (as is typical, e.g, for linear elasticity). As a consequence, deformation is infinitesimal so that changes in the body's shape due to deformation are not considered when calculating distances in the kinematic relations (e.g., the centripetal force is calculated based on the distance from the center of rotation to the *undeformed* center of mass, etc.)

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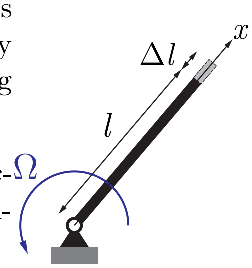
Example 5.5. Rotating elastic bar (using local balance equations)

Let us approximate a rotor blade by a slender bar of initial length l, mass m and cross-sectional area A, made of a linear elastic material with Young's modulus E. The bar is assumed to rotate in 2D at a constant angular velocity Ω about its end point O. How does the length of the bar change during rotation? What is the maximum stress inside the bar?



Since Ω = const., the acceleration at a point x inside the bar (with the x-coordinate measured from the hinge, as shown) is given by the radial and tangential components

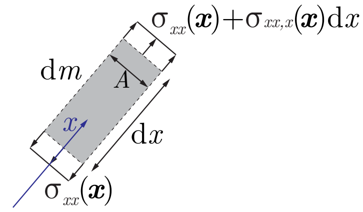
$$a_r(x) = -x\dot{\varphi}^2 = a_r(x) = -x\Omega^2, \qquad a_t(x) = x\ddot{\varphi} = 0. \tag{5.41}$$

Therefore, points on the bar are accelerated only in the radial direction, which results in an axial extension of the bar (but not transverse forces or bending torques resulting in bending of the beam, as in Example 5.4). Consequently, this is a 1D problem and the only non-zero stress component is $\sigma_{xx} = \sigma_{xx}(x)$.

The local balance of linear momentum, $\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f} = \rho\boldsymbol{a}$ here simplifies considerably: first, there are no body forces ($\boldsymbol{f} = \boldsymbol{0}$); second, there is only one non-zero stress component (viz. σ_{xx}); third, all deformation and motion depends only on the axial coordinate x. Therefore, linear momentum balance at every point of the rod, introducing the mass density $\rho = m/(Al)$, reduces to

$$\frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}x} = \rho a_r(x). \tag{5.42}$$

As an alternative derivation, we could also consider an infinitesimal segment of the bar of mass $\mathrm{d}m = \rho A\,\mathrm{d}x$, for which linear momentum balance reads



$$\mathrm{d}m\,a_r(x) = A\left[\sigma_{xx}(x) + \sigma_{xx,x}(x)\,\mathrm{d}x\right] - A\sigma_{xx}(x), \tag{5.43}$$

so that dividing by $A\,\mathrm{d}x$ gives again

$$\rho\,a_r(x) = \frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}x}(x). \tag{5.44}$$

Either way, inserting the acceleration (5.41) into the local linear momentum balance and assuming linear elasticity with

$$\sigma_{xx} = E\varepsilon_{xx} \qquad \text{and} \qquad \varepsilon_{xx} = \frac{\mathrm{d}u_x}{\mathrm{d}x} \tag{5.45}$$

leads to

$$-x\rho\Omega^2 = \frac{\mathrm{d}}{\mathrm{d}x}(E\varepsilon_{xx}) = \frac{\mathrm{d}}{\mathrm{d}x}\left(E\frac{\mathrm{d}u_x}{\mathrm{d}x}\right). \tag{5.46}$$

Assuming that Young's modulus E is constant along the length of the rod gives an ODE for $u(x)$, whose solution is obtained by integrating twice:

$$E\frac{\mathrm{d}^2u_x}{\mathrm{d}x^2} = -x\rho\Omega^2 \qquad \Rightarrow \qquad u_x(x) = -\frac{\rho\Omega^2}{6E}x^3 + c_1x + c_2. \tag{5.47}$$

Using the boundary conditions of vanishing displacements at the hinge (the bar is fixed here) and zero tractions at the free end (i.e., $\sigma_{xx}(l) = 0$) shows that

$$u_x(0) = c_2 = 0, \qquad E\frac{\mathrm{d}u_x}{\mathrm{d}x}(l) = -\frac{\rho\Omega^2}{2E}l^2 + c_1 = 0 \quad \Leftrightarrow \quad c_1 = \frac{\rho\Omega^2}{2E}l^2, \tag{5.48}$$

so that we finally obtain

$$u_x(x) = \frac{\rho\Omega^2}{6E}(3l^2x - x^3). \tag{5.49}$$

The total change of length Δl of the rod is identical to the displacement at the tip:

$$\Delta l = u_x(l) = \frac{\rho\Omega^2l^3}{3E}. \tag{5.50}$$

The stress inside the bar follows as

$$\sigma_{xx}(x) = E\frac{\mathrm{d}u_x}{\mathrm{d}x} = \frac{\rho\Omega^2}{2}(l^2 - x^2). \qquad \Rightarrow \qquad \sigma_{xx,\max} = \sigma_{xx}(0) = \frac{\rho\Omega^2 l^2}{2}. \tag{5.51}$$

As could have been expected, the maximum (tensile) stress is reached at the hinge. Both the maximum stress and the bar's elongation grow quadratically with angular velocity Ω.

As an alternative, the elongation could also have been obtained from the inner forces determined in Example 5.4. From the inner axial force obtained there, see Eq. (5.34), we conclude

$$N(x) = m\frac{l-x}{l}\frac{x+l}{2}\dot{\varphi}^2(t) \qquad \Rightarrow \qquad \sigma_{xx}(x) = \frac{N(x)}{A(x)} = m\frac{l^2 - x^2}{2Al}\Omega^2 = \frac{l^2 - x^2}{2}\rho\Omega^2. \tag{5.52}$$

From the stress we obtain the axial strain and the resulting change of length as

$$\varepsilon_{xx}(x) = \frac{\sigma_{xx}(x)}{E} = \frac{l^2 - x^2}{2E}\rho\Omega^2 \qquad \Rightarrow \qquad \Delta l = \int_0^l \varepsilon_{xx}(x)\,\mathrm{d}x = \frac{\rho\Omega^2 l^3}{3E}. \tag{5.53}$$

We conclude by noting that the above derivations were based on the assumption $\Omega = \text{const}$. If the angular velocity of the rod was not constant, then we would also need to account for a tangential acceleration component $a_t = x\dot{\Omega}$, resulting in an inner transverse force $Q(x)$. The latter, in turn, by angular momentum balance produces an inner torque $M(x)$. Overall, these inner reactions will be analogous to those in Example 5.4 in additionally produce bending deformation of the rod.

5.3 Waves and Vibrations in Slender Rods

In Section 5.1 we discussed how deformable bodies that are approximately massless can support vibrations by acting as effective linear springs. There, the mass of the system was concentrated into particles while the deformable bodies (e.g., bars and beams) only provided the system with stiffness against motion. Here, we consider the case were the deformable body is *not* approximately massless but its mass matters and affects the vibrational behavior of the system. As an introductory example, we consider **slender rods** in this section which may show either axial, longitudinal deformation (i.e., stretching and compression), flexural deformation (i.e., bending), or torsional deformation. The deformation behavior of slender rods was well described in Mechanics 2, however, there with the assumption of static equilibrium and without considering a time-dependent system response.

5.3.1 Longitudinal wave motion and vibrations

Let us first consider a long and slender bar undergoing axial deformation in the form of linear elastic extension and compression. The local balance of linear momentum, as already discussed in Example 5.5, is obtained from an infinitesimal volume element of length $\mathrm{d}x$, having an infinitesimal mass $\mathrm{d}m = \rho(x)A(x)\,\mathrm{d}x$ and a (possibly varying) cross-sectional area $A(x)$:

$$\mathrm{d}m\, a(x) = N(x + \mathrm{d}x, t) - N(x, t), \tag{5.54}$$

where $N(x)$ is the axial force inside the bar. Here, we do not assume any particular motion of the bar (other than that it is one-dimensional), so that we generally characterize the motion and deformation of the bar by the 1D displacement field $u(x,t)$, such that the *undeformed position* x at time $t=0$ and the *deformed position* $y(x,t)$ of that same point at time t are linked via

$$y(x,t) = x + u(x,t). \tag{5.55}$$

The acceleration of any material point $x = \text{const.}$ thus follows as

$$a(x) = \frac{\mathrm{d}}{\mathrm{d}t} y(x,t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[x + u(x,t)\right] = \ddot{u}(x,t). \tag{5.56}$$

Insertion into (5.54) along with $\mathrm{d}m = \rho(x)A(x)\,\mathrm{d}x$ and taking the limit of $\mathrm{d}x \to 0$ yields

$$\rho(x)A(x)\ddot{u}(x,t) = \lim_{\mathrm{d}x\to 0} \frac{N(x+\mathrm{d}x,t) - N(x,t)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} N(x,t) = N_{,x}(x,t). \tag{5.57}$$

We further know that $N(x,t) = A(x)\,\sigma(x,t)$ with the axial stress σ depending on position x and time t, so that

$$\rho(x)A(x)\ddot{u}(x,t) = \left[A(x)\sigma(x,t)\right]_{,x}. \tag{5.58}$$

For a **linear elastic** bar with Young's modulus $E(x)$ we know $\sigma(x,t) = E(x)\varepsilon(x,t) = E(x)u_{,x}(x,t)$. This turns the above linear momentum balance into

$$\rho(x)A(x)\ddot{u}(x,t) = \left[E(x)A(x)u_{,x}(x,t)\right]_{,x}. \tag{5.59}$$

Finally, let us assume the special case of a *homogeneous* rod with *constant* modulus $E(x) = E$, density $\rho(x) = \rho$ and cross-sectional area $A(x) = A$. In this case we arrive at:

$$\rho\ddot{u}(x,t) = E u_{,xx}(x,t) \qquad \Rightarrow \qquad \boxed{\ddot{u}(x,t) = c^2 u_{,xx}(x,t) \qquad \text{with} \qquad c = \sqrt{\frac{E}{\rho}}} \tag{5.60}$$

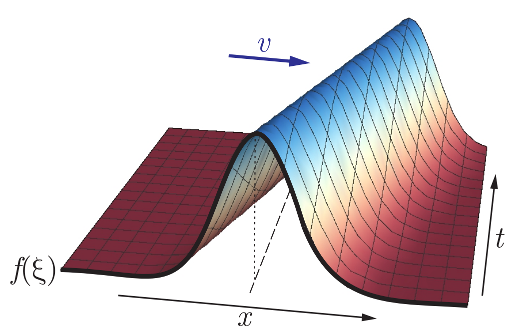
This is the one-dimensional **wave equation**, a linear, homogeneous partial differential equation (PDE) of second order in both time and space. This equation is analogous to (5.47) in the example of the rotating bar, except that here we make no assumption about the acceleration of the bar (unlike in the case of the rotation example where the acceleration was given by the centripetal acceleration). The above wave equation is sufficiently general to study 1D waves and vibrations in long and slender bars.

Consider, e.g., the general form of a **wave** moving in the x-direction with a speed v, for which the displacement is assumed to be expressed as

$$u(x,t) = f(x - vt) = f(\xi) \qquad \text{with} \qquad \xi = x - vt, \tag{5.61}$$

where f is any smooth, differentiable function describing the invariant shape of the wave, and we introduce the coordinate ξ for convenience. Note that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \qquad \text{and} \qquad \frac{\partial}{\partial t} = -v\frac{\partial}{\partial \xi}. \tag{5.62}$$

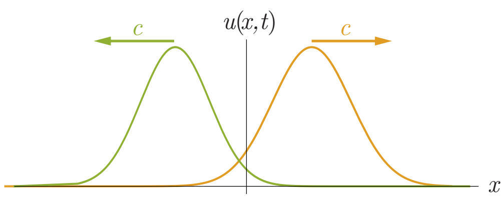


Inserting the general from (5.61) into (5.60) and using the relations (5.62) leads to

$$v^2 f_{,\xi\xi}(\xi) = c^2 f_{,\xi\xi}(\xi) \qquad \Rightarrow \qquad v = \pm c. \tag{5.63}$$

That is, any longituginal wave in the 1D bar must move with the speed c, which is the **longitudinal wave speed** in the bar[21]. The solution is thus of the general form (with smooth, differentiable functions $f_i(\xi)$)

$$u(x,t) = f_1(x - ct) + f_2(x + ct). \tag{5.64}$$



The solution involves so-called **characteristics** moving in opposite directions at the longitudinal wave speed c. The exact form of f_1 and f_2 depends on the initial/boundary conditions. For example, if a bar is excited mechanically at one of its ends by an applied force or displacement, then a wave travels through the bar at speed c. If the bar is excited at an interior point, waves of speed c will travel in both directions from the point of excitation.

Obviously, the specific solution depends on the boundary and initial conditions. When waves travel in objects of *finite* size, e.g., in a rod of length l, then they interact with the ends of the rod. At the end, waves are reflected, which – over time – results in a superposition of waves that may turn into a **standing wave** (depending on the boundary conditions applied to the ends of the bar), whose eigenfrequencies and mode shapes are analogous to those of vibrations discussed before. For example, consider hitting a guitar string to produce music: initially (in the first fraction of a second) waves propagate through the string, reflect at the ends and travel back. But what is ultimately observed is the string vibrating in a standing wave – the ends being fixed and the string's center moving periodically at approximately fixed amplitude. Let us identify such standing waves (and neglect the exact initial conditions that have produced the latter in a transient manner).

If the boundary conditions do not depend on time (e.g., when a rod is fixed at both ends so that $u(0,t) = u(l,t) = 0$ for all times t), then the PDE and boundary conditions admit a separable solution as follows. In order to solve the wave equation

$$\ddot{u}(x,t) = c^2 u_{,xx}(x,t), \tag{5.65}$$

we make the separable ansatz $u(x,t) = \hat{u}(x)q(t)$ with unknown functions $\hat{u}(x)$ and $q(t)$. Insertion into the wave equation yields

$$\hat{u}(x)\ddot{q}(t) = c^2 \hat{u}_{,xx}(x)q(t) \qquad \Leftrightarrow \qquad \frac{\ddot{q}(t)}{q(t)} = c^2 \frac{\hat{u}_{,xx}(x)}{\hat{u}(x)}. \tag{5.66}$$

Note that the fraction on the left-hand side of the equation can only depend on t, while the fraction on the right-hand side can only depend on x. This implies that both fractions are independent of both x and t and must therefore be constant. If we call that constant $-\omega^2$, then

$$\frac{\ddot{q}(t)}{q(t)} = -\omega^2 = \text{const.} \qquad \text{and} \qquad c^2 \frac{\hat{u}_{,xx}(x)}{\hat{u}(x)} = -\omega^2 = \text{const.} \tag{5.67}$$

[21] As an example, consider steel with $E \approx 210$ GPa, $\rho \approx 7870$ kg/m^3 so that $c = \sqrt{E/\rho} \approx 5166$ m/s.

Now we can independently solve each of these two ODEs, starting with

$$\ddot{q}(t)+\omega^2 q(t)=0 \qquad \text{and} \qquad \hat{u}_{,xx}(x)+\frac{\omega^2}{c^2}\hat{u}(x)=0. \tag{5.68}$$

These second-order ODEs are of the same type as the equation of motion for a single-DOF free vibration (discussed in Section 4.3), so the solution is of harmonic type. We thus write the **general solution** as

$$\boxed{\begin{aligned}&u(x,t)=\hat{u}(x)q(t) \qquad \text{with}\\ &\hat{u}(x)=B_1\cos\left(\frac{\omega}{c}x\right)+B_2\sin\left(\frac{\omega}{c}x\right) \qquad \text{and} \qquad q(t)=A_1\cos(\omega t)+A_2\sin(\omega t)\end{aligned}} \tag{5.69}$$

with coefficients $A_1,\ A_2,\ B_1,\ B_2\in\mathbb{R}$ which, along with ω, depend on the boundary and initial conditions.

In general, we will find more than one **eigenfrequency** ω (as shown in the examples below). In fact, we may expect an infinite set of eigenfrequencies $\{\omega_1,\omega_2,\dots\}$. Since the wave equation is a *linear* PDE, the complete solution can be expressed as as sum

$$u(x,t)=\sum_{n=1}^{\infty}\hat{u}_n(x)q_n(t), \tag{5.70}$$

in which each term in the (generally infinite) sum is of the form (5.69), viz.

$$\hat{u}_n(x)=B_{n,1}\cos\left(\frac{\omega_n}{c}x\right)+B_{n,2}\sin\left(\frac{\omega_n}{c}x\right) \quad \text{and} \quad q_n(t)=A_{n,1}\cos(\omega_n t)+A_{n,2}\sin(\omega_n t). \tag{5.71}$$

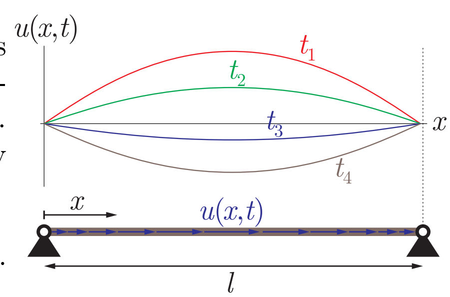
The **complete solution** thus becomes

$$\boxed{\begin{aligned}u(x,t)&=\sum_{n=1}^{\infty}\hat{u}_n(x)q_n(t)\\ &=\sum_{n=1}^{\infty}\left[B_{n,1}\cos\left(\frac{\omega_n}{c}x\right)+B_{n,2}\sin\left(\frac{\omega_n}{c}x\right)\right]\left[A_{n,1}\cos(\omega_n t)+A_{n,2}\sin(\omega_n t)\right]\end{aligned}} \tag{5.72}$$

All coefficients $A_{n,i},B_{n,i}\in\mathbb{R}$ as well as the frequencies $\omega_n\geq 0$ are to be found from the boundary and initial conditions. The lowest non-zero frequency ω_1 is known as the **natural frequency** (or **fundamental frequency**).

Example 5.6. Vibrating bar in the pinned-pinned configuration

Consider a vibrating bar of length l that is fixed at both ends (this is known as the **pinned-pinned** or **fixed-fixed** configuation). The general solution to the problem is given by (5.69). To find the coefficients as well as ω, let us use the boundary conditions



$$u(0,t)=u(l,t)=0 \quad \forall\, t \quad \Rightarrow \quad \hat{u}(0)q(t)=\hat{u}(l)q(t)=0 \quad \forall\, t.$$

Since the boundary conditions hold for all times t, we can directly enforce $\hat{u}(0) = \hat{u}(l) = 0$. By applying the boundary conditions to $\hat{u}(x)$ from (5.69), we obtain a system of equations which is written in matrix form as

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{u}(0) \\ \hat{u}(l) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \cos\left(\frac{\omega}{c}l\right) & \sin\left(\frac{\omega}{c}l\right) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \tag{5.73}$$

The first equation requires that $B_1 = 0$. The second one has either a trivial solution ($B_2 = 0$) or it requires that

$$\sin\left(\frac{\omega}{c}l\right) = 0 \qquad \Leftrightarrow \qquad \frac{\omega}{c}l = n\pi \qquad \text{for any integer } n. \tag{5.74}$$

We have hence indeed obtained an infinite set of **eigenfrequencies of a longitudinal bar** in its pinned-pinned configuration, viz.

$$\omega_n = \frac{cn\pi}{l} \qquad \text{with} \qquad n = 1, 2, \ldots, \infty. \tag{5.75}$$

Note that the above procedure can be generalized to other boundary conditions, in which case the *non-trivial solution* is found from requiring

$$\det \begin{pmatrix} 1 & 0 \\ \cos\left(\frac{\omega}{c}l\right) & \sin\left(\frac{\omega}{c}l\right) \end{pmatrix} = 0 \qquad \Rightarrow \qquad \omega_n = \frac{cn\pi}{l}. \tag{5.76}$$

With the above infinite set of eigenfrequencies, the complete solution $u(x,t) = \hat{u}(x)q(t)$ becomes

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(n\pi\frac{x}{l}\right)\left[A_{n,1}\cos\left(n\pi\frac{ct}{l}\right) + A_{n,2}\sin\left(n\pi\frac{ct}{l}\right)\right] \quad \text{with} \quad A_{n,1}, A_{n,2} \in \mathbb{R}, \tag{5.77}$$

whose coefficients $A_{n,1}$ and $A_{n,2}$ (with $n = 1, \ldots, \infty$) depend on the initial conditions. The lowest (non-zero) solution involves the **natural frequency** $\omega_1 = \pi\frac{c}{l}$. Like for the vibrations discussed previously, each **eigenfrequency** $\omega_n = \frac{cn\pi}{l}$ has an associated **mode shape**

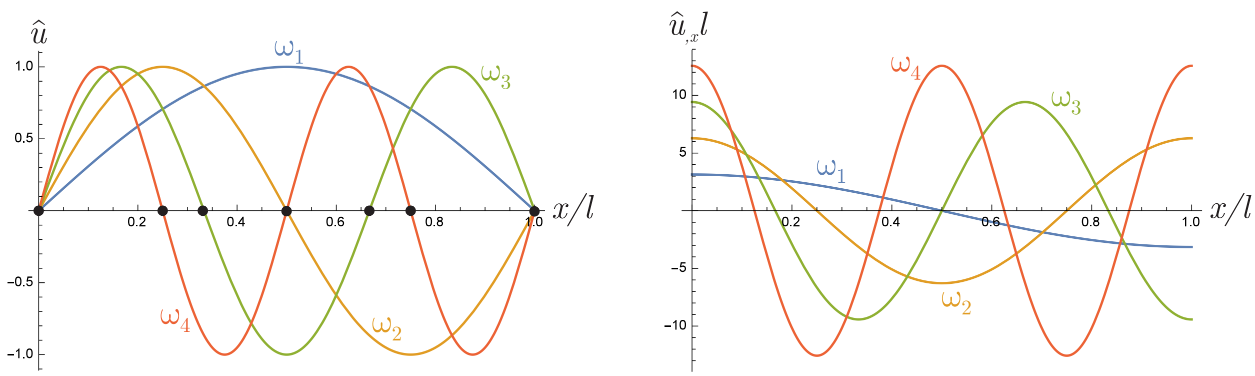
$$\hat{u}_n(x) = \sin\left(n\pi\frac{x}{l}\right). \tag{5.78}$$

A mode shape is often referred to also as a **normal mode** (or, especially in the context of acoustic waves, as a **harmonic**). That is, the nth term in the solution (5.77) has the shape of $\hat{u}_n(x)$ whose amplitude is time-dependent and given by the term in brackets in (5.77) (dependent on the initial conditions). The mode shapes for the first four (non-zero) eigenfrequencies are shown below on the left. For general boundary conditions, usually more than one of the modes are excited, so that the superposition in (5.77) indeed becomes the solution, with several (or many) modes overlaid.

Each mode shape comes with so-called **nodes**, i.e., points at which $\hat{u}_n(x) = 0$ (shown above as black dots). For example, for the pinned-pinned configuration we know that $\hat{u}(0) = \hat{u}(l) = 0$, so we must have at least two nodes at the ends of the bar. For higher-order modes, internal nodes appear as well. The **wave length** λ_n of the nth mode is obtained from

$$\lambda_n \frac{\omega_n}{2\pi} = c \qquad \Rightarrow \qquad \lambda_n = \frac{2l}{n}, \tag{5.79}$$

and it describes the length of a complete period of each mode shape.



pinned-pinned eigenmodes

From the solution, we can also calculate the axial stress $\sigma(x)$ inside the bar for each mode shape, which for a homogeneous bar with $E(x) = E$ leads to

$$\sigma(x,t) = E\varepsilon(x,t) = Eu_{,x}(x,t) = E\hat{u}_{,x}(x)q(t) \;\propto\; \hat{u}_{,x}. \tag{5.80}$$

The shape of the (normalized) stress distribution $\hat{u}_{,x}l$ is included in the above figure on the right. We can identify *stress nodes* at which the stress vanishes. Due to the pinned-pinned configuration, there are no stress nodes at the two ends (since there are reaction forces).

Example 5.7. Vibrating bar in the pinned-free configuration

As another example, consider the case of a **pinned-free** (or **fixed-free**) configuration. As a practical example, this scenario describes the air column in a long and slender pipe (such as those found in musical wind instruments like flutes and in organ pipes). When excited by the player, the air column develops a standing wave as a superposition of eigenmodes, which is how the length of the pipe is uniquely linked to its tone (the latter being determined by the frequency of vibration).

Here, the boundary value problem reads

$$\ddot{u}(x,t) = c^2 u_{,xx}(x,t) \qquad \text{with} \qquad u(0,t) = 0, \quad u_{,x}(l,t) = 0 \quad \forall\, t, \tag{5.81}$$

where the second condition was obtained from requiring a zero axial force $N(l) = EAu_{,x}(l) = 0$ at the free end at $x = l$. We exploit again that the boundary conditions are independent of time, so that we must enforce $\hat{u}(0) = \hat{u}_{,x}(l) = 0$. Writing the system of equations in matrix form gives

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{u}(0) \\ \hat{u}_{,x}(l) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{\omega}{c}\sin\left(\frac{\omega}{c}l\right) & \frac{\omega}{c}\cos\left(\frac{\omega}{c}l\right) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \tag{5.82}$$

As in the pinned-pinned configuration, we obtain $B_1 = 0$, which is now complemented by the characteristic equation

$$\frac{\omega}{c}\cos\left(\frac{\omega}{c}l\right) = 0 \qquad \Leftrightarrow \qquad \frac{\omega}{c}l = (2n-1)\frac{\pi}{2} \qquad \text{for any integer } n. \tag{5.83}$$

The infinite set of eigenfrequencies in the pinned-free configuration is therefore

$$\omega_n = \frac{c(2n-1)\pi}{2l}, \qquad \text{with} \qquad n = 1, 2, \ldots, \infty, \tag{5.84}$$

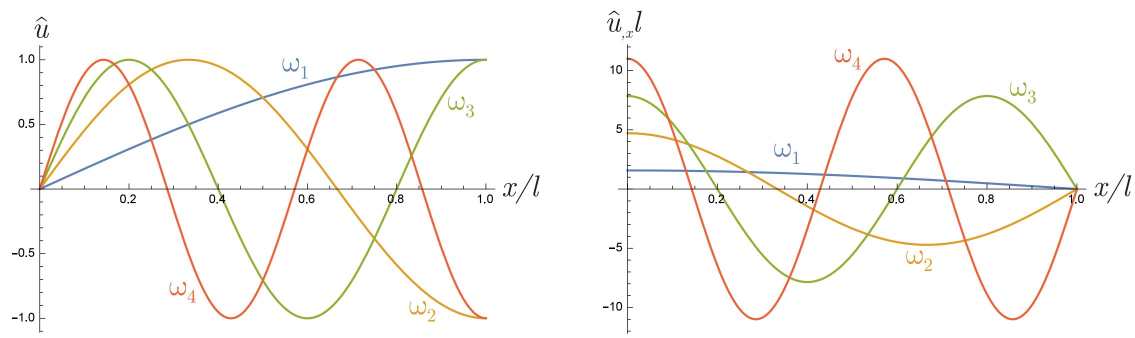
and the complete solution, with $A_{n,1}, A_{n,2} \in \mathbb{R}$ for all $n = 1 \ldots, n$, becomes

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi}{2}\frac{x}{l}\right)\left[A_{n,1}\cos\left(\frac{(2n-1)\pi}{2}\frac{ct}{l}\right) + A_{n,2}\sin\left(\frac{(2n-1)\pi}{2}\frac{ct}{l}\right)\right]. \tag{5.85}$$

Here, the **natural frequency** is $\omega_1 = \frac{\pi}{2}\frac{c}{l}$. Each eigenfrequency $\omega_n = c(2n-1)\pi/2l$ has an associated **mode shape** and **wave length** of, respectively,

$$\hat{u}_n(x) = \sin\left(\frac{(2n-1)\pi}{2}\frac{x}{l}\right) \qquad \text{and} \qquad \lambda_n = \frac{4l}{2n-1}. \tag{5.86}$$

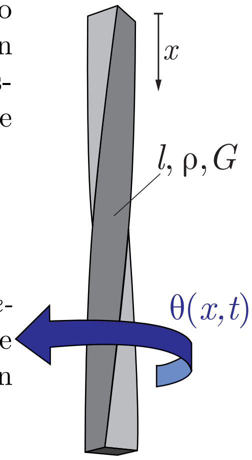
The mode shapes for the first four (non-zero) eigenfrequencies are shown below, along with the stress profiles associated with those normal modes.



pinned-free eigenmodes

For any other configuration, e.g., the free-free configuration, the eigenfrequencies and mode shapes can be derived analogously. We here restrict our discussion to the above two examples.

5.3.2 Torsional wave motion and vibrations



Torsional waves in long and slender rods can be analyzed in complete analogy to the previous section. Now using the twist angle $\theta(x)$ to describe the deformation of the rod (specifically, the in-plane rotation around the central axis in the cross-sectional plane), the governing equation of local angular momentum balance here is

$$I_0(x)\ddot{\theta}(x,t) = \left[G(x)J(x)\theta_{,x}(x,t)\right]_{,x}. \qquad \text{where} \qquad I_0(x) = \int_A \rho \bar{r}^2 \, \mathrm{d}a$$

is the moment of inertia per unit thickness. For the special case of a *homogeneous* rod with constant properties, this reduces significantly. Especially, the moment of inertia I_0 and the polar area moment J are related by equation (3.113) so that $I_0 = \rho J$.
This simplies the above into

$$\rho J\,\ddot{\theta}(x,t) = GJ\,\theta_{,xx}(x,t) \qquad \Rightarrow \qquad \boxed{\ddot{\theta}(x,t) = c_\mathrm{T}^2\theta_{,xx}(x,t) \qquad \text{with} \qquad c_\mathrm{T} = \sqrt{\frac{G}{\rho}}} \tag{5.87}$$

Notice that this equation, the **torsional wave equation**, has the exact same form as Eq. (5.60), which governed longitudinal waves in slender rods – the only differences being the replacement of the displacement $u(x)$ by the twist angle $\theta(x)$, and the replacement of the wave speed c by the **torsional wave speed** c_T, also known as the **shear wave speed**.

Recall that $G = E/2(1+\nu)$, so the shear wave speed is usually smaller than the longitudinal bar wave speed:

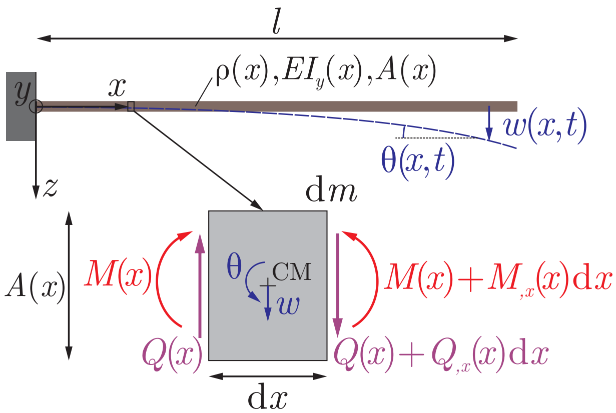
$$c_\mathrm{T} = \sqrt{G/\rho} \quad < \quad c = \sqrt{E/\rho}. \tag{5.88}$$

The analogy with longitudinal waves also means that there is no need to re-derive all those relations obtained for standing waves and mode shapes: since the governing equation is of the identical form, **all solutions derived in Section 5.3.1 for longitudinal waves also apply to torsional waves, if we make the replacements**

$$c \to c_\mathrm{T}, \qquad u(x) \to \theta(x). \tag{5.89}$$

The general solution of a standing wave is therefore again of type (5.69), whose coefficients and frequencies are to be found from the specific boundary conditions being applied.

5.3.3 Flexural wave motion and vibrations



Bending (or flexural) deformation is the final type we would like to consider. Unfortunately, this case is more involved than the previous ones due to the nature of the governing equations. From Mechanics 2 we know that bending of a beam (whose axis is along the $x = x_1$-direction) involves both a transverse force $Q(x,t)$ acting in the $z = x_3$-direction and a bending torque $M(x,t)$ acting around the $y = x_2$-axis, as shown on the left for a beam of length l and an infinitesimal beam element of length $\mathrm{d}x$ and mass $\mathrm{d}m$.

For a slender, linear elastic Euler-Bernoulli beam we assume that the deformation is captured by the (small) deflection $w(x,t) = u_3(x,t)$ in the z-direction, and the local change in angle of the beam follows as $\theta(x,t) = w_{,x}(x,t)$.

Let us consider an infinitesimal beam element of length $\mathrm{d}x$ and mass $\mathrm{d}m = \rho(x)A(x)\,\mathrm{d}x$, whose transverse forces Q and torques M (approximated by a Taylor expansion for a small beam element, as before) are shown above. Linear momentum balance in the z-direction yields:

$$\mathrm{d}m\,\ddot{w}(x,t) = Q(x,t) - Q(x,t) + Q_{,x}(x,t)\,\mathrm{d}x \qquad \Rightarrow \qquad \rho(x)A(x)\ddot{w}(x,t) = Q_{,x}(x,t). \tag{5.90}$$

Similarly, angular momentum balance around the center of mass of the beam element yields

$$\mathrm{d}I(x)\ddot{\theta}(x,t) = M(x,t) - M(x,t) + M_{,x}(x,t)\,\mathrm{d}x - 2Q(x,t)\frac{\mathrm{d}x}{2} - Q_{,x}(x,t)\frac{(\mathrm{d}x)^2}{2}, \tag{5.91}$$

where $\mathrm{d}I(x) = \mathrm{d}m\,I_y(x) = \rho(x)I_y(x)\,\mathrm{d}x$ is the moment of inertia and I_y is the area moment of inertia (which also governs the bending stiffness). Dividing (5.91) by $\mathrm{d}x$ leads to

$$\rho(x)I_y(x)\ddot{\theta}(x,t) = M_{,x}(x,t) - Q(x,t) - Q_{,x}(x,t)\frac{\mathrm{d}x}{2}. \tag{5.92}$$

The last term on the right-hand side (multiplying $\mathrm{d}x$ with $|\mathrm{d}x| \ll 1$) is significantly smaller than the others and is therefore neglected in the following. In order to insert (5.90), we differentiate (5.92) with respect to x, so that insertion of (5.90) yields

$$[\rho(x)I_y(x)\ddot{\theta}(x,t)]_{,x} = M_{,xx}(x,t) - Q_{,x}(x,t) = M_{,xx}(x,t) - \rho(x)A(x)\ddot{w}(x,t). \tag{5.93}$$

Rotational inertia is commonly much less significant than translational inertia for beam bending (deflections are more severe than axis rotations, so $|I_y\ddot{\theta}| \ll |A\ddot{w}|$). Therefore, one may also drop the term on the left-hand side of the above equation, so that

$$-M_{,xx}(x,t) + \rho(x)A(x)\ddot{w}(x,t) = 0. \tag{5.94}$$

Finally, we recall the constitutive relation of elastic beam bending,

$$M = -EI_y w_{,xx}, \tag{5.95}$$

which transforms (5.94) into

$$[E(x)I_y(x)w_{,xx}(x,t)]_{,xx} + \rho(x)A(x)\ddot{w}(x,t) = 0. \tag{5.96}$$

For the special case of a *homogeneous* beam with constant Young's modulus $E(x) = E$ and mass density $\rho(x) = \rho$, and constant cross-section $A(x) = A$ such that also $I_y(x) = I_y =$ const., the above reduces to

$$\boxed{w_{,xxxx}(x,t) + \frac{\rho A}{EI_y}\ddot{w}(x,t) = 0} \tag{5.97}$$

This is a homogeneous PDE of fourth order in space and second order in time, which is to be solved for the deflection $w(x,t)$ for specific initial and boundary conditions.

As for the previous cases of longitudinal and torsional waves, we may seek solutions in the form of a **standing wave**. If the boundary conditions do not depend on time, then we may again make a separable ansatz and insert it into (5.97):

$$w(x,t) = \hat{w}(x)\,q(t) \qquad \Rightarrow \qquad \hat{w}_{,xxxx}(x)q(t) + \frac{\rho A}{EI_y}\hat{w}(x)\ddot{q}(t) = 0. \tag{5.98}$$

As discussed for longitudinal deformation, we observe that the following two fractions must be constant, so we define

$$\frac{\hat{w}_{,xxxx}(x)}{\hat{w}(x)} = -\frac{\rho A}{EI_y}\frac{\ddot{q}(t)}{q(t)} = k^4 = \text{const.}, \tag{5.99}$$

which leads to two ODEs in space and time, respectively:

$$\hat{w}_{,xxxx}(x) - k^4\hat{w}(x) = 0 \qquad \text{and} \qquad \ddot{q}(t) + k^4\frac{EI_y}{\rho A}q(t) = 0. \tag{5.100}$$

The time-dependent part, as before, yields to a harmonic solution

$$q(t) = A_1\cos(\omega t) + A_2\sin(\omega t) \qquad \text{with} \qquad \omega^2 = k^4\frac{EI_y}{\rho A}. \tag{5.101}$$

The mode shape $\hat{w}(x)$ is more complex here due to the fourth-order ODE. The solution is obtained from the ansatz

$$\hat{w}(x) = B\exp(\lambda t) \qquad \Rightarrow \qquad \lambda^4\hat{w} - k^4\hat{w} = 0 \qquad \Rightarrow \qquad \lambda = \pm k \;\vee\; \lambda = \pm ik. \tag{5.102}$$

With these four possible solutions, we thus arrive at[22]

$$\hat{w}(x) = B_1\cos(kx) + B_2\sin(kx) + B_3\cosh(kx) + B_4\sinh(kx). \tag{5.103}$$

[22] We here exploit that, as discussed for single-DOF vibrations, solutions of the type $\exp(ikx)$ and $\exp(-ikx)$ can be converted into solutions $\cos(kx)$ and $\sin(kx)$. Likewise, we here use that $\cosh(kx) = \frac{1}{2}[\exp(kx) + \exp(-kx)]$ and $\sinh(kx) = \frac{1}{2}[\exp(kx) - \exp(-kx)]$ to transform the remaining two terms in the solution. Also, as before we may replace $A_1\cos(\omega t) + A_2\sin(\omega t)$ everywhere by $A\cos(\omega t + \varphi)$, if convenient.

The **general solution** of the flexural vibration problem follows as

$$\boxed{w(x,t) = \hat{w}(x)\, q(t)} \tag{5.104}$$

with

$$\boxed{\begin{aligned} &\hat{w}(x) = B_1 \cos(kx) + B_2 \sin(kx) + B_3 \cosh(kx) + B_4 \sinh(kx) \qquad \text{and} \\ &q(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t), \qquad \omega^2 = k^4 \frac{EI_y}{\rho A} \end{aligned}} \tag{5.105}$$

where coefficients A_1, A_2 and B_1, B_2, B_3, $B_4 \in \mathbb{R}$ are to be obtained from initial/boundary conditions. We observe that the mode shape $\hat{w}(x)$ and the eigenfrequency ω are linked through k. To determine the eigenfrequencies and associated mode shapes, we need to supply the above general solution with boundary conditions.

The **complete solution** follows as the superposition of all modes:

$$\boxed{\begin{aligned} w(x,t) &= \sum_{n=1}^{\infty} \hat{w}_n(x)\, q_n(t) \\ &= \sum_{n=1}^{\infty} \left[B_{n,1} \cos(k_n x) + B_{n,2} \sin(k_n x) + B_{n,3} \cosh(k_n x) + B_{n,4} \sinh(k_n x)\right] \\ &\qquad \times \left[A_{n,1} \cos(\omega_n t) + A_{n,2} \sin(\omega_n t)\right], \qquad \omega_n = k_n^2 \sqrt{\frac{EI_y}{\rho A}} \end{aligned}} \tag{5.106}$$

Example 5.8. Vibration of a cantilever beam

As an example, we consider the free vibration of a **cantilever beam** in its **clamped-free configuration**. If the beam is clamped at $x = 0$, then both the deflection and the tilt angle must vanish there for all times t:

$$w(0,t) = 0, \quad w_{,x}(0,t) = 0 \quad \forall\, t \qquad \Rightarrow \qquad \hat{w}(0) = 0, \quad \hat{w}_{,x}(0) = 0. \tag{5.107}$$

A free end at $x = l$ implies that both the bending torque and the transverse force vanish at that end. Since we know from beam theory that $M = -EIw_{,xx} \sim w_{,xx}$ and $Q = M_{,x} = EIw_{,xxx} \sim w_{,xxx}$, we require that

$$w_{,xx}(l,t) = 0, \quad w_{,xxx}(l,t) = 0 \quad \forall\, t \qquad \Rightarrow \qquad \hat{w}_{,xx}(l) = 0, \quad \hat{w}_{,xxx}(l) = 0. \tag{5.108}$$

The four conditions (5.107) and (5.108) applied to $\hat{w}(x)$ from (5.105) become in matrix form

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & k & 0 & k \\ -k^2 \cos(kl) & -k^2 \sin(kl) & k^2 \cosh(kl) & k^2 \sinh(kl) \\ k^3 \sin(kl) & -k^3 \cos(kl) & k^3 \sinh(kl) & k^3 \cosh(kl) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix}. \tag{5.109}$$

The non-trivial solution is obtained from the characteristic equation

$$0 = \det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & k & 0 & k \\ -k^2\cos(kl) & -k^2\sin(kl) & k^2\cosh(kl) & k^2\sinh(kl) \\ k^3\sin(kl) & -k^3\cos(kl) & k^3\sinh(kl) & k^3\cosh(kl) \end{pmatrix} = 2k^6\left[1+\cos(kl)\cosh(kl)\right]. \tag{5.110}$$

Even though not possible analytically, one can numerically find the roots of $1+\cos(kl)\cosh(kl) = 0$, which yields an infinite set of solutions:

$$k_n l = 1.8751,\ 4.6941,\ 7.8548,\ 10.9955,\ \dots \tag{5.111}$$

The eigenfrequencies are obtained via

$$\omega_n = k_n^2\sqrt{\frac{EI_y}{\rho A}} = (k_n l)^2\sqrt{\frac{EI_y}{\rho A l^4}} = (k_n l)^2\sqrt{\frac{EI_y}{ml^3}} \qquad \text{for} \quad n = 1,\dots,\infty \tag{5.112}$$

and evaluate to

$$\omega_n = 1.8751^2\sqrt{\frac{EI_y}{ml^3}}, \quad 4.6941^2\sqrt{\frac{EI_y}{ml^3}}, \quad 7.8548^2\sqrt{\frac{EI_y}{ml^3}}, \quad 10.9955^2\sqrt{\frac{EI_y}{ml^3}}, \quad \dots \tag{5.113}$$

For each eigenfrequency ω_n, we can find the corresponding mode shape by inserting k_n into the coefficient matrix in (5.109) and solving for the eigenvector. This can be performed numerically.

For example, for ω_1 we find the coefficients (not normalized)

$$(B_{1,1}, B_{1,2}, B_{1,3}, B_{1,4})^{\mathrm{T}} = (1, -0.734096, -1, 0.734096)^{\mathrm{T}}, \tag{5.114}$$

while the mode shape associated with ω_2 has the coefficients

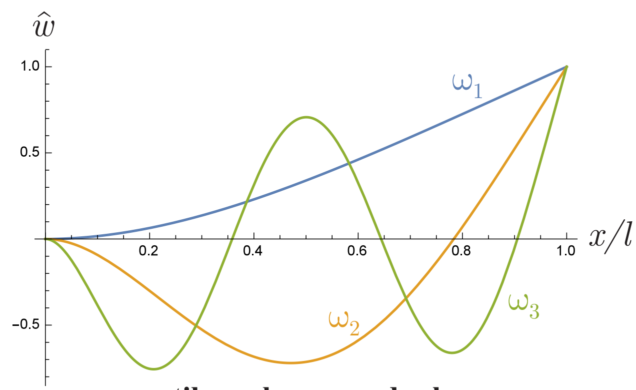
$$(B_{2,1}, B_{2,2}, B_{2,3}, B_{2,4})^{\mathrm{T}} = (1, -1.01847, -1, 1.01847)^{\mathrm{T}}, \tag{5.115}$$

and the third mode shape has

$$(B_{3,1}, B_{3,2}, B_{3,3}, B_{3,4})^{\mathrm{T}} = (1, -1, -1, 1)^{\mathrm{T}}. \tag{5.116}$$

Inserting the B_i-coefficients as well as the corresponding solution for k from (5.111) into $\hat{w}(x)$ from (5.105) provides the sought mode shapes. Note that the above coefficient vectors are, in principle, only defined up to a constant, since the complete solution defines the amplitude of each mode through $q_n(t)$.

The first three modes of the cantilever beam (normalized such that $\hat{w}(l) = 1$) are plotted below. As before, the complete solution is a superposition of all modes whose amplitudes depend on the initial conditions.



cantilever beam mode shapes

Other beam configurations, such as the **free-free** configuration (zero bending torques and forces at both ends) or the **pinned-pinned** configuration (zero deflection and zero torques at both ends), can be treated analogously. The respective boundary conditions are applied to the general solution, and the resulting system of equations admits a non-trivial solution for an infinite set of eigenfrequencies. There are various experimental demonstrations of beam mode shapes (see, e.g., this movie).

Finally, notice that – unlike in the cases of longitudinal and torsional waves – traveling waves of the form $w(x,t) = f_1(x - ct) + f_2(x + ct)$ are *not* solutions of equation (5.97), since it is of fourth order in space and second in time. The flexural wave speed is therefore dependent on the frequency of the wave (to be discussed elsewhere). Moreover, the wave speed in bending depends not only on material properties (such as E, G and ρ), but it also depends on the geometry of the beam through the factor $EI_y/\rho A$ since

$$\ddot{w}(x,t) + \frac{EI_y}{\rho A} w_{,xxxx}(x,t) = 0 \qquad \text{and} \qquad \frac{EI_y}{\rho A} = c^2 \frac{I_y}{A}. \tag{5.117}$$

5.4 Summary of Key Relations

effective stiffness of a rod in **extension**, **bending**, and **torsion**:

$$k_{\text{eff}} = \frac{F}{\Delta l} = \frac{EA}{l}, \qquad k_{\text{eff}} = \frac{F}{w} = \frac{3EI}{l^3}, \qquad k_{\text{eff}} = \frac{M}{\Delta\theta} = \frac{GJ}{l}$$

global balance of linear momentum for bodies and sub-bodies:

$$\sum_i \boldsymbol{F}_i^{\text{ext}} = M\boldsymbol{a}_{\text{CM}}$$

local balance of linear momentum:

$$\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f} = \rho\boldsymbol{a} \qquad \text{or} \qquad \sum_{j=1}^{3} \frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}x_j} + f_i = \rho a_i \qquad \text{for} \quad i = 1, 2, 3$$

longitudinal wave equation for stretching/compression of a homogeneous slender bar:

$$\ddot{u}(x,t) = c^2 u_{,xx}(x,t) \qquad \text{with} \qquad c = \sqrt{\frac{E}{\rho}}$$

torsional wave equation for twisting of a homogeneous slender bar:

$$\ddot{\theta}(x,t) = c_{\text{T}}^2 \theta_{,xx}(x,t) \qquad \text{with} \qquad c_{\text{T}} = \sqrt{\frac{G}{\rho}}$$

general solution for **longitudinal vibrations** (and **torsion** analogously):

$$u(x,t) = \hat{u}(x)q(t) \qquad \text{with}$$

$$\hat{u}(x) = B_1 \cos\left(\frac{\omega}{c}x\right) + B_2 \sin\left(\frac{\omega}{c}x\right) \qquad \text{and} \qquad q(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t)$$

flexural wave equation for bending of a homogeneous slender bar:

$$w_{,xxxx}(x,t) + \frac{\rho A}{EI_y}\ddot{w}(x,t) = 0$$

general solution for **flexural vibrations**:

$$w(x,t) = \hat{w}(x)q(t) \qquad \text{with} \qquad q(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t)$$

$$\text{and} \qquad \hat{w}(x) = B_1 \cos(kx) + B_2 \sin(kx) + B_3 \cosh(kx) + B_4 \sinh(kx), \quad \omega^2 = k^4 \frac{EI_y}{\rho A}$$

complete solution for longitudinal and flexural vibrations, respectively:

$$u(x,t) = \sum_{n=1}^{\infty} \hat{u}_n(x) q_n(t), \qquad w(x,t) = \sum_{n=1}^{\infty} \hat{w}_n(x) q_n(t)$$

Appendix

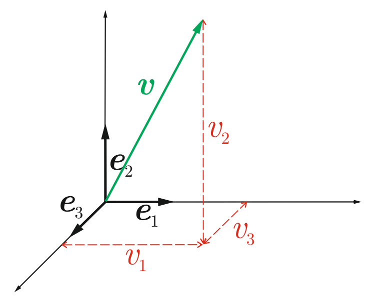
A What is a tensor?

In mechanics, we frequently need the concept of a *tensor*, which may be nebulous at first. We here try to summarize what a tensor is, how to work with it, and what the difference between a tensor and a matrix is. In this context, it is best to start with something we understand well: vectors.

A.1 What is a vector?

A *vector* is an object characterized by a magnitude and a direction. If we define a reference frame $\mathcal{C}$ with a Cartesian basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ in 3D, then we may define a vector $\boldsymbol{v}$ as

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 = \sum_{i=1}^{3} v_i\boldsymbol{e}_i, \tag{A.1}$$



where $\{v_1, v_2, v_3\}$ are the *components* of vector $\boldsymbol{v}$ in this basis.

For any given basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, one can obtain these components, e.g., by projecting vector $\boldsymbol{v}$ onto the base vectors, since $v_i = \boldsymbol{v} \cdot \boldsymbol{e}_i$ (for $i = 1, 2, 3$).

Although we often refer to a vector $\boldsymbol{v}$ by its components

$$[\boldsymbol{v}]_{\mathcal{C}} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \tag{A.2}$$

these three components are insufficient to describe the vector. Without knowing the basis of $\mathcal{C}$ being used here, the information $\{v_1, v_2, v_3\}$ is useless. It may reveal the magnitude of the vector as $\sqrt{v_1^2 + v_2^2 + v_3^2}$ but, without knowing the chosen basis, it does not imply a particular direction. Therefore, we differentiate between the *vector* $\boldsymbol{v}$ (see (A.1)) and its *components* $[\boldsymbol{v}]$ (see (A.2)) – the vector being generally understood and independent of a chosen basis, while its components make sense only within a particular reference frame.

Note that, if we know vector $\boldsymbol{v}$, we can compute its components in *any* Cartesian frame; e.g., in a frame $\mathcal{D}$ with basis $\{\overline{\boldsymbol{e}}_1, \overline{\boldsymbol{e}}_2, \overline{\boldsymbol{e}}_3\}$, since $[v_i]_{\mathcal{D}} = \boldsymbol{v} \cdot \overline{\boldsymbol{e}}_i$. Hence, $\boldsymbol{v}$ contains the complete information needed to find its components in any given basis (it contains much richer information than its components alone). If we know the bases of two different frames, say $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\{\overline{\boldsymbol{e}}_1, \overline{\boldsymbol{e}}_2, \overline{\boldsymbol{e}}_3\}$, we can also relate the vector components in these two frames via a rotation matrix (see Section 3.4.1).

A.2 What is a matrix?

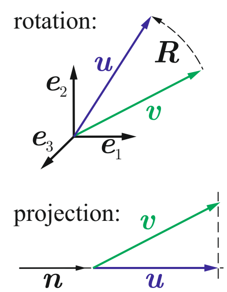
Next, we quickly introduce a *matrix* as a two-dimensional array of numerical values. For a matrix T having n rows and m columns, we write $T \in \mathbb{R}^{n\times m}$ and represent matrix T (e.g., for $n = m = 3$)

as

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}, \tag{A.3}$$

where T_{ij} denotes the component in the ith row and the jth column of the matrix. Therefore, matrix T is nothing but an ordered collection of $3 \times 3 = 9$ numerical values.

A.3 What is a tensor?



In many applications we need *linear mappings* from vectors onto vectors; simply put, a linear mapping is a function (or a black box) which takes a vector as an input and returns a vector as its output. For example, imagine projecting vectors $\boldsymbol{v}$ onto a certain direction $\boldsymbol{n}$ or into a plane. You provide a vector $\boldsymbol{v}$, and the projection finds a new vector $\boldsymbol{u}$ (which is the projection of $\boldsymbol{v}$ onto the given direction $\boldsymbol{n}$ or into the given plane). Another example are active rotations: for any given vector $\boldsymbol{v}$, we rotate $\boldsymbol{v}$ about some given axis by a given angle, resulting in a new vector $\boldsymbol{u}$ pointing in a different direction (see Section 3.4.1). Such linear mappings map vectors onto vectors.

The simplest linear mapping we know is the *identity* $\boldsymbol{I}$: it maps any vector $\boldsymbol{v}$ onto itself, i.e.,

$$\boldsymbol{I}\boldsymbol{v} = \boldsymbol{v}, \tag{A.4}$$

which we read as "$\boldsymbol{I}$ applied to $\boldsymbol{v}$ yields $\boldsymbol{v}$". For more complex mappings (such as the projections or rotations mentioned above), we introduce a linear mapping operator $\boldsymbol{T}$ – analogous to the above identity – such that

$$\boldsymbol{T}\boldsymbol{v} = \boldsymbol{u}, \tag{A.5}$$

meaning "$\boldsymbol{T}$ applied to $\boldsymbol{v}$ yields $\boldsymbol{u}$" (where $\boldsymbol{u} \neq \boldsymbol{v}$ in general).

If we now introduce our $\mathcal{C}$-frame with a specific Cartesian basis, we may write any linear mapping from $\boldsymbol{v}$ to $\boldsymbol{u}$ in 3D as

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \qquad \text{or} \qquad \sum_{j=1}^{3} T_{ij} v_j = u_i \quad \text{for } i = 1,2,3. \tag{A.6}$$

Because the mapping is *linear*, we know that we must be able to express it in the above form (the new vector $\boldsymbol{u}$ must be some linear combination of the components of the original vector $\boldsymbol{v}$). For example, for the identity mapping $\boldsymbol{I}$ we know its components

$$[I_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{so that} \qquad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{A.7}$$

or, for short, $u_i = \sum_{j=1}^{3} I_{ij} v_j = v_i$ for $i = 1,2,3$.

It is important to realize that eqns. (A.5) and (A.6) aim to give the same message – yet they contain different information. Analogous to our discussion of vectors above, (A.5) is a general statement: vector $\boldsymbol{v}$ is mapped onto vector $\boldsymbol{u}$, irrespective of whatever reference frame one wants to use. By contrast, (A.6) only applies in a particular chosen reference frame $\mathcal{C}$, in which $(v_1, v_2, v_3) = [\boldsymbol{v}]_\mathcal{C}$ and $(u_1, u_2, u_3) = [\boldsymbol{u}]_\mathcal{C}$ are the components of vectors $\boldsymbol{v}$ and $\boldsymbol{u}$, respectively. Analogously, we interpret the matrix with components T_{ij} in (A.6) as the *components* of a *tensor* $\boldsymbol{T}$ in the chosen $\mathcal{C}$-frame, so that we may write

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = [\boldsymbol{T}]_\mathcal{C}. \tag{A.8}$$

If we wanted to use a different frame with a different basis $\mathcal{D}$ to calculate $\boldsymbol{u}$, then the components of vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ would change, which implies that also the components of $\boldsymbol{T}$ would need to change. Just like $[\boldsymbol{v}]_\mathcal{C}$ is valid only in the $\mathcal{C}$-frame, $[\boldsymbol{T}]_\mathcal{C}$ are the components of $\boldsymbol{T}$ only the $\mathcal{C}$-frame, which would need to change when going to a different frame. Most importantly, the mapping in (A.6) only makes sense if the components of $\boldsymbol{T}$, $\boldsymbol{v}$ and $\boldsymbol{u}$ are all with respect to the same basis.

To understand the intricacies of a tensor, let us go one step futher and introduce the following vector operation. We denote by $\otimes$ a *tensor product* (also known as *outer product* or *dyadic product*), which can operate between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and is defined by the relation

$$\boxed{(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{v} = (\boldsymbol{b} \cdot \boldsymbol{v})\boldsymbol{a}} \tag{A.9}$$

In other words, $\boldsymbol{a} \otimes \boldsymbol{b}$ acting on vector $\boldsymbol{v}$ results in a new vector which points in the direction of $\boldsymbol{a}$ and has the magnitude $(\boldsymbol{b} \cdot \boldsymbol{v})|\boldsymbol{a}|$. This defines a linear mapping from vectors onto vectors: for a given vector $\boldsymbol{v}$, the object $\boldsymbol{a} \otimes \boldsymbol{b}$ turns it into a new vector $(\boldsymbol{b} \cdot \boldsymbol{v})\boldsymbol{a}$ (which is linear in $\boldsymbol{v}$). Therefore, $\boldsymbol{a} \otimes \boldsymbol{b}$ must be a *tensor*. As an example, consider $\boldsymbol{n} \otimes \boldsymbol{n}$ for some unit vector $\boldsymbol{n}$. When applied to any vector $\boldsymbol{v}$, this leads to

$$(\boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{v} = (\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n} \qquad (\text{ with } |\boldsymbol{n}| = 1), \tag{A.10}$$

which is nothing but the projection of vector $\boldsymbol{v}$ onto the $\boldsymbol{n}$-direction (this defines the *projection tensor* $\boldsymbol{T} = \boldsymbol{n} \otimes \boldsymbol{n}$).

To understand what $\boldsymbol{a} \otimes \boldsymbol{b}$ means, let us write the right-hand side of (A.9), using $\boldsymbol{b} \cdot \boldsymbol{v} = b_1 v_1 + b_2 v_2 + b_3 v_3$, in some Cartesian frame as

$$[(\boldsymbol{b} \cdot \boldsymbol{v})\boldsymbol{a}] = \begin{pmatrix} (b_1v_1 + b_2v_2 + b_3v_3)a_1 \\ (b_1v_1 + b_2v_2 + b_3v_3)a_2 \\ (b_1v_1 + b_2v_2 + b_3v_3)a_3 \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{A.11}$$

But since (A.9) tells us $(\boldsymbol{b} \cdot \boldsymbol{v})\boldsymbol{a} = (\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{v}$ and hence

$$[(\boldsymbol{b} \cdot \boldsymbol{v})\boldsymbol{a}] = [(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{v}] = [\boldsymbol{a} \otimes \boldsymbol{b}][\boldsymbol{v}], \tag{A.12}$$

we can identify the components of tensor $\boldsymbol{a} \otimes \boldsymbol{b}$ from (A.11) as

$$[\boldsymbol{a} \otimes \boldsymbol{b}] = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}. \tag{A.13}$$

Therefore, the components of $[\boldsymbol{a} \otimes \boldsymbol{b}]$ form a 3×3-matrix with $a_i b_j$ being the component in the $i$the row and jth column. For example, we can evaluate the following outer products (using the Cartesian frame with basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ so $[\boldsymbol{e}_1] = (1, 0, 0)$, $[\boldsymbol{e}_2] = (0, 1, 0)$, etc.), e.g.,

$$[\boldsymbol{e}_1 \otimes \boldsymbol{e}_1] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad [\boldsymbol{e}_1 \otimes \boldsymbol{e}_2] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad [\boldsymbol{e}_2 \otimes \boldsymbol{e}_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ etc.} \tag{A.14}$$

Note that this allows us to write for any matrix

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = T_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + T_{12} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \ldots = \sum_{i,j=1}^{3} T_{ij} [\boldsymbol{e}_i \otimes \boldsymbol{e}_j], \tag{A.15}$$

which motivates the definition of a tensor $\boldsymbol{T}$, using (A.9), as

$$\boxed{\boldsymbol{T} = \sum_{i,j=1}^{3} T_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j} \tag{A.16}$$

We can also verify the action of this tensor $\boldsymbol{T}$ onto a vector $\boldsymbol{v} = \sum_{k=1}^{3} v_k \boldsymbol{e}_k$ as

$$\boldsymbol{T}\boldsymbol{v} = \left(\sum_{i,j=1}^{3} T_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \right) \sum_{k=1}^{3} v_k \boldsymbol{e}_k = \sum_{i,j,k=1}^{3} T_{ij} v_k (\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \boldsymbol{e}_k = \sum_{i,j,k=1}^{3} T_{ij} v_k \boldsymbol{e}_i (\boldsymbol{e}_j \cdot \boldsymbol{e}_k). \tag{A.17}$$

Realizing that for orthonormal basis vectors we must have

$$\boldsymbol{e}_j \cdot \boldsymbol{e}_k = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{else,} \end{cases} \tag{A.18}$$

(A.17) simplifies correctly to

$$\boldsymbol{T}\boldsymbol{v} = \sum_{i,j,k=1}^{3} T_{ij} v_k \boldsymbol{e}_i (\boldsymbol{e}_j \cdot \boldsymbol{e}_k) = \sum_{i,j=1}^{3} T_{ij} v_j \boldsymbol{e}_i, \tag{A.19}$$

so that $\boldsymbol{T}\boldsymbol{v}$ is indeed a vector, whose components are given by $[\boldsymbol{T}\boldsymbol{v}] = T_{ij} v_j$.

A.4 Brief summary

While a vector is defined as

$$\boldsymbol{v} = \sum_{i=1}^{3} v_i \boldsymbol{e}_i \qquad \text{with components} \qquad [\boldsymbol{v}] = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \tag{A.20}$$

a tensor is defined as

$$\boldsymbol{T} = \sum_{i,j=1}^{3} T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad \text{with components} \qquad [\boldsymbol{T}] = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}. \tag{A.21}$$

Both $\boldsymbol{v}$ and $\boldsymbol{T}$ are independent of the chosen basis, while their components $v_i = [\boldsymbol{v}]_\mathcal{C}$ and $T_{ij} = [\boldsymbol{T}]_\mathcal{C}$ do depend on the chosen basis $\mathcal{C}$. If we want to switch between different bases, then we may use *transformation rules* to transform the components of either a vector or a tensor from one basis to another. (For the transformation of vector components, see the discussin of passive rorations in Section 3.4.1; for transformations of tensor components, we refer to the discussion in Mechanics 2 around Mohr's Circle).

Technically speaking, a tensor $\boldsymbol{T}$ provides a mapping from vectors $\boldsymbol{v}$ onto vectors $\boldsymbol{T}\boldsymbol{v}$. An example from mechanics is the stress tensor $\boldsymbol{\sigma}$: for any given surface normal $\boldsymbol{n}$, it provides the stress (or traction) vector on that surface as $\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$. Another example is the (mass) moment of inertia tensor $\boldsymbol{I}_\mathrm{B}$: for any angular velocity vector $\boldsymbol{\omega}$, it provides the resulting angular momentum vector $\boldsymbol{H}_\mathrm{B} = \boldsymbol{I}_\mathrm{B}\boldsymbol{\omega}$ (assuming that B is the center of mass or a stationary point).

Finally, what we called a "*tensor*" here is, strictly speaking, a *second-order tensor*, as the definition in (A.21) involves two basis vectors ($\boldsymbol{e}_i \otimes \boldsymbol{e}_j$). A vector, as defined in (A.20), is also referred to as a *first-order tensor* (a single basis vector is required), while a scalar can be viewed as a *zeroth-order tensor*; and one can also defined higher-order tensors, e.g., a third-order tensor as $\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c}$ or the fourth-order stiffness tensor $\mathbb{C}$ (which is needed in Continuum Mechanics).

Index

Glossary

acceleration "*Beschleunigung*": the acceleration is the rate of change of the velocity of any particle or point. 'Acceleration' generally refers to the acceleration vector $\boldsymbol{a}$, but it is also sometimes used for its magnitude $a = |\boldsymbol{a}|$. 4

amplitude "*Amplitude*": the peak-to-peak distance (i.e., the difference between peak (highest value) and trough (lowest value) of a harmonically varying quantity, e.g., the DOF of a vibration. 160, 167

angular acceleration "*Winkelbeschleunigung*": the angular acceleration $\dot{\omega} = \ddot{\varphi}$ (written in 2D) is the rate of change of the angular velocity per time. 10

angular momentum "*Drehimpuls*": the angular momentum vector is the rotational equivalent of the linear momentum vector. 30, 80, 81

angular momentum balance "*Drehimpulsbilanzgleichung*": the balance of angular momentum states that any angular momentum applied to a system through external torques is converted, without losses, into a change of the angular momentum of the system. 30, 52

angular velocity "*Winkelgeschwindigkeit*": the angular velocity $\omega = \dot{\varphi}$ (written in 2D) is the rate of angle change per time. 10, 66, 117

apparent forces "*Trägheitskräfte*": forces observed by a moving observer (e.g., on a rotating system) due to the motion of the reference frame. 122, 229

area moment of inertia "*Flächenträgheitsmoment*": a geometrical property of the cross-sectional area of a body, which characterizes how its shape is distributed with regard to an arbitrary axis (needed in the analysis of **bending**). 85, 87

bending "*Biegung*": flexural deformation of a slender body (especially of beams or plates). 226, 229

binormal vector "*Binormalvektor*": unit vector perpendicular to both the velocity vector and the principal normal vector. 13

body "*Körper*": a body is an object of finite size and having a specific shape (unlike a particle which is an infinitesimally small point mass). 3, 64

body force "*Volumenkraft*": a distributed force that acts throughout the volume of a body. 199

center of mass "*Massenmittelpunkt, Massenschwerpunkt*": the average position of all particles of a system, weighted according to their masses; for a rigid body with uniform density, the center of mass is the centroid. 64

center of percussion "*Stosszentrum, Aufschlagmittelpunkt*": the center of percussion of a rigid body attached to a hinge is that point for which a perpendicular impact onto the body will produce no reactive forces at the pivot/hinge (note that this is our use of the definition, while there are others out there). 109, 110

centrifugal force "*Zentrifugalkraft*": apparent/fictitious force of a body due to a rotating reference frame, pushing the body outward from the axis of rotation. 121

centripetal "*zentripetal*": pointing radially away from the center (of a rotation). 11

centripetal acceleration "*Zentripetalbeschleunigung*": acceleration of a body due to a rotating reference frame, accelerating the body inward towards the axis of rotation. 13, 121

centrode "*Polbahn*": locus of all ICRs of a moving body. 68

characteristic equation "*charakteristische Gleichung*": equation obtained from equating the characteristic polynomial of a matrix to zero. 159

characteristics "*Charakteristik*": for a PDE such as the wave equation, a characteristic is a a line $x = x(t)$ along which the field $u(x, t) = \text{const.}$. 206

coefficient of restitution "*Stossziffer, Stosszahl*": the ratio of impulsive force magnitudes during restitution and compression periods, or the negatvie ratio of the normal relative velocity components of impacting particles after and before the collision. 39, 106

component form "*Komponentenform, Komponentenschreibweise*": we may express any vector (or tensor) $\boldsymbol{v}$ in a given coordinate frame $\mathcal{C}$ through its components $[\boldsymbol{v}]_{\mathcal{C}}$ measured in that frame. 4

conservation of energy "*Energieerhaltung*": conservative systems maintain a constant total energy (sum of kinetic and potential energy) over time. 23, 101

conservation of linear momentum "*Impulserhaltung*": the total linear momentum of a particle or body remains constant if no external forces are acting. 80

conservative "*konservativ*": a force is conservative if it derives from a potential; a system is conservative if all forces are conservative. 101, 154

conservative force "*konservative/eingeprägte Kraft*": a force that derives from an energy potential and which along any closed path does not perform any total work. 23, 50

conservative system "*konservatives System*": a system that maintains a constant total energy (i.e., the sum of kinetic and potential energy remains constant). 22, 23, 50

constitutive law "*Konstitutivgesetz*": the material-dependent relation between stresses and strains. 200

constrained "*eingeschränkt, beschränkt*": if a particle or body cannot move freely but is forced to move in a particular fashion, its motion is *constrained*. 7

constraint force see **reaction force**. 7, 46

Coriolis acceleration "*Coriolisbeschleunigung*": acceleration of a body due to a rotating reference frame, stemming from a relative velocity with respect to the moving frame. 121

Coriolis force "*Corioliskraft*": apparent/fictitious force of a body due to a rotating reference frame, stemming from a relative velocity with respect to the moving frame. 121

damping matrix "*Dämpfungsmatrix*": the matrix multiplying $\dot{\boldsymbol{x}}$ in the linearized equations of motion, associated with viscous damping. 175

deformable "*deformierbar*": the ability of a body to change its shape (the opposite of rigid). 193

degrees of freedom "*Freiheitsgrade*": a body or particle has d degrees of freedom (DOFs), if its configuration in space can uniquely be described by d independent state variables (e.g., the position vector $\boldsymbol{x} = \{x_1, \ldots, x_d\}$ of a particle moving through space, or the position vector $\boldsymbol{x} = \{x_1, \ldots, x_d\}$ and its current rotation $\boldsymbol{\theta}$. 4

deviatoric moment "*deviatorisches Flächenträgheitsmoment*": an off-diagonal component of the area moment of inertia tensor. 88

displacement "*Verschiebung*": the vector pointing from the undeformed position of a point to its deformed position. 200

eccentric collision "*exzentrischer Stoss*": a collision whose impulsive forces' line of action does not go through the centers of mass of the participating bodies. 105

eigenfrequency "*Eigenfrequenz*": a frequency at which a body or system tends to oscillate without any applied forcing or damping. 160, 179, 207, 229, 231

eigenmode see **mode shape** . 179

eigenvalue problem "*Eigenwertproblem*": a problem of the type $\boldsymbol{K}\boldsymbol{x} = \lambda\boldsymbol{x}$ to be solved for eigenvectors ("*Eigenvektoren*") $\boldsymbol{x}$ and eigenvalues "*Eigenwertwerte*" λ. 178

equation of motion "*Bewegungsgleichung*": a differential equation in time which describes the mechanical behavior of a system in terms of its time-dependent motion. Provided initial conditions, the equation of motion is solved to determine the motion of the system. 33

equilibrium "*Gleichgewicht, Gleichgewichtslage*": a configuration of a system in which the resultant forces and torques vanish. 1, 154, 155

Euler acceleration "*Eulerbeschleunigung (Azimutalbeschleunigung)*": acceleration of a body due to a rotating reference frame, stemming from a rate of change of the angular velocity of the moving frame. 121

Euler angles "*Eulersche Winkel*": introduced by Leonhard Euler, these three angles describe the orientation of a rigid body with respect to a fixed 3D reference frame (used, e.g., for spinning tops whose motion is described uniquely by the three angles). 141

Euler force "*Eulerkraft*": apparent/fictitious force of a body due to a rotating reference frame, stemming from a rate of change of the angular velocity of the moving frame. 121

Euler's equations "*Eulersche (Kreisel-)Gleichungen*": a vectorial first-order system of ODEs describing the rotation of a rigid body in a rotating reference frame $\hat{\mathcal{M}}$ (which rotates with the body and whose basis aligns with the body's principal axes of inertia. 133

excitation frequency "*Anregerfrequenz, Erregungsfrequenz*": the frequency at which a forced vibration is induced through an externally applied force or torque. 165

external forces "*äussere Kräfte*": forces applied externally to a system (i.e., not forces between particles or bodies in systems, or forces within deformable bodies). 48

fictitious forces "*Trägheitskräfte*": see **apparent forces**. 122

flexural "*Biege-*": having to do with **bending** (e.g., flexural deformation implies bending deformation). 194

forced vibration "*erzwungene Schwingung*": a vibration witho an applied periodic forcing (applied force, displacement, torque, etc.). 158, 165, 184

free vibration "*freie Schwingung*": a vibration without any applied forcing. 158, 159, 183

fundamental frequency (lowest) **eigenfrequency** of a system. 160, 207

generalized degrees of freedom "*generalisierte Koordinaten*": independent DOFs describing the motion of a system uniquely (considering any imposed constraints). 152

generalized force "*generalisierte Kraft*": a force that is conjugate to a generalized DOF. 153

generalized velocity "*generalisierte Geschwindigkeit*": time derivative of a generalized degree of freedom. 153

gyroscope "*Gyroskop, Kreiselinstrument, Kreiselstabilisator*": a fast spinning axisymmetric wheel, whose axis of rotation is free to assume any orientation in space. 146

gyroscopic limit "*gyroskopisches System*": the limit $|\dot{\psi}| \gg |\dot{\varphi}$, i.e., the special case of a *fast-spinning* top which behaves like gyrosope. 145

harmonic see **mode shape**. 208

Hessian "*Hesse-Matrix*": the Hessian matrix (or simply *Hessian*) is a square matrix containing all second-order partial derivatives of a scalar function. 155

holonomic constraint "*holonome Nebenbedingung*": a holonomic constraint is a kinematic constraint which can be defined as $f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, t) = 0$, i.e., which depends only on the degrees of freedom and time (not their derivatives). 45, 152

Hooke's law "*Hooke'sches Gesetz*": linear relation between stresses and strains (e.g., $\sigma = E\varepsilon$ in 1D, $\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}$ in higher dimensions), cf. **linear elasticity**. 200, 230

impulsive force "*Kraftstoss*": the change of momentum due to a collision; in other words, a force (integrated over time) which acts on an object for a very short time interval during a collision. 38

inertial frame "*Inertialsystem*": an inertial frame is an absolute frame of reference whose basis and origin are independent of time. Only in inertial frames do Newton's axioms hold. 4, 15, 232

inertial reference frame "*Inertialsystem*": a frame of reference that is fixed and not moving (so that Newton's axioms apply). 111

instantaneous axis of rotation "*Momentanachse*": the axis around which each point on a rigid body undergoes a pure rotation at a given instance of time. 71

instantaneous center of rotation "*Momentanpol*": the point with respect to which each point on a rigid body undergoes a pure rotation at a given instance of time in 2D. 67

intermediate axis theorem "*Dschanibekow-Effekt*": a theorem describing the motion and tumbling of a rigid body with three distinct principal moments of inertia. 148, 235

internal forces "*innere Kräfte*": forces not applied externally to a system; e.g., forces between particles or bodies in systems of particles/rigid bodies, or forces within deformable bodies. 48

kinematic constraint "*kinematische Nebenbedingung*": a kinematic constraint is a mathematical relation that restricts the degrees of freedom of a dynamic system (e.g., a rigid ground imposes the kinematic constraint that a particle or body cannot penetrate the ground, or two particles connected by a rigid bar cannot change their distance. 7, 45

kinematics "*Kinematik*": the concepts of describing the relation between the geometric state of a system (i.e., its motion and possibly deformation), on the one hand, and the applied forces and torques that cause the motion (and deformation), on the other hand. 1, 3

kinetic energy "*kinetische Energie*": the energy stored in a particle of body at a given time due to its motion. 20, 101

kinetic friction coefficient "*kinetischer (oder dynamischer) Reibkoeffizient*": the kinetic friction coefficient is analogous to the static friction coefficient and characterizes the ratio between friction and normal force magnitudes ($\mu = |\boldsymbol{R}|/\|\boldsymbol{N}|$) during dynamic frictional sliding. 17, 18, 234

kinetics "*Kinetik*": the concepts of describing the geometric state of a system, i.e., its motion (and deformation, in general). 1, 3

linear elastic "*linear elastisch*": a material behavior that is characterized by a linear relation between the (infinitesimal) stress and strain components. 193

linear elasticity "*lineare Elastizitt*": material behavior characterized by a linear relation between stresses and strains (cf. **Hooke's law**). 200, 229

linear momentum "*(linearer) Impuls*": linear momentum of a body or particle is the product of its mass m and the velocity of its center of mass, i.e., $\boldsymbol{P} = m\boldsymbol{v}_{\text{CM}}$. Please do not confuse "*momentum*" with the German "*Moment*" (= "*torque*" in our terminology). 15, 79

linear momentum balance "*Impulsbilanzgleichung*": the balance of linear momentum states that any linear momentum applied to a system through external forces is converted, without losses, into a change of the linear momentum of the system. 16, 49, 80

longitudinal "*Längs-*": running lengthwise/along the axis; having to do with axial deformation. 195

longitudinal wave speed "*Longitudinalwellengeschwindigkeit*": speed at which longitudinal (axial) waves propagate in a 1D linear elastic rod. 206

mass accretion "*Massenzunahme*": mass accretion describes the continuous acquisition of additional mass of a body over time (e.g., a droplet growing while absorbing more fluid from damp air, or a snow ball rolling down a snowy ground). 59

mass density "*Massendichte*": mass per volume of a continuous body. 64

mass loss "*Massenverlust*": mass loss describes the continuous reduction of a body's mass over time (e.g., a rocket losing mass in the form of exhaust gas). 60

mass matrix "*Massenmatrix*": the matrix multiplying $\ddot{\boldsymbol{x}}$ in the linearized equations of motion, associated with kinetics/inertia. 175

metastable "*metastabil*": metastable equilibria corresponds to saddle points in the potential energy landscape. 155

mode shape "*Eigenmode, Normalmode*": the eigenvector associated with an eigenfrequency of a linearized dynamical system. 179, 208, 228, 229, 232

moment of inertia "*Massenträgheitsmoment*": the moment of inertia characterizes a particle's or body's inertia against rotational acceleration (analogous to the mass characterizing inertia against linear acceleration. 31

moment of inertia tensor "*Massenträgheitsmomententensor*": this tensor characterizes a rigid body's moment of inertia around all possible axes of rotation. 81, 85

natural frequency (lowest) **eigenfrequency** of a system. 160, 167, 207

nodes "*Knoten*": in the context of standing waves, nodes are points of constant zero motion (i.e., points at which the mode shape vanishes). 208

non-conservative force "*nicht-konservative Kraft*": a force whose work, when applied to a moving particle, depends on the particle's path and cannot be derived from a position-dependent potential. 24

non-conservative system "*Arbeitssatz*": a system that reduces its total energy (i.e., the sum of kinetic and potential energy) over time due to the action of non-conservative forces such as friction or viscous drag. 21

non-inertial frame "*Nicht-Inertialsystem*": a non-inertial frame is a moving frame of reference whose basis and/or origin depend on time; see also **inertial frame**. 8, 11, 111

normal mode see **mode shape**. 208

nutation "*Neigung, Nutation*": the tilt of a body away from the normal to the ground; varying nutation results in a rocking or swaying motion of the axis of rotation of the body. 141, 142

orthonormal basis "*orthonormale Basis*": a set of vectors $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is orthonormal, if each vector is of unit length (i.e., $|\boldsymbol{e}_i| = 1$) and if each pair of vectors is orthogonal (i.e., $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$, or $\boldsymbol{e}_i \perp \boldsymbol{e}_j$ for $i \neq j$). 3, 4

osculating circle "*Schmiegekreis*": the circle which best approximates a space curve at a given point (goes through the point and has the same curvature). 13

parallel axis theorem "*Steinerscher Satz, Satz von Steiner*": relates moment of inertia tensors with respect to two different reference points to each other. 89, 235

particle "*Massenpunkt, Punktmasse*": a body whose extensions are negligibly small compared to its path through space can be assumed to be a *particle*; particles only have translational degrees of freedom and no rotational degrees of freedom. 3

path length "*Pfadlänge*": the distance travelled by a particle since some initial reference time. 11

period "*Periode*": the time of a complete vibration/oscillation cycle. 160

period of compression "*Kompressionsphase*": the (initial) period of time $[t_-, t_0)$ during a collision, within which the two colliding objects move towards each other in the normal direction. 38, 56

period of restitution "*Restitutionsphase*": the (later) period of time $[t_0, t_+]$ during a collision, within which the two colliding objects move away from each other in the normal direction. 38, 56

phase delay "*Phasenverschiebung*": the time delay between an applied forcing and the resultant response of the system. 167

planetary gear "*Planetengetriebe*": also known as *epicyclic gear*, a gear system consisting of one or more outer *planet gears* revolving around a central *sun gear*. 94

polar coordinates "*Polarkoordinaten*": a 2D coordinate system in which a point on a plane is determined by a distance r from an origin and an angle φ (or θ) from a reference direction. 8

polar moment of area "*polares Flächenträgheitsmoment*": a geometrical property of the cross-sectional area of a body, which characterizes how its shape is distributed with regard to its central axis (needed in the analysis of torsion). 87

position "*Position, Lage*": a position in space is generally defined with respect to a basis and an origin, e.g., $\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$ in a Cartesian frame of reference. 3, 4

potential energy "*potentielle Energie*": the position-dependent energy from which a (sum of) conservative force(s) derives as $-\mathrm{d}V/\mathrm{d}\boldsymbol{r}$. 22

precession "*Präzession*": the change in the orientation of the rotational axis of a rotating body (such as a spinning top). 141

principal axes "*Hauptachsen*": the coordinate axes within which the components a second-order tensor (e.g., the moment of inertia, stress or strain tensor) are diagonal (i.e., all off-diagonal components vanish). The principal axes are defined by the eigenvectors of the tensor (the principal values, i.e., the corresponding eigenvalues are the tensor's diagonal entries in that frame of reference). 88

principal normal vector "*Hauptnormalenvektor*": unit vector perpendicular to the velocity vector and pointing towards the center of curvature. 12

reaction force "*Reaktionskraft*": a force that arises in response to a kinematic constraint; e.g., if the trajectory of a particle is guided by a rigid ground, slope or slide, then the ground exerts a reaction force onto the particle which is normal to the ground. Similarly, a particle attached to a rigid stick or string experiences a reaction force along the direction of the stick or string. 7, 227

resonance "*Resonanz*": excitation of a system at an eigenfrequency, causing a large-amplitude system response. 167, 189

resultant force "*Resultierende*": the sum of all forces acting on a particle or body (i.e., the net force). 15

resultant torque "*resultierendes Moment*": the sum of all torques acting on a particle or body with respect to a given point (i.e., the net torque). 30, 52

rigid "*starr, unverformbar*": a rigid body or link is not deformable, i.e., it maintains its shape and size under arbitrary deformation (e.g., a *rigid link* is a connecting bar/beam that is infinitely stiff and hence cannot extend or shrink. 3, 7, 45, 50

rigid body "*Starrörper*": a rigid body is not deformable (or infinitely stiff). 64

rigid system "*starres System*": in a rigid system of particles, the relative distances between each pair of particles remains constant at all times. 45, 51

rigid-body motion "*Starrkörperbewegung*": motion of a system (or body) without changing the relative distances between any of its particles (or material points), hence producing no change in potential energy. 179

second-order tensor "*Tensor zweiter Stufe*": the result of a dyadic/outer product of two vectors (or a linear mapping of vectors onto vectors), defined by a matrix of components *and* and associated basis. 85

shear "*Schub*": loosely speaking, deformation associated with angle changes; characterized by the off-diagonal components of the stress and strain tensors. 199

shear wave speed "*Scherwellengeschwindigkeit*": speed at which shear waves propagate in a linear elastic medium. 211, 235

slider crank "*Schubkurbel(getriebe)*": arrangement of hinged mechanical parts designed to convert straight-line motion into circular motion. 68

sliding friction "*Gleitreibung*": sliding friction refers to the process of a two bodies sliding past in which, whose contact is charaterized by both normal forces and friction forces (the latter opposing the motion). The ratio between friction and normal forces is the **kinetic friction coefficient** μ. 18

space curve "*Raumkurve*": the trajectory of a particle through space. 11

special orthogonal group "*Inertialsystem*": group of all second-order tensors satisfying $\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}$ (to be mathematically exact, this is the group of all distance-preserving isometries, where the group operation is given by composing transformations). 114

speed "*Schnelligkeit, Geschwindigkeit*": the speed v refers to the magnitude of the velocity vector $\boldsymbol{v}$, i.e., $v = |\mathbf{v}|$. 4

spin "*Spin*": rotation of a body about one of its principal axes, leaving the principal axis stationary (often but not necessarily of rotational symmetry). 141, 142

spin tensor "*Spintensor*": a second-order tensor mapping any vector onto the cross product of the angular velocity vector and that vector. 117

spinning top "*Kreisel*": a (often but not necessarily rotational symmetric) body that is spun quickly around a fixed point (inertia causes it to remain precisely balanced on its tip in the presence of gravity). 111, 135, 141

stable "*stabil*": loosely speaking, an equilibrium is stable if a small perturbation of the system does not deviate far from the equilibrium; mathematically speaking, a stable equilibrium of a conservative system is a potential energy minimum. 155

standing wave "*stehende Welle, Stehwelle*": also known as a *stationary wave*, a standing wave is a wave which oscillates in time but whose peak amplitude profile does not move in space. 206, 213

state space representation "*Zustandsraumdarstellung*": description of a physical system as a first-order ODE with state variables described as functions of time. 158

static "*statisch*": a problem is (quasi)static if inertial effects are negligible, so the kinetic energy can be neglected. 154

Steiner's theorem "*Steinerscher Satz, Satz von Steiner*": see **parallel axis theorem**. 89

stiffness matrix "*Steifigkeitsmatrix*": the matrix multiplying $\boldsymbol{x}$ in the linearized equations of motion, associated with linear spring connections. 175

strain "*Dehnung*": deformation per original length. 85, 200

stress "*Spannung*": force per original area. 85

tennis racket theorem see **intermediate axis theorem**. 148

thrust "*Schubkraft, Vorschub*": the thrust $\dot{m}\boldsymbol{v}_m$ is the propulsive force of a rocket, produced by gas leaving the rocket at a mass stream $\dot{m}$ and relative velocity $\boldsymbol{v}_m$. 61

time-invariant "*zeitlich invariant*": not changing with time. 4

torque "*Moment, Drehmoment*": the product of force times lever arm with respect a given point B (note that in American English the term '*moment*' is usually preferred over '*torque*'). 30

torsion "*Torsion*": the twisting of a deformable body (e.g., a shaft) due to applied torques. 235

torsional "*Torsions-*": having to do with **torsion**. 195

torsional wave speed see **shear wave speed**). 211

unstable "*instabil*": loosely speaking, an equilibrium is unstable if a small perturbation of the system leads to a response that deviates far from the equilibrium; mathematically speaking, a stable equilibrium of a conservative system is a potential energy maximum. 155

velocity "*Geschwindigkeit*": the velocity is the rate of change of the position of any particle or point. 'Velocity' generally refers to the velocity vector $\mathbf{v}$, whereas *speed* ("*Schnelligkeit*") refers to the magnitude $v = |\mathbf{v}|$ of the velocity vector. 4

velocity transfer formula expression relating the velocities of two points on the same rigid body by the distance vector and the angular velocity vector. 66

vibration "*Vibration, Schwingung*": a small-amplitude oscillation around a stable equilibrium. 158

wave length "*Wellenlänge*": the spatial period of a periodic wave (i.e., the spatial distance over which the wave's shape repeats). 208

work "*Arbeit*": amount of energy transferred to a system by moving it through the action of applied forces. 20

work–energy balance "*Arbeitssatz*": the change in kinetic energy between two states equals the work performed on the particle or body by external forces. 20, 101