ECE5463: Introduction to Robotics

Lecture Note 11: Dynamics of a Single Rigid Body

Prof. Wei Zhang

Department of Electrical and Computer Engineering
Ohio State University
Columbus, Ohio, USA

Spring 2018

Outline

- Kinetic Energy of a Rigid Body
- Rotational Inertia Matrix
- Newton Euler Equation
- Twist-Wrench Formulation of Rigid-Body Dynamics

Robot Dynamic Model Can be Complicated

- Dynamic model of PUMA 560 Arm:

$I_2 = I_{zz2} + m_2 * (r_{x2}^2 + r_{y2}^2) + (m_3 + m_4 + m_5 + m_6) * a_2^2$;
$I_3 = -I_{xx2} + I_{yy2} + (m_3 + m_4 + m_5 + m_6) * a_2^2 + m_2 * r_{x2}^2 - m_2 * r_{y2}^2$;
$I_4 = m_2 * r_{y2} * (d_2 + r_{z2}) + m_3 * a_2 * r_{z3} + (m_3 + m_4 + m_5 + m_6) * a_2 * (d_2 + d_3)$;
$I_5 = -m_3 * a_2 * r_{y3} + (m_4 + m_5 + m_6) * a_2 * d_4 + m_4 * a_2 * r_{z4}$;
$I_6 = I_{xx3} + m_3 * r_{y3}^2 + m_4 * a_3^2 + m_4 * (d_4 + r_{z4})^2 + I_{yy4} + m_5 * a_3^2 + m_5 * d_4^2 + I_{zz5} + m_6 * a_3^2 + m_6 * d_4^2 + m_6 * r_{z6}^2 + I_{zz6}$;
$I_7 = m_3 * r_{y3}^2 + I_{zz3} - I_{yy3} + m_4 * r_{z4}^2 + 2 * m_4 * d_4 * r_{z4} + (m_4 + m_5 + m_6) * (d_4^2 - a_3^2) + I_{yy4} - I_{zz4} + I_{zz5} - I_{yy5} + m_6 * r_{z6}^2 - I_{zz6} + I_{zz6}$;
$I_8 = -m_4 * (d_2 + d_3) * (d_4 + r_{z4}) - (m_5 + m_6) * (d_2 + d_3) * d_4 \; m_3 * r_{y3} * r_{z3} + m_3 * (d_2 + d_3) * r_{y3}$;
$I_9 = m_2 * r_{y2} * (d_2 + r_{z2})$;
$I_{10} = 2 * m_4 * a_3 * r_{z4} + 2 * (m_4 + m_5 + m_6) * a_3 * d_4$;
$I_{11} = -2 * m_2 * r_{x2} * r_{y2}$;
$I_{12} = (m_4 + m_5 + m_6) * a_2 * a_3$;
$I_{13} = (m_4 + m_5 + m_6) * a_3 * (d_2 + d_3)$;
$I_{14} = I_{zz4} + I_{yy5} + I_{zz6}$;
$I_{15} = m_6 * d_4 * r_{z6}$;
$I_{16} = m_6 * a_2 * r_{z6}$;
$I_{17} = I_{zz5} + I_{zz6} + m_6 * r_{z6}^2$;
$I_{18} = m_6 * (d_2 + d_3) * r_{z6}$;
$I_{19} = I_{yy4} - I_{zz4} + I_{zz5} - I_{yy5} + m_6 * r_{z6}^2 + I_{zz6} - I_{zz6}$;
$I_{20} = I_{yy5} - I_{zz5} - m_6 * r_{z6}^2 + I_{zz6} - I_{zz6}$;
$I_{21} = I_{zz4} - I_{yy4} + I_{zz5} - I_{zz5}$;
$I_{22} = m_6 * a_3 * r_{z6}$;
$I_{23} = I_{zz6}$;

Part II. Gravitional Constants

$g_1 = -g * ((m_3 + m_4 + m_5 + m_6) * a_2 + m_2 * r_{x2})$;
$g_2 = g * (m_3 * r_{y3} - (m_4 + m_5 + m_6) * d_4 - m_4 * r_{z4})$;
$g_3 = g * m_2 * r_{y2}$;
$g_4 = -g * (m_4 + m_5 + m_6) * a_3$;
$g_5 = -g * m_6 * r_{z6}$;

Table A3. Computed Values for the Constants Appearing in the Equations of Forces of Motion.
(Inertial constants have units of kilogram meters-squared)

I_1 =	1.43	±0.05	I_2 =	1.75	±0.07
I_3 =	1.38	±0.05	I_4 =	6.90×10^{-1}	$\pm0.20\times10^{-1}$
I_5 =	3.72×10^{-1}	$\pm0.31\times10^{-1}$	I_6 =	3.33×10^{-1}	$\pm0.16\times10^{-1}$
I_7 =	2.98×10^{-1}	$\pm0.29\times10^{-1}$	I_8 =	-1.34×10^{-1}	$\pm0.14\times10^{-1}$
I_9 =	2.38×10^{-2}	$\pm1.20\times10^{-2}$	I_{10} =	-2.13×10^{-2}	$\pm0.22\times10^{-2}$
I_{11} =	-1.42×10^{-2}	$\pm0.70\times10^{-2}$	I_{12} =	-1.10×10^{-2}	$\pm0.11\times10^{-2}$
I_{13} =	-3.79×10^{-3}	$\pm0.90\times10^{-3}$	I_{14} =	1.64×10^{-3}	$\pm0.07\times10^{-3}$
I_{15} =	1.25×10^{-3}	$\pm0.30\times10^{-3}$	I_{16} =	1.24×10^{-3}	$\pm0.30\times10^{-3}$
I_{17} =	6.42×10^{-4}	$\pm3.00\times10^{-4}$	I_{18} =	4.31×10^{-4}	$\pm1.30\times10^{-4}$
I_{19} =	3.00×10^{-4}	$\pm14.0\times10^{-4}$	I_{20} =	-2.02×10^{-4}	$\pm8.00\times10^{-4}$
I_{21} =	-1.00×10^{-4}	$\pm6.00\times10^{-4}$	I_{22} =	-5.80×10^{-5}	$\pm1.50\times10^{-5}$
I_{23} =	4.00×10^{-5}	$\pm2.00\times10^{-5}$			
I_{m1} =	1.14	±0.27	I_{m2} =	4.71	±0.54
I_{m3} =	8.27×10^{-1}	$\pm0.93\times10^{-1}$	I_{m4} =	2.00×10^{-1}	$\pm0.16\times10^{-1}$
I_{m5} =	1.79×10^{-1}	$\pm0.14\times10^{-1}$	I_{m6} =	1.93×10^{-1}	$\pm0.16\times10^{-1}$

(Gravitational constants have units of newton meters)

g_1 =	−37.2	±00.5	g_2 =	−8.44	±0.20
g_3 =	1.02	±0.50	g_4 =	2.49×10^{-1}	$\pm0.25\times10^{-1}$
g_5 =	-2.82×10^{-2}	$\pm0.56\times10^{-2}$			

$b_{224} = 2 * \{-I_{16} * C3 * S4 * S5 + I_{20} * SC4 * SS5 + I_{21} * SC4 - I_{22} * S4 * S5\}$;
$\approx -2.48\times10^{-3} * C3 * S4 * S5$.
$b_{225} = 2 * \{-I_{15} * S5 + I_{16} * (C3 * C4 * C5 - S3 * S5) + I_{20} * SS4 * SC5 + I_{22} * C4 * C5\}$;
$\approx -2.50\times10^{-3} * S5 + 2.48\times10^{-3} * (C3 * C4 * C5 - S3 * S5)$.
$b_{226} = 0$. $b_{234} = b_{224}$.
$b_{235} = b_{225}$. $b_{236} = 0$.
$b_{245} = 2 * \{-I_{15} * S4 * C5 - I_{16} * S3 * S4 * C5\} - I_{17} * S4 + I_{20} * S4 * (1 - 2 * SS5)$; ≈ 0 .
$b_{246} = I_{23} * C4 * S5$; ≈ 0 .
$b_{256} = I_{23} * S4 * C5$; ≈ 0 .
$b_{312} = 0$. $b_{313} = 0$.
$b_{314} = 2 * \{-I_{15} * C23 * C4 * S5 + I_{22} * S23 * C4 * S5 + I_{20} * (S23 * (CC5 * CC4 - 0.5) + C23 * C4 * SC5)\} + I_{14} * S23 + I_{19} * S23 * (1 - (2 * SS4))$;
$\approx -2.50\times10^{-3} * C23 * C4 * S5 + 1.64\times10^{-3} * S23 + 0.30\times10^{-3} * S23 * (1 - 2 * SS4)$.
$b_{315} = 2 * \{-I_{15} * C23 * S4 * C5 + I_{22} * S23 * S4 * C5\} - I_{17} * C23 * S4 + I_{20} * S4 * (C23 * (1 - 2 * SS5) - 2 * S23 * C4 * SC5)$;
$\approx -2.50\times10^{-3} * C23 * S4 * C5 - 6.42\times10^{-4} * C23 * S4$.
$b_{316} = -b_{136}$. $b_{323} = 0$.
$b_{324} = 2 * \{I_{20} * SC4 * SS5 + I_{21} * SC4 - I_{22} * S4 * S5\}$;
≈ 0 .
$b_{325} = 2 * \{-I_{15} * S5 + I_{20} * SS4 * SC5 + I_{22} * C4 * C5\}$;
$\approx -2.50\times10^{-3} * S5$.
$b_{326} = 0$. $b_{334} = b_{324}$.
$b_{335} = b_{325}$. $b_{336} = 0$.
$b_{345} = -I_{15} * 2 * S4 * C5 - I_{17} * S4 + I_{20} * S4 * (1 - 2 * SS5)$;
$\approx -2.50\times10^{-3} * S4 * C5$.
$b_{346} = b_{246}$. $b_{356} = b_{256}$. $b_{412} = -b_{214}$.
$b_{413} = -b_{314}$. $b_{414} = 0$.
$b_{415} = -I_{20} * (S23 * C4 * (1 - 2 * SS5) + 2 * C23 * SC5) - I_{17} * S23 * C4$;
$\approx -6.42\times10^{-4} * S23 * C4$.
$b_{416} = -b_{146}$. $b_{423} = -b_{324}$. $b_{424} = 0$.
$b_{425} = I_{17} * S4 + I_{20} * S4 * (1 - 2 * SS5)$;
$\approx 6.42\times10^{-4} * S4$.
$b_{426} = -b_{246}$. $b_{434} = 0$.
$b_{435} = b_{425}$. $b_{436} = -b_{346}$.
$b_{445} = -I_{20} * 2 * SC5$; ≈ 0 .
$b_{446} = 0$;
$b_{456} = -I_{23} * S5$; ≈ 0 .
$b_{512} = -b_{215}$. $b_{513} = -b_{315}$. $b_{514} = -b_{415}$.
$b_{515} = 0$. $b_{516} = -b_{156}$. $b_{523} = -b_{325}$.
$b_{524} = -b_{425}$. $b_{525} = 0$. $b_{526} = -b_{256}$.
$b_{534} = b_{524}$. $b_{535} = 0$. $b_{536} = -b_{356}$.
$b_{545} = 0$. $b_{546} = -b_{456}$. $b_{556} = 0$.
$b_{612} = b_{126}$. $b_{613} = b_{136}$. $b_{614} = b_{146}$.
$b_{615} = b_{156}$. $b_{616} = 0$. $b_{623} = 0$.
$b_{624} = b_{246}$. $b_{625} = b_{256}$. $b_{626} = 0$.
$b_{634} = b_{624}$. $b_{635} = b_{625}$. $b_{636} = 0$.
$b_{645} = b_{456}$. $b_{646} = 0$. $b_{656} = 0$.

Table A6. The expressions for the terms of the centrifugal matrix.
(The Abbreviated Expressions have units of kg-m².)

$c_{11} = 0$.
$c_{12} = +I4 * C2 - I8 * S23 - I9 * S2 + I13 * C23 + I_{15} * S23 * S4 * S5 + I_{16} * C2 * S4 * S5 + I_{18} * (C23 * C4 * S5 + S23 * C5) + I_{19} * C23 * SC4 + I_{20} * S4 * (C23 * C4 * cC5 - S23 * SC5) + I_{22} * C23 * S4 * S5$;
$\approx 6.90\times10^{-1} * C2 + 1.34\times10^{-1} * S23 - 2.38\times10^{-2} * S2$.
$c_{13} = 0.5 * b_{123}$.
$c_{14} = -I_{15} * S23 * S4 * S5 - I_{16} * C2 * S4 * S5 + I_{18} * C23 * C4 * S5 + I_{20} * S23 * S4 * SC5 - I_{22} * C23 * S4 * S5$; ≈ 0 .
$c_{15} = -I_{15} * S23 * S4 * S5 - I_{16} * C2 * S4 * S5$; $+I_{18} * (S23 * C5 + C23 * C4 * S5) - I_{22} * C23 * S4 * S5$
≈ 0 .
$c_{16} = 0$. $c_{21} = -0.5 * b_{112}$.
$c_{22} = 0$. $c_{23} = 0.5 * b_{223}$.
$c_{24} = -I_{15} * C4 * S5 - I_{16} * S3 * C4 * S5 + I_{20} * C4 * SC5$;
≈ 0 .
$c_{25} = -I_{15} * C4 * S5 + I_{16} * (C3 * C5 - S3 * C4 * S5) + I_{22} * C5$; ≈ 0 .
$c_{26} = 0$. $c_{31} = -0.5 * b_{113}$.
$c_{32} = -c_{23}$. $c_{33} = 0$.
$c_{34} = -I_{15} * C4 * S5 + I_{20} * C4 * SC5$;
$\approx -1.25\times10^{-3} * C4 * S5$.
$c_{35} = -I_{15} * C4 * S5 + I_{22} * C5$; $\approx c_{34}$.
$c_{36} = 0$. $c_{41} = -0.5 * b_{114}$. $c_{42} = -0.5 * b_{224}$.
$c_{43} = 0.5 * b_{423}$. $c_{44} = 0$. $c_{45} = 0$.
$c_{46} = 0$. $c_{51} = -0.5 * b_{115}$. $c_{52} = -0.5 * b_{225}$.
$c_{53} = 0.5 * b_{523}$. $c_{54} = -0.5 * b_{445}$. $c_{55} = 0$.
$c_{56} = 0$. $c_{61} = 0$. $c_{62} = 0$.
$c_{63} = 0$. $c_{64} = 0$. $c_{65} = 0$.
$c_{66} = 0$.

Table A7. Gravity Terms.
(The Abbreviated Expressions have units of newton-meters.)

$g_1 = 0$.
$g_2 = g1 * C2 + g2 * S23 + g3 * S2 + g4 * C23$; $+g5 * (S23 * C5 + C23 * C4 * S5)$
$\approx -37.2 * C2 - 8.4 * S23 + 1.02 * S2$.
$g_3 = g2 * S23 + g4 * C23 + g5 * (S23 * C5 + C23 * C4 * S5)$;
$\approx -8.4 * S23 + 0.25 * C23$.
$g_4 = -g5 * S23 * S4 * S5$;
$\approx 2.8\times10^{-2} * S23 * S4 * S5$.
$g_5 = g5 * (C23 * S5 + S23 * C4 * C5)$;
$\approx -2.8\times10^{-2} * (C23 * S5 + S23 * C4 * C5)$.
$g_6 = 0$.

Table A4. The expressions giving the elements of the kinetic energy matrix.
(The Abbreviated Expressions have units of kg-m².)

$a_{11} = I_{m1} + I_1 + I_3 * CC2 + I_7 * SS23 + I_{10} * SC23 + I_{11} * SC2 + I_{20} * (SS5 * (SS23 * (1 + CC4) - 1) - 2 * SC23 * C4 * SC5) + I_{21} * SS23 * CC4 + 2 * \{I_5 * C2 * S23 + I_{12} * C2 * C23 + I_{15} * (SS23 * C5 + SC23 * C4 * S5) + I_{16} * C2 * (S23 * C5 + C23 * C4 * S5) + I_{18} * S4 * S5 + I_{22} * (SC23 * C5 + CC23 * C4 * S5)\}$;
$\approx 2.57 + 1.38 * CC2 + 0.30 * SS23 + 7.44\times10^{-1} * C2 * S23$.
$a_{12} = I_4 * S2 + I_8 * C23 + I_9 * C2 + I_{13} * S23 - I_{15} * C23 * S4 * S5 + I_{16} * S2 * S4 * S5 + I_{18} * (S23 * C4 * S5 - C23 * C5) + I_{19} * S23 * SC4 + I_{20} * S4 * (S23 * C4 * CC5 + C23 * SC5) + I_{22} * S23 * S4 * S5$;
$\approx 6.90\times10^{-1} * S2 - 1.34\times10^{-1} * C23 + 2.38\times10^{-2} * C2$.
$a_{13} = I_8 * C23 + I_{13} * S23 - I_{15} * C23 * S4 * S5 + I_{19} * S23 * SC4 + I_{18} * (S23 * C4 * S5 - C23 * C5) + I_{22} * S23 * S4 * S5 + I_{20} * S4 * (S23 * C4 * CC5 + C23 * SC5)$;
$\approx -1.34\times10^{-1} * C23 + -3.97\times10^{-3} * S23$.
$a_{14} = I_{14} * C23 + I_{15} * S23 * C4 * S5 + I_{16} * C2 * C4 * S5 + I_{18} * C23 * S4 * S5 - I_{20} * (S23 * C4 * SC5 + C23 * SS5) + I_{22} * C23 * C4 * S5$; ≈ 0 .
$a_{15} = I_{15} * S23 * S4 * C5 + I_{16} * C2 * S4 * C5 + I_{17} * S23 * S4 + I_{18} * (S23 * S5 - C23 * C4 * C5) + I_{22} * C23 * S4 * C5$;
≈ 0 .
$a_{16} = I_{23} * (C23 * C5 - S23 * C4 * S5)$; ≈ 0 .
$a_{22} = I_{m2} + I_2 + I_6 + I_{20} * SS4 * SS5 + I_{21} * SS4 + 2 * \{I_5 * S3 + I_{12} * C3 + I_{15} * C5 + I_{16} * (S3 * C5 + C3 * C4 * S5) + I_{22} * C4 * S5\}$;
$\approx 6.79 + 7.44\times10^{-1} * S3$.
$a_{23} = I_5 * S3 + I_6 + I_{12} * C3 + I_{16} * (S3 * C5 + C3 * C4 * S5) + I_{20} * SS4 * SS5 + I_{21} * SS4 + 2 * \{I_{15} * C5 + I_{22} * C4 * S5\}$;
$\approx .333 + 3.72\times10^{-1} * S3 - 1.10\times10^{-2} * C3$.
$a_{24} = -I_{15} * S4 * S5 - I_{16} * S3 * S4 * S5 + I_{20} * S4 * SC5$;
≈ 0 .
$a_{25} = I_{15} * C4 * C5 + I_{16} * (C3 * S5 + S3 * C4 * C5) + I_{17} * C4 + I_{22} * S5$; ≈ 0 .
$a_{26} = I_{23} * S4 * S5$; ≈ 0 .
$a_{33} = I_{m3} + I_6 + I_{20} * SS4 * SS5 + I_{21} * SS4 + 2 * \{I_{15} * C5 + I_{22} * C4 * S5\}$; ≈ 1.16 .
$a_{34} = -I_{15} * S4 * S5 + I_{20} * S4 * SC5$;
$\approx -1.25\times10^{-3} * S4 * S5$.
$a_{35} = I_{15} * C4 * C5 + I_{17} * C4 + I_{22} * S5$;
$\approx 1.25\times10^{-3} * C4 * C5$.
$a_{36} = I_{23} * S4 * S5$; ≈ 0 .
$a_{44} = I_{m4} + I_{14} - I_{20} * SS5$; ≈ 0.20 .
$a_{45} = 0.$
$a_{46} = I_{23} * C5$; ≈ 0 .
$a_{55} = I_{m5} + I_{17}$; ≈ 0.18 .
$a_{56} = 0.$
$a_{66} = I_{m6} + I_{23}$; ≈ 0.19 .

Table A5. The expressions giving the elements of the Coriolis matrix.
(The Abbreviated Expressions have units of kg-m².)

$b_{112} = 2 * \{-I_3 * SC2 + I_5 * C223 + I_7 * SC23 - I_{12} * S223 + I_{15} * (2 * SC23 * C5 + (1 - 2 * SS23) * C4 * S5) + I_{16} * (C223 * C5 - S223 * C4 * S5) + I_{21} * SC23 * CC4 + I_{20} * ((1 + CC4) * SC23 * SS5 - (1 - 2 * SS23) * C4 * SC5) + I_{22} * ((1 - 2 * SS23) * C5 - 2 * SC23 * C4 * S5)\} + I_{10} * (1 - 2 * SS23) + I_{11} * (1 - 2 * SS2)$;
$\approx -2.76 * SC2 + 7.44\times10^{-1} * C223 + 0.60 * SC23 - 2.13\times10^{-2} * (1 - 2 * SS23)$.
$b_{113} = 2 * \{I_5 * C2 * C23 + I_7 * SC23 - I_{12} * C2 * S23 + I_{15} * (2 * SC23 * C5 + (1 - 2 * SS23) * C4 * S5) + I_{16} * C2 * (C23 * C5 - S23 * C4 * S5) + I_{21} * SC23 * CC4 + I_{20} * ((1 + CC4) * SC23 * SS5 - (1 - 2 * SS23) * C4 * SC5) + I_{22} * ((1 - 2 * SS23) * C5 - 2 * SC23 * C4 * S5)\} + I_{10} * (1 - 2 * SS23)$;
$\approx 7.44\times10^{-1} * C2 * C23 + 0.60 * SC23 + 2.20\times10^{-2} * C2 * S23 - 2.13\times10^{-2} * (1 - 2 * SS23)$.
$b_{114} = 2 * \{-I_{15} * SC23 * S4 * S5 - I_{16} * C2 * C23 * S4 * S5 + I_{18} * C4 * S5 - I_{20} * (SS23 * SS5 * SC4 - SC23 * S4 * SC5) - I_{22} * CC23 * S4 * S5 - I_{21} * SS23 * SC4\}$;
$\approx -2.50\times10^{-3} * SC23 * S4 * S5 + 8.60\times10^{-4} * C4 * S5 - 2.48\times10^{-3} * C2 * C23 * S4 * S5$.
$b_{115} = 2 * \{I_{20} * (SC5 * (CC4 * (1 - CC23) - CC23) - SC23 * C4 * (1 - 2 * SS5)) - I_{15} * (SS23 * S5 - SC23 * C4 * C5) - I_{16} * C2 * (S23 * S5 - C23 * C4 * C5) + I_{18} * S4 * C5 + I_{22} * (CC23 * C4 * C5 - SC23 * S5)\}$;
$\approx -2.50\times10^{-3} * (SS23 * S5 - SC23 * C4 * C5) - 2.48\times10^{-3} * C2 * (S23 * S5 - C23 * C4 * C5) + 8.60\times10^{-4} * S4 * C5.$
$b_{116} = 0$.
$b_{123} = 2 * \{-I_8 * S23 + I_{13} * C23 + I_{15} * S23 * S4 * S5 + I_{18} * (C23 * C4 * S5 + S23 * C5) + I_{19} * C23 * SC4 + I_{20} * S4 * (C23 * C4 * CC5 - S23 * SC5) + I_{22} * C23 * S4 * S5\}$;
$\approx 2.67\times10^{-1} * S23 - 7.58\times10^{-3} * C23$.
$b_{124} = -I_{18} * 2 * S23 * S4 * S5 + I_{19} * S23 * (1 - (2 * SS4)) + I_{20} * S23 * (1 - 2 * SS4 * CC5) - I_{14} * S23$; ≈ 0 .
$b_{125} = I_{17} * C23 * S4 + I_{18} * 2 * (S23 * C4 * C5 + C23 * S5) + I_{20} * S4 * (C23 * (1 - 2 * SS5) - S23 * C4 * 2 * SC5)$;
≈ 0 .
$b_{126} = -I_{23} * (S23 * C5 + C23 * C4 * S5)$; ≈ 0 .
$b_{134} = b_{124}$. $b_{135} = b_{125}$. $b_{136} = b_{126}$.
$b_{145} = 2 * \{I_{15} * S23 * C4 * C5 + I_{16} * C2 * C4 * C5 + I_{18} * C23 * S4 * C5 + I_{22} * C23 * C4 * C5\} + I_{17} * S23 * C4 - I_{20} * (S23 * C4 * (1 - 2 * SS5) + 2 * C23 * SC5)$;
≈ 0 .
$b_{146} = I_{23} * S23 * S4 * S5$; ≈ 0 .
$b_{156} = -I_{23} * (C23 * S5 + S23 * C4 * C5)$; ≈ 0 .
$b_{212} = 0$. $b_{213} = 0$.
$b_{214} = I_{14} * S23 + I_{19} * S23 * (1 - (2 * SS4)) + 2 * \{-I_{15} * C23 * C4 * S5 + I_{16} * S2 * C4 * S5 + I_{20} * (S23 * (CC5 * CC4 - 0.5) + C23 * C4 * SC5) + I_{22} * S23 * C4 * S5\}$;
$\approx 1.64\times10^{-3} * S23 - 2.50\times10^{-3} * C23 * C4 * S5 + 2.48\times10^{-3} * S2 * C4 * S5 + 0.30\times10^{-3} * S23 * (1 - (2 * SS4))$.
$b_{215} = 2 * \{-I_{15} * C23 * S4 * C5 + I_{22} * S23 * S4 * C5 + I_{16} * S2 * S4 * C5\} - I_{17} * C23 * S4 + I_{20} * (C23 * S4 * (1 - 2 * SS5) - 2 * S23 * SC4 * SC5)$;
$\approx -2.50\times10^{-3} * C23 * S4 * C5 + 2.48\times10^{-3} * S2 * S4 * C5 - 6.42\times10^{-4} * C23 * S4$.
$b_{216} = -b_{126}$.
$b_{223} = 2 * \{-I_{12} * S3 + I_5 * C3 + I_{16} * (C3 * C5 - S3 * C4 * S5)\}$;
$\approx 2.20\times10^{-2} * S3 + 7.44\times10^{-1} * C3$.

Kinetic Energy

- Consider a point mass $\mathtt{\bar{m}}$ with {s}-frame coordinate $p(t)$, its kinetic energy is given by

$$\mathcal{K} = \frac{1}{2}\mathtt{\bar{m}}\|\dot{p}\|^2$$

- Note: m denotes *moment* (vector) and $\mathtt{\bar{m}}$ denotes *mass* (scalar).

- Question: given a moving rigid body with configuration $T(t) = (R(t), p(t))$, how to compute its kinetic energy?
  - Rigid body consists of infinitely many "particles" with different {s}-frame velocities

$$\mathcal{K} \approx \frac{1}{2}\sum_i \mathtt{\bar{m}}_i\|\dot{p}_i\|^2$$

  - Velocities of particles $\dot{p}_i$ are caused by the rigid body velocity (twist)
  - The overall kinetic energy should depend on the rigid body velocity as well as the geometry and mass distribution of the body

Recall: Rigid Body Velocity

Given rigid body $T(t) = (R(t), p(t))$:

- Spatial twist: $\mathcal{V}_s = (\omega_s, v_s)$, $[\omega_s] = \dot{R}R^T$, $v_s = \dot{p} - \omega_s \times p$

- Body twist: $\mathcal{V}_b = (\omega_b, v_b)$, $[\omega_b] = R^T\dot{R}$, $v_b = R^T\dot{p}$

$$\omega_b = R^T\omega_s$$

Recall: Rigid Body Velocity

- Consider a particle i on the body with $\{b\}$-frame coordinate r_i and $\{s\}$-frame coordinate p_i

$p_i = R r_i + p$

  - Velocity of particle i:

$v_{s,i} = \dot{p}_i$

$v_{b,i} = R^T v_{s,i} = R^T \dot{p}_i$

we also know its relation to twist:

$v_{s,i} = \dot{p}_i = \omega_s \times p_i + v_s$

$v_{b,i} = \omega_b \times r_i + v_b$

  - Acceleration of particle i:

$a_{s,i} = \ddot{p}_i$

$a_{b,i} = R^T a_{s,i} = R^T \ddot{p}_i$

  - Velocity of the origin of $\{b\}$:

$\{s\}$-frame: $\dot{p} \neq v_s$

Let $r_i = 0 \Rightarrow p_i = p$

$v_{s,i} = \dot{p}_i = \dot{p}$

$v_{b,i} = R^T \dot{p} = v_b$

Rigid Body Kinetic Energy

- **Kinetic Energy:** Given a rigid body $T(t) = (R(t), p(t))$ with body twist $\mathcal{V}_b = (\omega_b, v_b)$. Suppose the {b}-frame origin coincides with the center of mass of the body. Then its kinetic energy is given by:

$$\mathcal{K} = \frac{1}{2}\bar{\mathtt{m}}\|v_b\|^2 + \frac{1}{2}\omega_b^T \mathcal{I}_b \omega_b, \qquad \text{with } \mathcal{I}_b = \int_{\mathcal{B}} \rho(r)[r]^T[r]dV \in \mathbb{R}^{3\times 3}$$

where $\mathcal{I}_b$ is the **rotational inertia matrix** in body frame

Derivation: Divide the body into small point masses, where point i has mass $\bar{\mathtt{m}}_i$, {b}-frame coordinate r_i, and {s}-frame coordinate p_i

origin of {b} = CoM $\iff$ $\sum \bar{m}_i r_i = 0$

$p_i = R r_i + p$

$$\mathcal{K} = \frac{1}{2}\sum_i \bar{m}_i \|\dot{p}_i\|^2 = \frac{1}{2}\sum \bar{m}_i \|\dot{p} + \dot{R} r_i\|^2 = \frac{1}{2}\sum \bar{m}_i \left(\|\dot{p}\|^2 + \|R[w_b]r_i\|^2 + 2\dot{p}^T(\dot{R}r_i)\right)$$

$\dot{R} = R[w_b]$

$$\mathcal{K} = \frac{1}{2} v_b^T \bar{m} v_b + \frac{1}{2} w_b^T \mathcal{I}_b w_b$$

term 2: $\|R[w_b]r_i\|^2$

$= \|R(-r_i)\times w_b\|^2 = \|R[r_i]w_b\|^2$

Derivation of Kinetic Energy (Continued)

- $\mathcal{K} = \text{term } 1 + \text{term } 2 + \text{term } 3$

$$\text{term } 3 = \frac{1}{2}\sum_i \bar{m}_i \, 2\dot{p}^T(\dot{R} r_i) = \frac{1}{2}\cdot 2 \cdot \dot{p}^T \dot{R} \sum \cancel{\bar{m}_i r_i} = 0$$

$$\text{term } 2 = \frac{1}{2}\sum \bar{m}_i \left(\omega_b^T [r_i]^T \underline{R^T \cdot R} [r_i] \omega_b\right) = \frac{1}{2}\omega_b^T \underbrace{\left(\sum_i \bar{m}_i [r_i]^T [r_i]\right)}_{\mathcal{I}_b} \omega_b$$

$$\text{term } 1 = \frac{1}{2}\sum \bar{m}_i \|\dot{p}\|^2 = \frac{1}{2}\left(\sum m_i\right) \| R v_b\|^2 = \frac{1}{2}\bar{M}\|v_b\|^2$$

$\Rightarrow$ desired result follows provided we have

$$\mathcal{I}_b \approx \sum_i \bar{m}_i [r_i]^T [r_i]$$

$$= \int_{\mathcal{B}} \rho(r) [r]^T [r] \, dV$$

$$= -\int_{\mathcal{B}} \rho(r) [r]^2 \, dV$$

Outline

- Kinetic Energy of a Rigid Body
- Rotational Inertia Matrix
- Newton Euler Equation
- Twist-Wrench Formulation of Rigid-Body Dynamics

Rotational Inertia Matrix in Body Frame

$$\mathcal{I}_b \triangleq \int_{\mathcal{B}} \rho(r)[r]^T[r]dV$$

positive semidefinite matrix

$\underset{\sim}{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- Individual entries of $\mathcal{I}_b$:

$$\mathcal{I}_b = \begin{bmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{yx} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{zx} & \mathcal{I}_{zy} & \mathcal{I}_{zz} \end{bmatrix}$$

where

$$\mathcal{I}_{xx} = \int_{\mathcal{B}} (y^2+z^2)\rho(x,y,z)dV, \quad \mathcal{I}_{yy} = \int_{\mathcal{B}} (x^2+z^2)\rho(x,y,z)dV$$

$$\mathcal{I}_{zz} = \int_{\mathcal{B}} (x^2+y^2)\rho(x,y,z)dV, \quad \mathcal{I}_{xy} = \mathcal{I}_{yx} = -\int_{\mathcal{B}} xy\rho(x,y,z)dV$$

$$\mathcal{I}_{xz} = \mathcal{I}_{zx} = -\int_{\mathcal{B}} xz\rho(x,y,z)dV \quad \mathcal{I}_{yz} = \mathcal{I}_{zy} = -\int_{\mathcal{B}} yz\rho(x,y,z)dV$$

$\rho(r) \equiv 1$

- If the body has a uniform density, then $\mathcal{I}_b$ is determined exclusively by the shape of the rigid body

Principal Axes of Inertia

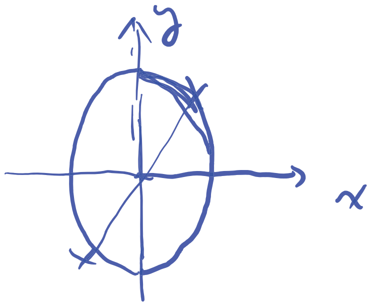
Let v_1, v_2, v_3 and $\lambda_1, \lambda_2, \lambda_3$ be the eigenvectors and eigenvalues of $\mathcal{I}_b$, respectively. They are called the **principal axes of inertia**

- The principal axes of inertia are in the directions of v_1, v_2, v_3

- The principal moments of inertia about these axes are $\lambda_1, \lambda_2, \lambda_3$

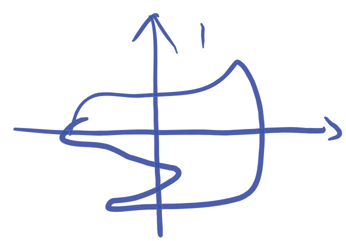
- All the eigenvalues are nonnegative. The largest one maximizes the moment of inertia among all the axes passing through the center of mass of the body.

- If the principal axes of inertia are aligned with the axes of $\{b\}$, the off-diagonal terms of $\mathcal{I}_b$ are all zero.

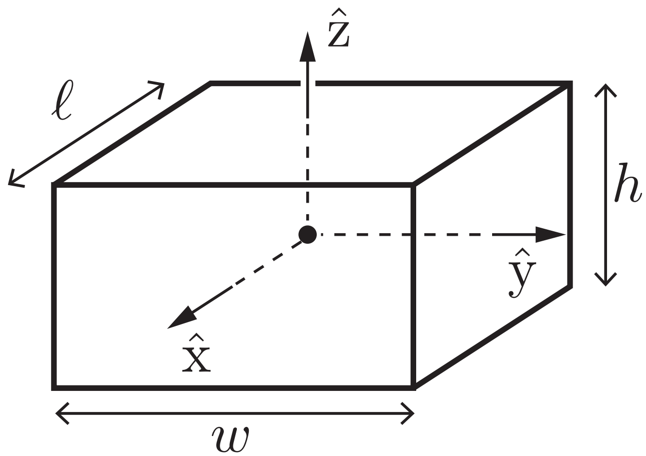


$\rho(r) = 1$

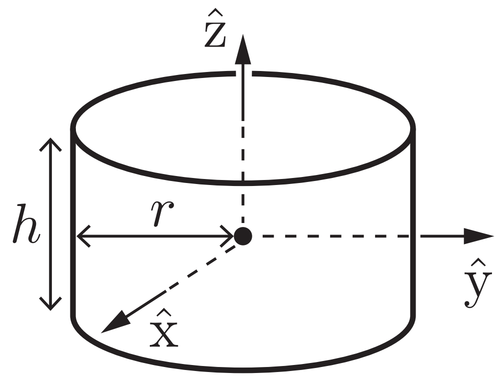
$\Rightarrow \quad I_{xy} = -\int_B xy \, dv$



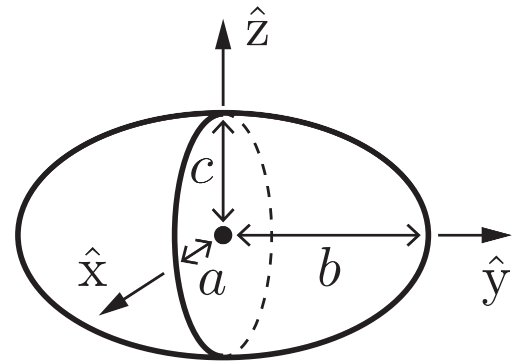
Examples of Inertia Matrix



rectangular parallelepiped:
volume $= abc$,
$\mathcal{I}_{xx} = \mathfrak{m}(w^2 + h^2)/12$,
$\mathcal{I}_{yy} = \mathfrak{m}(\ell^2 + h^2)/12$,
$\mathcal{I}_{zz} = \mathfrak{m}(\ell^2 + w^2)/12$



circular cylinder:
volume $= \pi r^2 h$,
$\mathcal{I}_{xx} = \mathfrak{m}(3r^2 + h^2)/12$,
$\mathcal{I}_{yy} = \mathfrak{m}(3r^2 + h^2)/12$,
$\mathcal{I}_{zz} = \mathfrak{m}r^2/2$



ellipsoid:
volume $= 4\pi abc/3$,
$\mathcal{I}_{xx} = \mathfrak{m}(b^2 + c^2)/5$,
$\mathcal{I}_{yy} = \mathfrak{m}(a^2 + c^2)/5$,
$\mathcal{I}_{zz} = \mathfrak{m}(a^2 + b^2)/5$

The principal axes and the inertia about the principal axes for uniform-density bodies

Inertia Matrix in a Different Frame

- Consider another frame $\{c\}$ with relative orientation R_{bc}
- The origin of both frames is located at the CoM of the body. The rotational inertia matrix in $\{c\}$ frame is defined as $\mathcal{I}_c = \int_{\mathcal{B}} \rho(r_c)[r_c]^T[r_c]dV$
- Kinetic energy is independent of reference frames $\Rightarrow \mathcal{I}_c = R_{bc}^T \mathcal{I}_b R_{bc}$

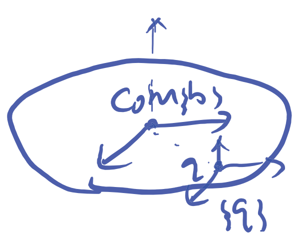
$$K = \frac{1}{2} w_c^T \mathcal{I}_c w_c = \frac{1}{2} w_b^T \mathcal{I}_b w_b = \frac{1}{2} (R_{bc} w_c)^T \mathcal{I}_b (R_{bc} w_c)$$

$$= \frac{1}{2} w_c^T R_{bc}^T \mathcal{I}_b R_{bc} w_c$$

$$\Rightarrow \quad \mathcal{I}_c = R_{bc}^T \mathcal{I}_b R_{bc} = R_{cb} \mathcal{I}_b R_{cb}^T$$

- **Steiner's Theorem:** The inertia matrix I_q about a frame aligned with $\{b\}$, but at a point $q = (q_x, q_y, q_z)$ in $\{b\}$, is related to $\mathcal{I}_b$ by

$R_{qb} = I$



$$\mathcal{I}_q = \mathcal{I}_b + \bar{\mathtt{m}} \left(\underbrace{q^T q}_{\text{scalar} = \|q\|^2} \overset{3\times 3}{I} - qq^T \right) \longrightarrow \text{matrix} \in \mathbb{R}^{3\times 3}$$

Outline

- Kinetic Energy of a Rigid Body
- Rotational Inertia Matrix
- Newton Euler Equation
- Twist-Wrench Formulation of Rigid-Body Dynamics

Newton Euler Equation

- Recall that for a point mass $\mathtt{\bar{m}}$ with a fixed-frame coordinate $p(t)$, Newton's second law of motion: $f = \mathtt{\bar{m}}\ddot{p}(t)$
- A rigid body consists of infinitely many point masses. The collective motion of these particles depend on the linear and rotational velocities of the body, and on the total force and moment acting on the body.
- **Euler-Newton Equation of Motion:** Given rigid body $T(t) = (R(t), p(t))$ with rotational inertia matrix $\mathcal{I}_b$ and body twist $\mathcal{V}_b = (\omega_b, v_b)$:

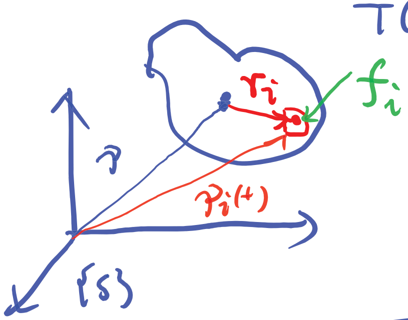
$$\begin{cases} m_b = \mathcal{I}_b \dot{\omega}_b + \omega_b \times \mathcal{I}_b \omega_b \\ f_b = \mathtt{\bar{m}}\dot{v}_b + \omega_b \times \mathtt{\bar{m}} v_b \end{cases} \tag{1}$$

  - $\mathtt{\bar{m}}$: mass of the body; assume origin of $\{b\}$ = CoM
  - f_b, m_b: total force and moment (expressed in $\{b\}$) acting on the body
  - $\mathtt{\bar{m}} v_b$: is the **linear momentum** of the body
  - $\mathcal{I}_b \omega_b$: is the **angular momentum** of the body

Derivation of Newton Euler Equation

$T(t)=(R(t),p(t))$

Divide the body into point masses.



$R^T\dot R = [\omega_b]$

$\dot R = R[\omega_b]$

$v_b = R^T\dot p$

→ mass is $\bar m_i$

$\dot p_i = \dot R r_i + \dot p = R[\omega_b]r_i + Rv_b$

point mass i	$\{s\}$-frame	$\{b\}$-frame
position:	$p_i(t)=R(t)r_i+p(t)$	r_i
velocity:	$v_{s,i}=\dot p_i$	$v_{b,i}=R^T\dot p_i=R^T v_{s,i}$
accelleration:	$a_{s,i}=\ddot p_i$	$a_{b,i}=R^T a_{s,i}=R^T\ddot p_i$
force:	$f_{s,i}=Rf_{b,i}$	$f_{b,i}=R^T f_{s,i}$
moment:	$m_{s,i}=p_i\times f_{s,i}$	$m_{b,i}=r_i\times f_{b,i}$

Whole body: $\bar m=\sum \bar m_i$, CoM $=\sum r_i\bar m_i=0$

total force: $f_b=\sum f_{b,i}$

$m_b=\sum m_{b,i}$

Derivation of Newton Euler Equation (Continued...)

We know Newton's Law: $f_{s,i} = \bar{m}_i \ddot{p}_i = \bar{m}_i \left[(R[\omega_b])' r_i + (R v_b)' \right]$

$$= \bar{m}_i \dot{R}[\omega_b] r_i + \bar{m}_i R[\dot{\omega}_b] r_i + \bar{m}_i \dot{R} v_b + \bar{m}_i R \dot{v}_b$$

$$f_{b,i} = R^T f_{s,i} = \bar{m}_i \Big\{ \underbrace{R^T\dot{R}}_{[\omega_b]}[\omega_b] r_i + R^T R [\dot{\omega}_b] r_i + \underbrace{R^T\dot{R}}_{[\omega_b]} v_b + R^T R \dot{v}_b \Big\}$$

$$\Rightarrow \boxed{f_{b,i} = \bar{m}_i \left([\omega_b]^2 r_i + [\dot{\omega}_b] r_i + [\omega_b] v_b + \dot{v}_b \right)}$$

same Newton's law in body frame

$$\Rightarrow f_b = \sum f_{b,i} = \underbrace{\sum \bar{m}_i [\omega_b]^2 r_i}_{= [\omega_b]^2 \cancel{\sum \bar{m}_i r_i}\ (0)} + \underbrace{\sum \bar{m}_i [\dot{\omega}_b] r_i}_{[\dot{\omega}_b] \sum \bar{m}_i r_i = 0} + \sum \bar{m}_i [\omega_b] v_b + \sum \bar{m}_i \dot{v}_b$$

$$\boxed{= \bar{m} [\omega_b] v_b + \bar{m} \dot{v}_b = \bar{m} \dot{v}_b + \omega_b \times (\bar{m} v_b)}$$

Now consider the rotational dynamics:

$$m_b = \sum m_{b,i} = \sum (r_i \times f_{b,i}) = \sum [r_i]\, \bar{m}_i \left([w_b]^2 r_i + [\dot{w}_b] r_i + [w_b] v_b + \dot{v}_b \right)$$

$$= \sum \bar{m}_i \left([r_i][w_b]^2 r_i + [r_i][\dot{w}_b] r_i \right) + \underbrace{\sum \bar{m}_i [r_i]}_{0} [w_b] v_b + \underbrace{\sum \bar{m}_i [r_i]}_{0} \dot{v}_b$$

$[r_i][w_b][w_b] r_i$

$$= \sum \bar{m}_i \left(-[r_i]^2 \dot{w}_b - [r_i][w_b][r_i] w_b \right)$$

$$= \Big(\underbrace{\sum \bar{m}_i [r_i]^T [r_i]}_{I_b} \Big) \dot{w}_b + [w_b] \Big(\underbrace{\sum \bar{m}_i [r_i]^T [r_i]}_{I_b} \Big) w_b$$

$$= I_b \dot{w}_b + w_b \times I_b w_b$$

Note: $a \times b = -b \times a$

$[a] b = -[b] a$

$[a]^T [b]^T = [a][b]$

$\Downarrow$

$([b][a])^T$

if $\sum \bar{m}_i r_i = 0$

$\Rightarrow \sum \bar{m}_i [r_i] = 0$

$[\alpha a + \beta b] = \alpha [a] + \beta [b]$

α, β: scalars

Fact: $[r_i \times w_b] = [r_i][w_b] - [w_b][r_i]$

(HW0: prob 6)

Outline

$$= \left([r_i \times \omega_b] + [\omega_b][r_i] \right) \cdot [r_i]\, \omega_b$$

$$= \underbrace{\cancel{[r_i \times \omega_b] \cdot (\underbrace{r_i}_{\in \mathbb{R}^3} \times \omega_b)}}_{0} + [\omega_b][r_i]^2\, \omega_b$$

$$= -[\omega_b][r_i]^T[r_i]\, \omega_b$$

- Kinetic Energy of a Rigid Body
- Rotational Inertia Matrix
- Newton Euler Equation
- Twist-Wrench Formulation of Rigid-Body Dynamics

Lie Bracket

- **Lie Bracket:** Given two twists $\mathcal{V}_1 = (\omega_1, v_1)$ and $\mathcal{V}_2 = (\omega_2, v_2)$, the Lie bracket of $\mathcal{V}_1$ and $\mathcal{V}_2$, written as $[\mathrm{ad}_{\mathcal{V}_1}]\,\mathcal{V}_2$, is defined as follows:

$$[\mathrm{ad}_{\mathcal{V}_1}]\,\mathcal{V}_2 = \begin{bmatrix} [\omega_1] & 0 \\ [v_1] & [\omega_1] \end{bmatrix} \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} \in \mathbb{R}^6$$

where $[\mathrm{ad}_{\mathcal{V}}] \triangleq \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix}$ for any $\mathcal{V} = (\omega, v) \in se(3)$

- Lie Bracket can be viewed as a generalization of the cross-product operation of two vectors to two twists

$$\omega_1 \times \omega_2 = [\omega_1]\,\omega_2 \in \mathbb{R}^3, \quad \xrightarrow{\text{generalize}} \quad \text{“}\mathcal{V}_1 \times \mathcal{V}_2\text{''} = [\mathrm{ad}_{\mathcal{V}_1}]\,\mathcal{V}_2 \in \mathbb{R}^6$$

- Given a twist $\mathcal{V} = (\omega, v)$ and a wrench $\mathcal{F} = (m, f)$, we define the mapping:

$$[\mathrm{ad}_{\mathcal{V}}]^T \mathcal{F} = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix}^T \begin{bmatrix} m \\ f \end{bmatrix}$$

Twist-Wrench Formulation

- Rigid body with body twist $\mathcal{V}_b = (\omega_b, v_b)$ and body wrench $\mathcal{F}_b = (m_b, f_b)$

$\in \mathbb{R}^{3\times 3}$

- Spatial inertia matrix $\mathcal{G}_b \in \mathbb{R}^{6\times 6}$: $\mathcal{G}_b = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \bar{m} I \end{bmatrix}$

3×3 identity matrix

- Spatial momentum $\mathcal{P}_b \in \mathbb{R}^6$: $\mathcal{P}_b = \mathcal{G}_b \mathcal{V}_b$

momentum: $\mathcal{P}_b = \begin{bmatrix} \mathcal{I}_b \omega_b \\ \bar{m} v_b \end{bmatrix}$

- Kinetic energy: $\mathcal{K} = \frac{1}{2}\mathcal{V}_b^T \mathcal{G}_b \mathcal{V}_b$

$$\mathcal{K} = \frac{1}{2}\bar{m}\|v_b\|^2 + \frac{1}{2}\omega_b^T \mathcal{I}_b \omega_b$$

$$= \frac{1}{2} v_b^T \bar{m} I v_b + \frac{1}{2}\omega_b^T \mathcal{I}_b \omega_b = \frac{1}{2}\begin{bmatrix} \omega_b \\ v_b \end{bmatrix}^T \begin{bmatrix} \mathcal{I}_b & \\ & \bar{m} I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

Twist-Wrench Formulation

- Newton-Euler Equation (1) can be written in twist-wrench form:

$$\boxed{\mathcal{F}_b = \mathcal{G}_b\dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^T\mathcal{P}_b = \underline{\mathcal{G}_b\dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^T\mathcal{G}_b\mathcal{V}_b}}$$

rotational dynamics: $m_b = \mathcal{I}_b\dot{w}_b + w_b \times \mathcal{I}_b w_b$

$= \mathcal{I}_b\dot{w}_b - [w_b]^T\mathcal{I}_b w_b$

$$\begin{cases} m_b = \mathcal{I}_b\dot{w}_b + w_b \times \mathcal{I}_b w_b = \mathcal{I}_b\dot{w}_b + [w_b]\mathcal{I}_b w_b + [v_b]\bar{m}I v_b \\ f_b = \bar{m}\dot{v}_b + w_b \times \bar{m}v_b = \bar{m}\dot{v}_b + [w_b]\bar{m}v_b \end{cases}$$

$= \bar{m}[v_b]v_b = 0$

$$\Rightarrow \mathcal{F}_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \bar{m}I \end{bmatrix}\begin{bmatrix} \dot{w}_b \\ \dot{v}_b \end{bmatrix} + \begin{bmatrix} [w_b] & [v_b] \\ 0 & [w_b] \end{bmatrix}\begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \bar{m}I \end{bmatrix}\begin{bmatrix} w_b \\ v_b \end{bmatrix}$$

$$= \mathcal{G}_b\dot{\mathcal{V}}_b - \begin{bmatrix} [w_b] & 0 \\ [v_b] & [w_b] \end{bmatrix}^T \mathcal{G}_b\mathcal{V}_b$$

$= [\mathrm{ad}_{\mathcal{V}_b}]$ different from $[\mathrm{Ad}_T]$

Dynamics in Other Frames

- Our derivation of dynamics relies on using CoM {b} frame. We can also write dynamics in another frame, say {a}, with relative configuration T_{ba}
- Kinetic energy is independent of reference frame: $\frac{1}{2}\mathcal{V}_b^T \mathcal{G}_b \mathcal{V}_b = \frac{1}{2}\mathcal{V}_a^T \mathcal{G}_a \mathcal{V}_a$
- This implies that the spatial inertia matrix $\mathcal{G}_a$ is related to $\mathcal{G}_b$ by

$$\mathcal{G}_a = [\mathrm{Ad}_{T_{ba}}]^T \mathcal{G}_b [\mathrm{Ad}_{T_{ba}}]$$

$\frac{1}{2}\mathcal{V}_b^T G_b \mathcal{V}_b$ we $\underline{\mathcal{V}_b = [\mathrm{Ad}_{T_{ba}}]\mathcal{V}_a}$

$= \frac{1}{2}\mathcal{V}_a^T \underbrace{[\mathrm{Ad}_{T_{ba}}]^T G_b [\mathrm{Ad}_{T_{ba}}]}_{= G_a} \mathcal{V}_a$

- One can show the Newton-Euler equation (1) can be written equivalently in frame {a} as:

$$\mathcal{F}_a = \mathcal{G}_a \dot{\mathcal{V}}_a - [\mathrm{ad}_{\mathcal{V}_a}]^T \mathcal{G}_a \mathcal{V}_a$$

the form of the dynamic equation does not change.

More Discussions

-

More Discussions

-