ECE5463: Introduction to Robotics

Lecture Note 4: General Rigid Body Motion

Prof. Wei Zhang

Department of Electrical and Computer Engineering
Ohio State University
Columbus, Ohio, USA

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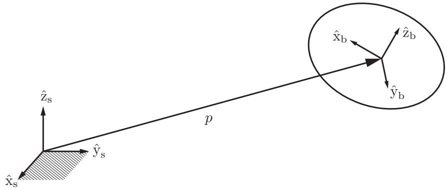
Outline

- Representation of General Rigid Body Motion
- Homogeneous Transformation Matrix
- Twist and $se(3)$
- Twist Representation of Rigid Motion
- Screw Motion and Exponential Coordinate

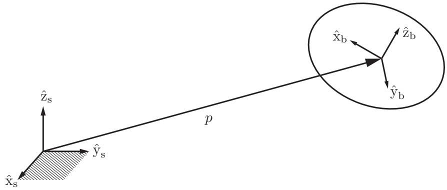
General Rigid Body Configuration



- General rigid body configuration includes both the orientation $R \in SO(3)$ and the position $p \in \mathbb{R}^3$ of the rigid body.
- Rigid body configuration can be represented by the pair (R, p)
- **Definition (Special Euclidean Group):**

$$SE(3) = \{(R, p) : R \in SO(3), p \in \mathbb{R}^3\} = SO(3) \times \mathbb{R}^3$$

Special Euclidean Group



- Let $(R, p) \in SE(3)$, where p is the coordinate of the origin of {b} in frame {s} and R is the orientation of {b} relative to {s}. Let q_s, q_b be the coordinates of a point q relative to frames {s} and {b}, respectively. Then

$$q_s = Rq_b + p$$

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Homogeneous Representation

- For any point $x \in \mathbb{R}^3$, its homogeneous coordinate is $\tilde{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$
- Similar, homogeneous coordinate for the origin is $\tilde{o} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
- Homogeneous coordinate for a vector v is:
- Some rules of syntax for homogeneous coordinates:

Homogeneous Transformation Matrix

- Associate each $(R, p) \in SE(3)$ with a 4×4 matrix:

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \text{ with } T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

- T defined above is called a homogeneous transformation matrix. Any rigid body configuration $(R, p) \in SE(3)$ corresponds to a homogeneous transformation matrix T.

- Equivalently, $SE(3)$ can be defined as the set of all homogeneous transformation matrices.

- Slight abuse of notation: $T = (R, p) \in SE(3)$ and $Tx = Rx + p$ for $x \in \mathbb{R}^3$

Uses of Transformation Matrices

- Representation of rigid body configuration (orientation and position)

- Change of reference frame in which a vector or a frame is represented

Uses of Transformation Matrices

- Rigid body motion operator that displaces a vector

Uses of Transformation Matrices

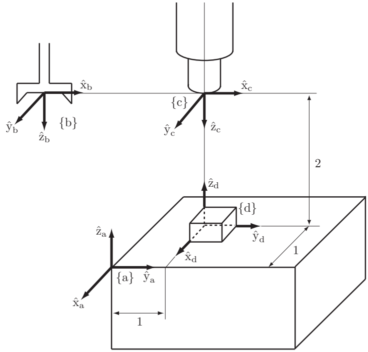
- Rigid body motion operator that displaces a frame

Example of Homogeneous Transformation Matrix

In terms of the coordinates of a fixed space frame {s}, frame {a} has its $\hat{\mathrm{x}}_{\mathrm{a}}$-axis pointing in the direction $(0, 0, 1)$ and its $\hat{\mathrm{y}}_{\mathrm{a}}$-axis pointing $(-1, 0, 0)$, and frame {b} has its $\hat{\mathrm{x}}_{\mathrm{b}}$-axis pointing $(1, 0, 0)$ and its $\hat{\mathrm{y}}_{\mathrm{b}}$-axis pointing $(0, 0, -1)$. The origin of {a} is at $(3, 0, 0)$ in {s} and the origin of {b} is at $(0, 2, 0)$ is {s}.

Example of Homogeneous Transformation Matrix

Fixed frame {a}; end effector frame {b}, the camera frame {c}, and the workpiece frame {d}. Suppose $\|p_c - p_b\| = 4$



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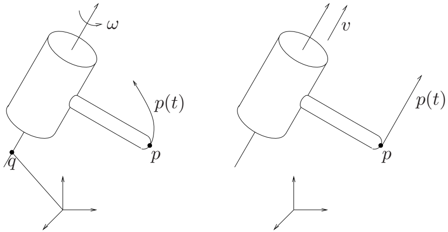
Towards Exponential Coordinate

- Recall: rotation matrix $R \in SO(3)$ can be represented in exponential coordinate $\hat{\omega}\theta$
  - $q(\theta) = \mathrm{Rot}(\hat{\omega}, \theta) q_0$ viewed as a solution to $\dot{q}(t) = [\hat{\omega}]q(t)$ with $q(0) = q_0$ at $t = \theta$.
  - The above relation requires that the rotation axis passes through the origin.
- We can find exponential coordinate for $T \in SE(3)$ using a similar approach (i.e. via differential equation)
- We first need to introduce some important concepts.

Differential Equation for Rigid Body Motion

- Rotation about axis that may not pass through the origin



- Translation

Differential Equation for Rigid Body Motion

- Consider the following differential equation in homogeneous coordinates

$$\dot{p}(t) = \omega \times p(t) + v \quad \Rightarrow \quad \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \qquad (1)$$

- The variable v contains all the constant terms (e.g. $-\omega \times q$ in the rotation example); thus, it may NOT be equal to the linear velocity of the origin of the rigid body.

- Solution to (1) is $\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = \exp\left(\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} t \right) \begin{bmatrix} p(0) \\ 1 \end{bmatrix}$

- Motion of this form is characterized by (ω, v) which is called spatial velocity or Twist.

Twist

- Angular velocity and "linear" velocity can be combined to form the *Spatial Velocity* or *Twist*

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

- Each twist $\mathcal{V}$ corresponds to a motion equation (1).

- For each twist $\mathcal{V} = (\omega, v)$, let $[\mathcal{V}]$ be its matrix representation

$$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

- With these notations, solution to (1) is given by

$$\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = e^{[\mathcal{V}]t} \begin{bmatrix} p_0 \\ 1 \end{bmatrix}$$

$se(3)$

- Similar to $so(3)$, we can define $se(3)$:

$$se(3) = \{([\omega], v) : [\omega] \in so(3), v \in \mathbb{R}^3\}$$

- $se(3)$ contains all matrix representation of twists or equivalently all twists.

- In some references, $[\mathcal{V}]$ is called a twist. We follow the textbook notation to call the spatial velocity $\mathcal{V} = (\omega, v)$ a twist.

- Sometimes, we may abuse notation by writing $\mathcal{V} \in se(3)$.

Example of Twist

- $\mathcal{V} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$ can have multiple different physical interpretations

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Exponential Map of $se(3)$: From Twist to Rigid Motion

Theorem 1.

For any $\mathcal{V} = (\omega, v)$ *and* $\theta \in \mathbb{R}$, *we have* $e^{[\mathcal{V}]\theta} \in SE(3)$

- *Case 1* ($\omega = 0$): $e^{[\mathcal{V}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$
- *Case 2* ($\omega \neq 0$): *without loss of generality assume* $\|\omega\| = 1$. *Then*

$$e^{[\mathcal{V}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}, \text{ with } G(\theta) = I\theta + (1 - \cos(\theta))[\omega] + (\theta - \sin(\theta))[\omega]^2 \quad (2)$$

Log of $SE(3)$: from Rigid-Body Motion to Twist

Theorem 2.

Given any $T = (R, p) \in SE(3)$*, one can always find twist* $\mathcal{V} = (\omega, v)$ *and a scalar* θ *such that*

$$e^{[\mathcal{V}]\theta} = T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Matrix Logarithm Algorithm:

- If $R = I$, then set $\omega = 0$, $v = p/\|p\|$, and $\theta = \|p\|$.
- Otherwise, use matrix logarithm on $SO(3)$ to determine ω and θ from R. Then v is calculated as $v = G^{-1}(\theta)p$, where

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cos\frac{\theta}{2}\right)[\omega]^2$$

Example of Exponential/Log

Quick Summary

- Angular and linear velocity can be combined to form a spatial velocity or twist $\mathcal{V} = (\omega, v)$

- Each twist $\mathcal{V} = (\omega, v)$ defines a motion such that any point p on the rigid body follows a trajectory generated by the following ODE:

$$\dot{p}(t) = \omega \times p(t) + v$$

- Solution to this ODE (in homogeneous coordinate): $\tilde{p}(t) = e^{[\mathcal{V}]t}\tilde{p}(0)$.

- For any twist $\mathcal{V} = (\omega, v)$ and $\theta \in \mathbb{R}$, its matrix exponential $e^{[\mathcal{V}]\theta} \in SE(3)$, i.e., it corresponds to a rigid body transformation. We have an analytical formula to compute the exponential (Theorem 1)

- For any $T \in SE(3)$, we also have analytical formula (Theorem 2) to find $\mathcal{V} = (\omega, v)$ and $\theta \in \mathbb{R}$ such at $e^{[\mathcal{V}]\theta} = T$.

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Screw Interpretation of Twist

- Given a twist $\mathcal{V} = (\omega, v)$, the associated motion (1) may have different interpretations (different rotation axes, linear velocities).

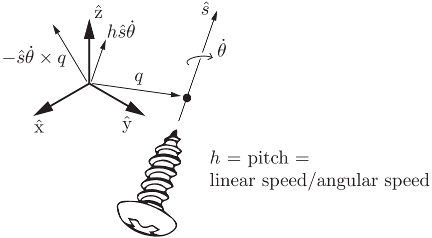
- We want to impose some nominal interpretable structure on the motion.

- Recall: an angular velocity vector ω can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about that axis

- Similarly, a twist (spatial velocity) $\mathcal{V}$ can be interpreted in terms of a **screw axis** $\mathcal{S}$ and a velocity $\dot{\theta}$ about the screw axis

Screw Motion: Definition

- Rotating about an axis while also translating along the axis



- Represented by screw axis $\{q, \hat{s}, h\}$ and rotation speed $\dot{\theta}$
  - $\hat{s}$: unit vector in the direction of the rotation axis
  - q: any point on the rotation axis
  - h: **screw pitch** which defines the ratio of the linear velocity along the screw axis to the angular velocity about the screw axis

Screw Motion as Solution to ODE

- Consider a point p on a rigid body under a screw motion with (rotation) speed $\dot{\theta}$. Let $p(t)$ be its coordinate at time t. The overall velocity is

$$\dot{p}(t) = \hat{s}\dot{\theta} \times (p(t) - q) + h\hat{s}\dot{\theta} \tag{3}$$

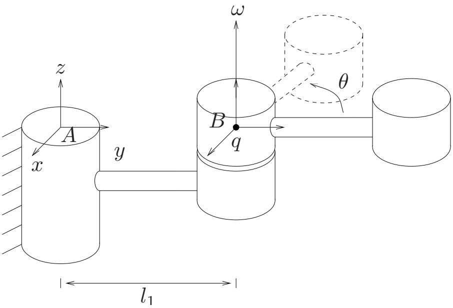
- Thus, any screw axis $\{q, \hat{s}, h\}$ with rotation speed $\dot{\theta}$ can be represented by a particular twist (ω, v) with $\omega = \hat{s}\dot{\theta}$ and $v = -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta}$.

From Twist to Screw Axis

- The converse is true as well: given any twist $\mathcal{V} = (\omega, v)$ one can always find $\{q, \hat{s}, h\}$ and $\dot{\theta}$ such that the corresponding screw motion (eq. (3)) coincides with the motion generated by the twist (eq. (1)).
  - If $\omega = 0$, then it is a pure translation ($h = \infty$)
  - If $\omega \neq 0$:

Examples Screw Axis and Twist

- What is the twist that corresponds to rotating about $\hat{z}_b$?



- What is the screw axis for twist $\mathcal{V} = (0, 2, 2, 4, 0, 0)$?

Implicit Definition of Screw Axis for a Given a Twist

- For any twist $\mathcal{V} = (\omega, v)$, we can always view it as a "screw velocity" that consists an screw axis $\mathcal{S}$ and the velocity $\dot{\theta}$ about the screw axis.

- Instead of using $\{q, \hat{s}, h\}$ to represent $\mathcal{S}$, we adopt a more convenient representation defined below:

- **Screw axis (corresponding to a twist):** Given any twist $\mathcal{V} = (\omega, v)$, its screw axis is defined as
  - If $\omega \neq 0$, then $\mathcal{S} := \mathcal{V}/\|\omega\| = (\omega/\|\omega\|, v/\|\omega\|)$.
  - If $\omega = 0$, then $\mathcal{S} := \mathcal{V}/\|v\| = (0, v/\|v\|)$

Unit Screw Axis

- **(unit) screw axis** $\mathcal{S}$ can be represented by

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

where either (i)$\|\omega\| = 1$ or (ii) $\omega = 0$ and $\|v\| = 1$

- We have used (ω, v) to represent both screw axis (where $\|\omega\|$ or $\|v\| = 1$ must be unity) and a twist (where there are no constrains on ω and v)

- $\mathcal{S} = (w, v)$ is called a screw axis, but we typically do not bother to explicitly find the corresponding $\{q, \hat{s}, h\}$. We can find them whenever needed.

Exponential Coordinates of Rigid Transformation

- Screw axis $\mathcal{S} = (\omega, v)$ is just a normalized twist; its matrix representation is

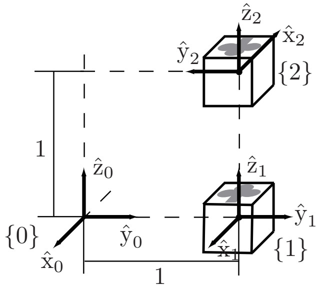
$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

- Therefore, a point started at $p(0)$ at time zero, travel along screw axis $\mathcal{S}$ at unit speed for time t will end up at $p(t) = e^{[\mathcal{S}]t} p(0)$

- Given $\mathcal{S}$ we can use Theorem 1 to compute $e^{[\mathcal{S}]t} \in SE(3)$;

- Given $T \in SE(3)$, we can use Theorem 2 to find $\mathcal{S} = (\omega, v)$ and θ such that $e^{[\mathcal{S}]\theta} = T$. We call $\mathcal{S}\theta$ the **Exponential Coordinate** of the homogeneous transformation $T \in SE(3)$

Example of Exponential Coordinates



More Discussions

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