EconS 425 - Sequential Move Games

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Industrial Organization

Introduction

- Today, we'll continue with our overview of game theory by looking at what happens when players take turns choosing their actions, rather than moving at the same time.
  - These are known as sequential move games.
- Sequential move games add another layer of strategy to the decision making of all agents involved.

Sequential Move Games

- As stated before, sequential move games are simply where the order of movement matters.
  - For example, suppose we had two players, and player 1 was able to choose their action before player 2 could choose theirs.
  - Player 2 is able to observe the action taken by player 1, then respond accordingly.
- Typically, one player will have an advantage over the other player in this case, but determining which player has that advantage depends on the game structure.

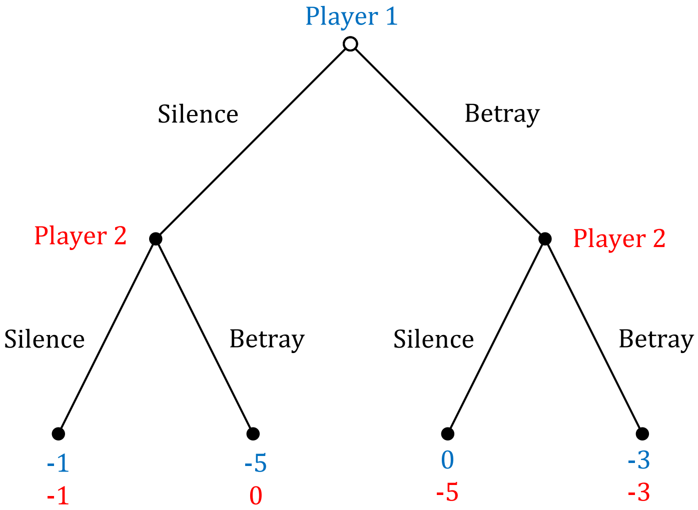
Sequential Move Games

- Let's return to the prisoner's dilemma.
  - This time, however, we will let player 1 decide whether to choose silence or betray first. Then let player 2 observe player 1's action and respond to it.
  - Everything else about the game remains the same.
- To model this game as a sequential move game, we must make use of the extensive form of the game (as opposed to the normal form that we have already seen).
  - This is represented by a series of decision trees with the outcomes and payoffs at the bottom.

Sequential Move Games

		Player 2: Silence	Player 2: Betray
Player 1	Silence	-1, -1	-5, 0
	Betray	0, -5	-3, -3

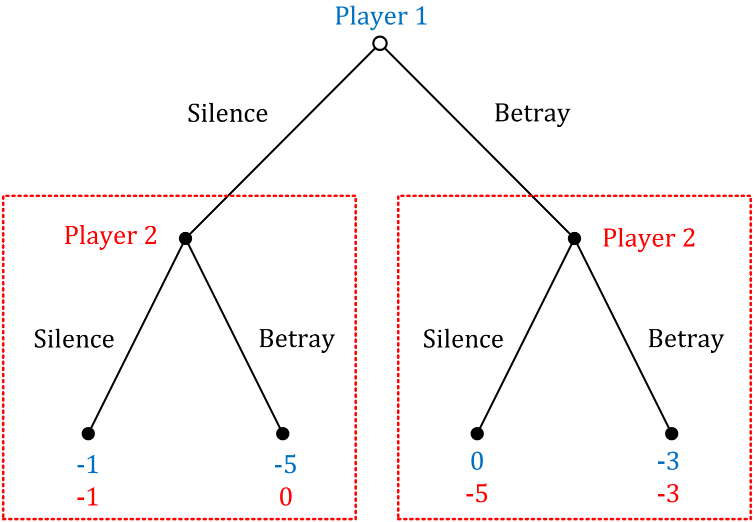
Sequential Move Games



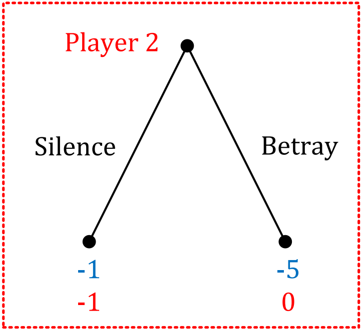
Sequential Move Games

- To analyze a sequential move game, we must make use of a technique known as backward induction.
  - We need to look at the actions that each player can make in order from the later actions until the earlier actions.
  - Essentially, we work backwards until we get to the top of the game tree.
- As we are able to determine the best responses for players, we can substitute them up the extensive form until we are left with one final choice.
  - We'll start with both of player 2's possible actions, since they occur at the end of the game.

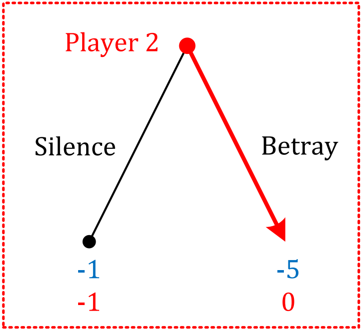
Sequential Move Games



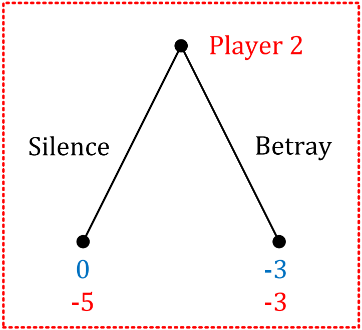
Sequential Move Games



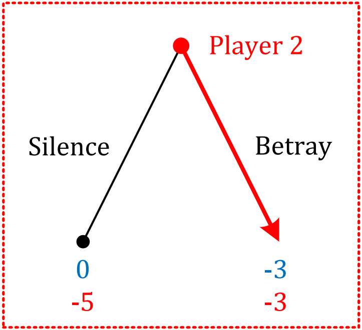
Sequential Move Games



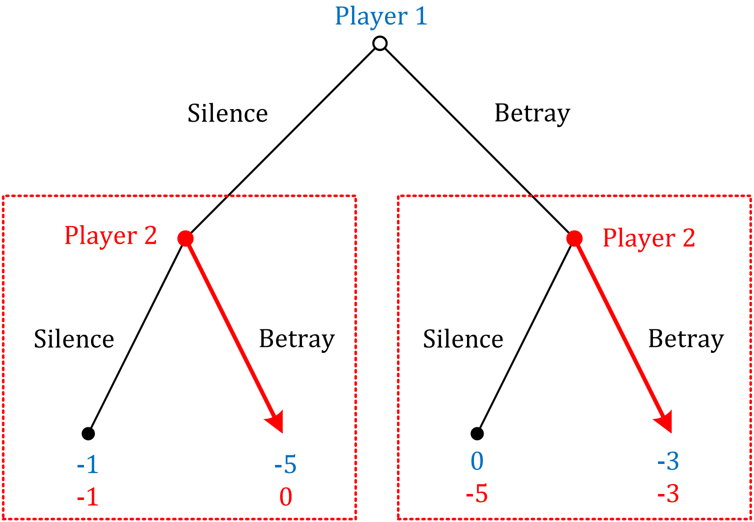
Sequential Move Games



Sequential Move Games



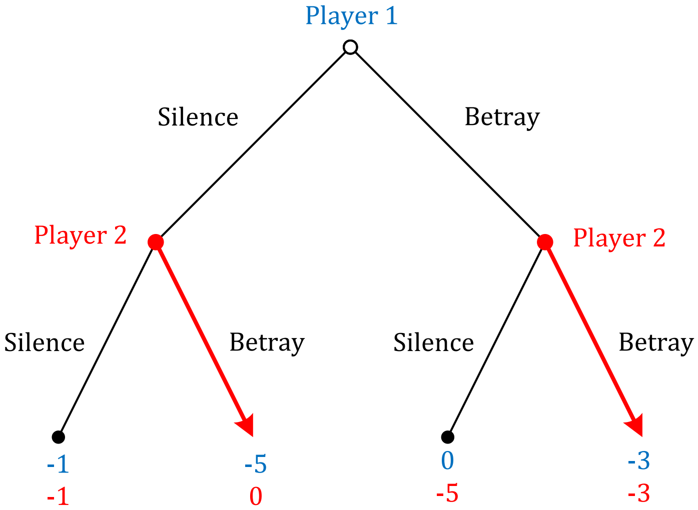
Sequential Move Games



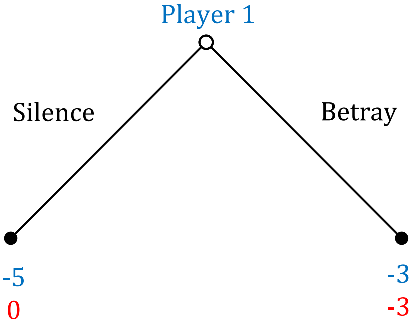
Sequential Move Games

- Now that we have determined player 2's best responses to every possible action we can move up the extensive form to player 1's action.
  - Since this is a game with perfect information (everyone knows everything about everyone), player 1 knows how player 2 will react to all of their possible actions.
  - Thus, player 1 will make their choice taking into consideration player 2's response.
- We can show this decision making process for player 1 by simply substituting up player 2's responses in the extensive form.
  - This is known as the reduced form.

Sequential Move Games



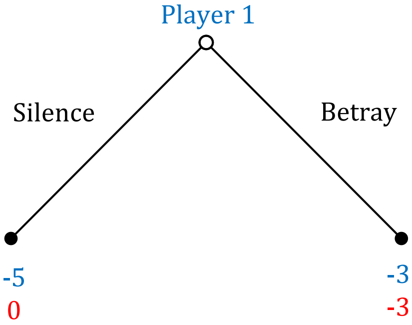
Sequential Move Games



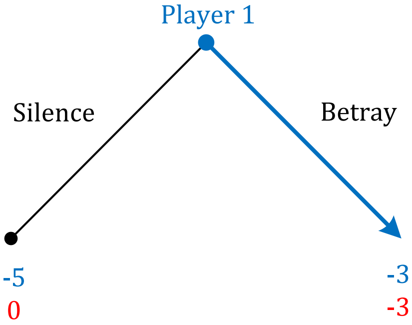
Sequential Move Games

- Now, player 1 simply chooses whichever of their actions yields the highest payoff, since player 2's responses are already taken into consideration.
- Once that is complete, we simply reassemble the extensive form of the game and can see all of the strategies for each player.

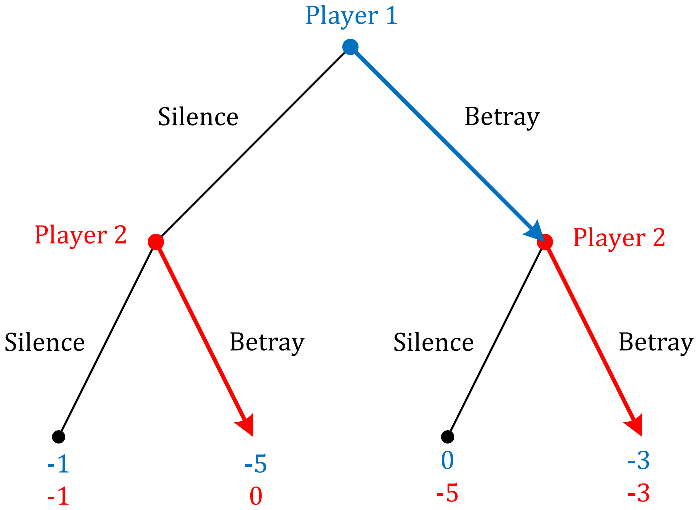
Sequential Move Games



Sequential Move Games



Sequential Move Games



Sequential Move Games

- As we can see, in equilibrium, player 1 will choose to betray player 2, and then player 2 will respond by betraying player 1.
  - This is the same outcome as in the simultaneous move game. This will always happen when a simultaneous move game only has a single Nash equilibrium.
- If I were being picky, I would say that the equilibrium strategy for player 1 is Betray, while the equilibrium strategy for player 2 is Betray/Betray.
  - Recall that a strategy is a collection of all the actions a player makes. Player 2 has two different actions in this game (one for each of player 1's possible choices), and a complete strategy must include all of them, even if they aren't on the equilibrium path.
  - I'm not too picky though in this class.

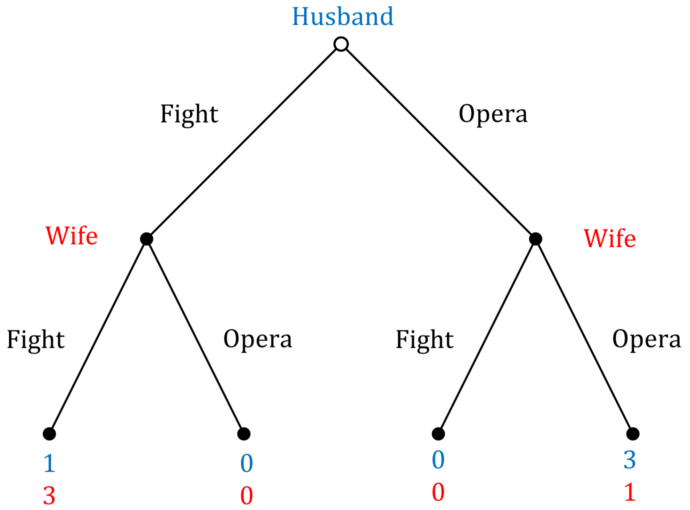
Sequential Move Games

- What if we had a game with more than one Nash equilibrium, like in "The Battle of the Sexes?"
  - Perhaps moving sequentially can help us determine which outcome we will arrive at.
- Let's first assume that the husband gets to make their choice first, then the wife gets to observe the husbands choice and make her own.
  - This basically breaks the original premise of the game. Most marriage problems can be solved (or created) with a simple text message, by the way.

Sequential Move Games

		Wife: Fight	Wife: Opera
Husband	Fight	1, 3	0, 0
Husband	Opera	0, 0	3, 1

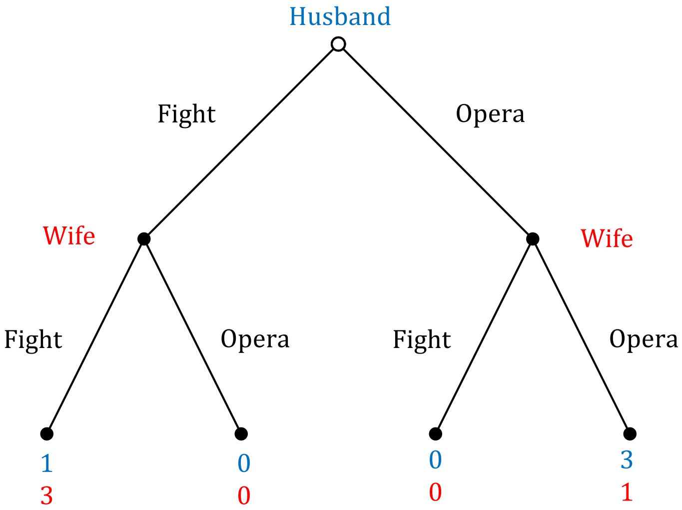
Sequential Move Games



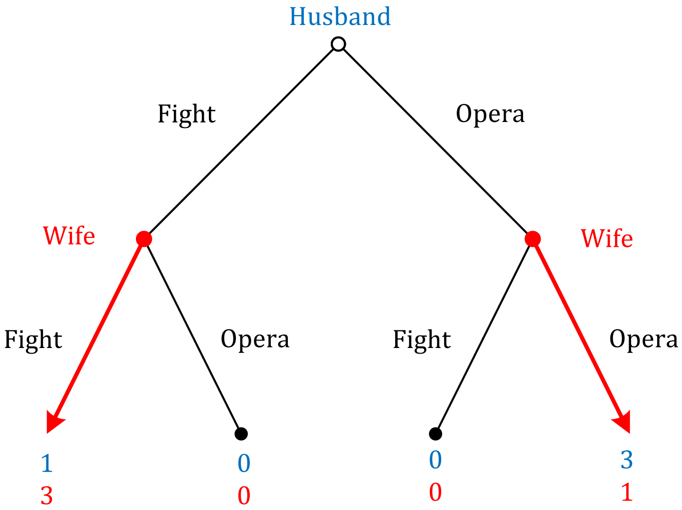
Sequential Move Games

- Again, we use the backward induction technique in order to find the equilibrium outcome for this game.
  - Since the wife moves last, we'll look at their best responses to all of the husband's possible choices.
  - Then we'll look at what the husband's best choice is, taking the wife's responses into account.
- To save a few slides, I'm just going to analyze the game as a whole, step by step.
  - This is usually quicker, too.

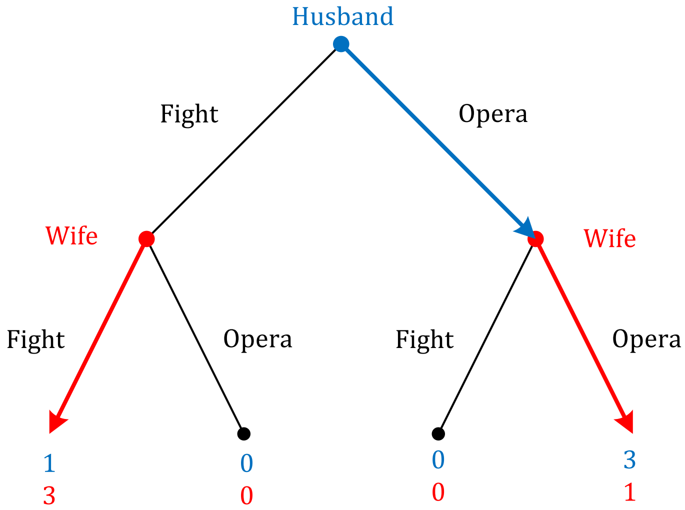
Sequential Move Games



Sequential Move Games



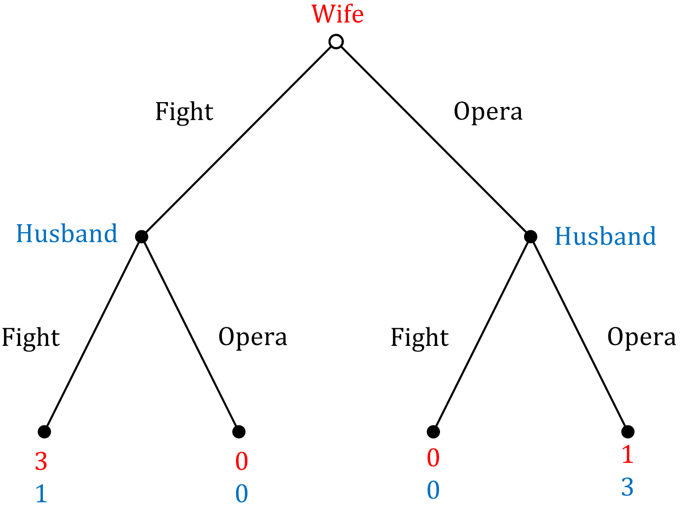
Sequential Move Games



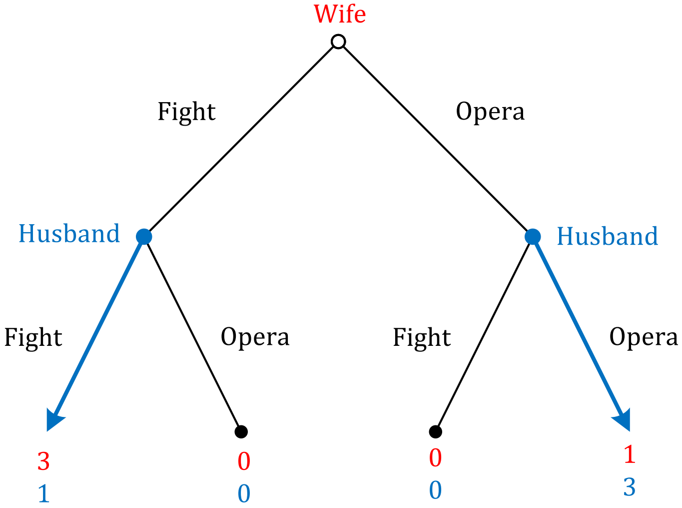
Sequential Move Games

- Notice how the wife's best response always led to one of the two possible Nash equilibria.
  - This should make sense. The husband and wife always got the highest payoffs when they attended the same event.
- Since the husband knows this, however, he can select his action knowing that whatever he chooses, the wife will follow him there.
  - So naturally, he chooses his most preferred activity; the opera in this case.
- What if the wife moved first?

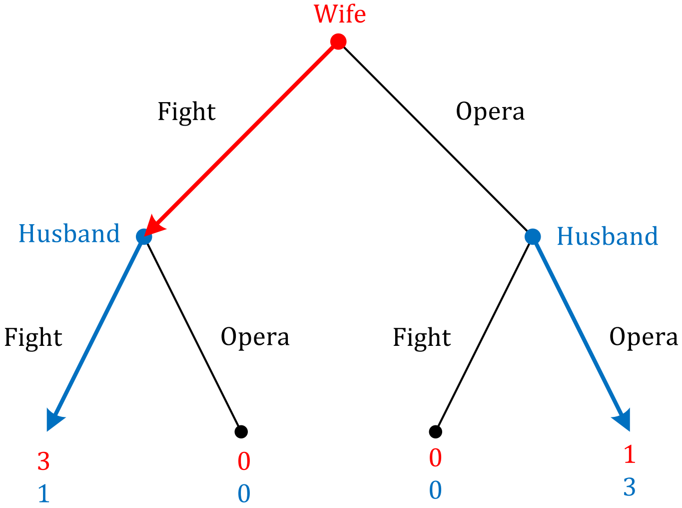
Sequential Move Games



Sequential Move Games



Sequential Move Games



Sequential Move Games

- Now we see the opposite result.
  - Since the wife knew that the husband would follow her wherever she chose to go, she was able to choose the activity that gave her the highest payoff; the boxing fight in this case.
- Depending on which player was able to move first, the Nash equilibrium we reached was different. Each player selected the Nash equilibrium that yielded them the highest payoff.
  - We call this Nash equilibrium a subgame perfect Nash equilibrium in this case.
  - A subgame perfect Nash equilibrium is simply a Nash equilibrium that survives backward induction.

Continuous Action Spaces

- Like our simultaneous move game counterpart, the majority of examples in this class use continuous action spaces.
- Let's look at our woolly mammoth hunter example again.
  - This time, hunter 1 gets to choose his effort level before hunter 2.
  - Intuitively, hunter 1 is able to set off for the hunt before hunter 2 is able to. By displaying his intended effort level through hunting equipment, traps, etc, hunter 2 is left to respond to hunter 1's effort level the next day.

Continuous Action Spaces

- The maximization problem for hunter i remains the same,

$$\max_{e_i} \quad e_i(1000 - e_1 - e_2) - 100e_i$$

- We can solve this problem using backward induction, just like we did with the earlier games.
  - Remember that we must start with the final mover (hunter 2), and work our way back up the tree until we reach the first mover (hunter 1).
  - We want to find a best response function for hunter 2, and substitute that into earlier stages of our game.

Continuous Action Spaces

$$\max_{e_2} \quad e_2(1000 - e_1 - e_2) - 100e_2$$

- We'll find that nothing changes for hunter 2. Taking a first-order condition with respect to e_2 yields,

$$\frac{\partial Meat}{\partial e_2} = 1000 - e_1 - 2e_2 - 100 = 0$$

and solving this expression for e_2 gives us our best response function for any given effort level of hunter 1,

$$e_2(e_1) = 450 - \frac{e_1}{2}$$

- This should make sense. For hunter 2, he is simply reacting to hunter 1's effort choice just like he was back in the simultaneous move game. Nothing has changed for him.

Continuous Action Spaces

- This is where things start to change.
  - Remember when we were looking at the earlier games that we would send the result of the later stages of the game up the tree to the earlier stages. Then the earlier player would pick their best choice taking that into consideration.
  - We can do that even without a formal "tree" to look at.
- Hunter 1's maximization problem is,

$$\max_{e_1} \quad e_1(1000 - e_1 - e_2) - 100e_1$$

but remember that hunter 1 gets to move first, and knows exactly how hunter 2 is going to react to his choice of effort. Intuitively, hunter 1 knows that hunter 2's effort is a function of his own effort, and he wants to factor that into his own maximization problem,

$$\max_{e_1} \quad e_1(1000 - e_1 - e_2(e_1)) - 100e_1$$

Continuous Action Spaces

$$\max_{e_1} \quad e_1(1000 - e_1 - e_2(e_1)) - 100e_1$$

$$e_2(e_1) = 450 - \frac{e_1}{2}$$

- We can simply substitute in the best response function for hunter 2 into hunter 1's maximization problem. This is equivalent to passing up the result of hunter 1's choice up the "tree,"

$$\begin{aligned} & \max_{e_1} \quad e_1\left(1000 - e_1 - \left(450 - \frac{e_1}{2}\right)\right) - 100e_1 \\ = \quad & \max_{e_1} \quad e_1\left(550 - \frac{e_1}{2}\right) - 100e_1 \end{aligned}$$

Continuous Action Spaces

$$\max_{e_1} \quad e_1 \left(550 - \frac{e_1}{2}\right) - 100e_1$$

- Taking a first-order condition with respect to e_1 yields,

$$\begin{aligned} 550 - e_1 - 100 &= 0 \\ e_1^* &= 450 \end{aligned}$$

and plugging this value back into the best response function for hunter 2 gives us,

$$e_2^* = 450 - \frac{e_1^*}{2} = 225$$

Continuous Action Spaces

$$e_1^* = 450 \qquad e_2^* = 225$$

- Interestingly, hunter 1 (the first mover) exerts twice as much effort as hunter 2 (the last mover).
  - Hunter 1 knows that his choice of a high effort level will deter hunter 2 from also exhibiting such a high effort level.
- With regard to their payoff level, in terms of meat, each hunter receives

$$\begin{aligned} Meat_i &= e_i^*(1000 - e_1^* - e_2^*) - 100e_i^* \\ Meat_1 &= 450(1000 - 450 - 225) - 100(450) = 101,250 \\ Meat_2 &= 225(1000 - 450 - 225) - 100(225) = 50,625 \end{aligned}$$

Continuous Action Spaces

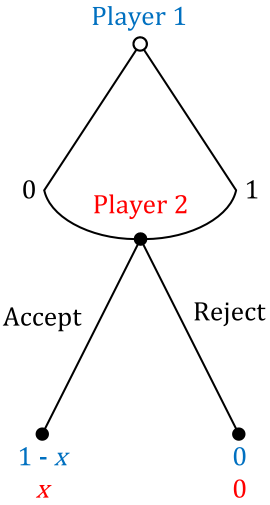
- Hunter 1 is able to obtain the same meat level he would if the hunters cooperated back in the simultaneous move game, while hunter 2 obtains a meager amount of meat.
  - This is a classic example of first mover's advantage, or simply, "The early bird gets the worm."
- Moving forward, we'll see examples of both first and second mover's advantage in our models.
  - Typically, this depends on whether the best response function is negatively or positively sloped.

Bargaining

- Let's look at one more application of sequential move games: Bargaining.
  - Bargaining is when one player makes an offer to another player, who can either reject or accept the offer.
  - Bargaining is a common game seen in the real world that many people do not utilize well enough.
- Bargaining makes use of both continuous and discrete action spaces, so we can actually draw game trees in this case.

Bargaining

- The traditional bargaining game involves two people deciding how to split a pie.
  - Player 1 offers some proportion of the pie, x, to player 2, where x can take any value from 0 (no pie) to 1 (the whole pie).
  - After observing the offer player 1 makes to them, player 2 then either rejects or accepts the offer.
  - If player 2 accepts, then both player 1 and player 2 receive their proportion of the pie as a payoff, $1 - x$ and x, respectively.
  - If player 2 rejects, both players receive a payoff of zero.


Player 1
0
1
Player 2
Accept
Reject
1 - x
x
0
0

Bargaining

- Again, we use backward induction to find our solution to this problem.
  - Starting at the bottom of the tree, we analyze player 2's decision.
  - Like in our other continuous move game, however, player 2 has an infinite amount of possible choices from player 1 to respond to.
  - Fortunately, we can simply partition them into the two possible choices that player 2 has, accept or reject.
- Player 2 will accept player 1's offer if

$$\begin{aligned} \text{Payoff from Accept} &\geq \text{Payoff from Reject} \\ x &\geq 0 \end{aligned}$$

  - *Note*: I am assuming that player 2 will accept if they are indifferent. This is a common assumption. We could say that they are offered a single crumb of pie such that they receive more than zero.

Bargaining

- Now, we can substitute this result up the tree. Player 1's payoff is

$$\begin{aligned} 1 - x \text{ if } x &\geq 0 \text{ (Player 2 Accepts)} \\ 0 \text{ if } x &< 0 \text{ (Player 2 Rejects)} \end{aligned}$$

- Obviously, player 1 will want to pick the smallest value of x that guarantees player 2 will accept the offer to maximize his own payoff.
    - Thus, player 1 offers $x = 0$, or the smallest number x possible that is greater than zero (a single crumb), and player 2 accepts the offer.
- Accepting a single crumb is better for player 2 than receiving no pie at all, regardless of whether this is a fair allocation or not.
    - Fairness is an opinion, anyways.

Bargaining

- In the real world, many experiments in this context have been done.
  - Unsurprisingly, people that are offered a "single crumb" of the pie often reject such unfair offers.
  - We get our results because we assume that players only care about how much pie they receive, and fairness isn't an issue (Perfect rationality).
  - If we wanted to, we could add fairness into the model, such as the following payoff for player 2,

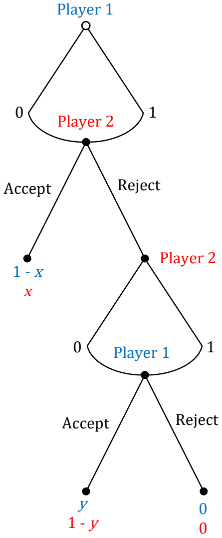
$$\underbrace{x}_{\text{Player 2's payoff}} - \alpha \underbrace{(1 - x - x)}_{\text{Difference between payoffs for players}}$$

  where $\alpha \geq 0$ is a parameter that specifies how important fairness is to player 2.
- I'll leave this analysis to another class, however.

Bargaining

- As it stands, this is an example of first mover's advantage.
  - What if player 2 were able to make a counter offer, though?
- Now, instead of both players receiving zero if player 2 rejects player 1's offer, player 2 now gets to pick some proportion y of the pie to offer to player 1.
  - If player 1 accepts the offer, both players receive payoffs of y and $1 - y$, respectively.
  - If player 1 rejects the offer, both players receive a payoff of 0.

Bargaining


Player 1
0
1
Player 2
Accept
Reject
1 - x
x
Player 2
0
1
Player 1
Accept
Reject
y
1 - y
0
0

Bargaining

- Let's perform backward induction,
  - Starting with player 1's final choice, it will accept player 2's offer if

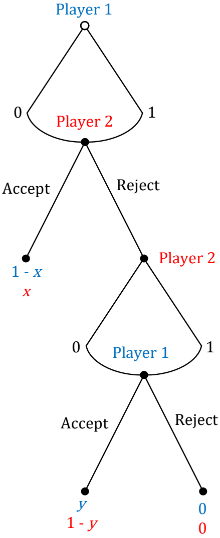
$$\begin{aligned} \text{Payoff from Accept} &\geq \text{Payoff from Reject} \\ y &\geq 0 \end{aligned}$$

  - And moving this up the tree, player 2's payoff will be

$$\begin{aligned} 1 - y \text{ if } y &\geq 0 \\ 0 \text{ if } y &< 0 \end{aligned}$$

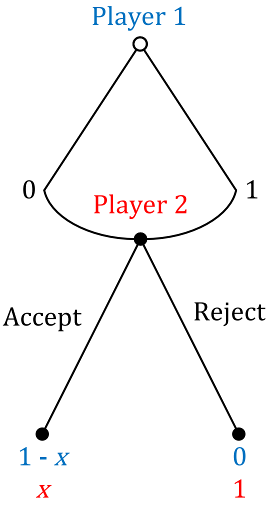
    so naturally, Player 2 will offer player 1 no pie and keep it all for himself. Player 2 receives a payoff of 1 while player 1 receives a payoff of zero.
  - This is the same result from our single round of bargaining.
- Now, we can move this result up the tree.

Bargaining



Player 1
0
1
Player 2
Accept
Reject
1 - x
x
Player 2
0
1
Player 1
Accept
Reject
y
1 - y
0
0

Bargaining



Bargaining

- Now, continuing with backward induction,
  - Player 2 will accept player 1's initial offer if

$$\text{Payoff from Accept} \geq \text{Payoff from Reject}$$
$$x \geq 1$$

  - Which leaves player 1's payoff as

$$1 - x \text{ if } x \geq 1$$
$$0 \text{ if } x < 1$$

- This works out bad for player 1. Since player 2 knows they can get the whole pie for themself in the second round, they will reject any offer that is less than the whole pie in the first round.
  - Thus, in equilibrium, player 1 has to give the whole pie to player 1 in the first round.

Bargaining

- A two round bargaining game like this is an example of second mover's advantage.
  - Since player 2 had the benefit of getting to make the last offer, they got to reap the spoils.
- In the real world, remember that most things can be bargained for, like wages, capital purchases, etc.
  - Bargaining is becoming a lost art.
- If someone is making you an offer, they are offering you the smallest value that they think you will accept.
  - Think about this when it comes to being offered a wage from your job and remember that you are worth more than their original offer.
  - They likely have a higher wage they are willing to offer you.

Summary

- Sequential move games allow players to take turns, using oberservations from previous rounds of the game to their advantage (or disadvantage).
    - It also solves the problem of multiple equilibria from the simultaneous move game.
- Bargaining is important in life. Be comfortable making counter offers to people!

Next Time

- Cournot Competition. What happens when two firms compete in quantities?
- Reading: 7.3

Homework 3-5

- Return to our two-period bargaining problem we covered today. Suppose now that both players are impatient. If player 1's initial offer is rejected, the payoffs that both players receive in the second round of bargaining (where player 2 makes an offer to player 1) are discounted by δ, where $0 < \delta < 1$ is the discount factor that we studied before.
  1. In the second round, how much of the pie does player 2 offer to player 1?
  2. In the first round, as a function of δ, how much pie does player 1 offer player 2?
  3. As player 2 becomes more patient (i.e., as δ increases), does the intial offer of pie they receive increase? Why?