ECON 715 - Mathematical Economics

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Contents

Chapter 1

Linear Models and Matrix Algebra

1.1 Introduction

The purpose of this chapter is to solve and analyze a system of m linear equations with n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= d_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= d_m \end{aligned} \tag{1.1}$$

In the above system, the unknown variables are $x_1, x_2, ..., x_n$, the parameters (or numbers) a_{ij} are the coefficients in equation i on variable x_j, and the constants $d_1, d_2, ..., d_m$ are given numbers. In a linear system, each of the m equations is a linear function of the unknown variables $x_1, x_2, ..., x_n$.

Example 1 *The following are examples of linear systems:*

(a) linear system with 3 equations and 3 unknowns $(m = 3,\ n = 3)$*:*

$$\begin{aligned} 6x_1 + 3x_2 + x_3 &= 22 \\ x_1 + 4x_2 - 2x_3 &= 12 \\ 4x_1 - x_2 + 5x_3 &= 10 \end{aligned}$$

(b) Linear system with 2 equations and 3 unknowns $(m = 2,\ n = 3)$*:*

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_2 - x_3 &= 0 \end{aligned}$$

(c) Linear system with 3 equations and 2 unknowns $(m = 3,\ n = 2)$*:*

$$\begin{aligned} 5x_1 + 3x_2 &= 10 \\ 2x_1 - x_2 &= 4 \\ 7x_1 + 2x_2 &= 14 \end{aligned}$$

Linear systems of equations are important because many popular models in economics are linear by design, such as the linear regression model, and its estimation by Ordinary Least Squares (OLS) requires solving a linear system of equations. Any quadratic optimization problem results in first order conditions which are a linear system of equations. Linear models are especially useful in finance and portfolio theory. Even if a model is not linear, an equilibrium can be approximated locally with a system of linear equations. Finally, non-linear models can be solved using computer algorithms, which use successive linear approximations.

1.1.1 Solutions to linear systems

Definition 1 *A* ***solution*** *to linear system (1.1) is n-tuple* $x = x_1, x_2, ..., x_n$*, which satisfies each of the equations in (1.1). The set of all the solutions to a linear system is called the* ***solution set****.*

The key questions we ask about solutions to linear systems are:

1. Does a solution exist?

2. Under what conditions there exists a unique solution?

3. How to compute the solutions, if they exist?

Example 2 *Consider the linear system:*

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_2 - x_3 &= 0 \end{aligned}$$

Verify that $[x_1 = 3, x_2 = 1, x_3 = 1]$ *is a solution to this system. Plugging the proposed solution into the equations, gives:*

$$\begin{aligned} 3 + 1 + 1 &= 5 \\ 1 - 1 &= 0 \end{aligned}$$

Indeed, both equations are satisfied, and therefore, $[x_1 = 3, x_2 = 1, x_3 = 1]$ *is a solution to the given linear system of equations. You can verify however, that the proposed solution is not unique, and there are many more solutions to this system. The solution set is in fact:*

$$\{(x_1, x_2, x_3)\,|x_1 = 5 - 2x_3, x_2 = x_3\}$$

Before dealing with large systems of linear equations, it is useful to analyze a single linear equation with one unknown:

$$a \cdot x = d \tag{1.2}$$

There are only three possibilities regarding the solution set of such equations:

1. Unique solution, when $a \neq 0$. In this case, the unique solution is given by $x = a^{-1}d$.

2. No solution, when $a = 0$ and $d \neq 0$. There is no x that satisfies $0 \cdot x = 5$ for example.

3. Infinitely many solution, when $a = 0$ and $d = 0$. Any x satisfies $0 \cdot x = 0$.

It turns out that the above discussion is general, and the solution set to any linear system with m equations and n unknowns either (i) contains a unique solution, (ii) empty (no solution), or (iii) contains infinitely many solutions.

1.1.2 Solution methods

There are three methods of solving systems of m linear equations with n unknowns:

1. Substitution,

2. Elimination (Gaussian elimination),

3. Matrix algebra.

Gaussian elimination is computationally the fastest method. Most of this chapter is dedicated to matrix algebra, because it allows complete characterization of solution sets to general linear systems of m equations and n unknowns. In this section we briefly illustrate the three methods.

With the substitution method we solve one of the equations in (1.1), say the first, for x_1:

$$x_1 = \frac{d_1 - (a_{12}x_2 + ... + a_{1n}x_n)}{a_{11}}$$

and substitute in the remaining $m-1$ equations. This gives a system of $m-1$ equations, and $n-1$ unknowns $x_2, x_3, ..., x_n$. Then solve for x_2 from one of these $m-1$ equations, and substitute in the remaining $m-2$ equations. Repeat this substitution until we are left with either one equation, or one unknown, or both.

Example 3 *(Substitution method). Consider the system of 3 equations and 3 unknowns.*

$$\begin{aligned} 6x_1 + 3x_2 + x_3 &= 22 \\ x_1 + 4x_2 - 2x_3 &= 12 \\ 4x_1 - x_2 + 5x_3 &= 10 \end{aligned}$$

Solve the second equation for x_1 in terms of x_2 and x_3, gives

$$x_1 = 12 - 4x_2 + 2x_3$$

Substitute this expression in the remaining two equations, gives a system with two equations and two unknowns (x_2, x_3):

$$\begin{aligned} 6(12 - 4x_2 + 2x_3) + 3x_2 + x_3 &= 22 \\ 4(12 - 4x_2 + 2x_3) - x_2 + 5x_3 &= 10 \end{aligned}$$

Simplifying,

$$\begin{aligned} 21x_2 - 13x_3 &= 50 \\ 17x_2 - 13x_3 &= 38 \end{aligned}$$

Next, solve for x_2 from the first equation above:

$$x_2 = \frac{50 + 13x_3}{21}$$

Substitute in the last equation

$$\begin{aligned} 17\left(\frac{50 + 13x_3}{21}\right) - 13x_3 &= 38 \\ 850 + 221x_3 - 273x_3 &= 798 \\ 52x_3 &= 52 \\ x_3 &= 1 \end{aligned}$$

Next, substitute $x_3 = 1$ into the expression for x_2

$$x_2 = \frac{50 + 13x_3}{21} = \frac{50 + 13 \cdot 1}{21} = 3$$

Finally, substitute $x_2 = 3$ and $x_3 = 1$ into the expression for x_1

$$x_1 = 12 - 4x_2 + 2x_3 = 12 - 4 \cdot 3 + 2 \cdot 1 = 2$$

Thus, the solution we found is $[x_1 = 2, x_2 = 3, x_3 = 1]$, which turns out to be the unique solution to the given system.

A very similar method is elimination. It requires the application of **elementary equation operations**, each of them creates an equivalent system of equations, which has the same solution as the original system. The elementary equation operations are:

1. Multiplying an equation by a non-zero scalar,

2. Interchanging two equations,

3. Adding two equations.

We will illustrate the elimination method in the next example.

Example 4 *(Elimination method). Consider the linear system*

$$\begin{aligned} 6x_1 + 3x_2 + x_3 &= 22 \\ x_1 + 4x_2 - 2x_3 &= 12 \\ 4x_1 - x_2 + 5x_3 &= 10 \end{aligned}$$

We can interchange the first and second equations:

$$\begin{aligned} x_1 + 4x_2 - 2x_3 &= 12 \\ 6x_1 + 3x_2 + x_3 &= 22 \\ 4x_1 - x_2 + 5x_3 &= 10 \end{aligned}$$

Then using the first equation, eliminate x_1 from the second and third equations. That is, add *to the second equation* $-6 \times eq1$ *and add to the third equation* $-4 \times eq1$*:*

$$\begin{aligned} x_1 + 4x_2 - 2x_3 &= 12 \\ -21x_2 + 13x_3 &= -50 \\ -17x_2 + 13x_3 &= -38 \end{aligned}$$

Next, use equation 2 to eliminate x_2 *from the last equation. That is, add* $-\frac{17}{21} \times eq2$ *to the last equation:*

$$\begin{aligned} x_1 + 4x_2 - 2x_3 &= 12 \\ -21x_2 + 13x_3 &= -50 \\ \left(13 - \frac{17}{21}13\right) x_3 &= -38 + \frac{17}{21} \cdot 50 \end{aligned}$$

The last equation simplifies to

$$52x_3 = 52,\ x_3 = 1$$

Then, substitute this solution into the second equation, containing x_2 *and* x_3*, and solve for* x_2*:*

$$-21x_2 + 13 \cdot 1 = -50$$

The solution is $x_2 = 3$*. Substitute* $x_2 = 3$ *and* $x_3 = 1$ *into equation containing these variables and* x_1*, and get*

$$x_1 + 4 \cdot 3 - 2 \cdot 1 = 12$$

The solution is $x_1 = 2$*. Thus, we found* $[x_1 = 2, x_2 = 3, x_3 = 1]$*.*

Finally, an alternative method to substitution and elimination is using matrix algebra. The linear system of equations in (1.1) can be written in matrix form as follows. Arrange all the coefficients a_{ij} in a **coefficient matrix** A, the unknown variables $x_1, ..., x_n$ in a vector x, and the constants $d_1, ..., d_n$ in vector d:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{m \times n}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}, \quad d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}_{n \times 1}$$

Then, the system in (1.1) can be written compactly as

$$Ax = d \tag{1.3}$$

For example, the system

$$\begin{aligned} 6x_1 + 3x_2 + x_3 &= 22 \\ x_1 + 4x_2 - 2x_3 &= 12 \\ 4x_1 - x_2 + 5x_3 &= 10 \end{aligned}$$

Can be written as $Ax = d$, where

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix}_{3\times 3}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad d = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}$$

Matrix algebra enables us to write the system in (1.1) as $Ax = d$, which is much shorter, but this is not the only advantage of matrix algebra. Matrix algebra also enables us to derive analytical solution to the system of m linear equations with n unknowns, *if such solution exists*. For example, if $Ax = d$ is a linear system with n equations and n unknowns, then it has a unique solution if and only if the **inverse** matrix A^{-1} of A exists. In such case, the unique solution is given by:

$$x^* = A^{-1}d \tag{1.4}$$

Suppose we calculated the inverse matrix A^{-1}, and we want to see how changes in the constant vector d affect the solution x^*. All we need to do, is multiply A^{-1} by the new vector d, which is a simple operation. If instead we did not use matrix algebra, and solved the system in (1.1) by elimination, then each time we change d would require solving the entire system all over again. Moreover, matrix algebra gives us easy-to-check conditions on the coefficient matrix A that guarantee existence of a unique solution to the linear system.

1.2 Matrices

A **matrix** is a rectangular array of elements (or terms). The **dimension** of a matrix is given by the number of rows and the number of columns. Thus, a matrix A of order $m \times n$ is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

The above matrix has dimension $m \times n$, sometimes indicated for clarity as above. The first dimension is always the number of rows, and the second dimension is always the number of columns. The dimension $m \times n$ reads "m by n". The typical element of a matrix is a_{ij}, which is the element in row i and column j.

We can think of matrices as generalization of numbers, since a single number (scalar) is a matrix with dimension 1×1. Just as with numbers, we need to define basic operations with matrices, such as addition, multiplication, and other operations which generalize the ones we are familiar with from number arithmetics.

1.2.1 Matrix addition and subtraction

Two matrices can be added or subtracted if and only if they have the same dimension. Then, the sum (difference) of the matrices requires adding (subtracting) the corresponding elements. For example, $A_{3\times 2} + B_{3\times 2}$ is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}_{3\times 2}$$

In other words, if $A+B=C$, then $c_{ij} = a_{ij} + b_{ij}$ for all i and j. Notice that $A+B = B+A$ and $A-B = -B+A$ i.e. the commutative law applies for matrix addition and subtraction.

1.2.2 Scalar multiplication

To multiply a matrix by a scalar (a number) we need to multiply each element in the matrix by that scalar. For example,

$$5\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 5a_{11} & 5a_{12} \\ 5a_{21} & 5a_{22} \\ 5a_{31} & 5a_{32} \end{bmatrix}$$

1.2.3 Multiplication of matrices

There are at least three different ways to define multiplication of two matrices: (i) matrix product, (ii) Hadamard product (also known as Schur product or elementwise product) and (iii) Kronecker product.

Matrix product

If matrix A has dimension $m \times n$ and matrix B has dimension $n \times p$, the product $AB = C$ has dimension $m \times p$ and is defined as follows:

$$c_{ij} = r_i(A) \cdot c_j(B) = \begin{bmatrix} a_{i1} & a_{i_2} & \cdots & a_{in} \end{bmatrix}_{1\times n} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}_{n\times 1} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

In other words, element c_{ij} in the product matrix $C = AB$, is equal to the sum of the products of row i in A and column j in B, denoted $r_i(A)$ and $c_j(B)$ respectively. Notice that the matrix product AB is defined only if the number of columns in A is the same as the number of rows in B. For example,

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}_{1\times 2}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}_{2\times 3} \\ C &= AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \end{bmatrix}_{1\times 3} \end{aligned}$$

Notice that in the above example, the product BA is not defined. Thus, unlike products of numbers, e.g. $3 \cdot 7 = 7 \cdot 3$, with matrices in general $AB \neq BA$. In Matlab, matrix product is computed as follows: `A*B`.

Hadamard product

For two matrices A, B of the same dimension $m \times n$, the Hadamard product (or element-wise product), $A \circ B = C$ is of the same dimension $m \times n$, with elements

$$c_{ij} = a_{ij} \cdot b_{ij}$$

For example

$$\begin{aligned} A \circ B &= C \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} \circ \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2} &= \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{22}b_{22} \\ a_{31}b_{31} & a_{32}b_{32} \end{bmatrix} \end{aligned}$$

For matrices of different dimension, the Hadamard product is not defined. In Matlab, Hadamard product is computed as follows: `A.*B` (notice the `.*` instead of `*`).

Kronecker product

For two matrices of any size, A is an $m \times n$ matrix and B is a $p \times q$ matrix, the Kronecker product $A \otimes B = C$ is the $mp \times nq$ block matrix with elements

$$c_{ij} = a_{ij}B$$

For example

$$\begin{aligned} A \otimes B &= C \\ A_{m\times n} \otimes B_{p\times q} &= \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}_{mp\times nq} \end{aligned}$$

Thus, if A is 7×5 and B is 3×10, then the dimension of the Kronecker product $A_{7\times5}\otimes B_{3\times10}$ is 21×50. In Matlab, Kronecker product is computed via `kron(A,B)`.

1.2.4 Transpose of a matrix

The transpose of a matrix A is the matrix A' constructed such that row i in A becomes column i in A'. Thus, if A is $m \times n$, its transpose A' is $n \times m$, and $a_{ij} = a_{ji}$.

For example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \end{bmatrix}_{2\times3}, \quad A' = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}_{3\times2}$$

Notice that the transpose of a scalar is the scalar itself, since scalar are 1×1 matrices.

1.2.5 Special matrices

1. A matrix with the same number of rows as the number of columns (i.e. $m = n$) is called **square.** For example, the following matrices are square:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}, \quad B = \begin{bmatrix} 1 & 0 & 3 \\ 7 & 5 & 0 \\ 9 & 2 & 4 \end{bmatrix}_{3\times3}$$

2. A matrix with one column ($n = 1$) is called **column vector**, and a matrix with only

one row ($m = 1$) is called **row vector**. A matrix with only one element ($m = n = 1$) is called a **scalar** or a number. For example,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad x' = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}_{1\times n}, \quad a = 5$$

x is a column vector (with n rows), x' is a row vector (with n columns), and a is a scalar (with one row and one column).

3. A square matrix A, that is equal to its transpose (i.e. $A = A'$), is called **symmetric**. For example, the next matrix is symmetric:

$$A = \begin{bmatrix} 1 & -1 & 4 \\ -1 & 0 & 3 \\ 4 & 3 & 2 \end{bmatrix}_{3\times 3} = A'$$

Thus, in a symmetric matrix, we must have $a_{ij} = a_{ji}\ \forall i, j$, and it must be square.

4. A square matrix that has all off-diagonal elements equal to zero, and at least one element on the diagonal is not zero, is called **diagonal**. For example, A is diagonal:

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

5. A diagonal matrix, with all elements on the diagonal being 1 is called the **identity matrix**, and denoted I or I_n:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

6. A matrix with all elements equal to zero is called a **null matrix**, and denoted by 0 or

$0_{m\times n}$:

$$0_{m\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times n}$$

7. A square matrix that has zeros everywhere above the diagonal is called **lower triangular** matrix, and a matrix that has zeros everywhere below the diagonal is called **upper triangular**. In the next example, A is lower triangular, and B is upper triangular:

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}_{3\times 3}$$

8. A square matrix A, for which $A \cdot A = A$ is called **idempotent**. For example,

$$A = \begin{bmatrix} 5 & -5 \\ 4 & -4 \end{bmatrix},$$

$$\begin{aligned} A \cdot A &= \begin{bmatrix} 5 & -5 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} 5 & -5 \\ 4 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 5\cdot 5 + (-5)\cdot 4 & 5\cdot(-5) + (-5)\cdot(-4) \\ 4\cdot 5 + (-4)\cdot 4 & 4\cdot(-5) + (-4)\cdot(-4) \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 4 & -4 \end{bmatrix} \end{aligned}$$

Exercise 1 *let $A_{n\times n}$ be a square matrix and I_n is identity matrix of the same size as A. Prove that*

$$IA = AI = A$$

That is, pre-multiplying or post-multiplying any square matrix by identity matrix does not change the original matrix.

1.2.6 Determinant of a square matrix

The determinant of a square matrix A, denoted by $|A|$ or $\det(A)$, is a uniquely defined scalar (number) associated with that matrix. Determinants are defined only for square matrices.

The determinant of a scalar (a 1×1 matrix) is the scalar itself. For a 2×2 matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad |A| = a_{11}a_{22} - a_{21}a_{12}$$

For a $n \times n$ matrix, the determinant can be computed by Laplase-expansion. For that, we need the concepts of minors and cofactors.

The ij **minor** of a square matrix A, denoted $|M_{ij}|$, is the determinant of a smaller square matrix, obtained by removing row i and column j from A. For example, let A be 3×3:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

Then, $|M_{11}|$ is the determinant of a smaller, 2×2 matrix, obtained by deleting the first row and the first column of A:

$$|M_{11}| = \begin{vmatrix} \square & \square & \square \\ \square & a_{22} & a_{23} \\ \square & a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

Similarly, $|M_{31}|$ is the determinant of a matrix obtained by deleting the 3rd row and the first column of A:

$$|M_{31}| = \begin{vmatrix} \square & a_{12} & a_{13} \\ \square & a_{22} & a_{23} \\ \square & \square & \square \end{vmatrix} = a_{12}a_{23} - a_{22}a_{13}$$

And $|M_{22}|$ is the determinant of a matrix obtained by deleting the 2nd row and 2nd column of A :

$$|M_{22}| = \begin{vmatrix} a_{11} & \square & a_{13} \\ \square & \square & \square \\ a_{31} & \square & a_{33} \end{vmatrix} = a_{11}a_{33} - a_{31}a_{13}$$

Notice that the minors of a 3×3 matrix are determinants of a 2×2 matrix, which we already defined.

The ij **cofactor** of a matrix A, is the ij minor with a prescribed sign, as follows:

$$|C_{ij}| = (-1)^{i+j} |M_{ij}|$$

Notice that when $i + j$ is even, then $(-1)^{i+j} = 1$, and the cofactor is equal to the minor.

However, when $i+j$ is odd, then $(-1)^{i+j} = -1$, and the cofactor has the opposite sign from the minor. Thus, the signs of $(-1)^{i+j}$ in a $n \times n$ matrix have the following pattern:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Using these concepts, the determinant of $A_{n\times n}$ can be computed as follows:

$$\begin{aligned} |A| &= \sum_{j=1}^{n} a_{ij}|C_{ij}| \quad \text{[expansion along the } i^{th} \text{ row]} \\ &= \sum_{i=1}^{n} a_{ij}|C_{ij}| \quad \text{[expansion along the } j^{th} \text{ column]} \end{aligned}$$

This means that for computing the determinant of a square matrix, one can pick any *row* or *any* column, and sum over the elements of the row or column, weighted by the cofactors of these elements. Of course, we would prefer to choose an "easy" row or column, with as many zeros as possible. Always remember that a determinant of a square matrix is a scalar (a number), not a matrix.

Example 5 *Use Laplace expansion to find the determinant of*

$$A = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

expanding along the 1^{st} *row, gives:*

$$\begin{aligned} |A| &= 6 \cdot \begin{vmatrix} -2 & 5 \\ 8 & 7 \end{vmatrix} - 1 \cdot \begin{vmatrix} 4 & 5 \\ 2 & 7 \end{vmatrix} + 1 \cdot \begin{vmatrix} 4 & -2 \\ 2 & 8 \end{vmatrix} \\ &= 6 \cdot (-2 \cdot 7 - 8 \cdot 5) - 1 \cdot (4 \cdot 7 - 2 \cdot 5) + 1 \cdot (4 \cdot 8 + 2 \cdot 2) \\ &= 6 \cdot (-54) - 18 + 36 = -306 \end{aligned}$$

1.2.7 Inverse of a square matrix

Recall that an inverse of a scalar a is a scalar a^{-1} such that $a^{-1}a = 1$ and $aa^{-1} = 1$. Such inverse exists if $a \neq 0$. We now generalize this concept to matrices. An inverse of a square matrix A, if it exists, is a matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I_n$$

In other words, the inverse of a square $(n \times n)$ matrix A (when exists), is another matrix, that when pre-multiplies or post-multiplies A, results in an identity matrix. If the inverse of a matrix A exists, we say that A is **nonsingular** (or **invertible**) matrix, and if the inverse does not exist, we say that A is **singular** (or **non-invertible**) matrix.

One way to compute the inverse of a square matrix (the **Adjoint** method), uses the cofactors defined previously. Let A be $n \times n$ matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

Step 1: Create the **cofactor matrix** $C = [|C_{ij}|]$, where each element a_{ij} is replaced by the ij cofactor:

$$C = \begin{bmatrix} |C_{11}| & |C_{12}| & \cdots & |C_{1n}| \\ |C_{21}| & |C_{22}| & \cdots & |C_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ |C_{n1}| & |C_{n2}| & \cdots & |C_{nn}| \end{bmatrix}_{n \times n}$$

Step 2: Transpose C, to create the so called **adjoint** matrix of A:

$$adj\,(A) = C' = \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \vdots & \vdots & \ddots & \vdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix}_{n \times n}$$

Step 3: The inverse of A is:

$$A^{-1} = \frac{1}{|A|} adj\,(A)$$

This step requires calculating the determinant of A.

Now we can see that an inverse of a square matrix A exists, i.e. A is nonsingular, if and only if $|A| \neq 0$, i.e. the determinant of the matrix is not zero. Thus, the determinant "determines" if the matrix is invertible. It is a good idea to start with the calculation of the determinant, and if it is not zero, continue with the steps of matrix inversion.

Example 6 *Find the inverse of a 2×2 matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Step 1: the matrix of cofactors is

$$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

Step 2: adjoint matrix of A:

$$adj\,(A) = C' = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Step 3: inverse of A:

$$A^{-1} = \frac{1}{|A|} adj\,(A) = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The above formula is well defined if $|A| = ad - cb \neq 0$. The above is a useful formula that can be applied each time you have to invert some 2×2 matrix.

Exercise 2 *Verify that*

$$\frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

is the inverse matrix of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Solution 1 *The product of the two matrices is*

$$\begin{aligned} \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \frac{1}{ad - cb} \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Example 7 *Find the inverse of a* 3×3 *matrix*

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix}$$

We start with calculating the determinant of A*, since we know that it is needed in step 3 of matrix inversion. Laplace expansion along the* 1^{st} *row, gives:*

$$\begin{aligned} |A| &= 6 \begin{vmatrix} 4 & -2 \\ -1 & 5 \end{vmatrix} - 3 \begin{vmatrix} 1 & -2 \\ 4 & 5 \end{vmatrix} + 1 \begin{vmatrix} 1 & 4 \\ 4 & -1 \end{vmatrix} \\ &= 6 \cdot 18 - 3 \cdot 13 - 17 = 52 \end{aligned}$$

Next, the cofactor matrix:

$$C = \begin{bmatrix} \begin{vmatrix} 4 & -2 \\ -1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 4 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ -1 & 5 \end{vmatrix} & \begin{vmatrix} 6 & 1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 6 & 3 \\ 4 & -1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} & -\begin{vmatrix} 6 & 1 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 6 & 3 \\ 1 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 18 & -13 & -17 \\ -16 & 26 & 18 \\ -10 & 13 & 21 \end{bmatrix}$$

Next, the adjoint matrix:

$$adj\,(A) = C' = \begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix}$$

Thus, the inverse matrix of A *is:*

$$A^{-1} = \frac{1}{|A|} adj\,(A) = \frac{1}{52} \begin{bmatrix} 18 & -16 & -10 \\ -13 & 26 & 13 \\ -17 & 18 & 21 \end{bmatrix} = \begin{bmatrix} 0.346\,15 & -0.307\,69 & -0.192\,31 \\ -0.25 & 0.5 & 0.25 \\ -0.326\,92 & 0.346\,15 & 0.403\,85 \end{bmatrix}$$

The Matlab commands that compute the inverse of a square matrix are `inv(A)` or `A^-1`, both use the same algorithm and both produce the same result.

1.3 Properties of Matrix Operations, Determinants and Inverses

1.3.1 Properties of matrix addition

Note that there is no need in separate discussion of matrix subtraction, because subtraction can be written as: $A - B = A + (-B)$. Thus, for matrices of the same size (conformable for addition), the following properties hold:

1. $A + B = B + A$, commutative law holds for addition.
2. $(A + B) + C = A + (B + C)$, associative law holds for matrix addition. This means that the order at which you add matrices does not matter.

1.3.2 Properties of matrix product

1. $AB \neq BA$, so in general commutative law does not hold for matrix product.
2. $(AB)\,C = A\,(BC) = ABC$, associative law holds for matrix product.
3. $A\,(B + C) = AB + AC$, and $(B + C)\,A = BA + CA$, distributive law holds for matrix product.

1.3.3 Properties of Hadamard product

1. $A \circ B = B \circ A$, commutative holds for Hadamard product.
2. $(A \circ B) \circ C = A \circ (B \circ C) = A \circ B \circ C$, associative law holds for Hadamard product.
3. $A \circ (B + C) = A \circ B + A \circ C$, distributive law holds for Hadamard product.

1.3.4 Properties of transposes

1. $(A')' = A$, transpose of a transpose is the original matrix.
2. $(A + B)' = A' + B'$, transpose of a sum is sum of transposes.
3. $(AB)' = B'A'$, transpose of a product is product of transposes, in *reverse order*.

Proof. We prove that $(AB)' = B'A'$. Let $A_{m\times n}$ and $B_{n\times p}$. Thus $(AB)_{m\times p}$, $(AB)'_{p\times m}$ and $\left(B'_{p\times n}A'_{n\times m}\right)_{p\times m}$, so the dimentions are correct.

Let $r_i(A)$ denote row i in matrix A and $c_j(B)$ denote column j in matrix B. Also denote by a_{ij}, b_{ij} be the elements in row i column j of A and B respectively. Similarly, a'_{ij}, b'_{ij} be the elements in row i column j of A' and B' respectively. Notice that $a'_{ij} = a_{ji}$ and $b'_{ij} = b_{ji}$. Then, observe that:

Matrix	Element ij
AB	$r_i(A) \cdot c_j(B) = \sum_{k=1}^{n} a_{ik}b_{kj}$
$(AB)'$	$r_j(A) \cdot c_i(B) = \sum_{k=1}^{n} a_{jk}b_{ki}$, i.e. ji element in AB
$B'A'$	$r_i(B') \cdot c_j(A') = \sum_{k=1}^{n} b'_{ik}a'_{kj} = \sum_{k=1}^{n} b_{ki}a_{jk}$

Notice that in the above we used the fact that $a'_{kj} = a_{jk}$ and $b'_{ik} = b_{ki}$. Thus,

$$(AB)'_{ij} = \sum_{k=1}^{n} a_{jk}b_{ki} = (B'A')_{ij}$$

■

1.3.5 Properties of inverses

1. $(A^{-1})^{-1} = A$, the inverse of an inverse is the original matrix.

2. $(AB)^{-1} = B^{-1}A^{-1}$, the inverse of a product is the product of inverses, in *reverse order*.

3. $(A')^{-1} = (A^{-1})'$, the inverse of the transpose is the transpose of the inverse.

Proof. (1) $(A^{-1})^{-1} = A$.

By definition of $(A^{-1})^{-1}$,

$$\left(A^{-1}\right)^{-1} A^{-1} = I$$

To see the above, denote $B = A^{-1}$, so the above is the definition of an inverse: $B^{-1}B = I$. Multiply both sides by A

$$\begin{aligned} \left(A^{-1}\right)^{-1} A^{-1}A &= IA \\ \left(A^{-1}\right)^{-1} &= A \end{aligned}$$

(2) $(AB)^{-1} = B^{-1}A^{-1}$.

Post-multiply $B^{-1}A^{-1}$ by AB (recall that all the matrices here are square, since inverses are only defined for square matrices, so all the products are well defined):

$$B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$$

The last equality says that the inverse of AB is $B^{-1}A^{-1}$, which implies that $B^{-1}A^{-1} = (AB)^{-1}$.

(3) $(A')^{-1} = (A^{-1})'$.

By definition of A^{-1},

$$A^{-1}A = I$$

Transposing both sides of the equation:

$$\begin{aligned} \left(A^{-1}A\right)' &= (I)' = I \\ A'\left(A^{-1}\right)' &= I \end{aligned}$$

On the left hand side, we used the property that transpose of a product is product of transposes, in reverse order. Therefore, the last equality says that the inverse of A' is $(A^{-1})'$, which implies that $(A^{-1})' = (A')^{-1}$. ■

1.3.6 Properties of determinants

1. $|A| = |A'|$, determinant of a matrix is the same as the determinant of its transpose.

2. $|A^{-1}| = \frac{1}{|A|}$, determinant of an inverse is one over the determinant of the original matrix.

3. $|AB| = |A| \cdot |B|$, determinant of a product is product of determinants.

4. $|kA| = k^n|A|$, multiplication of $n \times n$ matrix by a constant k, changes the determinant by a factor k^n.

5. Multiplication of only one row or one column by a constant k, will result in determinant $k|A|$.

6. If one of the rows is a linear combination of other rows, the determinant is zero. Same is true for columns. In this case, we say that the matrix is singular or noninvertible.

7. Interchanging any two rows or any two columns, will change the sign of the determinant.

8. Adding a scalar multiple of one row to another row leaves the determinant unchanged. Same is true for columns.

Exercise 3 *Prove that the determinant of a triangular matrix, is equal to the product of the elements on its diagonal.*

Solution 2 *Consider a lower-triangular matrix A:*

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

Using Laplace expansion along the 1^{st} *row:*

$$|A| = a_{11} \cdot \begin{vmatrix} a_{22} & 0 & \cdots & 0 \\ a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdot \begin{vmatrix} a_{33} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{n3} & \cdots & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$$

For upper-triangular matrix, we can expand along the 1^{st} *column.*

Exercise 4 *Prove that the determinant of an identity matrix is 1. That is,* $|I_n| = 1$.

Solution 3 *Since identity matrix is a special case of triangular matrix, we have*

$$|I_n| = \underbrace{1 \cdot 1 \cdot \ldots \cdot 1}_{n \ times} = 1$$

Exercise 5 *Calculate the determinants of the following matrices:*

$$A = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}, \ B = \begin{bmatrix} 4 & -2 & 5 \\ 6 & 1 & 1 \\ 2 & 8 & 7 \end{bmatrix}$$

Solution 4 *Expanding along the* 1^{st} *row of A, gives:*

$$\begin{aligned} |A| &= 6 \cdot \begin{vmatrix} -2 & 5 \\ 8 & 7 \end{vmatrix} - 1 \cdot \begin{vmatrix} 4 & 5 \\ 2 & 7 \end{vmatrix} + 1 \cdot \begin{vmatrix} 4 & -2 \\ 2 & 8 \end{vmatrix} \\ &= 6 \cdot (-2 \cdot 7 - 8 \cdot 5) - 1 \cdot (4 \cdot 7 - 2 \cdot 5) + 1 \cdot (4 \cdot 8 + 2 \cdot 2) \\ &= 6 \cdot (-54) - 18 + 36 = -306 \end{aligned}$$

expanding along the 2^{nd} *row of* B*, gives:*

$$|B| = -6 \cdot \begin{vmatrix} -2 & 5 \\ 8 & 7 \end{vmatrix} + 1 \cdot \begin{vmatrix} 4 & 5 \\ 2 & 7 \end{vmatrix} - 1 \cdot \begin{vmatrix} 4 & -2 \\ 2 & 8 \end{vmatrix} = 306$$

Notice that the only difference between $|A|$ *and* $|B|$ *is the sign of the cofactors - all the signs changed dues to the fact that row* 1 *in* A *is now row* 2 *in* B*.*

1.4 Systems of Equations

Consider first systems with first n linear equations with n unknowns, a special case of (1.1), with $m = n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= d_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= d_n \end{aligned}$$

We can represent the above system in matrix form, as

$$Ax = d$$

where A is $n \times n$, x is $n \times 1$ and d is $n \times 1$. Following our previous discussion about determinant and inverses, we have an important result about existence of a unique solution to this system:

Theorem 1 ***(Existence of a unique solution to linear system of equations).*** *Suppose* $Ax = d$*, and* A *is* $n \times n$ *coefficient matrix. Then, the following are equivalent:*

1. $|A| \neq 0$*, i.e. the rows and columns of the matrix* A *are independent,*
2. A *is nonsingular,*
3. A^{-1} *exists (*A *is invertible),*
4. *a unique solution* $x^* = A^{-1}d$ *exists.*

The above theorem generalizes to systems of m equations with n unknowns. Such systems are often encountered in financial economics, where we often solve linear systems with $m > n$, for example 7 equations and 3 unknowns. Although these systems are beyond the scope of this course, we present here the general theorem about linear systems, with the number of equations and unknowns is not necessarily the same. The general theorem uses the concept of a **rank** of a matrix, which is the number of independent columns or rows in a matrix (must be the same number). The rank of a matrix is easily computed using any modern mathematical software. For example, in Matlab `rank(A)` gives the rank of any matrix.

Theorem 2 ***(Solution set to linear systems).*** *Suppose $Ax = d$, and A is $m \times n$ coefficient matrix (possibly $m \neq n$). Let the matrix $[A|d]$ be the* ***augmented matrix*** *of coefficients, with additional column containing the constants d. The next four conditions characterize the possible solution set.*

1. *If $rank(A) \neq rank\left([A|d]\right)$, then the system has no solution*[1].

2. *If $rank(A) = rank\left([A|d]\right)$, then there exists at least one solution to the system (and possibly infinitely many solutions).*

3. *If $rank(A) = rank\left([A|d]\right) = n$, then there exists a unique solution to the system.*

4. *If $rank(A) = rank\left([A|d]\right) < n$, then there exist infinitely many solutions to the system.*

So far, we have two methods of solving a system of linear equation $Ax = d$: (i) solving by elimination (substitution), or (ii) finding the inverse of the coefficient matrix and premultiplying the constant vector: $x^* = A^{-1}d$. One advantage of the second method is that we can easily find solutions to systems for many different d vectors. Once we calculated A^{-1}, it is easy to multiply it by many different d. Yet another method of solving a linear system of equations is by Cramer's rule. While solving the system by inverting A gives the solution to all x, Cramer's rule allows solving for each unknown value separately.

1.4.1 Cramer's rule

Cramer's rule allows solving for each unknown value separately. Let the vector of unknowns be

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

[1]When adding a column (or row) to a matrix, the rank of the augment matrix is either unchanged (if the new column can be expressed as a linear combination of previous columns) or increases by 1 (if the new column is independent of the previous columns).

The solution to unknown j is:

$$x_j^* = \frac{|A_j|}{|A|}$$

where A_j is the same as the coefficient matrix A obtained after replacing column j by the constant vector d, and as usual, $|A_j|$ is the determinant of A_j. That is,

$$|A_j| = \underbrace{\begin{vmatrix} a_{11} & a_{12} & \cdots & d_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & d_2 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & d_n & \cdots & a_{nn} \end{vmatrix}}_{\text{column } j \text{ is replaced by } d}$$

1.5 Applications

1.5.1 Solving linear system of equations

Example 8 *Express the following system in matrix form, and solve it using (i) matrix inversion and (ii) Cramer's rule.*

$$\begin{aligned} 7x_1 - x_2 - x_3 &= 0 \\ 10x_1 - 2x_2 + x_3 &= 8 \\ 6x_1 + 3x_2 - 2x_3 &= 7 \end{aligned}$$

Solution 5 *The system in matrix form ($Ax = d$) is:*

$$\underbrace{\begin{bmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 8 \\ 7 \end{bmatrix}}_{d}$$

Whether we use inversion of A or Cramer's rule, we must find the determinant of A. We use Laplace expansion along the first row:

$$|A| = 7 \cdot |C_{11}| - 1 \cdot |C_{12}| - 1 \cdot |C_{13}|$$

Here $|C_{ij}|$ is the ij cofactor:

$$\begin{aligned}
|C_{11}| &= (-1)^{1+1}\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} = 1\cdot[(-2)\cdot(-2)-3\cdot 1] = 1 \\
|C_{12}| &= (-1)^{1+2}\begin{vmatrix} 10 & 1 \\ 6 & -2 \end{vmatrix} = (-1)\cdot[10\cdot(-2)-6\cdot 1] = 26 \\
|C_{13}| &= (-1)^{1+3}\begin{vmatrix} 10 & -2 \\ 6 & 3 \end{vmatrix} = 1\cdot[10\cdot 3-6\cdot(-2)] = 42
\end{aligned}$$

Later, we will need the other cofactors as well, so in the same manner we find:

$$\begin{aligned}
|C_{21}| &= (-1)^{2+1}\begin{vmatrix} -1 & -1 \\ 3 & -2 \end{vmatrix} = -5 \\
|C_{22}| &= (-1)^{2+2}\begin{vmatrix} 7 & -1 \\ 6 & -2 \end{vmatrix} = -8 \\
|C_{23}| &= (-1)^{2+3}\begin{vmatrix} 7 & -1 \\ 6 & 3 \end{vmatrix} = -27
\end{aligned}$$

$$\begin{aligned}
|C_{31}| &= (-1)^{3+1}\begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} = -3 \\
|C_{32}| &= (-1)^{3+2}\begin{vmatrix} 7 & -1 \\ 10 & 1 \end{vmatrix} = -17 \\
|C_{33}| &= (-1)^{3+3}\begin{vmatrix} 7 & -1 \\ 10 & -2 \end{vmatrix} = -4
\end{aligned}$$

Thus, the determinant of A is:

$$|A| = 7\cdot 1 - 1\cdot 26 - 1\cdot 42 = -61$$

Next, create the matrix of cofactors (each element in A is replaced by its cofactor).

$$C = \begin{bmatrix} |C_{11}| & |C_{12}| & |C_{13}| \\ |C_{21}| & |C_{22}| & |C_{23}| \\ |C_{31}| & |C_{32}| & |C_{33}| \end{bmatrix} = \begin{bmatrix} 1 & 26 & 42 \\ -5 & -8 & -27 \\ -3 & -17 & -4 \end{bmatrix}$$

The adjoint matrix of A is the transpose of the matrix of cofactors:

$$adj\,(A) = C' = \begin{bmatrix} 1 & -5 & -3 \\ 26 & -8 & -17 \\ 42 & -27 & -4 \end{bmatrix}$$

Thus, the inverse of A is:

$$A^{-1} = \frac{1}{|A|} adj\,(A) = \frac{1}{-61} \begin{bmatrix} 1 & -5 & -3 \\ 26 & -8 & -17 \\ 42 & -27 & -4 \end{bmatrix}$$

Finally, the solution to the linear system is

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = A^{-1}d = \frac{1}{-61} \begin{bmatrix} 1 & -5 & -3 \\ 26 & -8 & -17 \\ 42 & -27 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Using Cramer's rule, the solution to the first unknown is:

$$x_1^* = \frac{|A_1|}{|A|} = \frac{1}{-61} \begin{vmatrix} \mathbf{0} & -1 & -1 \\ \mathbf{8} & -2 & 1 \\ \mathbf{7} & 3 & -2 \end{vmatrix} = 1$$

Here, A_j is a matrix obtained from A when column j is replaced by d. Calculation of the above determinant involves the same steps as in $|A|$, and therefore I skip these steps. Similarly,

$$x_2^* = \frac{|A_2|}{|A|} = \frac{1}{-61} \begin{vmatrix} 7 & \mathbf{0} & -1 \\ 10 & \mathbf{8} & 1 \\ 6 & \mathbf{7} & -2 \end{vmatrix} = 3$$

$$x_3^* = \frac{|A_3|}{|A|} = \frac{1}{-61} \begin{vmatrix} 7 & -1 & \mathbf{0} \\ 10 & -2 & \mathbf{8} \\ 6 & 3 & \mathbf{7} \end{vmatrix} = 4$$

1.5.2 Market equilibrium (supply and demand)

Given demand and supply

$$\begin{aligned} Q_d &= a - bP \qquad a, b > 0 \\ Q_s &= -c + dP \quad c, d > 0 \end{aligned}$$

Find the equilibrium (Q^*, P^*).

Equilibrium requires $Q_d = Q_s = Q$. Thus, the system to be solved is:

$$\begin{aligned} Q &= a - bP \\ Q &= -c + dP \end{aligned}$$

or

$$\begin{aligned} Q + bP &= a \\ Q - dP &= -c \end{aligned}$$

In matrix form:

$$\begin{aligned} \begin{bmatrix} 1 & b \\ 1 & -d \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix} &= \begin{bmatrix} a \\ -c \end{bmatrix} \\ Ax &= d \end{aligned}$$

Recall that:

$$A^{-1} = \frac{1}{|A|} adj\,(A)$$

Determinant of A:

$$|A| = -d - b \neq 0$$

The matrix of cofactors:

$$C = \begin{bmatrix} -d & -1 \\ -b & 1 \end{bmatrix}$$

The adjoint matrix:

$$adj\,(A) = C' = \begin{bmatrix} -d & -b \\ -1 & 1 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} Q^* \\ P^* \end{bmatrix} = \frac{1}{-d-b} \begin{bmatrix} -d & -b \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ -c \end{bmatrix} = \begin{bmatrix} \frac{ad-bc}{b+d} \\ \frac{a+c}{b+d} \end{bmatrix}$$

Using Cramer's rule is left as an exercise.

1.5.3 General equilibrium (two or more markets)

Consider two markets, indexed 1 and 2, with the following linear demand and supply:

$$\begin{gathered}\text{Market 1}\\ Q_{d1} = a_1 + a_{11}P_1 + a_{12}P_2\\ Q_{s1} = b_1 + b_{11}P_1 + b_{12}P_2\\ \text{Market 2}\\ Q_{d2} = a_2 + a_{21}P_1 + a_{22}P_2\\ Q_{s2} = b_2 + b_{21}P_1 + b_{22}P_2\end{gathered}$$

Thus, a_{ij} is the coefficient in demand for good i on the price of good j. Recall that two goods can be related as complements ($a_{ij} < 0$), substitutes ($a_{ij} > 0$) or unrelated ($a_{ij} = 0$). Using the market clearing conditions $Q_{d1} = Q_{s1} = Q_1$ and $Q_{d2} = Q_{s2} = Q_2$, we have a system of 4 equations and 4 unknowns (Q_1, Q_2, P_1, P_2):

$$\begin{aligned} Q_1 &= a_1 + a_{11}P_1 + a_{12}P_2\\ Q_1 &= b_1 + b_{11}P_1 + b_{12}P_2\\ Q_2 &= a_2 + a_{21}P_1 + a_{22}P_2\\ Q_2 &= b_2 + b_{21}P_1 + b_{22}P_2\end{aligned}$$

Rearranging,

$$\begin{array}{ccccc} Q_1 & & -a_{11}P_1 & -a_{12}P_2 & = a_1\\ Q_1 & & -b_{11}P_1 & -b_{12}P_2 & = b_1\\ & Q_2 & -a_{21}P_1 & -a_{22}P_2 & = a_2\\ & Q_2 & -b_{21}P_1 & -b_{22}P_2 & = b_2\end{array}$$

In matrix form:

$$\underbrace{\begin{bmatrix} 1 & 0 & -a_{11} & -a_{12}\\ 1 & 0 & -b_{11} & -b_{12}\\ 0 & 1 & -a_{21} & -a_{22}\\ 0 & 1 & -b_{21} & -b_{22}\end{bmatrix}}_{A}\underbrace{\begin{bmatrix} Q_1\\ Q_2\\ P_1\\ P_2\end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} a_1\\ b_1\\ a_2\\ b_2\end{bmatrix}}_{d}$$

Solving this system, by inverting $A_{4\times 4}$ or using Cramer's rule, is very tedious. Alternatively, we subtract supply from demand in each market, and eliminate the quantities Q_1 and

Q_2. This will result in a system with only prices as unknowns (P_1 and P_2):

$$\begin{aligned}(b_{11} - a_{11}) P_1 + (b_{12} - a_{12}) P_2 &= a_1 - b_1 \\ (b_{21} - a_{21}) P_1 + (b_{22} - a_{22}) P_2 &= a_2 - b_2\end{aligned}$$

Written compactly,

$$\begin{aligned}c_{11}P_1 + c_{12}P_2 &= d_1 \\ c_{21}P_1 + c_{22}P_2 &= d_2\end{aligned}$$

where

$$\begin{array}{ll}c_{11} = b_{11} - a_{11}, & c_{12} = b_{12} - a_{12} \\ c_{21} = b_{21} - a_{21} & c_{22} = b_{22} - a_{22}\end{array} \quad \text{and} \quad \begin{array}{l}d_1 = a_1 - b_1 \\ d_2 = a_2 - b_2\end{array}$$

In matrix form, the system is:

$$\underbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}}_{d}$$

This system has only 2 equations with 2 unknowns, and can be easily solved with matrix inversion or using Cramer's rule. We then substitute the equilibrium prices into either demand or supply in the two markets.

1.5.4 Keynesian model with taxes

Goods market:

$$\begin{aligned}\text{[Equilibrium in closed economy]} &: Y = C + I + G \\ \text{[Demand for consumption]} &: C = C_0 + b(Y - T) \\ \text{[Taxes]} &: T = T_0 + tY \\ \text{[Planned investment]} &: I = I_0 \\ \text{[Government spending]} &: G = G_0 + gY\end{aligned}$$

This system has 5 unknown endogenous variables: Y, C, T, I, G. It is easy to solve this entire system without linear algebra, by substituting equations 2 - 4 into the first equation, and solving for equilibrium output Y^*. But in order to demonstrate application of matrix algebra, we will reduce this system into 3 equations with unknowns Y, C, T, by substituting the last two equations into the first one:

$$\begin{aligned} Y &= C + I_0 + G_0 + gY \\ C &= C_0 + b\,(Y - T) \\ T &= T_0 + tY \end{aligned}$$

Rearranging the above for matrix form:

$$\begin{array}{cccl} (1-g)\,Y & -C & & = I_0 + G_0 \\ -bY & +C & +bT & = C_0 \\ -tY & & T & = T_0 \end{array}$$

And the matrix form is:

$$\underbrace{\begin{bmatrix} 1-g & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Y \\ C \\ T \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} I_0 + G_0 \\ C_0 \\ T_0 \end{bmatrix}}_{d}$$

Once again, we have a system of the $Ax = d$ form, which can be solved using matrix algebra (either inverting A or using Cramer's rule).

1.5.5 IS/LM model with taxes

This model adds the money market to the Keynesian model in the previous section. With the money market, there are too many equations in the model, we typically reduce the system to 2 equations: The IS curve and the LM curve. The goods market:

$$\begin{aligned} \text{[Equilibrium in closed economy]} &: \; Y = C + I + G \\ \text{[Demand for consumption]} &: \; C = C_0 + b\,(Y - T) \\ \text{[Taxes]} &: \; T = T_0 + tY \\ \text{[Planned investment]} &: \; I = I_0 - hi \\ \text{[Government spending]} &: \; G = G_0 + gY \end{aligned}$$

From the goods market, we derive the IS curve (the combinations of Y and i that constitute equilibrium in the goods market), by substituting equations 2 - 5 into the first equation:

$$\begin{aligned} Y &= \underbrace{C_0 + b\left(Y - T_0 - tY\right)}_{C} + \underbrace{I_0 - hi}_{I} + \underbrace{G_0 + gY}_{G} \\ Y - b\left(1-t\right)Y - gY &= C_0 - bT_0 + I_0 + G_0 - hi \\ Y &= \frac{C_0 - bT_0 + I_0 + G_0}{1 - b\left(1-t\right) - g} - \frac{h}{1 - b\left(1-t\right) - g} i \end{aligned}$$

The IS curve is therefore:

$$[IS] : Y = \frac{A}{1 - b\left(1-t\right) - g} - \frac{h}{1 - b\left(1-t\right) - g} i$$

There are many variants of the Keynesian goods market. For example, if government spending is just constant $G = G_0$, and if there is no proportional tax, then the IS curve simplifies to:

$$[IS] : Y = \frac{A}{1-b} - \frac{h}{1-b} i$$

Another variation is to allow investment to depend on output: $I = I_0 + dY - hi$. As an exercise, derive the IS curve with this investment function. In all variations however, the IS curve is a linear relationship between Y and i.

The money market:

$$\begin{aligned} [\text{Money demand}] &: M_d = kY - li \\ [\text{Money supply}] &: M_s = M_0 \end{aligned}$$

From the money market we derive the LM curve:

$$[LM] : Y = \frac{M_0}{k} + \frac{l}{k} i$$

Thus, the goal is to use the IS and LM (linear) equations to solve for equilibrium output and interest rate (Y^*, i^*). Here we choose the simplest IS curve.

$$\begin{aligned} [IS] &: Y = \frac{A}{1-b} - \frac{h}{1-b} i \\ [LM] &: Y = \frac{M_0}{k} + \frac{l}{k} i \end{aligned}$$

As always, the above system must be rearranged into matrix form $Ax = d$. Thus, we write

$$\begin{aligned} Y + \frac{h}{1-b} i &= \frac{A}{1-b} \\ Y - \frac{l}{k} i &= \frac{M_0}{k} \end{aligned}$$

or

$$\underbrace{\begin{bmatrix} 1 & \frac{h}{1-b} \\ 1 & -\frac{l}{k} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Y \\ i \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} \frac{A}{1-b} \\ \frac{M_0}{k} \end{bmatrix}}_{d}$$

Once again, we obtain a familiar linear system, which can be solved using matrix inversion or Cramer's rule. You should think of the IS/LM model as a relatively simple macroeconomic model, which allows analyzing the effects of fiscal and monetary policy on the economy. Fiscal policy in this model is represented by government spending parameters G_0 and g, and tax policy parameters T_0 and t. The monetary policy is represented by the money supply M_0. Thus, with the tools you have developed in this chapter, we can change these exogenous policy parameters and calculate the impact on endogenous variables, such as output, consumption, government deficit $(G - T)$, investment.

1.5.6 Leontief Input-Output model

This is a model designed for command economy, i.e. instead of letting the markets determine the quantities of goods produced (based on supply and demand), the central planners decide on the levels of production of every good and service. There are n industries. Each industry produces a single output, using as inputs the products produced by other industries or its own product. For example, the steel industry requires steel as one of its inputs, as well as machines and other inputs. The production function is of the fixed proportions type, i.e. to produce one unit of good j you need particular amounts of inputs. Let a_{ij} be the amount of good i needed to produce 1 unit of good j. Since the units of goods are all different, it is convenient to convert them all to dollars. So $a_{32} = 0.35$ means that 35 cents worth of good 3 is needed to produce 1 dollar worth of good 2. The $[a_{ij}]$ is called the *input-output coefficient* *matrix*. We can summarize all the input coefficients in a matrix:

$$\begin{array}{cc} & \text{Output} \\ & \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \cdots & \mathbf{n} \end{array} \\ \text{Input} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \vdots \\ \mathbf{n} \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \end{array}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

There are also consumers in this model, and we let d_i be the demand of consumers for product i. Thus, for all products $i = 1, 2, ..., n$, we must have:

$$\begin{aligned} x_1 &= a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + d_1 \\ x_2 &= a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n + d_2 \\ &\vdots \\ x_n &= a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n + d_n \end{aligned}$$

The left hand side of a typical equation, say x_i, is the total output (in dollars) produced by industry i. This output is used by all other industries : $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n$ (as intermediate good), as well as consumed by the consumers d_i (as final good). The above system can be rewritten as:

$$\begin{aligned} (1 - a_{11})\, x_1 - a_{12}x_2 - ... - a_{1n}x_n &= d_1 \\ -a_{21}x_1 + (1 - a_{22})\, x_2 - ... - a_{2n}x_n &= d_2 \\ &\vdots \\ -a_{n1}x_1 - a_{n2}x_2 - ... + (1 - a_{nn})\, x_n &= d_n \end{aligned}$$

In matrix form, the above becomes:

$$\begin{bmatrix} 1 - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1 - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1 - a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}_{n\times 1} \tag{1.5}$$

The matrix on the left is called *technology matrix* or *Leontief matrix*, and can be written as

$$T = I - A$$

Then, for any given demand d, we want to find out the necessary output x in each industry, which will satisfy simultaneously the consumer demand and the inter-industry demand. These required industry output levels can be found by solving the system in (1.5), or in matrix notation:

$$Tx = d$$

We already know that if T is nonsingular, there is a unique solution:

$$x^* = T^{-1}d$$

It turns out that if the input-output coefficient matrix A has all nonnegative values and the sum of entries in each column is less than 1 (i.e. $\sum_{i=1}^{n} a_{ij} < 1 \; \forall j$), then the inverse $(I - A)^{-1}$ exists, and contains only nonnegative values (which implies that the industry outputs are also nonnegative). The condition $\sum_{i=1}^{n} a_{ij} < 1 \; \forall j$ means that the cost of producing 1 dollar worth of good j is less than 1 dollar. If this condition is not satisfied, the production of good j is not economically justifiable.

Example 9 *Suppose the input-output matrix is*

$$A = \begin{bmatrix} 0.15 & 0.5 & 0.25 \\ 0.3 & 0.1 & 0.4 \\ 0.15 & 0.3 & 0.2 \end{bmatrix}$$

Suppose that consumer demand fluctuates between

$$d = \begin{bmatrix} 20 \\ 20 \\ 10 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} 10 \\ 20 \\ 20 \end{bmatrix}$$

Find the corresponding industry outputs that simultaneously satisfy the consumer demands and the inter-industry input requirements.

Solution 6

$$T = I - A = \begin{bmatrix} 1 - 0.15 & -0.5 & -0.25 \\ -0.3 & 1 - 0.1 & -0.4 \\ -0.15 & -0.3 & 1 - 0.2 \end{bmatrix}$$

Need to solve

$$Tx = d$$

The solution is:

$$x = T^{-1}\begin{bmatrix} 20 \\ 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 84.774 \\ 75.72 \\ 56.79 \end{bmatrix}$$

$$x = T^{-1}\begin{bmatrix} 10 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 79.012 \\ 79.506 \\ 69.630 \end{bmatrix}$$

1.5.7 Ordinary Least Squares

Another important application of linear algebra is in statistics, specifically in multiple regression analysis. We assume that the data is generated with the following model:

$$Y_i = \beta_1 + \beta_2 X_{2,i} + \beta_3 X_{3,i} + ... + \beta_k X_{k,i} + u_i \tag{1.6}$$

Here Y_i is the value of the dependent variable (i.e. variable of interest, which we want to predict) for subject i, $X_{2,i}, ..., X_{k,i}$ are values of independent variables (regressors) for subject i, $\beta_1, ..., \beta_k$ are unknown coefficients to be estimated, and u_i is unobserved error term representing all influences on the dependent variable other than the regressors. The only assumption about u_i needed here is that it has mean zero, i.e. $E(u_i) = 0$. There are a number of ways to estimate the unknown parameters $\beta_1, ..., \beta_k$, but the most popular method is Ordinary Least Squares (OLS). Suppose that we have a random sample of size n on the dependent variable and the regressors.

Definition 2 *(OLS estimator). Let $b_1, ..., b_k$ be some estimators of the unknown coefficients $\beta_1, ..., \beta_k$ in (1.6). The fitted (or predicted) values of the dependent variable are*

$$\hat{Y}_i = b_1 + b_2 X_{2,i} + b_3 X_{3,i} + ... + b_k X_{k,i} \tag{1.7}$$

and residual (or prediction error) for observation i:

$$e_i = Y_i - \hat{Y}_i = Y_i - b_1 - b_2 X_{2,i} - b_3 X_{3,i} - ... - b_k X_{k,i} \tag{1.8}$$

The OLS estimator of $\beta_1, ..., \beta_k$ is the vector $\left(b_1^{OLS}, ..., b_k^{OLS}\right)$ which minimizes the sum of squared residuals

$$RSS = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (Y_i - b_1 - b_2 X_{2,i} - b_3 X_{3,i} - ... - b_k X_{k,i})^2 \tag{1.9}$$

Mathematically, OLS estimator is defined as[2]

$$\left(b_1^{OLS}, ..., b_k^{OLS}\right) = \arg\min_{b_1,...,b_k} \sum_{i=1}^{n} \left(Y_i - b_1 - b_2 X_{2,i} - b_3 X_{3,i} - ... - b_k X_{k,i}\right)^2$$

In order to solve for the OLS estimator we need to solve the first order conditions

$$\frac{\partial RSS}{\partial b_j} = 0, \text{ for } j = 1, ..., k \tag{1.10}$$

Notice that since the objective function is quadratic, the first order conditions are linear. Thus, solving for OLS estimator boils down to solving a system of k linear equations and k unknowns. This task is fairly simple for $k = 2$ and leads to easy to interpret solution:

$$\begin{aligned} b_2 &= \frac{\sum_{i=1}^{n}\left(X_{2,i} - \bar{X}_2\right)\left(Y_i - \bar{Y}\right)}{\sum_{i=1}^{n}\left(X_{2,i} - \bar{X}_2\right)^2} \\ b_1 &= \bar{Y} - b_2 \bar{X}_2 \end{aligned}$$

where $\bar{X}_2$ and $\bar{Y}_2$ are sample averages of X_2 and Y: $\bar{X}_2 = \frac{1}{n}\sum_{i=1}^{n} X_{2,i}$ and $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. When $k = 3$ solving 1.10 without linear algebra is very messy and leads to equations that would barely fit this page. With $k = 4$ the task of solving for the OLS estimator without linear algebra is nearly impossible. Next, we demonstrate how the OLS estimation problem can be elegantly stated and solved using linear algebra tools discussed in this chapter.

OLS with linear algebra

First, all the data on regressors can be written as an n by k matrix:

$$X = \begin{bmatrix} 1 & X_{2,1} & X_{3,1} & \cdots & X_{k,1} \\ 1 & X_{2,2} & X_{3,2} & \cdots & X_{k,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{2,n} & X_{3,n} & \cdots & X_{k,n} \end{bmatrix}_{n \times k}$$

There are n observations (rows), and k regressors (columns), where the first regressor X_1 is just a vector of 1s. The data on the dependent variable is just an n by 1 vector, as well as

[2] $\arg\min$ of some function $f(x)$ is the argument or the element x^* which minimizes the function f.

the error term:

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}_{n\times 1}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}_{n\times 1}$$

Finally, the unknown coefficients, and their estimators are k by 1 vectors:

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}_{k\times 1}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}_{k\times 1}$$

With this notation, the multiple regression model in 1.6 can be written as

$$Y = X\beta + u,$$

the fitted equation is

$$\hat{Y} = Xb,$$

the residuals

$$e = Y - \hat{Y} = Y - Xb,$$

and the sum of squared residuals becomes

$$RSS = e'e = (Y - Xb)'(Y - Xb)$$

To see why the last step is correct, notice that

$$e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

and

$$e'e = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \sum_{i=1}^{n} e_i^2$$

Thus, the OLS problem we need to solve is

$$\min_b RSS = (Y - Xb)' (Y - Xb)$$

Using rules of transpose (transpose of a product = product of transposes in reverse order), we have $(Y - Xb)' = Y' - b'X'$. Thus, the OLS problem is:

$$\begin{aligned} \min_b RSS &= (Y' - b'X') (Y - Xb) \\ &= Y'Y - Y'Xb - b'X'Y + b'X'Xb \end{aligned}$$

Observe that the two middle terms are transposes of each other and they have dimension 1×1 (i.e. scalar): $Y'_{1\times n} X_{n\times k} b_{k\times 1}$. Thus, the two middle terms are the same because transpose of a scalar is the scalar itself. Thus, the OLS problem is reduced to

$$\min_b RSS = Y'Y - 2b'X'Y + b'X'Xb$$

There are two terms with b - the first one is linear $-2b'X'Y$ and the second one is quadratic $b'X'Xb$ (to be discussed in section 3.5.2 on quadratic forms). It is easy to verify the following rules of derivatives (gradient vector) with respect to vector $x_{k\times1}$, where a is $k \times 1$ vector (in our case $a = X'Y$), A is $k \times k$ matrix:

$$\begin{aligned} \frac{\partial}{\partial x} (a'x) &= \frac{\partial}{\partial x} (x'a) = a \\ \frac{\partial}{\partial x} (x'Ax) &= 2Ax \end{aligned}$$

Applying these rules gives us the first order necessary conditions for the OLS problem:

$$\begin{aligned} \frac{\partial RSS}{\partial b} &= -2X'Y + 2X'Xb = 0 \\ X'Xb &= X'Y \end{aligned}$$

As we will see in chapter 3, the above condition characterizes a unique global minimum. The solution, if it exists, is given by:

$$b^{OLS} = (X'X)^{-1} X'Y \tag{1.11}$$

From 1.11 we see that a unique solution exists if and only if $X'X$ is invertible. It turns out that this is equivalent to all of the k columns of X being linearly independent. If two or more columns are linearly dependent, the problem is called *perfect multicollinearity*, and in this case the OLS estimator cannot be calculated. Also observe, that $X'X$ is $k \times k$ matrix. While the data size may be large (n can be millions of observations), the number of estimated parameters k is usually no more than a few dozens. Most of the computational power is spent on matrix multiplications $X'X$ and $X'Y$.

In Matlab, it is possible to find OLS estimators with `b = inv(X'*X)*X'*Y`, however Matlab will issue a warning saying that matrix inversion is slower and less accurate and suggesting that we use `b = (X'*X)\(X'*Y)`, which uses Gaussian elimination to solve the system of first order conditions $X'Xb = X'Y$. If you simply type `b = X\Y`, matlab will understand that you are interested in `b = (X'*X)\(X'*Y)`, but will spend a few seconds rearranging `X\Y` as `(X'*X)\(X'*Y)`, and will lose some accuracy in the process.

Chapter 2

Limits and Differential Calculus

2.1 Functions, Limits, Continuity and Derivatives

In this section, we review basic concepts of functions, their graphs and their inverses, limits and derivatives. All these concepts should be familiar from your calculus course.

2.1.1 Set basics

Definition of a set

A **set** is any collection of elements. For example:

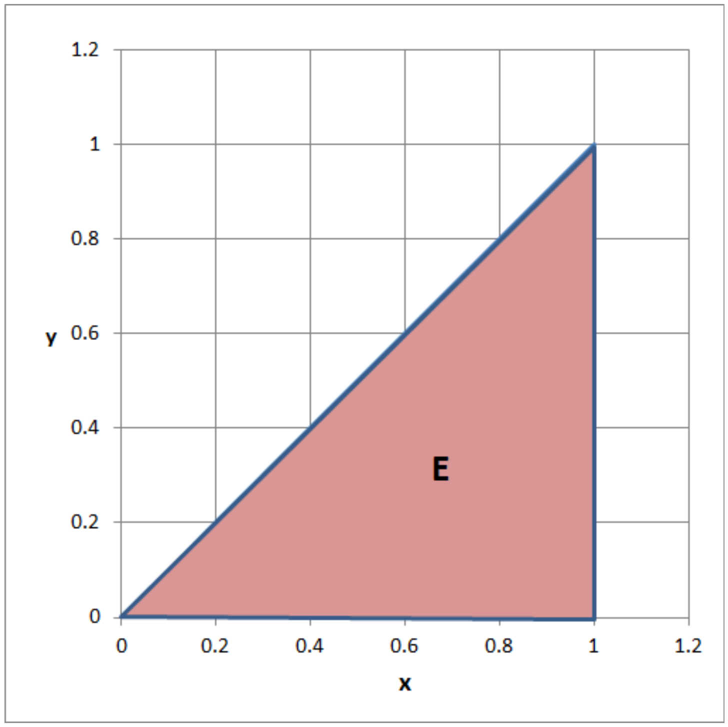
1. $A = \{0, 2, 4, 6, 8, 10\}$ - the set of even numbers between zero and 10.

2. $B = \{\text{red, white, blue}\}$ - the set of colors on the U.S. national flag.

3. $C = \{\text{SFSU students}|\text{female}, GPA \geq 3.2\}$ - the set of SFSU students that satisfy the conditions listed after the vertical bar, i.e., female and GPA at least 3.2.

4. $D = \{(x, y) \in \mathbb{R}^2 | x = y\}$ - the set of vectors in the two dimensional Euclidean space, such that the x-coordinate is equal to the y-coordinate.

5. $E = \{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq 1, 0 \leq y \leq 1, x \geq y\}$ - the set of vectors in the two dimensional Euclidean space such that both coordinates are between 0 and 1 and the x-coordinate is greater or equal to the y-coordinate. The next figure illustrates this set graphically.



6. $F = \{\}$ is an empty set, also denoted by $\emptyset$, is a set which does not contain any elements.

Set operations

Elements **contained** in set A are denoted by $a \in A$ and elements that are **not contained** in A are denoted $b \notin A$. For example, if $A = \{1, 2, 3\}$, then $3 \in A$ and $7 \notin A$.

B is a **subset** of set A, denoted $B \subseteq A$, if every element in B is also contained in A. For example, if $A = \{1, 2, 3, 4, 5\}$, then both $B = \{1, 5\}$ and $C = \{1, 2, 3, 4, 5\}$ are subsets of A, but $D = \{4, 5, 10\}$ is not a subset of A because $10 \notin A$.

Complement of set A is the set A^c, which contains all the elements that are not in A. For example, if $A = \{1, 2, 3\}$, then $5 \in A^c$, that is $5 \notin A$.

Cartesian product of sets A and B is the set $A \times B$ of all ordered pairs such that the first element belongs to A and the second belongs to B. For example, if $A = \{1, 2, 3\}$ and $B = \{7, 8\}$, then $A \times B = \{(1, 7), (1, 8), (2, 7), (2, 8), (3, 7), (3, 8)\}$.

Union of sets A and B is the set $A \cup B$ of elements that are either in A or in B. For example, if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

Intersection of sets A and B is the set $A \cap B$ that contains only the elements that are in both A and B. For example, if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cap B = \{3\}$.

Set subtraction, denoted $A - B$ or $A \backslash B$, is the set of elements in A, that are not in B. That is, $A \backslash B = \{a \in A \cap B^c\}$. For example, if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \backslash B = \{1, 2\}$. The set $\mathbb{R} \backslash \{0\}$ denotes the set of all real numbers, except zero.

Special sets

Convex sets: the set B is convex if $\forall x, y \in B$, we have $\alpha x + (1-\alpha) y \in B$, $\forall \alpha \in [0,1]$. In words, a linear combination of any two elements in the set, also belongs to the set. For example, the set of real numbers is convex since a linear combination of any two real numbers is convex. However, the set of integers $\mathbb{Z}$ is not convex, since 1 and 4 is 2.5, not an integer.

The set of **real numbers** is denoted by $\mathbb{R}$, and it contains all numbers in $(-\infty, \infty)$. The Cartesian product $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ contains all the two-dimensional vectors in the real plane: $\{(x, y) \,|x \in \mathbb{R}, y \in \mathbb{R}\}$. Similarly, $\mathbb{R}^3$,$\mathbb{R}^4$,...,$\mathbb{R}^n$ represent spaces of real numbers (Euclidean spaces) of 3-dimensions, 4-dimenstions, etc. The set $\mathbb{R}_+$ contains all non-negative numbers, i.e. $[0, \infty)$ and the set $\mathbb{R}_{++}$ contains all positive real numbers $(0, \infty)$. The set of extended real numbers is $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, which contains the real numbers, as well as $-\infty$ and ∞.

2.1.2 Basic concepts of functions

Definition 3 *A **function** $f : A \to B$ consists of the domain set A, the codomain set B, and a rule that assigns to every element in the domain, a unique element in the codomain*[1]*. We can say that the function f maps from A into B.*

Example 10 *Let $F : \mathbb{R}_+^2 \to \mathbb{R}$ be a function, with the rule $F(K, L) = AK^\theta L^{1-\theta}$, $A > 0$, $0 < \theta < 1$. The domain is $\mathbb{R}_+^2 = \mathbb{R}_+ \times \mathbb{R}_+$, meaning that $K \in \mathbb{R}_+$ and $L \in \mathbb{R}_+$ ($K \geq 0$ and $L \geq 0$). The codomain is $\mathbb{R}$ - the set of real numbers. Thus, this function maps pairs of non-negative numbers (K, L) into a real number $AK^\theta L^{1-\theta}$. For example, if $A = 2$, and $\theta = 0.3$, then the point $(K, L) = (10, 5)$ is mapped into $2 \cdot 10^{0.3} \cdot 5^{1-0.3} = 12.311$. We say that 12.311 is the image of $(10, 5)$ under F or the value of F at $(10, 5)$.*

Often, when describing a function, we simply write the mapping rule, without specifying the domain and the codomain, which are supposed to be clear from the context.

Definition 4 *The **image** of a function $f : A \to B$ is the set*

$$\mathrm{Im}(f) = \mathrm{Im}(A) = \{y \in B | y = f(x), x \in A\}$$

or in short

$$\mathrm{Im}(f) = \{f(x) \in B | x \in A\}$$

*The term **range** of a function is often used as synonym of image, and sometimes used to refer to the codomain.*

[1] A mapping that assigns possibly more than one value to every element in the domain is called **correspondence**.

Example 11 *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function, with the rule $f(x) = \frac{x}{1+x}$. The domain of f is $\mathbb{R}_+$, its codomain is $\mathbb{R}$, but the image of f is the set $[0, 1)$.*

Figure (2.1) is a graphical illustration of domain A, codomain B and an image Im, of some abstract function.

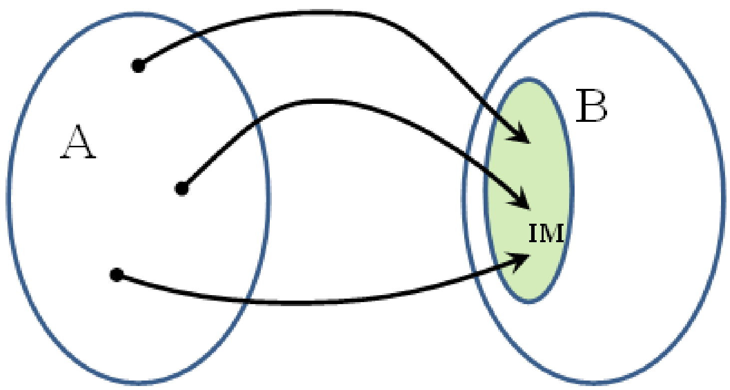


Figure 2.1: Domain A, codomain B, and image IM.

Definition 5 *The* ***graph*** *of a function $f : A \to B$ is the set of all ordered pairs of the form (x, y) such that $x \in A$ and $y = f(x) \in B$. Formally*

$$Gr(f) = \{(x, y) \in A \times B | y = f(x)\}$$

Example 12 *The graph of the function $f : \mathbb{R} \to \mathbb{R}$, with rule $f(x) = x^3$, is:*

$$Gr(f) = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} | y = x^3\right\}$$

The graph of $F(K, L) = AK^{\theta}L^{1-\theta}$ is:

$$Gr(F) = \left\{(K, L, Y) \in \mathbb{R}_+^2 \times \mathbb{R} | Y = AK^{\theta}L^{1-\theta}\right\}$$

Definition 6 *Let $f : A \to B$ be a function. Then f has an* ***inverse*** *if there exists a function $g : \operatorname{Im}(f) \to A$ such that*

$$f(x) = y \quad \Longleftrightarrow \quad g(y) = x$$

for all $x \in A$ and all y in $\operatorname{Im}(f)$. In this case, g is the inverse of f, and is denoted by f^{-1}.

It is easy to verify that an inverse of a function f exists if for every y in the image of f there is a unique x in the domain of f. This property is satisfied if f is **strictly monotone.**

A function f is **strictly increasing** if for any x_1 and x_2 in the domain of f, we have

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) > f(x_2)$$

and **strictly decreasing**, if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) < f(x_2)$$

In either of these cases, the inverse f^{-1} exists.

Example 13 *Let $f(x) = x^3$. Notice that f is strictly increasing. The inverse of f is $f^{-1}(y) = y^{1/3}$.*

In the above example, the domain of the function $f(x) = x^3$ is the set of real numbers $\mathbb{R}$, (also called the real line or the unidimensional Euclidian space). The codomain of this function is also $\mathbb{R}$. This is an example of a function of one variable - x. We often encounter functions of several variables, for example, utility function can depend on quantities consumed of several goods: $u(x,y) = x^\alpha y^\beta$, $\alpha, \beta > 0$. Here the utility u depends on quantities of two goods, and the domain of this function is $\mathbb{R}_+ \times \mathbb{R}_+$, or $\mathbb{R}_+^2$, where the $+$ indicates that quantities consumed are restricted to non-negative numbers. The value of utility associated with each pair (x,y) is $x^\alpha y^\beta$, which is a real number. Thus, the codomain of u is $\mathbb{R}$ or $\mathbb{R}_+$. Thus, we can write $u : \mathbb{R}_+^2 \to \mathbb{R}$, i.e., the utility function maps elements (pairs) from the two-dimensional Euclidian space into the real line.

Example 14 *The following function $f : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $(x,y) \to (x+y, x^2 - y^2, x^3)$ is a function that maps every element in $\mathbb{R}^2$ to an element in $\mathbb{R}^3$.*

The most general real-valued functions are $f : \mathbb{R}^m \to \mathbb{R}^n$, i.e. mapping m-dimensional Euclidian spaces into n-dimensional Euclidian spaces.

2.1.3 Limit of a function

There are several ways to define a limit of a function, and here we present just one way, that I find the most convenient for this course. This definition is based on *one-sided* limits.

Definition 7 *The **left limit** of the function f at point x_0 is L, written as,*

$$\lim_{x \nearrow x_0} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $x_0 - x < \delta \Rightarrow |f(x) - L| < \varepsilon$. The notation $x \nearrow x_0$ means that x approaches x_0 from below (from the left). In words, L is the left limit if the function can approach arbitrarily close to L (i.e. $|f(x) - L| < \varepsilon$) when x approaches close enough to x_0 from below (i.e. $x_0 - x < \delta$).

*Similarly, the **right limit** of the function f at point x_0 is L, written as,*

$$\lim_{x \searrow x_0} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $x - x_0 < \delta \Rightarrow |f(x) - L| < \varepsilon$. The notation $x \searrow x_0$ means that x approaches x_0 from above (from the right). In words, L is the right limit if the function can approach arbitrarily close to L ($|f(x) - L| < \varepsilon$) when x approaches close enough to x_0 from above (i.e. $x - x_0 < \delta$).

*If both of these limits are the same, we say that the function f has the **limit** L at point x_0, and write:*

$$\lim_{x \to x_0} f(x) = L$$

*Conversely, if the left limit is not the same as the right limit, then we say that the limit of f at point x_0 **does not exist**.*

Example 15 *Figure (2.2) shows a function f such that the left limit L_1 is not equal to the right limit L_2, and therefore, the limit of the function at point x_0 does not exist.*

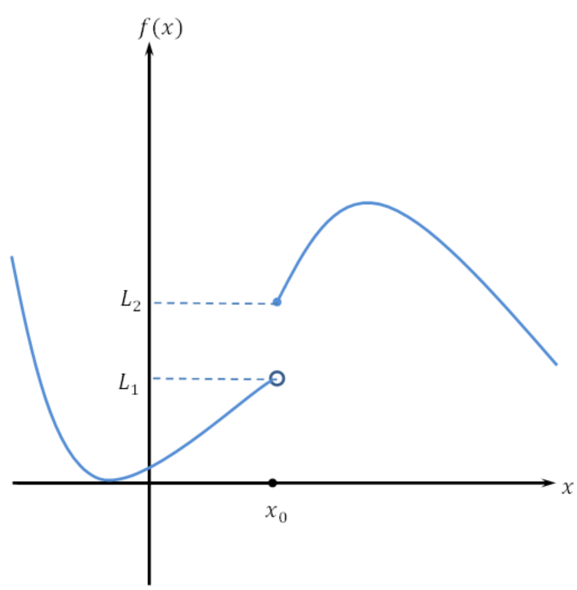


Figure 2.2: The limit of f at x_0 does not exist.

Limits involving infinity

A word of caution is needed here. In the above definition, the point x_0 as well as the limit L are *real numbers*. The set of real numbers is denoted by $\mathbb{R}$. Often we need limits of functions where x_0 is infinity or minus infinity (∞ or $-\infty$). In other cases, the limit itself, L, can be ∞ or $-\infty$. The definition we presented must be modified, to accommodate such cases. For such cases, we define the extended real line as $\bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, \infty\}$, i.e. the union of the set of real numbers with the two objects ∞ and $-\infty$. Since these objects are not numbers, the difference $x - x_0$ is not defined when x is a number and x_0 is either ∞ or $-\infty$. The definition of a limit of a function at x_0, when x_0 is infinity or minus infinity, goes as follows:

Definition 8 *The **limit of the function** f **as** x **approaches** ∞ **is** L, written as,*

$$\lim_{x \to \infty} f(x) = L$$

if for every $\varepsilon > 0$, there exists a number $M > 0$, such that $x > M \Rightarrow |f(x) - L| < \varepsilon$. In words, we can always find a large enough positive *number M such that for $x > M$ the function $f(x)$ gets arbitrarily close to L.*

*Similarly, The **limit of the function** f **as** x **approaches** $-\infty$ **is** L, written as,*

$$\lim_{x \to -\infty} f(x) = L$$

if for every $\varepsilon > 0$, there exists a number $M < 0$, such that $x < M \Rightarrow |f(x) - L| < \varepsilon$. In words, we can always find a large enough negative *number M such that for $x < M$ the function $f(x)$ gets arbitrarily close to L.*

The definition of limit of a function at x_0, when the limit L is infinity or minus infinity, goes as follows:

Definition 9 *The **limit of the function** f **at point** x_0 **is** ∞, written as,*

$$\lim_{x \to x_0} f(x) = \infty$$

if for every $M > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow f(x) > M$. In words, for any positive *number M, we can always make the function $f(x)$ exceed this number, if we choose x close enough to x_0.*

*Similarly, the **limit of the function** f **at point** x_0 **is** $-\infty$, written as,*

$$\lim_{x \to x_0} f(x) = -\infty$$

if for every $M < 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow f(x) < M$. In words, for any negative *number M, we can always make the function $f(x)$ be smaller than this number, if we choose x close enough to x_0.*

We could add to the above the definitions of limits of the type: $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$, but these are straightforward and omitted. The next table summarizes our limits definitions:

$x_0 \in \mathbb{R}$, $L \in \mathbb{R}$
$\lim_{x\nearrow x_0} f(x) = L_1$ if $\forall\varepsilon > 0,\ \exists\delta > 0$ s.t. $x_0 - x < \delta \Rightarrow |f(x) - L| < \varepsilon$
$\lim_{x\searrow x_0} f(x) = L_2$ if $\forall\varepsilon > 0,\ \exists\delta > 0$ s.t. $x - x_0 < \delta \Rightarrow |f(x) - L| < \varepsilon$
$\lim_{x\to x_0} f(x) = L$ if $L_1 = L_2 = L$
$L_1 \neq L_2 \Rightarrow \nexists \lim_{x\to x_0} f(x)$
$x_0 \in \{-\infty, \infty\}$, $L \in \mathbb{R}$
$\lim_{x\to\infty} f(x) = L$ if $\forall\varepsilon > 0,\ \exists M > 0$ s.t. $x > M \Rightarrow |f(x) - L| < \varepsilon$
$\lim_{x\to-\infty} f(x) = L$ if $\forall\varepsilon > 0,\ \exists M < 0$ s.t. $x < M \Rightarrow |f(x) - L| < \varepsilon$
$x_0 \in \mathbb{R}$, $L \in \{-\infty, \infty\}$
$\lim_{x\to x_0} f(x) = \infty$ if $\forall M > 0,\ \exists\delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow f(x) > M$
$\lim_{x\to x_0} f(x) = -\infty$ if $\forall M < 0,\ \exists\delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow f(x) < M$

The next two examples illustrate the use of the definitions of limits to prove that a function has a given limit at a given point.

Example 16 *Consider the function $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = x^2$. Prove that*

$$\lim_{x\nearrow 5} x^2 = 25$$

That is, prove that the left limit of f at 5 is 25.

Solution 7 *We need to find a number $\delta > 0$, such that $5 - x < \delta \Rightarrow |x^2 - 25| < \varepsilon$ for any $\varepsilon > 0$. In words, we can make the function x^2 approach 25 arbitrarily close, by letting x approach close enough to 5 from below. Since $x \nearrow 5$, we know that $x < 5$, and $x^2 - 25 < 0$. Therefore, $|x^2 - 25| = 25 - x^2$ by definition of absolute value. Thus, we need to find $\delta > 0$ such that*

$$[x > 5 - \delta] \quad \Rightarrow \quad \left[25 - x^2 < \varepsilon\right]$$

or

$$[x > 5 - \delta] \quad \Rightarrow \quad \left[25 - \varepsilon < x^2\right]$$

or

$$[x > 5 - \delta] \quad \Rightarrow \quad \left[x > \sqrt{25 - \varepsilon}\right]$$

We can see that the required $\delta > 0$ is such that

$$\begin{aligned} 5-\delta &\geq \sqrt{25-\varepsilon} \\ &\quad or \\ 0 &< \delta \leq 5-\sqrt{25-\varepsilon} \end{aligned}$$

Notice that the interval containing the desired δ is not empty for any $\varepsilon > 0$ because the upper bound $5-\sqrt{25-\varepsilon} > 0$. Thus, we proved that for any $\varepsilon > 0$, we can find $\delta > 0$ such that $5-x < \delta \Rightarrow |x^2-25| < \varepsilon$. For example, if $\varepsilon = 0.01$, we can pick

$$\delta = 5-\sqrt{25-0.01} = 0.0010001$$

and this δ is guaranteed to satisfy the definition of lower limit.

Example 17 *Prove that*

$$\lim_{x\to\infty} \frac{x}{1+x} = 1$$

Solution 8 *We need to find a number $M > 0$ such that $x > M \Rightarrow \left|\frac{x}{1+x}-1\right| < \varepsilon$ for any $\varepsilon > 0$. In words, we can make the function $\frac{x}{1+x}$ approach arbitrarily close to 1, by choosing x big enough (bigger than some $M > 0$). Notice that for any $x > 0$, we have $\frac{x}{1+x} < 1$ and $\frac{x}{1+x} - 1 < 0$. Thus, the absolute value is $\left|\frac{x}{1+x}-1\right| = 1-\frac{x}{1+x}$. Therefore, the desired $M > 0$ is such that:*

$$\begin{aligned} [x > M] &\Rightarrow \left[1-\frac{x}{1+x} < \varepsilon\right] \\ [x > M] &\Rightarrow \left[\frac{1}{1+x} < \varepsilon\right] \\ [x > M] &\Rightarrow [1 < \varepsilon + \varepsilon x] \\ [x > M] &\Rightarrow \left[x > \frac{1-\varepsilon}{\varepsilon}\right] \end{aligned}$$

Thus, the desired M is $M \geq \frac{1-\varepsilon}{\varepsilon}$. For example, if $\varepsilon = 0.01$, we can choose $M = \frac{1-0.01}{0.01} = 99$. Larger M than that will obviously work as well. For example, $M = \frac{1}{\varepsilon}$.

Properties of limits

Limits of sums, differences, products and ratios of functions, are sums, differences, products and ratios. Formally, let f and g be two functions, such that $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ both exist. Then,

1. $$\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

2. $$\lim_{x \to x_0} (f(x) - g(x)) = \lim_{x \to x_0} f(x) - \lim_{x \to x_0} g(x)$$

3. $$\lim_{x \to x_0} (f(x) \cdot g(x)) = \left(\lim_{x \to x_0} f(x)\right) \cdot \left(\lim_{x \to x_0} g(x)\right)$$

4. $$\lim_{x \to x_0} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$$

   This property holds provided that the denominator is not zero.

5. **Substitution rule (chain rule).** Suppose that $\lim_{x \to x_0} f(x) = c$, and $\lim_{y \to c} g(y)$ exists. Then,

   $$\lim_{x \to x_0} g(f(x)) = \lim_{y \to c} g(y),$$

   provided that at least one of the following additional conditions hold:

   (a) $g(y)$ is continuous at c.[2]

   (b) $f(x) \neq c$ for all x in some open interval around c, except at $x = x_0$.

   This means that, in most cases, if you have a composite function $g(f(x))$. Then you can calculate the limit of $f(x)$ first (suppose this limit is c), and then calculate the limit of $g(y)$ as $y \to c$.

Example 18 *(Substitution rule). Find*

$$\lim_{x \to \infty} \frac{2x}{1+x}$$

Notice that this limit is of the form $\frac{\infty}{\infty}$*, and in the future we will find such limits with the help of L'Hôpital's rule. Here we show an alternative way, which utilizes the substitution rule. Dividing the numerator and denominator by* x*, gives*

$$\lim_{x \to \infty} \frac{2x}{1+x} = \lim_{x \to \infty} \frac{2}{1/x + 1}$$

[2] Continuity of functions is defined in the next section.

Let $f(x) = \frac{1}{x}$ and $g(y) = \frac{2}{y+1}$. Then, using the substitution rule

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x} = 0$$

and

$$\lim_{y\to 0} g(y) = \lim_{y\to 0} \frac{2}{y+1} = 2$$

2.1.4 Continuous functions

Definition 10 *The function f is* ***continuous at point*** *x_0 of its domain, if*

$$\lim_{x\to x_0} f(x) = f(x_0)$$

This means that (i) the function f has to be defined at point x_0, (ii) the limit as x approaches x_0 exists (a finite number), and (iii) this limit is equal to the value of the function at point x_0. If the function f is continuous at every point at its domain, then we say that f is a ***continuous function****.*

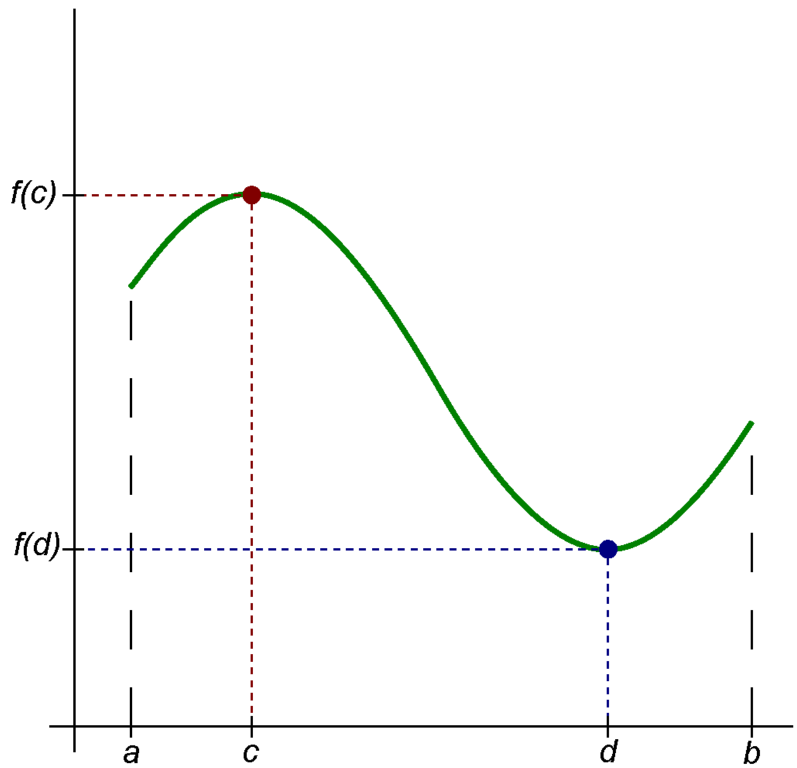
Notice that if we know in advance that a function f is continuous at x_0, then we can immediately find the limit by simply plugging in the point x_0 into the function. For example, if we know that e^x is continuous, then $\lim_{x\to 0} e^x = e^0 = 1$.

Continuous functions are very important in optimization (finding maximum or minimum). In Economics, many problems involve finding minimum or maximum of a function (e.g. profit maximization, utility maximization, cost minimization, minimum least squares, etc.). The next theorem says that all continuous function have at least one minimum and at least one maximum.

Theorem 3 *(**Extreme Value Theorem**) if a real-valued function f is continuous on a closed and bounded interval $[a, b]$, then f must attain its maximum and minimum value, each at least once. That is, there exist numbers c and d in $[a, b]$ such that:*

$$f(c) \geq f(x) \geq f(d) \quad \text{for all } x \in [a, b]$$

The next figure illustrates the Extreme Value Theorem:



The Extreme Value Theorem extends to continuous functions of any number of variables.[3]

2.1.5 Derivative of a function

Intuitively, the derivative of a function at a point, gives the slope of the function at that point. In other words, $f'(x)$ is the answer to the question "what is the change in the function f that results from a small change in its input x?" For example, if the function $C(q)$ gives the cost of producing q units of output, the derivative of the cost function, $C'(q)$ gives the change in cost resulting from producing additional unit of q, and we call this derivative the *marginal cost*. The marginal cost can be seen as the slope of the cost function with respect to quantity.

In addition, we often use derivatives to find maximum or minimum of a function. If the slope is positive, we know that the function f is increasing, and finding maximum requires increasing the value of x. Similarly, if the slope is negative, we know that the function f is decreasing, and finding maximum requires decreasing the value of x. Formally, we define derivative of a function with one variable, as follows.

[3] A continuous real-valued function on a nonempty compact space is bounded above and attains its supremum.

Definition 11 *The **derivative** of a function f at point x_0 is*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

*If the above limit exists, we say that the function f is **differentiable** at point x_0. If the derivative of f exists at each point of its domain, we say that f is a **differentiable function**.*

The above limit can be equivalently expressed as follows (letting $x = x_0 + h$):

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

The denominator is the change on the horizontal axis, Δx, and the numerator is the change on the vertical axis, Δy. Thus, the derivative gives the change of the function value (vertical axis) resulting from a small change in the function argument (horizontal axis) - a slope. Therefore, the derivative of a function at a point indicates whether the function is increasing or decreasing with the value of its argument, in the neighborhood of that point.

Another commonly used notation for the derivative of a function of one variable is:

$$f'(x) = \frac{df(x)}{dx} \quad \text{or} \quad \frac{d}{dx} f(x)$$

As with the definition of limits, there are cases where the slope of the function at a given point is infinity or minus infinity. In the above definition, the words "limit exists" mean that the limit is a real number. With such definition, which restricts derivatives to be real numbers, we can prove that a differentiable function must be continuous.

Theorem 4 *(**Differentiability implies continuity**). If f is differentiable at x_0, then f is continuous at x_0.*

Proof. The definition of continuity at point x_0 can be written as

$$\lim_{x \to x_0} [f(x) - f(x_0)] = 0$$

We need to prove that the above definition is satisfied whenever f has derivative at x_0, which means that the following limit exists:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Multiply and divide the term in the brackets by $x - x_0$:

$$\begin{aligned}
\lim_{x \to x_0} \left[f(x) - f(x_0) \right] &= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \right] \\
&= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \right] \cdot \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \cdot 0 = 0
\end{aligned}$$

The second step uses the property that limit of a product = product of limits. The last step uses the given, that f is differentiable at x_0, so the derivative $f'(x_0)$ exists and is finite. ■

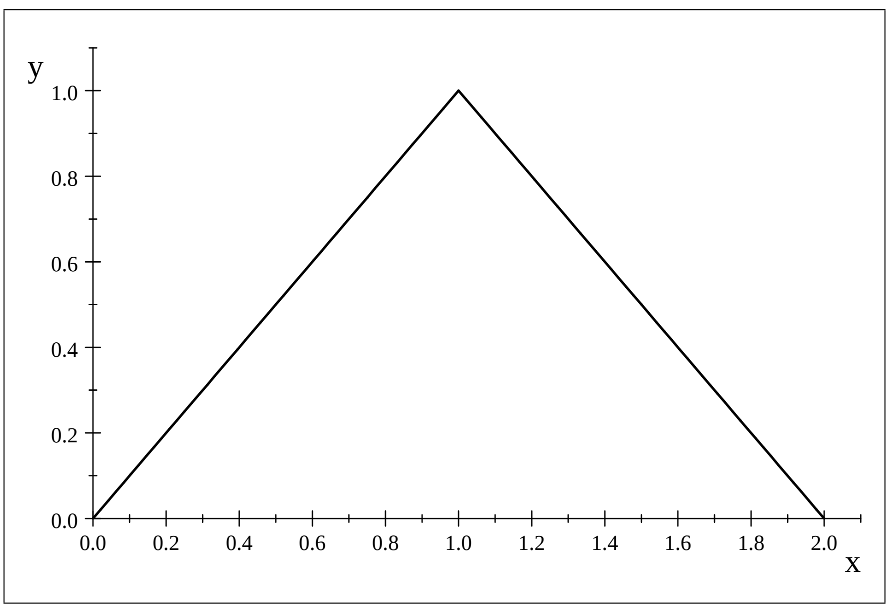
Example 19 *The following is a continuous function at $x = 1$, but not differentiable.*

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } 1 \le x \le 2 \end{cases}$$

Notice that

$$\lim_{x \to 1} f(x) = f(1) = 1$$

However, $f(x)$ is not differentiable at $x = 1$. The left derivative is 1 and the right derivative is -1, as seen the next figure.



2.1.6 Rules of differentiation

The derivative of a function f at a point x, $f'(x)$, is itself a function. Finding the derivative function from the original function f is called *differentiation*. The definition of derivatives in the previous section, can be used to prove the following rules of differentiation:

1. For constant a,
$$\frac{d}{dx}a = 0$$

2. Power rule:
$$\frac{d}{dx}\left(ax^b\right) = abx^{b-1}$$
The first two rules imply that for linear functions
$$\frac{d}{dx}(a + bx) = b$$

3. Sums or differences rule:
$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

4. Product rule:
$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)\, g(x) + f(x)\, g'(x)$$

5. Quotient rule:
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{g(x)^2}$$

6. Chain rule:
$$\frac{d}{dx}[f(g(x))] = f'(g(x))\, g'(x)$$

7. Logarithmic function:
$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

8. Exponential function:
$$\begin{aligned} \frac{d}{dx}a^x &= \ln(a)\, a^x \\ \text{in particular, } \frac{d}{dx}e^x &= e^x \end{aligned}$$

9. Inverse function rule: let $y = f(x)$ and $x = f^{-1}(y)$, then
$$\frac{dy}{dx} = \frac{df(x)}{dx} = \frac{1}{df^{-1}(y)/dy} = \frac{1}{dx/dy}$$

Example 20 *As example for the last rule, consider $y = e^x$, so the inverse is $x = f^{-1}(y) =$*

$\ln(y)$. *Observe that*

$$\begin{aligned}\frac{dy}{dx} &= e^x \quad \textit{[rule 8]}\\ \frac{dx}{dy} &= \frac{1}{y} = \frac{1}{e^x} \quad \textit{[rule 7]}\end{aligned}$$

Since $y = e^x$, *we have* $\frac{dy}{dx} = \frac{1}{dx/dy}$.

2.1.7 L'Hôpital's rule

Another important result, which sometimes helps to compute limits of ratios of functions, is known as the L'Hôpital's rule.

Theorem 5 ***(L'Hôpital's rule).*** *Suppose* $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ *or* $\pm\infty$, *and*

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} \text{ exists, and } g'(x) \neq 0 \text{ for all } x \neq x_0$$

then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

The L'Hôpital's rule sometimes helps computing limits of ratios, of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\frac{-\infty}{\infty}$ or $\frac{\infty}{-\infty}$, if the numerator and denominator are simplified by differentiations.

Example 21 *Consider the function*

$$f(x) = \frac{a^{1-x} - 1}{1 - x}, \quad x > 0,\, a > 0$$

We would like to compute the limit $\lim_{x\to 1} f(x)$, *and indeed when* $x \to 1$, *both the numerator and denominator approach zero (i.e., we have a limit of the form* $\frac{0}{0}$*). Using the L'Hôpital's rule*

$$\lim_{x\to 1} \frac{a^{1-x} - 1}{1 - x} = \lim_{x\to 1} \frac{-\ln(a)\, a^{1-x}}{-1} = \ln(a)$$

Notice that we used the differentiation of exponential function, rule 8, as well as the chain rule, which resulted in the "$-$" in front of the $\ln$.

Example 22 *Consider the function* $f(x) = x\ln(x)$. *We would like to find the limit* $\lim_{x\searrow 0} f(x)$, *using L'Hôpital's rule. Recall that* $\lim_{x\searrow 0}$ *is the right limit, when* x *approaches* 0 *from the right (from above), also denoted* $\lim_{x\to 0^+}$. *The* $\ln(\cdot)$ *function is defined only for strictly positive numbers, and therefore there can only be right limit here (the left limit does not exist).*

At first glance this does not look like a quotient for which the rule can be applied. But notice that

$$x \ln(x) = \frac{\ln(x)}{x^{-1}}$$

and $\lim_{x \searrow 0} \ln(x) = -\infty$, $\lim_{x \searrow 0} x^{-1} = \infty$. *Thus, we have a limit of the form* $\frac{-\infty}{\infty}$, *and we can apply L'Hôpital's rule.*

$$\lim_{x \searrow 0} \frac{\ln(x)}{x^{-1}} = \lim_{x \searrow 0} \frac{x^{-1}}{-x^{-2}} = \lim_{x \searrow 0} -x = 0$$

Thus, we proved that $\lim_{x \searrow 0} x \ln(x) = 0$.

Exercise 6 *Using L'Hôpital's rule, prove that*

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

Exercise 7 *Using L'Hôpital's rule, prove that*

$$\lim_{x \to \infty} \frac{x^2}{e^x} = 0$$

Exercise 8 *Using L'Hôpital's rule, prove that*

$$\lim_{x \to 0} x^x = 1$$

Hint: use the chain rule with $f(x) = \ln(x^x)$ *and* $g(y) = e^y$.

Exercise 9 *Let the production function be* $F(K, L) = A\left[\theta K^\rho + (1-\theta) L^\rho\right]^{\frac{1}{\rho}}$, $\quad \theta \in (0,1)$, $\rho \leq 1$, $A > 0$. *Using L'Hôpital's rule, prove that*

$$\lim_{\rho \to 0} A\left[\theta K^\rho + (1-\theta) L^\rho\right]^{\frac{1}{\rho}} = AK^\theta L^{1-\theta}$$

2.1.8 Higher order derivatives

Since the derivative of f, $f'(x)$, is itself a function of x, we can define the derivative of $f'(x)$. This derivative of derivative is the second derivative of f, and denoted

$$f''(x) \quad \text{or} \quad \frac{d^2 f(x)}{dx^2}$$

and defined as usual as the regular derivative, but instead of f we use f' in the definition:

$$\lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

More generally, the n-order derivative of f is denoted

$$\frac{d^n f(x)}{dx^n}$$

and is obtained by differentiating the function f, n times. If derivative of order n exists, we say that f is n-times differentiable.

Exercise 10 *let $u(x)$ be utility from wealth. We define the Arrow-Pratt* ***coefficient of relative risk aversion*** *as*[4]

$$RRA = -\frac{u''(x)}{u'(x)}x$$

Calculate the coefficient of relative risk aversion for

$$u(x) = \frac{x^{1-\gamma} - 1}{1-\gamma}, \ \gamma \neq 1$$

2.1.9 Partial derivatives

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is given by $y = f(x_1, ..., x_n)$, i.e. f is a function of n variables. The derivative of f with respect to x_i is defined as

$$\frac{\partial}{\partial x_i} f(x_1, ..., x_n) = \lim_{h\to 0} \frac{f(x_1, ..., x_i + h, ..., x_n) - f(x_1, ..., x_n)}{h}$$

Other notations

$$\frac{\partial}{\partial x_i} f(x_1, ..., x_n) \equiv f_{x_i}(x_1, ..., x_n) \equiv f_i(x_1, ..., x_n)$$

There is nothing special about partial derivative, compared to the regular derivative. When we calculate the partial derivative of a function of several variables, with respect to variable x_i, we treat the function as a function of that variable only, and all other variables are held constant. The intuitive meaning of the partial derivative $\frac{\partial}{\partial x_i} f(x_1, ..., x_n)$ is the change in the value of the function, per small unit increase in the variable x_i, *holding all other variables constant.*

[4]Higher RRA means the investors are willing to invest smaller fraction of their portfolio in risky assets.

Chain rule

All the rules of derivatives for functions of one variable, apply to partial derivatives. The chain rule for multivariate functions is similar to the one for univariate function. Suppose that the function f is a function of two variables, g and h, which are functions of t. Then,

$$\frac{\partial}{\partial t} f\left(g\left(t\right), h\left(t\right)\right) = \frac{\partial f}{\partial g} g'\left(t\right) + \frac{\partial f}{\partial h} h'\left(t\right)$$

Thus, changing t affects f through its effect on g and through its effect on h. The total effect is the sum of these two *partial* effects. In general, let $y = f\left(x_1, x_2, ..., x_n\right)$ and $x_1, x_2, ..., x_n$ are functions of another variable, say t, we can write $y\left(t\right) = f\left(x_1\left(t\right), x_2\left(t\right), ..., x_n\left(t\right)\right)$ and the chain rule becomes

$$y'\left(t\right) = \frac{\partial f}{\partial x_1} x_1'\left(t\right) + \frac{\partial f}{\partial x_2} x_2'\left(t\right) + ... + \frac{\partial f}{\partial x_n} x_n'\left(t\right)$$

In some texts, the above equation is called **total derivative**. Notice that the total effect of t is the sum of all the partial effects, through $x_1, x_2, ..., x_n$. This is in contrast to partial derivative, which measures only one of these effects, holding all others constant.

Gradient vector

It is often convenient to collect all the partial derivatives of a function $f\left(x_1, ..., x_n\right)$ in a vector called the **gradient vector**, and defined as follows:

$$\nabla f\left(x_1, ..., x_n\right) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

where f_i is the partial derivative of the function $f\left(x_1, ..., x_n\right)$ with respect to variable x_i.

2.2 Applications of Derivatives to Economics

2.2.1 Average v.s. marginal cost functions

Suppose a firm has a total cost function, which depends on the quantity of output produced Q. The typical, short run cost function, is

$$TC(Q) = FC + VC(Q)$$

The total cost consist of fixed cost FC, which the firm already committed to pay, and variable cost $VC(Q)$ which increases in the quantity produced. The average cost per unit is important to the firm, because the average cost determines whether the firm is making profit or not (if the average cost is greater then the price it gets per unit, the firm has a loss). The average total cost is:

$$ATC(Q) = \frac{TC(Q)}{Q} = \frac{FC + VC(Q)}{Q} = \underbrace{\frac{FC}{Q}}_{AFC} + \underbrace{\frac{VC(Q)}{Q}}_{AVC(Q)}$$

Here $AVC(Q)$ is the average variable cost, and AFC is the average fixed cost.

The marginal cost, $MC(Q)$, is the change in the total cost when the firm increases output by a small unit:

$$MC(Q) = \frac{d}{dQ}TC(Q) = \frac{d}{dQ}VC(Q)$$

It turns out that any average quantity is increasing (decreasing) when the marginal is above (below) it. Consider the relationship between your average grades (GPA) and your marginal grade (the grade in the last course you took). Whenever the marginal is above the average, it will drive the average up, and whenever the marginal is below the average, the average will decline. The next proposition proves this for average and marginal cost functions.

Proposition 1 *Suppose that the total cost is differential function $TC(Q)$, with the associated average and marginal cost functions $ATC(Q) = \frac{TC(Q)}{Q}$ and $MC(Q) = TC'(Q)$. Then*

$$\frac{d}{dQ}ATC(Q) \gtreqless 0 \quad \Longleftrightarrow \quad MC(Q) \gtreqless AVC(Q)$$

Proof. Differentiating the average total cost, $ATC(Q) = \frac{TC(Q)}{Q}$, with respect to Q, requires using the quotient rule:

$$\frac{d}{dQ}ATC(Q) = \frac{TC'(Q)\,Q - TC(Q)}{Q^2} = \frac{TC'(Q) - TC(Q)/Q}{Q}$$

Note that

$$\frac{d}{dQ}ATC(Q) \gtreqless 0 \quad \Longleftrightarrow \quad \underbrace{TC'(Q)}_{MC(Q)} - \underbrace{TC(Q)/Q}_{ATC(Q)} \gtreqless 0 \quad \Longleftrightarrow \quad MC(Q) \gtreqless ATC(Q)$$

Thus, the average total cost function $ATC(Q)$ is increasing whenever the marginal cost $MC(Q)$ is above it, and $ATC(Q)$ is decreasing whenever $MC(Q)$ is below it. The same is true for the relationship between the average variable cost and marginal cost functions:

$$\frac{d}{dQ}AVC(Q) \gtreqless 0 \quad \Longleftrightarrow \quad MC(Q) \gtreqless AVC(Q)$$

Simply repeat the previous proof, with $VC(Q)$ instead of $TC(Q)$. ■

Exercise 11 *Given the total cost function*

$$TC(Q) = Q^3 - 5Q^2 + 12Q + 75,$$

write out the variable cost function, $VC(Q)$, the marginal cost function $MC(Q)$, the average total cost function $ATC(Q)$, and the average variable cost function $AVC(Q)$.

Exercise 12 *Consider the average variable cost function, defined as*

$$AVC(Q) = \frac{VC(Q)}{Q}$$

Prove that

$$\lim_{Q \to 0} AVC(Q) = MC(0)$$

where $MC(Q)$ is the marginal cost function.

2.2.2 Marginal utility and marginal products

Preferences of individuals are often described with utility functions. For example, if individuals derive utility from quantities of two goods, their utility function if $u(x,y)$. The **marginal utilities** from the two goods are the partial derivatives:

$$\begin{aligned} [\text{Marginal utility from } x] \quad &: \quad MU_x = \frac{\partial}{\partial x}u(x,y) = u_x(x,y) \\ [\text{Marginal utility from } y] \quad &: \quad MU_y = \frac{\partial}{\partial y}u(x,y) = u_y(x,y) \end{aligned}$$

The marginal utility from good x is the change in the utility per small unit increase in x, *holding the quantity of y fixed.* Marginal utility from y has similar interpretation. If the marginal utility from a good is positive, say $u_x(x,y) > 0$, it means that utility is increasing in that good.

The ratio of marginal utility from x to that of y is known as the Marginal Rate of Substitution between x and y, and is also the absolute value of the slope of indifference curves:

$$MRS_{x,y} = \frac{u_x(x,y)}{u_y(x,y)}$$

The $MRS_{x,y}$ describes the rate at which the consumers are willing to substitute good y for good x. For example, if $MRS_{x,y} = 2$, then the consumer is willing to give up 2 units of good y for one unit of good x, and remain on the same indifference curve (i.e. attain the same utility). The Marginal Rate of Substitution between y and x is therefore $MRS_{y,x} = \frac{u_y(x,y)}{u_x(x,y)}$, and represents the rate at which the consumer is willing to substitute good x for good y.

The above marginal utilities can be differentiated once again, to obtain the second derivatives of the utility function.

$$\begin{aligned}
u_{xx}(x,y) &= \frac{\partial}{\partial x} u_x(x,y) \\
u_{yy}(x,y) &= \frac{\partial}{\partial y} u_y(x,y) \\
u_{xy}(x,y) &= u_{yx}(x,y) = \frac{\partial}{\partial y} u_x(x,y)
\end{aligned}$$

The second derivatives of the utility function describe how the *marginal utilities* change when the quantities of x and y change. For example, if $u_{xx}(x,y) < 0$, then we say that there is diminishing marginal utility from good x. Thus, $u_x(x,y) > 0$ and $u_{xx}(x,y) < 0$ mean that utility is increasing with x, but at diminishing rate. For example, suppose x is chocolate, then $u_x(x,y) > 0$ means that your utility is increasing in the quantity of chocolate consumed, but $u_{xx}(x,y) < 0$ means that each additional bite of chocolate does not increase utility as much as the previous, because you are getting full. At some point you could have $u_x(x,y) < 0$ because you ate too much chocolate.

Firm's technology is often described by production function. Formally, the **production function** gives the maximal output level that can be produced with given inputs. Suppose output is Y and there are two inputs, K and L (capital and labor). Then, the production function is written as $Y = F(K,L)$. In analogy to the marginal utilities, we define the

marginal products:

$$\begin{aligned}
[\text{Marginal product of } K] &: \; MP_K = \frac{\partial}{\partial K} F(K,L) = F_K(K,L) \\
[\text{Marginal product of } L] &: \; MP_L = \frac{\partial}{\partial L} F(K,L) = F_L(K,L)
\end{aligned}$$

The marginal product of an input gives the change in output per small unit increase in that input, *holding all other inputs fixed*. For example, the marginal product of labor gives the increase in output due to hiring additional worker, *holding the capital fixed*. In fact, the marginal product of labor determines how many workers are hired by a firm. A competitive firm, will hire workers as long as the value of their marginal product is greater than their wage. If the value of production contributed by additional worker is smaller than the wage paid by the firm, it is not in the firm's best interest to employ that worker.

Similar to MRS in consumer's utility, the ratio of marginal products give the Marginal Rate of Technical Substitution, which describes the rate at which inputs can be substituted in production, without changing the output level:

$$\begin{aligned}
MRTS_{K,L} &= \frac{F_K(K,L)}{F_L(K,L)} \\
MRTS_{L,K} &= \frac{F_L(K,L)}{F_K(K,L)}
\end{aligned}$$

For example, if $MRTS_{L,K} = \frac{F_L(K,L)}{F_K(K,L)} = 2$, this means that one worker can be substituted with 2 machines, without changing the total output (i.e. remaining on the same isoquant).

Typically, we expect the marginal products of all inputs to be positive, $MP_K, MP_L > 0$, but due to congestion, we expect that the marginal products themselves be decreasing. This requires computing the second derivatives:

$$\begin{aligned}
\frac{\partial^2}{\partial K^2} F(K,L) &= F_{KK}(K,L) \\
\frac{\partial^2}{\partial L^2} F(K,L) &= F_{LL}(K,L)
\end{aligned}$$

If these are negative, then the production function exhibits decreasing marginal products of K and L.

Example 23 *Consider the Constant Elasticity of Substitution (CES) utility function:*

$$u(x,y) = [\alpha x^\sigma + (1-\alpha) y^\sigma]^{\frac{1}{\sigma}}, \; \sigma \le 1$$

Prove that the marginal utility of x and y is positive and diminishing.

Solution 9 *The marginal utility of x:*

$$u_x = \left[\alpha x^\sigma + (1-\alpha) y^\sigma\right]^{\frac{1-\sigma}{\sigma}} \alpha x^{\sigma-1} > 0$$

The second derivative:

$$\begin{aligned}
u_{xx} &= (1-\sigma)\left[\alpha x^\sigma + (1-\alpha) y^\sigma\right]^{\frac{1-\sigma}{\sigma}-1} \alpha^2 x^{2\sigma-2} - (1-\sigma)\left[\alpha x^\sigma + (1-\alpha) y^\sigma\right]^{\frac{1-\sigma}{\sigma}} \alpha x^{\sigma-2} \\
&= (1-\sigma)\left[\alpha x^\sigma + (1-\alpha) y^\sigma\right]^{\frac{1-\sigma}{\sigma}} \alpha x^{\sigma-2} \left\{\left[\alpha x^\sigma + (1-\alpha) y^\sigma\right]^{-1} \alpha x^\sigma - 1\right\} \\
&= (1-\sigma)\left[\alpha x^\sigma + (1-\alpha) y^\sigma\right]^{\frac{1-\sigma}{\sigma}} \alpha x^{\sigma-2} \left\{\frac{\alpha x^\sigma}{\alpha x^\sigma + (1-\alpha) y^\sigma} - 1\right\} < 0
\end{aligned}$$

The fraction in the curly brackets is smaller than 1, and therefore the sign of the brackets is negative. The proof for y is identical.

2.2.3 Comparative statics analysis

Comparative statics is the analysis of the effect on equilibrium of some model (endogenous variables), of changes in exogenous variables or parameters. For example, consider the system of supply and demand:

$$\begin{aligned}
\text{[Demand]} &: Q_d = a - bP \\
\text{[Supply]} &: Q_s = -c + dP
\end{aligned}$$

where $a, b, c, d > 0$, Q is quantity and P is price. The equilibrium requires $Q_d = Q_s = Q$. Solving the model for equilibrium price and quantity gives:

$$\begin{aligned}
P^* &= \frac{a+c}{b+d} \\
Q^* &= \frac{ad-bc}{b+d}
\end{aligned}$$

Suppose that we want to assess the effect of one small unit increase in a on equilibrium. This is called comparative statics, and simply requires computing

$$\begin{aligned}
\frac{\partial}{\partial a} P^* &= \frac{1}{b+d} \\
\frac{\partial}{\partial a} Q^* &= \frac{d}{b+d}
\end{aligned}$$

Without knowing the values of the parameters, we can only say that both these effects are positive.

2.3 Differential

A closely related to derivatives, and often misunderstood concept, is the **differential**. It has enormous importance in applications to economics. For example, differential is very useful in comparative statics involving systems of non-linear equations, such as first order conditions.

2.3.1 Univariate functions

Definition 12 *Suppose $y = f(x)$ is a function of one variable. We denote its derivative as:*

$$\frac{dy}{dx} = f'(x)$$

The quantity

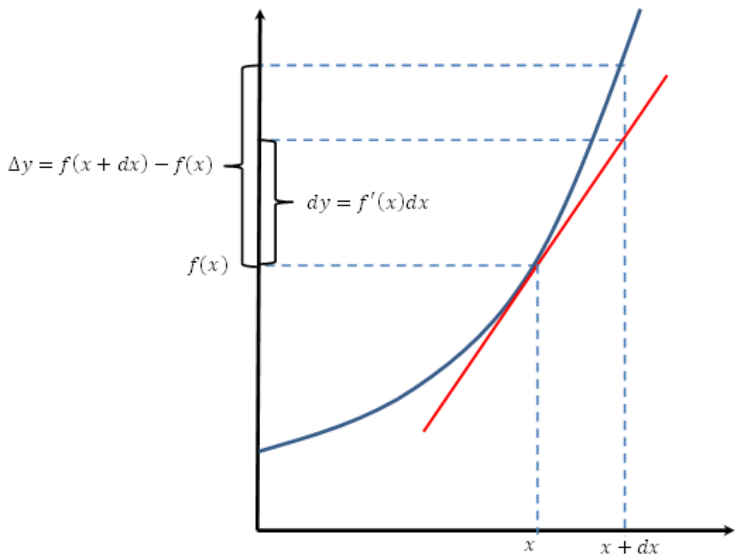
$$dy = f'(x)\,dx$$

*is called the **differential of** f **at** x**, with increment** dx.*

Thus, the differential gives the *approximate* change in y, resulting from a change dx in x. The reason why we say that this is an approximation, is because the slope $f'(x)$ is the slope of the tangent line to the function f at point x. So the differential gives the change in y as we move *along the tangent line* to the function f and not along the function f itself. The exact change in y as x increases by dx is

$$\Delta y = f(x + dx) - f(x) = \frac{f(x + dx) - f(x)}{dx}dx$$

Comparing this to the differential, $dy = f'(x)\,dx$, we see that as dx becomes small, the quotient in the above expression approaches the derivative $f'(x)$, and the approximation is closer - small change along the function $f(x)$ is close to the change along the tangent line to the function. The next figure illustrates the differential $dy = f'(x)\,dx$ vs. the exact change $\Delta y = f(x + dx) - f(x)$ in some function, due to increment dx.



Example 24 *Suppose we want to approximate the change in* $y = f(x) = x^2$ *at* $x = 2$*, when* x *changes by increment* $dx = 0.1$*. The exact change in* y *is*

$$\Delta y = f(2.1) - f(2) = 2.1^2 - 2^2 = 0.41$$

Now, suppose that we want to approximate this change with differential. The derivative at the initial point is $f'(x) = 2x = 4$*. So the differential of* f *at* $x = 2$*, with increment* $dx = 0.1$ *is*

$$dy = f'(x)\,dx = 4 \cdot 0.1 = 0.4$$

In the above example, the exact change is $\Delta y = 0.41$ and the approximate change based on differential is $dy = 0.4$. These are very close because the increment $dx = 0.1$ is small, so the tangent line is close to the actual function.

In terms of notation, if we are given a function $y = f(x)$, the following objects are often used interchangeably:

$$dy,\ df,\ df(x)$$

and they all denote the differential of y or of f at x, with increment dx, i.e. $f'(x)\,dx$.

2.3.2 Multivariate functions

Now suppose that we have a function of n variables, $y = f(x_1, ..., x_n)$. The **total differential of** f **at** $x_1, ..., x_n$**, with increments** $dx_1, ..., dx_n$, is

$$dy = \frac{\partial}{\partial x_1} f(x_1, ..., x_n)\,dx_1 + \frac{\partial}{\partial x_1} f(x_1, ..., x_n)\,dx_2 + ... + \frac{\partial}{\partial x_n} f(x_1, ..., x_n)\,dx_n$$

or, in more compact notation,

$$dy = f_1 dx_1 + ... + f_n dx_n$$

where f_i is the partial derivative with respect to x_i.

The total differential approximates the change in the value of the multivariate function y, resulting from small changes in its arguments $x_1, ..., x_n$. In the case of a univariate function, $f(x)$, the differential dy approximates the actual change in y moving along a *tangent line*. In the multivariate case, the differential dy approximates the actual change in y by moving on a *tangent plane*.

Example 25 *Suppose the profit of a firm depends on its output* x_1 *and advertising* x_2 *according to*

$$y = f(x_1, x_2) = 10x_1 - x_1^2 + 20x_2 - x_2^2$$

Suppose that initially the firm produces 4 *units of output and* 11 *units of advertising. What will be the change in the firm's profit if it increases the output by* 0.1 *units and reduces advertising by* 0.05 *units? Notice that this question examines simultaneous change in two arguments of the function, in contrast to the usual "ceteris paribus", by which we change one thing at a time, holding all else constant. One way to answer the question is to calculate the profit before and after the change:*

$$\begin{aligned} f(4, 11) &= 10 \cdot 4 - 4^2 + 20 \cdot 11 - 11^2 = 123 \\ f(4 + 0.1, 11 - 0.05) &= 10 \cdot 4.1 - 4.1^2 + 20 \cdot 10.95 - 10.95^2 = 123.29 \end{aligned}$$

Thus, the ***exact*** *change in profit is* $\Delta y = 123.29 - 123 = 0.29$.

Now, suppose we want to approximate the above change using ***total differential****. The partial derivatives at the initial point are:*

$$\begin{aligned} f_1(x_1, x_2) &= 10 - 2x_1 = 10 - 8 = 2 \\ f_2(x_1, x_2) &= 20 - 2x_2 = 20 - 22 = -2 \end{aligned}$$

As a side comment, the above derivatives indicate that the firm's profit is increasing in output but decreasing in advertisement. This suggests that the firm should increase output and reduce it advertising level. The proposed changes in output and advertising are: $dx_1 = 0.1$*, and* $dx_2 = -0.05$*. Thus, the approximate change in profit, based on the total differential is:*

$$dy = f_1 \cdot dx_1 + f_2 \cdot dx_2 = 2 \cdot 0.1 + (-2) \cdot (-0.05) = 0.3$$

In the above example, the exact change in profit $\Delta y = 0.29$ and the approximate change based on total differential is $dy = 0.3$. Since the increments dx_1, dx_2 are small, the approximation is close to the exact change in y.

2.3.3 Rules of differentials

Since the concept of differential is closely related to the derivative (recall that the differential of f is $dy = f'(x)\,dx$, where $f'(x)$ is the derivative of f), it is not surprising that the rules of differentials are very similar to the rules of derivatives. For example, suppose that $y = f(x) = \ln(x)$. Then,

$$\begin{aligned} [\text{Derivative}] &: \quad f'(x) = \frac{1}{x} \\ [\text{Differential}] &: \quad dy = \frac{1}{x}dx \end{aligned}$$

Thus, let a, b be a constants and u and v are two functions of variables x_1 and x_2 respectively. Then,

1. For constant a,

$$da = 0$$

2. Power rule:

$$d\left(au^b\right) = abu^{b-1}du$$

The first two rules imply that for linear functions

$$d(a + bu) = bdu$$

3. Sums or differences rule:

$$d[u \pm v] = du \pm dv$$

4. Product rule:

$$d[u \cdot v] = du \cdot v + u \cdot dv$$

5. Quotient rule:

$$d\left[\frac{u}{v}\right] = \frac{du \cdot v - u \cdot dv}{v^2}$$

6. Logarithmic function:

$$d\ln(x) = \frac{1}{x}dx$$

7. Exponential function:

$$\begin{aligned} da^x &= \ln(a)\, a^x dx \\ \text{in particular, } de^x &= e^x dx \end{aligned}$$

2.3.4 Elasticities

Economics is about change - prices, quantities, profits, unemployment, wealth, all change over time. The **rate of change** in some variable from initial value y to new value y_1 is defined to be

$$\hat{y} = \frac{y_1 - y}{y} = \frac{\Delta y}{y} \tag{2.1}$$

where Δ (delta) means "change". The **percentage change** in a variable is the rate of change, multiplied by 100:

$$\frac{\Delta y}{y} = \left(100\frac{\Delta y}{y}\right)\% \tag{2.2}$$

Example 26 *The price of pair of shoes changed from* \$100 *to* \$105*. What is the* ***rate of change*** *in the price?*

$$\hat{p} = \frac{105 - 100}{100} = \frac{5}{100} = 0.05$$

We can express the change in percentages, by multiplying the rate of change by 100*, so the* ***percentage change*** *in the price is*

$$(100\hat{p})\,\% = (100 \cdot 0.05)\,\% = 5\%$$

Point elasticity

Economic theory often describes a relationship between a pair of variables, such as quantity demanded of a good and the price of that good, or the quantity demanded of a good and income of buyers. The theory tells us that when the price of a good goes up, then under some conditions, the quantity demanded of that good will go down. Economics is a quantitative discipline, so to make the relationship between variables quantitative, we ask what is the percentage change in the quantity demanded of a good resulting from 1% increase in the price of a good, holding all other factors affecting the demand constant. The answer to that question is the **price elasticity of demand**. Similarly, we would like to measure the percentage change in the quantity demanded of a good resulting from a 1% increase in buyers' income, holding all other factors affecting the demand constant. The answer to this question is **income elasticity of demand**. In general, the **elasticity of y with respect**

to x is defined:

$$\eta_{y,x} = \frac{\text{\% change in } y}{\text{\% change in } x} = \frac{\%\Delta y}{\%\Delta x}$$

In words, this is the prcentage change in y, resulting from a 1% increase in x.

Using the definition of percentage change (2.2), the elasticity of y with respect to x is:

$$\eta_{y,x} = \frac{\left(100\frac{\Delta y}{y}\right)\%}{\left(100\frac{\Delta x}{x}\right)\%} = \frac{\left(\frac{\Delta y}{y}\right)}{\left(\frac{\Delta x}{x}\right)}$$

When these changes are small, Δy and Δx by the differentials dy and dx. Recall that differentials of functions approximate the change in the function using a linear approximation, which is close when the increments in the function's arguments are small. This leads to the definition of the **point elasticity of** y **with respect to** x:

$$\eta_{y,x} = \frac{\left(\frac{dy}{y}\right)}{\left(\frac{dx}{x}\right)} \tag{2.3}$$

Thus, for small changes in the variables, the quantities $\frac{dy}{y}$ and $\frac{dx}{x}$ represent the rates of change in these variables, and multiplication by 100 will turn them into percentage changes. Rearranging the point elasticity definition (2.3), gives the practical formula we use to calculate point elasticities:

$$\eta_{y,x} = \frac{dy}{dx}\frac{x}{y}$$

The first term above is the derivative of y with respect to x, $\frac{dy}{dx} = f'(x)$. If the relationship between y and x involves other variables, then the derivative symbol d becomes the partial derivative symbol ∂. So in general, it is perfectly fine to write the point elasticity formula as:

$$\eta_{y,x} = \frac{\partial y}{\partial x}\frac{x}{y}$$

Recall that $d\ln(y) = \frac{1}{y}dy$. Using this result, we can obtain another formula for point elasticity:

$$\eta_{y,x} = \frac{d\ln(y)}{d\ln(x)} \tag{2.4}$$

Here the numerator is $d\ln(y) = \frac{1}{y}dy$ and denominator is $d\ln(x) = \frac{1}{x}dx$, and the formula is exactly the same as the definition of point elasticity (2.3). This result is the reason why in econometrics, a very popular specification is the log-log model, in which the estimated coefficients are elasticities.

Arc elasticity

Note that the point elasticity of demand is valid for small changes Δy and Δx. If these changes are large (and say y changed between y_1 and y_2 while x changed between x_1 and x_2), we define the **arc elasticity of** y **with respect to** x, where the rates of change are usually calculated using the midpoint formula:

$$\eta_{y,x}^{arc} = \frac{\left(\frac{\Delta y}{y}\right)}{\left(\frac{\Delta x}{x}\right)}, \text{ where } \frac{\Delta y}{y} = \frac{y_2 - y_1}{(y_2 + y_1)/2}, \; \frac{\Delta x}{x} = \frac{x_2 - x_1}{(x_2 + x_1)/2} \tag{2.5}$$

The name midpoint formula comes from the fact that we calculate the rate of change of a variable y relative to the midpoint between y_1 and y_2. This way, the arc elasticity becomes independent of arbitrary choice of the *initial point*, which could be either y_1 or y_2. The discussion of arc elasticity here is a sidestep from the main topic of differentials, and brought here so that you have a complete picture of all there is to know about elasticities. Moreover, when arc elasticity formula is applied to small changes, the result approaches point elasticity.

Most applied work in economics calculates or estimates some kind of elasticities, so the importance of the topic cannot be overstated. Suppose the individual demand function for some good

$$x = f(p_x, p_y, I)$$

where x is the quantity demanded of the good, p_x is the price of the good, p_y is the price vector of other related goods (substitutes or complements), and I is income of the buyer. We can define the following elasticities:

1. Price elasticity of demand:
$$\eta_{x,p_x} = \frac{\partial x}{\partial p_x}\frac{p_x}{x}$$
The price elasticity of demand gives the percentage change in the quantity demanded x due to a 1% increase in the price of the good p_x.

2. Cross price elasticity of x with respect to price of y:
$$\eta_{x,p_y} = \frac{\partial x}{\partial p_y}\frac{p_y}{y}$$
This elasticity gives the percentage change in the quantity demanded x due to a 1% increase in the price of another good, p_y. If the rise in price of y leads to an increase in quantity demanded x (i.e. $\eta_{x,p_y} > 0$), we say that x is a *gross substitute* of y. If the rise in price of y leads to a decrease in quantity demanded x (i.e. $\eta_{x,p_y} < 0$),

we say that x is a *gross complement* of y. The word "gross" is there because a rise in the price of any good embodies two effects (i) substitution effect, and (ii) income effect, discussed in principles and intermediate microeconomics classes. Thus, gross substitution or complementarity is determined by both effects. Similarly, $\eta_{y,p_x} = \frac{\partial y}{\partial p_x}\frac{p_x}{y}$ is the cross price elasticity of x with respect to the price of y.

3. Income elasticity of demand:

$$\eta_{x,I} = \frac{\partial x}{\partial I}\frac{I}{x}$$

This elasticity gives the percentage change in the quantity demanded x due to a 1% increase in buyers' income. Recall from principles and intermediate microeconomics classes, if $\eta_{x,I} > 0$ then x is a *normal good* and if $\eta_{x,I} < 0$, then x is *inferior good.*

Example 27 *Suppose the demand curve is*

$$q = 100 - 2p$$

where q is the quantity demanded of the good, p is the price of the good. Calculate the price elasticity of demand.

$$\eta_{q,p} = \frac{dq}{dp}\frac{p}{q} = -2\frac{p}{q} = -2\frac{p}{100-2p} = -\frac{p}{50-p}$$

The (point) price elasticity of demand in this example depends on the particular point on the demand curve at which it is calculated. This is the usual case with linear demand curves. You can verify that

$$\begin{aligned} \eta_{q,p}\left(p=0\right) &= 0 \\ \eta_{q,p}\left(p=25\right) &= -1 \\ \eta_{q,p}\left(p=45\right) &= -9 \\ \lim_{p\nearrow 50}\eta_{q,p} &= -\infty \end{aligned}$$

Example 28 *Suppose the demand curve is*

$$q = \frac{A}{p^{\varepsilon}}, \quad \varepsilon > 0$$

where q is the quantity demanded of the good, p is the price of the good. Calculate the price elasticity of demand.

Notice that the demand can be written as $q = Ap^{-\varepsilon}$*, and the price elasticity is:*

$$\eta_{q,p} = \frac{dq}{dp}\frac{p}{q} = -A\varepsilon p^{-\varepsilon-1}\frac{p}{Ap^{-\varepsilon}} = -\varepsilon$$

This demand curve has constant price elasticity of demand (same elasticity at every point), equal to $-\varepsilon$*.*

Alternatively, we can use (2.4) formula to calculate elasticity. Taking ln *of the demand:*

$$\ln q = \ln A - \varepsilon \ln p$$

and

$$\eta_{q,p} = \frac{d\ln q}{d\ln p} = -\varepsilon$$

Exercise 13 *Suppose the production function of a firm is* $Y = AK^{\theta}L^{1-\theta}$*, where* $0 < \theta < 1$*,* Y *is output,* A *is productivity parameter,* K *is physical capital, and* L *is labor. This is the Cobb-Douglas production function.*

(a) Suppose that initially the firm employes $K = 400$ *machines and* $L = 900$ *workers, and assume that* $A = 2$*, and* $\theta = 0.5$*. Find the exact change in output resulting from employing additional machine and one more worker (i.e.* $K = 401$*, and* $L = 901$*).*

(b) Using total differential, calculate the approximate change in output for the same increments as in the previous section (i.e. $dK = 1$*, and* $dL = 1$*). Still assume that* $\theta = 0.5$*.*

(c) Prove that the elasticity of output with respect to capital is θ*, and the elasticity of output with respect to labor is* $1-\theta$*. Here use* θ *as unknown parameter, without assuming particular value.*

So far we calculated the elasticity of some functions, with respect to one of the arguments, while holding all other arguments fixed. Alternatively, we can calculate the "total" elasticity of a function, i.e. the % change in the value of the function, resulting from any percent change in all its arguments: $\%\Delta x_1, \%\Delta x_2, ..., \%\Delta x_n$.

Proposition 2 *Suppose* $y = f(x_1, x_2, ..., x_n)$ *is a differentiable function of* n *variables. Then,*

$$\%\Delta y \approx \eta_{y,x_1} \cdot \%\Delta x_1 + \eta_{y,x_2} \cdot \%\Delta x_2 + ... + \eta_{y,x_n} \cdot \%\Delta x_n$$

where η_{y,x_i} *is the elasticity of* y *with respect to* x_i *and* $\%\Delta$ *reads as percent change.*

Proof. Totally differentiating the function, gives:

$$dy = \frac{\partial f}{\partial x_1} \cdot dx_1 + \frac{\partial f}{\partial x_2} \cdot dx_2 + ... + \frac{\partial f}{\partial x_n} \cdot dx_n$$

Dividing all terms by y

$$\frac{dy}{y} = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{y} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{y} + ... + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{y}$$

Multiplying and dividing each term i on the right hand side by x_i:

$$\frac{dy}{y} = \frac{\partial f}{\partial x_1} \frac{x_1}{y} \cdot \frac{dx_1}{x_1} + \frac{\partial f}{\partial x_2} \frac{x_2}{y} \cdot \frac{dx_2}{x_2} + ... + \frac{\partial f}{\partial x_n} \frac{x_n}{y} \cdot \frac{dx_n}{x_n}$$

The calculus formula of point elasticity of y with respect to x_i is:

$$\eta_{y,x_i} = \frac{\partial f}{\partial x_i} \frac{x_i}{y}$$

Thus, we have

$$\frac{dy}{y} = \eta_{y,x_1} \cdot \frac{dx_1}{x_1} + \eta_{y,x_2} \cdot \frac{dx_2}{x_2} + ... + \eta_{y,x_2} \cdot \frac{dx_n}{x_n}$$

The term $\frac{dy}{y}$ is the approximate percent change in y, since dy is differential (linear approximation of the actual change in y), while the terms $\frac{dx_i}{x_i}$ are % change in x_i. Thus, the exact percent change in y, i.e. $\%\Delta y$, is only approximated by the right hand side of the last equation:

$$\%\Delta y \approx \eta_{y,x_1} \cdot \%\Delta x_1 + \eta_{y,x_2} \cdot \%\Delta x_2 + ... + \eta_{y,x_n} \cdot \%\Delta x_n$$

■

2.3.5 Elasticity of Substitution

The elasticities we discussed so far measured the response of one variable (in %) to a 1% increase in another variable. Another type of elasticity is designed to measure the willingness of consumers to substitute one good for another in utility, and how easy it is to substitute inputs in production. In this section we focus on utility, and the treatment of production functions is analogous.

Definition 13 *Let $u(x, y)$ be a utility function. The the elasticity of substitution between x and y is*

$$ES_{y,x} = \frac{\%\Delta (y/x)}{\%\Delta MRS_{x,y}}$$

Thus, the elasticity of substitution between y and x is the percentage change in the ratio of y/x that results from a 1% change in the slope of the indifference curves. The range of elasticity of substitution is $ES_{y,x} \in [0, \infty)$, with $ES_{y,x} > 1$ defines x and y to be substitutes and $ES_{y,x} < 1$ defines x and y to be complements. There are several equivalent calculus formulas for the ES:

$$ES_{y,x} = \left(\frac{\partial\left(y/x\right)}{y/x}\right) / \left(\frac{\partial MRS_{x,y}}{MRS_{x,y}}\right) = \frac{\partial\left(y/x\right)}{\partial MRS_{x,y}} \cdot \frac{MRS_{x,y}}{y/x} = \frac{\partial \ln\left(y/x\right)}{\partial \ln\left(MRS_{x,y}\right)}$$

In equilibrium, the Marginal Rate of Substitution is equal to the ratio of prices, $MRS_{x,y} = p_x/p_y$, and elasticity of substitution can be defined in equilibrium as follows:

$$ES_{y,x} = \frac{\%\Delta\left(y/x\right)}{\%\Delta\left(p_x/p_y\right)}$$

The cross price elasticities, η_{x,p_x} and η_{y,p_y}, which we discussed earlier, measure the response of quantity demanded of *one good* to changes in price of *another good*. The elasticity of substitution on the other hand measures the response of the relative demand y/x to change in relative price p_x/p_y. The purpose of the concept of elasticity of substitution may seem obscure at first. Afterall, we do have the cross price elasticities already. Recall however that consumers' optimal consumption mix depends on ratio of goods prices ($MRS_{x,y} = p_x/p_y$), and producers' optimal input mix depends on inputs price ratio ($MRTS_{K,L} = p_K/p_L$).

The concept of elasticity of substitution was first introduced by Hicks 1932 [2] "The Theory of Wages" when he studied the distribution of national income between various factors of production. In particular, let the ratio of capital to labor income be

$$\frac{p_K \cdot K}{w \cdot L}$$

Suppose that the relative price of labor to capital in the economy increases, i.e. $(w/p_K)\uparrow$. What happens to the relative share of labor income in the economy (or in some industry)? Hicks showed that the answer to this question depends on the elasticity of substitution between the two factors, Labor and Capital. The share of capital will increase if and only if

$$\begin{aligned} \%\Delta\left(\frac{p_K \cdot K}{w \cdot L}\right) &> 0 \\ &\iff \%\Delta\left(\frac{K/L}{w/p_k}\right) > 0 \\ &\iff \frac{\%\Delta\left(K/L\right)}{\%\Delta\left(w/p_k\right)} > 1 \\ &\iff ES_{K,L} > 1 \end{aligned}$$

That is, the share of capital will increase if K and L are substitutes, i.e. if it is easy to replace workers with machines. However, if labor and capital are complements, i.e. $ES_{K,L} < 1$ (not easy to replace workers with machines), then the share of capital will actually decline when relative wages increase.

Exercise 14 *Suppose that consumers spend their budget on housing h and consumption c. Show that in response to rising relative housing prices, $(p_h/p_c)\uparrow$, the share spent on housing will increase, if the elasticity of substitution between housing and consumption is less than one, i.e. $ES_{h,c} < 1$.*

Solution 10 *The share of housing in the budget will increase if and only if the ratio of spending on consumption to spending on housing decreases:*

$$\left(\frac{p_c \cdot c}{p_h \cdot h}\right)\downarrow$$

Thus,

$$\begin{aligned} \%\Delta\left(\frac{p_c \cdot c}{p_h \cdot h}\right) &< 0 \\ &\iff \%\Delta\left(\frac{c/h}{p_h/p_c}\right) < 0 \\ &\iff \frac{\%\Delta\left(c/h\right)}{\%\Delta\left(p_h/p_c\right)} < 1 \\ &\iff ES_{h,c} < 1 \end{aligned}$$

Thus, if consumers cannot easily replace housing with other consumption goods (say, not willing to switch from 3 bedroom to 2 bedroom when rent increases), then the relative share spent on housing will increase.

Exercise 15 *Prove that $ES_{x,y} = ES_{y,x}$.*

In applied economics reserach, the following utility (and analogous production) function is widely used.

Definition 14 *Constant Elasticity of Substitution (CES) utility function of 2 goods is*

$$u(x,y) = \left[\alpha x^{\sigma} + (1-\alpha)\, y^{\sigma}\right]^{\frac{1}{\sigma}}, \ \sigma \leq 1$$

When $\sigma = 0$, the above utility is defined to be Cobb-Douglas $x^{\alpha}y^{1-\alpha}$.

The reason for defining the CES utility with $\sigma = 0$ as Cobb-Douglas $x^{\alpha}y^{1-\alpha}$, is the following result.

Proposition 3 *(Cobb-Douglas limit). The limiting case as $\sigma \to 0$ is the Cobb-Douglas utility:*

$$\lim_{\sigma\to 0}\left[\alpha x^{\sigma}+(1-\alpha)\,y^{\sigma}\right]^{\frac{1}{\sigma}}=x^{\alpha}y^{1-\alpha}$$

Proof. Using the chain rule of limits, we take log of the CES utility and then taking limit as $\sigma \to 0$ gives

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\log\left[\alpha x^{\sigma}+(1-\alpha)\,y^{\sigma}\right]$$

This is a limit of the form $\frac{0}{0}$, so we will use L'Ĥopital's rule (differentiating with respect to σ).

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\log\left[\alpha x^{\sigma}+(1-\alpha)\,y^{\sigma}\right]=\lim_{\sigma\to 0}\frac{\alpha x^{\sigma}\ln x+(1-\alpha)\,y^{\sigma}\ln y}{\alpha x^{\sigma}+(1-\alpha)\,y^{\sigma}}=\alpha\ln x+(1-\alpha)\ln y$$

Therefore,

$$\lim_{\sigma\to 0}\left[\alpha x^{\sigma}+(1-\alpha)\,y^{\sigma}\right]^{\frac{1}{\sigma}}=x^{\alpha}y^{1-\alpha}$$

■

Marginal utility

We prove that the above CES function exhibits diminishing marginal utility.

$$\begin{aligned}
u_x &= \left[\alpha x^{\sigma}+(1-\alpha)\,y^{\sigma}\right]^{\frac{1-\sigma}{\sigma}}\alpha x^{\sigma-1}>0\\
u_{xx} &= (1-\sigma)\left[\alpha x^{\sigma}+(1-\alpha)\,y^{\sigma}\right]^{\frac{1-\sigma}{\sigma}-1}\alpha^2x^{2\sigma-2}-(1-\sigma)\left[\alpha x^{\sigma}+(1-\alpha)\,y^{\sigma}\right]^{\frac{1-\sigma}{\sigma}}\alpha x^{\sigma-2}\\
&= (1-\sigma)\left[\alpha x^{\sigma}+(1-\alpha)\,y^{\sigma}\right]^{\frac{1-\sigma}{\sigma}}\alpha x^{\sigma-2}\left\{\left[\alpha x^{\sigma}+(1-\alpha)\,y^{\sigma}\right]^{-1}\alpha x^{\sigma}-1\right\}\\
&= (1-\sigma)\left[\alpha x^{\sigma}+(1-\alpha)\,y^{\sigma}\right]^{\frac{1-\sigma}{\sigma}}\alpha x^{\sigma-2}\left\{\frac{\alpha x^{\sigma}}{\alpha x^{\sigma}+(1-\alpha)\,y^{\sigma}}-1\right\}<0
\end{aligned}$$

Derivation of elasticity of substitution

$$\begin{aligned}
MRS_{x,y} &= \frac{\alpha\sigma x^{\sigma-1}}{(1-\alpha)\,\sigma y^{\sigma-1}}=\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{y}{x}\right)^{1-\sigma}\\
\frac{1}{ES_{y,x}} &= \frac{\partial MRS_{x,y}}{\partial\,(y/x)}\cdot\frac{y/x}{MRS_{x,y}}=\left(\frac{\alpha}{1-\alpha}\right)(1-\sigma)\left(\frac{y}{x}\right)^{-\sigma}\cdot\frac{y/x}{\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{y}{x}\right)^{1-\sigma}}=1-\sigma\\
ES_{y,x} &= \frac{1}{1-\sigma}
\end{aligned}$$

The range of parameters are: $\sigma \le 1$ and $\eta \ge 0$. and we use the following classifications :

1. Perfect substitutes	$\sigma = 1$	$ES = \infty$
2. Substitutes	$0 < \sigma < 1$	$1 < ES < \infty$
3. Cobb-Douglas	$\sigma = 0$	$ES = 1$
4. Complements	$-\infty < \sigma < 0$	$0 < ES < 1$
5. Perfect complements	$\sigma = -\infty$	$ES = 0$

2.4 Implicit Functions

So far we have seen functions of the form $y = f(x)$. This equation describes *explicitly* the dependence of the endogenous variable y on another, exogenous variable, x. For example, $y = 2 + 5x^3$ describes the rule by which y is determined for every given value of x, and for this reason it is called an *explicit function*, since y is explicitly expressed in terms of x. In other words, y is *endogenous* variable while x is *exogenous* variable. Alternatively, the same relationship between x and y can be written as $y - 5x^3 = 2$. Here we do not make a distinction between exogenous and endogenous variables, and such an equation is therefore called an **implicit function**. In general, an implicit function of two variables, is written as

$$F(x, y) = c$$

where c is some constant. An implicit function of n variables is

$$F(x_1, x_2, ..., x_n) = c$$

For example,

$$3x^2yz + xyz^2 = 30$$

is an implicit function of 3 variables, x, y, z. Sometimes we do know which variables are endogenous and which ones are exogenous. In the last equation, if we know that y is endogenous variable, depending on two exogenous variables x and z, then we can say that the above equation implicitly defines y as a function of x and z. In the above example, it is easy to express y explicitly as a function x and z:

$$y = \frac{30}{3x^2z + xz^2}$$

In other cases, it is more difficult, or impossible to solve explicitly for a variable from an implicitly given relationship. For example,

$$y^5 - 5xy + 4x^2 = 0$$

cannot be solved analytically for y as a function of x.

In economics, we encounter implicit functions very frequently. For example, $u(x,y)=\bar{u}$, where u is utility function and $\bar{u}$ is a constant, is an implicit function which describes an *indifference curve*, i.e. the combinations of x and y which generate a utility level of $\bar{u}$. As another example, $F(K,L)=\bar{Y}$, where F is a production function and $\bar{Y}$ is a constant, is an implicit function, which defines an *isoquant*, i.e. the combinations of K and L which produce an output of $\bar{Y}$ units. Yet another example is the familiar budget constraint:

$$p_x x + p_y y = I$$

The left hand side is the spending on goods x and y (with prices p_x, p_y), and the right hand side is the income. The above budget constraint is written as implicit function of x and y, and it is possible to derive y as explicit function of x:

$$y = \frac{I}{p_y} - \frac{p_x}{p_y}x$$

Implicit functions also arise in optimization, as first order conditions that have a form of $F(x_1, x_2, ..., x_n) = 0$ (i.e. a derivative of some function equals to zero).

2.4.1 Derivatives of implicit functions

Even in cases where we cannot (or don't want to) solve for a particular variable as explicit function of other variables, it is still possible to find the derivative of one variable in terms of another. For example, suppose that an indifference curve is given as an implicit function $u(x,y)=\bar{u}$, where u is some utility function, and $\bar{u}$ is a constant number. Taking full differential of this equation:

$$du(x,y) = u_x(x,y)\,dx + u_y(x,y)\,dy = d\bar{u} = 0$$

Notice that $d\bar{u}=0$ because $\bar{u}$ is a constant. Thus, the derivative of y with respect to x (slope of the indifference curve) is

$$\frac{dy}{dx} = -\frac{u_x(x,y)}{u_y(x,y)}$$

Notice however that the above derivative does not exist if $u_y(x,y)=0$.

The last example is an illustration (and a proof for special case of 2 variables) of the *Implicit Function Theorem.*

Theorem 6 ***(Implicit Function Theorem 1).*** *Let $F(x_1, x_2, ..., x_n, y)$ a function with* *continuous partial derivatives (i.e. continuously differentiable function, denoted C^1) around some point $(x_1^*, x_2^*, ..., x_n^*, y^*)$, and suppose that*

$$F(x_1^*, x_2^*, ..., x_n^*, y^*) = c$$

and that

$$\frac{\partial}{\partial y} F(x_1^*, x_2^*, ..., x_n^*, y^*) \neq 0$$

Then, there is a C^1 function $y = f(x_1, x_2, ..., x_n)$ defined on an open neighborhood B of $(x_1^*, x_2^*, ..., x_n^*)$, such that:

(a) $F(x_1, x_2, ..., x_n, f(x_1, x_2, ..., x_n)) = c$ for all $(x_1, x_2, ..., x_n) \in B$,

(b) $y^* = f(x_1^*, x_2^*, ..., x_n^*)$,

(c) for each $i = 1, 2, ..., n$, we have

$$\frac{\partial y}{\partial x_i}(x_1^*, x_2^*, ..., x_n^*) = -\frac{F_{x_i}(x_1^*, x_2^*, ..., x_n^*, y^*)}{F_y(x_1^*, x_2^*, ..., x_n^*, y^*)}$$

The above theorem establishes sufficient conditions for the existence of y as a function of all other variables $x_1, x_2, ..., x_n$, and also shows how to calculate the derivative of y with respect to any other variable (part (c) of the theorem). The last result is the most common use of the implicit function theorem.

Example 29 *Find the slope of the budget constraint*

$$p_x x + p_y y = I$$

using the implicit function theorem.

Let $F(x, y) = p_x x + p_y y$, so the implicit function can be written as $F(x, y) = I$. Then, using the implicit function theorem,

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{p_x}{p_y}$$

To verify this result, we can write the budget constraint in explicit function form:

$$y = \frac{I}{p_y} - \frac{p_x}{p_y} x$$

and use the regular derivatives to find that

$$\frac{dy}{dx} = -\frac{p_x}{p_y}$$

The advantage of implicit function theorem is that it enables us to calculate such derivatives even when it is impossible to express y explicitly in terms of x.

2.4.2 General comparative statics (systems of nonlinear equations)

In chapter (1) we applied linear algebra to solve systems of linear equations. In most practical applications, we deal with systems of nonlinear equations, but such systems can be approximated locally with linear functions, using *total differentials*. We will now demonstrate how we can perform *local* comparative statics analysis (recall that comparative statics is the analysis of the effect on endogenous variables, of changes in exogenous variables or parameters). We begin with an example of two nonlinear equations, and then generalize the method n nonlinear equations.

For example, the following is a nonlinear system:

$$\begin{aligned} x^2 + axy + y^2 &= 1 \\ x^2 + y^2 - a^2 &= -3 \end{aligned} \tag{2.6}$$

This is a nonlinear system of 2 equations, with 2 endogenous variables x, y and an exogenous parameter a. For example, x and y could be consumption levels of some goods, and a can be a tax rate or subsidy, which are outside of consumer's control. We would like to evaluate the effect of small changes in a on the endogenous variables x and y, around some point (locally). Think of each equation as implicit function of x, y and a. Taking total differentials, with increments dx, dy and da:

$$\begin{aligned} 2xdx + aydx + axdy + xyda + 2ydy &= 0 \\ 2xdx + 2ydy - 2ada &= 0 \end{aligned}$$

Notice that the above system imposes restrictions on the increments, and they cannot be arbitrary. Collecting terms, and moving terms with da to the right:

$$\begin{aligned} (2x + ay)\, dx + (ax + 2y)\, dy &= -xyda \\ 2xdx + 2ydy &= 2ada \end{aligned}$$

In matrix form, the above is:

$$\begin{bmatrix} 2x+ay & ax+2y \\ 2x & 2y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} -xy \\ 2a \end{bmatrix} da$$

Suppose that initially $(x, y, a) = (0, 1, 2)$ (verify that this point is a solution to the system (2.6)). Then,

$$\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} da$$

$$\begin{aligned} \begin{bmatrix} dx \\ dy \end{bmatrix} &= \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 4 \end{bmatrix} da \\ \begin{bmatrix} dx \\ dy \end{bmatrix} &= \begin{bmatrix} -2 \\ 2 \end{bmatrix} da \end{aligned}$$

This result means that if a increases from 2 to 2.01, i.e. $da = 0.01$, the resulting *approximate* changes in x and y are:

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} -0.02 \\ 0.02 \end{bmatrix}$$

The above results are approximate because we use the total differentials instead of actually solving the system (2.6). If we do that (difficult without special computer package), we find $x = -0.019565$ and $y = 1.0197$, so the exact changes are $\Delta x = -0.019565$ and $\Delta y = 0.0197$. Using total differentials gave us a good approximation of the effect on the endogenous variables of a small change in exogenous variable, in a nonlinear system of equations, without the need to solve the system exactly.

Now we would like to generalize the result from the above example of two endogenous variables and one exogenous variable, to n endogenous variables and m exogenous variables (or parameters). Suppose we have a system of n nonlinear equations, with n endogenous variables $y_1, ..., y_n$ and m exogenous variables (or parameters) $x_1, ..., x_m$:

$$\begin{aligned} F_1(y_1, ..., y_n; x_1, ..., x_m) &= c_1 \\ F_2(y_1, ..., y_n; x_1, ..., x_m) &= c_2 \\ &\vdots \\ F_n(y_1, ..., y_n; x_1, ..., x_m) &= c_n \end{aligned} \tag{2.7}$$

Notice that each equation is an implicit function. Taking total differentials[5]:

$$\begin{aligned}
\frac{\partial F_1}{\partial y_1}dy_1 + \frac{\partial F_1}{\partial y_2}dy_2 + ... + \frac{\partial F_1}{\partial y_n}dy_n + \frac{\partial F_1}{\partial x_1}dx_1 + \frac{\partial F_1}{\partial x_2}dx_2 + ... + \frac{\partial F_1}{\partial x_m}dx_m &= 0 \\
\frac{\partial F_2}{\partial y_1}dy_1 + \frac{\partial F_2}{\partial y_2}dy_2 + ... + \frac{\partial F_2}{\partial y_n}dy_n + \frac{\partial F_2}{\partial x_1}dx_1 + \frac{\partial F_2}{\partial x_2}dx_2 + ... + \frac{\partial F_2}{\partial x_m}dx_m &= 0 \\
&\vdots \\
\frac{\partial F_n}{\partial y_1}dy_1 + \frac{\partial F_n}{\partial y_2}dy_2 + ... + \frac{\partial F_n}{\partial y_n}dy_n + \frac{\partial F_n}{\partial x_1}dx_1 + \frac{\partial F_n}{\partial x_2}dx_2 + ... + \frac{\partial F_n}{\partial x_m}dx_m &= 0
\end{aligned}$$

This can be written in matrix form:

$$\underbrace{\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \cdots & \frac{\partial F_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial y_1} & \frac{\partial F_n}{\partial y_2} & \cdots & \frac{\partial F_n}{\partial y_n} \end{bmatrix}}_{\substack{n\times n \\ J_y}} \begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_n \end{bmatrix}_{n\times 1} = - \underbrace{\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_m} \end{bmatrix}}_{\substack{n\times m \\ J_x}} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{bmatrix}_{m\times 1}$$

Or, written compactly:

$$\begin{aligned}
J_y \cdot dy &= -J_x \cdot dx \\
dy &= -J_y^{-1} J_x \cdot dx
\end{aligned}$$

Suppose that we want to evaluate the effect of a small change in **one** of the exogenous variables, x_i, on all the endogenous variables $y_1, ..., y_n$, while keeping all other exogenous variables x_j, $j \neq i$ constant. That is, $dx_j = 0$ for all $j \neq i$, and the above system becomes:

$$\underbrace{\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \cdots & \frac{\partial F_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial y_1} & \frac{\partial F_n}{\partial y_2} & \cdots & \frac{\partial F_n}{\partial y_n} \end{bmatrix}}_{\substack{n\times n \\ J_y}} \begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_n \end{bmatrix}_{n\times 1} = - \underbrace{\begin{bmatrix} \frac{\partial F_1}{\partial x_i} \\ \frac{\partial F_2}{\partial x_i} \\ \vdots \\ \frac{\partial F_n}{\partial x_i} \end{bmatrix}}_{\substack{n\times 1 \\ \frac{\partial F}{\partial x_i}}} dx_i$$

[5]This step is often referred to as *linearizing* the system of equations.

Or, written compactly:

$$\begin{aligned} J_y \cdot dy &= -\frac{\partial F}{\partial x_i} dx_i \\ &\Rightarrow \; dy = -J_y^{-1} \frac{\partial F}{\partial x_i} dx_i \end{aligned}$$

The $n \times n$ matrix of partial derivatives of the functions $F_1, ..., F_n$, with respect to the endogenous variables $y_1, ..., y_n$, is called the **Jacobian Matrix**, and denoted by J_y. Notice that to solve for the changes in endogenous variables $dy_1, dy_2, ..., dy_n$, resulting from dx_i change in x_i, we need to invert the Jacobian matrix, which is possible if and only if the **Jacobian Determinant** $|J_y|$ is not zero. If we are interested in the partial derivatives, $\frac{\partial y_1}{\partial x_i}, \frac{\partial y_2}{\partial x_i}, ..., \frac{\partial y_n}{\partial x_i}$, we divide the above system by dx_i, and switching from d to ∂, to denote partial derivatives:

$$\frac{\partial y}{\partial x_i} = \begin{bmatrix} \left(\frac{\partial y_1}{\partial x_i}\right) \\ \left(\frac{\partial y_2}{\partial x_i}\right) \\ \vdots \\ \left(\frac{\partial y_n}{\partial x_i}\right) \end{bmatrix} = -J_y^{-1} \frac{\partial F}{\partial x_i}$$

Once again, the key condition for the above solution to exist is $|J| \neq 0$, i.e. nonzero Jacobian determinant. Alternatively, one can use Gaussian elimination or Cramer's rule to solve for $\frac{\partial y}{\partial x_i}$ from

$$J_y \frac{\partial y}{\partial x_i} = -\frac{\partial F}{\partial x_i}$$

The conclusions of the above steps can be summarized in the most general Implicit Function Theorem, for n implicit functions.

Theorem 7 ***(Implicit Function Theorem 2).*** *Let $F_1, ..., F_n : \mathbb{R}^{n+m} \to \mathbb{R}$ be C^1 functions (i.e. all partial derivatives exist, and are continuous functions). Consider the system of equations*

$$\begin{aligned} F_1 (y_1, ..., y_n; x_1, ..., x_m) &= c_1 \\ F_2 (y_1, ..., y_n; x_1, ..., x_m) &= c_2 \\ &\vdots \\ F_n (y_1, ..., y_n; x_1, ..., x_m) &= c_n \end{aligned}$$

Suppose $(y^*, x^*) = (y_1^*, ..., y_n^*; x_1^*, ..., x_m^*)$ is a solution to this system, and let J_y be the $n \times n$

Jacobian matrix with respect to y

$$J_y = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \cdots & \frac{\partial F_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial y_1} & \frac{\partial F_n}{\partial y_2} & \cdots & \frac{\partial F_n}{\partial y_n} \end{bmatrix}$$

evaluated at the solution (y^*, x^*). *If the Jacobian determinant* $|J| \neq 0$, *then*

(a) There exist C^1 *functions*

$$\begin{aligned} y_1 &= f_1(x_1, ..., x_m) \\ y_2 &= f_2(x_1, ..., x_m) \\ &\vdots \\ y_n &= f_n(x_1, ..., x_m) \end{aligned}$$

defined in a neighborhood B *of* $(y_1^*, ..., y_n^*; x_1^*, ..., x_m^*)$, *such that for all* $x = (x_1, ..., x_m) \in B$, *we have:*

$$\begin{aligned} F_1(f_1(x), ..., f_n(x); x_1, ..., x_m) &= c_1 \\ F_2(f_1(x), ..., f_n(x); x_1, ..., x_m) &= c_2 \\ &\vdots \\ F_n(f_1(x), ..., f_n(x); x_1, ..., x_m) &= c_n \end{aligned}$$

(b) The partial derivatives of $y_1, ..., y_n$ *with respect to* $x_1, ..., x_m$ *can be computed by (i) matrix inversion*

$$\frac{\partial y}{\partial x_i} = -J_y^{-1} \frac{\partial F}{\partial x_i}$$

or (ii) using Gaussian elimination

$$\frac{\partial y}{\partial x_i} = -J_y \backslash \frac{\partial F}{\partial x_i}$$

or (iii) using Cramer's rule

$$\frac{\partial y_h}{\partial x_i} = -\frac{|J_{y,h}|}{|J_y|}$$

where $J_{y,h}$ *is the Jacobian matrix with column* h *replaced by* $\frac{\partial F}{\partial x_i}$.

The general Implicit Function Theorem provides sufficient conditions under which a system with n nonlinear equations (2.7), can be solved for the n endogenous unknown variables $y_1, ..., y_n$. It also provides a way of computing the partial derivatives of the endogenous variables with respect to any of the exogenous variables.

Going back to the system in (2.6),

$$\begin{aligned} x^2 + axy + y^2 &= 1 \\ x^2 + y^2 - a^2 &= -3 \end{aligned}$$

Here x and y are the endogenous variables, and a is exogenous variable. We can write this system as

$$\begin{aligned} F_1(x, y; a) &= 1 \\ F_2(x, y; a) &= -3 \end{aligned}$$

Applying the general Implicit Function Theorem, we can compute the partial derivatives of x and y with respect to a by solving:

$$\begin{aligned} \underbrace{\begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}}_{J} \begin{bmatrix} \frac{\partial x}{\partial a} \\ \frac{\partial y}{\partial a} \end{bmatrix} &= -\begin{bmatrix} \frac{\partial F_1}{\partial a} \\ \frac{\partial F_2}{\partial a} \end{bmatrix} \\ \begin{bmatrix} 2x + ay & ax + 2y \\ 2x & 2y \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial a} \\ \frac{\partial y}{\partial a} \end{bmatrix} &= -\begin{bmatrix} xy \\ -2a \end{bmatrix} \end{aligned}$$

and the solution to the unknown partial derivatives is given by

$$\begin{bmatrix} \frac{\partial x}{\partial a} \\ \frac{\partial y}{\partial a} \end{bmatrix} = -\begin{bmatrix} 2x + ay & ax + 2y \\ 2x & 2y \end{bmatrix}^{-1} \begin{bmatrix} xy \\ -2a \end{bmatrix}$$

Suppose that initially $(x, y, a) = (0, 1, 2)$. Then, the partial derivatives at this point are

$$\begin{bmatrix} \frac{\partial x}{\partial a} \\ \frac{\partial y}{\partial a} \end{bmatrix} = -\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

To summarize, in this section we learned how to compute derivatives of variables that are related to each other trough implicit function. The main result is the Implicit Function Theorem. We then extended the analysis to systems of nonlinear equations, that can be viewed as n implicit functions. The general implicit function theorem allows us to perform comparative statics - i.e. evaluating the marginal effects of any exogenous variable, on any endogenous variable in a nonlinear system of equations.

Chapter 3

Optimization

3.1 Introduction

By definition, Economics is a discipline that deals with optimization: "the study of choices under scarcity". Every choice that people make can be viewed as part of some optimization problem. For example, when someone decides to earn a college degree, this decision is made to achieve some goals subject to time and resource constraints - an optimization problem. Similarly, decisions made by firms are an attempt to optimize, for example maximize the dividends for shareholders. It is not surprising that almost all economic theories are derived from some mathematical optimization problems. Even when economists are simply looking at patterns in the data, and estimating some relationships, the statistical methods used are derived from optimization problems. For example, Ordinary Least Squares estimation minimizes the sum of squared deviations of the statistical relationship and the actual data. Another popular estimation method is Maximum Likelihood Estimator, which maximizes the likelihood (chances) of obtaining the observed data with the proposed statistical model.

To make things precise, *optimization theory* is the field of mathematics that analyzes *minimum* and *maximum* of functions. We call an optimization problem, any problem that attempts to find a minimum or a maximum of some function, possibly under constraints. The general structure of an optimization problem is:

$$\begin{aligned} &\max_{x_1,\ldots,x_n} f\left(x_1,\ldots,x_n\right) \quad \text{or} \quad \min_{x_1,\ldots,x_n} f\left(x_1,\ldots,x_n\right) \\ &s.t. \;\; \text{constraints on } x_1,\ldots,x_n \end{aligned}$$

The function f is called the **objective function**, and $x_1, \ldots, x_n$ are called the **choice variables** or **decision variables**. The choice variables are so called because the person solving the optimization problem wants to choose them to maximize or to minimize the objective function. The number of choice variables n can be quite large, and often infinite. The shorthand $s.t.$ means "subject to" or "such that". The constraints usually restrict the range of admissible values of the choice variables. For example, quantities and prices of goods are often restricted to be nonnegative. A specific example of an optimization problem is consumer's choice problem:

$$\max_{x_1,...,x_n} u(x_1,...,x_n)$$
$$s.t.$$
$$p_1x_1 + p_2x_2 + ... + p_nx_n = I$$
$$x_1, x_2, ..., x_n \geq 0$$

In the above problem the objective function is the utility, and the choice variables are quantities of n goods. These quantities are restricted to satisfy the budget constraint (prices are $p_1,...,p_n$ and income is I). Moreover, the chosen quantities are restricted to be nonnegative.

The two most important questions related to optimization problems, are:

1. **Does an optimum exist?** For example, what is the largest number smaller than 100? Such number does not exist, because for any number $x < 100$, there is always another number x', such that $x' > x$ and $x' < 100$. So not every optimization problem has a solution. Thus, *existence* of a solution is the most fundamental issue in optimization.

2. **Is the solution unique?** Often, a solution to an optimization problem is the prediction of the model. If the solution is unique, it means that the model has a unique prediction, and if there are many (sometimes infinitely many) solutions, it means that the theory is indecisive about the issue being studied. For example, a theory that predicts that the effect of a given policy on growth is anything between -10% and 10% with equal probabilities, is not very useful theory - i.e. a theory that says anything can happen is not useful. Therefore, *uniqueness* of a solution is often something that economists are interested in.

Besides existence and uniqueness, we need to distinguish between local and global optimum (in the next section). In this chapter we will study mathematical techniques for solving optimization problems. These are mainly calculus techniques. In practice, many optimization problems can only be solved with the help of computers, and applied economists need to be familiar with some *numerical* optimization techniques (computer algorithms). Numerical methods for optimization problems are beyond the scope of this course, and they are addressed in most texts on numerical analysis.

3.1.1 Local v.s. global optima

A function f has a **local maximum** (or relative maximum) at a point x^* if there exists a neighborhood B of x^* such that $f(x^*) \geq f(x)$ for all $x \in B$. Similarly, a function f has a **local minimum** (or relative minimum) at a point x^* if there exists a neighborhood B of x^* such that $f(x^*) \leq f(x)$ for all $x \in B$.

A function f has a **global maximum** (or absolute maximum) at a point x^* if $f(x^*) \geq f(x)$ for all x in the (possibly restricted) domain of f. Similarly, a function f has a **global minimum** (or absolute minimum) at a point x^* if $f(x^*) \leq f(x)$ for all x in the (possibly restricted) domain of f. Notice that any global maximum (minimum) is also a local maximum (minimum).

The general name for maximum or minimum is **extremum** (extrema in plural). The points at which the objective function attains its global maximum are referred to as arg max or maximizer, and we write $x^* = \arg\max_x f(x)$. Similarly, the points at which the objective function attains its global minimum referred to as arg min, or minimizer, and we write $x^* = \arg\min_x f(x)$.

Figure (3.1) shows the graph of the function $f(x) = x^3 - 12x^2 + 36x + 8$, with restricted domain $-1 \leq x \leq 9$. All the values $-1 < x < 9$ are called *interior* points, while $x = -1$ and $x = 9$ are *boundary* values.

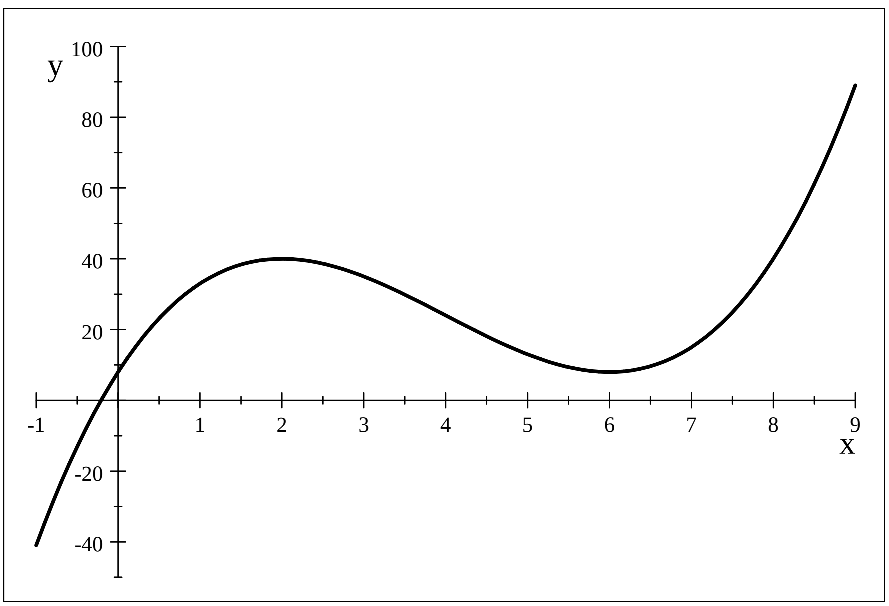


Figure 3.1: $f(x) = x^3 - 12x^2 + 36x + 8$

In the above graph, the objective function f has local maximum at $x_1^* = 2$ and local minimum at $x_2^* = 6$. Notice that there is neighborhood of $x_1^* = 2$, say $B_1 = (1.9, 2.1)$, such that $f(2) \geq x$ for all $x \in B_1$, which makes $f(2)$ a local maximum. Similarly, there is a neighborhood of $x_2^* = 6$, say $B_2 = (5.9, 6.1)$, such that $f(6) \leq f(x)$ for all $x \in B_2$, which makes $f(6)$ a local minimum.

Figure (3.1) also illustrates that the function f has a global minimum at $x_3^* = -1$ and a global maximum at $x_4^* = 9$. We write

$$\begin{aligned} -1 &= \arg\min_x f(x) \\ 9 &= \arg\max_x f(x) \end{aligned}$$

These global extrema in figure (3.1) occur at the *boundaries* of the domain $\{-1, 9\}$, while the local extrema occur at the *interior* of the domain $(-1, 9)$.

Although the figure (3.1) illustrates extrema of a function of one variable, the above definitions of local and global extrema apply to multivariate functions as well. The next section deals with calculus techniques for finding local extrema of differentiable functions of one variable, and also with identifying its type (maximum, minimum).

3.2 Optimizing Functions of One Variable

In this section we present calculus criteria for finding and characterizing local extrema (minima and maxima) of differentiable functions. Looking at figure (3.1), and recalling that the derivative $f'(x)$ represents the slope of the function f at point x, leads to the conclusion that if a function f has an interior extremum at point x^*, then it is necessary that $f'(x^*) = 0$. If this was not true, and suppose $f'(x^*) > 0$, then $f(x^* + \varepsilon) > f(x^*)$ for some small $\varepsilon > 0$, which means that $f(x^*)$ cannot be a local maximum. Similarly, if $f'(x^*) < 0$, then $f(x^* - \varepsilon) > f(x^*)$ for some small $\varepsilon > 0$, which means that $f(x^*)$ cannot be a local minimum. Thus, if either $f'(x^*) > 0$ or $f'(x^*) < 0$, then $f(x^*)$ cannot be a local maximum or a local minimum. Thus, we proved the following theorem.

Theorem 8 *(First-order necessary condition for interior extrema). If a differentiable function f has a local maximum or a local minimum at an interior point x of its domain, then it is* ***necessary*** *that*

$$f'(x) = 0$$

In mathematical symbols, we write

$$f(x) \ \textit{is local extremum} \ \Rightarrow f'(x) = 0$$

The condition $f'(x) = 0$ is necessary for local maximum or local minimum, but this condition is not *sufficient*. That is, if we have $f'(x) = 0$, this does not imply that the function f attains either a maximum or a minimum at point x. The next definition provides the somewhat confusing terminology related to first order conditions.

Definition 15 *Let x be a point in the domain of the function f such that $f'(x) = 0$ or $f'(x)$ doesn't exist.*

*(i). x is called **critical point** or **critical value**,*

*(ii). $f(x)$ is called the **stationary value of** f or **critical value of** f,*

*(iii). $(x, f(x))$ is called a **stationary point** or **critical point**,*

*(iv). a stationary point $(x, f(x))$ that is not a local extremum (max or min), is called **saddle point**.*

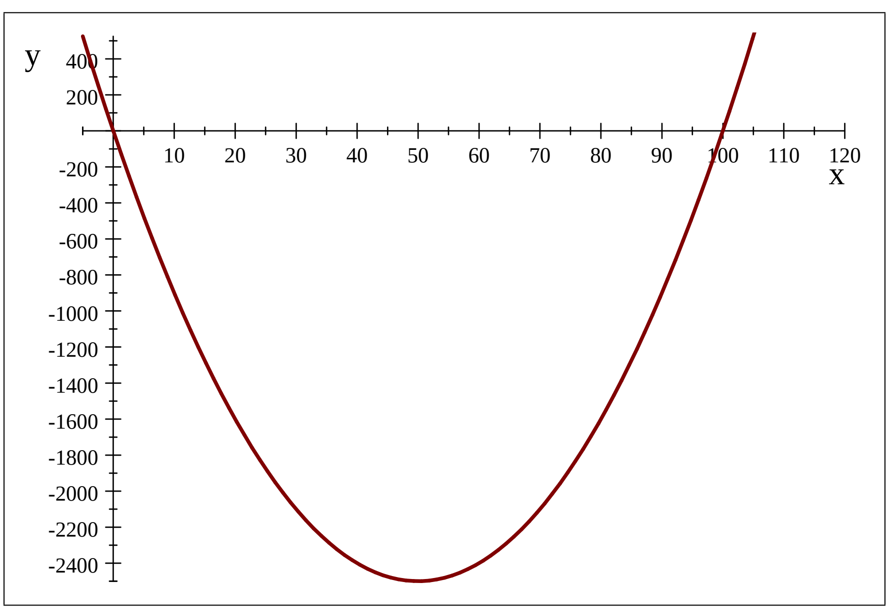
As you can see from definition 15, there is unfortunately no unique definition of concepts such as critical point and critical value, that all texts agree on.

Critical values are not always extreme values (i.e. maxima or minima), as the following example illustrates.

Example 30 *The function $f(x) = x^2 - 100x$ has one critical value, found by solving $f'(x) = 0$:*

$$\begin{aligned} 2x - 100 &= 0 \\ \Rightarrow & \quad x = 50 \end{aligned}$$

Thus, $x = 50$ is the only critical value of this function. In the next figure we see that this function has a local (and global) minimum at the point $x = 50$.

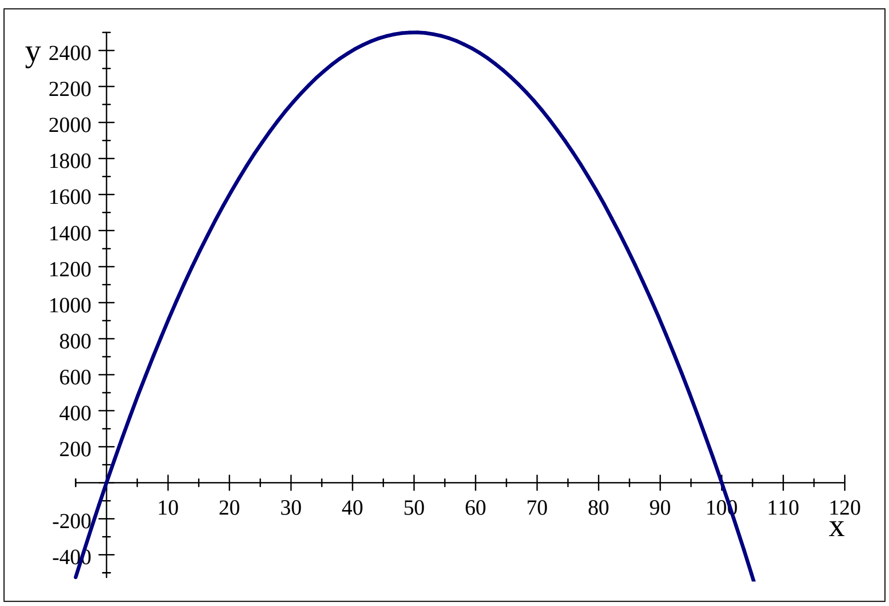


$f(x) = x^2 - 100x$

The function $f(x) = -x^2 + 100x$ *has one critical value, found by solving* $f'(x) = 0$*:*

$$\begin{aligned} -2x + 100 &= 0 \\ &\Rightarrow x = 50 \end{aligned}$$

Thus, $x = 50$ *is the only critical value of this function. In the next figure we see that this function has a local (and global) maximum at the point* $x = 50$*.*

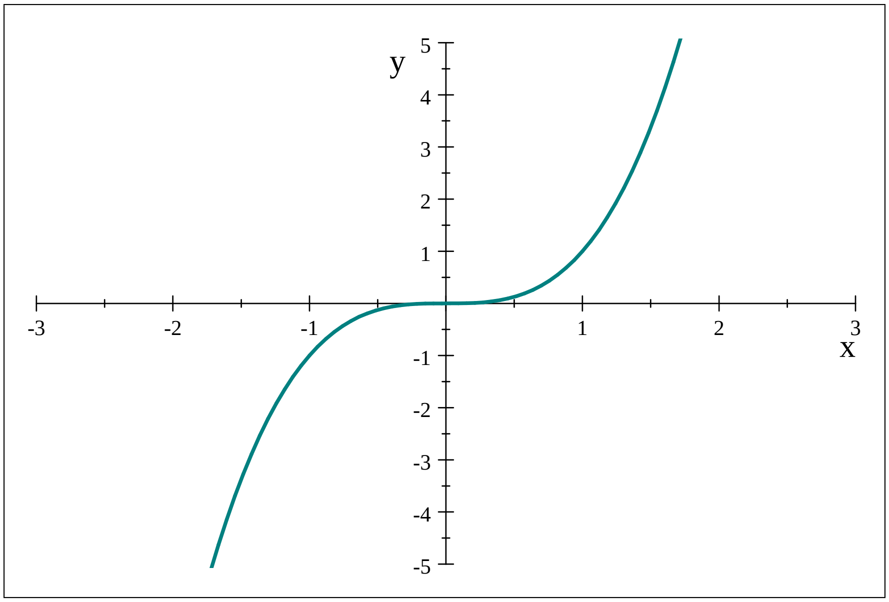


$f(x) = -x^2 + 100x$

The function $f(x) = x^3$ *has one critical value, found by solving* $f'(x) = 0$*:*

$$\begin{aligned} 3x^2 &= 0 \\ &\Rightarrow x = 0 \end{aligned}$$

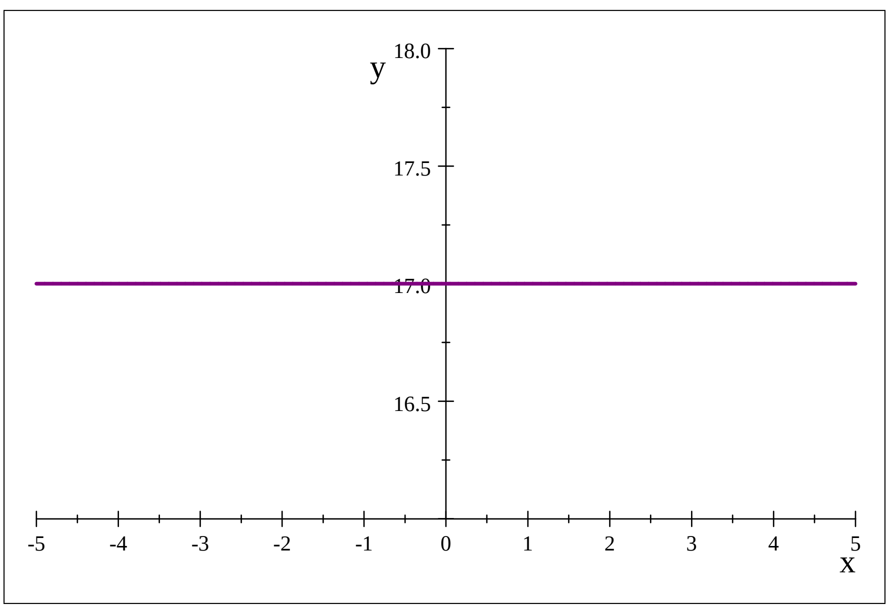
Thus, $x = 0$ *is the only critical value of this function. In the next figure we see that at* $x = 0$ *the function does not have neither local minimum nor local maximum, despite the fact that* $f'(0) = 0$*.*



$f(x) = x^3$

The point $(0,0)$ *is stationary, but not a local extremum, and is therefore a saddle point.*

The function $f(x) = 17$ *has* $f'(x) = 0$ *for all* $x \in \mathbb{R}$*, so all* x *values in the domain are critical values. All the values are both global minima and global maxima.*



$f(x) = 17$

The last example illustrates an important point that a critical value does not imply that the function has an extremum at that point. But every value x^* at which a differentiable function attains an extremum, must be a critical value at which the derivative is zero, $f'(x^*) = 0$. This is precisely what we mean when we say that a first-order *necessary* condition for maximum or minimum is $f'(x^*) = 0$. The name "first-order" is from the fact that it utilizes the first-order derivative, which we usually call "first derivative" for short.

We should emphasize again that the first-order necessary condition can be used for differentiable functions (functions that have first-order derivatives). Figure (3.2) illustrates a non-differentiable function, with local maximum at a and local minimum at point b. These points however cannot be found by solving $f'(x) = 0$, since the function f does not have derivatives at these points. In practice, economists try to use differentiable functions in

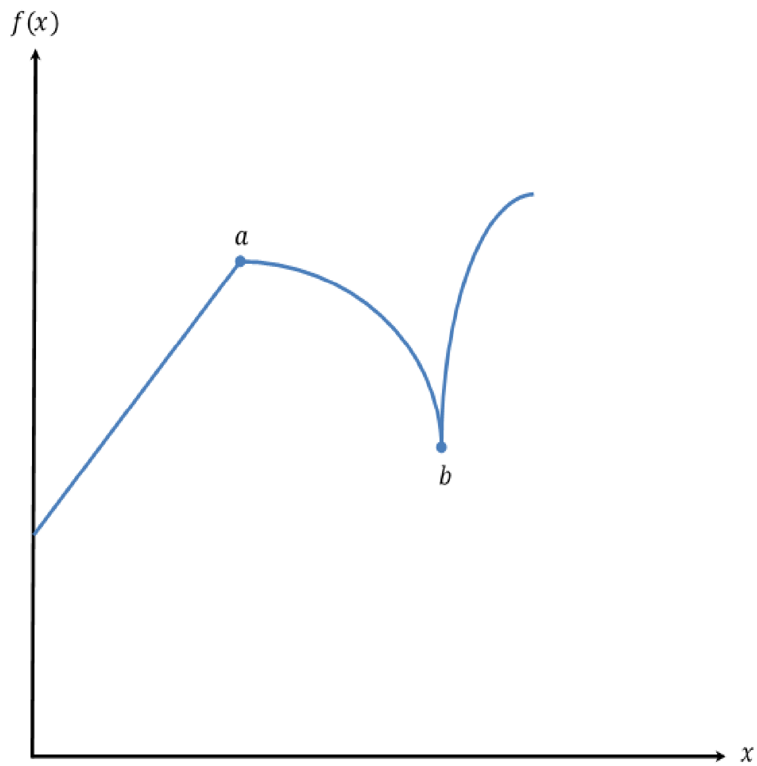


Figure 3.2: Local Extrema of non-differentiable function.

their models, but sometimes we do encounter cases like the one in figure (3.2). There are numerical techniques that allow finding local extrema without relying on derivatives (these are called "derivative-free" methods).

After finding critical values x such that $f'(x) = 0$, there are four possibilities:

1. $f(x)$ is a local maximum,

2. $f(x)$ is a local minimum,

3. $f(x)$ is both local minimum and local maximum,

4. $f(x)$ is neither local maximum nor minimum.

How do we know which one of the three cases occurs at a given critical value? Once again, looking at figure (3.1), we see that at a local maximum the slope of the graph decreases as we increase x a little bit, while at a local minimum, the slope of the graph increases with x. Recall that the derivative of a function f' determines whether it is increasing or decreasing at a point x: if $f'(x) > 0$ then f is increasing and if $f'(x) < 0$ the function f is decreasing at point x. Similarly, the second derivative $f''(x)$ determines whether f' is increasing or decreasing (because the second derivative of f is the first derivative of f', and so it determines the slope of f'). Thus, if we found a critical point x, at which $f'(x) = 0$, the function f has a local maximum at that point if $f''(x) < 0$ and f has a local minimum if $f''(x) > 0$. These conclusions are formally stated in the following theorem.

Theorem 9 *(Second-order sufficient conditions for interior extrema). Suppose that the function f is twice differentiable, and x is a critical value of f, i.e. $f'(x) = 0$.*

(i) if $f''(x) < 0$ then $f(x)$ is a local maximum,

(ii) if $f''(x) > 0$ then $f(x)$ is a local minimum.

(iii) if $f''(x) = 0$ then $f(x)$ could be a local maximum, or minimum, or neither (the second-order test is inconclusive).

*Thus, we say that $f''(x) < 0$ is a **second-order sufficient condition** for local maximum at critical value x, and $f''(x) > 0$ is a **second-order sufficient condition** for local minimum at critical value x.*

Notice that the second-order sufficient conditions are applicable only after the first-order necessary conditions are satisfied (i.e. applicable to critical values only). We should also mention that the above conditions are not *necessary*, i.e. it is possible that at some critical point x we have $f''(x) = 0$ (so the second-order sufficient conditions do not hold), and yet the function has a local maximum or minimum at x. Finally, the name "second-order" is due to the fact that we utilize second-order derivatives.

Example 31 *The function $f(x) = x^2 - 100x$ has one critical value $x = 50$, and $f''(50) = 2 > 0$, so based on the second-order sufficient condition we conclude that $f(50)$ is a local minimum.*

The function $f(x) = -x^2 + 100x$ has one critical value at $x = 50$, and $f''(50) - 2 < 0$, so based on the second-order sufficient condition we conclude that $f(50)$ is a local maximum.

The function $f(x) = x^3$ has one critical value at $x = 0$, and $f''(0) = 0$, so the second-order sufficient condition does not tell us whether $f(0)$ is a local maximum or a local minimum or neither. In fact, when you plot the graph of this function, you see that this function *does not have a local or global extrema. The point* $(0,0)$ *is* ***inflection point****, where the function switches from convex to concave.*

The function $f(x) = 17$ *has all* $x \in \mathbb{R}$ *critical values. Also,* $f''(x) = 0$ *for all* $x \in \mathbb{R}$*, thus the second-order sufficient condition does not tell us whether* $f(x)$ *is a local maximum or a local minimum or neither for any* $x \in \mathbb{R}$*. By looking at the graph of the function, we do see that all* $f(x)$ *are both global maxima and minima.*

The above example illustrates that if $f''(x) = 0$ at a critical value x, then the second-order sufficient condition does not determine whether $f(x)$ is a maximum, minimum or neither. Thus, we say that the second-order test is inconclusive.

Example 32 *Given the function* $f(x) = x^3 - 12x^2 + 36x + 8$, $x \in \mathbb{R}$*,*

(i) Find the critical values and stationary points of f*.*

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 24x + 36 &= 0 \end{aligned}$$

$$x_1, x_2 = \frac{24 \pm \sqrt{24^2 - 4 \cdot 3 \cdot 36}}{2 \cdot 3} = \frac{24 \pm 12}{6}$$

Critical values are $x_1 = 2$, $x_2 = 6$*. Stationary points are* $(2, f(2)) = (2, 40)$ *and* $(6, f(6)) = (6, 8)$*.*

(ii) Using the second-order sufficient condition, determine whether the function f *has a local maximum, a local minimum or if the second-order test is inconclusive.*

$$\begin{aligned} f''(x) &= 6x - 24 \\ f''(2) &= 6 \cdot 2 - 24 = -12 < 0 \quad \Rightarrow f(2) \text{ is local maximum} \\ f''(6) &= 6 \cdot 6 - 24 = 12 > 0 \quad \Rightarrow f(6) \text{ is local minimum} \end{aligned}$$

The next table summarizes the first-order necessary and second-order sufficient conditions for local extrema of univariate (one variable) functions.

Condition	Maximum	Minimum
First-order necessary	$f'(x) = 0$	$f'(x) = 0$
Second-order sufficient†	$f''(x) < 0$	$f''(x) > 0$
†Applicable only after the first-order necessary condition is satisfied		

In cases of $f''(x) = 0$, the second-order test is inconclusive, and higher order derivatives can be used. We will not discuss higher order tests here.

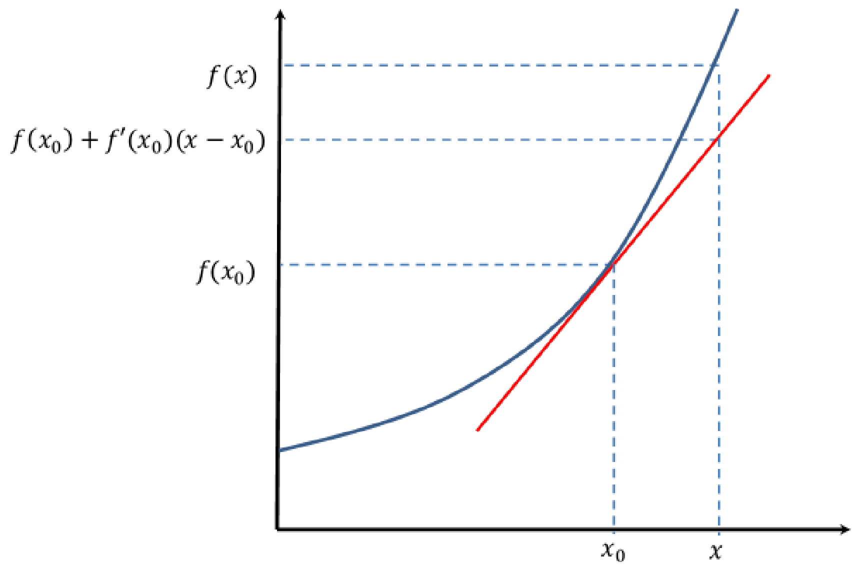


Figure 3.3: Linear Approximation

3.3 Taylor Approximation

The previous section discussed first-order necessary and second-order sufficient conditions for local extrema. In practice, sometimes we encounter a difficult optimization problem, that can only be solved numerically. What computers often do, in order to solve complex optimization problems, is use an approximation of the objective function with some simpler functions - usually polynomials. We have already encountered linear approximation to a function of one variable, when we discussed the differential. Let's recall that given $y = f(x)$, the differential of y at x_0, when x increments by dx, is

$$dy = f'(x_0)\, dx$$

The differential is in fact a linear approximation of the change in y, using the tangent line to the original function, instead of the function f itself. Suppose that $dx = x - x_0$, where x_0 is the initial value of x, and the actual change in y is $\Delta y = f(x) - f(x_0)$. We can write

$$f(x) = f(x_0) + \Delta y \approx f(x_0) + dy = f(x_0) + f'(x_0)(x - x_0) \equiv P_1(x)$$

The linear approximation is illustrated in figure (3.3).

Thus, the value of a function at point x can be linearly approximated near some point x_0 by using the tangent line to the function at x_0, provided that the derivative at x_0 exists. Linear approximation can be viewed as approximation with a polynomial of degree 1. The same function $f(x)$ can also be approximated with a polynomial of degree 2 near point x_0, provided that it is *twice differentiable*. It turns out that if we require that the approximating polynomial should have the same value at x_0 as the approximated function, i.e. $P_2(x_0) = f(x_0)$, as well as first and second derivatives ($P_2'(x_0) = f'(x_0)$ and $P_2''(x_0) = f''(x_0)$), then the approximation is:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \equiv P_2(x)$$

It turns out that any function that is n-times differentiable at point x_0 can be approximated around that point (locally) by a polynomial of degree n. This result is stated precisely in the following theorem.

Theorem 10 *(Taylor's Theorem for functions of one variable). Suppose that f is n-times differentiable at point x_0. Then*

$$\begin{aligned} f(x) &= \left[f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \right] + R_n(x) \\ &\equiv P_n(x) + R_n(x) \\ \textit{and } \lim_{x \to x_0} R_n(x) &= 0 \end{aligned}$$

where $n! = 1 \cdot 2 \cdot ... \cdot n$ (read "n factorial"), and $R_n(x)$ is a remainder.

Thus, when a function is approximated "close" to the given point, $x \to x_0$, the approximation is more precise and the remainder disappears.

Example 33 *Approximate the function $f(x) = e^x$ around $x_0 = 0$ with Taylor polynomial of degree 3, and compare this approximation to the actual value of f at $x = 1$.*

$$\begin{aligned} P_3(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 \\ &= e^{x_0} + \frac{e^{x_0}}{1}x + \frac{e^{x_0}}{2}x^2 + \frac{e^{x_0}}{6}x^3 \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} = 1 + 1 + \frac{1}{2} + \frac{1}{6} = 2.6667 \end{aligned}$$

The actual value of $f(1) = e^1 = 2.7183$.

You might ask yourself, why approximate the function e^x when any calculator can give you "exact" value of this function at any x. The point is that your calculator uses Taylor polynomials to give you an approximate value of functions such as e^x, $\ln(x)$, $\sin(x)$, $\cos(x)$ and many others.

3.4 Exponential and Logarithmic Functions

Many problems in economics involve decisions over time - dynamic choices. For example, if you are saving money for retirement, a deposit of y_0 dollars today at interest rate r per year will grow to $y_t = y_0(1+r)^t$ after t periods. This was an example of future value. Similarly, the present value of an amount FV received t periods from today, and given the interest rate r per period, is $PV = FV(1+r)^{-t}$. Such calculations of future and present values involve **exponential functions**, and their inverses, the **logarithmic functions**. Due to their vast importance in economics, these functions deserve the special attention in this chapter.

Definition 16 *An exponential function with base $b > 0$, $b \neq 1$ has the form*

$$y = f(x) = b^x, \quad x \in \mathbb{R}$$

Notice that the domain of exponential functions consists of all real numbers. Examples include

$$y = 2^x,\ y = e^x,\ y = 10^x,\ y = 0.5^x,\ y = 0.9^x$$

where $e = 2.718281828459046...$ Figure (3.4) plots the graphs of $y = 2^x$ (solid thin), $y = e^x$ (dashed) and $y = 4^x$ (solid thick). All the functions in figure (3.4) are monotone increasing,

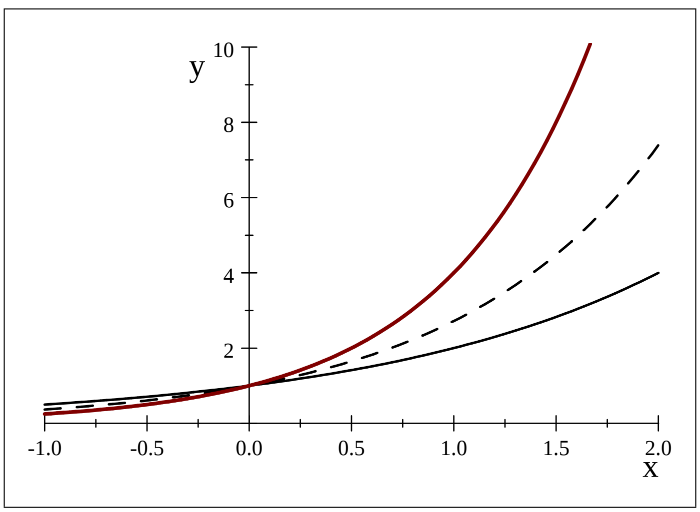


Figure 3.4: $y = 2^x$, $y = e^x$, $y = 4^x$

which is the case with all exponential functions with base $b > 1$. Moreover, all exponential functions are positive valued functions (i.e. $f(x) > 0$ for all x). Also notice that as $x \to -\infty$, the value of exponential function with base $b > 1$ approaches 0.

Figure (3.5) plots the graphs of $y = 0.2^x$ (solid thin), $y = 0.5^x$ (dashed) and $y = 0.7^x$ (solid thick). All the functions in figure (3.5) are monotone decreasing, which is the case

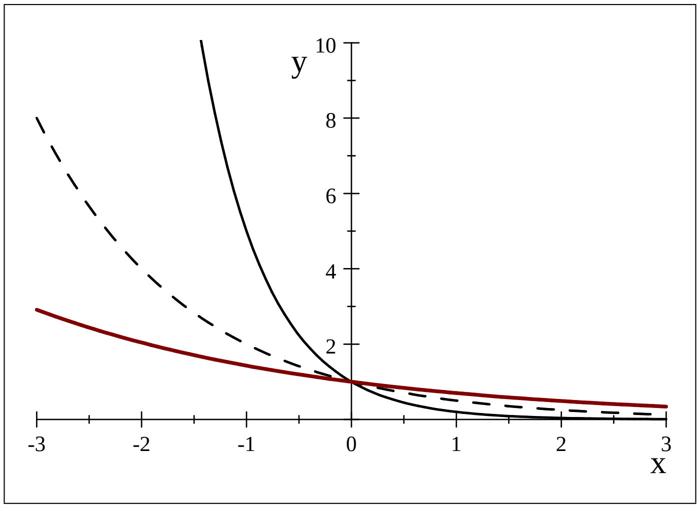


Figure 3.5: $y = 0.2^x$, $y = 0.5^x$, $y = 0.7^x$

with all exponential functions with base $b < 1$. Moreover, all exponential functions are positive valued functions (i.e. $f(x) > 0$ for all x). Also notice that as $x \to \infty$, the value of exponential function with base $b < 1$ approaches 0.

The definition of exponential functions excludes non-positive bases and $b = 1$. In the case of $b = 1$, the function $y = 1^x = 1$ is just a constant function, and is not classified as exponential function. In the case of $b \leq 0$, functions like $y = 0^x$ or $y = (-2)^x$ are not defined for some x or do not always attain a real value. Thus, exponential functions restrict the base to $b > 0$, $b \neq 1$.

Exponential functions should not be confused with *power functions*, such as $y = 5x^2$, $y = 2x^{-0.5}$ or $y = -1.5x^5$, or in general $y = cx^r$ with $c, r \in \mathbb{R}$. A power function raises the variable x to some constant power, while in exponential functions the variable x is the exponent. Thus, in a power function the input x is the base, while in exponential function the input x is the exponent.

Since exponential functions are monotone, there is only one value of x which is mapped into each (positive) value of y. Thus, for every exponential function $y = f(x) = b^x$, there exists an inverse function $f^{-1}(y) : \mathbb{R}_{++} \to \mathbb{R}$. These inverse functions are called **logarithmic functions** and are defined as follows.

Definition 17 *Let $y = f(x) = b^x$, with $b > 0$, $b \neq 1$ and $x \in \mathbb{R}$ be an exponential function. The inverse of f is denoted $f^{-1}(y) = \log_b y$ (which reads "log base b of y") is defined as follows*[1]*:*

$$y = b^x \iff x = \log_b y, \quad y > 0$$

[1] Sometimes we write $\log_b(y)$ or without the parenthesis $\log_b y$.

The logarithmic function therefore gives the value of x such that b raised to the power of x is equal to y. For example, the next table presents some exponential functions and their inverses (logarithmic functions).

Exponential function $y = f(x)$	Logarithmic function $x = f^{-1}(y)$
$y = 2^x$	$x = \log_2 y$
$y = e^x$	$x = \log_e y = \ln y$
$y = 5^x$	$x = \log_5 y$
$y = 10^x$	$x = \log_{10} y = \log y$

The base $e = 2.71828...$ is called the *natural base* and we denote $\log_e y = \ln y$ (natural logarithm). The base of $b = 10$ is called the *common base* and we denote $\log_{10} y = \log y$ (the common logarithm). In Matlab the notation is different, and it uses $\log y$ to denote the natural logarithm. Thus, in Matlab, if we want to compute $\ln(7)$, we type `log(7)`. In fact, all other exponents and logarithms are rarely used, and can all be represented in terms of base e exponents and logarithms.

Figure (3.6) illustrates the graphs of $y = e^x$ (solid) and $y = \ln(x)$ (dashed). The two

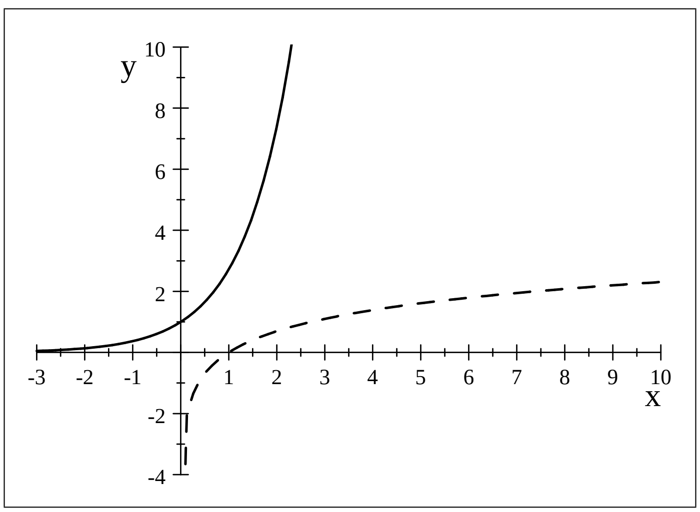


Figure 3.6: $y = e^x$, $y = \ln(x)$

graphs are mirror images of each other, and symmetric about the 45^0 line.

3.4.1 The natural base e and base conversion

It is very surprising that of all the exponential functions $y = 2^x$, $y = 1.5^x$, $y = 5^x$, $y = 10^x$, the most useful one is $y = e^x = (2.718281828459046...)^x$. The number e is not a rational number, i.e. cannot be written as a ratio of two integers, so we can't even write the entire number e. This number has infinitely many digits, and the first 50 are as follows:

$$e = 2.71828182845904523536028747135266249775724709369995...$$

In practice, we can only represent an approximation to this number. Surely, a base of $b = 2$ looks more "neat" than the natural base e. It turns out that when it comes to calculations involving growth or discounting, the natural base is indeed "natural", as the next example illustrates.

Suppose you deposit \$1 for a year at interest rate of $r = 100\%$. How much money will you have after one year? We are tempted to say that the answer is \$2, but this is the case only when interest does not compound during the year, i.e. if you withdraw the money after 6 months, you still get the \$1 deposited. Suppose that interest compounds every 6 moths, i.e. twice a year. Then, after the first 6 moths you have $1 \cdot \left(1 + \frac{100\%}{2}\right) = (1 + 0.5) = \$1.5$ and at the end of the year we will have

$$\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{2}\right) = \left(1 + \frac{1}{2}\right)^2 = \$2.25$$

Notice that after the first 6 months you have \$1.5, so in the second 6 months interest accumulates on this new amount.

What if you want to withdraw your money after a quarter (3 months)? If the interest compounds only two times a year, and you withdraw after 3 months, then you don't get any interest and simply get back your \$1. However, if interest compounds every quarter of a year, then after one quarter you get $1 \cdot \left(1 + \frac{100\%}{4}\right) = \$1.25$ and after a year you get

$$\left(1 + \frac{1}{4}\right)^4 = \$2.4414$$

We can go on and ask what happens if you withdraw the money after one month. If interest compounds monthly, then after one month you will receive $\left(1 + \frac{100\%}{12}\right)$ and after a year

$$\left(1 + \frac{1}{12}\right)^{12} = \$2.613$$

Similarly, if interest compounds weekly, assuming 52 weeks per year, then after a year your initial deposit of \$1 becomes

$$\left(1 + \frac{1}{52}\right)^{52} = \$2.692\,6$$

If interest compounds daily, then after a year (365 days) your initial deposit of \$1 becomes[2]

$$\left(1+\frac{1}{365}\right)^{365} = \$2.714\,6$$

With hourly compounding, (approximately 8766 hours per year), then after a year your initial deposit of \$1 becomes

$$\left(1+\frac{1}{8766}\right)^{8766} = 2.718126797795577$$

We can go on and calculate the amount after one year with interest compounding every minute, every second, every millisecond, etc. The results are summarized in the next table

Frequency of compounding n	Gross return on \$1 after year $\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
4	2.4414
12	2.613035290224676
52	2.692596954437168
365	2.714567482021973
8766 (hours per year)	2.718126797795577
525960 (minutes per year)	2.718126797795577
$31,557,600$ (seconds per year)	2.718281776412525
$\vdots$	$\vdots$
∞	$e = 2.718281828459046...$

Thus, one of several ways to define the number e is as follows.

Definition 18 *The number* $e = 2.718281828459046...$ *is defined as*

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

The economic interpretation of the number e is the gross return on \$1 after one year at annual interest rate of 100%, with *continuous compounding* (i.e. interest compounds

[2] In real world banking, the interest usually compounds daily. For example, your credit card agreement might say that Annual Percentage Rate is 10%, with daily compounding. Thus, on every \$1 of debt held for a year, will become $e^{0.1} = 1.105\,2$, i.e. will accumulate 10.52% Annual Percentage Yield. Mortgage interest usually compounds monthly.

$n = \infty$ times per year, or every instant). Therefore, the number e indeed arises naturally in economics.

Suppose that the annual interest rate is not 100% but is $r \neq 0$, and the interest compounds n times per year. Then, after one year, the gross return on \$1 is

$$\left(1 + \frac{r}{n}\right)^n$$

Taking the limit as $n \to \infty$, gives

$$\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = \lim_{n\to\infty} \left[\left(1 + \frac{1}{n/r}\right)^{\frac{n}{r}}\right]^r$$

Letting $x = n/r$, the above limit becomes

$$\lim_{x\to\infty} \left[\left(1 + \frac{1}{x}\right)^x\right]^r = \lim_{y\to e} y^r = e^r$$

Here we used the substitution rule of limits, since the limit of the term in the squared brackets is, by definition, the number e. Thus, the function e^r gives the return on \$1 invested at annual interest rate r and continuous compounding. If, instead of \$1 you invest \$$A$, then the gross return is Ae^r after one year and Ae^{rt} after t years.

We will now show that any exponential function with base a can be represented as another exponential function with base b. Similarly, any logarithmic function with base a can be represented in terms of any other base b. This means that for all practical purposes, you only need one base, and since base e arises naturally, we might as well use only the natural base. Moreover, as you already know, $y = e^x$ is the only exponential function whose derivative is the function itself e^x. Therefore, the only exponential and logarithmic function that one needs are e^x and $\ln(x)$.

Let $y = a^x$ be exponential function with base a. This function can be written as

$$y = a^x = (b^r)^x$$

We simply write a as some other base b to some power r, which is yet unknown, but we know that it exists. Thus, by definition of logarithmic function,

$$a = b^r \iff r = \log_b a$$

Thus, the original exponential function with base a can be written as

$$y = b^{(\log_b a)x}$$

which is another exponential function with base b.

Next, write the two exponential functions together:

$$\begin{aligned} y &= a^x \iff x = \log_a y \\ y &= b^{(\log_b a)x} \iff (\log_b a)\, x = \log_b y \end{aligned}$$

Those two together imply

$$\begin{aligned} (\log_b a) \log_a y &= \log_b y \\ \log_a y &= \frac{\log_b y}{\log_b a} \end{aligned}$$

To summarize, we derived two base-conversion formulas, which are summarized in the following proposition.

Proposition 4 *(Base-conversion for exponential and logarithmic functions). Given an exponential function* $y = a^x$ *and its inverse (the logarithmic function)* $x = \log_a y$ *(both with base* a*), we can represent them as exponential and logarithmic functions of base* b*:*

$$\begin{aligned} y &= a^x = b^{(\log_b a)x} \\ \log_a y &= \frac{\log_b y}{\log_b a} \end{aligned}$$

In particular, suppose $b = e$*, the natural base. Then the conversion to the natural base becomes:*

$$\begin{aligned} y &= a^x = e^{(\ln a)x} \\ \log_a y &= \frac{\ln y}{\ln a} \end{aligned}$$

Example 34 *Suppose we want to calculate* $\log_5 125$*. We know that this is* 3*, because* $\log_5 125$ *is the answer to the question "to which power we should raise* 5 *to get* 125*"? But suppose that don't know the answer, and have a calculator that can only calculate natural logarithms* $\ln(x)$*. Thus, by the base-conversion theorem, we have*

$$\log_5 125 = \frac{\ln 125}{\ln 5} = \frac{4.8283}{1.6094} = 3$$

Example 35 *Suppose we want to calculate* $\log_2 1024$*, but your calculator has only the natural logarithm function* $\ln(x)$*. Show how you can compute* $\log_2 1024$ *using the* $\ln(x)$ *function only.*

3.4.2 Rules of exponents and logarithms

For completeness, I present the rules of exponents and logarithms here.

Rules of exponents

1. $x^a x^b = x^{a+b}$
2. $(x^a)^b = x^{ab}$
3. $x^{-a} = \frac{1}{x^a}$
4. $x^0 = 1$
5. $(xy)^a = x^a y^a$

The above rules apply to power functions as well as exponential functions.

Rules of logarithms

1. $\ln(x^a) = a\ln(x)$ (log turns power into a product).
2. $\ln(xy) = \ln(x) + \ln(y)$ (log of a product = sum of logs)
3. $\ln(x/y) = \ln(x) - \ln(y)$ (log of a ratio = difference of logs)

The above rules are proved using the rules of exponent.

Exercise 16 *Prove the rules of logarithms using the rules of exponential functions.*

Derivatives of exponential and logarithmic functions

You are already familiar with the following rules from the chapter about derivatives:Logarithmic function:

1. Logarithmic function:
$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

2. Exponential function:

$$\begin{aligned} \frac{d}{dx}a^x &= \ln(a)\,a^x \\ \text{in particular, } \frac{d}{dx}e^x &= e^x \end{aligned}$$

In this section we provide a proof of these rules.

Proof. (i) $\frac{d}{dx}\ln(x) = \frac{1}{x}$.

$$\begin{aligned} \frac{d}{dx}\ln(x) &= \lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h} \\ &= \lim_{h\to 0}\ln\left[\left(\frac{x+h}{x}\right)^{\frac{1}{h}}\right] \\ &= \lim_{h\to 0}\ln\left[\left(1+\frac{h}{x}\right)^{\frac{1}{h}}\right] \\ &= \lim_{h\to 0}\ln\left[\left(1+\frac{1/x}{1/h}\right)^{\frac{1}{h}}\right] \end{aligned}$$

Let $n = 1/h$. The above limit becomes

$$\begin{aligned} \frac{d}{dx}\ln(x) &= \lim_{n\to\infty}\ln\left[\left(1+\frac{1/x}{n}\right)^{n}\right] \\ &= \ln\lim_{n\to\infty}\left[\left(1+\frac{1/x}{n}\right)^{n}\right] \\ &= \ln e^{1/x} = \frac{1}{x} \end{aligned}$$

The second step used the substitution rule of limits.

(ii) $\frac{d}{dx}e^x = e^x$.

Let $y = e^x$ and the inverse function is $x = \ln(y)$. From the the inverse function rule of derivatives,

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{1/y} = y = e^x$$

(iii) $\frac{d}{dx}a^x = \ln(a)\,a^x$.

Let $y = a^x$, and take ln of both sides:

$$\ln y = x\ln(a)$$

Differentiate with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \ln(a)$$

Rearranging:

$$\frac{dy}{dx} = \ln(a)\, y = \ln(a)\, a^x$$

■

3.4.3 Application: growth rates

In economics, we often want to analyze the change in variables over time: growth of GDP, growth of prices (inflation), growth of sales, etc. Exponential and logarithmic functions were designed to analyze anything that grows. In this section, we distinguish between discrete time, $t = ... - 2, -1, 0, 1, 2, ...$ (t can only be integer) and continuous time $t \in \mathbb{R}$ (t can be any real number). A variable that evolves over time in discrete jumps is denoted by y_t and a variable that changes continuously is denoted $y(t)$. The rates of growth are then

$$\begin{aligned}
[t \text{ discrete}] &: \quad \hat{y} = \frac{y_{t+1} - y_t}{y_t} \\
[t \text{ continuous}] &: \quad \hat{y} = \frac{dy(t)/dt}{y(t)} = \frac{d}{dt}\ln y(t)
\end{aligned}$$

If a variable grows at constant rate g, then the value at time t is given by:

$$\begin{aligned}
[t \text{ discrete}] &: \quad y_t = y_0 (1+g)^t \\
[t \text{ continuous}] &: \quad y(t) = y(0)\, e^{gt}
\end{aligned}$$

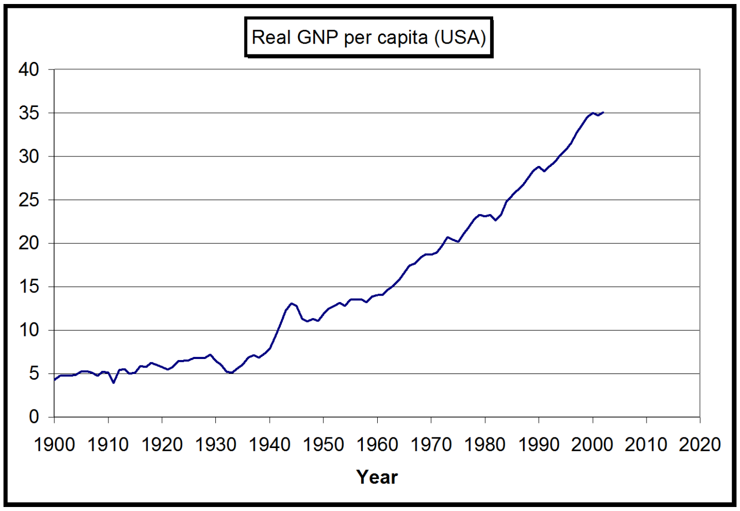
To verify that the above formulas for y_t and $y(t)$ generate a constant growth rate of these variables, we use the corresponding growth rate formulas:

$$\begin{aligned}
[t \text{ discrete}] &: \quad \hat{y} = \frac{y_{t+1} - y_t}{y_t} = \frac{y(0)(1+g)^{t+1} - y(0)(1+g)^t}{y(0)(1+g)^t} = g \\
[t \text{ continuous}] &: \quad \hat{y} = \frac{d}{dt}\ln y(t) = \frac{d}{dt}(\ln y(0) + gt) = g
\end{aligned}$$

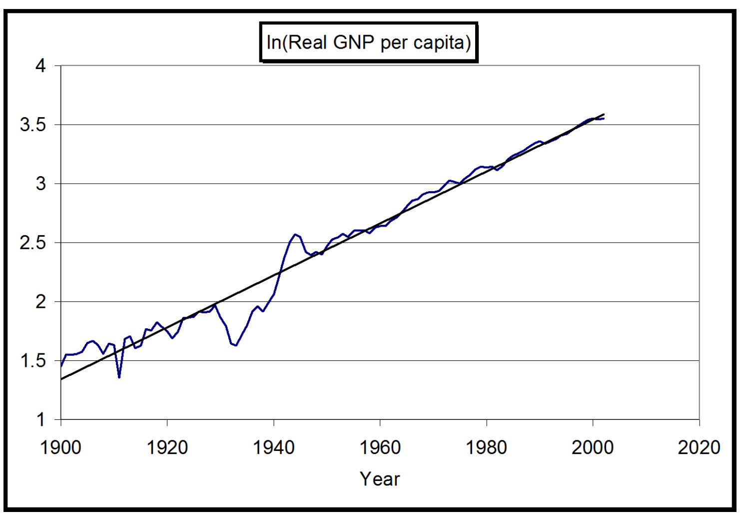
Notice that if a continuous time variable $y(t)$ is growing at constant rate g, then $\ln y(t) = \ln y(0) + gt$ is a linear function of time with slope g. If a discrete time variable y_t is growing at constant rate g, then $\ln y_t = \ln y_0 + t\ln(1+g)$ is a linear function of time with slope $\ln(1+g)$, which is approximately g for small g.[3] An important application of this result is

[3] Recall that we proved $\lim_{g\to 0} \frac{\ln(1+g)}{g} = 1$.

for plots of time series data that exhibits growth. Instead of plotting the actual variable y_t as a function of time, we better plot its natural logarithm as a function of time. This way we can see immediatly if the long run growth trend has constant or changing growth rate, and we can apprioximately find the growth rate as the slope of the curve. For example, the next figure plots the real GNP per capita in the U.S.A. during 1900-2000.



The next figure shows the ln of the same real GNP per capita time series. Notice that from the graph of the ln(RGNP) we can see that during the 20th century the growth trend exhibited constant growth rate (ln(RGNP) is a linear function of time), and the growth rate was about $\frac{3.5-1.5}{100} = 2\%$ per year. None of that can be seen from looking at the original time series data on real GNP.



Example 36 *Calculate the growth rate of $y(t) = 0.01t$.*

First, the notation $y(t)$ means that time is continuous, and we must use the continuous growth formula.

$$\hat{y} = \frac{d}{dt} \ln y(t) = \frac{1}{0.01t} 0.01 = \frac{1}{t}$$

Thus, $y(t)$ is not growing at constant rate. In fact, it is growing at a diminishing rate, so after 10 year the growth rate is 10% but after 20 years the growth rate is 5%.

Example 37 *(Doubling time). Suppose China's GDP per capita is growing at 7%. How long will it take for Chinese GDP per capita to double? We will answer this question under the two assumptions of discrete and continuous growth. If growth is discrete, then we need to solve the following equation for the unknown t:*

$$\begin{aligned}
2y_0 &= y_0 (1+g)^t \\
2 &= (1+g)^t \\
\ln 2 &= t \ln (1+g) \\
t^* &= \frac{\ln 2}{\ln (1+g)} = \frac{0.69315}{0.067659} = 10.245
\end{aligned}$$

If growth is continuous, then we need to solve the following equation for the unknown t:

$$\begin{aligned}
2y(0) &= y(0) e^{gt} \\
2 &= e^{gt} \\
\ln 2 &= gt \\
t^{**} &= \frac{\ln 2}{g} = \frac{0.69315}{0.07} = 9.9021
\end{aligned}$$

Notice that for small growth rate g, the two doubling times are very similar. Even for $g = 7\%$ we got pretty similar t^* and t^{**}. This is because we proved that

$$\lim_{g \to 0} \frac{\ln (1+g)}{g} = 1$$

The last example gives rise to the famous **rule of 70** for calculating approximate doubling time, which works for small g. Using the continuous growth doubling time

$$t = \frac{\ln 2}{g} = \frac{0.69315}{g} \approx \frac{0.7}{g} = \frac{70}{100 \cdot g}$$

Thus, the approximate doubling time is given by 70 divided by the percentage growth rate. In the above example of China, the approximate doubling time with the rule of 70 is:

$$[\text{Rule of } 70] : t^{***} = \frac{70}{7} = 10$$

Notice that even with fast growth rate of 7%, the rule of 70 gives a good approximation to the exact doubling times (compare with t^* and t^{**}).

Growth rate of product and ratio

In economics, many variables of interest are products or ratios of other variables. For example, revenue $R = P \cdot Q$ is a product of price and quantity and GDP per capita $y = \frac{GDP}{POP}$ is a ratio of GDP to population. If we know the growth rates of the underlying variables, what is the growth rate of their products and rations? We first answer this question in continuous time, i.e. the variables $x(t)$ and $y(t)$ are continuous (and differentiable) functions of time $t \in \mathbb{R}$.

$$\begin{aligned}
\widehat{xy} &= \frac{d}{dt} \ln (x(t)\, y(t)) \\
&= \frac{d}{dt} [\ln x(t) + \ln y(t)] \\
&= \frac{d}{dt} \ln x(t) + \frac{d}{dt} \ln y(t) \\
&= \hat{x} + \hat{y}
\end{aligned}$$

Thus, the growth rate of a product is the sum of growth rates. Similarly, the growth of a ratio is the difference of growth rates:

$$\begin{aligned}
\widehat{\left(\frac{x}{y}\right)} &= \frac{d}{dt} \ln (x(t)/y(t)) \\
&= \frac{d}{dt} [\ln x(t) - \ln y(t)] \\
&= \frac{d}{dt} \ln x(t) - \frac{d}{dt} \ln y(t) \\
&= \hat{x} - \hat{y}
\end{aligned}$$

For example, if the price of a good grows at 2% rate and the quantity sold grows at 3%, then the revenue grows at 5%, when growth is continuous. Similarly, if GDP grows at 4% and population grows at 1%, the GDP per capita grows at 3% rate, when growth is continuous.

In discrete time, we get similar results, which hold only approximately, for small growth rates. Thus, when time is discrete $t = ... - 2, -1, 0, 1, 2, ...$, we have

$$\begin{aligned}
\widehat{xy} &= \frac{x_{t+1}y_{t+1} - x_t y_t}{x_t y_t} \\
1 + \widehat{xy} &= (1 + \hat{x})(1 + \hat{y}) \\
\underbrace{\ln\left(1 + \widehat{xy}\right)}_{\approx \widehat{xy}} &= \underbrace{\ln(1 + \hat{x})}_{\approx \hat{x}} + \underbrace{\ln(1 + \hat{y})}_{\approx \hat{y}} \\
\widehat{xy} &\approx \hat{x} + \hat{y}
\end{aligned}$$

If all growth rates are small, then we can use the result $\lim_{g \to 0} \ln(1 + g)/g = 1$ in the last step.

Similarly, the growth of a ratio, when time is discrete, is approximately the difference of growth rates.

$$\begin{aligned}
\widehat{x/y} &= \frac{\left(\frac{x_{t+1}}{y_{t+1}}\right) - \left(\frac{x_t}{y_t}\right)}{\left(\frac{x_t}{y_t}\right)} \\
1 + \widehat{x/y} &= \frac{(1 + \hat{x})}{(1 + \hat{y})} \\
\underbrace{\ln\left(1 + \widehat{x/y}\right)}_{\approx \widehat{x/y}} &= \underbrace{\ln(1 + \hat{x})}_{\approx \hat{x}} - \underbrace{\ln(1 + \hat{y})}_{\approx \hat{y}} \\
\widehat{x/y} &\approx \hat{x} - \hat{y}
\end{aligned}$$

Again, the last step uses the result $\lim_{g \to 0} \ln(1 + g)/g = 1$, which is justified if all growth rates are small.

To summarize, if time is continuous, the growth rate of a product is the sum of growth rates, and the growth rate of a ratio is the difference of growth rates. If time is discrete, these results hold approximately, for small growth rates.

Growth rate of sum and difference

Skipped.

3.4.4 Application: present value

Suppose that you deposit an amount of PV in savings account. Then, after t periods with continuous compounding, you will have a future value of

$$FV(t) = PV \cdot e^{rt}$$

The present value of a payment $FV(t)$ received at time t is then

$$PV = \frac{FV(t)}{e^{rt}} = FV(t)\, e^{-rt}$$

Similarly, with compounting at discrete intervals, the future (time t) value of current amount of PV is

$$FV_t = PV(1+r)^t$$

The present value of payment FV_t received at time $t$is

$$PV = \frac{FV_t}{(1+r)^t}$$

Example 38 *(Annuities). An annuity is a sequence of equal payments at regular intervals. An example of annuity is a mortgage or loan, that pays to its holder (bank, financial institution) an amount of PMT at the end of every year, for the next T years. Thus, the present value of this annuity (mortgage) is:*

$$\begin{aligned} PV &= \frac{PMT}{(1+r)^1} + \frac{PMT}{(1+r)^2} + ... + \frac{PMT}{(1+r)^T} \\ &= \sum_{t=1}^{T} \frac{PMT}{(1+r)^t} = PMT \sum_{t=1}^{T} \left(\frac{1}{1+r}\right)^t = PMT \cdot R(r,T) \end{aligned}$$

Using the summation formula

$$\sum_{t=1}^{T} q^t = \frac{q - q^{T+1}}{1-q}$$

with $q = 1/(1+r)$ we have for $r \neq 0$

$$\sum_{t=1}^{T} q^t = \frac{q - q^{T+1}}{1-q} = \frac{\left(\frac{1}{1+r}\right) - \left(\frac{1}{1+r}\right)^{T+1}}{1 - \left(\frac{1}{1+r}\right)} = \frac{1 - \left(\frac{1}{1+r}\right)^T}{r}$$

If $r = 0$, the above is not defined, and we have

$$\sum_{t=1}^{T} \left(\frac{1}{1+r}\right)^t = \sum_{t=1}^{T} 1 = T$$

Thus,

$$R(r,T) = \sum_{t=1}^{T} \left(\frac{1}{1+r}\right)^t = \begin{cases} \left[1 - \left(\frac{1}{1+r}\right)^T\right]/r & \text{if } r \neq 0 \\ T & \text{if } r = 0 \end{cases}$$

Thus,

$$PV = PMT \cdot R(r,T)$$

If the amount of loan is given, and we wish to calculate the constant payment, then

$$PMT = \frac{PV}{R(r,T)}$$

3.4.5 Application: optimal holding time

Suppose that the market value of an asset, at time t, is given by $V(t)$ - a continuous and differentiable function of time. Even if this value always increases over time, one needs to take into account the interest that we give up by not selling the asset and investing the money in the bank. Suppose that interest rate is r per period. Then the present value of the asset, when it is sold at time t, is

$$PV(t) = V(t)\, e^{-rt}$$

The owner of the asset wishes to maximize the present value of the asset, by selling it at the right time. Thus, the optimization problem to be solved is:

$$\max_t PV(t) = V(t)\, e^{-rt}$$

The first order necessary condition for maximum is:

$$\begin{aligned} \frac{d}{dt} PV(t) &= V'(t)\, e^{-rt} - rV(t)\, e^{-rt} = 0 \\ \text{or } \frac{V'(t)}{V(t)} &= r \end{aligned}$$

The interpretation is that you keep the asset until the growth rate of it's value equalizes to the interest rate.

The second order sufficient condition for maximum is:

$$\frac{d^2 PV(t)}{dt^2} = \frac{d}{dt}\left[V'(t)\, e^{-rt} - rV(t)\, e^{-rt}\right] < 0$$

Recognizing that $V(t)\, e^{-rt} = PV(t)$, we compute the 2nd order sufficient condition

$$\begin{aligned} \frac{d}{dt}\left[V'(t)\, e^{-rt} - rPV(t)\right] &= V''(t)\, e^{-rt} - rV'(t)\, e^{-rt} - r\underbrace{\frac{d}{dt} PV(t)}_{=0} \\ &= V''(t)\, e^{-rt} - rV'(t)\, e^{-rt} \end{aligned}$$

The last term is zero because the second order sufficient condition is checked at the point where the first order necessary condition holds. Without knowing the function $V(t)$, we cannot tell if the second derivative is negative or not. But we can say that in order of the second order sufficient condition for maximum to hold, we need

$$\begin{aligned} V''(t)e^{-rt} - rV'(t)e^{-rt} &< 0 \\ \frac{V''(t)}{V'(t)} &< r \end{aligned}$$

The left hand side is the growth of $V'(t)$, and the condition says that this growth must be small enough. Otherwise, we would never want to sell the asset and invest in interest bearing financial asset.

Example 39 *Suppose that you have an asset with initial value of \$$K$, and over time the value evolves according to*

$$V(t) = Ke^{\sqrt{t}}$$

The interest rate is r, and is constant over time. The present value of the asset, if it is sold at time t, is:

$$PV(t) = Ke^{\sqrt{t}}e^{-rt} = Ke^{\sqrt{t}-rt}$$

Thus, the optimization problem to be solved is:

$$\max_t PV(t) = Ke^{\sqrt{t}-rt}$$

The first order necessary condition is:

$$\begin{aligned} \frac{d}{dt}PV(t) &= Ke^{\sqrt{t}-rt}\left(0.5t^{-0.5} - r\right) = 0 \\ 0.5t^{-0.5} &= r \\ 0.5 &= rt^{0.5} \\ t^* &= \left(\frac{0.5}{r}\right)^2 = \frac{0.25}{r^2} \end{aligned}$$

Notice that optimal selling time decreases in interest rate. If interest rate is higher, then the opportunity cost of holding the asset is also higher, and therefore we would sell the asset earlier. Also notice that the initial value of the asset $V(0) = K$ does not affect the optimal holding time; what maters is the growth rate of the value of the asset, compared to the interest rate (growth rate of the alternative financial asset).

The left hand side of the first order condition, $0.5t^{-0.5}$*, based on our general discussion,* *is supposed to represent the growth rate of the value of the asset. Let's verify this:*

$$\frac{d}{dt}\ln V(t) = \frac{d}{dt}\ln\left(Ke^{\sqrt{t}}\right) = \frac{d}{dt}\left[\ln K + t^{0.5}\right] = 0.5t^{-0.5}$$

As always, we need to check the second order sufficient condition for maximum:

$$\frac{d^2}{dt^2}PV(t) < 0$$

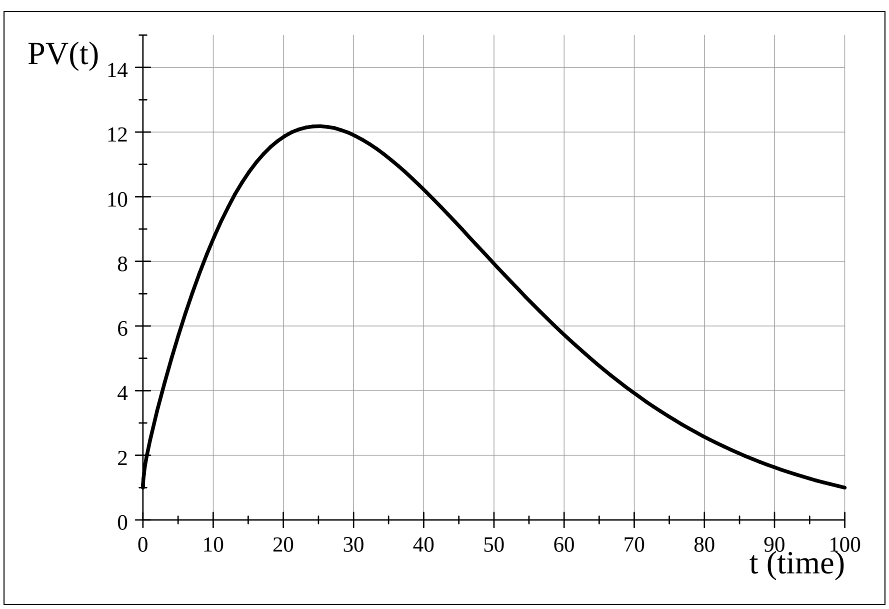
Recognizing that $PV(t) = Ke^{\sqrt{t}-rt}$*, we compute*

$$\frac{d}{dt}PV(t)\left(0.5t^{-0.5} - r\right) = \underbrace{PV'(t)}_{=0}\left(0.5t^{-0.5} - r\right) - PV(t)\,0.25t^{-1.5} < 0$$

Notice that in the above we used the first order necessary condition $PV'(t) = 0$*, because the second order sufficient condition applies at the critical value* $t = t^*$*. Thus, we say that the second order condition for maximum holds, and therefore,* $t = t^* = 0.25/r^2$ *indeed is an optimal time to sell the asset. For example, if the interest rate is* 10%*, we will hold the asset for*

$$t^* = \frac{0.25}{0.1^2} = 25 \text{ years}$$

The objective function in the last example, with $r = 10\%$ *is displayed in the next figure.*



Present Value of Asset

3.5 Unconstrained Optimization of Multivariate Functions

In section (3.2) we derived first order necessary and second order sufficient conditions for *local* extrema of differentiable functions of one variable, summarized in the next table:

Condition	Maximum	Minimum
First-order necessary	$f'(x) = 0$	$f'(x) = 0$
Second-order sufficient†	$f''(x) < 0$	$f''(x) > 0$
†Applicable only after the first-order necessary condition is satisfied		

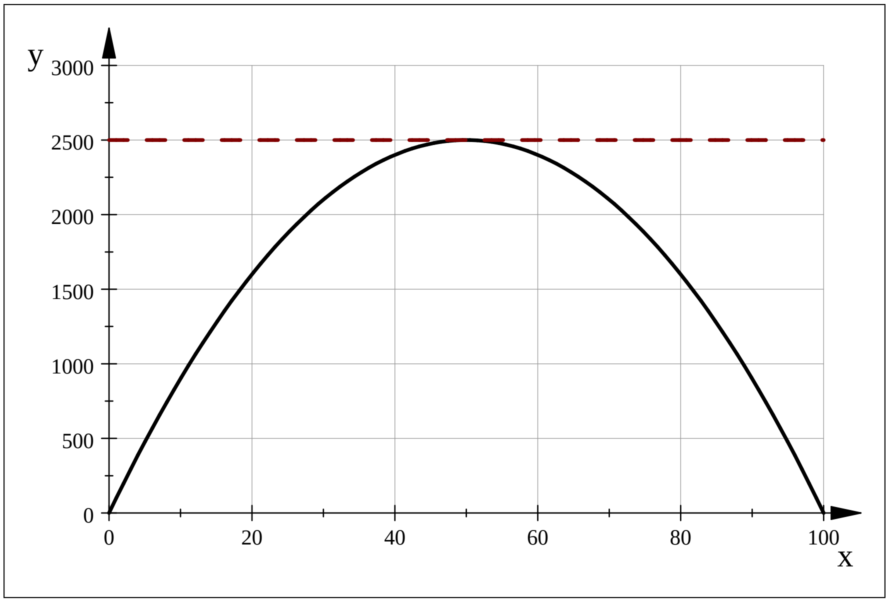
We applied these tests and solved profit maximization problems of firms that produce only one output, or for finding the optimal timing to sell an asset. Now we extend the analysis to functions of several variables, and derive necessary and sufficient conditions for problems of the following type:

$$\max_{x_1,...,x_n} f(x_1,...,x_n) \quad \text{or} \quad \min_{x_1,...,x_n} f(x_1,...,x_n)$$

Examples include profit maximization of multiproduct and many inputs firm.

3.5.1 First order necessary conditions for extrema

To derive first order necessary condition for multivariate function, it is convenient to use differential, instead of derivatives. In one variable case, the first order necessary condition of extremum of $y = f(x)$ is $f'(x) = 0$. Recall that an x that satisfies this condition is called *critical value*. An equivalent necessary condition for extremum can be presented with differential, namely $dy = f'(x)\,dx = 0$ for any increment $dx \neq 0$. Intuitively, at an extremum (minimum or maximum), the tangent line is flat, so any change in x in either direction ($dx > 0$ or $dx < 0$) should not change the value of y ($dy = 0$) along the tangent line. Obviously this condition is satisfied for all $dx \neq 0$ if an only if $f'(x) = 0$.



Now consider a function of two variables, $y = f(x_1, x_2)$. The *total differential* with increments dx_1,dx_2 is

$$dy = f_1(x_1, x_2)\,dx_1 + f_2(x_1, x_2)\,dx_2$$

where $f_i(x_1, x_2)$ is the partial derivative with respect to x_i, $i = 1, 2$. The total differential approximates the change in y with a tangent plane, as illustrated in figure (3.7). At a minimum or maximum, we expect that changes in any direction, along the tangent plane, not to change the value of the function, i.e. $dy = 0$, $\forall dx_1, dx_2$ not both zero. Figure (3.7) shows a function $y = f(x_1, x_2) = 100 - x_1^2 - x_2^2$, which has a maximum point at $x_1 = x_2 = 0$. Notice that any movement in any direction along the tangent plane ($dx_1 > 0$, $dx_1 < 0$, $dx_2 > 0$ or $dx_2 < 0$), must result in $dy = 0$, if the these changes occur at an extremum point. This is precisely the meaning of total differential equal to zero - the approximating plane tangent to the dome in figure (3.7) at a maximum, must be flat. In order for $dy = f_1(x_1, x_2)\,dx_1 + f_2(x_1, x_2)\,dx_2 = 0$ to hold for any dx_1, dx_2, not both zero, both partial derivatives must be zero: $f_1(x_1, x_2) = f_2(x_1, x_2) = 0$. This result extends to functions of n variables, as stated in the next theorem.

Theorem 11 *(First-order necessary condition for extrema of multivariate function). If a function $y = f(x_1, ..., x_n)$ is differentiable (i.e. has all partial derivatives), and if f has a local minimum or maximum at $x_1, x_2, ..., x_3$, then it is* ***necessary*** *that*

$$f_1(x_1, ..., x_n) = f_2(x_1, ..., x_n) = ... = f_n(x_1, ..., x_n) = 0$$

That is, the first order necessary condition for maximum or minimum of a multivariate function, is that all partial derivatives must be zero at that point.

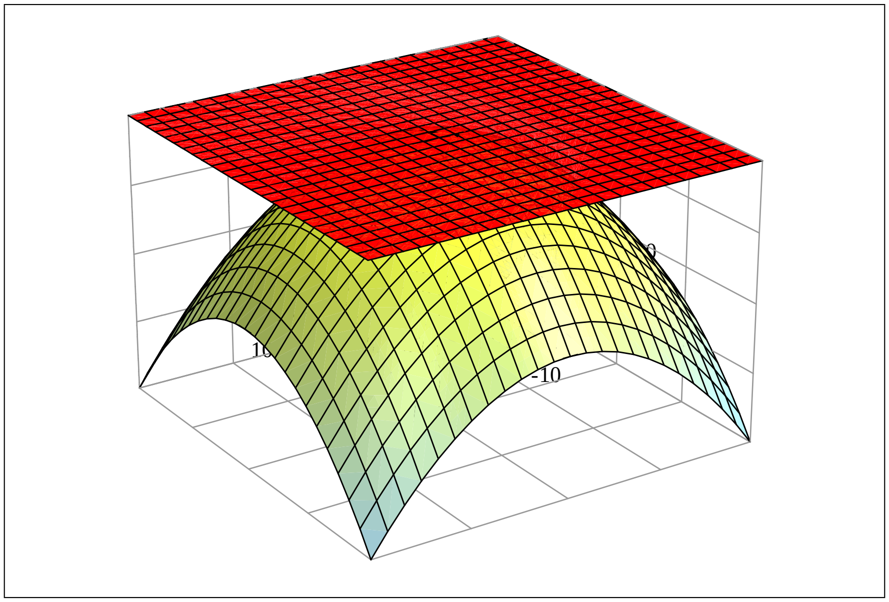


Figure 3.7: Maximum of $y = 100 - x_1^2 - x_2^2$

3.5.2 Second order sufficient conditions for local extrema

Recall that for a function of one variable, $y = f(x)$, the second order sufficient condition for a maximum, at critical value x, is $f''(x) < 0$ and for minimum $f''(x) > 0$. These second order conditions can be equivalently stated in terms of *second-order* differential, which is the differential of the first differential:

$$d^2y = d(dy) = d(f'(x)\,dx) = f''(x)\,dxdx = f''(x)\,dx^2$$

Note that in the above differentiation, the increment dx is treated as some constant change in x (can be negative or positive, say $dx = -0.01$). The second order sufficient condition for maximum can now be stated as $d^2y < 0$, which holds for any $dx \neq 0$ if and only if $f''(x) < 0$. This condition requires that the first derivative (slope of the function) be decreasing locally, if the objective function has a maximum at that point. Similarly, $f''(x) > 0$ means that the first derivative is increasing if the objective function has a minimum at that point.

Now consider a function of two variables, $y = f(x_1, x_2)$ and its first-order total differential $dy = f_1(x_1, x_2)\,dx_1 + f_2(x_1, x_2)\,dx_2$. The second order sufficient condition for maximum, just like in the univariate function case, is $d^2y < 0$ and for minimum $d^2y > 0$. Both conditions of course are checked at critical values (after the first order necessary conditions are satisfied).

The *second-order total differential* is

$$\begin{aligned} d^2y &= d\left[f_1\left(x_1,x_2\right)dx_1\right]+d\left[f_2\left(x_1,x_2\right)dx_2\right] \\ &= f_{11}dx_1dx_1+f_{12}dx_2dx_1+f_{21}dx_1dx_2+f_{22}dx_2dx_2 \\ &= f_{11}dx_1^2+2f_{12}dx_1dx_2+f_{22}dx_2^2 \end{aligned}$$

The increments dx_1 and dx_2 are some exogenous constants, say $dx_1=-0.01$ and $dx_2=0.007$. The derivatives $f_{12}=\frac{\partial^2 f}{\partial x_1 \partial x_2}$ and $f_{21}=\frac{\partial^2 f}{\partial x_2 \partial x_1}$ are called **cross partial** derivatives, and they measure the change in the first derivative of one variable with respect to change in the other variable. Following Young's theorem, the two partial derivative are equal to each other, $f_{12}=f_{21}$, under some conditions.[4] The second order sufficient condition for maximum is that $d^2y<0$ *for all* dx_1 and dx_2, not both equal to zero. Similarly, the second order sufficient condition for minimum is $d^2y>0$ *for all* dx_1 and dx_2, not both equal to zero. What restrictions on the second order derivatives are implied by these conditions? For maximum, we have:

$$d^2y=f_{11}dx_1^2+2f_{12}dx_1dx_2+f_{22}dx_2^2<0$$

For this to hold for all dx_1 and dx_2, not both equal to zero, we must have $f_{11}<0$ and $f_{22}<0$. To see why, notice that the above inequality must hold in particular for $dx_1=0$ and $dx_2\neq 0$, as well as for $dx_1\neq 0$ and $dx_2=0$. Next, we would like to see what restrictions on the cross partial derivative, f_{12}, would guarantee that $d^2y<0$ *for all* dx_1 and dx_2, not both equal to zero. We can add and subtract $f_{12}^2dx_2^2/f_{11}$ to second-order differential, and rearrange as follows:

$$\begin{aligned} d^2y &= f_{11}dx_1^2+2f_{12}dx_1dx_2+\frac{f_{12}^2dx_2^2}{f_{11}}+f_{22}dx_2^2-\frac{f_{12}^2dx_2^2}{f_{11}} \\ &= f_{11}\left(dx_1^2+\frac{2f_{12}}{f_{11}}dx_1dx_2+\frac{f_{12}^2}{f_{11}^2}dx_2^2\right)+\left(f_{22}-\frac{f_{12}^2}{f_{11}}\right)dx_2^2 \\ &= f_{11}\left(dx_1+\frac{f_{12}}{f_{11}}dx_2\right)^2+\left(\frac{f_{11}f_{22}-f_{12}^2}{f_{11}}\right)dx_2^2<0 \end{aligned}$$

In order for this inequality to hold for all dx_1 and dx_2, both coefficients on the squared terms must be negative, i.e. $f_{11}<0$ and

$$\left(\frac{f_{11}f_{22}-f_{12}^2}{f_{11}}\right)<0$$

[4] The theorem states that if a function $f:\mathbb{R}\to\mathbb{R}$ has continuous partial derivatives, then $f_{ij}\left(x_1,...,x_n\right)=f_{ji}\left(x_1,...,x_n\right)$ for all i,j.

The last inequality holds if and only if $f_{11}f_{22} - f_{12}^2 > 0$ (since $f_{11} < 0$). Thus, the second order sufficient condition for maximum is

$$f_{11} < 0 \text{ and } f_{11}f_{22} - f_{12}^2 > 0$$

Notice that the above implies that $f_{22} < 0$, since otherwise it would never be possible to satisfy $f_{11}f_{22} - f_{12}^2 > 0$.

Similarly, the second order sufficient condition for minimum is

$$d^2y = f_{11}\left(dx_1 + \frac{f_{12}}{f_{11}}dx_2\right)^2 + \left(\frac{f_{11}f_{22} - f_{12}^2}{f_{11}}\right)dx_2^2 > 0$$

which must hold for all dx_1 and dx_2, not both equal to zero. This requirement is met if and only if both coefficients on the quadratic terms are strictly positive, $f_{11} > 0$ and

$$\left(\frac{f_{11}f_{22} - f_{12}^2}{f_{11}}\right) > 0$$

The last inequality holds if and only if $f_{11}f_{22} - f_{12}^2 > 0$ (since $f_{11} > 0$). Therefore, the second order sufficient condition for minimum is

$$f_{11} > 0 \text{ and } f_{11}f_{22} - f_{12}^2 > 0$$

Thus, we derived the second order sufficient conditions for maximum and minimum of a two-variable function, and it took quite a few steps. The derivation of second order sufficient conditions for functions of 3 variable is much more messy, and things become intractable if we try to derive them for a function of n variables in this way. An easier way to derive the second order conditions for a function of n variables, requires the concept of **quadratic form**. Before we proceed to a formal discussion, notice that the second differential of two-variable function, $y = f(x_1, x_2)$, can be written in matrix form as follows:

$$\begin{aligned} d^2y &= f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2 \\ &= \begin{bmatrix} dx_1 & dx_2 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \end{aligned}$$

Viewed as a function of the increments dx_1 and dx_2, the above is called a quadratic form (because the sum of exponents on the variables is 2 in every term). Our second order sufficient conditions require the above form to be always negative (for maximum) or always positive (for minimum), for any dx_1 and dx_2, not both equal to zero. The matrix of second derivatives in the middle of the above quadratic form is called the **Hessian**. It turns out that there are theorems that provide conditions under which a quadratic form is always positive (positive definite) or always negative (negative definite).

Quadratic forms and definite matrices

Definition 19 *A quadratic form on $\mathbb{R}^n$ is a real valued function*

$$Q(x) = x'Ax$$

where x is a nonzero $n \times 1$ vector and A is $n \times n$ symmetric matrix:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

Recall that symmetric matrix means that $a_{ij} = a_{ji}\ \forall i, j$, or $A' = A$. Also note that $Q(x)$ is a scalar. Our interest in quadratic forms stems from the fact that second order differential, which determines the second order sufficient conditions for maximum and minimum, is a quadratic form. We want to know under what conditions this form is always (definitely) positive or always (definitely) negative. What determines the sign of a quadratic form $x'Ax$ is the matrix A, since x is any nonzero vector. Therefore, we can classify the matrix A as follows:

Definition 20 *(Definite matrices). Let A be $n \times n$ symmetric matrix. Then A is:*

(a) ***positive definite*** *if $x'Ax > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$.*

(b) ***negative definite*** *if $x'Ax < 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$.*

(c) ***positive semidefinite*** *if $x'Ax \geq 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$.*

(d) ***negative semidefinite*** *if $x'Ax \leq 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$.*

(e) ***indefinite*** *if $x'Ax < 0$ for some $x \in \mathbb{R}^n$, and $x'Ax > 0$ for some other $x \in \mathbb{R}^n$.*

Now consider the function of n variables, $y = f(x_1, ..., x_n)$. As we have seen with a function of two variables, the second-order differential of a function of n variables can be written in the following quadratic form:

$$d^2y = \begin{bmatrix} dx_1 & dx_2 & \cdots & dx_n \end{bmatrix} \underbrace{\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}}_{H} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}$$

The matrix of second derivatives in the middle is the Hessian matrix H, which is symmetric due to Young's theorem ($f_{ij} = f_{ji}$ when these cross partial derivatives are continuous functions). Thus, the second order sufficient condition for maximum is equivalent to the Hessian matrix H being *negative definite* and the second order sufficient condition for minimum is equivalent to the Hessian matrix H being *positive definite*. What are the conditions on symmetric matrices, that guarantee them to be definite? These conditions are stated in terms of **leading principal minors** of the matrix. Recall that the ij minor of a square matrix A, denoted $|M_{ij}|$, is the determinant of a smaller square matrix, obtained by removing row i and column j from A. A submatrix obtained from $n \times n$ matrix A by keeping the first k rows and the first k columns, is called the **kth order leading principal matrix**. Its determinant is called the kth order leading principal minor of A. For example, consider a 3×3 matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The 3 leading principle minors are

$$\begin{aligned}
1^{\text{st}} \text{ leading principle minor} &: \quad |A_1| = |a_{11}| \\
2^{\text{nd}} \text{ leading principal minor} &: \quad |A_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\
3^{\text{rd}} \text{ leading principal minor} &: \quad |A_3| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Theorem 12 *(Characterization of definite matrices by leading principal minors). Let A be $n \times n$ symmetric matrix. Then,*

(a) A is ***positive definite*** *if and only if all its n leading principal minors are positive:*

$$|A_1| > 0, \ |A_2| > 0, \ |A_3| > 0, \ldots$$

(b) A is ***negative definite*** *if and only if its n leading principal minors alternate in sign as follows:*

$$|A_1| < 0,\ \ |A_2| > 0,\ \ |A_3| < 0, ...$$

(c) A is ***positive semidefinite*** *if and only if all its n leading principal* ≥ 0.

(d) A is ***negative semidefinite*** *if and only if its n leading principal minors alternate in sign as follows:*

$$|A_1| \leq 0,\ \ |A_2| \geq 0,\ \ |A_3| \leq 0, ...$$

(e) A is ***indefinite*** *if the signs of its leading principal minors do not fit any of the above patterns.*

The above theorem, applied to the Hessian matrix, gives us the second order sufficient conditions for local minimum or maximum, as summarized in the next theorem.

Theorem 13 *(Second order sufficient conditions for* **local** *extrema of n-variable functions). Let* $f : U \to \mathbb{R}$ *be* C^2 *(twice continuously differential) function, whose domain is an open set* $U \subseteq \mathbb{R}^n$*, and suppose that* $x^* = \begin{bmatrix} x_1^* & x_2^* & \cdots & x_n^* \end{bmatrix}'$ *is a critical point, i.e.*

$$f_i\left(x_1^*, ..., x_n^*\right) = 0 \quad \forall i = 1, 2, ..., n$$

Let the Hessian matrix (i.e. matrix of all second derivatives), evaluated at x^**, be*

$$H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

(a) If H is ***negative definite****, i.e. if the leading principal minors of H alternate sign as follows:*

$$|H_1| = |f_{11}| < 0,\ \ |H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0,\ \ |H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} < 0, ...$$

then $f(x^*)$ *is a* ***local maximum****. Put differently, H negative definite is a second order sufficient condition for local maximum.*

*(b) If H is **positive definite**, i.e. if the leading principal minors of H are all positive:*

$$|H_1| = |f_{11}| > 0,\ |H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0,\ |H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} > 0, \ldots$$

*then $f(x^*)$ is a **local minimum**. Put differently, H positive definite is a second order sufficient condition for local minimum.*

*(c) If H is **indefinite**, i.e. some leading principle minor (or some pair of them) **is not zero** and violates **both** sign patterns in (a) and (b), then x^* is a **saddle point**.*

*(d) If the leading principal minors of H violate the patterns of (a) or (b) **only by being zero**, then the second order test is inconclusive, and x^* could be anything (maximum, minimum saddle or something else).*

In this course, we are primarily interested in maxima and minima, so we will check for patterns in (a) and (b), i.e. check for sufficient conditions for minima or maxima. If these conditions are violated, either as in part (c) of the theorem, or as in part (d) of the theorem, it is enough for us to say that the second order sufficient conditions do not hold. It is not important for us to classify the critical point into saddle or something else. What we do need to remember about second order *sufficient conditions* is that they are not necessary, and if they are violated, the point can still, in principle, be minimum or maximum or neither, depending on the type of the violation.

Equipped with the above theorem, we can solve some examples.

Example 40 *Let $y = f(x_1, x_2) = 100 - x_1^2 - x_2^2$. Find the critical values of this function, and classify them as local max, local min or saddle point.*

The critical values:

$$\begin{aligned} f_1(x_1, x_2) &= -2x_1 = 0 \\ f_2(x_1, x_2) &= -2x_2 = 0 \end{aligned}$$

The solution is $(x_1^*, x_2^*) = (0, 0)$.

The Hessian matrix is

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

The leading principal minors are

$$\begin{aligned}|H_1| &= |f_{11}| = -2 < 0 \\ |H_2| &= \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = (-2)(-2) - 0 = 4 > 0\end{aligned}$$

Thus, the leading principle minors alternate sign in a way that makes the Hessian matrix negative definite, which is the second order sufficient condition for maximum. Thus, by theorem (13), the function has a local maximum at $(x_1^*, x_2^*) = (0, 0)$. *The graph of the function in this example is plotted in figure (3.7).*

Example 41 *Let* $y = f(x_1, x_2) = x_1^2 + x_2^2$. *Find the critical values of this function, and classify them as local max, local min or saddle point.*

The critical values:

$$\begin{aligned}f_1(x_1, x_2) &= 2x_1 = 0 \\ f_2(x_1, x_2) &= 2x_2 = 0\end{aligned}$$

The solution is $(x_1^*, x_2^*) = (0, 0)$.

The Hessian matrix is

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The leading principal minors are

$$\begin{aligned}|H_1| &= |f_{11}| = 2 > 0 \\ |H_2| &= \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = 2 \cdot 2 - 0 = 4 > 0\end{aligned}$$

All leading principle minors are positive, and therefore the Hessian matrix is positive definite, which is the second order sufficient condition for minimum. Thus, by theorem (13), the function has a local minimum at $(x_1^*, x_2^*) = (0, 0)$. *The graph of the function in this example is plotted in figure (3.8).*

Example 42 *Let* $y = f(x_1, x_2) = x_1^2 - x_2^2$. *Find the critical values of this function, and classify them as local max, local min or saddle point.*

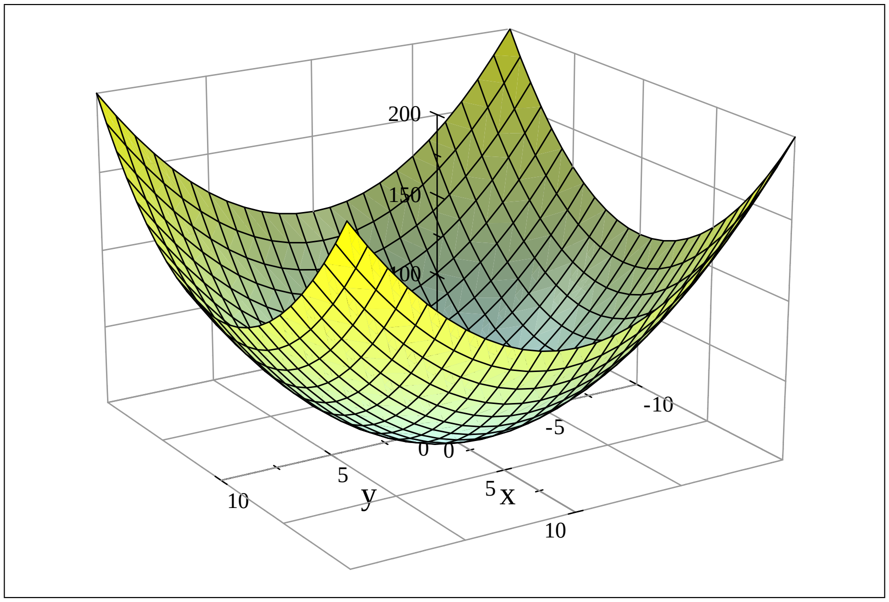


Figure 3.8: $y = x_1^2 + x_2^2$

The critical values:

$$
\begin{aligned}
f_1(x_1, x_2) &= 2x_1 = 0 \\
f_2(x_1, x_2) &= -2x_2 = 0
\end{aligned}
$$

The solution is $(x_1^*, x_2^*) = (0, 0)$*.*

The Hessian matrix is

$$
H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}
$$

The leading principal minors are

$$
\begin{aligned}
|H_1| &= |f_{11}| = 2 > 0 \\
|H_2| &= \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = 2 \cdot (-2) - 0 = -4 < 0
\end{aligned}
$$

The leading principle minors violate the pattern for either negative definiteness or positive definiteness, and therefore the Hessian matrix is indefinite. Thus, by theorem 13, the function has a saddle point at $(x_1^*, x_2^*) = (0, 0)$*, because of* $|H_2|$ *(it is not zero, and violates both patterns (a) and (b) of the theorem). The graph of the function in this example is plotted in figure (3.9).*

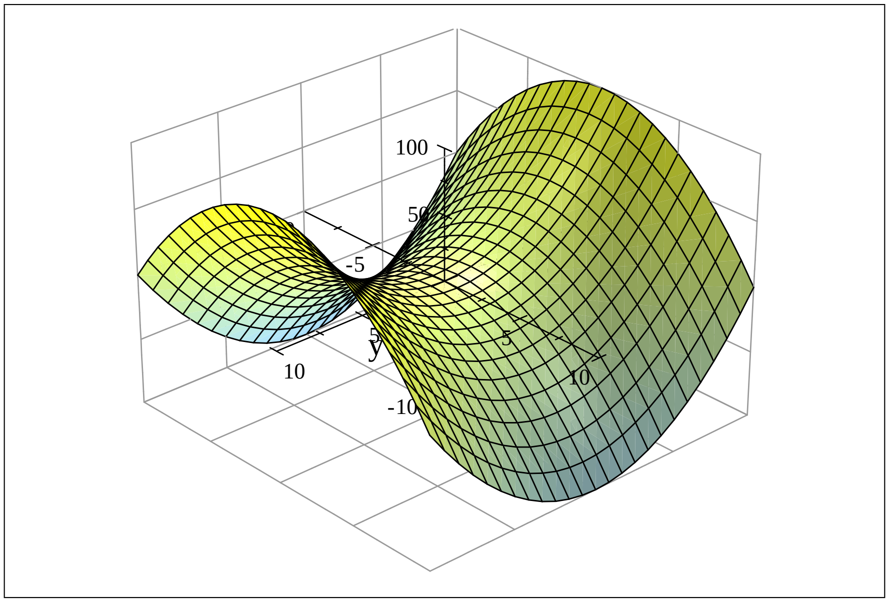


Figure 3.9: $y = x_1^2 - x_2^2$

Example 43 *Let* $y = f(x_1, x_2, x_3) = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 2$. *Find the critical values of this function, and classify them as local max, local min or saddle point.*

The critical values:

$$\begin{aligned} f_1 &= 4x_1 + x_2 + x_3 = 0 \\ f_2 &= x_1 + 8x_2 = 0 \\ f_3 &= x_1 + 2x_3 = 0 \end{aligned}$$

The solution is $(x_1^*, x_2^*, x_3^*) = (0, 0, 0)$.

The Hessian matrix is

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

The leading principal minors are:

$$
\begin{aligned}
|H_1| &= |f_{11}| = 4 > 0 \\
|H_2| &= \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 1 & 8 \end{vmatrix} = 31 > 0 \\
|H_3| &= \begin{vmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 54 > 0
\end{aligned}
$$

Thus, by theorem 13 we conclude that the function has a local minimum at the critical point $(x_1^*, x_2^*, x_3^*) = (0, 0, 0)$. *The function in this example cannot be graphed, because it has 4 dimensions.*

3.5.3 Second order sufficient conditions for global extrema

In economic applications, we usually hope to have a unique global (absolute) maximum or minimum. This is because the solution of the model is its prediction, and we want there to be a particular prediction, instead of many or none. Notice that the local maximum we found for $y = f(x_1, x_2) = 100 - x_1^2 - x_2^2$, and plotted in figure (3.7), is not only local maximum, but also **global maximum**. The Hessian matrix for this function is

$$
H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}
$$

The leading principal minors are

$$
\begin{aligned}
|H_1| &= |f_{11}| = -2 < 0 \\
|H_2| &= \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = (-2)(-2) - 0 = 4 > 0
\end{aligned}
$$

Notice that the Hessian matrix and the sign pattern of the leading principle minors, do not depend on particular point at which they are computed. In other words, the leading principle minors *always* alternate sign as follows: $|H_1| < 0$, $|H_2| > 0$, for all (x_1, x_2) in the domain of the function. It turns out that this is a sufficient condition for global maximum. Similarly, the sufficient condition for global minimum is that all leading principal minors should be positive. The following theorem summarizes this discussion.

Theorem 14 *(Second order sufficient conditions for* ***global*** *extrema of n-variable func-*

tions). Let $f : U \rightarrow \mathbb{R}$ be C^2 (twice continuously differential) function, whose domain is an open set $U \subseteq \mathbb{R}^n$, and suppose that $x^* = \begin{bmatrix} x_1^* & x_2^* & \cdots & x_n^* \end{bmatrix}'$ is a critical point, i.e.

$$f_i(x_1^*, ..., x_n^*) = 0 \quad \forall i = 1, 2, ..., n$$

Let the Hessian matrix (i.e. matrix of all second derivatives) be

$$H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

*(a) If the Hessian matrix is **negative semidefinite** for all $x \in U$, then f is **concave,** and $f(x^*)$ is a **global maximum** (but not necessarily unique).*

*(b) If the Hessian matrix is **positive semidefinite** for all $x \in U$, then f **is convex,** and $f(x^*)$ is a **global minimum** (but not necessarily unique).*

*(c) If the Hessian matrix is **negative definite** for all $x \in U$, then f is **strictly concave,** and $f(x^*)$ is a **unique global maximum**.*

*(d) If the Hessian matrix is **positive definite** for all $x \in U$, then f **is strictly convex,** and $f(x^*)$ is a **unique global minimum**.*

Theorem 14 says that if we found a critical value on strictly concave function, then we are guaranteed that the function has a unique global maximum at that point. This is a very good reason why economists want to construct models where they maximize a strictly concave objective function or minimize a strictly convex function - the first order conditions give a unique global optimum. It turns out that the checking the sign pattern of leading principal minors is often very complicated. Fortunately, there are powerful theorems that make the task of identifying concave functions easier. For example, the sum of concave functions is concave, so if we maximize $F(x_1, x_2) = f(x_1, x_2) + g(x_1, x_2)$, and f, g are concave, then their sum is also concave. The next section provides a brief review of concave functions and presents the most useful results needed for optimization.

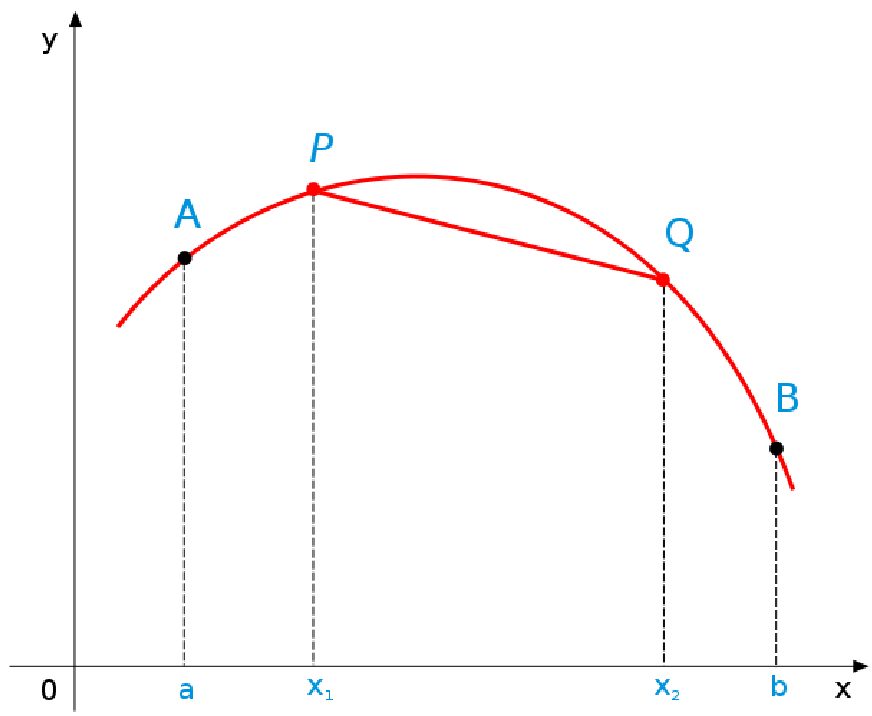
3.5.4 Concave functions

Concave functions are best illustrated graphically for a one-variable case. Notice that according to theorem 14, strict concavity of a one-variable twice continuously differentiable function $f(x)$ is characterized by $f''(x) < 0$ for all x in the domain of f. The general definition of a concave function does not rely on second derivatives, or even on first derivatives.

Definition 21 *(Concave function). A real valued function on a convex subset U of $\mathbb{R}^n$ is **concave** if for all $x_1, x_2 \in U$ and for all $\theta \in (0, 1)$,*

$$f(\theta x_1 + (1-\theta) x_2) \geq \theta f(x_1) + (1-\theta) f(x_2)$$

*The function is **strictly concave** if the above inequality is strict. The next figure plots the graph of a concave function of one variable.*



The graphical interpretation of a concave function is that any cord connecting two points on the graph of the function must be below the graph. **Convex** function is defined similarly, with the inequality reversed. A convex function of one variable is such that any cord connection two points on the graph, should be above the graph of the function. **Linear** functions are both concave and convex.

For differentiable functions of one variable, concavity can be characterized by saying that any tangent line to the graph of the function lies above the graph of the function. Mathematically, for all x_0 in the domain of a concave function f, and any x in the domain, with $x \neq x_0$, we have

$$f(x_0) + f'(x_0)(x - x_0) \geq f(x_0)$$

For strictly concave functions the above inequality becomes strict, and for convex functions the direction of the inequality is reversed.

Finally, twice continuously differentiable concave functions are characterized as in theorem 14. For example, a function $y = 10 - x^2$ is strictly concave function in one variable. The easiest way to check it is using the second derivative test (since this function is twice differentiable), that is $f''(x) = -2 < 0 \; \forall x$. A function of two variables, $y = 100 - x_1^2 - x_2^2$, with a graph plotted in figure (3.7) is strictly concave, as we saw earlier, with negative definite Hessian matrix:

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

The leading principal minors are

$$\begin{aligned} |H_1| &= |f_{11}| = -2 < 0 \\ |H_2| &= \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = (-2)(-2) - 0 = 4 > 0 \end{aligned}$$

Thus $y = f(x_1, x_2) = 100 - x_1^2 - x_2^2$ is strictly concave function. This means that at the critical value $(x_1^*, x_2^*) = (0, 0)$, the function f attains its unique global maximum.

The next theorem provides useful properties of concave functions, which help us test for concavity without calculating the Hessian and leading principal minors.

Theorem 15 *(Properties of concave functions).*

1. *A function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ (n-variable real valued function) is concave if and only if $-f$ is convex.*

2. *Let $f_1, ..., f_k : U \subseteq \mathbb{R}^n \to \mathbb{R}$ (n-variable real valued functions), and let $\alpha_1, ..., \alpha_k$ be positive numbers. If $f_1, ..., f_k$ are all concave then the sum $\alpha_1 f_1 + ... + \alpha_k f_k$ is also concave, and if the functions are convex, then the sum is convex.*

3. *An increasing and concave transformation of a concave function, is a concave function.*

Using the above properties, we can say that a sufficient condition for the profit function $\pi(Q_1, Q_2, Q_3) = R_1(Q_1) + R_2(Q_2) + R_3(Q_3) - C(Q_1 + Q_2 + Q_3)$ to be concave (which guarantees that a critical point is a unique global maximum), is that $R_1(Q_1) + R_2(Q_2) + R_3(Q_3)$ are concave, and $C(Q_1 + Q_2 + Q_3)$ is convex (which means that $-C(Q_1 + Q_2 + Q_3)$ is concave).

Proof. The easiest and most general proof of the above uses the definition of concave functions (21).

1. Multiply the definition by -1, gives:

$$\begin{aligned} f\left(\theta x_1+(1-\theta)\,x_2\right) &\geq \theta f\left(x_1\right)+(1-\theta)\,f\left(x_2\right) \\ -f\left(\theta x_1+(1-\theta)\,x_2\right) &\leq \theta\cdot\left(-f\left(x_1\right)\right)+(1-\theta)\cdot\left(-f\left(x_2\right)\right) \end{aligned}$$

   The last inequality is exactly the definition of $-f$ being a convex function.

2. Let $f_1,...,f_k$ be concave functions. By definition (21), we have:

$$\begin{aligned} f_1\left(\theta x_1+(1-\theta)\,x_2\right) &\geq \theta f_1\left(x_1\right)+(1-\theta)\,f_1\left(x_2\right) \\ f_2\left(\theta x_1+(1-\theta)\,x_2\right) &\geq \theta f_2\left(x_1\right)+(1-\theta)\,f_2\left(x_2\right) \\ &\vdots \\ f_k\left(\theta x_1+(1-\theta)\,x_2\right) &\geq \theta f_k\left(x_1\right)+(1-\theta)\,f_k\left(x_2\right) \end{aligned}$$

   Multiplying the above inequalities by positive numbers $\alpha_1,...,\alpha_k$ (which preserves the direction of inequality), and summing, gives:

$$\begin{aligned} \sum_{i=1}^{k}\alpha_i f_i\left(\theta x_1+(1-\theta)\,x_2\right) &\geq \sum_{i=1}^{k}\alpha_i\left[\theta f_i\left(x_1\right)+(1-\theta)\,f_i\left(x_2\right)\right] \\ &= \theta\sum_{i=1}^{k}\alpha_i f_i\left(x_1\right)+(1-\theta)\sum_{i=1}^{k}\alpha_i f_i\left(x_2\right), \end{aligned}$$

   which is exactly the definition of $\alpha_1 f_1+...+\alpha_k f_k$ being concave.

3. Let f be concave function, and let g be another concave function, which is also increasing. Then

$$\begin{aligned} f\left(\theta x_1+(1-\theta)\,x_2\right) &\geq \theta f\left(x_1\right)+(1-\theta)\,f\left(x_2\right) \\ g\left(f\left(\theta x_1+(1-\theta)\,x_2\right)\right) &\geq g\left(\theta f\left(x_1\right)+(1-\theta)\,f_2\left(x_2\right)\right) \quad g\text{ is increasing, \& } f \text{ is concave} \\ &\geq \theta g\left(f(x_1\right)+(1-\theta)\,g\left(f\left(x_2\right)\right) \quad \text{concavity of } g \end{aligned}$$

■

Exercise 17 *The above proof is general, in that it does not make any assumptions about differentiability. As an exercise, prove theorem 15, given that the functions in question are twice continuously differentiable. Hint, use the definition of definite matrices (definition 20), and prove that*

1. *If H, the Hessian matrix of function f, is negative definite, then $-H$ is positive definite.*

2. *If $H_1, ..., H_k$ are negative definite Hessian matrices of $f_1, ..., f_k$, then the sum $\alpha_i H_1 + ... + \alpha_k H_k$ is also negative definite matrix (for positive numbers $\alpha_1, ..., \alpha_k$), proving that $\alpha_1 f_1 + ... + \alpha_k f_k$ is also concave.*

Now looking back at the function $f(x_1, x_2) = 100 - x_1^2 - x_2^2$, recognizing that x^2 is strictly convex, we can say that based on theorem 15, $-x^2$ is strictly concave, and the sum $-x_1^2 - x_2^2$ is strictly concave. Notice that addition of constants does not affect concavity. As another example, if we maximize profit of a firm, when profit is the difference between revenue and cost, $\pi(Q) = R(Q) - C(Q)$, then concavity of $R(Q)$ and convexity of $C(Q)$ (concavity of $-C(Q)$), imply concavity of $\pi(Q)$.

3.5.5 Application: multiproduct competitive firm

Suppose a competitive firm produces quantities of two goods, Q_1 and Q_2, and sells them at given prices P_1 and P_2. Recall that a competitive firm takes the prices as given, and can only choose the quantities produced. Then the revenue of the firm is given by:

$$R(Q_1, Q_2) = P_1 Q_1 + P_2 Q_2$$

In addition it is given that the cost function of the firm is given by:

$$C(Q_1, Q_2) = 2Q_1^2 + Q_1 Q_2 + 2Q_2^2$$

Notice that such cost function indicates that the production lines of the two products are not separate. That is, the marginal cost of each product depends on the quantities produced of *both* products:

$$\begin{aligned} MC_1 &= \frac{\partial}{\partial Q_1} C(Q_1, Q_2) = 4Q_1 + Q_2 \\ MC_2 &= \frac{\partial}{\partial Q_2} C(Q_1, Q_2) = 4Q_2 + Q_1 \end{aligned}$$

Observe that higher production of one good, makes it more expensive to produce the other good, perhaps because some factors, while producing one good, cannot be used in the production of the other good at the same time.

In any case, with the given revenue and cost functions, the firm maximizes the profit $\pi = R - C$:

$$\max_{Q_1, Q_2} \pi(Q_1, Q_2) = P_1 Q_1 + P_2 Q_2 - 2Q_1^2 - Q_1 Q_2 - 2Q_2^2$$

The first order necessary conditions are:

$$\begin{aligned}\pi_1 (Q_1, Q_2) &= P_1 - 4Q_1 - Q_2 = 0 \\ \pi_2 (Q_1, Q_2) &= P_2 - 4Q_2 - Q_1 = 0\end{aligned}$$

where π_i is a shorthand notation for $\partial\pi/\partial Q_i$. Notice that these conditions equate the marginal revenues (P_1 and P_2) with the marginal costs of each product. Solving the first order necessary conditions, give the unique critical values:

$$\begin{aligned}Q_1^* &= \frac{4P_1 - P_2}{15} \\ Q_2^* &= \frac{4P_2 - P_1}{15}\end{aligned}$$

Notice that the quantity produced of one product increases in the price of that product, but decreases in the price of the other product. This is of course provided that Q_1^* and Q_2^* indeed maximize the profit. For example, if $P_1 = 12$ and $P_2 = 18$, then $Q_1^* = 2$, $Q_2^* = 4$ and $\pi (Q_1^*, Q_2^*) = 48$.

To verify that the critical point does represent maximum, we need to check the second order condition. The Hessian of the profit function is

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ -1 & -4 \end{bmatrix}$$

The leading principal minors are

$$\begin{aligned}|H_1| &= -4 < 0 \\ |H_2| &= 15 > 0\end{aligned}$$

Thus, the Hessian is negative definite, and the signs of the leading principal minors do not depend on where they are evaluated (do not depend on Q_1 and Q_2), by theorem 14 we conclude that the profit function is strictly concave, and $\pi (Q_1^*, Q_2^*)$ is the **unique global maximum.**

Notice that $\pi_{12} = \pi_{21} < 0$, which means that the marginal profit of good i decreases as the firm produces more of the other good. This is because the marginal cost of good i increase with the production of the other good, as discussed above.

3.5.6 Application: multiproduct monopoly

Now suppose we examine a monopoly (a single seller), that sells two goods, Q_1 and Q_2. The price will be determined by these chosen quantities, according to the inverse demand curves for these products:

$$\begin{aligned} P_1 &= 55 - Q_1 - Q_2 \\ P_2 &= 70 - Q_1 - 2Q_2 \end{aligned}$$

Notice that higher quantity sold of good i lowers the price the monopoly receives for that good. Moreover, higher quantity sold of good i also leads to lower price for the other good j, which means that the two goods are substitutes for the consumers. The revenue is then given by

$$\begin{aligned} R(Q_1, Q_2) &= (55 - Q_1 - Q_2) Q_1 + (70 - Q_1 - 2Q_2) Q_2 \\ &= 55Q_1 + 70Q_2 - 2Q_1Q_2 - Q_1^2 - 2Q_2^2 \end{aligned}$$

Let the cost function be:

$$C(Q_1, Q_2) = Q_1^2 + Q_1Q_2 + Q_2^2$$

The monoply's profit is therefore:

$$\begin{aligned} \pi(Q_1, Q_2) &= 55Q_1 + 70Q_2 - 2Q_1Q_2 - Q_1^2 - 2Q_2^2 - \left(Q_1^2 + Q_1Q_2 + Q_2^2\right) \\ &= 55Q_1 + 70Q_2 - 3Q_1Q_2 - 2Q_1^2 - 3Q_2^2 \end{aligned}$$

The monopoly's problem is:

$$\max_{Q_1, Q_2} \pi(Q_1, Q_2) = 55Q_1 + 70Q_2 - 3Q_1Q_2 - 2Q_1^2 - 3Q_2^2$$

The first order necessary conditions are:

$$\begin{aligned} \pi_1(Q_1, Q_2) &= 55 - 3Q_2 - 4Q_1 = 0 \\ \pi_2(Q_1, Q_2) &= 70 - 3Q_1 - 6Q_2 = 0 \end{aligned}$$

The solution is

$$(Q_1^*, Q_2^*) = \left(8, 7\frac{2}{3}\right)$$

The implied prices and profit are:

$$
\begin{aligned}
P_1^* &= 55 - Q_1^* - Q_2^* = 55 - 8 - 7\frac{2}{3} = \frac{118}{3} = 39\frac{1}{3} \\
P_2^* &= 70 - Q_1^* - Q_2^* = 70 - 8 - 2 \cdot 7\frac{2}{3} = \frac{140}{3} = 46\frac{2}{3} \\
\pi^* &= 488\frac{1}{3}
\end{aligned}
$$

To verify that we indeed found the maximal profit of the monopoly, we need to check the second order conditions. The Hessian of the profit function is:

$$
H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -4 & -3 \\ -3 & -6 \end{bmatrix}
$$

The leading principal minors are:

$$
\begin{aligned}
|H_1| &= -4 < 0 \\
|H_2| &= 15 > 0
\end{aligned}
$$

This pattern, according to theorem 14, implies that the Hessian matrix is negative definite, and that the profit function is strictly concave, regardless of the point at which the Hessian is evaluated. Therefore, the solution that we found with the first order necessary conditions is the unique global maximum.

3.5.7 Application: price discriminating monopoly

Now suppose that a monopoly sells a single good, to 3 separated (or segmented) markets. Separation of markets is another way of saying that buyers cannot perform arbitrage - buy in a lower price market and sell at a higher price market. The total revenue from all markets is then the sum of the revenues in each market:

$$
R(Q_1, Q_2, Q_3) = R_1(Q_1) + R_2(Q_2) + R_3(Q_3)
$$

The cost function depends on the total amount of the single good produced, regardless of where the good is sold:

$$
C(Q) = C(Q_1 + Q_2 + Q_3)
$$

The firm's problem is

$$\max_{Q_1,Q_2,Q_3} \pi\left(Q_1,Q_2,Q_3\right) = R_1\left(Q_1\right) + R_2\left(Q_2\right) + R_3\left(Q_3\right) - C\left(Q_1+Q_2+Q_3\right)$$

The first order necessary conditions are:

$$\begin{aligned}
\pi_1\left(Q_1,Q_2,Q_3\right) &= R_1'\left(Q_1\right) - C'\left(Q\right) = 0 \\
\pi_2\left(Q_1,Q_2,Q_3\right) &= R_2'\left(Q_2\right) - C'\left(Q\right) = 0 \\
\pi_3\left(Q_1,Q_2,Q_3\right) &= R_3'\left(Q_3\right) - C'\left(Q\right) = 0
\end{aligned}$$

In other words, the marginal revenue in all markets must be the same, and equal to the common marginal cost:

$$MR_i\left(Q_i\right) = MC\left(Q\right), \quad i = 1,2,3$$

We will not analyze the second order calculus conditions for maximum from theorem (14). These conditions are complicated and do not reveal much about this problem. Instead, we notice that $\pi\left(Q_1,Q_2,Q_3\right)$ is a sum of 4 functions, and we have useful results about sums of concave and convex functions (theorem 15). Then, if we know that all 4 functions $R_1\left(Q_1\right)$, $R_2\left(Q_2\right)$, $R_3\left(Q_3\right)$, $-C\left(Q_1+Q_2+Q_3\right)$ are concave, then we can conclude that the objective function $\pi\left(Q_1,Q_2,Q_3\right)$ is also concave, and the first order necessary conditions determine a global maximum. If concavity is strict, then the global maximum is unique.

The condition $MR\left(Q\right) = MC\left(Q\right)$ can be written in such a way that relates the price to marginal cost, and helps understand how firms with market power price their products in segmented markets. Let the revenue be $R\left(Q\right) = P\left(Q\right)\cdot Q$, where $P\left(Q\right)$ is the inverse demand. The marginal revenue is then

$$MR\left(Q\right) = R'\left(Q\right) = P'\left(Q\right)Q + P\left(Q\right) = P\left(Q\right)\left[1 + P'\left(Q\right)\frac{Q}{P\left(Q\right)}\right]$$

Since the price elasticity of demand is

$$\eta = \frac{dQ}{dP}\frac{P}{Q}$$

we see that the marginal revenue can be written as

$$MR\left(Q\right) = P\left(Q\right)\left[1 + \frac{1}{\eta}\right]$$

Since the demand is in general decreasing, the price elasticity of demand is a negative number,

$\eta < 0$, and we can write

$$MR(Q) = P(Q)\left[1 - \frac{1}{|\eta|}\right]$$

The first order condition says that this marginal revenue must be equal to the marginal cost:

$$P(Q)\left[1 - \frac{1}{|\eta|}\right] = MC(Q)$$

If the firm is competitive and takes the market price as given, $P'(Q) = 0$ and the demand it faces is perfectly elastic $|\eta| = \infty$. In such case, the first order necessary condition reduces to $P = MC$ (price = marginal cost). However, if the firm can affect the price by its quantity, then $P'(Q) \neq 0$ and the general condition $MR(Q) = MC(Q)$ holds.

In fact, we can show that the price will be higher than the marginal cost. Notice that no firm will sell units of output for negative marginal revenue, because these units are produced at non-negative cost. Thus, we must have

$$\begin{aligned} MR(Q) &> 0 \\ P(Q)\left[1 - \frac{1}{|\eta|}\right] &> 0 \\ &\Rightarrow |\eta| > 1 \end{aligned}$$

This means that a firm with market power will operate on the elastic portion of any demand. The price can be written as

$$P(Q) = \left[1 + \frac{1}{|\eta| - 1}\right] MC(Q) = (1 + \mu) MC(Q)$$

because the term in the square brackets is bigger than 1. The term $\mu = \frac{1}{|\eta|-1}$ is called the *markup rate*. For example, suppose that $|\eta| = 3$, then the selling price is

$$P(Q) = \left[1 + \frac{1}{3 - 1}\right] MC(Q) = 1.5MC(Q)$$

so the markup is 50% above cost per unit. If the elasticity of demand is $|\eta| = 2$ (in absolute value), then

$$P(Q) = \left[1 + \frac{1}{2 - 1}\right] MC(Q) = 2MC(Q)$$

so the markup is 100% above cost per unit. This math suggests that a firm which sells products to distinct markets, will charge a *higher price* in a market with *lower price elasticity* of demand (in absolute value). Intuitively, low elasticity of demand means that buyers do not respond to price increases with large decline in quantity bought, perhaps because the product does not have good substitutes. The practice of price discrimination between distinct markets, is called *third degree price discrimination.*

Exercise 18 *Suppose that Pfizer can produce a medical drug at constant marginal cost of \$20. The company sells the drug in Canada and the U.S., with price elasticities of demand in the two countries being* -4 *in Canada and* -1.5 *in the U.S. Assuming that arbitrage between these countries is not possible (due to regulations), find the drug prices and markup rates in the two countries.*

Solution 11 *Prices:*

$$\begin{aligned} P_{CAN} &= \left[1+\frac{1}{|\eta|-1}\right] MC = \left(1+\frac{1}{4-1}\right) 20 = \$26\frac{2}{3} \\ P_{USA} &= \left[1+\frac{1}{|\eta|-1}\right] MC = \left(1+\frac{1}{1.5-1}\right) 20 = \$60 \end{aligned}$$

Markup rates:

$$\begin{aligned} \mu_{CAN} &= \frac{P_{CAN}}{MC} - 1 = \frac{26\frac{2}{3}}{20} - 1 = 0.333... = 33\frac{1}{3}\% \\ \mu_{USA} &= \frac{P_{USA}}{MC} - 1 = \frac{60}{20} - 1 = 2 = 200\% \end{aligned}$$

Remark: the markup rates can also be calculated using the formula:

$$\mu = \frac{1}{|\eta|-1}$$

3.6 Constrained Optimization

In the last section we dealt with optimization problems of the form:

$$\max_{x_1,...,x_n} f(x_1,...,x_n) \quad \text{or} \quad \min_{x_1,...,x_n} f(x_1,...,x_n)$$

We developed a procedure for solving these problems, using first order necessary and second order sufficient conditions. In the above problem, the choice variables $x_1,...,x_n$ are unconstrained, and can attain any real value. Many economics problems however involve constraints on the choice variables. For example, consumer's choice can be represented as the following constrained optimization problem:

$$\max_{x_1,...,x_n} u(x_1,...,x_n)$$
$$s.t.$$
$$p_1x_1 + p_2x_2 + ... + p_nx_n = I$$
$$x_1, x_2, ..., x_n \geq 0$$

The consumer chooses quantities of goods bought, $x_1, ..., x_n$, that maximize the utility function $u(x_1,...,x_n)$ (the objective function), subject to the budget constraint, and subject to non-negativity (no short selling) constraints on the quantities of goods bought. The general structure of a constrained optimization problem is:

$$\max_{x_1,...,x_n} f(x_1,...,x_n) \quad \text{or} \quad \min_{x_1,...,x_n} f(x_1,...,x_n)$$
$$s.t.$$

$$m \text{ equality constraints: } \begin{cases} g^1(x_1,...,x_n) = c_1 \\ \vdots \\ g^m(x_1,...,x_n) = c_m \end{cases}$$

$$k \text{ inequality constraints: } \begin{cases} h^1(x_1,...,x_n) \leq d_1 \\ \vdots \\ h^k(x_1,...,x_n) \leq d_k \end{cases}$$

Minimization problems are written in similar way, but with "min" instead of "max". Such problems are extremely complex, but mathematicians have developed variety of necessary and sufficient conditions for them. The proofs of such conditions are very complex, and our discussion will mostly state some important results from optimization theory, and show their applications to economics.

3.6.1 Equality constraints

We start with a problem of optimizing a multivariate function $f(x_1, ..., x_n)$ subject to one equality constraint. That is, we start with problems of the form:

$$\max_{x_1,...,x_n} f(x_1, ..., x_n) \quad \text{or} \quad \min_{x_1,...,x_n} f(x_1, ..., x_n)$$
$$s.t.$$
$$g(x_1, ..., x_n) = c$$

A specific example of a problem stated above is the problem of maximizing the area of a rectangular $x \times y$ frame, subject to the constraint that all sides must add up to 100 feet:

$$\begin{aligned} &\max_{x,y} xy \\ &s.t. \\ &2x + 2y = 100 \end{aligned} \tag{3.1}$$

Here the constraint is $g(x, y) = 2x+2y$, and $c = 100$. Problems like the above can sometimes be solved by substituting the constraint into the objective. The constraint implies $y = 50-x$. Plugging into the objective function gives the unconstrained problem:

$$\max_{x} x(50 - x)$$

The first order necessary condition is

$$50 - 2x = 0$$

and the critical value is

$$(x^*, y^*) = (25, 25)$$

The above example shows that sometimes we can substitute the constraints into the objective, and create an unconstrained optimization problem. The substitution method can work for simple constraints, but becomes impossible when we cannot solve explicitly for one of the choice variables, or complicated for more complex problems. An alternative way to solving (3.1) is to write the corresponding **Lagrange** function, of 3 unknown variables:

$$\mathcal{L}(x, y, \lambda) = xy - \lambda[2x + 2y - 100] \tag{3.2}$$

The idea is to replace the constrained optimization problem in (3.1), with 2 unknowns

(x, y), by an "unconstrained" optimization problem (3.2) with 3 unknowns (x, y, λ). The object $\mathcal{L}(x, y, \lambda)$ is called the **Lagrange function**, and the coefficient λ is called the **Lagrange multiplier**. The reason for the quotation marks is that we are not exactly solving $\max_{x,y,\lambda} \mathcal{L}(x, y, \lambda)$, but the first order necessary conditions are the same, *as if* we actually solved $\max_{x,y,\lambda} \mathcal{L}(x, y, \lambda)$. These first order necessary conditions are:

$$\begin{aligned}
\mathcal{L}_\lambda(x, y, \lambda) &= -(2x + 2y - 100) = 0 \\
\mathcal{L}_x(x, y, \lambda) &= y - 2\lambda = 0 \\
\mathcal{L}_y(x, y, \lambda) &= x - 2\lambda = 0
\end{aligned}$$

These are 3 equations with 3 unknowns x, y, λ. The first condition $\mathcal{L}_\lambda(x, y, \lambda) = \frac{\partial}{\partial \lambda}\mathcal{L}(x, y, \lambda) = 0$ guarantees that the constraint is satisfied. The other two conditions imply that

$$\lambda = \frac{y}{2} = \frac{x}{2}$$

Thus, $x = y$. Plugging this into the constraint, gives the same critical point of the Lagrange as we found by substitution:

$$(x^*, y^*, \lambda^*) = (25, 25, 12.5)$$

Notice that $x^* = 25$ is the critical point of the objective function we obtained via the substitution method. It turns out that, although we were originally interested in finding only x and y, the value of the Lagrange multiplier has important economic meaning that we will explain later.

First order necessary conditions

The next theorem provides the general first order necessary conditions for optimization with one equality constraint.

Theorem 16 *(Lagrange Theorem1 - First Order Necessary Conditions for optimization with one equality constraint). Let f and g be C^1 functions. Suppose that $x^* = (x_1^*, x_2^*, ..., x_n^*)$ is a solution to*

$$\begin{gathered}
\max_{x_1,\ldots,x_n} f(x_1, ..., x_n) \quad or \quad \min_{x_1,\ldots,x_n} f(x_1, ..., x_n) \\
s.t. \\
g(x_1, ..., x_n) = c
\end{gathered} \tag{3.3}$$

Suppose further that $x^* = (x_1^*, x_2^*, ..., x_n^*)$ *is not a critical point of* g*. Then, there is a real number* λ^* *such that* $(x_1^*, x_2^*, ..., x_n^*, \lambda^*)$ *is a critical point of the Lagrange function*

$$\mathcal{L}(x_1, ..., x_n, \lambda) = f(x_1, ..., x_n) - \lambda [g(x_1, ..., x_n) - c]$$

In other words, the first order necessary condition for the problem in (3.3) is

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{\partial \mathcal{L}}{\partial x_1} = ... = \frac{\partial \mathcal{L}}{\partial x_n} = 0$$

In order to get some intuition about the above theorem, consider the optimization problem:

$$\begin{aligned} &\max_{x,y} u(x,y) \\ &s.t. \\ &g(x,y) = c \end{aligned} \tag{3.4}$$

The standard utility maximization problem, subject to a budget constraint, is a special case of the above. The solution to the above problem can be described as a tangency condition between a level curve $u(x,y) = \bar{u}$ and the constraint $g(x,y) = c$.

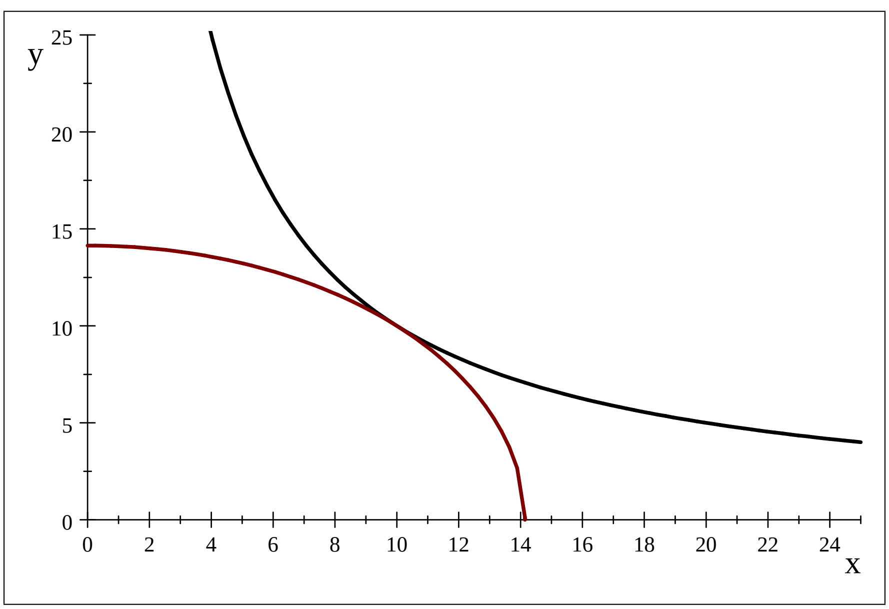


Illustration of Lagrange Theorem.

Notice that at the optimum, the level curve of the objective function (indifference curve) is tangent to the constraint:

$$\frac{u_x(x^*, y^*)}{u_y(x^*, y^*)} = \frac{g_x(x^*, y^*)}{g_y(x^*, y^*)}$$

If $g_x(x^*, y^*), g_y(x^*, y^*) \neq 0$, i.e. (x^*, y^*) is not a critical value of $g(x, y)$, the above can be written as

$$\frac{u_x(x^*, y^*)}{g_x(x^*, y^*)} = \frac{u_y(x^*, y^*)}{g_y(x^*, y^*)} = \lambda,$$

where λ is some constant. Rewriting the last equation as two equations, gives

$$\begin{aligned} u_x(x^*, y^*) - \lambda g_x(x^*, y^*) &= 0 \\ u_y(x^*, y^*) - \lambda g_y(x^*, y^*) &= 0 \end{aligned}$$

Plus, the constraint must be satisfied, i.e. $g(x^*, y^*) = c$. But these are exactly the first order conditions given by lagrange Theorem (16). Suppose we wrote the lagrange function associated with problem (3.4):

$$\mathcal{L}(x, y, \lambda) = u(x, y) - \lambda[g(x, y) - c]$$

According to the Lagrange Theorem (16), the first order necessary condition for maximum or minimum is:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \lambda} &= -g(x^*, y^*) + c = 0 \\ \frac{\partial \mathcal{L}}{\partial x} &= u_x(x^*, y^*) - \lambda g_x(x^*, y^*) = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= u_y(x^*, y^*) - \lambda g_y(x^*, y^*) = 0 \end{aligned}$$

Theorem 17 *(Lagrange Theorem2 - First Order Necessary Conditions for optimization with multiple equality constraint). Suppose that we are solving a problem with $m < n$ constraints,* *such as:*[5]

$$\max_{x_1,...,x_n} f(x_1,...,x_n) \quad or \quad \min_{x_1,...,x_n} f(x_1,...,x_n)$$
$$s.t.$$
$$g^1(x_1,...,x_n) = c_1$$
$$\vdots$$
$$g^m(x_1,...,x_n) = c_m$$

The corresponding Lagrange function requires adding a Lagrange multiplier for every constraint:

$$\mathcal{L}(x_1,...,x_n,\lambda_1,...,\lambda_m) = f(x_1,...,x_n) - \sum_{i=1}^{m} \lambda_i \left[g^i(x_1,...,x_n) - c_i\right]$$

The first order necessary conditions require taking derivatives with respect to all the variables in the Lagrange function:

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = ... = \frac{\partial \mathcal{L}}{\partial \lambda_m} = \frac{\partial \mathcal{L}}{\partial x_1} = ... = \frac{\partial \mathcal{L}}{\partial x_n} = 0$$

Second order sufficient conditions for local optimum

The second order sufficient conditions are not exactly the same as for unconstrained problem $\max_{x_1,...,x_n,\lambda} \mathcal{L}(x_1,...,x_n,\lambda)$. We start with the simplest case of optimizing a two variable function with one constraint: $\max_{x,y} f(x,y)$ $s.t.$ $g(x,y) = c$. The Lagrange function for this problem is

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda[g(x,y) - c]$$

The first order necessary conditions are:

$$\begin{aligned}
\mathcal{L}_\lambda(x,y,\lambda) &= -g(x,y) + c = 0 \\
\mathcal{L}_x(x,y,\lambda) &= f_x - \lambda g_x = 0 \\
\mathcal{L}_y(x,y,\lambda) &= f_y - \lambda g_y = 0
\end{aligned}$$

We suspect that the Hessian of the Lagrange function $\mathcal{L}$ (or more precisely, its leading principal minors) should play a role in the second order conditions, just like in unconstrained optimization. Indeed, the Hessian matrix of the Lagrange function $\mathcal{L}$, when we order the

[5] The condition $m < n$ is required for the so called Nondegenerate Constraint Qualification (NDSQ). Just like in the one constraint case, we required that the optimal solution is not a critical point of the constraint, now we require that it is not a critical point of all m equality constraints. This cannot happen when $m \geq n$ (See Simon and Blume, section 18.2).

variables as λ, x, y, is:

$$H = \begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda x} & \mathcal{L}_{\lambda y} \\ \mathcal{L}_{x\lambda} & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ \mathcal{L}_{y\lambda} & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix} = \begin{bmatrix} 0 & -g_x & -g_y \\ -g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ -g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix}$$

The second order conditions however are stated in terms of the **bordered Hessian:**

$$\bar{H} = \begin{bmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix}$$

Recall from the properties of determinants, that multiplication of only one row or one column by a constant k, will result in determinant $k|A|$. Thus, multiplying the first row and the first column of H by -1, will not change the determinant of H. In fact, all principal minors of H will be the same as the corresponding principal minors of $\bar{H}$. So in stating the second order conditions for constrained optimization, we will use the bordered Hessian $\bar{H}$ and not the Hessian of the Lagrange function H, but the results will be the same. The name *bordered* comes from the fact that the matrix of second derivatives with respect to the original unknowns x, y, i.e.

$$\begin{bmatrix} \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix}$$

is bordered (surrounded) by the first derivatives of the constraints, from above and from the left.

The leading principal minors of the bordered Hessian are $\bar{H}$:

$$\begin{aligned} |\bar{H}_1| &= 0 \\ |\bar{H}_2| &= \begin{vmatrix} 0 & g_x \\ g_x & \mathcal{L}_{xx} \end{vmatrix} = -g_x^2 < 0 \\ |\bar{H}_3| &= \begin{vmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{vmatrix} \end{aligned}$$

The first principal minor of $\bar{H}$ is always zero, which is o.k. since it involves derivatives with respect to λ, and we are neither maximizing nor minimizing the Lagrange function with respect to λ. Then $|\bar{H}_2|$ is always negative. Thus, our second order sufficient condition involves only $|\bar{H}_3|$ or higher, in problems with more choice variables.

Theorem 18 *(Second order sufficient condition for local constrained extremum, with* 2 *unknowns, and* 1 *equality constraint). Suppose that we are solving*

$$\max_{x,y} f(x,y) \quad \textit{or} \quad \min_{x,y} f(x,y)$$
$$s.t.$$
$$g(x,y) = c$$

The corresponding Lagrange function is

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda[g(x,y) - c]$$

Suppose that (x^*, y^*, λ^*) *is a critical value of* $\mathcal{L}$*, i.e. satisfies the first order necessary conditions:*

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = 0$$

Let the bordered Hessian and the 3rd principal minor be:

$$\bar{H} = \begin{bmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix}, \quad |\bar{H}_3| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{vmatrix}$$

(a) If $|\bar{H}_3| > 0$*, then* (x^*, y^*) *is a local maximum of* f *subject to the constraint* $g(x,y) = c$.
(b) If $|\bar{H}_3| < 0$*, then* (x^*, y^*) *is a local minimum of* f *subject to the constraint* $g(x,y) = c$.

As usual, second order *sufficient conditions* are not necessary, and if they are violated, we simply conclude that the second order test had failed and that the critical point can be minimum, maximum, or neither. Sufficient conditions *guarantee* a certain result, but if they are violated, the result can still hold.

We now generalize the above result to optimization with n variables, but still with one equality constraint.

Theorem 19 *(Second order sufficient condition for local constrained extremum, with* n *unknowns, and* 1 *equality constraint). Suppose that we are solving*

$$\max_{x_1,...,x_n} f(x_1, ..., x_n) \quad \textit{or} \quad \min_{x_1,...,x_n} f(x_1, ..., x_n)$$
$$s.t.$$
$$g(x_1, ..., x_n) = c$$

The corresponding Lagrange function is

$$\mathcal{L}(x_1, ..., x_n, \lambda) = f(x_1, ..., x_n) - \lambda [g(x_1, ..., x_n) - c]$$

Suppose that $(x_1^*, ..., x_n^*, \lambda^*)$ *is a critical value of* $\mathcal{L}$*, i.e. satisfies the first order necessary conditions:*

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{\partial \mathcal{L}}{\partial x_1} = ... = \frac{\partial \mathcal{L}}{\partial x_n} = 0$$

Let the bordered Hessian be:

$$\begin{aligned} \bar{H} &= \begin{bmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1n} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & \mathcal{L}_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & \mathcal{L}_{n1} & \mathcal{L}_{n2} & \cdots & \mathcal{L}_{nn} \end{bmatrix}_{(n+1)\times(n+1)} \\ \text{where } \mathcal{L}_{ij} &= \frac{\partial^2}{\partial x_i \partial x_j} \mathcal{L} \end{aligned}$$

The leading principal minors, starting from 3rd (notice that there are $n-1$ *of them) are:*

$$|\bar{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} \end{vmatrix}, |\bar{H}_4| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ g_3 & \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{vmatrix}, ..., |\bar{H}_{n+1}|$$

(a) If $|\bar{H}_3| > 0$, $|\bar{H}_4| < 0$, $|\bar{H}_5| > 0$,*..., then* $(x_1^*, ..., x_n^*)$ *is a* ***local maximum*** *of* f *subject to the constraint* $g(x_1, ..., x_n) = c$.

(b) If $|\bar{H}_3| < 0$, $|\bar{H}_4| < 0$, $|\bar{H}_5| < 0$,*...,then* $(x_1^*, ..., x_n^*)$ *is a* ***local minimum*** *of* f *subject to the constraint* $g(x_1, ..., x_n) = c$.

Finally, we generalize the last theorem to optimization with n variables and $m < n$ constraints. Notice that we need to introduce a Lagrange multiplier for each constraint.

Theorem 20 *(Second order sufficient condition for local constrained extremum, with n un-*

knowns, and $m < n$ constraints). Suppose that we are solving

$$\max_{x_1,...,x_n} f(x_1,...,x_n) \quad or \quad \min_{x_1,...,x_n} f(x_1,...,x_n)$$
$$s.t.$$
$$g^1(x_1,...,x_n) = c_1$$
$$\vdots$$
$$g^m(x_1,...,x_n) = c_m$$

The corresponding Lagrange function is

$$\mathcal{L}(x_1,...,x_n,\lambda_1,...,\lambda_m) = f(x_1,...,x_n) - \sum_{i=1}^{m} \lambda_i \left[g^i(x_1,...,x_n) - c_i\right]$$

Suppose that $(x_1^*,...,x_n^*,\lambda_1^*,...,\lambda_m^*)$ is a critical value of $\mathcal{L}$, i.e. satisfies the first order necessary conditions:

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = ... = \frac{\partial \mathcal{L}}{\partial \lambda_m} = \frac{\partial \mathcal{L}}{\partial x_1} = ... = \frac{\partial \mathcal{L}}{\partial x_n} = 0$$

Let the bordered Hessian (evaluated at the critical point) be:

$$\bar{H} = \begin{bmatrix} 0 & \cdots & 0 & g_1^1 & \cdots & g_n^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & g_1^m & \cdots & g_n^m \\ g_1^1 & \cdots & g_1^m & \mathcal{L}_{11} & \cdots & \mathcal{L}_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \cdots \\ g_n^1 & \cdots & g_n^m & \mathcal{L}_{n1} & \cdots & \mathcal{L}_{nn} \end{bmatrix}_{(n+m)\times(n+m)}$$

$$\text{where } \mathcal{L}_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}\mathcal{L},\ g_j^i = \frac{\partial g^i}{\partial x_j}$$

*(a) If sign $\left|\bar{H}_{2m+1}\right| = (-1)^{m+1}$, and all the higher order leading principal minor of $\bar{H}$ alternate sign, then $(x_1^*,...,x_n^*)$ is a **local maximum** of f subject to the constraints.*

*(b) If sign $\left|\bar{H}_{2m+1}\right| = (-1)^{m}$, and all the higher order leading principal minors of $\bar{H}$ have this sign too, then $(x_1^*,...,x_n^*)$ is a **local minimum** of f subject to the constraints.*

Based on the last theorem, notice that in order to check the second order conditions, we need to calculate $n-m$ leading principal minors, of order $2m+1$ and higher. The first $m \times m$ principal submatrix matrix has all zeros. The following m leading principal minors are zeros. This is why we only need to calculate the leading principal minors of order

$2m+1, 2m+2, ..., n+m$. Also notice that it makes a difference whether the number of constraints m is odd or even because (-1) raised to odd power gives to opposite sign to the case of an even power. Finally, verify that the last theorem applied to the case of $m=1$ is consistent with the the second order condition in case of a single constraint.

Example 44 *Suppose we want to solve the following optimization problem,*

$$\max_{x_1,x_2,x_3,x_4} x_1^2 - x_2^2 + x_3^2 + x_4^2 + 4x_2x_3 - 2x_1x_4$$
$$s.t.$$
$$x_2 + x_3 + x_4 = 0$$
$$x_1 - 9x_2 + x_4 = 0$$

The Lagrange function associated with this optimization problem is:

$$\begin{aligned}\mathcal{L}(x_1,x_2,x_3,x_4,\lambda_1,\lambda_2) &= x_1^2 - x_2^2 + x_3^2 + x_4^2 + 4x_2x_3 - 2x_1x_4 \\ &\quad -\lambda_1\left[x_2 + x_3 + x_4 - 0\right] \\ &\quad -\lambda_2\left[x_1 - 9x_2 + x_4 - 0\right]\end{aligned}$$

The first order necessary conditions are:

$$\begin{aligned}\mathcal{L}_{\lambda_1} &= x_2 + x_3 + x_4 = 0 \\ \mathcal{L}_{\lambda_2} &= x_1 - 9x_2 + x_4 = 0 \\ \mathcal{L}_1 &= 2x_1 - 2x_4 - \lambda_2 = 0 \\ \mathcal{L}_2 &= -2x_2 + 4x_3 - \lambda_1 + 9\lambda_2 = 0 \\ \mathcal{L}_3 &= 2x_3 + 4x_2 - \lambda_1 = 0 \\ \mathcal{L}_4 &= 2x_4 - 2x_1 - \lambda_1 - \lambda_2 = 0\end{aligned}$$

This is a linear system with 6 *unknowns* $(x_1,x_2,x_3,x_4,\lambda_1,\lambda_2)$*, and and* 6 *equations. If there is no dependence among the equations, there must be a unique solution to the system. We can see that this solution must be such that all variables are zero:* $x_1 = x_2 = x_3 = x_4 = \lambda_1 =$

$\lambda_2 = 0$. *The bordered Hessian matrix is*

$$\bar{H} = \begin{bmatrix} 0 & 0 & g_1^1 & g_2^1 & g_3^1 & g_4^1 \\ 0 & 0 & g_1^2 & g_2^2 & g_3^2 & g_4^2 \\ g_1^1 & g_1^2 & \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} & \mathcal{L}_{14} \\ g_2^1 & g_2^2 & \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} & \mathcal{L}_{24} \\ g_3^1 & g_3^2 & \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} & \mathcal{L}_{34} \\ g_4^1 & g_4^2 & \mathcal{L}_{41} & \mathcal{L}_{42} & \mathcal{L}_{43} & \mathcal{L}_{44} \end{bmatrix}_{6\times 6} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -9 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & -2 \\ 1 & -9 & 0 & -2 & 4 & 0 \\ 1 & 0 & 0 & 4 & 2 & 0 \\ 1 & 1 & -2 & 0 & 0 & 2 \end{bmatrix}$$

The second order conditions require the leading principal minors, starting from $2m + 1 = 5$. *Thus, we have to evaluate two leading principal minors:*

$$\left|\bar{H}_5\right| = \begin{vmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & -9 & 0 & -2 & 4 \\ 1 & 0 & 0 & 4 & 2 \end{vmatrix} = 154 > 0$$

and

$$\left|\bar{H}_6\right| = \begin{vmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -9 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & -2 \\ 1 & -9 & 0 & -2 & 4 & 0 \\ 1 & 0 & 0 & 4 & 2 & 0 \\ 1 & 1 & -2 & 0 & 0 & 2 \end{vmatrix} = 96 > 0$$

Both signs are positive, so the second order sufficient condition for maximum does not hold (the signs must alternate). The sufficient condition for minimum is that all signs are the same as $(-1)^m = (-1)^2 > 0$, *so we conclude that the critical value* $x_1 = x_2 = x_3 = x_4 = 0$ *is a local minimum of* $x_1^2 - x_2^2 + x_3^2 + x_4^2 + 4x_2x_3 - 2x_1x_4$ *subject to the constraints* $x_2 + x_3 + x_4 = 0$ *and* $x_1 - 9x_2 + x_4 = 0$.

Sufficient conditions for global optimum

Our main interest in economics is to find global optimum, not local. The second order sufficient conditions for local optimum presented above, are therefore not very useful. In this section we present important results that will allow us to determine whether a critical point is a *global* maximum or minimum. We will illustrate the main point with the consumer's utility maximization problem:

$$\max_{x,y} u(x,y)$$
$$s.t.$$
$$p_x x + p_y y = I$$

The first order necessary conditions:

$$\begin{aligned}
\mathcal{L}_\lambda &= -(p_x x + p_y y - I) = 0 \\
\mathcal{L}_x &= u_x(x,y) - \lambda p_x = 0 \\
\mathcal{L}_y &= u_y(x,y) - \lambda p_y = 0
\end{aligned}$$

Notice that the last two conditions imply that

$$\frac{u_x(x,y)}{p_x} = \frac{u_y(x,y)}{p_y} = \lambda$$

The ratios u_x/p_x and u_y/p_y represent the marginal utility per dollar spent on the two goods. At the optimum, the condition says that the utility generated from the last dollar spent on x should be the same as the utility from the last dollar spent on y.

The first order conditions also imply

$$\frac{u_x(x,y)}{u_y(x,y)} = \frac{p_x}{p_y}$$

The left hand side is the Marginal Rate of Substitution between x and y, which is the absolute value of the slope of indifference curves. The right hand side is the price ratio, i.e. the absolute value of the slope of the budget constraint. Thus, the necessary condition for optimal bundle (the bundle which maximizes the utility subject to the budget constraint) requires tangency between indifference curves and the budget constraint. Figure 3.10 illustrates the optimal bundle. The solid indifference curve represents the highest utility that can be achieved within the budget constraint.

The question we want to ask is how do we know if the first order condition indeed gives a unique global maximum? In figure 3.10 it seems that indeed the tangency point characterizes a unique global maximum. There is no other point on the budget constraint that can give higher utility. But it is possible to draw indifference curves, that are not of the "standard" shape, such that a point of tangency with the budget constraint gives only a local maximum. You probably realize at this point that the key for unique global maximum of utility, subject

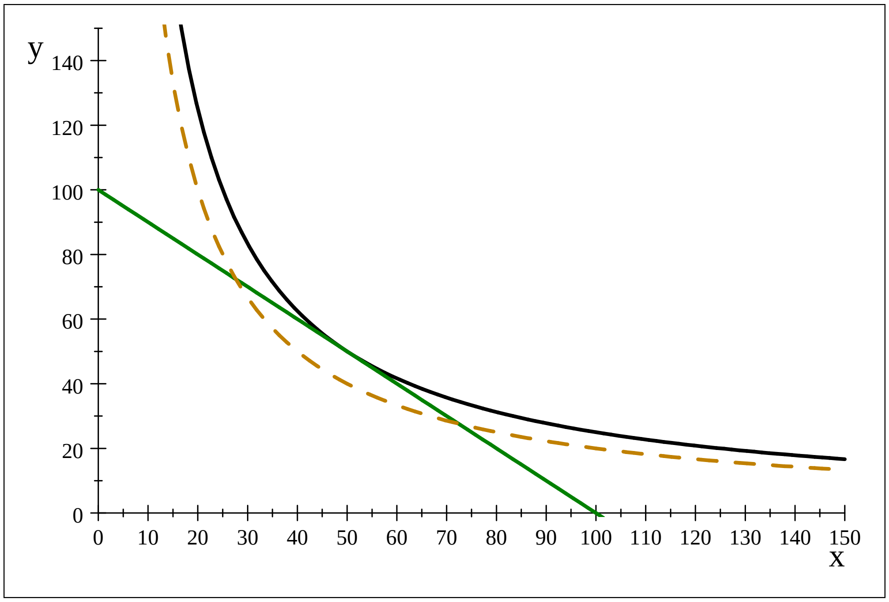


Figure 3.10: Optimal consumption bundle

to linear budget constraint, lies in the shape of the indifference curves. In particular, with linear budget constraint, the sufficient condition for unique global maximum of utility is that indifference curves be **strictly convex**.

What kind of utility functions have strictly convex indifference curves? For starters, **strictly concave** utility functions have **strictly convex** indifference curves. For example, $u(x, y) = x^{0.25}y^{0.25}$. This is strictly concave utility function, which you can verify using its Hessian matrix:

$$H = \begin{bmatrix} -\frac{3}{16}x^{-1.75}y^{0.25} & \frac{1}{8}x^{-0.75}y^{-0.75} \\ \frac{1}{8}x^{-0.75}y^{-0.75} & -\frac{3}{16}x^{0.25}y^{-1.75} \end{bmatrix}$$

The leading principal minors are:

$$\begin{aligned} |H_1| &= -\frac{3}{16}x^{-1.75}y^{0.25} < 0 \\ |H_2| &= \left(\frac{3}{16}\right)^2 x^{-1.5}y^{-1.5} - \left(\frac{2}{16}\right)^2 x^{-1.5}y^{-1.5} > 0 \end{aligned}$$

Thus, $u(x, y) = x^{0.25}y^{0.25}$ is strictly concave because the pattern of the signs is consistent with negative definiteness of the Hessian matrix. Looking at a generic indifference curve, $x^{0.25}y^{0.25} = \bar{u}$, or explicitly

$$y = \frac{\bar{u}^4}{x}$$

One can see that the shape of this indifference curve is like in figure 3.10.

But strictly concave utility functions are not the only ones that have strictly convex indifference curves. For example, consider $u(x,y) = xy$, which is the utility function in the previous example, raised to the power of 4. This utility function is not strictly concave. The Hessian matrix is:

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The leading principal minors are:

$$\begin{aligned} |H_1| &= 0 \\ |H_2| &= -1 < 0 \end{aligned}$$

This pattern is not consistent with negative definiteness of the Hessian matrix. Nevertheless, the indifference curves have the same exact shape as in figure 3.10: $xy = \bar{u}$, or explicitly

$$y = \frac{\bar{u}}{x}$$

This is why figure 3.10 illustrates the solution to

$$\max_{x,y} x^{0.25} y^{0.25}$$
$$s.t.$$
$$x + y = 100$$

and also the solution to

$$\max_{x,y} xy$$
$$s.t.$$
$$x + y = 100$$

In fact, the same solution is obtained when we maximize any utility function which a monotone transformation of $x^{0.25}y^{0.25}$, subject to the same budget constraint. For example, $u(x,y) = x^2y^2$ or $u(x,y) = 0.5\ln x + 0.5\ln y$. All these functions have the same shape of indifference curves, and all lead to the same optimal bundle.

For additional illustration, figure 3.11 plots the utility function of $u(x,y) = x^{0.25}y^{0.25}$. When we cut the utility at certain level, given by the plane at height 1, we obtain an indifference curve on which all bundles (x,y) give utility of 1. Figure 3.12 illustrates an indifference curve of $u(x,y) = xy$, obtained by cutting this utility at some fixed level. Interestingly, both utility functions have very different shapes, but the indifference curves

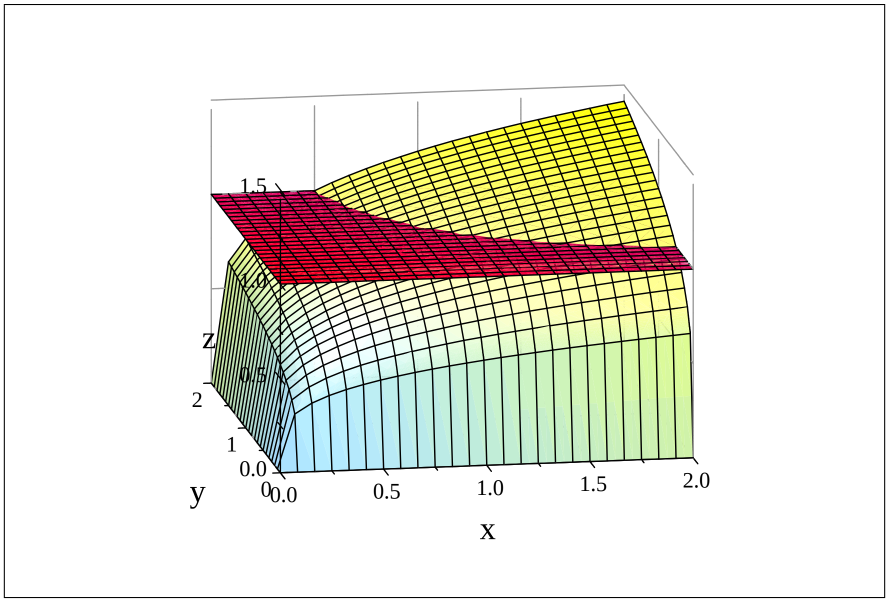


Figure 3.11: $u(x,y) = x^{0.25}y^{0.25}$

look similar, a result we showed above mathematically.

Such utility functions, that have the "right" shape of indifference curves, are called **strictly quasiconcave.** In general discussion of constrained optimization we talk about **level curves** instead of indifference curves. So strictly quasiconcave objective functions, have strictly convex level curves. In general optimization problems, instead of a budget constraint, we have a **constraint set**, which is the set of all the choice variables that satisfy the constraints. The general sufficient condition for a unique global maximum in a constrained optimization problem is **strict quasiconcave objective function and convex constraint set**.

Definition 22 *A set C is* ***convex*** *if for any two elements in the set, $x_1, x_2 \in C$, any linear combination of these elements is also an element in the set:*

$$\alpha x_1 + (1-\alpha) x_2 \in C \quad \forall \alpha \in [0,1].$$

A set C is ***strictly convex*** *if any linear combination $\alpha x_1 + (1-\alpha) x_2$, $\alpha \in (0,1)$, belongs to the* ***interior*** *of the set.*[6]

Definition 23 *(i) The* ***upper contour*** *set of a function $f : X \to \mathbb{R}$ at level $\alpha \in \mathbb{R}$ is the*

[6] Interior of a set S contains all the elements that are not on the **boundary** of the set. A point p is said to be a boundary point of set S, if any open neighborhood containing the point, also contains elements outside the set. The boundary of a set S, denoted ∂S, contains all the boundary points.

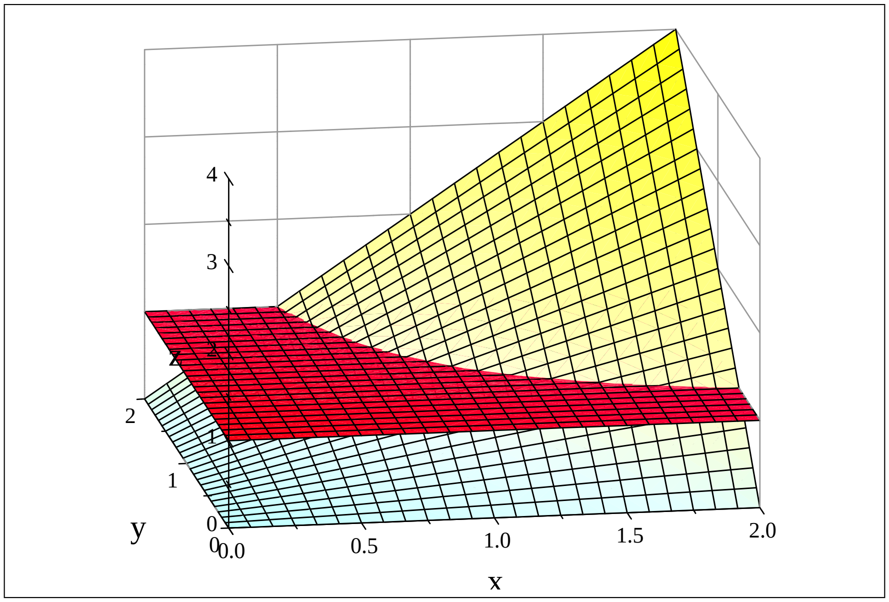


Figure 3.12: $u(x, y) = xy$

set

$$U(\alpha; f) \equiv \{x \in X | f(x) \geq \alpha\}$$

*(ii) The **lower contour** set of a function $f : X \to \mathbb{R}$ at level $\alpha \in \mathbb{R}$ is the set*

$$L(\alpha; f) \equiv \{x \in X | f(x) \leq \alpha\}$$

Graphically, in the case of consumer choice model (see figure 3.10), an upper contour set of a utility function u, at level $\bar{u}$, is the set of all the bundles that are on, or above the indifference curve with utility level $\bar{u}$. The lower contour set of a utility function u, at level $\bar{u}$ is the collection of all the bundles on, or below the indifference curve with utility level $\bar{u}$. Thus, the upper contour sets of utility function $u(x, y)$ at level $\bar{u}$, are

$$\begin{aligned} U(\bar{u}; u) &= \left\{(x, y) \in \mathbb{R}^2 | u(x, y) \geq \bar{u}\right\} \\ L(\bar{u}; u) &= \left\{(x, y) \in \mathbb{R}^2 | u(x, y) \leq \bar{u}\right\} \end{aligned}$$

Definition 24 *(i) A function f is **quasiconcave** if and only if for all $\alpha \in \mathbb{R}$, the upper contour set $U(\alpha; f)$ is convex.*

*(ii) A function f is **strictly quasiconcave** if and only if for all $\alpha \in \mathbb{R}$, the upper contour set $U(\alpha; f)$ is strictly convex.*

*(iii) A function f is **quasiconvex** if and only if for all $\alpha \in \mathbb{R}$, the lower contour set $L(\alpha; f)$ is convex.*

*(iv) A function f is **strictly quasiconvex** if and only if for all $\alpha \in \mathbb{R}$, the lower contour set $L(\alpha; f)$ is strictly convex.*

The following is the most important and the most general theorem characterizing global constrained optima.

Theorem 21 *(Sufficient conditions for constrained global optimum). Let $f : X \to \mathbb{R}$ be a function, defined on a convex subset X of $\mathbb{R}^n$. Let $C \subseteq X$ be a **convex** constraint set. Suppose we want to solve*

$$\max_x f(x) \quad \text{or} \quad \min_x f(x)$$
$$s.t. \quad x \in C$$

Let x^* be a critical point f subject to the constraint set. Then

*(i) if f is **quasiconcave**, then x^* is constrained **global maximum**,*

*(ii) if f is **strictly quasiconcave**, or C is **strictly convex**, then x^* is a **unique constrained global maximum**,*

*(ii) if f is **quasiconvex**, then x^* is constrained **global minimum**,*

*(iv) if f is **strictly quasiconvex**, or C is **strictly convex**, then x^* is a **unique constrained global minimum**.*

In other words, any local maximum with convex constraint set and quasiconcave objective, is also a global maximum. Similarly, and any local minimum with convex constraint set and quasiconvex objective, is also a global minimum. Next, we would like to be able to check if some function is quasiconcave/quasiconvex. One easy way of recognizing quasiconcave functions, without using calculus, is based on the following two theorems.

Theorem 22 *(Concavity implies quasiconcavity). Any concave function is quasiconcave, and any convex function is quasiconvex. Similarly, strictly concave function is strictly quasiconcave, and any strictly convex function is strictly quasiconcave.*

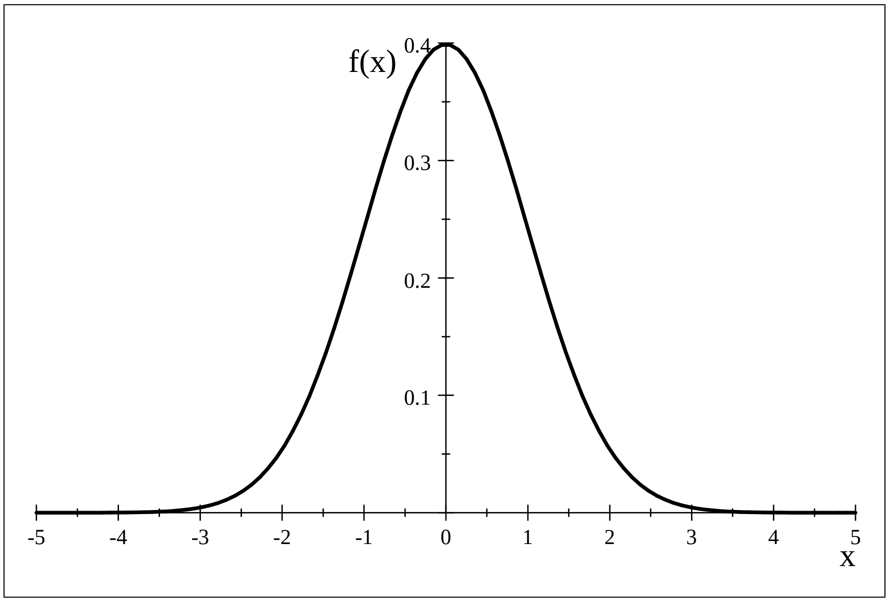
Theorem 23 *(Monotone increasing transformations of concave/convex functions are quasiconcave/quasiconvex). Let f be concave function and g an increasing monotone function. Then the composite function $g \circ f$ is quasiconcave. Similarly, if f is convex, then $g \circ f$ is quasiconvex.*

The next examples illustrate how we can easily recognize quasiconcave functions, as monotone transformations of concave functions.

Example 45 *The probability density function (pdf) of normal random variable, is quasi-concave function:*

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

The graph of the function is:



Notice that the quadratic function $(x-\mu)^2 = x^2 - 2x\mu + \mu^2$ *is convex (the coefficient on the quadratic term is positive). The minus makes the function in the curly brackets concave (*f *is concave if and only if* $-f$ *is convex). The exponential function* $\exp(x) = e^x$ *is monomtone increasing, and* $\frac{1}{\sqrt{2\pi}\sigma}$ *is a positive constant. Thus, by theorem 23, the entire function* $f(x)$ *is quisiconcave. In fact, all the concavities and quasiconcavities in the above example are strict.*

Example 46 *The Cobb-Douglas utility function* $u(x,y) = x^\alpha y^\beta$*,* $\alpha, \beta > 0$*, is quasiconcave. Notice that*

$$x^\alpha y^\beta = \exp\{\alpha \log(x) + \beta \log(y)\}$$

The logarithmic functions are concave, and sum of concave functions is concave. Thus, the term in the curly brackets is concave, and exponential function $\exp(x) = e^x$ *is a monotone increasing transformation. Again, by theorem 23, the entire function* $f(x)$ *is quasiconcave (strictly). Thus, when maximizing Cobb-Douglas utility over a convex constraint set, the critical point is a unique global maximum.*

Example 47 *The CES utility function* $u(x,y) = [\alpha x^\sigma + (1-\alpha) y^\sigma]^{\frac{1}{\sigma}}$*,* $\sigma \leq 1$ *is quasiconcave. If* $\sigma = 1$*, then* $u(x,y)$ *is linear function, which is both concave and convex. Thus,*

$u(x,y)$ in this case is both quasiconcave and quasiconvex. The case of $\sigma = 0$ is Cobb-Douglas, and was discussed in the previous example. Consider first the case of $0 < \sigma < 1$. The functions αx^{σ} and $(1-\alpha) y^{\sigma}$ are concave, and their sum $g(x,y) = \alpha x^{\sigma} + (1-\alpha) y^{\sigma}$ is concave. Thus, $u = g^{\frac{1}{\sigma}}$ is a monotone increasing transformation of a concave function, so u is quasiconcave. Finally, when $\sigma < 0$, the functions αx^{σ} and $(1-\alpha) y^{\sigma}$ are convex, so $-\alpha x^{\sigma}$ and $-(1-\alpha) y^{\sigma}$ are concave, so their sum $h(x,y) = -\alpha x^{\sigma} - (1-\alpha) y^{\sigma}$ is also concave. Then $u(x,y) = (-h)^{\frac{1}{\sigma}}$ is a monotone increasing transformation of a concave function ($\frac{d}{dh}(-h)^{\frac{1}{\sigma}} = -\frac{1}{\sigma}(-h)^{\frac{1}{\sigma}-1} = -\frac{1}{\sigma}(\alpha x^{\sigma} + (1-\alpha) y^{\sigma})^{\frac{1}{\sigma}-1} > 0$). Notice that the sign is positive since $\sigma < 0$ and $-\frac{1}{\sigma} > 0$.

The last examples illustrated how we can detect quasiconcave functions, without using calculus. There are also easy to check calculus criteria that help detect quasiconcave and quasiconvex functions. These conditions are based on yet another type of functions, **pseudoconcae (**and **pseudocnvex).** The details are provided in Simon and Blume 1994 (theorem 21.19) and omitted here.

In summary, this section provided useful results about sufficient conditions for global optima: convexity of constraint set + quasiconcavity of objective function are sufficient for global maximum, and convexity of constraint set + quasiconvexity of objective function are sufficient for global minimum. Notice that convexity of constraint set is always a nice property to have, but unfortunately, in some research problems we can encounter non-convex constraint sets - a problem named *nonconvexities*. Also observe that theorem 21 provides sufficient conditions for *constrained* global optimum via quasiconcave/quasiconvex functions. With unconstrained optimization, we learned that a critical point of a concave function is a global maximum, while a critical point of a convex function is a global minimum. The same is not true for quasiconcave and quasiconvex functions. For example, $f(x) = x^3$ is quasiconcave and also quasiconvex function. The critical point $x^* = 0$ however is neither global nor local maximum or minimum. The function in example 45, the pdf of Normal random variable, is an example where the critical point is indeed a global maximum, but this example cannot be generalized, as the $f(x) = x^3$ demonstrates. So keep it is important to keep in mind that quasiconcave and quasiconvex functions are mainly used in constrained optimization.

3.6.2 The meaning of Lagrange multipliers

Consider a simple maximization problem

$$\max_{x,y} f(x,y)$$
$$s.t.$$
$$g(x,y)=c$$

Solving this problem with Lagrange method, gives

$$\mathcal{L}=f(x,y)-\lambda\left[g(x,y)-c\right]$$

The first order necessary conditions are

$$\begin{aligned}\mathcal{L}_\lambda &= -g(x,y)+c=0\\ \mathcal{L}_x &= f_x(x,y)-\lambda g_x(x,y)=0\\ \mathcal{L}_y &= f_y(x,y)-\lambda g_y(x,y)=0\end{aligned}$$

Suppose that the first order necessary conditions indeed characterize the maximum. This problem has one parameter, c, and we are often interested in finding how the maximized value of the objective function changes when we change the value of some parameter. For example, c could represent a scarce resource, and we would like to measure its social value. The solution to the problem depends on the parameter c, and we therefore write the solution as:

$$x(c),y(c),\lambda(c)$$

The maximized value of the objective function also depends on the parameter c, and it is called the **maximum value function**, defined as:

$$V(c)\equiv f(x(c),y(c))$$

We are interested in the following derivative:

$$\frac{\partial}{\partial c}V(c)=\frac{\partial}{\partial c}f(x(c),y(c))=f_x(x(c),y(c))\frac{\partial x(c)}{\partial c}+f_y(x(c),y(c))\frac{\partial y(c)}{\partial c}$$

This says that changes in c affect the objective function through x and y, which readjust (recall section 2.1.9).

We can also express the Lagrange function at the optimum as:

$$\mathcal{L}(x(c), y(c), \lambda(c)) = f(x(c), y(c)) - \lambda(c)[g(x(c), y(c)) - c]$$

Notice that at the optimum, since the constraint holds, the term in the squared brackets is zero, and we must have

$$\mathcal{L}(x(c), y(c), \lambda(c)) = f(x(c), y(c)) = V(c)$$

So, we can find the desired effect of changes in c on the objective function, also through calculating the effect of changes in c on the Lagrange function. Thus, differentiating the Lagrange function, gives:

$$\begin{aligned}
\frac{\partial}{\partial c}\mathcal{L}(c) &= f_x \frac{\partial x(c)}{\partial c} + f_y \frac{\partial y(c)}{\partial c} - \lambda'(c)\underbrace{[g(x(c), y(c)) - c]}_{=0,\text{ by FONC}} \\
&\quad -\lambda(c)\left[g_x \frac{\partial x(c)}{\partial c} + g_y \frac{\partial y(c)}{\partial c} - 1\right] \\
&= \underbrace{[f_x - \lambda(c) g_x]}_{=0,\text{ by FONC}} \frac{\partial x(c)}{\partial c} + \underbrace{[f_y - \lambda(c) g_y]}_{=0,\text{ by FONC}} \frac{\partial y(c)}{\partial c} + \lambda(c) \\
&= \lambda(c)
\end{aligned}$$

Thus, we get

$$\frac{\partial}{\partial c}\mathcal{L}(c) = \frac{\partial}{\partial c}V(c) = \lambda(c)$$

Thus, λ is the marginal effect of a shift in constraint through the constant c, on the maximized value of the objective function. Notice that although the parameter c affects the maximum value function through x and y, we can find this effect by taking the *partial* derivative of the Lagrange function, i.e. ignoring the effect of the parameter on the choice variables x and y. This result is an example of a class of theorems, called *envelope theorems*, that study the impact of changes in parameters of an optimization problem, on the optimized value of the objective function.

3.6.3 Envelope theorems

The last example was a special case of an envelope theorem. Consider a parametrized optimization problem where the parameter a affects both the objective and the constraint:

$$\max_{x_1,\ldots,x_n} f\left(x_1,\ldots,x_n;a\right) \quad \text{or} \quad \min_{x_1,\ldots,x_n} f\left(x_1,\ldots,x_n;a\right)$$

$$s.t.$$

$$g\left(x_1,\ldots,x_n;a\right)=0$$

Let the solution to this problem be $x_1^*\left(a\right),\ldots,x_n^*\left(a\right)$, and the optimized value of the objective function is:

$$V\left(a\right)=f\left(x_1^*\left(a\right),\ldots,x_n^*\left(a\right);a\right)$$

The envelope theorem states

$$\frac{\partial}{\partial a}V\left(a\right)=\frac{\partial}{\partial a}\mathcal{L}$$

where $\mathcal{L}$ is the Lagrange function associated with the optimization problem,

$$\mathcal{L}=f\left(x_1,\ldots,x_n;a\right)-\lambda g\left(x_1,\ldots,x_n;a\right)$$

The next example in an application of the above envelope theorem to consumer's choice model.

Example 48 *Consider consumer's problem:*

$$\max_{x,y} u\left(x,y\right)$$

$$s.t.$$

$$p_x x+p_y y=I$$

Find the marginal effect of changes in income and prices on the maximized utility function (also called ***inderect utility****). The lagrange function is*

$$\mathcal{L}=u\left(x,y\right)-\lambda\left[p_x x+p_y y-I\right]$$

Using the envelope theorem, we need to take the partial derivatives the Lagrange function *with respect to income and prices:*

$$\begin{aligned}
\frac{\partial V}{\partial I} &= \frac{\partial \mathcal{L}}{\partial I} = \lambda \\
\frac{\partial V}{\partial p_x} &= \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda x \\
\frac{\partial V}{\partial p_y} &= \frac{\partial \mathcal{L}}{\partial p_y} = -\lambda y
\end{aligned}$$

Notice that the indirect utility is increasing in income, and decreasing in prices - higher prices lower the maximized value of utility. The size of the impact of a unit increase in prices depends on the amount of the good consumed. For example, $\lambda = 2$ and and the consumer originally bought 5 units, the impact of a one unit increase in price is $-2 \cdot 5 = -10$, same as the impact of 5 units decrease in income. The last result is called *Roy's Identity* after René Roy (1947)[3]. That is,

$$\begin{aligned}
\frac{\partial V}{\partial p_x} &= -\lambda x \\
&\Rightarrow \; x = -\frac{\frac{\partial V}{\partial p_x}}{\lambda} \\
\text{or } x &= -\frac{\frac{\partial V}{\partial p_x}}{\frac{\partial V}{\partial I}}
\end{aligned}$$

The last expression is the more common form of Roy's identity, which shows how to derive the Marshallian (uncompensated) demand from derivatives of the indirect utility function. In words, the demand for x is equal to the derivative of the indirect utility function with respect to p_x, normalized by the the marginal value of income.

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