ECE 329 – Fall 2022

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

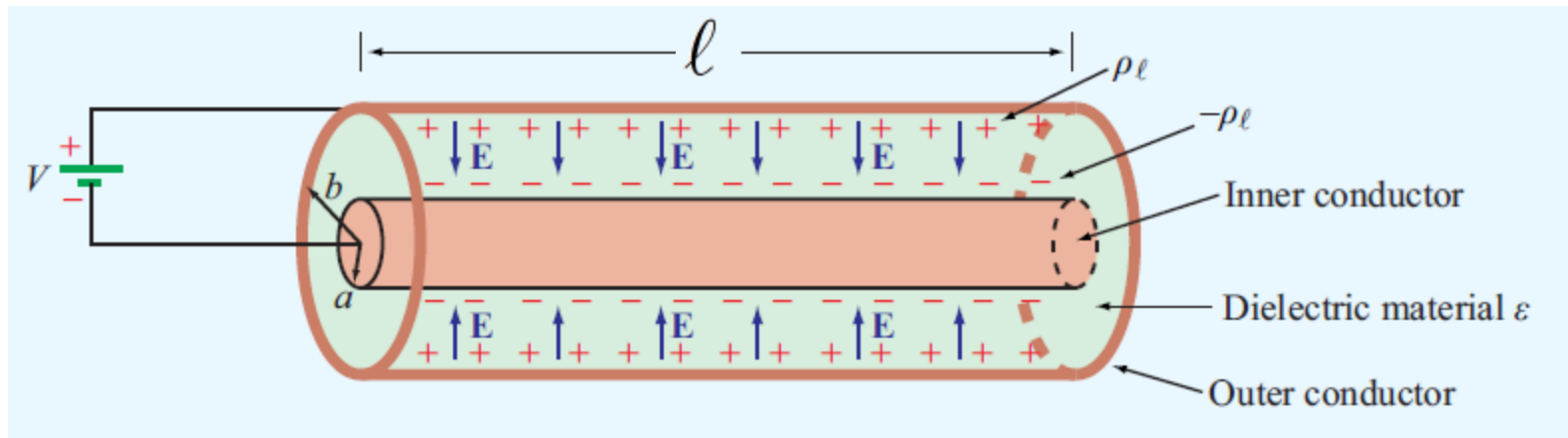From Lecture 10

Remember

HKN review session

Saturday 9/17, 3-5pm – ECEB 1013

Monday 9/19

No official class but I will be here to take questions/discuss exam

Coaxial Capacitor

Cylindrical symmetry – **For** $\ell \gg b$ **we can neglect fringing and assume perfectly radial field**
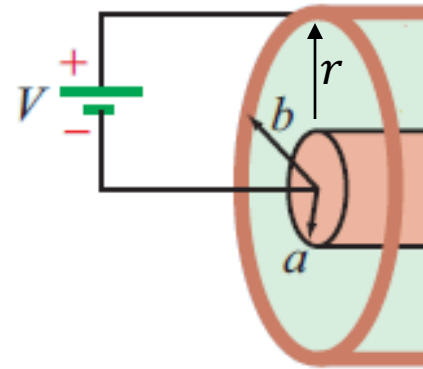

Consider a concentric Gaussian surface between inner and outer conductor with radius $a < r < b$

Gauss Law

$$\epsilon \oint_S \mathbf{E} \cdot d\mathbf{S} = Q_V \implies \epsilon E_r 2\pi r \ell = Q$$

Then we have

$$E_r = \frac{Q}{2\pi\epsilon\ell r}$$

$$V = \int_{r=a}^{b} E_r dr = \int_{r=a}^{b} \frac{Q}{2\pi\epsilon\ell r} dr = \frac{Q}{2\pi\ell\epsilon} \int_{r=a}^{b} \frac{dr}{r} = \frac{Q}{2\pi\ell\epsilon} \ln\frac{b}{a}$$

Using

$$\boxed{Q = CV} \qquad \boxed{V = \frac{Q}{2\pi\ell\epsilon} \ln\frac{b}{a}}$$

we obtain the capacitance for the coaxial structure

$$C = \frac{2\pi}{\ln\frac{b}{a}}\ell\epsilon$$

The capacitance per unit length is simply

$$\mathcal{C} = \frac{2\pi}{\ln \frac{b}{a}}\epsilon$$

Since for a capacitor with imperfect dielectric

$$RC = \frac{\epsilon}{\sigma} \quad \text{or} \quad G = \frac{\sigma}{\epsilon}C$$
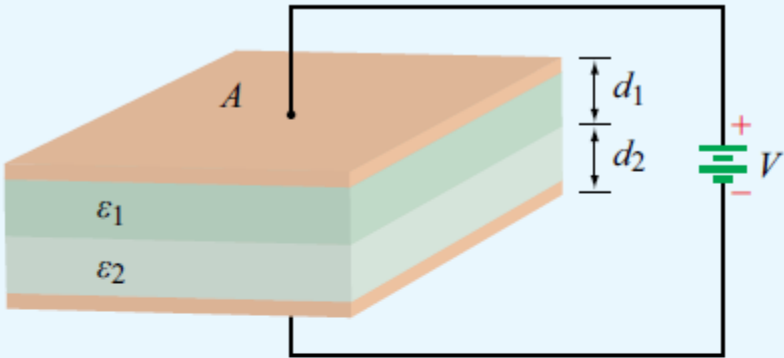
we can express the conductance per unit length as

$$\mathcal{G} = \frac{2\pi}{\ln \frac{b}{a}}\sigma$$

Suggested reading about realistic electrolytic capacitors:
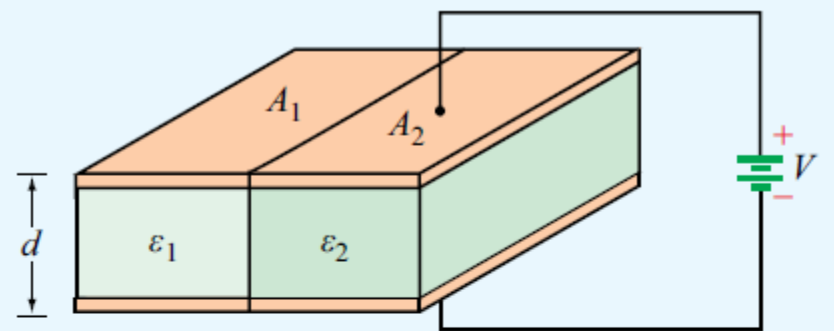https://eepower.com/technical-articles/electrolytic-capacitor-leakage-current/
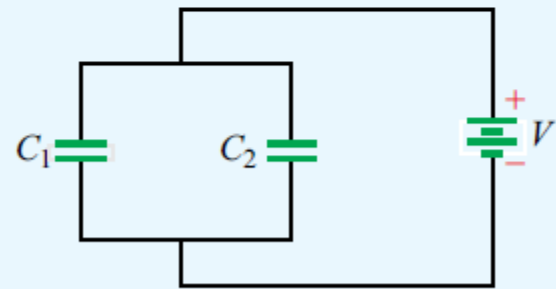
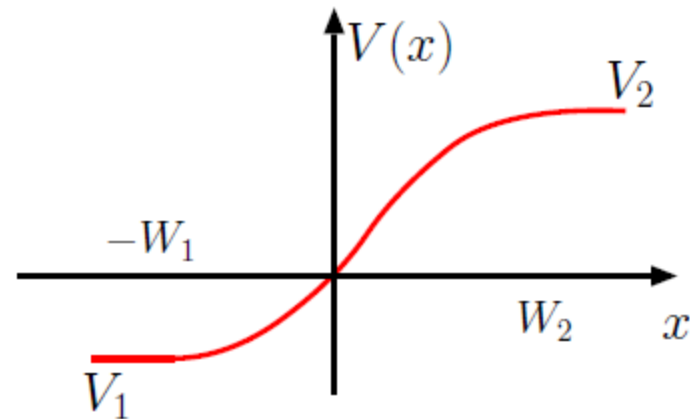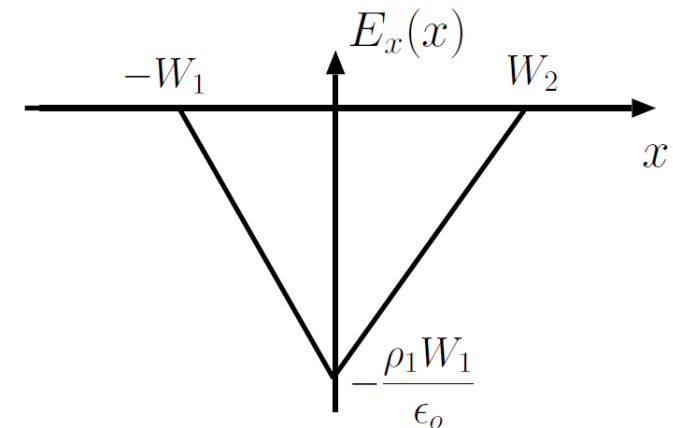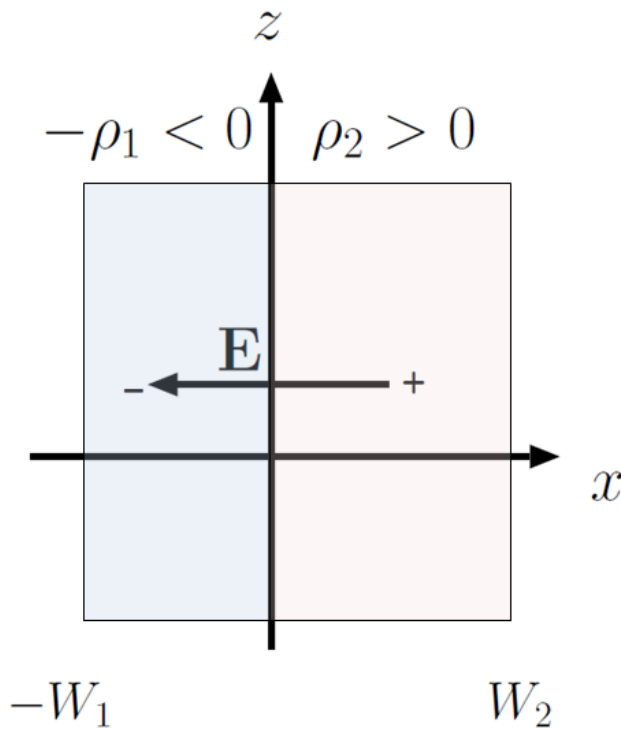# Capacitors with multiple dielectrics

(a)

(b)

(a)

(b)

Capacitance of diode junction

We have discussed earlier the case of two slabs with opposite volumetric charge

The potential across the junction was obtained as

$$V_{21} = V_2 - V_1 = \frac{\rho_2 W_2^2 + \rho_1 W_1^2}{2\epsilon_o}$$

The charge neutrality assumption $W_1\rho_1 = W_2\rho_2$ **gives**

$$V = \frac{\rho_2 W_2(W_1 + W_2)}{2\epsilon_o} = \frac{\rho_1 W_1(W_1 + W_2)}{2\epsilon_o}$$

and from simple manipulations:

$$W_1 = \frac{2\epsilon_o V}{(W_1 + W_2)\rho_1} \qquad W_2 = \frac{2\epsilon_o V}{(W_1 + W_2)\rho_2}$$

$$\Rightarrow \quad W_1 + W_2 = \sqrt{2\epsilon_o V \frac{\rho_1 + \rho_2}{\rho_1 \rho_2}}$$

Consider now a finite cross-section of area A for the structure. We have total positive charge for $x > 0$

$$Q = \rho_2 W_2 A$$

From the previous results $W_1 + W_2 = \sqrt{2\epsilon_o V \dfrac{\rho_1 + \rho_2}{\rho_1 \rho_2}}$

$$V = \frac{\rho_2 W_2 (W_1 + W_2)}{2\epsilon_o} = \frac{Q\sqrt{2\epsilon_o V \dfrac{\rho_1 + \rho_2}{\rho_1 \rho_2}}}{2\epsilon_o A}$$

$$Q = A\sqrt{\frac{2\epsilon_o \rho_1 \rho_2}{\rho_1 + \rho_2}} \sqrt{V}$$

non-linear charge-voltage relation

We define the small-signal capacitance differentiating

$$Q = A\sqrt{\frac{2\epsilon_o \rho_1 \rho_2}{\rho_1 + \rho_2}}\sqrt{V}$$

and obtaining

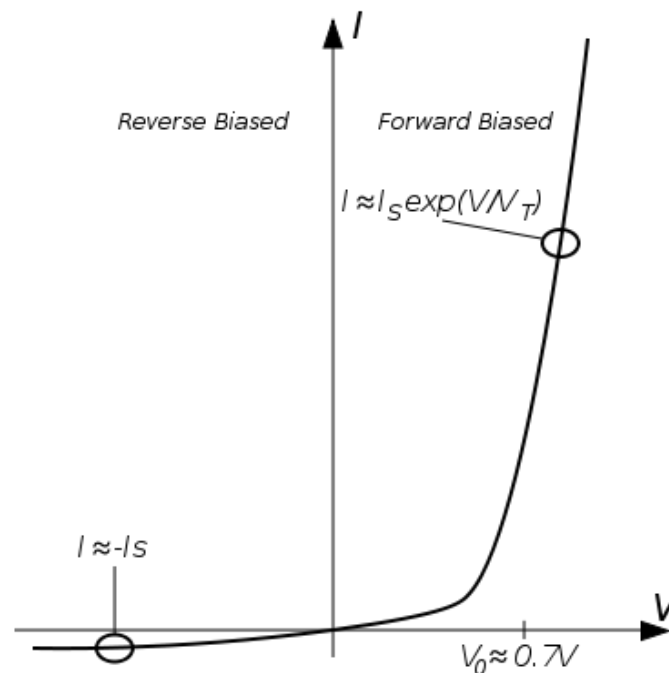$$C = \frac{dQ}{dV} = A\sqrt{\frac{\epsilon_o}{2V}\frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)}}$$

which depends on the potential as $C \propto V^{-1/2}$.

In the case of parallel plate capacitor, the expression is linear and the value of the capacitance is constant

$$C = \frac{Q}{V}$$

NOTE: In ECE 340 you will learn how the potential across this "space charge region" varies when the applied battery voltage is changed.

This will require some elementary knowledge of semiconductor theory and it will justify the expression for the $I\text{-}V$ curve of a diode, which you have seen in ECE 110.

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Lecture 11

Lecture 11 – Outline

- **Conductivity as random motion of charges**
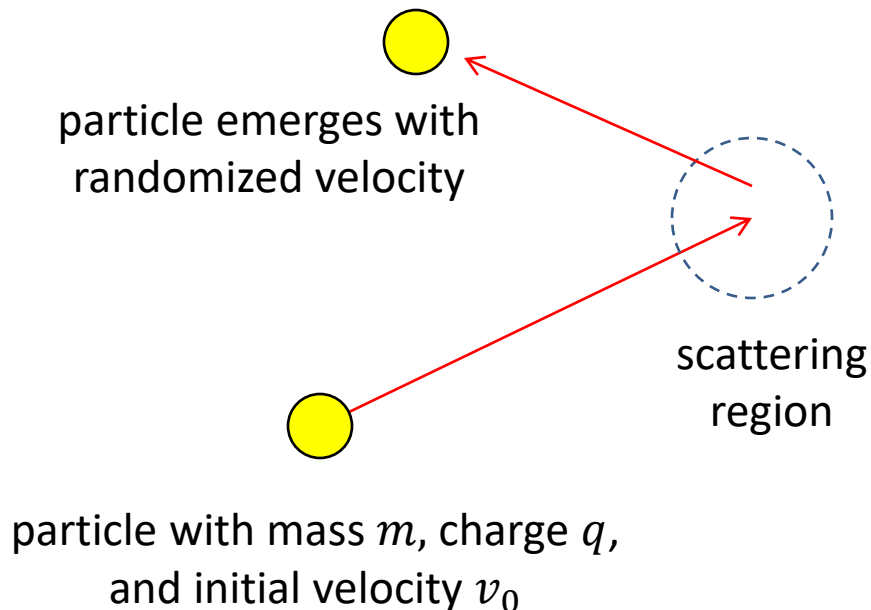- **Drude models for conductivity and susceptibility**

Reading assignment

Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:

11) Lorentz-Drude models for conductivity and susceptibility and polarization current

Random motion of charges

Simple microscopic model by Drude, extended by Lorentz (established before the development of quantum mechanics).

Mobile charges are treated with Newton's dynamic laws as classical particles which are scattered by atoms.

particle emerges with
randomized velocity

scattering
region

In the absence of electric field, as a result of many consecutive random collisions, the time-average velocity is zero.

particle with mass m, charge q,
and initial velocity v_0

Equation of motion in electric field

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} - m\frac{\mathbf{v}}{\tau}$$

mass × acceleration = electric force − friction due to collisions

$\mathbf{V}$ = macroscopic average velocity (drift velocity)

τ = average time between collisions

$\nu = \dfrac{1}{\tau}$ = collisions per unit time (frequency of collisions)

These parameters represent the average behavior of a large ensemble of particles at any given time

15

In the absence of electric field

turn field off

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} - m\frac{\mathbf{v}}{\tau}$$
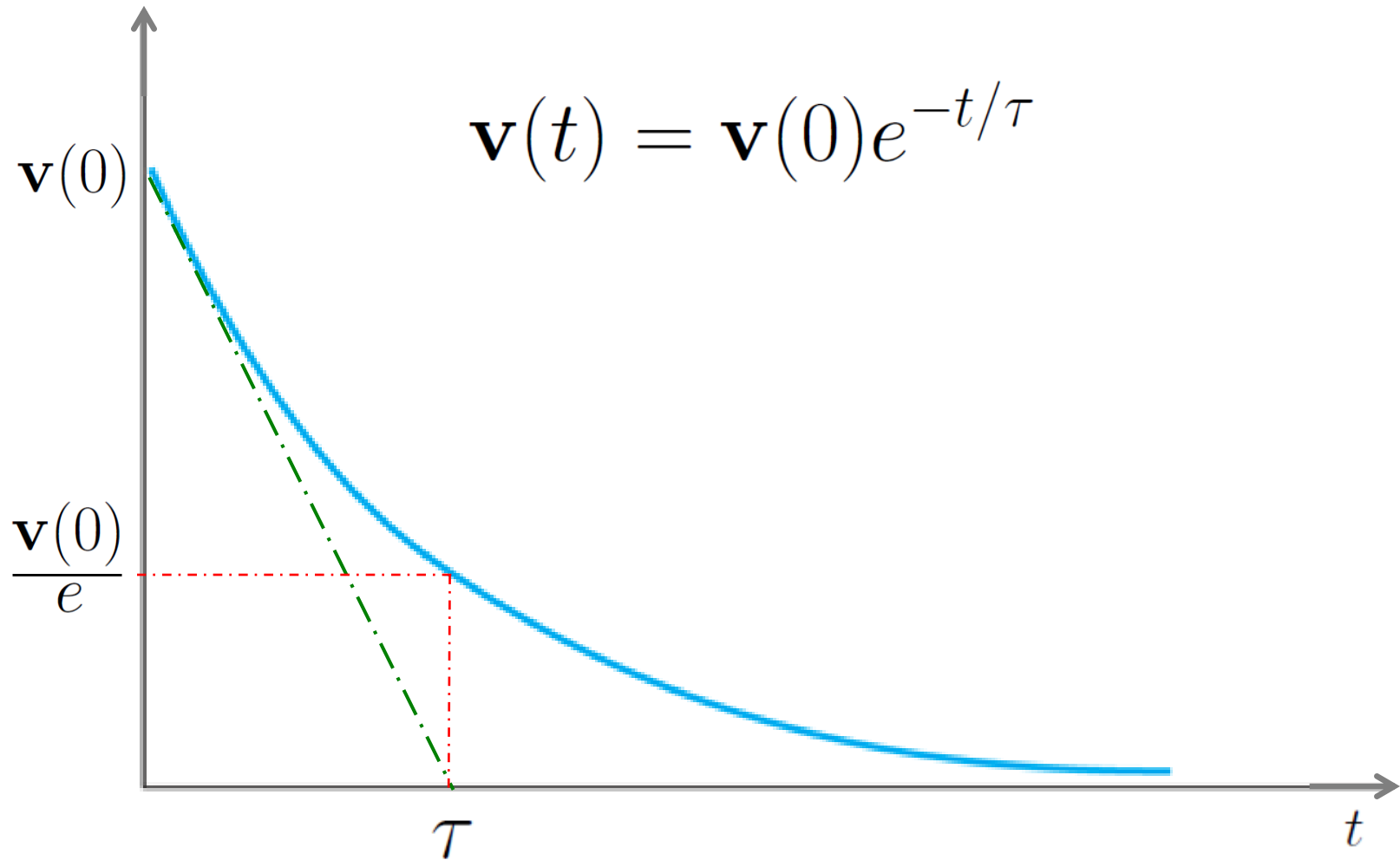
$$\boxed{\begin{array}{l} \mathbf{E} = 0 \\ t = 0 \end{array}}$$

$$\frac{d\mathbf{v}}{dt} = -\frac{\mathbf{v}}{\tau}$$

solution is an exponential decay

$$\mathbf{v}(t) = \mathbf{v}(0)e^{-t/\tau}$$

Exponential decay

$$\mathbf{v}(t) = \mathbf{v}(0)e^{-t/\tau}$$

Constant Applied Electric Field (DC)

A steady state solution of the equation of motion is reached

$$m\frac{d\mathbf{v}}{dt} = 0 = q\mathbf{E} - m\frac{\mathbf{v}}{\tau}$$

$$\boxed{\mathbf{v} = \frac{q\tau}{m}\mathbf{E}}$$ **steady-state velocity**

$$\mu = \left|\frac{q\tau}{m}\right|$$ **is called mobility**

Typical electron mobility values

Metals (e.g., Al, Au, Cu, Ag)   **10 – 50** $\dfrac{\text{cm}^2}{\text{V·s}}$

Pure crystalline silicon   **1,400** $\dfrac{\text{cm}^2}{\text{V·s}}$

Polycristalline silicon   **~ 100** $\dfrac{\text{cm}^2}{\text{V·s}}$

Amorphous silicon   **~ 1** $\dfrac{\text{cm}^2}{\text{V·s}}$

Carbon nanotube   **~ 70,000** $\dfrac{\text{cm}^2}{\text{V·s}}$ **(room T)**

Graphene   **~ 200,000** $\dfrac{\text{cm}^2}{\text{V·s}}$ **(low T)**

AlGaAs/GaAs heterojunction   **~ 30,000,000** $\dfrac{\text{cm}^2}{\text{V·s}}$

Current Density

Assume N charge carriers per unit volume (carrier density) moving at the steady-state average velocity (drift velocity) in the direction of the electric field.
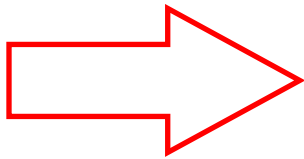
This corresponds to an average flux of charge density (current density)

$$\mathbf{J} = Nq\mathbf{v} = \frac{Nq^2\tau}{m}\mathbf{E} = \frac{Nq^2}{m\,\nu}\mathbf{E} \quad \left[\frac{C/s}{m^2}\right]$$

Current density can also be defined equivalently as number of charges crossing a unit area per second.

$$\mathbf{J} = Nq\mathbf{v} = \frac{Nq^2\tau}{m}\mathbf{E} = \frac{Nq^2}{m\,\nu}\mathbf{E} \quad \left[\frac{C/s}{m^2}\right]$$

$$\frac{Nq^2\tau}{m} = \frac{Nq^2}{m\,\nu} = \sigma \qquad \textbf{Conductivity}$$

$$\Longrightarrow \boxed{\mathbf{J} = \sigma\mathbf{E}}$$

This is the expression we used earlier in Lecture 8

Conductivity of some common materials at room temperature

Material	Conductivity, σ (S/m)
Conductors	
Silver	6.2×10^7
Copper	5.8×10^7
Gold	4.1×10^7
Aluminum	3.5×10^7
Iron	10^7
Mercury	10^6
Carbon	3×10^4
Semiconductors	
Pure germanium	2.2
Pure silicon	4.4×10^{-4}
Insulators	
Glass	10^{-12}
Paraffin	10^{-15}
Mica	10^{-15}
Fused quartz	10^{-17}

Question

If metals have such a low mobility, why do they have very high conductivity?

If there are different conducting species participating in the current density, the different conductivities are simply summed up

$$\sigma = \sum_s \sigma_s$$

where

$$\sigma_s = \frac{N_s q_s^2}{m_s \nu_s}$$

Time-varying Electric Field (AC)

This regime can be studied in the frequency domain using phasor techniques, for instance:

$$\mathbf{E}(t) = \mathrm{Re}\{\tilde{\mathbf{E}}e^{j\omega t}\} \qquad \Longleftrightarrow \qquad \tilde{\mathbf{E}}$$

$$\mathbf{J}(t) = \mathrm{Re}\{\tilde{\mathbf{J}}e^{j\omega t}\} \qquad \Longleftrightarrow \qquad \tilde{\mathbf{J}}$$

Phasor transformed force balance equation

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} - m\frac{\mathbf{v}}{\tau} \quad \Rightarrow \quad mj\omega\tilde{\mathbf{v}} = q\tilde{\mathbf{E}} - m\frac{\tilde{\mathbf{v}}}{\tau}$$

$$\tilde{\mathbf{J}} = \sigma\tilde{\mathbf{E}}$$

$$\tilde{\mathbf{J}} = \sigma \tilde{\mathbf{E}}$$

The conductivity should be a function of frequency

$$\sigma = \sum_s \sigma_s \quad \text{and} \quad \sigma_s = \frac{N_s q_s^2}{m_s(\nu_s + j\omega)}$$

However, in many cases the DC conductivity can still be a good approximation of AC conductivity if the frequency of operation is much smaller than the collision frequencies.

Polarization effects in dielectrics

As we have seen, all dielectric materials, including perfect dielectrics, can be polarized and possess:

susceptibility

$$\chi_e \neq 0$$

relative permittivity

$$\epsilon_r = 1 + \chi_e > 1$$

In a perfect dielectric there are no free carriers and $\sigma = 0$. In a realistic dielectric there are some free carriers with $\sigma \neq 0$ (but very small) which can support some DC current.

In contrast to metals, where charges are free to move throughout the material, in dielectrics most charges are attached to specific atoms and molecules.

These polarization charges are known as bound charges.

Bound charges are able, however, to be displaced within an atom or a molecule. Such cumulative microscopic displacements account for the characteristic behavior of dielectric materials.
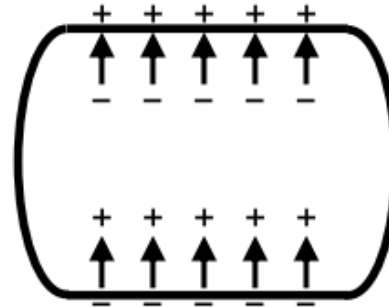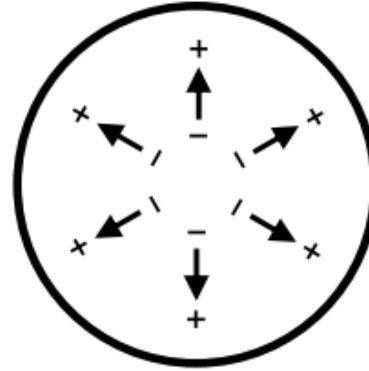
Since the polarized bound charge is not free to move from atom to atom, it cannot contribute to DC current. However, it can "oscillate" and contribute to AC current.

Some authors distinguish charge density as "unpaired" (free to move, contributing to DC current) and "paired" (bound charge with dipole configuration, due to polarization)

$$\rho = \rho_u + \rho_p \qquad \text{same as} \qquad \rho = \rho_{free} + \rho_p$$

There are two contributions to bound charge ρ_p:

- **Volume charge density**

- **Surface charge density**

The volume density of bound charge can be expressed as

$$\rho_v(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r})$$

Note: here the vector $\mathbf{r}$ represents (x, y, z)

The surface density of bound charge can be expressed as

$$\rho_s(\mathbf{r}) = \mathbf{P}(\mathbf{r}) \cdot \mathbf{n}$$

This is actually a result of the formula above with an abrupt change of $\mathbf{P}$ at the surface

To summarize, for a dielectric we can write

$$\rho_{total} = \rho_{free} + \rho_P = \rho - \nabla \cdot \mathbf{P}$$

The divergence of the electric field is

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\left[\rho - \nabla \cdot \mathbf{P}\right]$$

and for the electric displacement

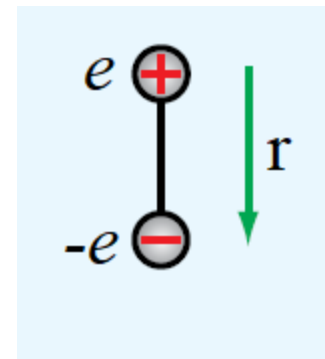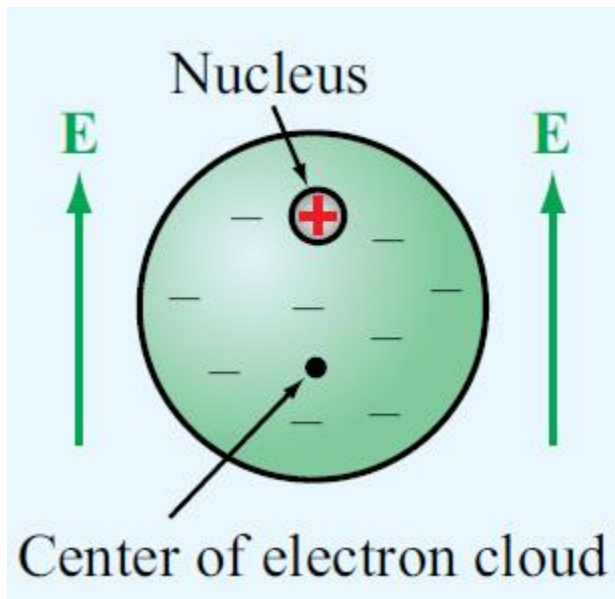$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\nabla \cdot \mathbf{D} = \rho$$

Susceptibility model

In the Lorentz-Drude model, each polarized atom (or molecule) is represented by a vector **dipole moment p**

$$\mathbf{p} = -e\mathbf{r}$$

displacement of electron cloud from nucleus
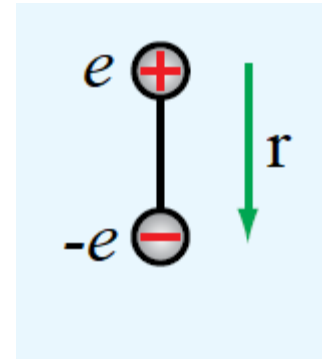
electronic charge

Equivalent dipole

We find two equivalent definitions in the literature (it can be confusing)

$$\mathbf{p} = -e\mathbf{r}$$
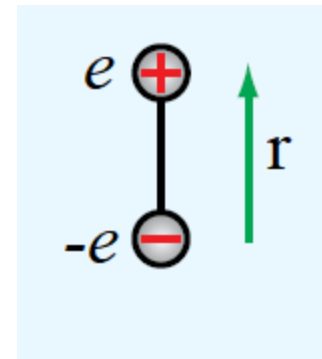
(This is what we have in our class notes)

vector *r* from positive to negative

$$\mathbf{p} = e\mathbf{r}$$

(e.g., in Prof. Rao's book)

vector *r* from negative to positive

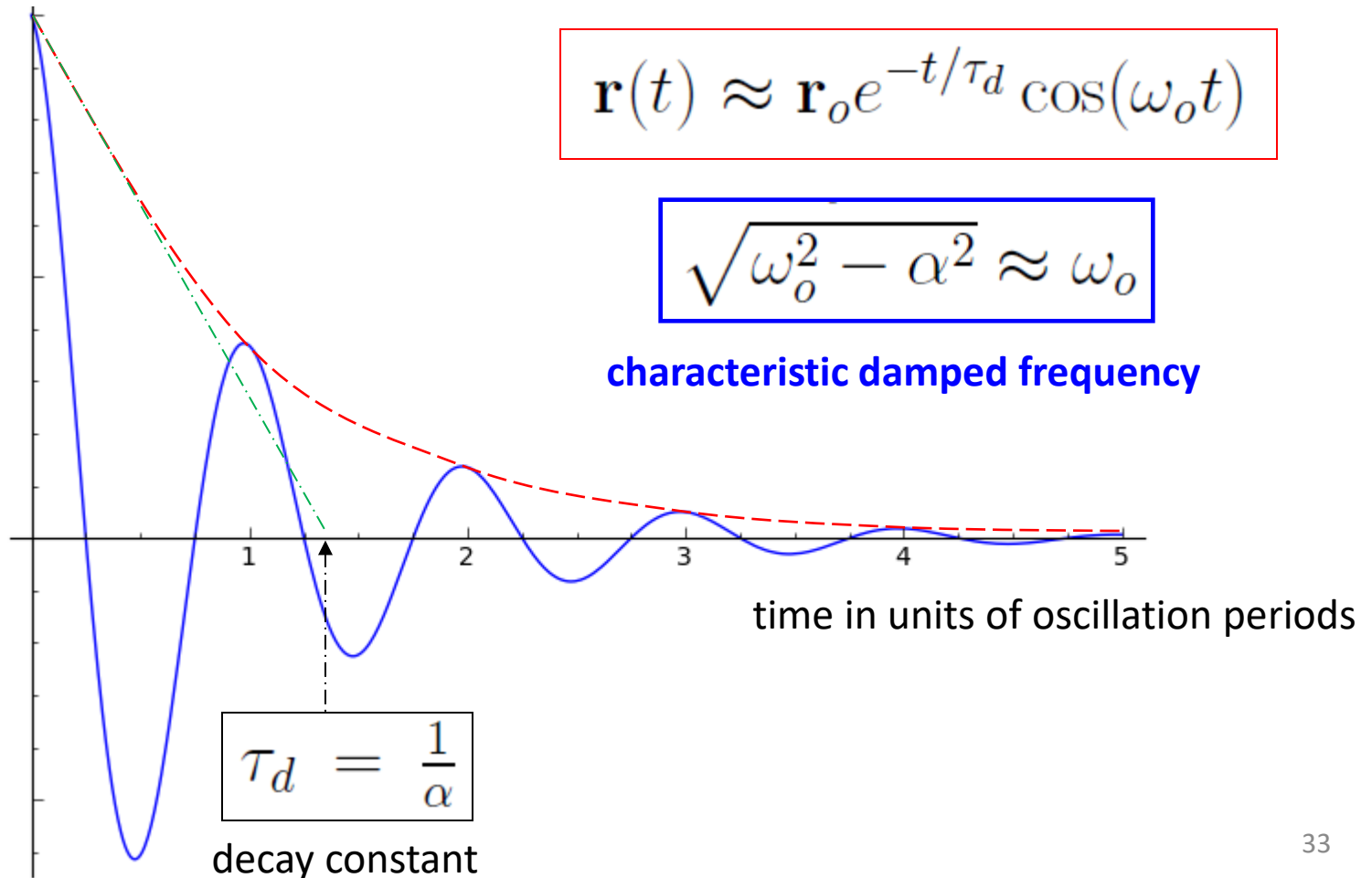**Release the polarizing force due to the electric field at t = 0**
⇒ the dipole charges relax to equilibrium causing the dipole field to decay following the observed damped co-sinusoidal oscillation

$\mathbf{E}_p \propto \mathbf{p} \propto \mathbf{r}$ dipole field proportional to charge displacement

$$\mathbf{r}(t) \approx \mathbf{r}_o e^{-t/\tau_d} \cos(\omega_o t)$$

$$\sqrt{\omega_o^2 - \alpha^2} \approx \omega_o$$

characteristic damped frequency

time in units of oscillation periods

$$\tau_d = \frac{1}{\alpha}$$

decay constant

The behavior can be described by the Lorentz oscillator equation for the displacement vector

$$m\frac{d^2\mathbf{r}}{dt^2} = -e\mathbf{E} - m\omega_o^2\mathbf{r} - m2\alpha\frac{d\mathbf{r}}{dt}$$

$t = 0$

friction dissipative force

"spring" binding force

external force

acceleration

mass

The same Lorentz oscillator equation may be used do describe the steady-state condition after a field has been applied

steady-state

steady-state

$$m\frac{d^2\mathbf{r}}{dt^2} = \underbrace{-e\mathbf{E}}_{\neq\ 0} - m\omega_o^2\mathbf{r} - m2\alpha\frac{d\mathbf{r}}{dt}$$

$$-e\mathbf{E} = m\omega_o^2\mathbf{r} \qquad \Longrightarrow \qquad \boxed{\mathbf{r} = -\frac{e}{m\omega_o^2}\mathbf{E}}$$

from which we can determine the atomic dipole moment

$$\mathbf{p} = -e\mathbf{r} = \frac{e^2}{m\omega_o^2}\mathbf{E}$$

Consider a medium consisting of N_d polarized particles per unit volume (dipole density)

$$\mathbf{p} = -e\mathbf{r} = \frac{e^2}{m\omega_o^2}\mathbf{E}$$

polarization field in the dielectric material

$$\mathbf{P} = N_d\mathbf{p} = \frac{N_d e^2}{m\omega_o^2}\mathbf{E}$$

From prior definition, we get an expression for the DC susceptibility

$$\mathbf{P} = \epsilon_o \chi_e \mathbf{E} \qquad \chi_e \equiv \frac{N_d e^2}{m\epsilon_o \omega_o^2}$$

AC Conditions: Polarization Current

Let's consider now a time-varying Electric Field in a dielectric medium. If the frequency of the field is $\omega \ll \omega_o$ the DC susceptibility is still a good approximation of the AC value

$$\chi_e \equiv \frac{N_d e^2}{m\epsilon_o\, \omega_o^2}$$

and we can continue to use also

$$\mathbf{r} = -\frac{e}{m\omega_o^2}\mathbf{E}$$

$$\mathbf{P} = \epsilon_o \chi_e \mathbf{E}$$

AC Conditions: Polarization Current

For a field $\mathbf{E}(t)$

$$\mathbf{r}(t) = -\frac{e}{m\omega_o^2}\mathbf{E}(t)$$

Electrons are displaced back and forth, with an effective spatial velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\frac{e}{m\omega_o^2}\frac{d\mathbf{E}}{dt}$$

This movement of bound charges in the dielectric medium generates an effective AC current density. With N_d bound electrons per unit volume

$$\mathbf{J}_p = -eN_d\mathbf{v} = \frac{N_d e^2}{m\omega_o^2}\frac{d\mathbf{E}}{dt} = \frac{d\mathbf{P}}{dt}$$

polarization current density (AC conditions)