ECE 301: Signals and Systems
Homework Assignment #4

Due on October 28, 2015

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Problem 1

Let $x[n]$ be a real periodic signal with period N and Fourier coefficients a_k.

(a) Show that if N is even, at least two of the Fourier coefficients within one period of a_k are real.

(b) Show that if N is odd, at least one of the Fourier coefficients within one period of a_k is real.

Solution

We have

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-j(2\pi/N)kn}$$

Noe that

$$a_0 = \frac{1}{N} \sum_{n=<N>} x[n]$$

which is real if $x[n]$ is real.

(a) If N is even, then

$$a_{N/2} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-j\pi n} = \frac{1}{N} \sum_{n=<N>} x[n](-1)^n$$

Clearly, $a_{N/2}$ is also if $x[n]$ is real.

(b) If N is odd, only a_0 is guaranteed to be real.

Problem 2

Consider the function

$$a[k] = \sum_{n=0}^{N-1} e^{j(2\pi/N)kn}$$

(a) Show that $a[k] = N$ for $k = 0, \pm N, \pm 2N, \pm 3N$, ...

(b) Show that $a[k] = 0$ whenever k is not an integer multiple of N. (*Hint*: Use the finite sum formula.)

(c) Repeat parts (a) and (b) if

$$a[k] = \sum_{n=<N>} e^{j(2\pi/N)kn}$$

where $n =< N >$ means any consecutive N integer numbers.

Solution

(a) Let $k = pN$, $p \in \mathbb{Z}$. Then,

$$a[pN] = \sum_{n=0}^{N-1} e^{j(2\pi/N)pNn} = \sum_{n=0}^{N-1} e^{j2\pi pn} = \sum_{n=0}^{N-1} 1 = N.$$

(b) Using the finite sum formula, we have

$$a[k] = \frac{1 - e^{j2\pi k}}{1 - e^{j(2\pi/N)k}} = 0, \text{ if } k \neq pN, p \in \mathbb{Z}.$$

(c)

$$a[k] = \sum_{n=q}^{q+N-1} e^{j(2\pi/N)kn}$$

where q is some arbitrary integer. By putting $k = pN$, we may again easily show that

$$a[pN] = \sum_{n=q}^{q+N-1} e^{j(2\pi/N)pNn} = \sum_{n=q}^{q+N-1} e^{j2\pi pn} = \sum_{n=q}^{q+N-1} 1 = N.$$

Now,

$$a[k] = e^{j(2\pi/N)kq} \sum_{n=0}^{N-1} e^{j(2\pi/N)kn}.$$

Using part (b), we may argue that

$$a[k] = e^{j(2\pi/N)kq} \frac{1 - e^{j2\pi k}}{1 - e^{j(2\pi/N)k}} = 0, \text{ if } k \neq pN, p \in \mathbb{Z}.$$

Problem 3

Let $x[n]$ be a periodic signal with fundamental period N and Fourier series coefficients a_k. In this problem, we derive the time-scaling property

$$x_{(m)}[n] = \begin{cases} x[\frac{n}{m}], & n = 0, \pm m, \pm 2m, ... \\ 0, & \text{elsewhere} \end{cases}$$

in the textbook.

(a) Show that $x_{(m)}[n]$ has period of mN.

(b) Show that if

$$x[n] = v[n] + w[n]$$

then

$$x_{(m)}[n] = v_{(m)}[n] + w_{(m)}[n]$$

(c) Assuming that $x[n] = e^{j2\pi k_0 n/N}$ for some integer k_0, verify that

$$x_{(m)}[n] = \frac{1}{m}\sum_{l=0}^{m-1} e^{j2\pi(k_0+lN)n/mN}$$

Noe that here you may use the results from the Problem 2.

(d) Using the results of parts (a), (b), (c), show that if $x[n]$ has the Fourier coefficients a_k, then $x_{(m)}[n]$ must have the Fourier coefficients $\frac{1}{m}a_k$.

Solution

(a) Note that

$$x_{(m)}[n+mN] = \begin{cases} x[\frac{n}{m}+N], & n = 0, \pm m, \pm 2m, ... \\ 0, & \text{elsewhere} \end{cases} = \begin{cases} x[\frac{n}{m}], & n = 0, \pm m, \pm 2m, ... \\ 0, & \text{elsewhere} \end{cases} = x_{(m)}[n]$$

Therefore, $x_{(m)}[n]$ is periodic with period mN.

(b) The time-scaling operation discussed in this problem is a linear operation. Therefore, if

$$x[n] = v[n] + w[n],$$

then

$$x_{(m)}[n] = \begin{cases} x[\frac{n}{m}], & n = 0, \pm m, \pm 2m, ... \\ 0, & \text{elsewhere} \end{cases} = \begin{cases} v[\frac{n}{m}] + w[\frac{n}{m}], & n = 0, \pm m, \pm 2m, ... \\ 0, & \text{elsewhere} \end{cases} = v_{(m)}[n]+w_{(m)}[n].$$

(c) Let us consider

$$y[n] = \frac{1}{m}\sum_{l=0}^{m-1} e^{j(2\pi/mN)(k_0+lN)n} = \frac{1}{m}e^{j(2\pi/mN)k_0 n}\sum_{l=0}^{m-1} e^{j(2\pi/m)ln}.$$

This may be written as

$$y[n] = \begin{cases} e^{j(2\pi/mN)k_0 n}, & n = 0, \pm N, \pm 2N, ... \\ 0, & \text{elsewhere} \end{cases}$$

Now, also note that by applying time-scaling on $x[n]$, we obtain

$$x_{(m)}[n] = \begin{cases} e^{j(2\pi/mN)k_0 n}, & n = 0, \pm N, \pm 2N, ... \\ 0, & \text{elsewhere} \end{cases}$$

Therefore, we obtain that $y[n] = x_{(m)}[n]$.

(d) We have

$$b_k = \frac{1}{mN} \sum_{n=0}^{mN-1} x_{(m)}[n] e^{-j(2\pi/mN)kn}.$$

We know that only every mth value in the above summation is nonzero. Therefore,

$$\begin{aligned} b_k &= \frac{1}{mN} \sum_{n=0}^{N-1} x_{(m)}[nm] e^{-j(2\pi/mN)kmn} \\ &= \frac{1}{mN} \sum_{n=0}^{N-1} x_{(m)}[nm] e^{-j(2\pi/N)kn} \end{aligned}$$

Note that $x_{(m)}[nm] = x[n]$. Therefore,

$$b_k = \frac{1}{mN} \sum_{n=0}^{N-1} x_{(m)}[nm] e^{-j(2\pi/N)kn} = \frac{a_k}{m}.$$

So that we have shown that if $x[n]$ has the Fourier coefficients a_k, then $x_{(m)}[n]$ must have the Fourier coefficients $\frac{1}{m}a_k$.

Problem 4

Compute the Fourier transform of each of the following signals

(a) $[e^{\alpha t} cos(w_0 t)]u(t)$, $a > 0$

(b) $e^{-3|t|} sin(2t)$

(c) $x(t)$ as show in Figure 1.

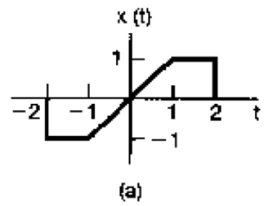

Figure 1: The graph of signal $x(t)$ in (c).

Solution

(a) The given signal is

$$e^{-at} cos(w_0 t)u(t) = \frac{1}{2}e^{-at}e^{jw_0 t}u(t) + \frac{1}{2}e^{-at}e^{-jw_0 t}u(t).$$

Therefore,

$$X(jw) = \frac{1}{2(\alpha - jw_0 + jw)} + \frac{1}{2(\alpha + jw_0 + jw)}$$

(b) We know that

$$e^{-3|t|} \xleftrightarrow{CSFT} \frac{6}{9 + w^2}$$
$$sin(2t) \xleftrightarrow{CSFT} \frac{\pi}{j}(\delta(w - w_0) - \delta(w + w_0))$$

Therefore, we obtain

$$\begin{aligned} X(jw) &= \frac{1}{2\pi}(\frac{6}{9 + w^2}) * \frac{\pi}{j}(\delta(w - w_0) - \delta(w + w_0)) \\ &= \frac{j3}{9 + (w + 2)^2} - \frac{j3}{9 + (w - 2)^2} \end{aligned}$$

(c)

$$\begin{aligned} X(jw) &= \int_{-\infty}^{\infty} x(t)e^{-jwt}dt \\ &= \int_{-2}^{-1} -e^{-jwt}dt + \int_{-1}^{1} te^{-jwt}dt + \int_{1}^{2} e^{-jwt}dt \\ &= -\frac{1}{jw}e^{2jw} - \frac{1}{jw}e^{-2jw} + \frac{1}{w^2}e^{-jw} - \frac{1}{w^2}e^{jw} \\ &= -\frac{2}{jw}(cos(2w) - \frac{sin(w)}{w}) \end{aligned}$$

Problem 5

Consider the signal $x(t)$ in Figure 2.

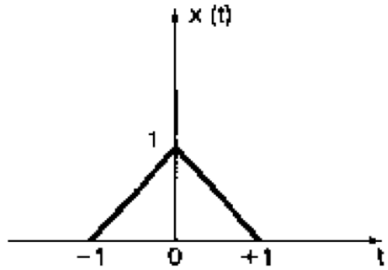

Figure 2: The graph of signal $x(t)$.

(a) Find the Fourier transform $X(jw)$ of $x(t)$.

(b) Sketch the signal

$$\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t - 4k).$$

(c) Find another signal $g(t)$ such that $g(t)$ is not the same as $x(t)$ and

$$\tilde{x}(t) = g(t) * \sum_{k=-\infty}^{\infty} \delta(t - 4k).$$

(d) Argue that, although $G(jw)$ is different from $X(jw)$, $G(j\frac{\pi k}{2}) = X(j\frac{\pi k}{2})$ for all integers k. You should not explicitly evaluate $G(jw)$ to answer the question.

Solution

(a) Note that

$$x(t) = x_1(t) * x_2(t)$$

where

$$x_1(t) = \begin{cases} 1, & |w| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Also, the Fourier transform $X_1(jw)$ of $x_1(t)$ is

$$X_1(jw) = 2\frac{sin(w/2)}{w}.$$

Using the convolution property we have

$$X(jw) = X_1(jw)X_1(jw) = [2\frac{sin(w/2)}{w}]^2$$

Problem 5 continued on next page. . .

(b) The signal of $\tilde{x}(t)$ is as ashown in Figure 3.

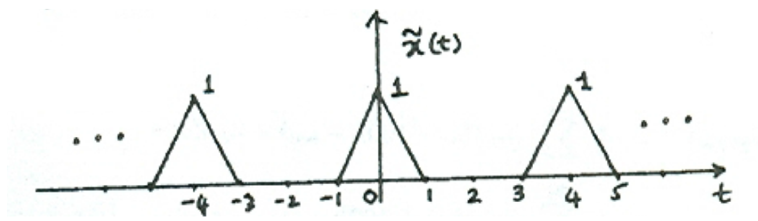

Figure 3: The graph of signal $\tilde{x}(t)$.

(c) One possible choice of $g(t)$ is as ashown in Figure 4.

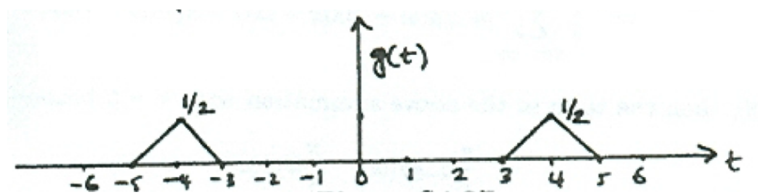

Figure 4: The graph of signal $g(t)$.

(d) Note that

$$\tilde{X}(jw) = X(jw)\frac{\pi}{2}\sum_{-\infty}^{\infty}\delta(j(w - k\frac{\pi}{2})) = G(jw)\frac{\pi}{2}\sum_{-\infty}^{\infty}\delta(j(w - k\frac{\pi}{2}))$$

This may also be written as

$$\tilde{X}(jw) = \frac{\pi}{2}\sum_{-\infty}^{\infty}X(j\pi k/2)\delta(j(w - k\frac{\pi}{2})) = \frac{\pi}{2}\sum_{-\infty}^{\infty}G(j\pi k/2)\delta(j(w - k\frac{\pi}{2}))$$

Clearly, this is possible only if

$$G(j\pi k/2) = X(j\pi k/2).$$