ECE5463: Introduction to Robotics

Lecture Note 12: Dynamics of Open Chains: Lagrangian Formulation

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Outline

- Introduction
- Euler-Lagrange Equations
- Lagrangian Formulation of Open-Chain Dynamics

From Single Rigid Body to Open Chains

- Recall Newton-Euler Equation for a single rigid body:

$$\mathcal{F}_b = \mathcal{G}_b \dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^T (\mathcal{G}_b \mathcal{V}_b)$$

- Open chains consist of multiple rigid links connected through joints

- Dynamics of adjacent links are coupled.

- We are concerned with modeling multi-body dynamics subject to constraints.

Preview of Open-Chain Dynamics

- Equations of Motion are a set of 2nd-order differential equations:

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$

  - $\theta \in \mathbb{R}^n$: vector of joint variables; $\tau \in \mathbb{R}^n$: vector of joint forces/torques
  - $M(\theta) \in \mathbb{R}^{n \times n}$: mass matrix
  - $h(\theta, \dot{\theta}) \in \mathbb{R}^n$: forces that lump together centripetal, Coriolis, gravity, and friction terms that depend on θ and $\dot{\theta}$

- **Forward dynamics:** Determine acceleration $\ddot{\theta}$ given the state $(\theta, \dot{\theta})$ and the joint forces/torques:

$$\ddot{\theta} = M^{-1}(\theta)(\tau - h(\theta, \dot{\theta}))$$

- **Inverse dynamics:** Finding torques/forces given state $(\theta, \dot{\theta})$ and desired acceleration $\ddot{\theta}$

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$

Lagrangian vs. Newton-Euler Methods

- There are typically two ways to derive the equation of motion for an open-chain robot: Lagrangian method and Newton-Euler method

Lagrangian Formulation

- Energy-based method
- Dynamic equations in closed form
- Often used for study of dynamic properties and analysis of control methods

Newton-Euler Formulation

- Balance of forces/torques
- Dynamic equations in numeric/recursive form
- Often used for numerical solution of forward/inverse dynamics

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- Euler-Lagrange Equations
- Lagrangian Formulation of Open-Chain Dynamics

Generalized Coordinates and Forces

- Consider k particles. Let f_i be the force acting on the ith particle, $\bar{\mathtt{m}}_i$ be its mass, p_i be its position. Newton's law: $f_i = \bar{\mathtt{m}}_i \ddot{p}_i, \quad i = 1, \ldots k$

- Now consider the case in which some particles are rigidly connected, imposing constraints on their positions

$$\alpha_j(p_1, \ldots, p_k) = 0, \quad j = 1, \ldots, n_c$$

- k particles in $\mathbb{R}^3$ under n_c constraints $\Rightarrow$ $3k - n_c$ degree of freedom

- Dynamics of this constrained k-particle system can be represented by $n \triangleq 3k - n_c$ independent variables q_i's, called the **generalized coordinates**

$$\begin{cases} \alpha_j(p_1, \ldots, p_k) = 0 \\ j = 1, \ldots, n_c \end{cases} \quad \Leftrightarrow \quad \begin{cases} p_i = \gamma_i(q_1, \ldots, q_n) \\ i = 1, \ldots, k \end{cases}$$

Generalized Coordinates and Forces

- To describe equation of motion in terms of generalized coordinates, we also need to express external forces applied to the system in terms components along generalized coordinates. These "forces" are called **generalized forces**.

- Generalized force f_i and coordinate rate $\dot{q}_i$ are dual to each other in the sense that $f^T \dot{q}$ corresponds to power

- The equation of motion of the k-particle system can thus be described in terms of $3k - n_c$ independent variables instead of the $3k$ position variables subject to n_c constraints.

- This idea of handling constraints can be extended to interconnected rigid bodies (open chains).

Euler-Lagrange Equation

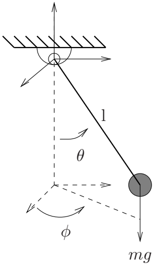
- Now let $q \in \mathbb{R}^n$ be the generalized coordinates and $f \in \mathbb{R}^n$ be the generalized forces of some constrained dynamical system.

- **Lagrangian function:** $\mathcal{L}(q, \dot{q}) = \mathcal{K}(q, \dot{q}) - \mathcal{P}(q)$
  - $\mathcal{K}(q, \dot{q})$: kinetic energy of system
  - $\mathcal{P}(q)$: potential energy

- **Euler-Lagrange Equations**:

$$f = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} \tag{1}$$

Example: Spherical Pendulum



Example: Spherical Pendulum (Continued)

Outline

- Introduction
- Euler-Lagrange Equations
- Lagrangian Formulation of Open-Chain Dynamics

Lagrangian Formulation of Open Chains

- For open chains with n joints, it is convenient and always possible to choose the joint angles $\theta = (\theta_1, \ldots, \theta_n)$ and the joint torques $\tau = (\tau_1, \ldots, \tau_n)$ as the generalized coordinates and generalized forces, respectively.
  - If joint i is revolute: θ_i joint angle and τ_i is joint torque
  - If joint i is prismatic: θ_i joint position and τ_i is joint force
- Lagrangian function: $\mathcal{L}(\theta, \dot{\theta}) = \mathcal{K}(\theta, \dot{\theta}) - \mathcal{P}(\theta, \dot{\theta})$
- Dynamic Equations:

$$\tau_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} - \frac{\partial \mathcal{L}}{\partial \theta_i}$$

- To obtain the Lagrangian dynamics, we need to derive the kinetic and potential energies of the robot in terms of joint angles θ and torques τ.

Some Notations

For each link $i = 1, \ldots, n$, Frame $\{\text{i}\}$ is attached to the center of mass of link i. All the following quantities are expressed in frame $\{\text{i}\}$

- $\mathcal{V}_i$: Twist of link $\{\text{i}\}$
- $\bar{\mathtt{m}}_i$: mass; $\mathcal{I}_i$: rotational inertia matrix;
- $G_i = \begin{bmatrix} \mathcal{I}_i & 0 \\ 0 & \bar{\mathtt{m}}_i I \end{bmatrix}$: Spatial inertia matrix
- Kinetic energy of link i: $\mathcal{K}_i = \frac{1}{2}\mathcal{V}_i^T \mathcal{G}_i \mathcal{V}_i$
- $J_b^{(i)} \in \mathbb{R}^{6\times i}$: body Jacobian of link i

$$J_b^{(i)} = \begin{bmatrix} J_{b,1}^{(i)} & \ldots & J_{b,i}^{(i)} \end{bmatrix}$$

where $J_{b,j}^{(i)} = \left[\mathrm{Ad}_{e^{-[\mathcal{B}_i]\theta_i} \cdots e^{-[\mathcal{B}_{j+1}]\theta_{j+1}}}\right] \mathcal{B}_j$, $j < i$ and $J_{b,i}^{(i)} = \mathcal{B}_i$

Kinetic and Potential Energies of Open Chains

- $J_{ib} = [J_b^{(i)} \quad 0] \in \mathbb{R}^{6\times n}$

- Total Kinetic Energy:

$$\mathcal{K}(\theta,\dot{\theta}) = \frac{1}{2}\sum\nolimits_{i=1}^{n} \mathcal{V}_i^T \mathcal{G}_i \mathcal{V}_i = \frac{1}{2}\dot{\theta}^T \left(\sum\nolimits_{i=1}^{n} \left(J_{ib}^T(\theta)\mathcal{G}_i J_{ib}(\theta)\right)\right)\dot{\theta} \triangleq \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta}$$

- Potential Energy:

$$\mathcal{P}(\theta) = \sum\nolimits_{i=1}^{n} \bar{\mathtt{m}}_i \mathtt{g} \mathtt{h}_i(\theta)$$

  - $\mathtt{h}_i(\theta)$: height of CoM of link i

Lagrangian Dynamic Equations of Open Chains

- Lagrangian: $\mathcal{L}(\theta, \dot{\theta}) = \mathcal{K}(\theta, \dot{\theta}) - \mathcal{P}(\theta)$

- $\tau_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} - \frac{\partial \mathcal{L}}{\partial \theta_i} \Rightarrow$

$$\tau_i = \sum_{i=j}^{n} M_{ij}(\theta)\ddot{\theta}_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\Gamma_{ijk}(\theta)\dot{\theta}_j\dot{\theta}_k + \frac{\partial \mathcal{P}}{\partial \theta_i}, \quad i = 1, \ldots, n$$

- $\Gamma_{ijk}(\theta)$ is called the **Christoffel symbols of the first kind**

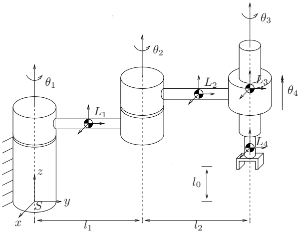
$$\Gamma_{ijk}(\theta) = \frac{1}{2}\left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{jk}}{\partial \theta_i}\right)$$

Lagrangian Dynamic Equations of Open Chains

- Dynamic equation in vector form:

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta)$$

  - $C_{ij}(\theta, \dot{\theta}) \triangleq \sum_{k=1}^{n} \Gamma_{ijk}\dot{\theta}_k$ is called the **Coriolis matrix**



$$\begin{aligned}
M_{11} &= I_{y2}s_2^2 + I_{y3}s_{23}^2 + I_{z1} + I_{z2}c_2^2 + I_{z3}c_{23}^2 \\
&\quad + m_2 r_1^2 c_2^2 + m_3(l_1c_2 + r_2c_{23})^2 \\
M_{12} &= 0 \\
M_{13} &= 0 \\
M_{21} &= 0 \\
M_{22} &= I_{x2} + I_{x3} + m_3l_1^2 + m_2r_1^2 + m_3r_2^2 + 2m_3l_1r_2c_3 \\
M_{23} &= I_{x3} + m_3r_2^2 + m_3l_1r_2c_3 \\
M_{31} &= 0 \\
M_{32} &= I_{x3} + m_3r_2^2 + m_3l_1r_2c_3 \\
M_{33} &= I_{x3} + m_3r_2^2.
\end{aligned}$$

$$\begin{aligned}
\Gamma_{112} &= (I_{y2} - I_{z2} - m_2r_1^2)c_2s_2 + (I_{y3} - I_{z3})c_{23}s_{23} \\
&\quad - m_3(l_1c_2 + r_2c_{23})(l_1s_2 + r_2s_{23}) \\
\Gamma_{113} &= (I_{y3} - I_{z3})c_{23}s_{23} - m_3r_2s_{23}(l_1c_2 + r_2c_{23}) \\
\Gamma_{121} &= (I_{y2} - I_{z2} - m_2r_1^2)c_2s_2 + (I_{y3} - I_{z3})c_{23}s_{23} \\
&\quad - m_3(l_1c_2 + r_2c_{23})(l_1s_2 + r_2s_{23}) \\
\Gamma_{131} &= (I_{y3} - I_{z3})c_{23}s_{23} - m_3r_2s_{23}(l_1c_2 + r_2c_{23}) \\
\Gamma_{211} &= (I_{z2} - I_{y2} + m_2r_1^2)c_2s_2 + (I_{z3} - I_{y3})c_{23}s_{23} \\
&\quad + m_3(l_1c_2 + r_2c_{23})(l_1s_2 + r_2s_{23}) \\
\Gamma_{223} &= -l_1m_3r_2s_3 \\
\Gamma_{232} &= -l_1m_3r_2s_3 \\
\Gamma_{233} &= -l_1m_3r_2s_3 \\
\Gamma_{311} &= (I_{z3} - I_{y3})c_{23}s_{23} + m_3r_2s_{23}(l_1c_2 + r_2c_{23}) \\
\Gamma_{322} &= l_1m_3r_2s_3
\end{aligned}$$

$$\begin{bmatrix} 0 \\ -(m_2gr_1 + m_3gl_1)\cos\theta_2 - m_3r_2\cos(\theta_2 + \theta_3)) \\ -m_3gr_2\cos(\theta_2 + \theta_3)) \end{bmatrix}$$

Lagrangian Dynamic Equations of Open Chains

- Dynamic model of PUMA 560 Arm:

$I_2 = I_{zz2} + m_2 * (r_{x2}^2 + r_{y2}^2) + (m_3 + m_4 + m_5 + m_6) * a_2^2$;
$I_3 = -I_{xx2} + I_{yy2} + (m_3 + m_4 + m_5 + m_6) * a_2^2 + m_2 * r_{x2}^2 - m_2 * r_{y2}^2$;
$I_4 = m_2 * r_{x2} * (d_2 + r_{z2}) + m_3 * a_2 * r_{z3} + (m_3 + m_4 + m_5 + m_6) * a_2 * (d_2 + d_3)$;
$I_5 = -m_3 * a_2 * r_{y3} + (m_4 + m_5 + m_6) * a_2 * d_4 + m_4 * a_2 * r_{z4}$;
$I_6 = I_{zz3} + m_3 * r_{y3}^2 + m_4 * a_3^2 + m_4 * (d_4 + r_{z4})^2 + I_{yy4} + m_5 * a_3^2 + m_5 * d_4^2 + I_{zz5} + m_6 * a_3^2 + m_6 * d_4^2 + m_6 * r_{z6}^2 + I_{zz6}$;
$I_7 = m_3 * r_{y3}^2 + I_{zz3} - I_{yy3} + m_4 * r_{z4}^2 + 2 * m_4 * d_4 * r_{z4} + (m_4 + m_5 + m_6) * (d_4^2 - a_3^2) + I_{yy4} - I_{zz4} + I_{zz5} - I_{yy5} + m_6 * r_{z6}^2 - I_{zz6} + I_{zz6}$;
$I_8 = -m_4 * (d_2 + d_3) * (d_4 + r_{z4}) - (m_5 + m_6) * (d_2 + d_3) * d_4 + m_3 * r_{y3} * r_{z3} + m_3 * (d_2 + d_3) * r_{y3}$;
$I_9 = m_2 * r_{y2} * (d_2 + r_{z2})$;
$I_{10} = 2 * m_4 * a_3 * r_{z4} + 2 * (m_4 + m_5 + m_6) * a_3 * d_4$;
$I_{11} = -2 * m_2 * r_{x2} * r_{y2}$;
$I_{12} = (m_4 + m_5 + m_6) * a_2 * a_3$;
$I_{13} = (m_4 + m_5 + m_6) * a_3 * (d_2 + d_3)$;
$I_{14} = I_{zz4} + I_{yy5} + I_{zz6}$;
$I_{15} = m_6 * d_4 * r_{z6}$;
$I_{16} = m_6 * a_2 * r_{z6}$;
$I_{17} = I_{zz5} + I_{zz6} + m_6 * r_{z6}^2$;
$I_{18} = m_6 * (d_2 + d_3) * r_{z6}$;
$I_{19} = I_{yy4} - I_{zz4} + I_{zz5} - I_{yy5} + m_6 * r_{z6}^2 + I_{xx6} - I_{zz6}$;
$I_{20} = I_{yy5} - I_{zz5} - m_6 * r_{z6}^2 + I_{zz6} - I_{xx6}$;
$I_{21} = I_{zz4} - I_{yy4} + I_{zz5} - I_{zz5}$;
$I_{22} = m_6 * a_3 * r_{z6}$;
$I_{23} = I_{zz6}$;

Part II. Gravitional Constants

$g_1 = -g * ((m_3 + m_4 + m_5 + m_6) * a_2 + m_2 * r_{x2})$;
$g_2 = g * (m_3 * r_{y3} - (m_4 + m_5 + m_6) * d_4 - m_4 * r_{z4})$;
$g_3 = g * m_2 * r_{y2}$;
$g_4 = -g * (m_4 + m_5 + m_6) * a_3$;
$g_5 = -g * m_6 * r_{z6}$;

Table A3. Computed Values for the Constants Appearing in the Equations of Forces of Motion.
(Inertial constants have units of kilogram meters-squared)

I_1 = 1.43	±0.05	I_2 = 1.75	±0.07
I_3 = 1.38	±0.05	I_4 = 6.90×10^{-1}	$\pm0.20\times10^{-1}$
I_5 = 3.72×10^{-1}	$\pm0.31\times10^{-1}$	I_6 = 3.33×10^{-1}	$\pm0.16\times10^{-1}$
I_7 = 2.98×10^{-1}	$\pm0.29\times10^{-1}$	I_8 = -1.34×10^{-1}	$\pm0.14\times10^{-1}$
I_9 = 2.38×10^{-2}	$\pm1.20\times10^{-2}$	I_{10} = -2.13×10^{-2}	$\pm0.22\times10^{-2}$
I_{11} = -1.42×10^{-2}	$\pm0.70\times10^{-2}$	I_{12} = -1.10×10^{-2}	$\pm0.11\times10^{-2}$
I_{13} = -3.79×10^{-3}	$\pm0.90\times10^{-3}$	I_{14} = 1.64×10^{-3}	$\pm0.07\times10^{-3}$
I_{15} = 1.25×10^{-3}	$\pm0.30\times10^{-3}$	I_{16} = 1.24×10^{-3}	$\pm0.30\times10^{-3}$
I_{17} = 6.42×10^{-4}	$\pm3.00\times10^{-4}$	I_{18} = 4.31×10^{-4}	$\pm1.30\times10^{-4}$
I_{19} = 3.00×10^{-4}	$\pm14.0\times10^{-4}$	I_{20} = -2.02×10^{-4}	$\pm8.00\times10^{-4}$
I_{21} = -1.00×10^{-4}	$\pm6.00\times10^{-4}$	I_{22} = -5.80×10^{-5}	$\pm1.50\times10^{-5}$
I_{23} = 4.00×10^{-5}	$\pm2.00\times10^{-5}$		
I_{m1} = 1.14	±0.27	I_{m2} = 4.71	±0.54
I_{m3} = 8.27×10^{-1}	$\pm0.93\times10^{-1}$	I_{m4} = 2.00×10^{-1}	$\pm0.16\times10^{-1}$
I_{m5} = 1.79×10^{-1}	$\pm0.14\times10^{-1}$	I_{m6} = 1.93×10^{-1}	$\pm0.16\times10^{-1}$

(Gravitational constants have units of newton meters)

g_1 = −37.2	±00.5	g_2 = −8.44	±0.20
g_3 = 1.02	±0.50	g_4 = 2.49×10^{-1}	$\pm0.25\times10^{-1}$
g_5 = -2.82×10^{-2}	$\pm0.56\times10^{-2}$		

Table A4. The expressions giving the elements of the kinetic energy matrix.
(The Abbreviated Expressions have units of kg-m².)

$a_{11} = I_{m1} + I_1 + I_3 * CC2 + I_7 * SS23 + I_{10} * SC23 + I_{11} * SC2 + I_{20} * (SS5 * (SS23 * (1 + CC4) - 1) - 2 * SC23 * C4 * SC5) + I_{21} * SS23 * CC4 + 2 * \{I_5 * C2 * S23 + I_{12} * C2 * C23 + I_{15} * (SS23 * C5 + SC23 * C4 * S5) + I_{16} * C2 * (S23 * C5 + C23 * C4 * S5) + I_{18} * S4 * S5 + I_{22} * (SC23 * C5 + CC23 * C4 * S5)\}$;
$\approx 2.57 + 1.38 * CC2 + 0.30 * SS23 + 7.44\times10^{-1} * C2 * S23$.

$a_{12} = I_4 * S2 + I_8 * C23 + I_9 * C2 + I_{13} * S23 - I_{15} * C23 * S4 * S5 + I_{16} * S2 * S4 * S5 + I_{18} * (S23 * C4 * S5 - C23 * C5) + I_{19} * S23 * SC4 + I_{20} * S4 * (S23 * C4 * CC5 + C23 * SC5) + I_{22} * S23 * S4 * S5$;
$\approx 6.90\times10^{-1} * S2 - 1.34\times10^{-1} * C23 + 2.38\times10^{-2} * C2$.

$a_{13} = I_8 * C23 + I_{13} * S23 - I_{15} * C23 * S4 * S5 + I_{19} * S23 * SC4 + I_{18} * (S23 * C4 * S5 - C23 * C5) + I_{22} * S23 * S4 * S5 + I_{20} * S4 * (S23 * C4 * CC5 + C23 * SC5)$;
$\approx -1.34\times10^{-1} * C23 + -3.97\times10^{-3} * S23$.

$a_{14} = I_{14} * C23 + I_{15} * S23 * C4 * S5 + I_{16} * C2 * C4 * S5 + I_{18} * C23 * S4 * S5 - I_{20} * (S23 * C4 * SC5 + C23 * SS5) + I_{22} * C23 * C4 * S5$; ≈ 0 .

$a_{15} = I_{15} * S23 * S4 * C5 + I_{16} * C2 * S4 * C5 + I_{17} * S23 * S4 + I_{18} * (S23 * S5 - C23 * C4 * C5) + I_{22} * C23 * S4 * C5$;
≈ 0 .

$a_{16} = I_{23} * (C23 * C5 - S23 * C4 * S5)$; ≈ 0 .

$a_{22} = I_{m2} + I_2 + I_6 + I_{20} * SS4 * SS5 + I_{21} * SS4 + 2 * \{I_5 * S3 + I_{12} * C3 + I_{15} * C5 + I_{16} * (S3 * C5 + C3 * C4 * S5) + I_{22} * C4 * S5\}$;
$\approx 6.79 + 7.44\times10^{-1} * S3$.

$a_{23} = I_5 * S3 + I_6 + I_{12} * C3 + I_{16} * (S3 * C5 + C3 * C4 * S5) + I_{20} * SS4 * SS5 + I_{21} * SS4 + 2 * \{I_{15} * C5 + I_{22} * C4 * S5\}$;
$\approx .333 + 3.72\times10^{-1} * S3 - 1.10\times10^{-2} * C3$.

$a_{24} = -I_{15} * S4 * S5 - I_{16} * S3 * S4 * S5 + I_{20} * S4 * SC5$;
≈ 0 .

$a_{25} = I_{15} * C4 * C5 + I_{16} * (C3 * S5 + S3 * C4 * C5) + I_{17} * C4 + I_{22} * S5$; ≈ 0 .

$a_{26} = I_{23} * S4 * S5$; ≈ 0 .

$a_{33} = I_{m3} + I_6 + I_{20} * SS4 * SS5 + I_{21} * SS4 + 2 * \{I_{15} * C5 + I_{22} * C4 * S5\}$; ≈ 1.16 .

$a_{34} = -I_{15} * S4 * S5 + I_{20} * S4 * SC5$;
$\approx -1.25\times10^{-3} * S4 * S5$.

$a_{35} = I_{15} * C4 * C5 + I_{17} * C4 + I_{22} * S5$;
$\approx 1.25\times10^{-3} * C4 * C5$.

$a_{36} = I_{23} * S4 * S5$; ≈ 0 .

$a_{44} = I_{m4} + I_{14} - I_{20} * SS5$; ≈ 0.20 .

$a_{45} = 0$.

$a_{46} = I_{23} * C5$; ≈ 0 .

$a_{55} = I_{m5} + I_{17}$; ≈ 0.18 .

$a_{56} = 0$.

$a_{66} = I_{m6} + I_{23}$; ≈ 0.19 .

Table A5. The expressions giving the elements of the Coriolis matrix.
(The Abbreviated Expressions have units of kg-m².)

$b_{112} = 2 * \{-I_3 * SC2 + I_5 * C223 + I_7 * SC23 - I_{12} * S223 + I_{15} * (2 * SC23 * C5 + (1 - 2 * SS23) * C4 * S5) + I_{16} * (C223 * C5 - S223 * C4 * S5) + I_{21} * SC23 * CC4 + I_{20} * ((1 + CC4) * SC23 * SS5 - (1 - 2 * SS23) * C4 * SC5) + I_{22} * ((1 - 2 * SS23) * C5 - 2 * SC23 * C4 * S5)\} + I_{10} * (1 - 2 * SS23) + I_{11} * (1 - 2 * SS2)$;
$\approx -2.76 * SC2 + 7.44\times10^{-1} * C223 + 0.60 * SC23 - 2.13\times10^{-2} * (1 - 2 * SS23)$.

$b_{113} = 2 * \{I_5 * C2 * C23 + I_7 * SC23 - I_{12} * C2 * S23 + I_{15} * (2 * SC23 * C5 + (1 - 2 * SS23) * C4 * S5) + I_{16} * C2 * (C23 * C5 - S23 * C4 * S5) + I_{21} * SC23 * CC4 + I_{20} * ((1 + CC4) * SC23 * SS5 - (1 - 2 * SS23) * C4 * SC5) + I_{22} * ((1 - 2 * SS23) * C5 - 2 * SC23 * C4 * S5)\} + I_{10} * (1 - 2 * SS23)$;
$\approx 7.44\times10^{-1} * C2 * C23 + 0.60 * SC23 + 2.20\times10^{-2} * C2 * S23 - 2.13\times10^{-2} * (1 - 2 * SS23)$.

$b_{114} = 2 * \{-I_{15} * SC23 * S4 * S5 - I_{16} * C2 * C23 * S4 * S5 + I_{18} * C4 * S5 - I_{20} * (SS23 * SS5 * SC4 - SC23 * S4 * SC5) - I_{22} * CC23 * S4 * S5 - I_{21} * SS23 * SC4\}$;
$\approx -2.50\times10^{-3} * SC23 * S4 * S5 + 8.60\times10^{-4} * C4 * S5 - 2.48\times10^{-3} * C2 * C23 * S4 * S5$.

$b_{115} = 2 * \{I_{20} * (SC5 * (CC4 * (1 - CC23) - CC23) - SC23 * C4 * (1 - 2 * SS5)) - I_{15} * (SS23 * S5 - SC23 * C4 * C5) - I_{16} * C2 * (S23 * S5 - C23 * C4 * C5) + I_{18} * S4 * C5 + I_{22} * (CC23 * C4 * C5 - SC23 * S5)\}$;
$\approx -2.50\times10^{-3} * (SS23 * S5 - SC23 * C4 * C5) - 2.48\times10^{-3} * C2 * (S23 * S5 - C23 * C4 * C5) + 8.60\times10^{-4} * S4 * C5$.

$b_{116} = 0$.

$b_{123} = 2 * \{-I_8 * S23 + I_{13} * C23 + I_{15} * S23 * S4 * S5 + I_{18} * (C23 * C4 * S5 + S23 * C5) + I_{19} * C23 * SC4 + I_{20} * S4 * (C23 * C4 * CC5 - S23 * SC5) + I_{22} * C23 * S4 * S5\}$;
$\approx 2.67\times10^{-1} * S23 - 7.58\times10^{-3} * C23$.

$b_{124} = -I_{18} * 2 * S23 * S4 * S5 + I_{19} * S23 * (1 - (2 * SS4)) + I_{20} * S23 * (1 - 2 * SS4 * CC5) - I_{14} * S23$; ≈ 0 .

$b_{125} = I_{17} * C23 * S4 + I_{18} * 2 * (S23 * C4 * C5 + C23 * S5) + I_{20} * S4 * (C23 * (1 - 2 * SS5) - S23 * C4 * 2 * SC5)$;
≈ 0 .

$b_{126} = -I_{23} * (S23 * C5 + C23 * C4 * S5)$; ≈ 0 .

$b_{134} = b_{124}$. $b_{135} = b_{125}$. $b_{136} = b_{126}$.

$b_{145} = 2 * \{I_{15} * S23 * C4 * C5 + I_{16} * C2 * C4 * C5 + I_{18} * C23 * S4 * C5 + I_{22} * C23 * C4 * C5\} + I_{17} * S23 * C4 - I_{20} * (S23 * C4 * (1 - 2 * SS5) + 2 * C23 * SC5)$;
≈ 0 .

$b_{146} = I_{23} * S23 * S4 * S5$; ≈ 0 .

$b_{156} = -I_{23} * (C23 * S5 + S23 * C4 * C5)$; ≈ 0 .

$b_{212} = 0$. $b_{213} = 0$.

$b_{214} = I_{14} * S23 + I_{19} * S23 * (1 - (2 * SS4)) + 2 * \{-I_{15} * C23 * C4 * S5 + I_{16} * S2 * C4 * S5 + I_{20} * (S23 * (CC5 * CC4 - 0.5) + C23 * C4 * SC5) + I_{22} * S23 * C4 * S5\}$;
$\approx 1.64\times10^{-3} * S23 - 2.50\times10^{-3} * C23 * C4 * S5 + 2.48\times10^{-3} * S2 * C4 * S5 + 0.30\times10^{-3} * S23 * (1 - (2 * SS4))$.

$b_{215} = 2 * \{-I_{15} * C23 * S4 * C5 + I_{22} * S23 * S4 * C5 + I_{16} * S2 * S4 * C5\} - I_{17} * C23 * S4 + I_{20} * (C23 * S4 * (1 - 2 * SS5) - 2 * S23 * SC4 * SC5)$;
$\approx -2.50\times10^{-3} * C23 * S4 * C5 + 2.48\times10^{-3} * S2 * S4 * C5 - 6.42\times10^{-4} * C23 * S4$.

$b_{216} = -b_{126}$.

$b_{223} = 2 * \{-I_{12} * S3 + I_5 * C3 + I_{16} * (C3 * C5 - S3 * C4 * S5)\}$;
$\approx 2.20\times10^{-2} * S3 + 7.44\times10^{-1} * C3$.

$b_{224} = 2 * \{-I_{16} * C3 * S4 * S5 + I_{20} * SC4 * SS5 + I_{21} * SC4 - I_{22} * S4 * S5\}$;
$\approx -2.48\times10^{-3} * C3 * S4 * S5$.

$b_{225} = 2 * \{-I_{15} * S5 + I_{16} * (C3 * C4 * C5 - S3 * S5) + I_{20} * SS4 * SC5 + I_{22} * C4 * C5\}$;
$\approx -2.50\times10^{-3} * S5 + 2.48\times10^{-3} * (C3 * C4 * C5 - S3 * S5)$.

$b_{226} = 0$. $b_{234} = b_{224}$.

$b_{235} = b_{225}$. $b_{236} = 0$.

$b_{245} = 2 * \{-I_{15} * S4 * C5 - I_{16} * S3 * S4 * C5\} - I_{17} * S4 + I_{20} * S4 * (1 - 2 * SS5)$; ≈ 0 .

$b_{246} = I_{23} * C4 * S5$; ≈ 0 .

$b_{256} = I_{23} * S4 * C5$; ≈ 0 .

$b_{312} = 0$. $b_{313} = 0$.

$b_{314} = 2 * \{-I_{15} * C23 * C4 * S5 + I_{22} * S23 * C4 * S5 + I_{20} * (S23 * (CC5 * CC4 - 0.5) + C23 * C4 * SC5)\} + I_{14} * S23 + I_{19} * S23 * (1 - (2 * SS4))$;
$\approx -2.50\times10^{-3} * C23 * C4 * S5 + 1.64\times10^{-3} * S23 + 0.30\times10^{-3} * S23 * (1 - 2 * SS4)$.

$b_{315} = 2 * (-I_{15} * C23 * S4 * C5 + I_{22} * S23 * S4 * C5) - I_{17} * C23 * S4 + I_{20} * S4 * (C23 * (1 - 2 * SS5) - 2 * S23 * C4 * SC5)$;
$\approx -2.50\times10^{-3} * C23 * S4 * C5 - 6.42\times10^{-4} * C23 * S4$.

$b_{316} = -b_{136}$. $b_{323} = 0$.

$b_{324} = 2 * \{I_{20} * SC4 * SS5 + I_{21} * SC4 - I_{22} * S4 * S5\}$;
≈ 0 .

$b_{325} = 2 * \{-I_{15} * S5 + I_{20} * SS4 * SC5 + I_{22} * C4 * C5\}$;
$\approx -2.50\times10^{-3} * S5$.

$b_{326} = 0$. $b_{334} = b_{324}$.
$b_{335} = b_{325}$. $b_{336} = 0$.

$b_{345} = -I_{15} * 2 * S4 * C5 - I_{17} * S4 + I_{20} * S4 * (1 - 2 * SS5)$;
$\approx -2.50\times10^{-3} * S4 * C5$.

$b_{346} = b_{246}$. $b_{356} = b_{256}$. $b_{412} = -b_{214}$.
$b_{413} = -b_{314}$. $b_{414} = 0$.

$b_{415} = -I_{20} * (S23 * C4 * (1 - 2 * SS5) + 2 * C23 * SC5) - I_{17} * S23 * C4$;
$\approx -6.42\times10^{-4} * S23 * C4$.

$b_{416} = -b_{146}$. $b_{423} = -b_{324}$. $b_{424} = 0$.

$b_{425} = I_{17} * S4 + I_{20} * S4 * (1 - 2 * SS5)$;
$\approx 6.42\times10^{-4} * S4$.

$b_{426} = -b_{246}$. $b_{434} = 0$.

$b_{435} = b_{425}$. $b_{436} = -b_{346}$.

$b_{445} = -I_{20} * 2 * SC5$; ≈ 0 .

$b_{446} = 0$;

$b_{456} = -I_{23} * S5$; ≈ 0 .

$b_{512} = -b_{215}$. $b_{513} = -b_{315}$. $b_{514} = -b_{415}$.

$b_{515} = 0$. $b_{516} = -b_{156}$. $b_{523} = -b_{325}$.

$b_{524} = -b_{425}$. $b_{525} = 0$. $b_{526} = -b_{256}$.

$b_{534} = b_{524}$. $b_{535} = 0$. $b_{536} = -b_{356}$.

$b_{545} = 0$. $b_{546} = -b_{456}$. $b_{556} = 0$.

$b_{612} = b_{126}$. $b_{613} = b_{136}$. $b_{614} = b_{146}$.

$b_{615} = b_{156}$. $b_{616} = 0$. $b_{623} = 0$.

$b_{624} = b_{246}$. $b_{625} = b_{256}$. $b_{626} = 0$.

$b_{634} = b_{624}$. $b_{635} = b_{625}$. $b_{636} = 0$.

$b_{645} = b_{456}$. $b_{646} = 0$. $b_{656} = 0$.

Table A6. The expressions for the terms of the centrifugal matrix.
(The Abbreviated Expressions have units of kg-m².)

$c_{11} = 0$.

$c_{12} = +I4 * C2 - I8 * S23 - I9 * S2 + I13 * C23 + I_{15} * S23 * S4 * S5 + I_{16} * C2 * S4 * S5 + I_{18} * (C23 * C4 * S5 + S23 * C5) + I_{19} * C23 * SC4 + I_{20} * S4 * (C23 * C4 * cC5 - S23 * SC5) + I_{22} * C23 * S4 * S5$;
$\approx 6.90\times10^{-1} * C2 + 1.34\times10^{-1} * S23 - 2.38\times10^{-2} * S2$.

$c_{13} = 0.5 * b_{123}$.

$c_{14} = -I_{15} * S23 * S4 * S5 - I_{16} * C2 * S4 * S5 + I_{18} * C23 * C4 * S5 + I_{20} * S23 * S4 * SC5 - I_{22} * C23 * S4 * S5$; ≈ 0 .

$c_{15} = -I_{15} * S23 * S4 * S5 - I_{16} * C2 * S4 * S5$; $+I_{18} * (S23 * C5 + C23 * C4 * S5) - I_{22} * C23 * S4 * S5$
≈ 0 .

$c_{16} = 0$. $c_{21} = -0.5 * b_{112}$.

$c_{22} = 0$. $c_{23} = 0.5 * b_{223}$.

$c_{24} = -I_{15} * C4 * S5 - I_{16} * S3 * C4 * S5 + I_{20} * C4 * SC5$;
≈ 0 .

$c_{25} = -I_{15} * C4 * S5 + I_{16} * (C3 * C5 - S3 * C4 * S5) + I_{22} * C5$; ≈ 0 .

$c_{26} = 0$. $c_{31} = -0.5 * b_{113}$.

$c_{32} = -c_{23}$. $c_{33} = 0$.

$c_{34} = -I_{15} * C4 * S5 + I_{20} * C4 * SC5$;
$\approx -1.25\times10^{-3} * C4 * S5$.

$c_{35} = -I_{15} * C4 * S5 + I_{22} * C5$; $\approx c_{34}$.

$c_{36} = 0$. $c_{41} = -0.5 * b_{114}$. $c_{42} = -0.5 * b_{224}$.

$c_{43} = 0.5 * b_{423}$. $c_{44} = 0$. $c_{45} = 0$.

$c_{46} = 0$. $c_{51} = -0.5 * b_{115}$. $c_{52} = -0.5 * b_{225}$.

$c_{53} = 0.5 * b_{523}$. $c_{54} = -0.5 * b_{445}$. $c_{55} = 0$.

$c_{56} = 0$. $c_{61} = 0$. $c_{62} = 0$.

$c_{63} = 0$. $c_{64} = 0$. $c_{65} = 0$.

$c_{66} = 0$.

Table A7. Gravity Terms.
(The Abbreviated Expressions have units of newton-meters.)

$\mathbf{g}_1 = 0$.

$\mathbf{g}_2 = g1 * C2 + g2 * S23 + g3 * S2 + g4 * C23$; $+g5 * (S23 * C5 + C23 * C4 * S5)$
$\approx -37.2 * C2 - 8.4 * S23 + 1.02 * S2$.

$\mathbf{g}_3 = g2 * S23 + g4 * C23 + g5 * (S23 * C5 + C23 * C4 * S5)$;
$\approx -8.4 * S23 + 0.25 * C23$.

$\mathbf{g}_4 = -g5 * S23 * S4 * S5$;
$\approx 2.8\times10^{-2} * S23 * S4 * S5$.

$\mathbf{g}_5 = g5 * (C23 * S5 + S23 * C4 * C5)$;
$\approx -2.8\times10^{-2} * (C23 * S5 + S23 * C4 * C5)$.

$\mathbf{g}_6 = 0$.

More Discussions

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