Edexcel
GCE A Level Maths
Further Maths 3
Matrices.

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6 Matrices

Basic definitions

Dimension of a matrix

A matrix with r rows and c columns has *dimension* $r \times c$.

Transpose and symmetric matrices

The *transpose*, $\boldsymbol{A}^T$, of a matrix, $\boldsymbol{A}$, is found by interchanging rows and columns

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{A}^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

$(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$ - note the change of order of A and B.

A matrix, $\boldsymbol{S}$, is *symmetric* if the elements are symmetrically placed about the leading diagonal,

or if $\boldsymbol{S} = \boldsymbol{S}^T$.

Thus, $\boldsymbol{S} = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$ is a symmetric matrix.

Identity and zero matrices

The *identity* matrix $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

and the *zero* matrix is $\boldsymbol{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Determinant of a 3 × 3 matrix

The *determinant* of a 3×3 matrix, $\boldsymbol{A}$, is

$$\det(\boldsymbol{A}) = \Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\Rightarrow \Delta = aei - afh - bdi + bfg + cdh - ceg$$

Properties of the determinant

1) A determinant can be expanded by any row or column using $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

e.g $\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - f\begin{vmatrix} a & b \\ g & h \end{vmatrix}$ using the middle row and leaving the value unchanged

2) Interchanging two rows changes the sign of the determinant

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$ which can be shown by evaluating both determinants

3) A determinant with two identical rows (or columns) has value 0.

$\Delta = \begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix}$ interchanging the two identical rows gives $\Delta = -\Delta \Rightarrow \Delta = 0$

4) $\det(\boldsymbol{AB}) = \det(\boldsymbol{A}) \times \det(\boldsymbol{B})$ this can be shown by multiplying out

Singular and non-singular matrices

A matrix, $\boldsymbol{A}$, is *singular* if its determinant is zero, $\det(\boldsymbol{A}) = 0$

A matrix, $\boldsymbol{A}$, is *non-singular* if its determinant is not zero, $\det(\boldsymbol{A}) \neq 0$

Inverse of a 3×3 matrix

This is tedious, but no reason to make a mistake if you are careful.

Cofactors

In $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ the cofactors of $a, b, c,$ etc. are A, B, C etc., where

$A = +\begin{vmatrix} e & f \\ h & i \end{vmatrix}$, $B = -\begin{vmatrix} d & f \\ g & i \end{vmatrix}$, $C = +\begin{vmatrix} d & e \\ g & h \end{vmatrix}$,

$D = -\begin{vmatrix} b & c \\ h & i \end{vmatrix}$, $E = +\begin{vmatrix} a & c \\ g & i \end{vmatrix}$ $F = -\begin{vmatrix} a & b \\ g & h \end{vmatrix}$,

$G = +\begin{vmatrix} b & c \\ e & f \end{vmatrix}$, $H = -\begin{vmatrix} a & c \\ d & f \end{vmatrix}$, $I = +\begin{vmatrix} a & b \\ d & e \end{vmatrix}$

These are the 2×2 matrices used in finding the determinant, together with the correct sign from $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Finding the inverse

1) Find the determinant, $\det(\boldsymbol{A})$.
   If $\det(\boldsymbol{A}) = 0$, then $\boldsymbol{A}$ is singular and has no inverse.

2) Find the matrix of cofactors $\boldsymbol{C} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$

3) Find the transpose of $\boldsymbol{C}$, $\boldsymbol{C}^T = \begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$

4) Divide $\boldsymbol{C}^T$ by $\det(\boldsymbol{A})$ to give $\boldsymbol{A}^{-1} = \dfrac{1}{\det(\boldsymbol{A})}\begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$

   See example 10 on page 148.

Properties of the inverse

1) $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$

2) $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ - note the change of order of A and B.

   Proof $(\boldsymbol{AB})^{-1}\boldsymbol{AB} = \boldsymbol{I}$ from definition of inverse

   $\Rightarrow \quad (\boldsymbol{AB})^{-1}\boldsymbol{AB}(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) = \boldsymbol{I}(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})$

   $\Rightarrow \quad (\boldsymbol{AB})^{-1}\boldsymbol{A}(\boldsymbol{BB}^{-1})\boldsymbol{A}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ $\qquad \Rightarrow \quad (\boldsymbol{AB})^{-1}\boldsymbol{A}\,\boldsymbol{I}\,\boldsymbol{A}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

   $\Rightarrow \quad (\boldsymbol{AB})^{-1}\boldsymbol{A}\,\boldsymbol{A}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ $\qquad \Rightarrow \quad (\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

3) $\det(\boldsymbol{A}^{-1}) = \dfrac{1}{\det(\boldsymbol{A})}$

Matrices and linear transformations

Linear transformations

$\boldsymbol{T}$ is a linear transformation on a set of vectors if

(i) $\quad \boldsymbol{T}(\underline{\boldsymbol{x}}_1 + \underline{\boldsymbol{x}}_2) = \boldsymbol{T}(\underline{\boldsymbol{x}}_1) + \boldsymbol{T}(\underline{\boldsymbol{x}}_2)$ $\qquad$ for all vectors $\underline{\boldsymbol{x}}$ and $\underline{\boldsymbol{y}}$

(ii) $\quad \boldsymbol{T}(k\underline{\boldsymbol{x}}) = k\boldsymbol{T}(\underline{\boldsymbol{x}})$ $\qquad$ for all vectors $\underline{\boldsymbol{x}}$

Example: Show that $\boldsymbol{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ x+y \\ -z \end{pmatrix}$ is a linear transformation.

Solution: (i)

$$\boldsymbol{T}(\underline{\boldsymbol{x}}_1 + \underline{\boldsymbol{x}}_2) = \boldsymbol{T}\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = \boldsymbol{T}\left(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 2(x_1+x_2) \\ x_1+x_2+y_1+y_2 \\ -z_1-z_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_1+y_1 \\ -z_1 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ x_2+y_2 \\ -z_2 \end{pmatrix} = \boldsymbol{T}(\underline{\boldsymbol{x}}_1) + \boldsymbol{T}(\underline{\boldsymbol{x}}_2)$$

$$\Rightarrow \quad \boldsymbol{T}(\underline{\boldsymbol{x}}_1 + \underline{\boldsymbol{x}}_2) = \boldsymbol{T}(\underline{\boldsymbol{x}}_1) + \boldsymbol{T}(\underline{\boldsymbol{x}}_2)$$

(ii)

$$\boldsymbol{T}(k\underline{\boldsymbol{x}}) = \boldsymbol{T}\left(k\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \boldsymbol{T}\begin{pmatrix} kx \\ ky \\ kz \end{pmatrix} = \begin{pmatrix} 2kx \\ kx+ky \\ -kz \end{pmatrix} = k\begin{pmatrix} 2x_1 \\ x_1+y_1 \\ -z_1 \end{pmatrix} = k\boldsymbol{T}(\underline{\boldsymbol{x}})$$

$$\Rightarrow \quad \boldsymbol{T}(k\underline{\boldsymbol{x}}) = k\boldsymbol{T}(\underline{\boldsymbol{x}})$$

Both (i) and (ii) are satisfied, and so $\boldsymbol{T}$ is a linear transformation.

All matrices can represent linear transformations.

Base vectors $\underline{\boldsymbol{i}}$, $\underline{\boldsymbol{j}}$, $\underline{\boldsymbol{k}}$

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{\boldsymbol{j}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{\boldsymbol{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Under the transformation with matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} a \\ d \\ g \end{pmatrix}$ the first column of the matrix

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} b \\ e \\ h \end{pmatrix}$ the second column of the matrix

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} c \\ f \\ i \end{pmatrix}$ the third column of the matrix

This is an important result, as it allows us to find the matrix for given transformations.

Example: Find the matrix for a reflection in the plane $y = x$

Solution: The z-axis lies in the plane $y = x$ so $\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\Rightarrow$ the third column of the matrix is $\begin{pmatrix}0\\0\\1\end{pmatrix}$

Also $\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\Rightarrow$ the first column of the matrix is $\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\Rightarrow$ the second column of the matrix is $\begin{pmatrix}1\\0\\0\end{pmatrix}$

$\Rightarrow$ the matrix for a reflection in $y = x$ is $\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}$.

Example: Find the matrix of the linear transformation, $\boldsymbol{T}$, which maps $(1, 0, 0) \to (3, 4, 2)$, $(1, 1, 0) \to (6, 1, 5)$ and $(2, 1, -4) \to (1, 1, -1)$.

Solution:

Firstly $\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}3\\4\\2\end{pmatrix}$ $\Rightarrow$ first column is $\begin{pmatrix}3\\4\\2\end{pmatrix}$

Secondly $\begin{pmatrix}1\\1\\0\end{pmatrix} \to \begin{pmatrix}6\\1\\5\end{pmatrix}$ but $\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}3\\4\\2\end{pmatrix} + \boldsymbol{T}\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\Rightarrow$ $\boldsymbol{T}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}6\\1\\5\end{pmatrix} - \begin{pmatrix}3\\4\\2\end{pmatrix} = \begin{pmatrix}3\\-3\\3\end{pmatrix}$ $\Rightarrow$ second column is $\begin{pmatrix}3\\-3\\3\end{pmatrix}$

Thirdly $\begin{pmatrix}2\\1\\-4\end{pmatrix} \to \begin{pmatrix}1\\1\\-1\end{pmatrix}$

but $\begin{pmatrix}2\\1\\-4\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} - 4\begin{pmatrix}0\\0\\1\end{pmatrix} \to 2\begin{pmatrix}3\\4\\2\end{pmatrix} + \begin{pmatrix}3\\-3\\3\end{pmatrix} - 4\boldsymbol{T}\begin{pmatrix}0\\0\\1\end{pmatrix}$

$\Rightarrow$ $2\begin{pmatrix}3\\4\\2\end{pmatrix} + \begin{pmatrix}3\\-3\\3\end{pmatrix} - 4\boldsymbol{T}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$

$\Rightarrow$ $\boldsymbol{T}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\2\end{pmatrix}$ $\Rightarrow$ third column is $\begin{pmatrix}2\\1\\2\end{pmatrix}$

$\Rightarrow$ $\boldsymbol{T} = \begin{pmatrix}3 & 3 & 2\\4 & -3 & 1\\2 & 3 & 2\end{pmatrix}$.

Image of a line

Example: Find the image of the line $\underline{r} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ under $\boldsymbol{T}$,

where $\boldsymbol{T} = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 3 & 4 \\ 2 & -1 & 1 \end{pmatrix}$.

Solution: As $\boldsymbol{T}$ is a linear transformation, we can find

$$\boldsymbol{T}(\underline{r}) = \boldsymbol{T}\left(\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}\right) = \boldsymbol{T}\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \boldsymbol{T}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{T}(\underline{r}) = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 3 & 4 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 & -2 & 1 \\ 1 & 3 & 4 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$\Rightarrow \quad \boldsymbol{T}(\underline{r}) = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 14 \\ 1 \\ 9 \end{pmatrix}$ and so the vector equation of the new line is

$$\underline{r} = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 14 \\ 1 \\ 9 \end{pmatrix}.$$

Image of a plane 1

Similarly the image of a plane $\underline{r} = \underline{a} + \lambda \underline{b} + \mu \underline{c}$, under a linear transformation, $\boldsymbol{T}$, is

$$\boldsymbol{T}(\underline{r}) = \boldsymbol{T}(\underline{a} + \lambda \underline{b} + \mu \underline{c}) = \boldsymbol{T}(\underline{a}) + \lambda \boldsymbol{T}(\underline{b}) + \mu \boldsymbol{T}(\underline{c}).$$

Image of a plane 2

To find the image of a plane with equation of the form $ax + by + cz = d$, first construct a vector equation.

Example: Find the image of the plane $3x - 2y + 4z = 7$ under a linear transformation, $\boldsymbol{T}$.

Solution: To construct a vector equation, put $x = \lambda$, $y = \mu$ and find z in terms of λ and μ.

$$\Rightarrow \quad 3\lambda - 2\mu + 4z = 7 \quad \Rightarrow \quad z = \frac{7-3\lambda+2\mu}{4}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda \\ \mu \\ \frac{7-3\lambda+2\mu}{4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 7/4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -3/4 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1/2 \end{pmatrix}$$

$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 7/4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ making the numbers nicer in the 'parallel' vectors

and now continue as for in image for a plane 1.

7 Eigenvalues and eigenvectors

Definitions

1) An *eigenvector* of a linear transformation, $\boldsymbol{T}$, is a non-zero vector whose *direction* is unchanged by $\boldsymbol{T}$.

   So, if $\underline{e}$ is an eigenvector of $\boldsymbol{T}$ then its image $\underline{e}'$ is parallel to $\underline{e}$, or $\underline{e}' = \lambda\, \underline{e}$

   $\Rightarrow \quad \underline{e}' = \boldsymbol{T}(\underline{e}) = \lambda\, \underline{e}$.

   $\underline{e}$ defines a line which maps onto itself and so is invariant *as a whole line*.

   If $\lambda = 1$ each point on the line remains in the same place, and we have a line of *invariant points*.

2) The *characteristic equation* of a matrix $\boldsymbol{A}$ is $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$

   $\boldsymbol{A}\underline{e} = \lambda\, \underline{e}$

   $\Rightarrow \quad (\boldsymbol{A} - \lambda \boldsymbol{I})\, \underline{e} = \underline{0}$ has non-zero solutions   eigenvectors are non-zero

   $\Rightarrow \quad \boldsymbol{A} - \lambda \boldsymbol{I}$ is a singular matrix

   $\Rightarrow \quad \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$

   $\Rightarrow$ the solutions of the characteristic equation are the eigenvalues.

2 × 2 matrices

Example: Find the eigenvalues and eigenvectors for the transformation with matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}.$$

Solution: The characteristic equation is $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$

$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (1-\lambda)(4-\lambda) + 2 = 0$

$\Rightarrow \quad \lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad \lambda_1 = 2$ and $\lambda_2 = 3$

For $\lambda_1 = 2$

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow \quad x + y = 2x \quad \Rightarrow \quad x = y$

and $\quad -2x + 4y = 2y \quad \Rightarrow \quad x = y$

$\Rightarrow$ eigenvector $\underline{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we could use $\begin{pmatrix} 3.7 \\ 3.7 \end{pmatrix}$, but why make things nasty

For $\lambda_2 = 3$

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow \quad x + y = 3x \quad \Rightarrow \quad 2x = y$

and $\quad -2x + 4y = 3y \quad \Rightarrow \quad 2x = y$

$\Rightarrow$ eigenvector $\underline{e}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ choosing easy numbers.

Orthogonal matrices

Normalised eigenvectors

A normalised eigenvector is an eigenvector of length 1.

In the above example, the normalized eigenvectors are $\underline{e}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, and $\underline{e}_2 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$.

Orthogonal vectors

A posh way of saying perpendicular, scalar product will be zero.

Orthogonal matrices

If the columns of a matrix form vectors which are

(i) mutually orthogonal (or perpendicular)
(ii) each of length 1

then the matrix is an *orthogonal* matrix.

Example:

$\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$ and $\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$ are both unit vectors, and

$\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \bullet \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \frac{-2}{5} + \frac{2}{5} = 0, \Rightarrow$ the vectors are orthogonal

$$\Rightarrow \quad \boldsymbol{M} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \quad \text{is an orthogonal matrix}$$

Notice that

$$\boldsymbol{M}^T\boldsymbol{M} = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and so ***the transpose of an orthogonal matrix is also its inverse***.

This is true for **all** orthogonal matrices   think of any set of perpendicular unit vectors

Another definition of an orthogonal matrix is

$\boldsymbol{M}$ is orthogonal $\quad \Leftrightarrow \quad \boldsymbol{M}^T\boldsymbol{M} = \boldsymbol{I} \quad \Leftrightarrow \quad \boldsymbol{M}^{-1} = \boldsymbol{M}^T$

Diagonalising a 2 × 2 matrix

Let $\boldsymbol{A}$ be a 2×2 matrix with eigenvalues λ_1 and λ_2,
and eigenvectors $\quad \underline{\boldsymbol{e}}_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and $\underline{\boldsymbol{e}}_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$

then $\boldsymbol{A}\,\underline{\boldsymbol{e}}_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 u_1 \\ \lambda_1 v_1 \end{pmatrix}$ and $\boldsymbol{A}\,\underline{\boldsymbol{e}}_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 u_2 \\ \lambda_2 v_2 \end{pmatrix}$

$$\Rightarrow \quad \boldsymbol{A}\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 u_1 & \lambda_2 u_2 \\ \lambda_1 v_1 & \lambda_2 v_2 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \ldots\ldots\ldots \boxed{\mathbf{I}}$$

Define $\boldsymbol{P}$ as the matrix whose columns are eigenvectors of $\boldsymbol{A}$, and $\boldsymbol{D}$ as the diagonal matrix, whose entries are the eigenvalues of $\boldsymbol{A}$

$$\boxed{\mathbf{I}} \quad \Rightarrow \quad \boldsymbol{P} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{AP} = \boldsymbol{PD} \quad \Rightarrow \quad \boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{D}$$

The above is the general case for diagonalising **any** matrix.

In this course we consider only diagonalising symmetric matrices.

Diagonalising 2 × 2 symmetric matrices

Eigenvectors of symmetric matrices

Preliminary result:

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The scalar product $\underline{x} \cdot \underline{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_1 + x_2 y_2$

but $(x_1 \;\; x_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_1 + x_2 y_2$

$$\Rightarrow \qquad \underline{x}^T \, \underline{y} = \underline{x} \cdot \underline{y}$$

This result allows us to use matrix multiplication for the scalar product.

Theorem: Eigenvectors, for different eigenvalues, of a symmetric matrix are orthogonal.

Proof: Let $\boldsymbol{A}$ be a symmetric matrix, then $\boldsymbol{A}^T = \boldsymbol{A}$

Let $\boldsymbol{A}\,\underline{e}_1 = \lambda_1 \underline{e}_1$, and $\boldsymbol{A}\,\underline{e}_2 = \lambda_2 \underline{e}_2$, $\qquad \lambda_1 \neq \lambda_2$.

$\lambda_1 \underline{e}_1{}^T = (\lambda_1 \underline{e}_1)^T = (\boldsymbol{A}\,\underline{e}_1)^T = \underline{e}_1{}^T \boldsymbol{A}^T = \underline{e}_1{}^T \boldsymbol{A} \qquad$ since $\boldsymbol{A}^T = \boldsymbol{A}$

$\Rightarrow \quad \lambda_1 \underline{e}_1{}^T = \underline{e}_1{}^T \boldsymbol{A}$

$\Rightarrow \quad \lambda_1 \underline{e}_1{}^T \underline{e}_2 = \underline{e}_1{}^T \boldsymbol{A}\,\underline{e}_2 = \underline{e}_1{}^T \lambda_2 \underline{e}_2 = \lambda_2 \underline{e}_1{}^T \underline{e}_2$

$\Rightarrow \quad \lambda_1 \underline{e}_1{}^T \underline{e}_2 = \lambda_2 \underline{e}_1{}^T \underline{e}_2$

$\Rightarrow \quad (\lambda_1 - \lambda_2)\, \underline{e}_1{}^T \underline{e}_2 = \underline{0}$

But $\quad \lambda_1 - \lambda_2 \neq 0 \quad \Rightarrow \quad \underline{e}_1{}^T \underline{e}_2 = \underline{0} \quad \Leftrightarrow \quad \underline{e}_1 \cdot \underline{e}_2 = 0$

$\Rightarrow$ the eigenvectors are orthogonal $\qquad$ or perpendicular

Diagonalising a symmetric matrix

The above theorem makes diagonalising a symmetric matrix, $\boldsymbol{A}$, easy.

1) Find eigenvalues, λ_1 and λ_2, and eigenvectors, $\underline{e}_1$ and $\underline{e}_2$

2) Normalise the eigenvectors, to give $\hat{\underline{e}}_1$ and $\hat{\underline{e}}_2$.

3) Write down the matrix $\boldsymbol{P}$ with $\hat{\underline{e}}_1$ and $\hat{\underline{e}}_2$ as columns.
$\boldsymbol{P}$ will now be an orthogonal matrix since $\hat{\underline{e}}_1$ and $\hat{\underline{e}}_2$ are orthogonal
$\Rightarrow \quad \boldsymbol{P}^{-1} = \boldsymbol{P}^T$

4) $\boldsymbol{P}^T \boldsymbol{A}\,\boldsymbol{P}$ will be the diagonal matrix $\boldsymbol{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Example: Diagonalise the symmetric matrix $\boldsymbol{A} = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}$.

Solution: The characteristic equation is $\begin{vmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (6-\lambda)(9-\lambda) - 4 = 0$

$\Rightarrow \quad \lambda^2 - 15\lambda + 50 = 0 \quad \Rightarrow \quad (\lambda-5)(\lambda-10) = 0$

$\Rightarrow \quad \lambda = 5 \text{ or } 10$

For $\lambda_1 = 5$

$$\begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow \quad 6x - 2y = 5x \quad \Rightarrow \quad x = 2y$

and $\quad -2x + 9y = 5y \quad \Rightarrow \quad x = 2y$

$\Rightarrow \quad \underline{\boldsymbol{e}}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

and normalising $\Rightarrow \quad \underline{\hat{\boldsymbol{e}}}_1 = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

For $\lambda_2 = 10$

$$\begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 10\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow \quad 6x - 2y = 10x \quad \Rightarrow \quad -2x = y$

and $\quad -2x + 9y = 10y \quad \Rightarrow \quad -2x = y$

$\Rightarrow \quad \underline{\boldsymbol{e}}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

and normalising $\Rightarrow \quad \underline{\hat{\boldsymbol{e}}}_2 = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$

Notice that the eigenvectors are orthogonal

$$\Rightarrow \quad \boldsymbol{P} = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{D} = \boldsymbol{P}^T\boldsymbol{A}\,\boldsymbol{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}.$$

3×3 matrices

All the results for 2×2 matrices are also true for 3×3 matrices (or $n \times n$ matrices). The proofs are either the same, or similar in a higher number of dimensions.

The techniques are shown in the example for diagonalising a 3×3 symmetric matrix.

Diagonalising 3×3 symmetric matrices

Example: $\boldsymbol{A} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}$.

Find an orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^T\boldsymbol{AP}$ is a diagonal matrix.

Solution:

1) Find eigenvalues

The characteristic equation is $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & -2 & 0 \\ -2 & 1-\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)[-\lambda(1-\lambda) - 4] + 2 \times [2\lambda - 0] + 0 = 0$$

$$\Rightarrow \quad \lambda^3 - 3\lambda^2 - 6\lambda + 8 = 0$$

By inspection $\lambda = -2$ is a root $\Rightarrow$ $(\lambda + 2)$ is a factor

$$\Rightarrow \quad (\lambda+2)(\lambda^2 - 5\lambda + 4) = 0$$

$$\Rightarrow \quad (\lambda+2)(\lambda-1)(\lambda-4) = 0$$

$$\Rightarrow \quad \lambda = -2,\ 1 \text{ or } 4.$$

2) Find normalized eigenvectors

$$\lambda_1 = -2 \quad \Rightarrow \quad \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2x - 2y &= -2x && \textbf{I} \\ -2x + y + 2z &= -2y && \textbf{II} \\ 2y &= -2z && \textbf{III} \end{aligned}$$

I $\Rightarrow$ $y = 2x$, and **III** $\Rightarrow$ $y = -z$ choose $x = 1$ and find y and z

$$\Rightarrow \quad \underline{e}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad \text{and} \quad |\underline{e}_1| = e_1 = \sqrt{9} = 3 \quad \Rightarrow \quad \hat{\underline{e}}_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

$$\lambda_2 = 1 \quad \Rightarrow \quad \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2x - 2y &= x && \mathbf{I} \\ -2x + y + 2z &= y && \mathbf{II} \\ 2y &= z && \mathbf{III} \end{aligned}$$

$\mathbf{I} \Rightarrow x = 2y$, and $\mathbf{II} \Rightarrow z = 2y$ choose $y = 1$ and find x and z

$$\Rightarrow \quad \underline{e}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad |\underline{e}_2| = e_2 = \sqrt{9} = 3$$

$$\Rightarrow \quad \underline{\hat{e}}_2 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

$$\lambda_3 = 4 \quad \Rightarrow \quad \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2x - 2y &= 4x && \mathbf{I} \\ -2x + y + 2z &= 4y && \mathbf{II} \\ 2y &= 4z && \mathbf{III} \end{aligned}$$

$\mathbf{I} \Rightarrow x = -y$, and $\mathbf{III} \Rightarrow y = 2z$ choose $z = 1$ and find x and y

$$\Rightarrow \quad \underline{e}_3 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad |\underline{e}_3| = e_3 = \sqrt{9} = 3$$

$$\Rightarrow \quad \underline{\hat{e}}_3 = \begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

3) Find orthogonal matrix, P

$$\Rightarrow \quad \boldsymbol{P} = (\underline{\hat{e}}_1 \quad \underline{\hat{e}}_2 \quad \underline{\hat{e}}_3)$$

$$\Rightarrow \quad \boldsymbol{P} = \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{pmatrix}$$ is required orthogonal matrix

4) Find diagonal matrix, D

$$\Rightarrow \quad \boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

A nice long question! But, although you will not be asked to do a complete problem, the examiners can test every step above!

1. $$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}.$$

(*a*) Verify that $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of **A** and find the corresponding eigenvalue.

(3)

(*b*) Show that 9 is another eigenvalue of **A** and find the corresponding eigenvector.

(5)

(*c*) Given that the third eigenvector of **A** is $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, write down a matrix **P** and a diagonal matrix **D** such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}.$$

(5)

[P6 June 2002 Qn 5]

2. $$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}$$

(*a*) Find the eigenvalues of **M**.

(4)

A transformation T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is represented by the matrix **M**. There is a line through the origin for which every point on the line is mapped onto itself under T.

(*b*) Find a cartesian equation of this line.

(3)

[P6 June 2003 Qn 3]

3. $$\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 5 & 3 & u \end{pmatrix}, \quad u \neq 1.$$

(*a*) Show that det $\mathbf{A} = 2(u - 1)$.

(2)

(*b*) Find the inverse of $\mathbf{A}$.

(6)

The image of the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ when transformed by the matrix $\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 5 & 3 & 6 \end{pmatrix}$ is $\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$.

(*c*) Find the values of a, b and c.

(3)

[P6 June 2003 Qn 6]

4. The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & p \\ a & b & c \end{pmatrix},$$

where p, a, b and c are constants and $a > 0$.

Given that $\mathbf{MM}^{\mathrm{T}} = k\mathbf{I}$ for some constant k, find

(*a*) the value of p,

(2)

(*b*) the value of k,

(2)

(*c*) the values of a, b and c,

(6)

(*d*) $|\det \mathbf{M}|$.

(2)

[P6 June 2004 Qn 5]

5. The transformation R is represented by the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

(*a*) Find the eigenvectors of **A**.

(5)

(*b*) Find an orthogonal matrix **P** and a diagonal matrix **D** such that

$$\mathbf{A} = \mathbf{PDP}^{-1}.$$

(5)

(*c*) Hence describe the transformation R as a combination of geometrical transformations, stating clearly their order.

(4)

[P6 June 2004 Qn 6]

6.

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & k \end{pmatrix}.$$

(*a*) Show that $\det \mathbf{A} = 20 - 4k$.

(2)

(*b*) Find $\mathbf{A}^{-1}$.

(6)

Given that $k = 3$ and that $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ is an eigenvector of **A**,

(*c*) find the corresponding eigenvalue.

(2)

Given that the only other distinct eigenvalue of **A** is 8,

(*d*) find a corresponding eigenvector.

(4)

[FP3/P6 June 2005 Qn 7]

7. A transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

Find

(*a*) the two eigenvalues of **A**,

(4)

(*b*) a cartesian equation for each of the two lines passing through the origin which are invariant under *T*.

(3)

[*FP3/P6 January 2006 Qn 3]

8. $$\mathbf{A} = \begin{pmatrix} k & 1 & -2 \\ 0 & -1 & k \\ 9 & 1 & 0 \end{pmatrix}, \text{ where } k \text{ is a real constant.}$$

(*a*) Find values of *k* for which **A** is singular.

(4)

Given that **A** is non-singular,

(*b*) find, in terms of *k*, $\mathbf{A}^{-1}$.

(5)

[FP3/P6 January 2006 Qn 4]

9. $$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove by induction, that for all positive integers *n*,

$$\mathbf{A}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2 + 3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

(5)

[FP3 June 2006 Qn 1]

10. The eigenvalues of the matrix **M**, where

$$\mathbf{M} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix},$$

are λ_1 and λ_2, where $\lambda_1 < \lambda_2$.

(*a*) Find the value of λ_1 and the value of λ_2.

(3)

(*b*) Find $\mathbf{M}^{-1}$.

(2)

(*c*) Verify that the eigenvalues of $\mathbf{M}^{-1}$ are λ_1^{-1} and λ_2^{-1}.

(3)

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix **M**. There are two lines, passing through the origin, each of which is mapped onto itself under the transformation T.

(*d*) Find cartesian equations for each of these lines.

(4)

[FP3 June 2006 Qn 5]

11. Given that $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix},$$

(*a*) find the eigenvalue of **A** corresponding to $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$,

(2)

(*b*) find the value of p and the value of q.

(4)

The image of the vector $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ when transformed by **A** is $\begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$.

(*c*) Using the values of p and q from part (*b*), find the values of the constants l, m and n.

(4)

[FP3 June 2007 Qn 3]

12.

$$\mathbf{M} = \begin{pmatrix} 1 & p & 2 \\ 0 & 3 & q \\ 2 & p & 1 \end{pmatrix},$$

where p and q are constants.

Given that $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of **M**,

(a) show that $q = 4p$.

(3)

Given also that $\lambda = 5$ is an eigenvalue of **M**, and $p < 0$ and $q < 0$, find

(b) the values of p and q,

(4)

(c) an eigenvector corresponding to the eigenvalue $\lambda = 5$.

(3)

[FP3 June 2008 Qn 2]

13.

$$\mathbf{M} = \begin{pmatrix} 6 & 1 & -1 \\ 0 & 7 & 0 \\ 3 & -1 & 2 \end{pmatrix}$$

(a) Show that 7 is an eigenvalue of the matrix **M** and find the other two eigenvalues of **M**.

(5)

(b) Find an eigenvector corresponding to the eigenvalue 7.

(4)

[FP3 June 2009 Qn 3]

14.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ k & 0 & 1 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

Given that $\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$ is an eigenvector of **M**,

(a) find the eigenvalue of **M** corresponding to $\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$,

(2)

(b) show that $k = 3$,

(2)

(c) show that **M** has exactly two eigenvalues.

(4)

A transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by **M**.

The transformation T maps the line l_1, with cartesian equations $\dfrac{x-2}{1} = \dfrac{y}{-3} = \dfrac{z+1}{4}$, onto the line l_2.

(*d*) Taking $k = 3$, find cartesian equations of l_2.

(5)

[FP3 June 2010 Qn 6]

15. The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} k & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}, \quad k \neq 1.$$

(*a*) Show that det $\mathbf{M} = 2 - 2k$.

(2)

(*b*) Find $\mathbf{M}^{-1}$, in terms of k.

(5)

The straight line l_1 is mapped onto the straight line l_2 by the transformation represented by the matrix

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}$$

The equation of l_2 is $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$, where $\mathbf{a} = 4\mathbf{i} + \mathbf{j} + 7\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

(*c*) Find a vector equation for the line l_1.

(5)

[FP3 June 2011 Qn 7]

16. The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}.$$

(*a*) Show that 4 is an eigenvalue of **M**, and find the other two eigenvalues.

(5)

(*b*) For the eigenvalue 4, find a corresponding eigenvector.

(3)

The straight line l_1 is mapped onto the straight line l_2 by the transformation represented by the matrix **M**.
The equation of l_1 is $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$, where $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

(*c*) Find a vector equation for the line l_2.

(5)

[FP3 June 2012 Qn 8]

17. It is given that $\begin{pmatrix}1\\2\\0\end{pmatrix}$ is an eigenvector of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix}4 & 2 & 3\\ 2 & b & 0\\ a & 1 & 8\end{pmatrix}$$

and a and b are constants.

(a) Find the eigenvalue of **A** corresponding to the eigenvector $\begin{pmatrix}1\\2\\0\end{pmatrix}$.

(3)

(b) Find the values of a and b.

(3)

(c) Find the other eigenvalues of **A**.

(5)

[FP3 June 2013_R Qn 6]

18. The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix}1 & 1 & a\\ 2 & b & c\\ -1 & 0 & 1\end{pmatrix}, \text{ where } a, b \text{ and } c \text{ are constants.}$$

(a) Given that $\mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{k}$ are two of the eigenvectors of **M**,

find

(i) the values of a, b and c,

(ii) the eigenvalues which correspond to the two given eigenvectors. **(8)**

(b) The matrix **P** is given by

$$\mathbf{P} = \begin{pmatrix}1 & 1 & 0\\ 2 & 1 & d\\ -1 & 0 & 1\end{pmatrix}, \text{ where } d \text{ is constant, } d \neq -1$$

Find

(i) the determinant of **P** in terms of d,

(ii) the matrix $\mathbf{P}^{-1}$ in terms of d. **(5)**

[FP3 June 2013 Qn 5]

19. The symmetric matrix **M** has eigenvectors $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

with eigenvalues 5, 2 and –1 respectively.

(*a*) Find an orthogonal matrix **P** and a diagonal matrix **D** such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{MP} = \mathbf{D}$$

(4)

Given that $\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$

(*b*) show that

$$\mathbf{M} = \mathbf{PDP}^{-1}$$

(2)

(*c*) Hence find the matrix **M**.

(5)

[FP3 June 2014_R Qn 6]

20.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix}$$

(*a*) Show that matrix **M** is not orthogonal.

(2)

(*b*) Using algebra, show that 1 is an eigenvalue of **M** and find the other two eigenvalues of **M**.

(5)

(*c*) Find an eigenvector of **M** which corresponds to the eigenvalue 1.

(2)

The transformation $M : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix **M**.

(*d*) Find a cartesian equation of the image, under this transformation, of the line

$$x = \frac{y}{2} = \frac{z}{-1}$$

(4)

[FP3 June 2014 Qn 2]

21. $$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

(*a*) Find the eigenvalues of **A**.

(5)

(*b*) Find a normalised eigenvector for each of the eigenvalues of **A**.

(5)

(*c*) Write down a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{D}$.

(2)

[FP3 June 2015 Qn 3]

22. $$\mathbf{A} = \begin{pmatrix} -2 & 1 & -3 \\ k & 1 & 3 \\ 2 & -1 & k \end{pmatrix}, \text{ where } k \text{ is a constant}$$

Given that the matrix **A** is singular, find the possible values of *k*. **(4)**

[FP3 June 2016 Qn 1]

23. $$\mathbf{M} = \begin{pmatrix} p & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & q \end{pmatrix},$$

where *p* and *q* are constants.

Given that $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix **M**,

(*a*) find the eigenvalue corresponding to this eigenvector,

(3)

(*b*) find the value of *p* and the value of *q*.

(3)

Given that 6 is another eigenvalue of **M**,
(*c*) find a corresponding eigenvector.

(2)

Given that $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ is a third eigenvector of **M** with eigenvalue 3,

(*d*) find a matrix **P** and a diagonal matrix **D** such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{MP} = \mathbf{D}.$$

(3)

[FP3 June 2016 Qn 6]

24. The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 1 & k & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}, \; k \in \mathbb{R}, \; k \neq \frac{1}{2}$$

(*a*) Show that det **M** = 1 – 2*k*.

(2)

(*b*) Find $\mathbf{M}^{-1}$ in terms of *k*.

(4)

The straight line l_1 is mapped onto the straight line l_2 by the transformation represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}$$

Given that l_2 has cartesian equation

$$\frac{x-1}{5} = \frac{y+2}{2} = \frac{z-3}{1}$$

(*c*) find a cartesian equation of the line l_1

(6)

[FP3 June 2017 Qn 6]

25. A non-singular matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 3 & k & 0 \\ k & 2 & 0 \\ k & 0 & 1 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

(*a*) Find, in terms of *k*, the inverse of the matrix **M**.

(5)

The point *A* is mapped onto the point (–5, 10, 7) by the transformation represented by the matrix

$$\begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(*b*) Find the coordinates of the point *A*.

(3)

[F3 IAL June 2014 Qn 4]

26.

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 9 \\ 1 & 4 & k \\ 1 & 0 & -3 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

Given that $\begin{pmatrix} 7 \\ 19 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix **M**,

(*a*) find the eigenvalue of **M** corresponding to $\begin{pmatrix} 7 \\ 19 \\ 1 \end{pmatrix}$,

(2)

(*b*) show that $k -= 7$,

(2)

(*c*) find the other two eignevalues of the matrix **M**.

(4)

The image of the vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ under the transformation represented by **M** is $\begin{pmatrix} -6 \\ 21 \\ 5 \end{pmatrix}$.

(*d*) Find the values of the constants p, q and r.

(4)

[F3 IAL June 2015 Qn 4]

27.

$$\mathbf{M} = \begin{pmatrix} 1 & k & 0 \\ -1 & 1 & 1 \\ 1 & k & 3 \end{pmatrix}, \text{ where } k \text{ is a constant}$$

(*a*) Find $\mathbf{M}^{-1}$ in terms of k.

(5)

Hence, given that $k = 0$

(*b*) find the matrix **N** such that

$$\mathbf{MN} = \begin{pmatrix} 3 & 5 & 6 \\ 4 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix}$$

(4)

[F3 IAL June 2016 Qn 4]

28.

$$\mathbf{A} = \begin{pmatrix} -1 & 3 & a \\ 2 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & 0 & 4 \\ 3 & -2 & 3 \\ 1 & 2 & b \end{pmatrix}$$

where a and b are constants.

(*a*) Write down $\mathbf{A}^{\mathrm{T}}$ in terms of a.

(1)

(*b*) Calculate **AB**, giving your answer in terms of a and b.

(2)

(*c*) Hence show that

$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

(3)

[F3 IAL June 2017 Qn 2]

29.

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

(*a*) Show that 6 is an eigenvalue of the matrix **M** and find the other two eigenvalues of **M**.

(4)

(*b*) Find a normalised eigenvector corresponding to the eigenvalue 6.

(4)

[F3 IAL June 2017 Qn 4]